Eigenvalues and Eigenvectors

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

October 29, 2018

Motivation

Here is a list of questions that can be answered using eigenvalues and eigenvectors.

- Let the probability of rain tomorrow depend only on whether there is rain today. If it rains today, the probability of rain tomorrow is 20%. If it is clear today, the probability of rain tomorrow is 10%. What is the average ratio of rainy days to clear days in this climate?
- Let a particle go on a random walk along a line between S_1 and S_n. How much of its time does it spend at S_i?
- The Fibonacci sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. It is used in many applications, for example population modeling. Is there an explicit (not recursive) formula for the n-th term?
- Given a matrix A, what is A^n for large n?
- Given a matrix B, what is a matrix C such that $C^2 = B$?

Eigenvalues and Eigenvectors

Consider a square matrix A. A real (or complex) number λ is an eigenvalue if and only if there exists an eigenvector $X \neq 0$ such that

$$AX = \lambda X \tag{1}$$

$AX = \lambda X$ is equivalent to the system of linear equations $(A - \lambda I)X = 0$, which has a non-zero solution if and only if $A - \lambda I$ is singular,

$$\det(A - \lambda I) = 0 \tag{2}$$

$\det(A - \lambda I)$ is a polynomial in λ. It is called the characteristic polynomial.

Characteristic Polynomial

The eigenvalues of a square matrix A are the roots (solutions) of the polynomial equation $\det(A - \lambda I) = 0$.

Exercise 1: Find the eigenvalues of

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \tag{3}$$

and find one eigenvector for each eigenvalue.

Solution: find the determinant of

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{bmatrix} \tag{4}$$

The characteristic polynomial is $\lambda^2 - 3\lambda + 2$. The eigenvalues of A are $\lambda = 2$ and $\lambda = 1$. Now solve the systems of linear equations for the eigenvectors:

$$(A - 2I)X = 0 \text{ for } \lambda = 2 \tag{5}$$

and

$$(A - I)X = 0 \text{ for } \lambda = 1 \tag{6}$$

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tag{7}$$

The solution set is

$$S = \left\{ X \in \mathbb{R}^2 \mid X = s_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, s_1 \in \mathbb{R} \right\} \tag{8}$$

S is called the eigenspace of $\lambda = 2$. All vectors except $X = 0$ in the eigenspace of λ are called eigenvectors belonging to λ. Find the eigenspace of $\lambda = 1$.

End of Lesson

Next Lesson: Axioms and Theorems of Probability