Matrix Basics

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

September 10, 2018

Matrix Definition

A matrix is a tabular arrangement of real numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \tag{1}$$

The number of rows is m, the number of columns is n. $m \times n$ is called the dimension or size of the matrix.

Matrix Addition

We can define operations on matrices just like we define operations on numbers. For example, we can add an $m \times n$ matrix to another one as follows,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \vdots \\ b_{m1} & & \cdots & b_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1}+b_{m1} & & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

Matrix Addition

Example 1: Adding and Subtracting Matrices.

$$\begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} -6 & 5 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 11 & -11 \\ -2 & -5 \end{bmatrix}$$

Matrix Scalar Multiplication

Next, we define what it means to multiply a matrix by a scalar, i.e. a real number (NOT a matrix).

$$k \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & \ddots & & \\ \vdots & & & \vdots \\ ka_{m1} & & \cdots & ka_{mn} \end{bmatrix}$$

Example 2: Multiplying a Matrix by a Scalar.

$$2 \cdot \begin{bmatrix} -5 & -3 \\ -7 & 8 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ -14 & 16 \end{bmatrix}$$

$$-\frac{1}{3} \cdot \begin{bmatrix} -1 & -3 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{7}{3} & -\frac{1}{3} \end{bmatrix}$$

Matrix Transpose

The columns of a transpose $A^\intercal$ are the rows of the matrix A. The rows of a transpose $A^\intercal$ are the columns of the matrix A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

$$A^\intercal = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & \ddots & & \\ \vdots & & & \vdots \\ a_{1m} & & \cdots & a_{nm} \end{bmatrix}$$

Matrix Transpose

Example 3: Transposing a Matrix.

$$\begin{bmatrix} -1 & 2 & 1 \\ 7 & -2 & -1 \\ 0 & 6 & 6 \\ 7 & 6 & 4 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} -1 & 7 & 0 & 7 \\ 2 & -2 & 6 & 6 \\ 1 & -1 & 6 & 4 \end{bmatrix} \tag{2}$$

Matrix Product

Finally, we define matrix multiplication. You can multiply an $m \times j$ matrix by a $j \times n$ matrix, which will give you an $m \times n$ matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mj} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \vdots \\ b_{j1} & & \cdots & b_{jn} \end{bmatrix} =$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & \ddots & & \\ \vdots & & & \vdots \\ c_{m1} & & \cdots & c_{mn} \end{bmatrix}$$

where $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{ij}b_{jk}$.

Matrix Product

Notice that c_{ik} is the product of the i-th row vector of A and the k-th column vector of B. The dot product of two vectors $\vec{v}$ and $\vec{w}$ is defined to be $\vec{v}^{\,\intercal} \cdot \vec{w}$.

Example 4: Multiplying Matrices.

$$\begin{bmatrix} -1 & 5 \\ 10 & 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -8 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 38 & 8 \\ 26 & -80 \end{bmatrix}$$

Exercise 1: Consider

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \tag{3}$$

Find $A \cdot B$ as well as $B \cdot A$ and determine whether matrix multiplication is commutative.

Identity Matrix

The identity matrix I with dimension $m \times m$ is a square matrix such that for all $m \times m$ matrices A it is true that

$$A \cdot I = I \cdot A = A \tag{4}$$

An identity matrix always has all 1's in the diagonal and all 0's elsewhere.

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix} \tag{5}$$

Matrix Inverse

The inverse matrix A^{-1} of a square matrix A is the matrix for which

$$A \cdot A^{-1} = A^{-1} \cdot A = I \tag{6}$$

Not all matrices have an inverse. Finding the inverse of a $m \times m$ matrix is equivalent to solving a system of $m \cdot m$ equations with $m \cdot m$ variables. For example, the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{is} \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{7}$$

Matrix Determinant

Considering the last slide, it is evident that a matrix has an inverse if and only if $ad - bc \neq 0$. Such a matrix is called invertible. If $ad - bc = 0$ then the matrix is singular and has no inverse (find some examples). It turns out that the number $ad - bc$ is so special for 2×2 matrices that it gets its own name: it is the determinant of the matrix. On the next slide, I will define the determinant of any square matrix using an inductive procedure.

Matrix Determinant

- The determinant of a 1×1 matrix A is $\det(A) = a_{11}$.
- The determinant of a $m \times m$ matrix with $m > 1$ is $\det(A) = c$.

