Vectors

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

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Vectors

A vector space V over a field F is a set on which two operations (addition and scalar multiplication) are defined. Some axioms need to be fulfilled, most relevantly closure with respect to addition and scalar multiplication:

- If $v, w \in V$, then $v + w \in V$
- If $a \in F, v \in V$, then $av \in V$

In this course, the field will always be $\mathbb{R}$ or $\mathbb{C}$, the real or the complex numbers.

Excursus: Complex Numbers

The following set

$$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\} \tag{1}$$

is called the set of complex numbers. Note that $\mathbb{R} \subset \mathbb{C}$. Operations (addition, multiplication, and so on) are defined on complex numbers the same way as on real numbers with one additional rule:

$$i^2 = -1 \tag{2}$$

Excursus: Complex Operators

Exercise 1: Find the determinant of the following matrix:

$$A = \begin{bmatrix} 1-4i & 3-i \\ -3i & 3+4i \end{bmatrix} \tag{3}$$

Exercise 2: A matrix that equals its conjugate transpose is called a Hermitian matrix. Calculate the determinate of the following example.

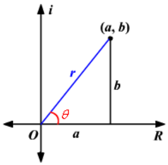
$$B = \begin{bmatrix} 2 & 2+i & 4 \\ 2-i & 3 & i \\ 4 & -i & 1 \end{bmatrix} \tag{4}$$

Exercise 3: Use expansion by conjugates to divide

$$\frac{7-2i}{3+4i} \tag{5}$$

Excursus: Polar Form

The complex numbers correspond to vectors in $\mathbb{R}^2$.



Instead of providing the coordinates (a, b) of a complex number, it is sometimes useful to provide the polar form (r, θ).

$$\begin{array}{ll} a = r\cos\theta & b = r\sin\theta \\ r^2 = a^2 + b^2 & \tan\theta = \frac{b}{a} \end{array} \tag{6}$$

A complex number $a + bi$ can always be written in its polar form $a + bi = r(\cos\theta + i\sin\theta)$.

Excursus: Euler's Formula

One of the most famous formulas in mathematics is Euler's formula

$$e^{ix} = \cos x + i \sin x \tag{7}$$

For the proof, we need some calculus. Recall the Maclaurin series expansions

$$e^{x} = \sum_{j=0}^{\infty} \frac{x^j}{j!} \tag{8}$$

$$\cos x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!} \tag{9}$$

$$\sin x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} \tag{10}$$

Calculus still works in the complex numbers, now try to find e^{ix}.

Excursus: Use of the Polar Form

Euler's formula makes multiplication, division, exponentiation and finding roots of complex numbers in polar form more simple.

Exercise 4: Multiply $(4, 60^\circ)$ by $(2, 20^\circ)$, where the given factors are complex numbers provided in polar form.

Exercise 5: Divide $(8, 100^\circ)$ by $(4, 65^\circ)$, where the given numbers are complex numbers provided in polar form.

Exercise 6: Find, using two alternative ways,

$$\frac{-2+5i}{-1-i} \text{ and } (2+3i)^5 \tag{11}$$

Excursus: Closeted Functions

You may remember that I once called trigonometric functions "closeted exponential functions." Here is the reason. Consider

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned} \tag{12}$$

Add and subtract these two equations for

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{13}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{14}$$

This is the definition of trigonometric functions on $\mathbb{C}$. The hyperbolic trigonometric functions are closeted sines and cosines, since $\cosh(x) = \cos(ix)$ and $i \sinh(x) = \sin(ix)$.

Excursus: de Moivre's Formula

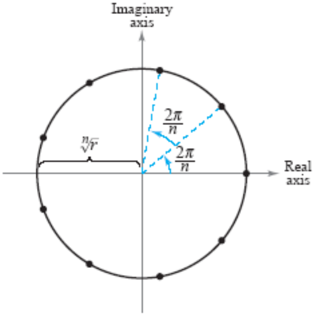
de Moivre's Formula

$\left(re^{i\theta}\right)^n = r^n e^{in\theta}$

Example 1: Cube Roots. Find the solution set for the following equation and $c = (27, 120^\circ)$, where c is a complex number provided in polar form.

$$x^3 = c \tag{15}$$

By de Moivre's formula, $x = (3, 40^\circ)$ is a solution. However, $c = (27, 480^\circ)$ and therefore, by de Moivre's formula again, $x = (3, 160^\circ)$ is also a solution. $c = (27, 840^\circ)$ provides the third solution, $x = (3, 280^\circ)$. Polynomial equations of degree n always have n solutions in $\mathbb{C}$.



