Cramer, Jordan, and Gauss

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

September 24, 2018

Topics Covered

This lesson covers the following topics:

1. Cramer's Rule
2. Echelon Forms and Gauss-Jordan Elimination
3. Least Squares Approximation

Cramer's Rule

Cramer's rule makes finding the solutions to systems of linear equations very simple, at the expense of understanding what's going on. It's black magic. Consider the following system of linear equations:

$$\begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 9 \\ 6 \end{bmatrix} \tag{1}$$

The inverse of the coefficient matrix is

$$-\frac{1}{7} \cdot \begin{bmatrix} -7 & 14 & 0 \\ 0 & -3 & 1 \\ -7 & 15 & 2 \end{bmatrix} \tag{2}$$

Therefore, the solution to the system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{7} \cdot \begin{bmatrix} -7 & 14 & 0 \\ 0 & -3 & 1 \\ -7 & 15 & 2 \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \tag{3}$$

Cramer's Rule

Finding the inverse, however, is time-consuming. Also, it gives us all three solutions, and we may only want the value of one of the variables and not all of them. Cramer's rule tells us that, for example,

$$y = \frac{\det(A_y)}{\det(A)} \tag{4}$$

where A is the coefficient matrix and A_y is the coefficient matrix with the second column (corresponding to y) replaced by the vector of constants. In other words,

$$y = \frac{\det\left(\begin{bmatrix} 3 & 20 & -2 \\ 1 & 9 & -1 \\ 3 & 6 & -3 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix}\right)} = \frac{-21}{-7} = 3 \tag{5}$$

Cramer's Rule

For the other two variables,

$$x = \frac{\det\left(\begin{bmatrix} 20 & 4 & -2 \\ 9 & 2 & -1 \\ 6 & -1 & -3 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix}\right)} = \frac{-14}{-7} = 2 \tag{6}$$

$$y = \frac{\det\left(\begin{bmatrix} 3 & 4 & 20 \\ 1 & 2 & 9 \\ 3 & -1 & 6 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix}\right)} = \frac{7}{-7} = -1 \tag{7}$$

Echelon Form

Consider the system of linear equations from the previous slides:

$$\begin{bmatrix} 3 & 4 & -2 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 9 \\ 6 \end{bmatrix} \tag{8}$$

Combine the coefficient matrix and the vector of constants to an augmented matrix:

$$\begin{bmatrix} 3 & 4 & -2 & 20 \\ 1 & 2 & -1 & 9 \\ 3 & -1 & -3 & 6 \end{bmatrix} \tag{9}$$

Now use elementary row operations to make sure that only zeroes populate the matrix below the diagonal. This is called the echelon form of the system.

$$\begin{bmatrix} 3 & 4 & -2 & 20 \\ 0 & -10 & 5 & -35 \\ 0 & 0 & 7 & -7 \end{bmatrix} \tag{10}$$

Echelon Form

Because the echelon form is the product of elementary row operations, the solutions of the system of linear equations associated with it are the same as the solutions of the original system of linear equations.

$$\begin{array}{rcrcrcr} 3x & + & 4y & - & 2z & = & 20 \\ & & -10y & + & 5z & = & -35 \\ & & & & 7z & = & -7 \end{array} \tag{11}$$

The last equation tells us that $z = -1$. Substituting $z = -1$, the middle equation tells us that $y = 3$. Substituting both of these results in the first equation tells us that $x = 2$.

Echelon Form

The echelon form provides another way to solve a system of linear equations. The elementary row operations are called Gaussian elimination or Gauss-Jordan elimination (there are technical details about the difference between these two elimination methods that we are not worried about right now). Gaussian or Gauss-Jordan elimination, however, is hard to do manually. We use the echelon form primarily to deal with pathological cases where the determinant of the coefficient matrix is zero.

Theory of Linear Systems

1. If some row of an echelon form has its first nonzero entry in the last column, then the system has no solution. The system is inconsistent.
2. If a system is consistent, it has a rank. The rank is the number of leading nonzero entries with respect to the rows of the echelon form.
3. If the rank of a system equals the number of rows (or the number of equations), then the system has exactly one solution.
4. If the rank of a system is strictly less than the number of rows (or the number of equations), then the system has infinitely many solutions.

End of Lesson

Next Lesson: Vectors