Complex Numbers

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

September 24, 2018

Vector Space

A vector space V over a field F is a set on which two operations (addition and scalar multiplication) are defined. Some axioms need to be fulfilled, most relevantly closure with respect to addition and scalar multiplication:

- If $v, w \in V$, then $v + w \in V$
- If $a \in F, v \in V$, then $av \in V$

In this course, the field will always be $\mathbb{R}$ or $\mathbb{C}$, the real or the complex numbers.

Complex Numbers

The following set

$$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\} \tag{1}$$

is called the set of complex numbers. Note that $\mathbb{R} \subset \mathbb{C}$. Operations (addition, multiplication, and so on) are defined on complex numbers the same way as on real numbers with one additional rule:

$$i^2 = -1 \tag{2}$$

Complex Operators

Exercise 1: Find the determinant of the following matrix:

$$A = \begin{bmatrix} 1-4i & 3-i \\ -3i & 3+4i \end{bmatrix} \tag{3}$$

Exercise 2: A matrix that equals its conjugate transpose is called a Hermitian matrix. Calculate the determinate of the following example.

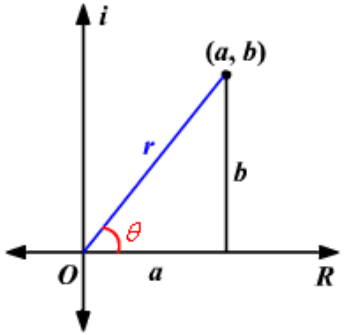
$$B = \begin{bmatrix} 2 & 2+i & 4 \\ 2-i & 3 & i \\ 4 & -i & 1 \end{bmatrix} \tag{4}$$

Exercise 3: Use expansion by conjugates to divide

$$\frac{7-2i}{3+4i} \tag{5}$$

Polar Form

The complex numbers correspond to vectors in $\mathbb{R}^2$.



Instead of providing the coordinates (a, b) of a complex number, it is sometimes useful to provide the polar form (r, θ).

$$\begin{array}{ll} a = r\cos\theta & b = r\sin\theta \\ r^2 = a^2 + b^2 & \tan\theta = \frac{b}{a} \end{array} \tag{6}$$

A complex number $a + bi$ can always be written in its polar form $a + bi = r(\cos\theta + i\sin\theta)$.

Euler's Formula

One of the most famous formulas in mathematics is Euler's formula

$$e^{ix} = \cos x + i \sin x \tag{7}$$

For the proof, we need some calculus. Recall the Maclaurin series expansions

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} \tag{8}$$

$$\cos x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!} \tag{9}$$

$$\sin x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} \tag{10}$$

Calculus still works in the complex numbers, now try to find e^{ix}.

Use of the Polar Form

Euler's formula makes multiplication, division, exponentiation and finding roots of complex numbers in polar form more simple.

Exercise 4: Multiply $(4, 60^\circ)$ by $(2, 20^\circ)$, where the given factors are complex numbers provided in polar form.

Exercise 5: Divide $(8, 100^\circ)$ by $(4, 65^\circ)$, where the given numbers are complex numbers provided in polar form.

Exercise 6: Find, using two alternative ways,

$$\frac{-2+5i}{-1-i} \text{ and } (2+3i)^5 \tag{11}$$

Closeted Functions

You may remember that I once called trigonometric functions "closeted exponential functions." Here is the reason. Consider

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned} \tag{12}$$

Add and subtract these two equations for

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{13}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{14}$$

This is the definition of trigonometric functions on $\mathbb{C}$. The hyperbolic trigonometric functions are closeted sines and cosines, since $\cosh(x) = \cos(ix)$ and $i \sinh(x) = \sin(ix)$.

de Moivre's Formula

de Moivre's Formula

$\left(re^{i\theta}\right)^n = r^n e^{in\theta}$

Example 1: Cube Roots. Find the solution set for the following equation and $c = (27, 120°)$, where c is a complex number provided in polar form.

$$x^3 = c \tag{15}$$

By de Moivre's formula, $x = (3, 40°)$ is a solution. However, $c = (27, 480°)$ and therefore, by de Moivre's formula again, $x = (3, 160°)$ is also a solution. $c = (27, 840°)$ provides the third solution, $x = (3, 280°)$. Polynomial equations of degree n usually have n solutions in $\mathbb{C}$.

Exponents and Square Roots

Example 2: Real Exponent. Find $x = (2 - 3i)^3$. Expanding the product gives us $-46 - 9i$. If we convert to polar form,

$$\left(\sqrt{13}\,(\cos(-56.31^\circ) + i\sin(-56.31^\circ))\right)^3 =$$

$$\left(\sqrt{13}\left(e^{i\cdot(-56.31^\circ)}\right)\right)^3 = 13^{\frac{3}{4}}\,(\cos(-168.93^\circ) + i\sin(-168.93^\circ)) =$$

$$-46 - 9i$$

Example 3: Square Root. Find $\sqrt{2-3i}$. Converting to polar form,

$$\left(\sqrt{13}\left(\cos(-56.31^\circ)+i\sin(-56.31^\circ)\right)\right)^{\frac{1}{2}} =$$

$$\left(\sqrt{13}\left(e^{i\cdot(-56.31^\circ)}\right)\right)^{\frac{1}{2}} = 13^{\frac{1}{4}}\left(\cos(-28.55^\circ)+i\sin(-28.55^\circ)\right) \approx$$

$$1.6741 - 0.89598i$$

However, notice that the last two lines could also be

$$\left(\sqrt{13}\left(e^{i\cdot(303.69^\circ)}\right)\right)^{\frac{1}{2}} = 13^{\frac{1}{4}}\left(\cos(151.85^\circ)+i\sin(151.85^\circ)\right) \approx$$

$$-1.6741 + 0.89598i$$

which is clearly a different number.