Calculate c by picking an arbitrary row, for example the i-th row. Then

$$c = \sum_{j=1}^{m} (-1)^{i+j} a_{ij} \det(A_{ij}) \tag{8}$$

where A_{ij} is the matrix that results when you delete the i-th row and the j-th column from A.

Adjugate Matrix

The adjugate matrix of a matrix A has as its elements the real numbers b_{ji} (note the switched indices) with

$$b_{ji} = (-1)^{i+j} \det(A_{ij}) \tag{9}$$

where A_{ij} is defined on the last slide. Consequently,

$$\det(A) \cdot I = \mathrm{adj}(A) \cdot A \tag{10}$$

for all square matrices A.

Calculating the Adjugate Matrix

Step 1: Determinants of Minor Square Matrices For each element of the matrix a_{ij}, delete the i-th row and the j-th column and calculate the determinant of the matrix that is left over. Put that determinant in a new matrix in a_{ij}'s place.

Step 2: Multiply by Checkerboard Matrix Multiply each element of the result matrix in step 1 by each element of the checkerboard matrix (see next slide). (This way of multiplying matrices is called Hadamard multiplication as opposed to matrix multiplication.)

Step 3: Transpose Now transpose the result matrix of step 2 in order to calculate the adjugate matrix.

Calculating the Adjugate Matrix Example

Step 1

$$\begin{bmatrix} 4 & 1 & -3 \\ 3 & 0 & 1 \\ 8 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -3 \\ -1 & 32 & -12 \\ 1 & 13 & -3 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 1 & -2 & -3 \\ -1 & 32 & -12 \\ 1 & 13 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 \\ 1 & 32 & 12 \\ 1 & -13 & -3 \end{bmatrix}$$

Checkerboard Matrix: $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ or, more simply, $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Step 3

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 32 & 12 \\ 1 & -13 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 32 & -13 \\ -3 & -12 & -3 \end{bmatrix}$$

Finding Inverse Using Adjugate

Right-multiply equation (10) by A^{-1} to see that

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \tag{11}$$

For example, the adjugate of

$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & 2 & 0 \\ 4 & 3 & 1 \end{bmatrix} \text{ is } \begin{bmatrix} 2 & -3 & 1 \\ 13 & -16 & -4 \\ -8 & 12 & 3 \end{bmatrix}^{\mathsf{T}} \tag{12}$$

Therefore, the inverse is

$$\frac{1}{7} \cdot \begin{bmatrix} 2 & 13 & -8 \\ -3 & -16 & 12 \\ 1 & -4 & 3 \end{bmatrix} \tag{13}$$

Exercise 2: Consider

$$B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \tag{14}$$

Calculate B^{-1} and show that $B \cdot B^{-1} = I$.

Exercise 3: Consider

$$D = \begin{bmatrix} 0 & -1 & 4 \\ 3 & 2 & 0 \\ 4 & 3 & -1 \end{bmatrix} \tag{15}$$

Calculate $\det(D)$. Then use software to calculate the inverse of D. What do you notice about $\det(D) \cdot D^{-1}$?

Systems of Linear Equations Introduced

Chaitali and Amulya go to a concession stand to buy fruit. Chaitali buys 5 bananas and 3 apples and spends \$13.50. Amulya buys 1 banana and 5 apples and spends 20 cents more than Chaitali. How much do bananas and apples cost at the concession stand?

Chaitali and Amulya go to a concession stand to buy fruit. Chaitali buys 5 bananas and 3 apples and spends \$13.50. Amulya buys 1 banana and 5 apples and spends 20 cents more than Chaitali. How much do bananas and apples cost at the concession stand?

$$\begin{aligned} 5x &+ 3y &= 13.5 \\ x &+ 5y &= 13.7 \end{aligned} \tag{16}$$

What Is a System of Linear Equations?

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{17}$$

This system of linear equations is the rule for the following set $S \subset \mathbb{R} \times \mathbb{R}$:

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} | 5x + 3y = 13.5 \text{ and } x + 5y = 13.7\} \tag{18}$$

$$
\begin{array}{rcrcl}
5x & + & 3y & = & 13.5 \\
x & + & 5y & = & 13.7
\end{array}
\tag{19}
$$

There are several ways to solve a system of equations like this.

