

Matrix Basics

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

September 10, 2018

Matrix Definition

A **matrix** is a tabular arrangement of real numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \quad (1)$$

The number of rows is m , the number of columns is n . $m \times n$ is called the **dimension** or **size** of the matrix.

Matrix Addition

We can define operations on matrices just like we define operations on numbers. For example, we can add an $m \times n$ matrix to another one as follows,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \vdots \\ b_{m1} & & \cdots & b_{mn} \end{bmatrix} =$$
$$\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} + b_{m1} & & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

Example 1: Adding and Subtracting Matrices.

$$\begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} -6 & 5 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 11 & -11 \\ -2 & -5 \end{bmatrix}$$

Matrix Scalar Multiplication

Next, we define what it means to multiply a matrix by a **scalar**, i.e. a real number (NOT a matrix).

$$k \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & \ddots & & \\ \vdots & & & \vdots \\ ka_{m1} & & \cdots & ka_{mn} \end{bmatrix}$$

Example 2: Multiplying a Matrix by a Scalar.

$$2 \cdot \begin{bmatrix} -5 & -3 \\ -7 & 8 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ -14 & 16 \end{bmatrix}$$

$$-\frac{1}{3} \cdot \begin{bmatrix} -1 & -3 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{7}{3} & -\frac{1}{3} \end{bmatrix}$$

Matrix Transpose

The columns of a **transpose** A^T are the rows of the matrix A . The rows of a transpose A^T are the columns of the matrix A .

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & \ddots & & \\ \vdots & & & \vdots \\ a_{1m} & & \cdots & a_{nm} \end{bmatrix}$$

Example 3: Transposing a Matrix.

$$\begin{bmatrix} -1 & 2 & 1 \\ 7 & -2 & -1 \\ 0 & 6 & 6 \\ 7 & 6 & 4 \end{bmatrix}^T = \begin{bmatrix} -1 & 7 & 0 & 7 \\ 2 & -2 & 6 & 6 \\ 1 & -1 & 6 & 4 \end{bmatrix} \quad (2)$$

Matrix Product

Finally, we define **matrix multiplication**. You can multiply an $m \times j$ matrix by a $j \times n$ matrix, which will give you an $m \times n$ matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & & \ddots & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mj} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & & \ddots & \\ \vdots & & & \vdots \\ b_{j1} & & \cdots & b_{jn} \end{bmatrix} =$$
$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & & \ddots & \\ \vdots & & & \vdots \\ c_{m1} & & \cdots & c_{mn} \end{bmatrix}$$

where $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{ij}b_{jk}$.

Notice that c_{ik} is the product of the i -th row vector of A and the k -th column vector of B . The dot product of two vectors \vec{v} and \vec{w} is defined to be $\vec{v}^T \cdot \vec{w}$. Note that

$$A^T B^T = (BA)^T \quad (3)$$

Example 4: Multiplying Matrices.

$$\begin{bmatrix} -1 & 5 \\ 10 & 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -8 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 38 & 8 \\ 26 & -80 \end{bmatrix}$$

Exercise 1: Consider

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad (4)$$

Find $A \cdot B$ as well as $B \cdot A$ and determine whether matrix multiplication is commutative.

Laws for Matrix Arithmetic

All the usual laws of arithmetic hold for matrix arithmetic, for example $A(B + C) = AB + AC$ or $c(AB) = (cA)B = A(cB)$.

There are two major exceptions:

- 1 The order of factors cannot be interchanged: $AB = BA$ is generally not true
- 2 $AB = 0$ does not imply that one of the matrices is the zero matrix, for example

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \cdot \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (5)$$

Identity Matrix

The **identity matrix** I with dimension $m \times m$ is a square matrix such that for all $m \times m$ matrices A it is true that

$$A \cdot I = I \cdot A = A \quad (6)$$

An identity matrix always has all 1's in the diagonal and all 0's elsewhere.

