Vectors

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

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A vector space V over a field F is a set on which two operations (addition and scalar multiplication) are defined. Some axioms need to be fulfilled, most relevantly closure with respect to addition and scalar multiplication:

- If $v, w \in V$, then $v + w \in V$
- If $a \in F, v \in V$, then $av \in V$

In this course, the field will always be $\mathbb{R}$ or $\mathbb{C}$, the real or the complex numbers.

Linear Combination and Span

If $T = \{v_1, \ldots, v_k\}$ is a set of vectors and $\{x_1, \ldots, x_k\} \subset F$ is a set of scalars, then

$$x_1 v_1 + \ldots + x_k v_k \tag{1}$$

is called a linear combination of $v_1, \ldots, v_k$. The set of all linear combinations of $v_1, \ldots, v_k$ is called the span of $v_1, \ldots, v_k$.

The span of a set of vectors is a subspace of V, meaning that it is closed with respect to addition and scalar multiplication. The solution set for a system of linear equations is always a subspace (if $u, v \in S$ and $a \in F$, then $u + v \in S$ and $au \in S$).

Linear Independence

A set of vectors $v_1, \ldots, v_k$ is linearly independent if and only if the vector equation

$$x_1 v_1 + \ldots + x_k v_k = 0 \tag{2}$$

has the unique solution $(x_1, \ldots, x_k) = (0, \ldots, 0)$.

Theorems of Linear Independence

1. $v_1, \ldots, v_k$ are linearly dependent if and only if at least one of these vectors is a linear combination of the others
2. The columns of a matrix A are linearly independent if and only if the rank of A is k (the number of columns).
3. A square matrix is invertible if and only if its columns (or rows) are linearly independent.

Basis and Dimension

If V is a vector space, then the set $B = \{b_1, \ldots, b_k\}$ is a basis of V if and only if $v_1, \ldots, v_k$ are linearly independent and the span of B is V.

Unique Representation

Every vector $u \in V$ is a linear combination $u = x_1 b_1 + \ldots + x_k b_k$ of basis vectors, if such a basis exists. The ordered set $(x_1, \ldots, x_k)$ is called the coordinates of u with respect to basis B.

There is a theorem that tells us that if two sets of vectors B_1 and B_2 are bases of vector space V, then the cardinality of B_1 equals the cardinality of B_2. Thus, if a finite base exists, it makes sense to define the dimension of the vector space to be the cardinality of that base.

Vector Spaces

A finite-dimensional vector space (let the dimension be n) over the real numbers corresponds to a set of ordered sets of real numbers (the coordinates of the vectors). In other words, it corresponds to $\mathbb{R}^n$.

Examples of vector spaces:

- three-dimensional space $\mathbb{R}^3$
- the set of all straight lines in two-dimensional space
- the set of all parabolas in two-dimensional space
- the set of all circles in two-dimensional space
- the set of all polynomials with degree $k \leq n$
- the set of all real-valued functions on $\mathbb{R}$

Try to determine the dimensions of these vector spaces.

Displacement Vectors

One way to interpret a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is to make it refer to a point in the *xy*-plane or *xyz*-three-dimensional space. The usual interpretation, however, is as a displacement vector with a direction and a length. Here is an example:

$$\vec{v} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \tag{3}$$

Vector Algebra

Vectors can be added, subtracted, and multiplied by a scalar (a real number).

$$\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ \pi \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 5+\pi \\ -7 \end{pmatrix} \tag{4}$$

$$1.5 \cdot \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 7.5 \\ -1.5 \end{pmatrix} \tag{5}$$

Length and Angle of Vector

Two-dimensional and three-dimensional vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ are easily visualized. They have a length and a direction (in two-dimensional space, the direction can be represented by an angle). We use the dot product to define length and angle for any vectors. Let V be an n-dimensional vector space with a basis. Let u and v be vectors whose coordinates with respect to the basis are $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$. Organize the coordinate vectors as one-column matrices $\vec{u}$ and $\vec{v}$. Then the dot product is defined using matrix multiplication.

$$u \cdot v = \vec{u}^{\mathsf{T}} \cdot \vec{v} \tag{6}$$

Length and Angle of Vector

Now verify that in the two-dimensional case, the length of a vector u is

$$\|u\|^2 = u \cdot u \tag{7}$$

and its angle is

$$\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|} \tag{8}$$

For vectors not in $\mathbb{R}^2$, let length and angle be so defined. Two vectors u and v are perpendicular, or orthogonal, if and only if $v \cdot w = 0$.

