Vectors
MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

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Vectors

A vector is an ordered pair or triplet of real numbers. One way to interpret it is to make it refer to a point in the xy-plane or xyz-three-dimensional space. The usual interpretation, however, is as a displacement vector with a direction and a length. Here is an example:

$$\vec{v} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \tag{1}$$

Vector Algebra

Vectors can be added, subtracted, and multiplied by a scalar (a real number).

$$\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ \pi \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 + \pi \\ -7 \end{pmatrix} \tag{2}$$

$$1.5 \cdot \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 7.5 \\ -1.5 \end{pmatrix} \tag{3}$$

Unit Vectors

All three-dimensional vectors can be expressed in components. For this expression we need unit vectors. Any three linearly-independent vectors would work, but it makes sense to use the following three:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

Vector Decomposition

For any vector $\vec{v}$ (assuming from now on three dimensions),

$$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k} \tag{5}$$

where $V = (v_x, v_y, v_z)$, and V is the point to which the origin $O = (0, 0, 0)$ would be displaced by vector

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \tag{6}$$

The length of vector $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{7}$$

The distance between two points P and Q is the length of a displacement vector between them. Let $\vec{OP}$ be the displacement vector from O to P and so on. Then

$$\vec{PQ} = \vec{PO} + \vec{OQ} = \vec{OQ} - \vec{OP} \tag{8}$$

and $\|\vec{PQ}\|$ is the distance between P and Q.

Dot Product

The following two definition of the dot product, or scalar product, $\vec{v} \cdot \vec{w}$ are equivalent:

geometric $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos \vartheta$ where ϑ is the angle between $\vec{v}$ and $\vec{w}$, $0 \leq \vartheta \leq \pi$.

algebraic $\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z$

The dot product is a number, not a vector.

Dot Product

Now we need to show that the two definitions are equivalent.
Consider a triangle PQR in three-dimensional space. Let
$\vec{v} = \vec{PQ}, \vec{w} = \vec{PR}$. Then

$$\vec{QR} = \vec{QP} + \vec{PR} = -\vec{v} + \vec{w} = \vec{w} - \vec{v} \qquad (9)$$

Here is the law of cosines for this triangle:

$$\|\vec{w} - \vec{v}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\| \cdot \|\vec{w}\| \cos \vartheta \qquad (10)$$

It follows that the two definitions are equivalent.

Dot Product

Perpendicularity and Dot Product

Two non-zero vectors $\vec{v}$ and $\vec{w}$ are perpendicular, or orthogonal, if and only if $\vec{v} \cdot \vec{w} = 0$.

Magnitude and Dot Product

Magnitude and dot product are related as follows: $\vec{v} \cdot \vec{v} = \|\vec{v}\|$.

Dot Product Exercise

Exercise 1: Find the angle between

$$\vec{v} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \qquad (11)$$

Consider the dot product

$$4 \cdot (-2) + 0 \cdot 1 + 7 \cdot 3 = 13 \qquad (12)$$

According to the two equivalent definitions of the dot product, this is equal to

$$\|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\vartheta = \sqrt{4^2 + 7^2} \cdot \sqrt{(-2)^2 + 1^2 + 3^2} \cdot \cos\vartheta \qquad (13)$$

Therefore,

$$\vartheta = \arccos \frac{13}{\sqrt{4^2 + 7^2} \cdot \sqrt{(-2)^2 + 1^2 + 3^2}} = 64.47^\circ \qquad (14)$$

Planes Again

The equation of the plane with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ and containing the point $P = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad (15)$$

Alternatively, for $d = ax_0 + by_0 + cz_0$

$$ax + by + cz = d \qquad (16)$$

Cross Product

The following two definitions of the **cross product** or **vector product** $\vec{v} \times \vec{w}$ are equivalent:

- **Geometric definition**

  If $\vec{v}$ and $\vec{w}$ are not parallel, then

  $$\vec{v} \times \vec{w} = \begin{pmatrix} \text{Area of parallelogram} \\ \text{with edges } \vec{v} \text{ and } \vec{w} \end{pmatrix} \vec{n} = (\|\vec{v}\|\|\vec{w}\|\sin\theta)\vec{n},$$

  where $0 \leq \theta \leq \pi$ is the angle between $\vec{v}$ and $\vec{w}$ and $\vec{n}$ is the unit vector perpendicular to $\vec{v}$ and $\vec{w}$ pointing in the direction given by the right-hand rule. If $\vec{v}$ and $\vec{w}$ are parallel, then $\vec{v} \times \vec{w} = \vec{0}$.

- **Algebraic definition**

  $$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2)\vec{i} + (v_3 w_1 - v_1 w_3)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}$$

  where $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

Cross Product

If you know what a determinant is, you can remember the algebraic definition as follows.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{17}$$

Note that $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$.

Cross Product Exercise

Exercise 2: Use the cross product to find the linear equation containing the three points

$$
\begin{aligned}
P &= (1, 3, 0) \\
Q &= (3, 4, -3) \\
R &= (3, 6, 2)
\end{aligned}
\tag{18}
$$

Cross Product Exercise Answer

One way to find the answer to the last exercise (without using the cross product) is to solve the following system of linear equations for the plane $x + ay + bz = c$,

$$\begin{aligned}
1 + 3a + 0b &= c \\
3 + 4a - 3b &= c \\
3 + 6a + 2b &= c
\end{aligned} \tag{19}$$

Change this to

$$\begin{aligned}
3a + 0b - c &= -1 \\
4a - 3b - c &= -3 \\
6a + 2b - c &= -3
\end{aligned} \tag{20}$$

Using matrices,

$$\begin{pmatrix} 3 & 0 & -1 \\ 4 & -3 & -1 \\ 6 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -3 \end{pmatrix} \tag{21}$$

Equation (21) yields the solution

$$x - \frac{10}{11}y + \frac{4}{11}z = -\frac{19}{11} \tag{22}$$

Now let's use the cross product instead, avoiding the matrices. Note that

$$\begin{aligned} \vec{PQ} &= 2\vec{i} + \vec{j} - 3\vec{k} \\ \vec{PR} &= 2\vec{i} + 3\vec{j} + 2\vec{k} \end{aligned} \tag{23}$$

The cross product, using the algebraic definition, is
$\vec{u} = \vec{PQ} \times \vec{PR} = 11\vec{i} - 10\vec{j} + 4\vec{k}$.

Cross Product Exercise Answer

Let $P = (x_0, y_0, z_0)$ be a fixed point on the plane with known coordinates. Since any point $S = (x, y, z)$ on the plane fulfills

$$\vec{PS} \cdot \vec{u} = 0 \tag{24}$$

this can be turned into the plane equation

$$u_x(x - x_0) + u_y(y - y_0) + u_z(z - z_0) = 0 \tag{25}$$

Therefore, using $P = (1, 3, 0)$, this translates into

$$11x - 10y + 4z = 19 \tag{26}$$

which is equivalent to (22). Notice how easy it is to find a linear equation when you have a point $P = (x_0, y_0, z_0)$ on the plane and a normal vector $\vec{u}$ to the plane $u_x \vec{i} + u_y \vec{j} + u_z \vec{k}$:

$$u_x x + u_y y + u_z z = u_x x_0 + u_y y_0 + u_z z_0 \tag{27}$$

Exercise 3: Find all interior angles for and the plane equation containing the triangle with points

$$P = (1, 4, -2), Q = (-1, 1, 2), R = (-1, 3, 1) \qquad (28)$$

Next Lesson: Least Squares Approximation