Exercise 7: Solve the equation

$$x^2 + 4x + 5 = 0 \tag{16}$$

in the complex numbers.

Excursus: Casus Irreducibilis

What is the use of complex numbers? There are many engineering examples, but here is one from solving cubic equations. Find the solutions for

$$x^3 - 3x + 1 = 0 \tag{17}$$

Cardano's formula gives us the three solutions,

$$x_k = w_k \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{-3}{4}}} + w_k^2 \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{-3}{4}}} \tag{18}$$

where $k = 1, 2, 3$ and (w_1, w_2, w_3) are the three cube roots of 1. You can look up online why Cardano's formula is true. The point is that all the solutions to this problem are real numbers, but we have to use complex algebra to calculate them.

Linear Combination and Span

If $T = \{v_1, \ldots, v_k\}$ is a set of vectors and $\{x_1, \ldots, x_k\} \subset F$ is a set of scalars, then

$$x_1 v_1 + \ldots + x_k v_k \tag{19}$$

is called a linear combination of $v_1, \ldots, v_k$. The set of all linear combinations of $v_1, \ldots, v_k$ is called the span of $v_1, \ldots, v_k$.

The span of a set of vectors is a subspace of V, meaning that it is closed with respect to addition and scalar multiplication. The solution set for a system of linear equations is always a subspace (if $u, v \in S$ and $a \in F$, then $u + v \in S$ and $au \in S$).

Linear Independence

A set of vectors $v_1, \ldots, v_k$ is linearly independent if and only if the vector equation

$$x_1 v_1 + \ldots + x_k v_k = 0 \tag{20}$$

has the unique solution $(x_1, \ldots, x_k) = (0, \ldots, 0)$.

Theorems of Linear Independence

1. $v_1, \ldots, v_k$ are linearly dependent if and only if at least one of these vectors is a linear combination of the others
2. The columns of a matrix A are linearly independent if and only if the rank of A is k (the number of columns).
3. A square matrix is invertible if and only if its columns (or rows) are linearly independent.

Basis and Dimension

If V is a vector space, then the set $B = \{b_1, \ldots, b_k\}$ is a basis of V if and only if $v_1, \ldots, v_k$ are linearly independent and the span of B is V.

Unique Representation

Every vector $u \in V$ is a linear combination $u = x_1 b_1 + \ldots + x_k b_k$ of basis vectors, if such a basis exists. The ordered set $(x_1, \ldots, x_k)$ is called the coordinates of u with respect to basis B.

There is a theorem that tells us that if two sets of vectors B_1 and B_2 are bases of vector space V, then the cardinality of B_1 equals the cardinality of B_2. Thus, if a finite base exists, it makes sense to define the dimension of the vector space to be the cardinality of that base.

Vector Spaces

A finite-dimensional vector space (let the dimension be n) over the real numbers corresponds to a set of ordered sets of real numbers (the coordinates of the vectors). In other words, it corresponds to $\mathbb{R}^n$.

Examples of vector spaces:

- three-dimensional space $\mathbb{R}^3$
- the set of all straight lines in two-dimensional space
- the set of all parabolas in two-dimensional space
- the set of all circles in two-dimensional space
- the set of all polynomials with degree $k \leq n$
- the set of all real-valued functions on $\mathbb{R}$

Try to determine the dimensions of these vector spaces.