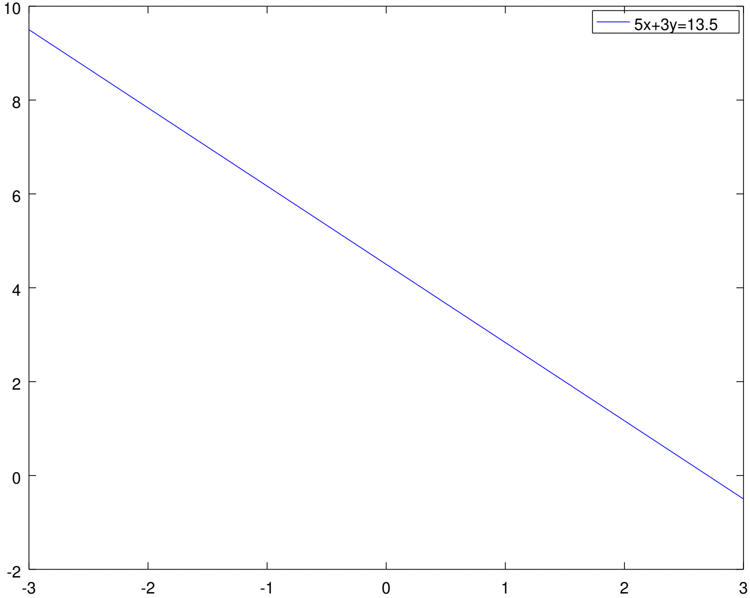
- Graphing
- Substitution
- Elimination
- Using a Matrix

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{20}$$

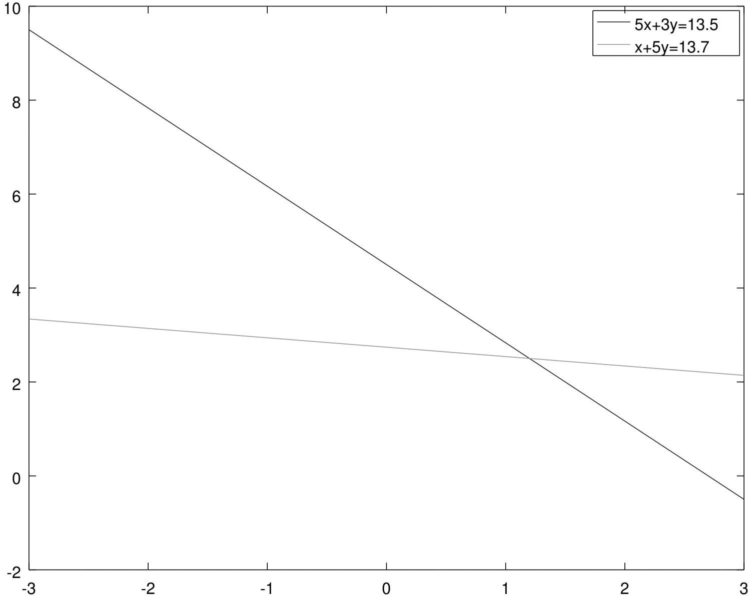
is equivalent to

$$\begin{array}{rcl} y & = & -\frac{5}{3}x + \frac{9}{2} \\ \\ y & = & -\frac{1}{5}x + \frac{137}{50} \end{array} \tag{21}$$

