Matrix Basics

MATH 3512, BCIT

Matrix Methods and Statistics for Geomatics

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Matrix Definition

A matrix is a tabular arrangement of real numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \tag{1}$$

The number of rows is m, the number of columns is n. $m \times n$ is called the dimension or size of the matrix.

Matrix Addition

We can define operations on matrices just like we define operations on numbers. For example, we can add an $m \times n$ matrix to another one as follows,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \vdots \\ b_{m1} & & \cdots & b_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1}+b_{m1} & & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

Matrix Addition

Example 1: Adding and Subtracting Matrices.

$$\begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} -6 & 5 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 11 & -11 \\ -2 & -5 \end{bmatrix}$$

Matrix Scalar Multiplication

Next, we define what it means to multiply a matrix by a scalar, i.e. a real number (NOT a matrix).

$$k \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & \ddots & & \\ \vdots & & & \vdots \\ ka_{m1} & & \cdots & ka_{mn} \end{bmatrix}$$

Matrix Scalar Multiplication

Example 2: Multiplying a Matrix by a Scalar.

$$2 \cdot \begin{bmatrix} -5 & -3 \\ -7 & 8 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ -14 & 16 \end{bmatrix}$$

$$-\frac{1}{3} \cdot \begin{bmatrix} -1 & -3 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{7}{3} & -\frac{1}{3} \end{bmatrix}$$

Matrix Transpose

The columns of a transpose $A^\intercal$ are the rows of the matrix A. The rows of a transpose $A^\intercal$ are the columns of the matrix A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

$$A^\intercal = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & \ddots & & \\ \vdots & & & \vdots \\ a_{1m} & & \cdots & a_{nm} \end{bmatrix}$$

Example 3: Transposing a Matrix.

$$\begin{bmatrix} -1 & 2 & 1 \\ 7 & -2 & -1 \\ 0 & 6 & 6 \\ 7 & 6 & 4 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} -1 & 7 & 0 & 7 \\ 2 & -2 & 6 & 6 \\ 1 & -1 & 6 & 4 \end{bmatrix} \tag{2}$$

Matrix Product

Finally, we define matrix multiplication. You can multiply an $m \times j$ matrix by a $j \times n$ matrix, which will give you an $m \times n$ matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mj} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \vdots \\ b_{j1} & & \cdots & b_{jn} \end{bmatrix} =$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & \ddots & & \\ \vdots & & & \vdots \\ c_{m1} & & \cdots & c_{mn} \end{bmatrix}$$

where $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{ij} b_{jk}$.

Matrix Product

Notice that c_{ik} is the product of the i-th row vector of A and the k-th column vector of B. The dot product of two vectors $\vec{v}$ and $\vec{w}$ is defined to be $\vec{v}^{\intercal} \cdot \vec{w}$.

Example 4: Multiplying Matrices.

$$\begin{bmatrix} -1 & 5 \\ 10 & 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -8 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 38 & 8 \\ 26 & -80 \end{bmatrix}$$

Matrix Product

Exercise 1: Consider

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \tag{3}$$

Find $A \cdot B$ as well as $B \cdot A$ and determine whether matrix multiplication is commutative.

Identity Matrix

The identity matrix I with dimension $m \times m$ is a square matrix such that for all $m \times m$ matrices A it is true that

$$A \cdot I = I \cdot A = A \tag{4}$$

An identity matrix always has all 1's in the diagonal and all 0's elsewhere.

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix} \tag{5}$$

Matrix Inverse

The inverse matrix A^{-1} of a square matrix A is the matrix for which

$$A \cdot A^{-1} = A^{-1} \cdot A = I \tag{6}$$

Not all matrices have an inverse. Finding the inverse of a $m \times m$ matrix is equivalent to solving a system of $m \cdot m$ equations with $m \cdot m$ variables. For example, the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{is} \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{7}$$

Matrix Determinant

Considering the last slide, it is evident that a matrix has an inverse if and only if $ad - bc \neq 0$. Such a matrix is called invertible. If $ad - bc = 0$ then the matrix is singular and has no inverse (find some examples). It turns out that the number $ad - bc$ is so special for 2×2 matrices that it gets its own name: it is the determinant of the matrix. On the next slide, I will define the determinant of any square matrix using an inductive procedure.

Matrix Determinant

- The determinant of a 1×1 matrix A is $\det(A) = a_{11}$.
- The determinant of a $m \times m$ matrix with $m > 1$ is $\det(A) = c$.

