Linear Equations

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

September 17, 2018

Chaitali and Amulya go to a concession stand to buy fruit. Chaitali buys 5 bananas and 3 apples and spends \$13.50. Amulya buys 1 banana and 5 apples and spends 20 cents more than Chaitali. How much do bananas and apples cost at the concession stand?

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$$\begin{aligned} 5x &+ 3y = 13.5 \\ x &+ 5y = 13.7 \end{aligned} \tag{1}$$

What Is a System of Linear Equations?

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{2}$$

This system of linear equations is the rule for the following set $S \subset \mathbb{R} \times \mathbb{R}$:

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} | 5x + 3y = 13.5 \text{ and } x + 5y = 13.7\} \tag{3}$$

Solution Methods

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{4}$$

There are several ways to solve a system of equations like this.

- Graphing
- Substitution
- Elimination
- Using a Matrix

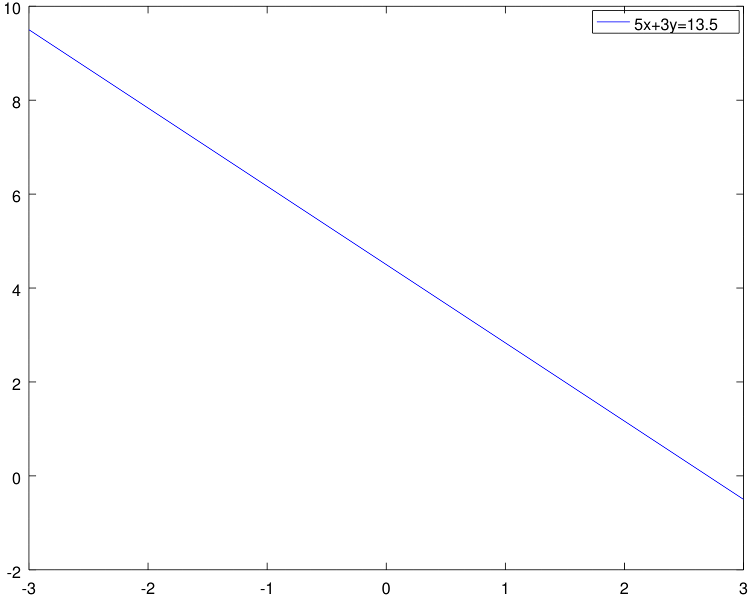
Graphing Method I

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{5}$$

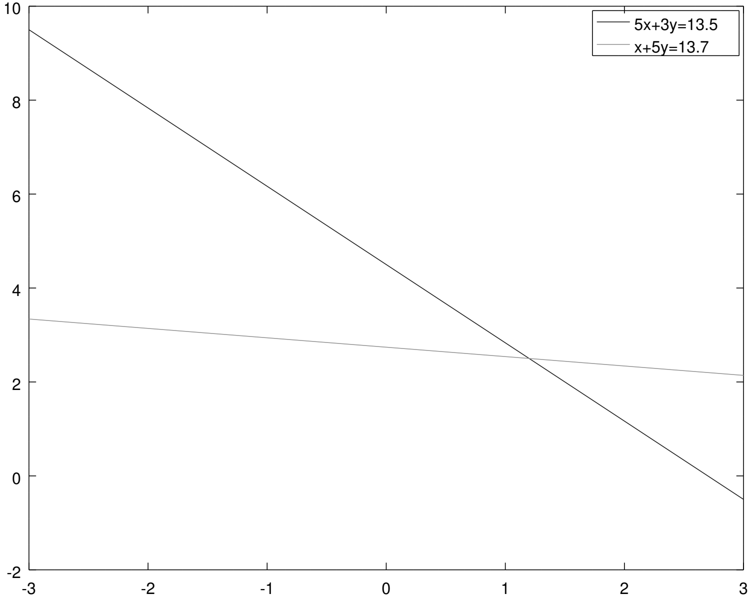
is equivalent to

$$\begin{array}{rcl} y & = & -\frac{5}{3}x + \frac{9}{2} \\ y & = & -\frac{1}{5}x + \frac{137}{50} \end{array} \tag{6}$$

