Eigenvalues and Eigenvectors

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

November 6, 2018

Motivation

Here is a list of questions that can be answered using eigenvalues and eigenvectors.

- Let the probability of rain tomorrow depend only on whether there is rain today. If it rains today, the probability of rain tomorrow is 20%. If it is clear today, the probability of rain tomorrow is 10%. What is the average ratio of rainy days to clear days in this climate?
- Let a particle go on a random walk along a line between S_1 and S_n. How much of its time does it spend at S_i?
- The Fibonacci sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. It is used in many applications, for example population modeling. Is there an explicit (not recursive) formula for the n-th term?
- Given a matrix A, what is A^n for large n?
- Given a matrix B, what is a matrix C such that $C^2 = B$?

Eigenvalues and Eigenvectors

Consider a square matrix A. A real (or complex) number λ is an eigenvalue if and only if there exists an eigenvector $X \neq 0$ such that

$$AX = \lambda X \tag{1}$$

$AX = \lambda X$ is equivalent to the system of linear equations $(A - \lambda I)X = 0$, which has a non-zero solution if and only if $A - \lambda I$ is singular,

$$\det(A - \lambda I) = 0 \tag{2}$$

$\det(A - \lambda I)$ is a polynomial in λ. It is called the characteristic polynomial.

Characteristic Polynomial

The eigenvalues of a square matrix A are the roots (solutions) of the polynomial equation $\det(A - \lambda I) = 0$.

Exercise 1: Find the eigenvalues of

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \tag{3}$$

and find one eigenvector for each eigenvalue.

Solution: find the determinant of

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{bmatrix} \tag{4}$$

The characteristic polynomial is $\lambda^2 - 3\lambda + 2$. The eigenvalues of A are $\lambda = 2$ and $\lambda = 1$. Now solve the systems of linear equations for the eigenvectors:

$$(A - 2I)X = 0 \text{ for } \lambda = 2 \tag{5}$$

and

$$(A - I)X = 0 \text{ for } \lambda = 1 \tag{6}$$

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tag{7}$$

The solution set is

$$S = \left\{ X \in \mathbb{R}^2 \mid X = s_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, s_1 \in \mathbb{R} \right\} \tag{8}$$

S is called the eigenspace of $\lambda = 2$. All vectors except $X = 0$ in the eigenspace of λ are called eigenvectors belonging to λ. Find the eigenspace of $\lambda = 1$.

Exercise 2: Find the eigenvalues and associated eigenvectors for

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

Eigenvalues and Eigenvectors

Suppose $\{V_1, \ldots, V_n\}$ is a basis for $\mathbb{R}^n$ and that each of these is an eigenvector for an $n \times n$ matrix A. $\{V_1, \ldots, V_n\}$ is called an eigenbasis with respect to A. Thus, we can write

$$AV_1 = \lambda_1 V_1, \ldots, AV_n = \lambda_n V_n \tag{10}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues. Let P be the matrix whose columns are the basis vectors $\{V_1, \ldots, V_n\}$. Then

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \tag{11}$$

This matrix D is called a diagonal form of A.

Not every $n \times n$ matrix A generates an eigenbasis. There is a theorem in linear algebra that tells us that such an eigenbasis is available if the characteristic polynomial has n distinct real roots.

Another theorem of linear algebra tells us that symmetric matrices have an associated eigenbasis. A matrix A is symmetric if and only if $A = A^{\mathsf{T}}$.

Exercise 3: Find a diagonal form D and an eigenbasis P for the matrix

$$A = \begin{bmatrix} 22 & 20 \\ -25 & -23 \end{bmatrix} \tag{12}$$

and show that $P^{-1}AP = D$.

Similarity

Similar Matrices

A and B are called similar if

$$B = P^{-1}AP$$

for some matrix P.

Similar matrices will have similar powers, transposes, and inverses, and will have equal determinants and characteristic polynomials.

If $B = P^{-1}AP$, then $B^k = P^{-1}A^kP$ for any positive integer k (show this for $k = 2$ and then think about how the idea generalizes).

Exercise 4: Find A^5 for

$$A = \begin{bmatrix} 19 & -12 \\ 24 & -15 \end{bmatrix} \tag{13}$$

The solution is

$$A^5 = \begin{bmatrix} 2179 & -1452 \\ 2904 & -1935 \end{bmatrix} \tag{14}$$

Exercise 5: Find a matrix C such that $C^2 = A$, where

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \tag{15}$$

Similarity

If $B = P^{-1}AP$, then $\det B = \det A$. To show this, you may remember that $\det(GH) = \det G \cdot \det H$. Therefore

$$\det B = \det(P^{-1}AP) = \det P^{-1} \det A \det P =$$

$$\det A \det P^{-1} \det P = \det A \det(P^{-1}P) = \det A$$

If there is an eigenbasis, then $\det A = \lambda_1 \cdot \ldots \cdot \lambda_n$ and $\mathrm{tr}(A) = \lambda_1 + \ldots + \lambda_n$, where $\mathrm{tr}(A)$ is the trace of A, which is the sum of its diagonal entries.

End of Lesson

Next Lesson: Axioms and Theorems of Probability