Dot Product

The following two definition of the dot product, or scalar product, $\vec{v} \cdot \vec{w}$ are equivalent:

geometric $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\vartheta$ where ϑ is the angle between $\vec{v}$ and $\vec{w}$, $0 \leq \vartheta \leq \pi$.

algebraic $\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z$

The dot product is a number, not a vector.

Dot Product

Now we need to show that the two definitions are equivalent. Consider a triangle PQR in three-dimensional space. Let $\vec{v} = \vec{PQ}, \vec{w} = \vec{PR}$. Then

$$\vec{QR} = \vec{QP} + \vec{PR} = -\vec{v} + \vec{w} = \vec{w} - \vec{v} \tag{9}$$

Here is the law of cosines for this triangle:

$$\|\vec{w} - \vec{v}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\| \cdot \|\vec{w}\| \cos\vartheta \tag{10}$$

It follows that the two definitions are equivalent.

Dot Product Exercise

Exercise 1: Find the angle between

$$\vec{v} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \tag{11}$$

Consider the dot product

$$4 \cdot (-2) + 0 \cdot 1 + 7 \cdot 3 = 13 \tag{12}$$

According to the two equivalent definitions of the dot product, this is equal to

$$\|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\vartheta = \sqrt{4^2 + 7^2} \cdot \sqrt{(-2)^2 + 1^2 + 3^2} \cdot \cos\vartheta \tag{13}$$

Therefore,

$$\vartheta = \arccos \frac{13}{\sqrt{4^2 + 7^2} \cdot \sqrt{(-2)^2 + 1^2 + 3^2}} = 64.47^\circ \tag{14}$$

Planes Again

The equation of the plane with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ and containing the point $P = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{15}$$

Alternatively, for $d = ax_0 + by_0 + cz_0$

$$ax + by + cz = d \tag{16}$$

Orthonormal Basis

If a basis consists of vectors which are

1. pairwise orthogonal and
2. all have length 1

then the basis is an orthonormal basis. A vector w can always be scaled to length 1 by multiplying it by $1/\|w\|$.

An orthogonal matrix Q consists of columns which form an orthonormal basis. It can be shown that $Q^\intercal Q = Q^\intercal Q = I$, therefore $Q^\intercal = Q^{-1}$.

Orthonormal Basis Exercise

Exercise 2: Find an orthonormal basis for the plane whose equation is

$$2x - 3y + 7z = 0 \tag{17}$$

QR Matrix Factorization

There are several matrix factorization theorems. For example, if you order the rows and columns of a square matrix A intelligently, then $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix. Computers use this factorization to find determinants and solutions for systems of linear equations.

QR Factorization Theorem

Any real square matrix A can be decomposed as $A = QR$, where Q is an orthogonal matrix and R is an upper triangular matrix.

The QR factorization theorem can also be used to solve a system of linear equations efficiently. Assume you have a QR factorization machine. How would you find the solutions for a system of linear equations?

An Infinite-Dimensional Vector Space

Consider the set $\mathcal{F} = \{f | f$ is a continuous function on $\mathbb{R}\}$. Define addition by $(f + g)(x) = f(x) + g(x)$ and scalar multiplication by $(af)(x) = af(x)$. Then the set $\mathcal{F}$ fulfills all the axioms for a vector space. Now define the dot product

$$f \cdot g = \int_0^1 f(x)g(x)\,dx \tag{18}$$

Show that $f(x) = \sin \pi x$ and $g(x) = \sin 2\pi x$ are orthogonal to each other. As it turns out,
$\{1, \sin 2\pi x, \cos 2\pi x, \sin 4\pi x, \cos 4\pi x, \ldots\}$ is a basis of orthogonal vectors for $\mathcal{F}$. Therefore, the dimension of $\mathcal{F}$ is infinite.

Tangent Planes

Assuming f is differentiable at (a, b), the equation of the tangent plane is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{19}$$

$f(x, y)$ can be approximated around (a, b) by the tangent plane,

$$f(x, y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \text{ near } (a, b) \tag{20}$$

Tangent Planes Exercise

Exercise 3: Find the equation of the tangent plane with respect to the unit circle at $P = (0.2, 0.3, \sqrt{0.87})$.

Hint: identify a vector that is orthogonal to the plane or use partial derivatives for the function whose function graph overlaps with the relevant part of the sphere.

Subspaces and Hyperplanes

Let V be a vector space with a subset $U \subset V$. U is a subspace of V if and only if U is also a vector space. A two-dimensional plane is a subspace of three-dimensional space. Distinguish, however, between sets of points and sets of vectors. The origin is not an element of all planes in $\mathbb{R}^3$, but the zero-vector is.