Displacement Vectors

One way to interpret a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is to make it refer to a point in the *xy*-plane or *xyz*-three-dimensional space. The usual interpretation, however, is as a displacement vector with a direction and a length. Here is an example:

$$\vec{v} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \tag{21}$$

Vectors can be added, subtracted, and multiplied by a scalar (a real number).

$$\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ \pi \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 5+\pi \\ -7 \end{pmatrix} \tag{22}$$

$$1.5 \cdot \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 7.5 \\ -1.5 \end{pmatrix} \tag{23}$$

Length and Angle of Vector

Two-dimensional and three-dimensional vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ are easily visualized. They have a length and a direction (in two-dimensional space, the direction can be represented by an angle). We use the dot product to define length and angle for any vectors. Let V be an n-dimensional vector space with a basis. Let u and v be vectors whose coordinates with respect to the basis are $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$. Organize the coordinate vectors as one-column matrices $\vec{u}$ and $\vec{v}$. Then the dot product is

$$u \cdot v = \vec{u}^{\mathsf{T}} \cdot \vec{v} \tag{24}$$

Unit Vectors

All three-dimensional vectors can be expressed in components. For this expression we need unit vectors. Any three linearly independent vectors would work, but it makes sense to use the following three:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{25}$$

Vector Decomposition

For any vector $\vec{v}$ (assuming from now on three dimensions),

$$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k} \tag{26}$$

where $V = (v_x, v_y, v_z)$, and V is the point to which the origin $O = (0, 0, 0)$ would be displaced by vector

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \tag{27}$$

The length of vector $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{28}$$

The distance between two points P and Q is the length of a displacement vector between them. Let $\vec{OP}$ be the displacement vector from O to P and so on. Then

$$\vec{PQ} = \vec{PO} + \vec{OQ} = \vec{OQ} - \vec{OP} \tag{29}$$

and $\|\vec{PQ}\|$ is the distance between P and Q.

Dot Product

The following two definition of the dot product, or scalar product, $\vec{v} \cdot \vec{w}$ are equivalent:

geometric $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\vartheta$ where ϑ is the angle between $\vec{v}$ and $\vec{w}$, $0 \leq \vartheta \leq \pi$.

algebraic $\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z$

The dot product is a number, not a vector.

Dot Product

Now we need to show that the two definitions are equivalent. Consider a triangle PQR in three-dimensional space. Let $\vec{v} = \vec{PQ}, \vec{w} = \vec{PR}$. Then

$$\vec{QR} = \vec{QP} + \vec{PR} = -\vec{v} + \vec{w} = \vec{w} - \vec{v} \tag{30}$$

Here is the law of cosines for this triangle:

$$\|\vec{w} - \vec{v}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\| \cdot \|\vec{w}\| \cos\vartheta \tag{31}$$

It follows that the two definitions are equivalent.

Dot Product

Perpendicularity and Dot Product

Two non-zero vectors $\vec{v}$ and $\vec{w}$ are perpendicular, or orthogonal, if and only if $\vec{v} \cdot \vec{w} = 0$.

Magnitude and Dot Product

Magnitude and dot product are related as follows: $\vec{v} \cdot \vec{v} = \|\vec{v}\|$.

Dot Product Exercise

Exercise 8: Find the angle between

$$\vec{v} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \tag{32}$$

Consider the dot product

$$4 \cdot (-2) + 0 \cdot 1 + 7 \cdot 3 = 13 \tag{33}$$

According to the two equivalent definitions of the dot product, this is equal to

$$\|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\vartheta = \sqrt{4^2 + 7^2} \cdot \sqrt{(-2)^2 + 1^2 + 3^2} \cdot \cos\vartheta \tag{34}$$

Therefore,

$$\vartheta = \arccos \frac{13}{\sqrt{4^2 + 7^2} \cdot \sqrt{(-2)^2 + 1^2 + 3^2}} = 64.47^\circ \tag{35}$$

Planes Again

The equation of the plane with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ and containing the point $P = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{36}$$

Alternatively, for $d = ax_0 + by_0 + cz_0$

$$ax + by + cz = d \tag{37}$$

Cross Product

The following two definitions of the **cross product** or **vector product** $\vec{v} \times \vec{w}$ are equivalent:

