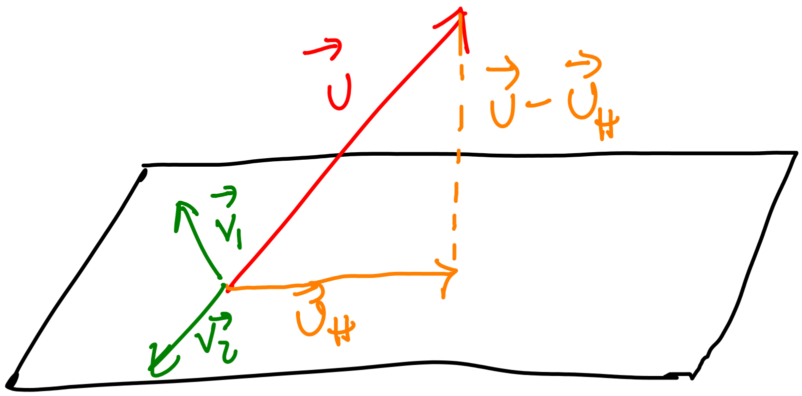
(1) What is the projection of $\vec{u} = (5, 2, 10)^\intercal$ onto the plane H characterized by $2x + y + 3z = 0$?



$$u = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix}$$

let $x_1 = 0 \quad y_1 = 1 \qquad 2 \cdot 0 + 1 + 3 \cdot z_1 = 0$

$$\hat{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{3} \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \qquad z_1 = -\frac{1}{3}$$

let $x_2 = 1 \quad y_2 = 0 \qquad 2 \cdot 1 + 0 + 3 \cdot z_2 = 0$

$$\hat{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -2/3 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \qquad z_2 = -\frac{2}{3}$$

let $u_H = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

(i) $\quad 2a + b + 3c = 0$

(ii) $\quad (5-a) \cdot 0 + (2-b) \cdot 3 + (10-c) \cdot (-1) = 0$

$\qquad 0a - 3b + c = 4$

(iii) $\quad (5-a) \cdot 3 + (2-b) \cdot 0 + (10-c) \cdot (-2) = 0$

$\qquad -3a + 0b + 2c = 5$

solve this system of linear equations for u=(-1,-1,1)

(2) Let u and v be some linearly independent vectors. Then the formula for u_H, where H is the hyperplane spanned by the basis $\{v\}$, is

$$u_H = \left(\frac{u \cdot v}{v \cdot v}\right) v \tag{1}$$

This only works for one-dimensional H! Show that it is true by writing $u_H = av$ for some $a \in \mathbb{R}$ and isolating a in

$$(u - av) \perp v$$

Let $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$(u - av) \perp v \Rightarrow$

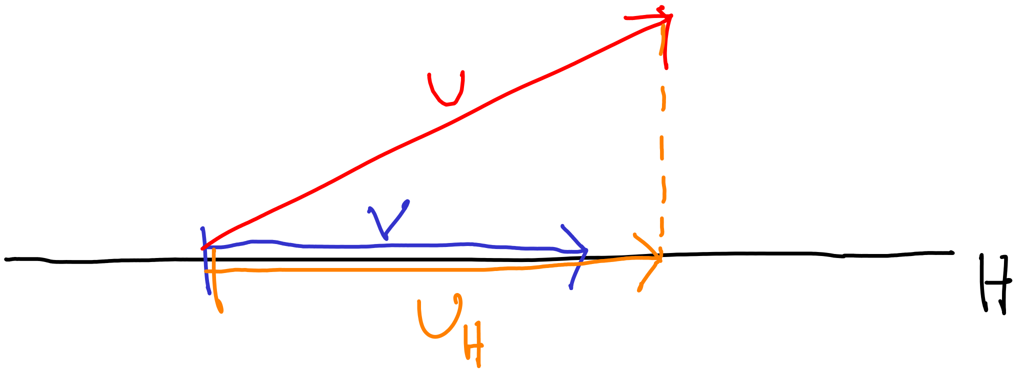
$$(u_1 - av_1)v_1 + (u_2 - av_2)v_2 + (u_3 - av_3)v_3 = 0$$

$$u_1v_1 + u_2v_2 + u_3v_3 - a(v_1v_1 + v_2v_2 + v_3v_3) = 0$$

$$u \cdot v - a \cdot (v \cdot v) = 0$$

$$a = \frac{u \cdot v}{v \cdot v}$$

Let $H = \text{span}(\{v\})$ with $v = (-2,3)^\intercal$. Find u_H for $u = (7,5)^\intercal$.