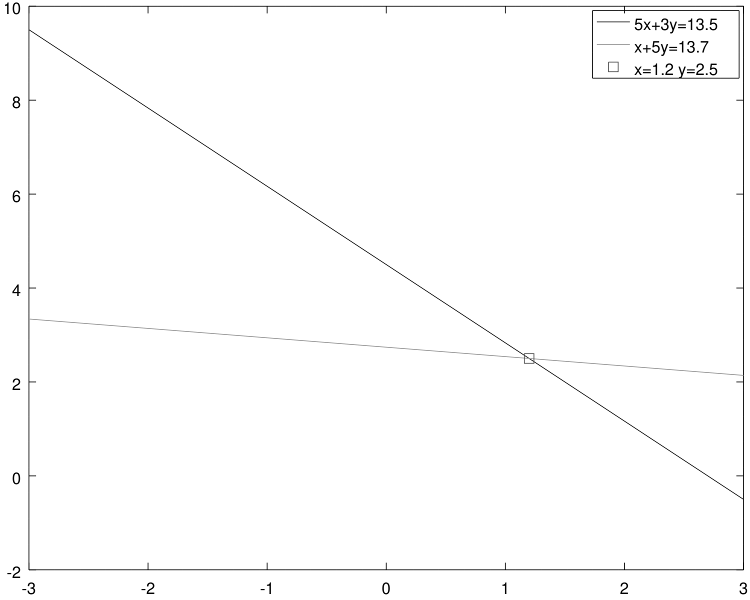
Graphing Method II



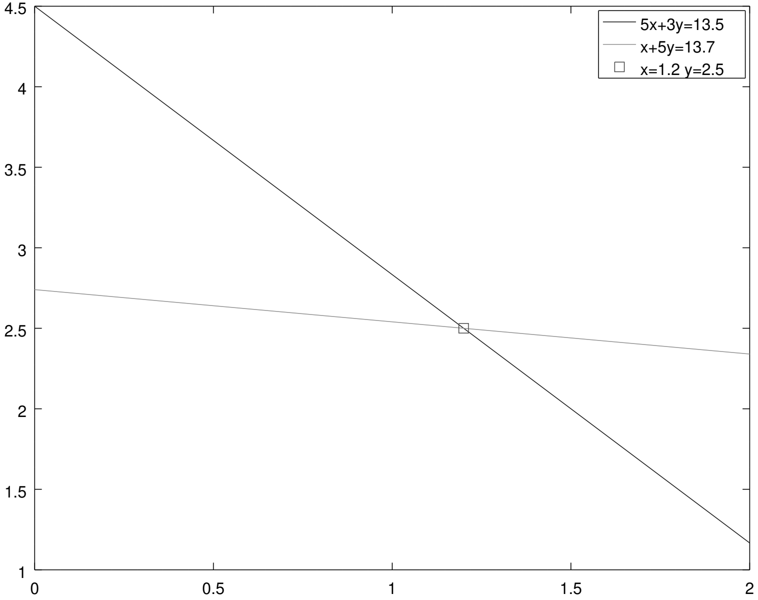
Graphing Method III



Graphing Method IV



Graphing Method V



Graphing Method Exercises

Find a solution to these systems of linear equations by graphing them and check your answer by substituting.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{7}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{8}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{9}$$

Substitution Method I

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{10}$$

The second equation yields $x = 13.7 - 5y$. Use this to substitute in the first equation

$$5 \cdot (13.7 - 5y) + 3y = 13.5 \tag{11}$$

therefore, $-22y = -55$ and $y = 5/2$. Now substitute $y = 5/2$ in the first equation (you could just as well use the second equation), so

$$5x + 3 \cdot \frac{5}{2} = 13.5 \tag{12}$$

which implies $x = 1.2$. A banana costs \$1.20; an apple costs \$2.50.

Substitution Method Exercises

Find a solution to these systems of linear equations by using the substitution method.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{13}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{14}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{15}$$

Elimination Method I

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{16}$$

is equivalent to

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ 5x & + & 25y & = & 68.5 \end{array} \tag{17}$$

Elimination Method II

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ 5x & + & 25y & = & 68.5 \end{array} \quad (18)$$

implies

$$(5x + 3y) - (5x + 25y) = 13.5 - 68.5 \quad (19)$$

therefore, $-22y = -55$ and $y = 5/2$. Now substitute $y = 5/2$ in the first equation (you could just as well use the second equation), so

$$5x + 3 \cdot \frac{5}{2} = 13.5 \quad (20)$$

which implies $x = 1.2$. A banana costs \$1.20; an apple costs \$2.50.