- **Geometric definition**

  If $\vec{v}$ and $\vec{w}$ are not parallel, then

$$\vec{v} \times \vec{w} = \begin{pmatrix} \text{Area of parallelogram} \\ \text{with edges } \vec{v} \text{ and } \vec{w} \end{pmatrix} \vec{n} = (\|\vec{v}\| \|\vec{w}\| \sin\theta)\vec{n},$$

  where $0 \leq \theta \leq \pi$ is the angle between $\vec{v}$ and $\vec{w}$ and $\vec{n}$ is the unit vector perpendicular to $\vec{v}$ and $\vec{w}$ pointing in the direction given by the right-hand rule. If $\vec{v}$ and $\vec{w}$ are parallel, then $\vec{v} \times \vec{w} = \vec{0}$.

- **Algebraic definition**

$$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2)\vec{i} + (v_3 w_1 - v_1 w_3)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}$$

  where $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

Cross Product

If you know what a determinant is, you can remember the algebraic definition as follows.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{38}$$

Note that $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$.

Cross Product Exercise

Exercise 9: Use the cross product to find the linear equation containing the three points

$$\begin{aligned} P &= (1, 3, 0) \\ Q &= (3, 4, -3) \\ R &= (3, 6, 2) \end{aligned} \tag{39}$$

Cross Product Exercise Answer

One way to find the answer to the last exercise (without using the cross product) is to solve the following system of linear equations for the plane $x + ay + bz = c$,

$$\begin{aligned} 1 + 3a + 0b &= c \\ 3 + 4a - 3b &= c \\ 3 + 6a + 2b &= c \end{aligned} \tag{40}$$

Change this to

$$\begin{aligned} 3a + 0b - c &= -1 \\ 4a - 3b - c &= -3 \\ 6a + 2b - c &= -3 \end{aligned} \tag{41}$$

Using matrices,

$$\begin{pmatrix} 3 & 0 & -1 \\ 4 & -3 & -1 \\ 6 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -3 \end{pmatrix} \tag{42}$$

Cross Product Exercise Answer

Equation (42) yields the solution

$$x - \frac{10}{11}y + \frac{4}{11}z = -\frac{19}{11} \tag{43}$$

Now let's use the cross product instead, avoiding the matrices. Note that

$$\begin{aligned} \vec{PQ} &= 2\vec{i} + \vec{j} - 3\vec{k} \\ \vec{PR} &= 2\vec{i} + 3\vec{j} + 2\vec{k} \end{aligned} \tag{44}$$

The cross product, using the algebraic definition, is $\vec{u} = \vec{PQ} \times \vec{PR} = 11\vec{i} - 10\vec{j} + 4\vec{k}$.

Cross Product Exercise Answer

Let $P = (x_0, y_0, z_0)$ be a fixed point on the plane with known coordinates. Since any point $S = (x, y, z)$ on the plane fulfills

$$\vec{PS} \cdot \vec{u} = 0 \tag{45}$$

this can be turned into the plane equation

$$u_x(x - x_0) + u_y(y - y_0) + u_z(z - z_0) = 0 \tag{46}$$

Therefore, using $P = (1, 3, 0)$, this translates into

$$11x - 10y + 4z = 19 \tag{47}$$

which is equivalent to (43). Notice how easy it is to find a linear equation when you have a point $P = (x_0, y_0, z_0)$ on the plane and a normal vector $\vec{u}$ to the plane $u_x\vec{i} + u_y\vec{j} + u_z\vec{k}$:

$$u_x x + u_y y + u_z z = u_x x_0 + u_y y_0 + u_z z_0 \tag{48}$$

Exercise 10: Find all interior angles for and the plane equation containing the triangle with points

$$P = (1, 4, -2), Q = (-1, 1, 2), R = (-1, 3, 1) \tag{49}$$

Exercise 11: Find the equation of the tangent plane with respect to the unit circle at $P = (0.2, 0.3, \sqrt{0.87})$.

Hint: identify a vector that is orthogonal to the plane or use partial derivatives for the function whose function graph overlaps with the relevant part of the sphere.

End of Lesson

Next Lesson: Least Squares Approximation