

# Matrix Basics

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

September 10, 2018

# Matrix Definition

A **matrix** is a tabular arrangement of real numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \quad (1)$$

The number of rows is  $m$ , the number of columns is  $n$ .  $m \times n$  is called the **dimension** or **size** of the matrix.

# Matrix Addition

We can define operations on matrices just like we define operations on numbers. For example, we can add an  $m \times n$  matrix to another one as follows,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \vdots \\ b_{m1} & & \cdots & b_{mn} \end{bmatrix} =$$
$$\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} + b_{m1} & & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

## Example 1: Adding and Subtracting Matrices.

$$\begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} -6 & 5 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 11 & -11 \\ -2 & -5 \end{bmatrix}$$

# Matrix Scalar Multiplication

Next, we define what it means to multiply a matrix by a **scalar**, i.e. a real number (NOT a matrix).

$$k \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & \ddots & & \\ \vdots & & & \vdots \\ ka_{m1} & & \cdots & ka_{mn} \end{bmatrix}$$

## Example 2: Multiplying a Matrix by a Scalar.

$$2 \cdot \begin{bmatrix} -5 & -3 \\ -7 & 8 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ -14 & 16 \end{bmatrix}$$

$$-\frac{1}{3} \cdot \begin{bmatrix} -1 & -3 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{7}{3} & -\frac{1}{3} \end{bmatrix}$$

# Matrix Transpose

The columns of a **transpose**  $A^T$  are the rows of the matrix  $A$ . The rows of a transpose  $A^T$  are the columns of the matrix  $A$ .

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & \ddots & & \\ \vdots & & & \vdots \\ a_{1m} & & \cdots & a_{nm} \end{bmatrix}$$

## Example 3: Transposing a Matrix.

$$\begin{bmatrix} -1 & 2 & 1 \\ 7 & -2 & -1 \\ 0 & 6 & 6 \\ 7 & 6 & 4 \end{bmatrix}^T = \begin{bmatrix} -1 & 7 & 0 & 7 \\ 2 & -2 & 6 & 6 \\ 1 & -1 & 6 & 4 \end{bmatrix} \quad (2)$$



# Matrix Product

Finally, we define **matrix multiplication**. You can multiply an  $m \times j$  matrix by a  $j \times n$  matrix, which will give you an  $m \times n$  matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & & \ddots & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mj} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & & \ddots & \\ \vdots & & & \vdots \\ b_{j1} & & \cdots & b_{jn} \end{bmatrix} =$$
$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & & \ddots & \\ \vdots & & & \vdots \\ c_{m1} & & \cdots & c_{mn} \end{bmatrix}$$

where  $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{ij}b_{jk}$ .

Notice that  $c_{ik}$  is the product of the  $i$ -th row vector of  $A$  and the  $k$ -th column vector of  $B$ . The dot product of two vectors  $\vec{v}$  and  $\vec{w}$  is defined to be  $\vec{v}^T \cdot \vec{w}$ . Note that

$$A^T B^T = (BA)^T \quad (3)$$

**Example 4: Multiplying Matrices.**

$$\begin{bmatrix} -1 & 5 \\ 10 & 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -8 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 38 & 8 \\ 26 & -80 \end{bmatrix}$$

**Exercise 1:** Consider

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad (4)$$

Find  $A \cdot B$  as well as  $B \cdot A$  and determine whether matrix multiplication is commutative.

# Laws for Matrix Arithmetic

All the usual laws of arithmetic hold for matrix arithmetic, for example  $A(B + C) = AB + AC$  or  $c(AB) = (cA)B = A(cB)$ .

There are two major exceptions:

- 1 The order of factors cannot be interchanged:  $AB = BA$  is generally not true
- 2  $AB = 0$  does not imply that one of the matrices is the zero matrix, for example

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \cdot \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (5)$$

# Identity Matrix

The **identity matrix**  $I$  with dimension  $m \times m$  is a square matrix such that for all  $m \times m$  matrices  $A$  it is true that

$$A \cdot I = I \cdot A = A \quad (6)$$

An identity matrix always has all 1's in the diagonal and all 0's elsewhere.