$$u_H = \left(\frac{u \cdot v}{v \cdot v}\right) v = \left(\frac{\begin{pmatrix}7\\5\end{pmatrix}\begin{pmatrix}-2\\3\end{pmatrix}}{\begin{pmatrix}-2\\3\end{pmatrix}\begin{pmatrix}-2\\3\end{pmatrix}}\right)\begin{pmatrix}-2\\3\end{pmatrix} =$$

$$\left(\frac{7\cdot(-2)+5\cdot 3}{(-2)(-2)+3\cdot 3}\right)\begin{pmatrix}-2\\3\end{pmatrix} = \frac{1}{13}\cdot\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}-\frac{2}{13}\\ \frac{3}{13}\end{pmatrix}$$

(4) Let's try **(1)** again with a different strategy: What is the projection of $\vec{u} = (5, 2, 10)^\intercal$ onto the plane H characterized by $2x + y + 3z = 0$?

$$v_1 = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \qquad \hat{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \text{ with } v_2 \perp v_1 \text{ and } v_2 \in H$$

(i) $0 \cdot x_2 + 3y_2 - z_2 = 0 \quad (v_2 \perp v_1)$

(ii) $2x_2 + y_2 + 3z_2 = 0 \quad (v_2 \in H)$

let $z_2 = 3 \Rightarrow y_2 = 1$

$$2x_2 + 1 + 3 \cdot 3 = 0 \qquad v_2 = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$$

$$2x_2 = -10$$

$$x_2 = -5$$

$$u_{H(v_1)} = \left(\frac{\begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}}{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}} \right) \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1.2 \\ 0.4 \end{pmatrix}$$

$$u_{H(v_2)} = \left(\frac{\begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}} \right) \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0.2 \\ 0.6 \end{pmatrix}$$

$$u_H = u_{H(v_1)} + u_{H(v_2)} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

(5) Consider the system of non-linear equations

$$\begin{array}{rcrcl} 3^x & - & y^2 & = & 2 \\ xy & + & \cos(y-5) & = & 16 \end{array} \tag{2}$$

To solve it numerically, we need to linearize the following function around an initial estimate of the solution $x = 4, y = 4$.

$$f((x,y)^\intercal) = \begin{pmatrix} e^{x \ln 3} - y^2 - 2 \\ xy + \cos(y-5) - 16 \end{pmatrix} \tag{3}$$

Find the Jacobian of f and derive the linearization of the function around $x = 4, y = 4$.

$$f_1(x,y) = e^{x\ln 3} - y^2 - 2 \qquad f_2(x,y) = xy + \cos(y-5) - 16$$

$$\frac{\partial f_1}{\partial x} = (\ln 3)e^{x \ln 3} \qquad \frac{\partial f_1}{\partial y} = -2y$$

$$\frac{\partial f_2}{\partial x} = y \qquad \frac{\partial f_2}{\partial y} = x + \sin(y-5)$$

$$f(x,y) \approx f(4,4) + J(4,4)\begin{pmatrix} x-4 \\ y-4 \end{pmatrix} =$$

$$\begin{bmatrix} 3^4 - 4^2 - 2 \\ 4\cdot 4 + \cos(-1) - 16 \end{bmatrix} + \begin{bmatrix} (\ln 3)\cdot 3^4 & -2\cdot 4 \\ 4 & 4 + \sin(-1) \end{bmatrix}\begin{bmatrix} x-4 \\ y-4 \end{bmatrix} =$$

$$\begin{bmatrix} 63 \\ 0.54030 \end{bmatrix} + \begin{bmatrix} 88.9876 & -8 \\ 4 & 3.1585 \end{bmatrix}\begin{bmatrix} x-4 \\ y-4 \end{bmatrix} =$$

$$\begin{bmatrix} 88.9876x - 8y - 260.9504 \\ 4x + 3.1585y - 28.0937 \end{bmatrix}$$

You can convert these back into a system of linear equations that approximates the original system of non-linear equations. Solve it to get a better approximation than x=4,y=4. Use the new approximation to repeat the process until you are satisfied with the precision of your solution. This is called Newton's method.