Projection

(1) What is the projection of $\vec{u} = (5, 2, 10)^\intercal$ onto the plane H characterized by $2x + y + 3z = 0$?

Procedure:

1. Find a vector in the plane. Choose x, y arbitrarily, then calculate z. Call this vector v_1.
2. Find another vector in the plane and make sure it is linearly independent of v_1. Call it v_2. Now you have a basis for H.
3. You know the following about u_H. Use it to find the coordinates of u_H by forming a system of linear equations.
   (a) u_H is in the plane
   (b) $u - u_H \perp v_1$
   (c) $u - u_H \perp v_2$

(2) Let u and v be some linearly independent vectors. Then the formula for u_H, where H is the hyperplane spanned by the basis $\{v\}$, is

$$u_H = \left(\frac{u \cdot v}{v \cdot v}\right) v \tag{1}$$

This only works for one-dimensional H! Show that it is true by writing $u_H = av$ for some $a \in \mathbb{R}$ and isolating a in

$$(u - av) \perp v$$

(3) Let $H = \text{span}(\{v\})$ with $v = (-2, 3)^\intercal$. Find u_H for $u = (7, 5)^\intercal$.

(4) Let's try **(1)** again with a different strategy: What is the projection of $\vec{u} = (5, 2, 10)^\intercal$ onto the plane H characterized by $2x + y + 3z = 0$?

Procedure:

1. Find an orthogonal basis for H. (In the above procedure, find v_2 such that $v_2 \perp v_1$.)
2. Note that the following is true (but only for a basis where the basis vectors v_i are pairwise orthogonal!): $u_H = u_{v_1} + \ldots + u_{v_n}$. Show that it is true for $n = 2$, i.e.

(a) $(u_{v_1} + u_{v_2}) \in H$ (trivial)

(b) $(u - (u_{v_1} + u_{v_2})) \perp v_1$ (use the fact that $v_1 \perp v_2$)

(c) $(u - (u_{v_1} + u_{v_2})) \perp v_2$ (same idea)

3. Use formula (1) to calculate u_{v_1} and u_{v_2}.

4. Use $u_{v_1} + u_{v_2} = u_H$ to find u_H.

(5) Consider the system of non-linear equations

$$\begin{array}{ccccc} 3^x & - & y^2 & = & 2 \\ xy & + & \cos(y-5) & = & 16 \end{array} \tag{2}$$

To solve it numerically, we need to linearize the following function around an initial estimate of the solution $x = 4, y = 4$.

$$f((x,y)^\intercal) = \begin{pmatrix} e^{x \ln 3} - y^2 - 2 \\ xy + \cos(y-5) - 16 \end{pmatrix} \tag{3}$$

Find the Jacobian of f and derive the linearization of the function around $x = 4, y = 4$.