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix} \quad (7)$$

The **inverse matrix**  $A^{-1}$  of a square matrix  $A$  is the matrix for which

$$A \cdot A^{-1} = A^{-1} \cdot A = I \quad (8)$$

Not all matrices have an inverse. Finding the inverse of a  $m \times m$  matrix is equivalent to solving a system of  $m \cdot m$  equations with  $m \cdot m$  variables. For example, the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (9)$$

# Inverse of Transpose

The inverse of the transpose is the transpose of the inverse, so

$$(A^{-1})^T = (A^T)^{-1} \quad (10)$$

This is true because

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I \quad (11)$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I \quad (12)$$

Note also that

$$(AB)^T = B^T A^T \quad (13)$$

# Matrix Determinant

Considering (9), it is evident that a matrix has an inverse if and only if  $ad - bc \neq 0$ . Such a matrix is called **invertible**. If  $ad - bc = 0$  then the matrix is **singular** and has no inverse (find some examples). It turns out that the number  $ad - bc$  is so special for  $2 \times 2$  matrices that it gets its own name: it is the **determinant** of the matrix. On the next slide, I will define the determinant of any square matrix using an inductive procedure.



# Matrix Determinant

- The determinant of a  $1 \times 1$  matrix  $A$  is  $\det(A) = a_{11}$ .
- The determinant of a  $m \times m$  matrix with  $m > 1$  is  $\det(A) = c$ .

Calculate  $c$  by picking an arbitrary row, for example the  $i$ -th row.

Then

$$c = \sum_{j=1}^m (-1)^{i+j} a_{ij} \det(A_{ij}) \quad (14)$$

$A_{ij}$  is the matrix that results when you delete the  $i$ -th row and the  $j$ -th column from  $A$ .

# Rules for Determinants

- 1 The determinant of a matrix equals the determinant of its transpose.
- 2 If a matrix has a row or a column of zeroes, the determinant is zero.
- 3 Interchanging two rows (or columns) changes the sign of the determinant.
- 4 The determinant of an upper triangular matrix is the product of its diagonal entries.

It follows from rule 3 that the determinant of a matrix, where two rows or columns are equal, is zero (why?). Rule 1 together with  $(A^{-1})^T = (A^T)^{-1}$  tells us that in much of what we have to say in the future, rows and columns are interchangeable.

# Adjugate Matrix

The **adjugate matrix** of a matrix  $A$  has as its elements the real numbers  $b_{ji}$  (note the switched indices) with

$$b_{ji} = (-1)^{i+j} \det(A_{ij}) \quad (15)$$

where  $A_{ij}$  is defined on a previous slide. Consequently,

$$\det(A) \cdot I = \text{adj}(A) \cdot A \quad (16)$$

for all square matrices  $A$ . This is true because the diagonal entries of  $\text{adj}(A) \cdot A$  correspond to the definition of  $\det(A)$ . The non-diagonal entries correspond to the definition of  $\det(\hat{A})$ , where  $\hat{A}$  is a matrix with two columns of  $A$  repeated.

# Calculating the Adjugate Matrix

- Step 1: Determinants of Minor Square Matrices** For each element of the matrix  $a_{ij}$ , delete the  $i$ -th row and the  $j$ -th column and calculate the determinant of the matrix  $A_{ij}$  that is left over. Put that determinant in a new matrix in  $a_{ij}$ 's place.
- Step 2: Multiply by Checkerboard Matrix** Multiply each element of the result matrix in step 1 by each element of the checkerboard matrix (see next slide). (This way of multiplying matrices is called Hadamard multiplication as opposed to matrix multiplication.)
- Step 3: Transpose** Now transpose the result matrix of step 2 in order to calculate the adjugate matrix.

# Calculating the Adjugate Matrix Example

Step 1

$$\begin{bmatrix} 4 & 1 & -3 \\ 3 & 0 & 1 \\ 8 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -3 \\ -1 & 32 & -12 \\ 1 & 13 & -3 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 1 & -2 & -3 \\ -1 & 32 & -12 \\ 1 & 13 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 1 & 32 & 12 \\ 1 & -13 & -3 \end{bmatrix}$$

Checkerboard Matrix:  $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$  or  $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Step 3

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 32 & 12 \\ 1 & -13 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 32 & -13 \\ -3 & -12 & -3 \end{bmatrix}$$

# Finding Inverse Using Adjugate

Right-multiply equation (16) by  $A^{-1}$  to see that

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \quad (17)$$

For example, the adjugate of

$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & 2 & 0 \\ 4 & 3 & 1 \end{bmatrix} \text{ is } \begin{bmatrix} 2 & -3 & 1 \\ 13 & -16 & -4 \\ -8 & 12 & 3 \end{bmatrix}^T \quad (18)$$

Therefore, the inverse is

$$\frac{1}{7} \cdot \begin{bmatrix} 2 & 13 & -8 \\ -3 & -16 & 12 \\ 1 & -4 & 3 \end{bmatrix} \quad (19)$$

**Exercise 2:** Consider

$$B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \quad (20)$$

Calculate  $B^{-1}$  and show that  $B \cdot B^{-1} = I$ .