Subspaces and Hyperplanes

Let V be an n-dimensional (therefore finite-dimensional) vector space with a subset $H \subset V$. Then H is a hyperplane in V if and only if H has a basis (all and only elements of H are a linear combination of a set of basis vectors in H). Lines and planes are hyperplanes. It is relatively easy to show that in the finite-dimensional case, all and only hyperplanes in V are subspaces of V.

Subspaces and Hyperplanes

The solution to a system of linear equations is always a hyperplane (of points, not of vectors) unless there is a unique solution. The reason is that whenever there are two distinct solutions $\vec{OP}$ and $\vec{OQ}$, any combination $\vec{OP} + s_1\vec{PQ}$ with $s_1 \in \mathbb{R}$ will also be a solution. Note that the empty set is a hyperplane.

Example 1: Solving a Dependent System of Equations.

$$\begin{array}{ccccccc} 3x & + & y & - & 5z & = & -4 \\ 4x & + & 2y & - & z & = & -5 \\ 6x & + & 4y & + & 7z & = & -7 \end{array} \tag{21}$$

Step 1. Find the determinant of the coefficient matrix.

$$\det\left(\begin{bmatrix} 3 & 1 & -5 \\ 4 & 2 & -1 \\ 6 & 4 & 7 \end{bmatrix}\right) = 0 \tag{22}$$

Step 2. The coefficient matrix is singular. Find the echelon form of the augmented matrix.

$$\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 2 & 17 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

The system is consistent with infinitely many solutions.

Step 3. From the second row, we learn that

$$y = \frac{1}{2} - \frac{17}{2}z \tag{24}$$

Plug this into the first row for

$$x = -\frac{3}{2} - \frac{25}{2}z \tag{25}$$

One solution is $(3, -8, 1)$. Another is $(-\frac{3}{2}, \frac{1}{2}, 0)$.

Step 4. z is here a free variable. Once the free variables are arbitrarily fixed, the other variables are determined as well. The solution space in this case is one-dimensional (a line in three-dimensional space). For an n-dimensional solution space and a solution v_0, express the solution space as a linear combination of its basis vectors $v_1, \ldots, v_n$ (make sure they are linearly independent).

$$S = \{v_0 + s_1 v_1 + \ldots + s_n v_n | s_1, \ldots, s_n \in \mathbb{R}\} \tag{26}$$

$$S = \left\{ \begin{pmatrix} 3 \\ -8 \\ 1 \end{pmatrix} + s_1 \begin{pmatrix} -9 \\ 17 \\ -2 \end{pmatrix} \middle| s_1 \in \mathbb{R} \right\} \tag{27}$$

Exercise 4: Solve the following system of linear equations:

$$\begin{array}{rcrcrcrcr} 4x_1 & + & x_2 & + & 5x_3 & - & 7x_4 & = & 14 \\ -3x_1 & + & 5x_2 & - & x_3 & - & 2x_4 & = & -18 \\ 11x_1 & - & 3x_2 & + & 11x_3 & - & 12x_4 & = & 46 \\ 10x_1 & - & 9x_2 & + & 7x_3 & - & 3x_4 & = & 50 \end{array} \tag{28}$$

Solution: create the coefficient matrix and calculate the determinant.

$$\begin{vmatrix} 4 & 1 & 5 & -7 \\ -3 & 5 & -1 & -2 \\ 11 & -3 & 11 & -12 \\ 10 & -9 & 7 & -3 \end{vmatrix} = 0 \tag{29}$$

The determinant is zero, therefore there are dependent equations. Find the echelon form of the augmented matrix.

$$\begin{bmatrix} 1 & 6 & 4 & -9 & -4 \\ 0 & 23 & 11 & -29 & -30 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

The system is consistent with infinitely many solutions, and there are two free variables. The solution space is a two-dimensional hyperplane in four-dimensional space.

Find three linearly independent solutions.

1. If $x_4 = 1, x_3 = 2$, then the top two rows of the echelon form determine $x_2 = -1, x_1 = 3$.
2. If $x_4 = 0, x_3 = 0$, then the top two rows of the echelon form determine $x_2 = -\frac{30}{23}, x_1 = \frac{88}{23}$.
3. If $x_4 = 1, x_3 = -1$, then the top two rows of the echelon form determine $x_2 = \frac{10}{23}, x_1 = \frac{147}{23}$.