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix} \quad (7)$$

The **inverse matrix** A^{-1} of a square matrix A is the matrix for which

$$A \cdot A^{-1} = A^{-1} \cdot A = I \quad (8)$$

Not all matrices have an inverse. Finding the inverse of a $m \times m$ matrix is equivalent to solving a system of $m \cdot m$ equations with $m \cdot m$ variables. For example, the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (9)$$

Inverse of Transpose

The inverse of the transpose is the transpose of the inverse, so

$$(A^{-1})^T = (A^T)^{-1} \quad (10)$$

This is true because

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I \quad (11)$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I \quad (12)$$

Matrix Determinant

Considering the last slide, it is evident that a matrix has an inverse if and only if $ad - bc \neq 0$. Such a matrix is called **invertible**. If $ad - bc = 0$ then the matrix is **singular** and has no inverse (find some examples). It turns out that the number $ad - bc$ is so special for 2×2 matrices that it gets its own name: it is the **determinant** of the matrix. On the next slide, I will define the determinant of any square matrix using an inductive procedure.

Matrix Determinant

- The determinant of a 1×1 matrix A is $\det(A) = a_{11}$.
- The determinant of a $m \times m$ matrix with $m > 1$ is $\det(A) = c$.

Calculate c by picking an arbitrary row, for example the i -th row.

Then

$$c = \sum_{j=1}^m (-1)^{i+j} a_{ij} \det(A_{ij}) \quad (13)$$

A_{ij} is the matrix that results when you delete the i -th row and the j -th column from A .

Rules for Determinants

- 1 The determinant of a matrix equals the determinant of its transpose.
- 2 If a matrix has a row or a column of zeroes, the determinant is zero.
- 3 Interchanging two rows (or columns) changes the sign of the determinant.
- 4 The determinant of an upper triangular matrix is the product of its diagonal entries.

It follows from rule 3 that the determinant of a matrix, where two rows or columns are equal, is zero (why?). Rule 1 together with $(A^{-1})^T = (A^T)^{-1}$ tells us that in much of what we have to say in the future, rows and columns are interchangeable.

Adjugate Matrix

The **adjugate matrix** of a matrix A has as its elements the real numbers b_{ji} (note the switched indices) with

$$b_{ji} = (-1)^{i+j} \det(A_{ij}) \quad (14)$$

where A_{ij} is defined on a previous slide. Consequently,

$$\det(A) \cdot I = \text{adj}(A) \cdot A \quad (15)$$

for all square matrices A . This is true because the diagonal entries of $\text{adj}(A) \cdot A$ correspond to the definition of $\det(A)$. The non-diagonal entries correspond to the definition of $\det(\hat{A})$, where \hat{A} is a matrix with two columns of A repeated.

Calculating the Adjugate Matrix

- Step 1: Determinants of Minor Square Matrices** For each element of the matrix a_{ij} , delete the i -th row and the j -th column and calculate the determinant of the matrix A_{ij} that is left over. Put that determinant in a new matrix in a_{ij} 's place.
- Step 2: Multiply by Checkerboard Matrix** Multiply each element of the result matrix in step 1 by each element of the checkerboard matrix (see next slide). (This way of multiplying matrices is called Hadamard multiplication as opposed to matrix multiplication.)
- Step 3: Transpose** Now transpose the result matrix of step 2 in order to calculate the adjugate matrix.

Calculating the Adjugate Matrix Example

Step 1

$$\begin{bmatrix} 4 & 1 & -3 \\ 3 & 0 & 1 \\ 8 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -3 \\ -1 & 32 & -12 \\ 1 & 13 & -3 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 1 & -2 & -3 \\ -1 & 32 & -12 \\ 1 & 13 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 1 & 32 & 12 \\ 1 & -13 & -3 \end{bmatrix}$$

Checkerboard Matrix: $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Step 3

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 32 & 12 \\ 1 & -13 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 32 & -13 \\ -3 & -12 & -3 \end{bmatrix}$$

Finding Inverse Using Adjugate

Right-multiply equation (15) by A^{-1} to see that

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \quad (16)$$

For example, the adjugate of

$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & 2 & 0 \\ 4 & 3 & 1 \end{bmatrix} \text{ is } \begin{bmatrix} 2 & -3 & 1 \\ 13 & -16 & -4 \\ -8 & 12 & 3 \end{bmatrix}^T \quad (17)$$

Therefore, the inverse is

$$\frac{1}{7} \cdot \begin{bmatrix} 2 & 13 & -8 \\ -3 & -16 & 12 \\ 1 & -4 & 3 \end{bmatrix} \quad (18)$$

Exercise 2: Consider

$$B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \quad (19)$$

Calculate B^{-1} and show that $B \cdot B^{-1} = I$.