**Exercise 3:** Consider

$$D = \begin{bmatrix} 0 & -1 & 4 \\ 3 & 2 & 0 \\ 4 & 3 & -1 \end{bmatrix} \quad (21)$$

Calculate  $\det(D)$ . Then use software to calculate the inverse of  $D$ . What do you notice about  $\det(D) \cdot D^{-1}$ ?

# Using Determinants to find Areas

The absolute value of the determinant of a  $2 \times 2$  matrix  $A$  is the area of the following parallelogram:

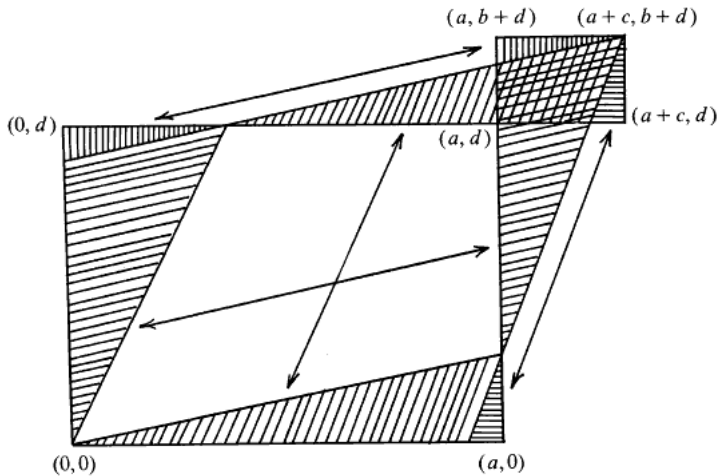
- 1 One vertex of the parallelogram is at the origin. Call it  $O$ .
- 2 Two vertices are at  $U = (a_{11}, a_{12})$  and  $V = (a_{21}, a_{22})$ .
- 3 The final vertex is at  $W = (a_{11} + a_{21}, a_{12} + a_{22})$ .

A proof without words is on the next slide. Alternatively, you can watch a video with the proof here:

[https://youtu.be/n-S63\\_goDFg](https://youtu.be/n-S63_goDFg).



# Using Determinants to find Areas



# Using Determinants to find Areas

The principle can be generalized to higher dimensions. The volume of a parallelepiped is the absolute value of the determinant of the three vectors spanning it put side by side in a matrix.

**Example 5: Parallelepiped.** Find the oriented volume of the parallelepiped built on

$a_1 = (2, 1, 0)^T$ ,  $a_2 = (0, 3, 11)^T$ ,  $a_3 = (1, 2, 7)^T$  in  $\mathbb{R}^3$ . Solution: the volume is

$$\left| \det \left( \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 11 \\ 1 & 2 & 7 \end{bmatrix} \right) \right| = 9 \quad (22)$$

# Using Determinants to find Areas

If the figure based on parallelograms has dimension  $k \leq n$  use the Gramian matrix. Its determinant is the square of the area, volume, etc. of the figure based on parallelograms.

**Example 6: Gramian Matrix.** Find the area of the parallelogram built on  $a = (1, 1, 2)^T$ ,  $b = (2, 0, 3)^T$ . Solution: the Gram determinant is

$$\det \left( \begin{bmatrix} 6 & 8 \\ 8 & 13 \end{bmatrix} \right) = 14 \quad (23)$$

Hence the area is  $\sqrt{14}$ .

**Exercise 4:** Find the area of the triangle with vertices  $(1, -4)$ ,  $(6, -6)$ ,  $(3, 2)$ .

Solution: Move the vertex  $(1, -4)$  to the origin. This puts the other two vertices at  $(5, -2)$  and  $(2, 6)$ . Calculate the determinant of

$$\det \left( \begin{bmatrix} 5 & -2 \\ 2 & 6 \end{bmatrix} \right) = 34 \quad (24)$$

The area of the triangle is half of the area of the parallelogram.  
The answer is 17.

# End of Lesson

Next Lesson: Linear Equations