Matrix Basics

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

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Matrix Definition

A matrix is a tabular arrangement of real numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \tag{1}$$

The number of rows is m, the number of columns is n. $m \times n$ is called the dimension or size of the matrix.

Matrix Addition

We can define operations on matrices just like we define operations on numbers. For example, we can add an $m \times n$ matrix to another one as follows,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \vdots \\ b_{m1} & & \cdots & b_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} + b_{m1} & & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

Matrix Addition

Example 1: Adding and Subtracting Matrices.

$$\begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} -6 & 5 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 11 & -11 \\ -2 & -5 \end{bmatrix}$$

Matrix Scalar Multiplication

Next, we define what it means to multiply a matrix by a scalar, i.e. a real number (NOT a matrix).

$$k \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & \ddots & & \\ \vdots & & & \vdots \\ ka_{m1} & & \cdots & ka_{mn} \end{bmatrix}$$

Matrix Scalar Multiplication

Example 2: Multiplying a Matrix by a Scalar.

$$2 \cdot \begin{bmatrix} -5 & -3 \\ -7 & 8 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ -14 & 16 \end{bmatrix}$$

$$-\frac{1}{3} \cdot \begin{bmatrix} -1 & -3 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{7}{3} & -\frac{1}{3} \end{bmatrix}$$

Matrix Transpose

The columns of a transpose $A^\intercal$ are the rows of the matrix A. The rows of a transpose $A^\intercal$ are the columns of the matrix A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

$$A^\intercal = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & \ddots & & \\ \vdots & & & \vdots \\ a_{1m} & & \cdots & a_{nm} \end{bmatrix}$$

Matrix Transpose

Example 3: Transposing a Matrix.

$$\begin{bmatrix} -1 & 2 & 1 \\ 7 & -2 & -1 \\ 0 & 6 & 6 \\ 7 & 6 & 4 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} -1 & 7 & 0 & 7 \\ 2 & -2 & 6 & 6 \\ 1 & -1 & 6 & 4 \end{bmatrix} \tag{2}$$

Matrix Product

Finally, we define matrix multiplication. You can multiply an $m \times j$ matrix by a $j \times n$ matrix, which will give you an $m \times n$ matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mj} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \vdots \\ b_{j1} & & \cdots & b_{jn} \end{bmatrix} =$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & \ddots & & \\ \vdots & & & \vdots \\ c_{m1} & & \cdots & c_{mn} \end{bmatrix}$$

where $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{ij}b_{jk}$.

Matrix Product

Notice that c_{ik} is the product of the i-th row vector of A and the k-th column vector of B. The dot product of two vectors $\vec{v}$ and $\vec{w}$ is defined to be $\vec{v}^{\mathsf{T}} \cdot \vec{w}$.

Example 4: Multiplying Matrices.

$$\begin{bmatrix} -1 & 5 \\ 10 & 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -8 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 38 & 8 \\ 26 & -80 \end{bmatrix}$$

Matrix Product

Exercise 1: Consider

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \tag{3}$$

Find $A \cdot B$ as well as $B \cdot A$ and determine whether matrix multiplication is commutative.

Laws for Matrix Arithmetic

All the usual laws of arithmetic hold for matrix arithmetic, for example $A(B+C) = AB + AC$ or $c(AB) = (cA)B = A(cB)$.
There are two major exceptions:

1. The order of factors cannot be interchanged: $AB = BA$ is generally not true
2. $AB = 0$ does not imply that one of the matrices is the zero matrix, for example

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \cdot \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4}$$

Identity Matrix

The identity matrix I with dimension $m \times m$ is a square matrix such that for all $m \times m$ matrices A it is true that

$$A \cdot I = I \cdot A = A \tag{5}$$

An identity matrix always has all 1's in the diagonal and all 0's elsewhere.