Consequently,

$$S = \left\{ \begin{pmatrix} 3 \\ -1 \\ 2 \\ 1 \end{pmatrix} + s_1 \begin{pmatrix} 19 \\ -7 \\ -46 \\ -23 \end{pmatrix} + s_2 \begin{pmatrix} 78 \\ 33 \\ -69 \\ 0 \end{pmatrix} \middle| \, s_1, s_2 \in \mathbb{R} \right\} \tag{31}$$

Cross Product

The following two definitions of the **cross product** or **vector product** $\vec{v} \times \vec{w}$ are equivalent:

- **Geometric definition**
  If $\vec{v}$ and $\vec{w}$ are not parallel, then

$$\vec{v} \times \vec{w} = \begin{pmatrix}\text{Area of parallelogram} \\ \text{with edges } \vec{v} \text{ and } \vec{w}\end{pmatrix} \vec{n} = (\|\vec{v}\| \|\vec{w}\| \sin\theta)\vec{n},$$

  where $0 \le \theta \le \pi$ is the angle between $\vec{v}$ and $\vec{w}$ and $\vec{n}$ is the unit vector perpendicular to $\vec{v}$ and $\vec{w}$ pointing in the direction given by the right-hand rule. If $\vec{v}$ and $\vec{w}$ are parallel, then $\vec{v} \times \vec{w} = \vec{0}$.

- **Algebraic definition**

$$\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}$$

  where $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

Cross Product

If you know what a determinant is, you can remember the algebraic definition as follows.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{32}$$

Note that $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$.

Cross Product Exercise

Exercise 5: Find a unit vector that is perpendicular to both $\vec{u} = (3, -1, 1)^\intercal$ and $\vec{v} = (2, 0, 7)^\intercal$

Cross Product Exercise

Exercise 6: Use the cross product to find the linear equation containing the three points

$$\begin{aligned} P &= (1,3,0) \\ Q &= (3,4,-3) \\ R &= (3,6,2) \end{aligned} \tag{33}$$

Cross Product Exercise Answer

One way to find the answer to the last exercise (without using the cross product) is to solve the following system of linear equations for the plane $x + ay + bz = c$,

$$\begin{aligned} 1 + 3a + 0b &= c \\ 3 + 4a - 3b &= c \\ 3 + 6a + 2b &= c \end{aligned} \tag{34}$$

Change this to

$$\begin{aligned} 3a + 0b - c &= -1 \\ 4a - 3b - c &= -3 \\ 6a + 2b - c &= -3 \end{aligned} \tag{35}$$

Using matrices,

$$\begin{pmatrix} 3 & 0 & -1 \\ 4 & -3 & -1 \\ 6 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -3 \end{pmatrix} \tag{36}$$

Cross Product Exercise Answer

Equation (36) yields the solution

$$x - \frac{10}{11}y + \frac{4}{11}z = -\frac{19}{11} \tag{37}$$

Now let's use the cross product instead, avoiding the matrices. Note that

$$\begin{aligned} \vec{PQ} &= 2\vec{i} + \vec{j} - 3\vec{k} \\ \vec{PR} &= 2\vec{i} + 3\vec{j} + 2\vec{k} \end{aligned} \tag{38}$$

The cross product, using the algebraic definition, is $\vec{u} = \vec{PQ} \times \vec{PR} = 11\vec{i} - 10\vec{j} + 4\vec{k}$.

Cross Product Exercise Answer

Let $P = (x_0, y_0, z_0)$ be a fixed point on the plane with known coordinates. Since any point $S = (x, y, z)$ on the plane fulfills

$$\vec{PS} \cdot \vec{u} = 0 \tag{39}$$

this can be turned into the plane equation

$$u_x(x - x_0) + u_y(y - y_0) + u_z(z - z_0) = 0 \tag{40}$$

Therefore, using $P = (1, 3, 0)$, this translates into

$$11x - 10y + 4z = 19 \tag{41}$$

which is equivalent to (37). Notice how easy it is to find a linear equation when you have a point $P = (x_0, y_0, z_0)$ on the plane and a normal vector $\vec{u}$ to the plane $u_x\vec{i} + u_y\vec{j} + u_z\vec{k}$:

$$u_x x + u_y y + u_z z = u_x x_0 + u_y y_0 + u_z z_0 \tag{42}$$

Exercise 7: Find all interior angles for and the plane equation containing the triangle with points

$$P = (1, 4, -2), Q = (-1, 1, 2), R = (-1, 3, 1) \tag{43}$$

End of Lesson

Next Lesson: Least Squares Approximation