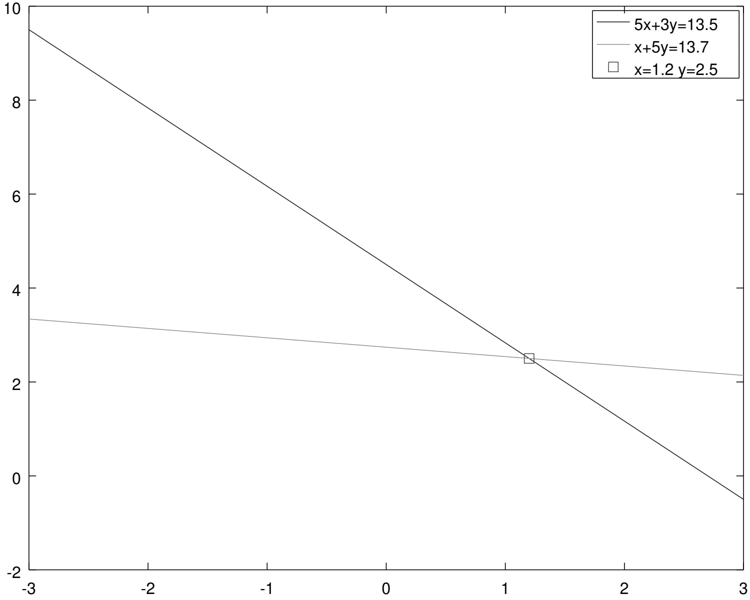
Graphing Method II



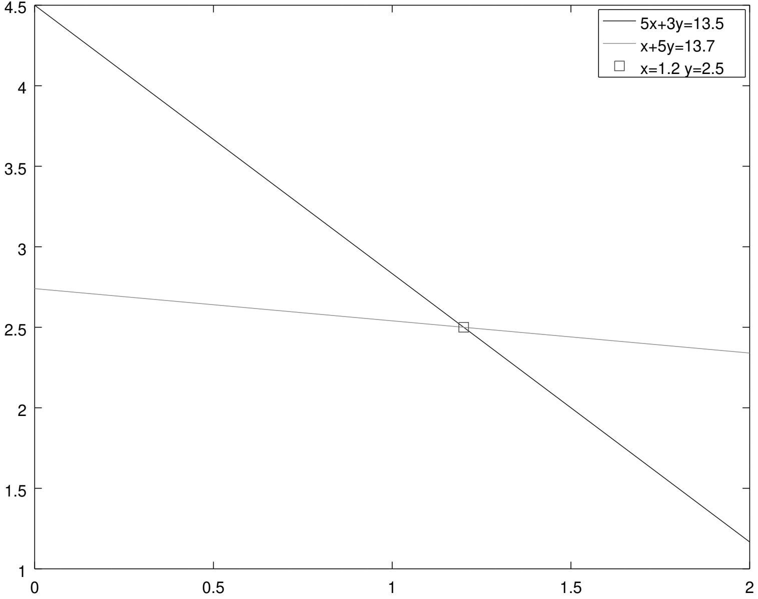
Graphing Method III



Graphing Method IV



Graphing Method V



Graphing Method Exercises

Find a solution to these systems of linear equations by graphing them and check your answer by substituting.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{22}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{23}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{24}$$

Substitution Method I

$$\begin{aligned} 5x &+ 3y &= 13.5 \\ x &+ 5y &= 13.7 \end{aligned} \tag{25}$$

The second equation yields $x = 13.7 - 5y$. Use this to substitute in the first equation

$$5 \cdot (13.7 - 5y) + 3y = 13.5 \tag{26}$$

therefore, $-22y = -55$ and $y = 5/2$. Now substitute $y = 5/2$ in the first equation (you could just as well use the second equation), so

$$5x + 3 \cdot \frac{5}{2} = 13.5 \tag{27}$$

which implies $x = 1.2$. A banana costs \$1.20; an apple costs \$2.50.

Substitution Method Exercises

Find a solution to these systems of linear equations by using the substitution method.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{28}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{29}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{30}$$

Elimination Method I

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{31}$$

is equivalent to

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ 5x & + & 25y & = & 68.5 \end{array} \tag{32}$$

Elimination Method II

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ 5x & + & 25y & = & 68.5 \end{array} \tag{33}$$

implies

$$(5x + 3y) - (5x + 25y) = 13.5 - 68.5 \tag{34}$$

therefore, $-22y = -55$ and $y = 5/2$. Now substitute $y = 5/2$ in the first equation (you could just as well use the second equation), so

$$5x + 3 \cdot \frac{5}{2} = 13.5 \tag{35}$$

which implies $x = 1.2$. A banana costs \$1.20; an apple costs \$2.50.

Elimination Method Exercises

Find a solution to these systems of linear equations by using the elimination method.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{36}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{37}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{38}$$

Matrices and Systems of Linear Equations I

Remember our system of linear equations.

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{39}$$

In matrix notation, we can write

$$\begin{bmatrix} 5 & 3 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13.5 \\ 13.7 \end{bmatrix}$$

Matrices and Systems of Linear Equations II

Let's call these three matrices A, v, b respectively. A and b are provided, and we are looking for v. If we had A^{-1}, we could go from

$$Av = b \tag{40}$$

to

$$A^{-1}Av = A^{-1}b \tag{41}$$

which is the same as

$$v = A^{-1}b \tag{42}$$

Matrix Row Operations

Another method to find the inverse of a matrix is using matrix row operations. There are three matrix row operations.

- Row Switching means you are allowed to switch two rows, for example $R_1 \leftrightarrow R_2$
- Row Multiplication means you are allowed to multiply all elements of a row by a real non-zero number, for example $\frac{2}{5}R_2 \rightarrow R_2$
- Row Addition means you are allowed to add one row to another and then replace one of the original rows by the sum of the two rows, for example $R_1 + R_2 \rightarrow R_1$

Row multiplication and row addition are often used together, for example $\frac{7}{8}R_1 - R_3 \rightarrow R_3$.