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix} \tag{6}$$

Matrix Inverse

The inverse matrix A^{-1} of a square matrix A is the matrix for which

$$A \cdot A^{-1} = A^{-1} \cdot A = I \tag{7}$$

Not all matrices have an inverse. Finding the inverse of a $m \times m$ matrix is equivalent to solving a system of $m \cdot m$ equations with $m \cdot m$ variables. For example, the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{8}$$

Matrix Determinant

Considering the last slide, it is evident that a matrix has an inverse if and only if $ad - bc \neq 0$. Such a matrix is called invertible. If $ad - bc = 0$ then the matrix is singular and has no inverse (find some examples). It turns out that the number $ad - bc$ is so special for 2×2 matrices that it gets its own name: it is the determinant of the matrix. On the next slide, I will define the determinant of any square matrix using an inductive procedure.

Matrix Determinant

- The determinant of a 1×1 matrix A is $\det(A) = a_{11}$.
- The determinant of a $m \times m$ matrix with $m > 1$ is $\det(A) = c$.

Calculate c by picking an arbitrary row, for example the i-th row. Then

$$c = \sum_{j=1}^{m} (-1)^{i+j} a_{ij} \det(A_{ij}) \tag{9}$$

A_{ij} is the matrix that results when you delete the i-th row and the j-th column from A.

Rules for Determinants

1. The determinant of a matrix equals the determinant of its transpose.
2. If a matrix has a row or a column of zeroes, the determinant is zero.
3. Interchanging two rows (or columns) changes the sign of the determinant.
4. The determinant of an upper triangular matrix is the product of its diagonal entries.

It follows from rule 3 that the determinant of a matrix, where two rows or columns are equal, is zero (why?).

Adjugate Matrix

The adjugate matrix of a matrix A has as its elements the real numbers b_{ji} (note the switched indices) with

$$b_{ji} = (-1)^{i+j} \det(A_{ij}) \tag{10}$$

where A_{ij} is defined on a previous slide. Consequently,

$$\det(A) \cdot I = \mathrm{adj}(A) \cdot A \tag{11}$$

for all square matrices A. This is true because the diagonal entries of $\mathrm{adj}(A) \cdot A$ correspond to the definition of $\det(A)$. The non-diagonal entries correspond to the definition of $\det(\hat{A})$, where $\hat{A}$ is a matrix with two columns of A repeated.

Calculating the Adjugate Matrix

Step 1: Determinants of Minor Square Matrices For each element of the matrix a_{ij}, delete the i-th row and the j-th column and calculate the determinant of the matrix A_{ij} that is left over. Put that determinant in a new matrix in a_{ij}'s place.

Step 2: Multiply by Checkerboard Matrix Multiply each element of the result matrix in step 1 by each element of the checkerboard matrix (see next slide). (This way of multiplying matrices is called Hadamard multiplication as opposed to matrix multiplication.)

Step 3: Transpose Now transpose the result matrix of step 2 in order to calculate the adjugate matrix.

Calculating the Adjugate Matrix Example

Step 1

$$\begin{bmatrix} 4 & 1 & -3 \\ 3 & 0 & 1 \\ 8 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -3 \\ -1 & 32 & -12 \\ 1 & 13 & -3 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 1 & -2 & -3 \\ -1 & 32 & -12 \\ 1 & 13 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 1 & 32 & 12 \\ 1 & -13 & -3 \end{bmatrix}$$

$$\text{Checkerboard Matrix: } \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Step 3

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 32 & 12 \\ 1 & -13 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 32 & -13 \\ -3 & -12 & -3 \end{bmatrix}$$

Finding Inverse Using Adjugate

Right-multiply equation (11) by A^{-1} to see that

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A) \tag{12}$$

For example, the adjugate of

$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & 2 & 0 \\ 4 & 3 & 1 \end{bmatrix} \text{ is } \begin{bmatrix} 2 & -3 & 1 \\ 13 & -16 & -4 \\ -8 & 12 & 3 \end{bmatrix}^{\mathsf{T}} \tag{13}$$

Therefore, the inverse is

$$\frac{1}{7} \cdot \begin{bmatrix} 2 & 13 & -8 \\ -3 & -16 & 12 \\ 1 & -4 & 3 \end{bmatrix} \tag{14}$$

Exercise 2: Consider

$$B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \tag{15}$$

Calculate B^{-1} and show that $B \cdot B^{-1} = I$.