Calculate c by picking an arbitrary row, for example the i-th row. Then

$$c = \sum_{j=1}^{m} (-1)^{i+j} a_{ij} \det(A_{ij}) \tag{8}$$

where A_{ij} is the matrix that results when you delete the i-th row and the j-th column from A.

Adjugate Matrix

The adjugate matrix of a matrix A has as its elements the real numbers b_{ij} with

$$b_{ij} = (-1)^{i+j} \det(A_{ij}) \tag{9}$$

where A_{ij} is defined on the last slide. Consequently,

$$\det(A) = \operatorname{adj}(A) \cdot A \tag{10}$$

for all square matrices A.

Matrix Determinants Exercises

Exercise 2: Consider

$$B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \tag{11}$$

Calculate B^{-1} and show that $B \cdot B^{-1} = I$.

Exercise 3: Consider

$$D = \begin{bmatrix} 0 & -1 & 4 \\ 3 & 2 & 0 \\ 4 & 3 & -1 \end{bmatrix} \tag{12}$$

Calculate $\det(D)$. Then use softward to calculate the inverse of D. What do you notice about $\det(D) \cdot D^{-1}$?

Matrix Method

$$
\begin{array}{rcrcl}
5x & + & 3y & = & 13.5 \\
x & + & 5y & = & 13.7
\end{array}
\tag{13}
$$

is the system of linear equations that we are trying to solve. A matrix is a rectangular arrangement of numbers, for example

$$
\begin{bmatrix} 5 & 3 & 13.5 \\ 1 & 5 & 13.7 \end{bmatrix}
\tag{14}
$$

There are many fascinating things you can do with matrices. The discipline that deals with matrices is called Linear Algebra.

Matrix Inverse

Matrix multiplication for an an $m \times j$ matrix by a $k \times n$ matrix is not defined when $j \neq k$. An inverse matrix A^{-1} of a square matrix A is defined to be the matrix

$$A \cdot A^{-1} = A^{-1} \cdot A = E \tag{15}$$

where

$$E = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & 1 & 0 \\ 0 & & \cdots & 0 & 1 \end{bmatrix}$$

Chaitali and Amulya go to a concession stand to buy fruit. Chaitali buys 5 bananas and 3 apples and spends \$13.50. Amulya buys 1 banana and 5 apples and spends 20 cents more than Chaitali. How much do bananas and apples cost at the concession stand?

Systems of Linear Equations Introduced

Chaitali and Amulya go to a concession stand to buy fruit. Chaitali buys 5 bananas and 3 apples and spends \$13.50. Amulya buys 1 banana and 5 apples and spends 20 cents more than Chaitali. How much do bananas and apples cost at the concession stand?

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{16}$$

What Is a System of Linear Equations?

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{17}$$

This system of linear equations is the rule for the following set $S \subset \mathbb{R} \times \mathbb{R}$:

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} | 5x + 3y = 13.5 \text{ and } x + 5y = 13.7\} \tag{18}$$

Solution Methods

$$
\begin{aligned}
5x &+ 3y &= 13.5 \\
x &+ 5y &= 13.7
\end{aligned}
\tag{19}
$$

There are several ways to solve a system of equations like this.

- Graphing
- Substitution
- Elimination
- Using a Matrix

Graphing Method I

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{20}$$

is equivalent to

$$\begin{array}{rcl} y & = & -\frac{5}{3}x + \frac{9}{2} \\ y & = & -\frac{1}{5}x + \frac{137}{50} \end{array} \tag{21}$$

Graphing Method II

./gm-03-SystemsEquations-01-eps-con

Graphing Method III

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Graphing Method IV

./gm-03-SystemsEquations-03-eps-con

Graphing Method V

./gm-03-SystemsEquations-04-eps-con

Graphing Method Exercises

Find a solution to these systems of linear equations by graphing them and check your answer by substituting.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{22}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{23}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{24}$$

Substitution Method I

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{25}$$

The second equation yields $x = 13.7 - 5y$. Use this to substitute in the first equation

$$5 \cdot (13.7 - 5y) + 3y = 13.5 \tag{26}$$

therefore, $-22y = -55$ and $y = 5/2$. Now substitute $y = 5/2$ in the first equation (you could just as well use the second equation), so

$$5x + 3 \cdot \frac{5}{2} = 13.5 \tag{27}$$

which implies $x = 1.2$. A banana costs \$1.20; an apple costs \$2.50.