Matrix Row Operations

To find the inverse of a square matrix, we combine A and E

$$\left[\begin{array}{cccc} 5 & 3 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right]$$

and apply matrix row operations until we get

$$\left[\begin{array}{cccc} 1 & 0 & x & y \\ 0 & 1 & z & w \end{array}\right]$$

where

$$A^{-1} = \left[\begin{array}{cc} x & y \\ z & w \end{array}\right]$$

Inverse Example

For our example,

$$\begin{bmatrix} 5 & 3 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 25/3 & 5 & 5/3 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 22/3 & 0 & 5/3 & -1 \\ 1 & 5 & 0 & 1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 22/3 & 0 & 5/3 & -1 \\ 22/3 & 110/3 & 0 & 22/3 \end{bmatrix} \longrightarrow \begin{bmatrix} 22/3 & 0 & 5/3 & -1 \\ 0 & 110/3 & -5/3 & 25/3 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & 5/22 & -3/22 \\ 0 & 1 & -1/22 & 5/22 \end{bmatrix}$$

Inverse Example

For step 1, we multiplied the first row by $5/3$ (row multiplication). For step 2, we subtracted the second row from the first row and replaced the first row by the result (row addition). For step 3, we multiplied the second row by $22/3$ (row multiplication). For step 4, we subtracted the first row from the second row and replaced the second row by the result (row addition). For the last step, we multiplied the first row by $3/22$ and the second row by $3/110$ (row multiplication applied twice).

Thus,

$$A^{-1} = \begin{bmatrix} 5/22 & -3/22 \\ -1/22 & 5/22 \end{bmatrix} = \frac{1}{22} \cdot \begin{bmatrix} 5 & -3 \\ -1 & 5 \end{bmatrix}$$

and

$$v = A^{-1}b = \begin{bmatrix} 5/22 & -3/22 \\ -1/22 & 5/22 \end{bmatrix} \cdot \begin{bmatrix} 13.5 \\ 13.7 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 2.5 \end{bmatrix}$$

Exercise 4: Marina had \$24,500 to invest. She divided the money into three different accounts. At the end of the year, she had made \$1,300 in interest. The annual yield on each of the three accounts was 4%, 5.5%, and 6%. If the amount of money in the 4% account was four %times the amount of money in the 5.5% account, how much had she %placed in each account?

Exercise 5: The currents running through an electrical system are given by the following system of equations. The three currents I_1, I_2, I_3 are measured in amps. Solve the system to find the currents in this circuit.

$$\begin{array}{rcrcrcl} I_1 & + & 2I_2 & - & I_3 & = & 0.425 \\ 3I_1 & - & I_2 & + & 2I_3 & = & 2.225 \\ 5I_1 & + & I_2 & + & 2I_3 & = & 3.775 \end{array} \tag{43}$$

Exercise 6: Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the following three points:
$(-2, 40), (1, 7), (3, 15)$.

Exercise 7: Billy's Restaurant ordered 200 flowers for Mother's Day. They ordered carnations at \$1.50 each, roses at \$5.75 each, and daisies at \$2.60 each. They ordered mostly carnations; and 20 fewer roses than daisies. The total order came to \$589.50. How many of each type of flower was ordered?

Exercise 8: The Arcadium arcade in Lynchburg, Tennessee uses 3 different colored tokens for their game machines. For $20 you can purchase any of the following mixtures of tokens: 14 gold, 20 silver, and 24 bronze; OR, 20 gold, 15 silver, and 19 bronze; OR, 30 gold, 5 silver, and 13 bronze. What is the monetary value of each token?

Exercise 9: In the position function for vertical height

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0 \tag{44}$$

$s(t)$ represents height in meters and t represents time in seconds.

1. Find the position function for a volleyball served at an initial height of one meter, with height of 6.275 meters 0.5 seconds after serve, and height of 9.1 meters one second after serve.
2. How long until the ball hits the ground on the other side of the net if everyone on that team completely misses it?

Exercise 10: Last Tuesday, Regal Cinemas sold a total of 8500 movie tickets. Proceeds totaled \$64,600. Tickets can be bought in one of 3 ways: a matinee admission costs \$5, student admission is \$6 all day, and regular admissions are \$8.50. How many of each type of ticket was sold if twice as many student tickets were sold as matinee tickets?

End of Lesson

Next Lesson: Vectors