Exercise 3: Consider

$$D = \begin{bmatrix} 0 & -1 & 4 \\ 3 & 2 & 0 \\ 4 & 3 & -1 \end{bmatrix} \tag{16}$$

Calculate $\det(D)$. Then use software to calculate the inverse of D. What do you notice about $\det(D) \cdot D^{-1}$?

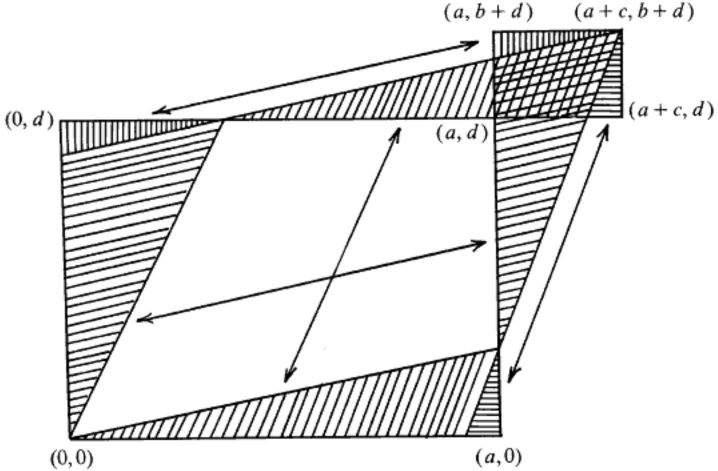
Using Determinants to find Areas

The absolute value of the determinant of a 2x2 matrix A is the area of the following parallelogram:

1. One vertex of the parallelogram is at the origin. Call it O.
2. Two vertices are at $U = (a_{11}, a_{12})$ and $V = (a_{21}, a_{22})$.
3. The final vertex is at $W = (a_{11} + a_{21}, a_{12} + a_{22})$.

A proof without words is on the next slide. Alternatively, you can watch a video with the proof here: `https://youtu.be/n-S63_goDFg`.

Using Determinants to find Areas



The principle can be generalized to higher dimensions. The volume of a parallelepiped is the absolute value of the determinant of the three vectors spanning it put side by side in a matrix.

Example 5: Parallelepiped. Find the oriented volume of the parallelepiped built on
$a_1 = (2, 1, 0)^\intercal, a_2 = (0, 3, 11)^\intercal, a_3 = (1, 2, 7)^\intercal$ in $\mathbb{R}^3$. Solution: the volume is

$$\left| \det \left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 11 \\ 1 & 2 & 7 \end{bmatrix} \right) \right| = 9 \tag{17}$$

Using Determinants to find Areas

If the figure based on parallelograms has dimension $k \leq n$ use the Gramian matrix. Its determinant is the square of the area, volume, etc. of the figure based on parallelograms.

Example 6: Gramian Matrix. Find the area of the parallelogram built on $a = (1, 1, 2)^\intercal, b = (2, 0, 3)^\intercal$. Solution: the Gram determinant is

$$\det\left(\begin{bmatrix} 6 & 8 \\ 8 & 13 \end{bmatrix}\right) = 14 \tag{18}$$

Hence the area is $\sqrt{14}$.

Area Exercise

Exercise 4: Find the area of the triangle with vertices $(1, -4), (6, -6), (3, 2)$.

Solution: Move the vertex $(1, -4)$ to the origin. This puts the other two vertices at $(5, -2)$ and $(2, 6)$. Calculate the determinant of

$$\det\left(\begin{bmatrix} 5 & -2 \\ 2 & 6 \end{bmatrix}\right) = 34 \tag{19}$$

The area of the triangle is half of the area of the parallelogram. The answer is 17.

End of Lesson

Next Lesson: Linear Equations