Substitution Method Exercises

Find a solution to these systems of linear equations by using the substitution method.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{28}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{29}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{30}$$

Elimination Method I

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{31}$$

is equivalent to

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ 5x & + & 25y & = & 68.5 \end{array} \tag{32}$$

Elimination Method II

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ 5x & + & 25y & = & 68.5 \end{array} \tag{33}$$

implies

$$(5x + 3y) - (5x + 25y) = 13.5 - 68.5 \tag{34}$$

therefore, $-22y = -55$ and $y = 5/2$. Now substitute $y = 5/2$ in the first equation (you could just as well use the second equation), so

$$5x + 3 \cdot \frac{5}{2} = 13.5 \tag{35}$$

which implies $x = 1.2$. A banana costs \$1.20; an apple costs \$2.50.

Elimination Method Exercises

Find a solution to these systems of linear equations by using the elimination method.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{36}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{37}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{38}$$

Matrices and Systems of Linear Equations I

Remember our system of linear equations.

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{39}$$

In matrix notation, we can write

$$\begin{bmatrix} 5 & 3 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13.5 \\ 13.7 \end{bmatrix}$$

Let's call these three matrices A, v, b respectively. A and b are provided, and we are looking for v. If we had A^{-1}, we could go from

$$Av = b \tag{40}$$

to

$$A^{-1}Av = A^{-1}b \tag{41}$$

which is the same as

$$v = A^{-1}b \tag{42}$$

The challenge is therefore to find A^{-1}. Scientific calculators and computers can find A^{-1} for you.

Matrix Inverse and Determinants

If you want to know how to find the inverse yourself, one method to use is calculating the determinant of a matrix. It takes a bit of time to understand determinants, and then it's still a complicated (and not very transparent) procedure to get to the inverse. For 2×2 matrices, however, the inverse is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{43}$$

for

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{44}$$

and the determinant is $\det A = ad - bc$.

Matrix Row Operations

Another method to find the inverse of a matrix is using matrix row operations. There are three matrix row operations.

- Row Switching means you are allowed to switch two rows, for example $R_1 \leftrightarrow R_2$
- Row Multiplication means you are allowed to multiply all elements of a row by a real non-zero number, for example $\frac{2}{5}R_2 \rightarrow R_2$
- Row Addition means you are allowed to add one row to another and then replace one of the original rows by the sum of the two rows, for example $R_1 + R_2 \rightarrow R_1$

Row multiplication and row addition are often used together, for example $\frac{7}{8}R_1 - R_3 \rightarrow R_3$.

Matrix Row Operations

To find the inverse of a square matrix, we combine A and E

$$\begin{bmatrix} 5 & 3 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix}$$

and apply matrix row operations until we get

$$\begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{bmatrix}$$

where

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Inverse Example

For our example,

$$\begin{bmatrix} 5 & 3 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 25/3 & 5 & 5/3 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 22/3 & 0 & 5/3 & -1 \\ 1 & 5 & 0 & 1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 22/3 & 0 & 5/3 & -1 \\ 22/3 & 110/3 & 0 & 22/3 \end{bmatrix} \longrightarrow \begin{bmatrix} 22/3 & 0 & 5/3 & -1 \\ 0 & 110/3 & -5/3 & 25/3 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & 5/22 & -3/22 \\ 0 & 1 & -1/22 & 5/22 \end{bmatrix}$$

For step 1, we multiplied the first row by $5/3$ (row multiplication). For step 2, we subtracted the second row from the first row and replaced the first row by the result (row addition). For step 3, we multiplied the second row by $22/3$ (row multiplication). For step 4, we subtracted the first row from the second row and replaced the second row by the result (row addition). For the last step, we multiplied the first row by $3/22$ and the second row by $3/110$ (row multiplication applied twice).

Matrices and Systems of Linear Equations III

Thus,

$$A^{-1} = \begin{bmatrix} 5/22 & -3/22 \\ -1/22 & 5/22 \end{bmatrix} = \frac{1}{22} \cdot \begin{bmatrix} 5 & -3 \\ -1 & 5 \end{bmatrix}$$

and

$$v = A^{-1}b = \begin{bmatrix} 5/22 & -3/22 \\ -1/22 & 5/22 \end{bmatrix} \cdot \begin{bmatrix} 13.5 \\ 13.7 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 2.5 \end{bmatrix}$$

End of Lesson

Next Lesson: Determinants and Inverse