Elimination Method Exercises

Find a solution to these systems of linear equations by using the elimination method.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{21}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{22}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{23}$$

Remember our system of linear equations.

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{24}$$

In matrix notation, we can write

$$\begin{bmatrix} 5 & 3 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13.5 \\ 13.7 \end{bmatrix}$$

Matrices and Systems of Linear Equations II

Let's call these three matrices A, v, b respectively. A and b are provided, and we are looking for v. If we had A^{-1}, we could go from

$$Av = b \tag{25}$$

to

$$A^{-1}Av = A^{-1}b \tag{26}$$

which is the same as

$$v = A^{-1}b \tag{27}$$

Matrix Row Operations

Another method to find the inverse of a matrix is using matrix row operations. There are three matrix row operations.

- Row Switching means you are allowed to switch two rows, for example $R_1 \leftrightarrow R_2$
- Row Multiplication means you are allowed to multiply all elements of a row by a real non-zero number, for example $\frac{2}{5}R_2 \rightarrow R_2$
- Row Addition means you are allowed to add one row to another and then replace one of the original rows by the sum of the two rows, for example $R_1 + R_2 \rightarrow R_1$

Row multiplication and row addition are often used together, for example $\frac{7}{8}R_1 - R_3 \rightarrow R_3$.

Matrix Row Operations

To find the inverse of a square matrix, we combine A and I

$$\begin{bmatrix} 5 & 3 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix}$$

and apply matrix row operations until we get

$$\begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{bmatrix}$$

where

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Inverse Example

For our example,

$$\left[\begin{array}{cccc} 5 & 3 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 25/3 & 5 & 5/3 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccc} 22/3 & 0 & 5/3 & -1 \\ 1 & 5 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccc} 22/3 & 0 & 5/3 & -1 \\ 22/3 & 110/3 & 0 & 22/3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 22/3 & 0 & 5/3 & -1 \\ 0 & 110/3 & -5/3 & 25/3 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccc} 1 & 0 & 5/22 & -3/22 \\ 0 & 1 & -1/22 & 5/22 \end{array}\right]$$

Inverse Example

For step 1, we multiplied the first row by $5/3$ (row multiplication). For step 2, we subtracted the second row from the first row and replaced the first row by the result (row addition). For step 3, we multiplied the second row by $22/3$ (row multiplication). For step 4, we subtracted the first row from the second row and replaced the second row by the result (row addition). For the last step, we multiplied the first row by $3/22$ and the second row by $3/110$ (row multiplication applied twice).

Thus,

$$A^{-1} = \begin{bmatrix} 5/22 & -3/22 \\ -1/22 & 5/22 \end{bmatrix} = \frac{1}{22} \cdot \begin{bmatrix} 5 & -3 \\ -1 & 5 \end{bmatrix}$$

and

$$v = A^{-1}b = \begin{bmatrix} 5/22 & -3/22 \\ -1/22 & 5/22 \end{bmatrix} \cdot \begin{bmatrix} 13.5 \\ 13.7 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 2.5 \end{bmatrix}$$

Matrices and Systems of Linear Equations IV

Why does this procedure work in order to give us the inverse matrix?

$$\begin{bmatrix} A & I \end{bmatrix} \xrightarrow{\text{elementary row operations}} \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

Because each elementary row operation corresponds to a matrix multiplication by some matrix E_i. This means that A changes into I by the following process:

$$A \cdot E_1 \cdot E_2 \cdot \ldots \cdot E_k = I \quad (28)$$

Therefore, $E_1 \cdot E_2 \cdot \ldots \cdot E_k = A^{-1}$ and as A is changed into I, I is changed to A^{-1}:

$$I \cdot E_1 \cdot E_2 \cdot \ldots \cdot E_k = A^1 \quad (29)$$

Exercise 1: Marina had \$24,500 to invest. She divided the money into three different accounts. At the end of the year, she had made \$1,300 in interest. The annual yield on each of the three accounts was 4%, 5.5%, and 6%. If the amount of money in the 4% account was four %times the amount of money in the 5.5% account, how much had she %placed in each account?