Exercise 3: Consider

$$D = \begin{bmatrix} 0 & -1 & 4 \\ 3 & 2 & 0 \\ 4 & 3 & -1 \end{bmatrix} \quad (20)$$

Calculate $\det(D)$. Then use software to calculate the inverse of D . What do you notice about $\det(D) \cdot D^{-1}$?

Using Determinants to find Areas

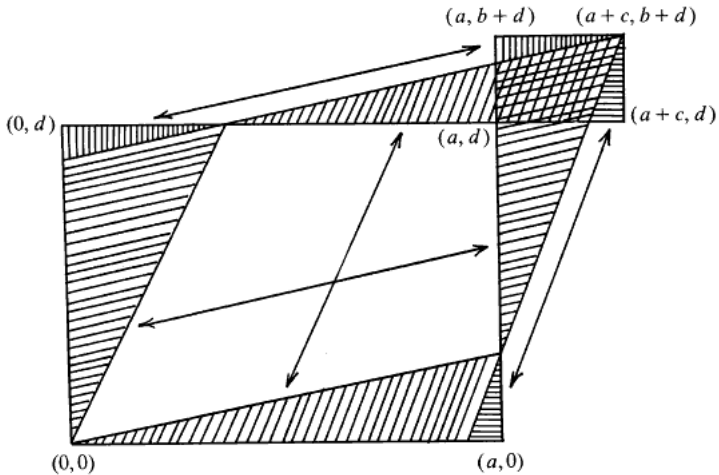
The absolute value of the determinant of a 2×2 matrix A is the area of the following parallelogram:

- 1 One vertex of the parallelogram is at the origin. Call it O .
- 2 Two vertices are at $U = (a_{11}, a_{12})$ and $V = (a_{21}, a_{22})$.
- 3 The final vertex is at $W = (a_{11} + a_{21}, a_{12} + a_{22})$.

A proof without words is on the next slide. Alternatively, you can watch a video with the proof here:

https://youtu.be/n-S63_goDFg.

Using Determinants to find Areas



Using Determinants to find Areas

The principle can be generalized to higher dimensions. The volume of a parallelepiped is the absolute value of the determinant of the three vectors spanning it put side by side in a matrix.

Example 5: Parallelepiped. Find the oriented volume of the parallelepiped built on

$a_1 = (2, 1, 0)^T$, $a_2 = (0, 3, 11)^T$, $a_3 = (1, 2, 7)^T$ in \mathbb{R}^3 . Solution: the volume is

$$\left| \det \left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 11 \\ 1 & 2 & 7 \end{bmatrix} \right) \right| = 9 \quad (21)$$

Using Determinants to find Areas

If the figure based on parallelograms has dimension $k \leq n$ use the Gramian matrix. Its determinant is the square of the area, volume, etc. of the figure based on parallelograms.

Example 6: Gramian Matrix. Find the area of the parallelogram built on $a = (1, 1, 2)^T$, $b = (2, 0, 3)^T$. Solution: the Gram determinant is

$$\det \left(\begin{bmatrix} 6 & 8 \\ 8 & 13 \end{bmatrix} \right) = 14 \quad (22)$$

Hence the area is $\sqrt{14}$.

Exercise 4: Find the area of the triangle with vertices $(1, -4)$, $(6, -6)$, $(3, 2)$.

Solution: Move the vertex $(1, -4)$ to the origin. This puts the other two vertices at $(5, -2)$ and $(2, 6)$. Calculate the determinant of

$$\det \left(\begin{bmatrix} 5 & -2 \\ 2 & 6 \end{bmatrix} \right) = 34 \quad (23)$$

The area of the triangle is half of the area of the parallelogram.
The answer is 17.

End of Lesson

Next Lesson: Linear Equations