Least Squares Adjustments with Non-Linear Equations

(1) Solve the following system of nonlinear equations numerically. Use $(x_0, y_0) = (0.6, 0.8)$ as your first approximation (the solution set is $S = \{(x, y) \in \mathbb{R}^2 | x \approx 0.64171, y \approx 0.80107\}$).

$$\begin{array}{rcrcl} \cos x & - & y & = & 0 \\ x & - & y^2 & = & 0 \end{array}$$

(2) Find a matrix C such that

$$C^2 = A \text{ where } A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

We will learn how to solve this problem using eigenvalues. For now, we are faced with a system of non-linear equations given

$$C = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

and

$$\begin{array}{rcrcr} x_1^2 & + & x_2x_3 & = & 3 \\ x_1x_2 & + & x_2x_4 & = & -1 \\ x_1x_3 & + & x_3x_4 & = & 2 \\ x_2x_3 & + & x_4^2 & = & 0 \end{array}$$

(3) You are trying to measure the coordinates of stations A and B. Your provisional estimate is $(8.3995, 3.0161)$ and $(-2.872, 1.4937)$. Then you observe the length between A and B to be 11.391. How would you report your least squares adjusted coordinates for A and B, given that you weigh equally the errors for A and B's coordinates as well as the distance between them?

Setting up the first four yave equations is simple. The fifth one, however, is non-linear.

$$\begin{array}{rcl} x & = & 8.3995 + \epsilon_1 \\ y & = & 3.0161 + \epsilon_2 \\ z & = & -2.872 + \epsilon_3 \\ w & = & 1.4937 + \epsilon_4 \\ \sqrt{(z-x)^2 + (w-y)^2} & = & 11.391 + \epsilon_5 \end{array}$$

Linearize the fifth equation using the Taylor polynomial expansion of the function $G(x, y, z, w) = \sqrt{(z-x)^2 + (w-y)^2}$.

(4) There are three points whose coordinates with measurement errors are

$$\begin{aligned} I &= (595.74, 537.76) \\ J &= (800.92, 658.44) \\ K &= (302.96, 168.88) \end{aligned} \tag{1}$$

From station I, you observe an angle of $158°49'21''$ instead of the expected $158°54'5.9107''$ between $\vec{IJ}$ and $\vec{IK}$. How should you least squares adjust the coordinates of I, J, K in light of your angle measurement? (Note that it is unnatural to give equal weight to the errors in coordinate measurements and angle measurements: this can be addressed by weight factors, but let us skip this step here for simplicity.)