Example 4: Cube Roots of 27. One is easy to find: the number 3, multiplied by itself three times, gives us 27. Where are the other two cube roots of 27? Consider

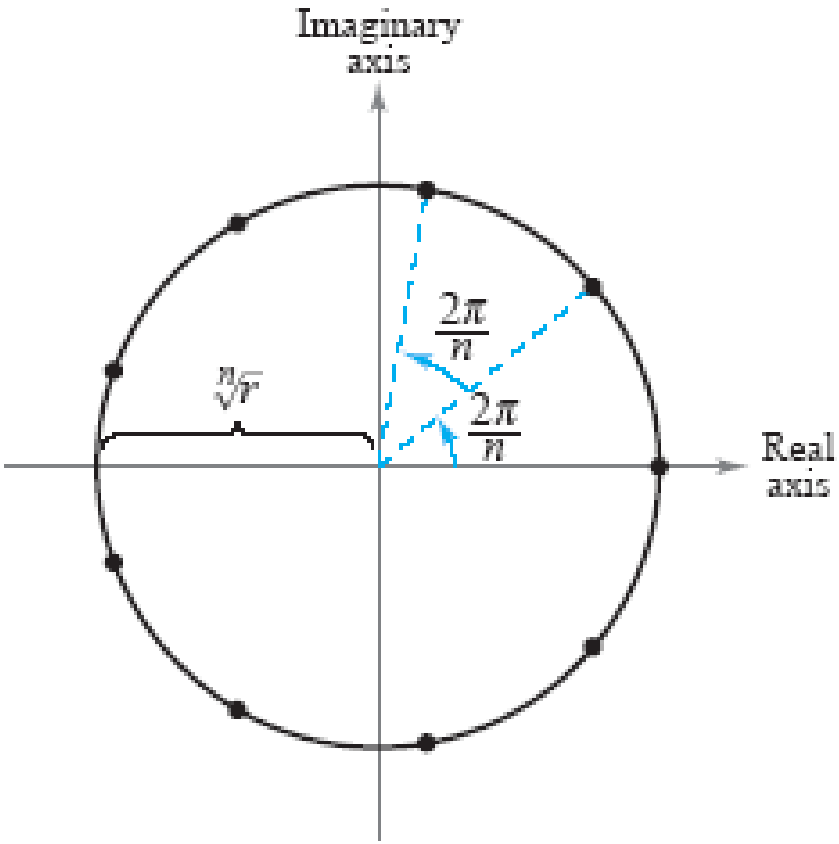
$$27^{\frac{1}{3}} = (27 + 0i)^{\frac{1}{3}} = (27(\cos)^{\circ} + i \sin 0^{\circ}))^{\frac{1}{3}}$$

$$\overset{(1)}{=} \left(27^{\frac{1}{3}} \left(e^{\frac{1}{3} i \cdot 0^{\circ}}\right)\right) = 3e^{i \cdot 0^{\circ}} = 3$$

$$\overset{(2)}{=} \left(27^{\frac{1}{3}} \left(e^{\frac{1}{3} i \cdot 360^{\circ}}\right)\right) = 3e^{i \cdot 120^{\circ}} = -1.5 + 1.5\sqrt{3} \cdot i$$

$$\overset{(3)}{=} \left(27^{\frac{1}{3}} \left(e^{\frac{1}{3} i \cdot 720^{\circ}}\right)\right) = 3e^{i \cdot 240^{\circ}} = -1.5 - 1.5\sqrt{3} \cdot i$$

Fundamental Theorem of Algebra



Fundamental Theorem of Algebra

Exercise 7: Solve the equation

$$x^2 + 4x + 5 = 0 \tag{16}$$

in the complex numbers.

Fundamental Theorem of Algebra

Every non-constant polynomial equation with complex coefficients has a complex solution (usually the number of solutions equals the degree of the polynomial). $\mathbb{C}$ is algebraically closed, while $\mathbb{R}$ is not.

Casus Irreducibilis

What is the use of complex numbers? There are many engineering examples, but here is one from solving cubic equations. Find the solutions for

$$x^3 - 3x + 1 = 0 \tag{17}$$

Cardano's formula gives us the three solutions,

$$x_k = w_k \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{-3}{4}}} + w_k^2 \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{-3}{4}}} \tag{18}$$

where $k = 1, 2, 3$ and (w_1, w_2, w_3) are the three cube roots of 1. You can look up online why Cardano's formula is true. The point is that all the solutions to this problem are real numbers, but we have to use complex algebra to calculate them.

End of Lesson

Next Lesson: Vector Spaces