

Eigenvalues and Eigenvectors

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

October 29, 2018

Here is a list of questions that can be answered using eigenvalues and eigenvectors.

- Let the probability of rain tomorrow depend only on whether there is rain today. If it rains today, the probability of rain tomorrow is 20%. If it is clear today, the probability of rain tomorrow is 10%. What is the average ratio of rainy days to clear days in this climate?
- Let a particle go on a random walk along a line between S_1 and S_n . How much of its time does it spend at S_i ?
- The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, ... It is used in many applications, for example population modeling. Is there an explicit (not recursive) formula for the n -th term?
- Given a matrix A , what is A^n for large n ?
- Given a matrix B , what is a matrix C such that $C^2 = B$?

Eigenvalues and Eigenvectors

Consider a square matrix A . A real (or complex) number λ is an **eigenvalue** if and only if there exists an **eigenvector** $X \neq 0$ such that

$$AX = \lambda X \quad (1)$$

$AX = \lambda X$ is equivalent to the system of linear equations $(A - \lambda I)X = 0$, which has a non-zero solution if and only if $A - \lambda I$ is singular,

$$\det(A - \lambda I) = 0 \quad (2)$$

$\det(A - \lambda I)$ is a polynomial in λ . It is called the **characteristic polynomial**.

Characteristic Polynomial

The eigenvalues of a square matrix A are the roots (solutions) of the polynomial equation $\det(A - \lambda I) = 0$.

Exercise 1: Find the eigenvalues of

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \quad (3)$$

and find one eigenvector for each eigenvalue.

Eigenvalues and Eigenvectors

Solution: find the determinant of

$$A - \lambda I = \begin{bmatrix} 3 - \lambda & -1 \\ 2 & -\lambda \end{bmatrix} \quad (4)$$

The characteristic polynomial is $\lambda^2 - 3\lambda + 2$. The eigenvalues of A are $\lambda = 2$ and $\lambda = 1$. Now solve the systems of linear equations for the eigenvectors:

$$(A - 2I)X = 0 \text{ for } \lambda = 2 \quad (5)$$

and

$$(A - I)X = 0 \text{ for } \lambda = 1 \quad (6)$$

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad (7)$$

The solution set is

$$S = \left\{ X \in \mathbb{R}^2 \mid X = s_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, s_1 \in \mathbb{R} \right\} \quad (8)$$

S is called the **eigenspace** of $\lambda = 2$. All vectors except $X = 0$ in the eigenspace of λ are called eigenvectors belonging to λ . Find the eigenspace of $\lambda = 1$.

Exercise 2: Find the eigenvalues and associated eigenvectors for

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad (9)$$

Eigenvalues and Eigenvectors

Suppose $\{V_1, \dots, V_n\}$ is a basis for \mathbb{R}^n and that each of these is an eigenvector for an $n \times n$ matrix A . $\{V_1, \dots, V_n\}$ is called an **eigenbasis** with respect to A . Thus, we can write

$$AV_1 = \lambda_1 V_1, \dots, AV_n = \lambda_n V_n \quad (10)$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues. Let P be the matrix whose columns are the basis vectors $\{V_1, \dots, V_n\}$. Then

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad (11)$$

This matrix D is called a **diagonal form** of A .

Eigenvalues and Eigenvectors

Not every $n \times n$ matrix A generates an eigenbasis. There is a theorem in linear algebra that tells us that such an eigenbasis is available if the characteristic polynomial has n distinct real roots.

Another theorem of linear algebra tells us that symmetric matrices have an associated eigenbasis. A matrix A is symmetric if and only if $A = A^T$.

Exercise 3: Find a diagonal form D and an eigenbasis P for the matrix

$$A = \begin{bmatrix} 22 & 20 \\ -25 & -23 \end{bmatrix} \quad (12)$$

and show that $P^{-1}AP = D$.

Similar Matrices

A and B are called **similar** if

$$B = P^{-1}AP$$

for some matrix P .

Similar matrices will have similar powers, transposes, and inverses, and will have equal determinants and characteristic polynomials.

If $B = P^{-1}AP$, then $B^k = P^{-1}A^kP$ for any positive integer k (show this for $k = 2$ and then think about how the idea generalizes).

Exercise 4: Find A^5 for

$$A = \begin{bmatrix} 19 & -12 \\ 24 & -15 \end{bmatrix} \quad (13)$$

The solution is

$$A^5 = \begin{bmatrix} 2179 & -1452 \\ 2904 & -1935 \end{bmatrix} \quad (14)$$

Exercise 5: Find a matrix C such that $C^2 = A$, where

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \quad (15)$$

If $B = P^{-1}AP$, then $\det B = \det A$. To show this, you may remember that $\det(GH) = \det G \cdot \det H$. Therefore

$$\det B = \det(P^{-1}AP) = \det P^{-1} \det A \det P =$$

$$\det A \det P^{-1} \det P = \det A \det(P^{-1}P) = \det A$$

If there is an eigenbasis, then $\det A = \lambda_1 \cdot \dots \cdot \lambda_n$ and $\operatorname{tr} A = \lambda_1 + \dots + \lambda_n$, where $\operatorname{tr} A$ is the **trace** of A , which is the sum of its diagonal entries.

End of Lesson

Next Lesson: Axioms and Theorems of Probability