Exercise 2: The currents running through an electrical system are given by the following system of equations. The three currents I_1, I_2, I_3 are measured in amps. Solve the system to find the currents in this circuit.

$$\begin{array}{rcrcrcl} I_1 & + & 2I_2 & - & I_3 & = & 0.425 \\ 3I_1 & - & I_2 & + & 2I_3 & = & 2.225 \\ 5I_1 & + & I_2 & + & 2I_3 & = & 3.775 \end{array} \tag{30}$$

Exercise 3: Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the following three points:
$(-2, 40), (1, 7), (3, 15)$.

Exercise 4: Billy's Restaurant ordered 200 flowers for Mother's Day. They ordered carnations at \$1.50 each, roses at \$5.75 each, and daisies at \$2.60 each. They ordered mostly carnations; and 20 fewer roses than daisies. The total order came to \$589.50. How many of each type of flower was ordered?

Exercise 5: The Arcadium arcade in Lynchburg, Tennessee uses 3 different colored tokens for their game machines. For $20 you can purchase any of the following mixtures of tokens: 14 gold, 20 silver, and 24 bronze; OR, 20 gold, 15 silver, and 19 bronze; OR, 30 gold, 5 silver, and 13 bronze. What is the monetary value of each token?

Exercise 6: In the position function for vertical height

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0 \tag{31}$$

$s(t)$ represents height in meters and t represents time in seconds.

1. Find the position function for a volleyball served at an initial height of one meter, with height of 6.275 meters 0.5 seconds after serve, and height of 9.1 meters one second after serve.
2. How long until the ball hits the ground on the other side of the net if everyone on that team completely misses it?

Exercise 7: Last Tuesday, Regal Cinemas sold a total of 8500 movie tickets. Proceeds totaled \$64,600. Tickets can be bought in one of 3 ways: a matinee admission costs \$5, student admission is \$6 all day, and regular admissions are \$8.50. How many of each type of ticket was sold if twice as many student tickets were sold as matinee tickets?

Exercise 8: Curve fitting. Determine the equation of the circle which passes through the three points $(1, 1), (2, 1), (1, 3)$.

Exercise 9: You receive a coded message. You know that each letter of the original message was replaced with a one- or two-digit number corresponding to its placement in the English alphabet, so "E" is represented by "5" and "W" by "23"; spaces in the message are indicated by zeroes. You also know that the message was transformed (encoded) left-multiplying the message by the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \tag{32}$$

Translate the coded message:

$$\begin{bmatrix} 105 & 62 & 53 & 107 & 88 & 37 & 52 & 31 \\ 99 & 76 & 19 & 92 & 81 & 16 & 49 & 14 \\ 149 & 25 & 189 & 185 & 131 & 124 & 75 & 99 \end{bmatrix} \tag{33}$$

Cramer's Rule

Cramer's rule makes finding the solutions to systems of linear equations very simple, at the expense of understanding what's going on. It's a bit of black magic. Recall the following equations:

$$\det(A) \cdot I = \operatorname{adj}(A) \cdot A \tag{34}$$

$$A \cdot v = b \tag{35}$$

(35) is a system of linear equations whose solution vector is v. Right-multiply (34) by v and divide by $\det(A)$ for

$$v = \frac{1}{\det(A)} \operatorname{adj}(A) \cdot b \tag{36}$$

Cramer's Rule

Let c_{ij} be the elements of the checkerboard matrix. $\mathrm{adj}(A) \cdot b$ is a vector whose elements x_i are

$$x_i = c_{i1} \det(A_{1i}) b_1 + c_{i2} \det(A_{2i}) b_2 + \ldots \tag{37}$$

Looking closely at (37), you will notice that x_i is also the determinant of A_i, where A_i is the matrix A with the i-th column replaced by b (choosing b as the column along which you calculate the matrix).

Consider the following system of linear equations:

$$\begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 9 \\ 6 \end{bmatrix} \tag{38}$$

The inverse of the coefficient matrix is

$$-\frac{1}{7} \cdot \begin{bmatrix} -7 & 14 & 0 \\ 0 & -3 & 1 \\ -7 & 15 & 2 \end{bmatrix} \tag{39}$$

Therefore, the solution to the system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{7} \cdot \begin{bmatrix} -7 & 14 & 0 \\ 0 & -3 & 1 \\ -7 & 15 & 2 \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \tag{40}$$

