Vectors

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

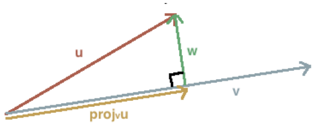
October 8, 2018

Projection

The projection u_H of a vector u onto a hyperplane H is the vector in the hyperplane that is "most similar" to u. The formal definition for u_H requires that

1. u is in H
2. $(u - u_H)$ is orthogonal to all basis vectors of H

Example 1: Finding a Projection. Let H be the line spanned by $\vec{v} = (-1, 1)^\intercal$ in $\mathbb{R}^2$. What is the projection $\vec{w}$ of $\vec{u} = (3, -2)^\intercal$?



Let $\vec{w} = (w_1, w_2)^\intercal$. Then (1) $\vec{u} - \vec{w}$ is orthogonal to $\vec{v}$ and (2) $\vec{w} = \alpha\vec{v}$ for some $\alpha \in \mathbb{R}$.

$$\begin{array}{ccccc} w_1 & - & w_2 & = & 5 \\ w_1 & + & w_2 & = & 0 \end{array} \qquad (1)$$

Cramer's rule tells us that $\vec{w} = (2.5, -2.5)^\intercal$.

Projection

Let $u = (u_1, \ldots, u_n)^\intercal$ be a vector and H be a k-dimensional hyperplane in the vector space $\mathbb{R}^n$. Let $x_1, \ldots, x_k$ be a basis for H. Then it is true for all vectors v in the hyperplane that

$$\|u - v\| \geq \|u - u_H\| \tag{2}$$

Proof: use the theorem of Pythagoras for

$$\|u - v\|^2 = \|u - u_H\|^2 + \|u_H - v\|^2 \geq \|u - u_H\|^2 \tag{3}$$

The claim follows. It illustrates what I mean when I say that u_H is the vector in H that is most similar to u.

Example 2: Finding Another Projection. What is the projection of $\vec{u} = (5, 2, 10)^\intercal$ onto the plane T characterized by $2x + y + 3z = 0$?

First we find two linearly independent vectors in H to form a basis of H, for example $\vec{v_1} = (1, 1, -1)^\intercal$ and $\vec{v_2} = (0, -3, 1)^\intercal$. The conditions

1. $u_H \in T$
2. $(u - u_H) \perp v_1$
3. $(u - u_H) \perp v_2$

give us the system of linear equations

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix} \tag{4}$$

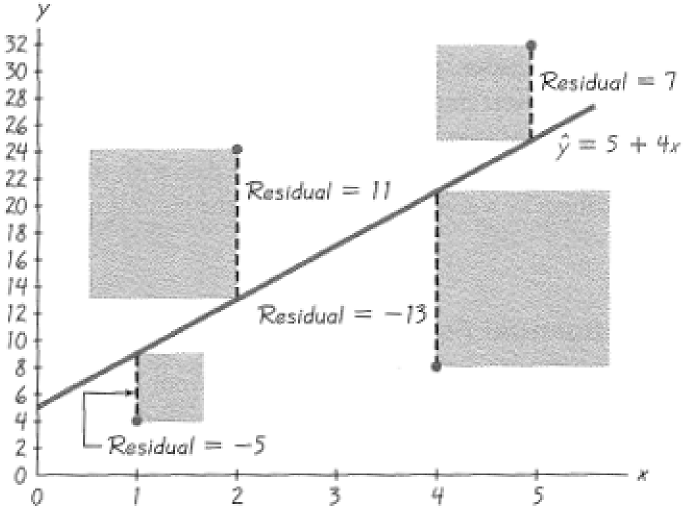
for which the solution is $u_H = (\hat{x}, \hat{y}, \hat{z})^\intercal = (-1, -1, 1)^\intercal$.

Consider the following table of measurements for the length of shoe prints and the height of the person wearing the shoes.

Shoe Print (cm)	Height (cm)
29.7	175.3
29.9	177.8
31.4	185.4
31.8	175.3
27.6	172.7

In the statistics portion of this course, we will learn whether the paired data provide evidence of a linear relationship. In the linear algebra portion, we will learn how to find the line which is closest to the data points in the least squares sense.

Least Squares Method



Least Squares Method

Least Squares Method

If L is a given line, the error for each data point is the vertical distance from that point to the line. The squared error is the sum of the squares of the errors. The line that best fits the data in the least squares sense is the line that minimizes the squared error.

You can find the regression line using calculus optimization. However, there is also an elegant method using linear algebra.

Least Squares Method

Let L be a line with slope m and y-intercept b. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be a set of paired data. Then the following equations hold:

$$\begin{array}{ccccccc} y_1 & = & mx_1 & + & b & + & \epsilon_1 \\ y_2 & = & mx_2 & + & b & + & \epsilon_2 \\ & & & \vdots & & & \\ y_n & = & mx_n & + & b & + & \epsilon_n \end{array} \tag{5}$$

where the ϵ_i are the errors $(i = 1, \ldots, n)$. This system is equivalent to the following vector equation,

$$Y = AV + E \tag{6}$$

where $Y = (y_1, \ldots, y_n)^\intercal, V = (m, b)^\intercal, E = (\epsilon_1, \ldots, \epsilon_n)^\intercal$. A is on the next slide.

Least Squares Method

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \tag{7}$$

End of Lesson

Next Lesson: Eigenvalues and Eigenvectors