Cramer's Rule

Finding the inverse, however, is time-consuming. Also, it gives us all three solutions, and we may only want the value of one of the variables and not all of them. Cramer's rule tells us that, for example,

$$y = \frac{\det(A_y)}{\det(A)} \tag{41}$$

where A is the coefficient matrix and A_y is the coefficient matrix with the second column (corresponding to y) replaced by the vector of constants. In other words,

$$y = \frac{\det\left(\begin{bmatrix} 3 & 20 & -2 \\ 1 & 9 & -1 \\ 3 & 6 & -3 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix}\right)} = \frac{-21}{-7} = 3 \tag{42}$$

Cramer's Rule

For the other two variables,

$$x = \frac{\det\left(\begin{bmatrix} 20 & 4 & -2 \\ 9 & 2 & -1 \\ 6 & -1 & -3 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix}\right)} = \frac{-14}{-7} = 2 \tag{43}$$

$$z = \frac{\det\left(\begin{bmatrix} 3 & 4 & 20 \\ 1 & 2 & 9 \\ 3 & -1 & 6 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix}\right)} = \frac{7}{-7} = -1 \tag{44}$$

Echelon Form

Consider the system of linear equations from the previous slides:

$$\begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 9 \\ 6 \end{bmatrix} \tag{45}$$

Combine the coefficient matrix and the vector of constants to an augmented matrix:

$$\begin{bmatrix} 3 & 4 & -2 & 20 \\ 1 & 2 & -1 & 9 \\ 3 & -1 & -3 & 6 \end{bmatrix} \tag{46}$$

Now use elementary row operations to make sure that only zeroes populate the matrix below the diagonal. This is called the echelon form of the system.

$$\begin{bmatrix} 3 & 4 & -2 & 20 \\ 0 & -10 & 5 & -35 \\ 0 & 0 & 7 & -7 \end{bmatrix} \tag{47}$$

Echelon Form

Because the echelon form is the product of elementary row operations, the solutions of the system of linear equations associated with it are the same as the solutions of the original system of linear equations.

$$\begin{array}{rcrcrcr} 3x & + & 4y & - & 2z & = & 20 \\ & & -10y & + & 5z & = & -35 \\ & & & & 7z & = & -7 \end{array} \tag{48}$$

The last equation tells us that $z = -1$. Substituting $z = -1$, the middle equation tells us that $y = 3$. Substituting both of these results in the first equation tells us that $x = 2$.

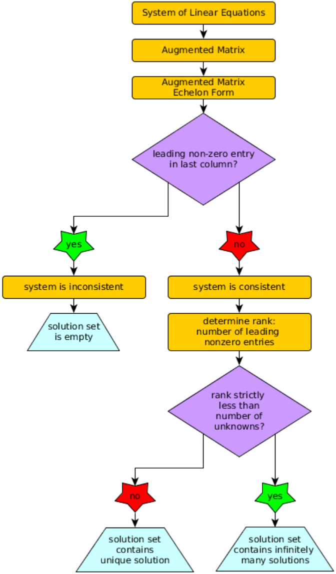
Echelon Form

The echelon form provides another way to solve a system of linear equations. The elementary row operations are called Gaussian elimination or Gauss-Jordan elimination (there are technical details about the difference between these two elimination methods that we are not worried about right now). Gaussian or Gauss-Jordan elimination, however, is hard to do manually. We use the echelon form primarily to deal with cases where

- the determinant of the coefficient matrix is zero
- the number of equations and the number of unknowns are not equal

1. If some row of an echelon form has its first nonzero entry in the last column, then the system has no solution. The system is inconsistent.
2. If a system is consistent, it has a rank. The rank is the number of leading nonzero entries with respect to the rows of the echelon form.
3. If the rank of a system equals the number of unknowns, then the system has exactly one solution.
4. If the rank of a system is strictly less than the number of unknowns, then the system has infinitely many solutions.

Theory of Linear Systems



Echelon Form Exercises

Exercise 10: Determine the number of solutions for the following echelon forms.

$$\begin{bmatrix} 5 & -3 & 1 & 4 \\ 0 & -\frac{7}{5} & \frac{4}{5} & \frac{1}{5} \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad \begin{bmatrix} 2 & \pi & 4 & 3 \\ 0 & 5 & 6 & -13 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & e^2 & -1 & 12 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & e^2 & -1 & 12 \\ 0 & 1 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 3 & e^2 & -1 & 12 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

End of Lesson

Next Lesson: Vectors