**(1)** Consider the vector space of quadratic polynomials

$$V = \{f | f(x) = ax^2 + bx + c\} \tag{1}$$

Are the following three quadratic polynomials a basis for $V$?

$$\begin{aligned} f_1(x) &= 3x^2 + x + 6 \\ f_2(x) &= 5x^2 + x + 11 \\ f_3(x) &= -2x^2 - 6x + 4 \end{aligned} \tag{2}$$

- If yes, find the coordinates in terms of this basis for $7x^2 - x - 4$.
- If no, express one of the $f_i$ by the others, $i = 1, 2, 3$.

$$f_1(x) \sim \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} \quad f_2(x) \sim \begin{pmatrix} 5 \\ 1 \\ 11 \end{pmatrix} \quad f_3(x) \sim \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}$$

$$\det\left( \begin{bmatrix} 3 & 5 & -2 \\ 1 & 1 & -6 \\ 6 & 11 & 4 \end{bmatrix} \right) = 0$$

therefore, the three vectors are linearly dependent and do NOT form a basis of V

$$\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} = a \cdot \begin{pmatrix} 5 \\ 1 \\ 11 \end{pmatrix} + b \cdot \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix} \Rightarrow \begin{aligned} 5a - 2b &= 3 \\ a - 6b &= 1 \\ 11a + 4b &= 6 \end{aligned}$$

$$\begin{bmatrix} 5 & -2 & 3 \\ 1 & -6 & 1 \\ 11 & 4 & 6 \end{bmatrix} \underset{\to R_2}{\overset{R_1 - 5R_2}{\sim}} \begin{bmatrix} 5 & -2 & 3 \\ 0 & 28 & -2 \\ 11 & 4 & 6 \end{bmatrix} \underset{\to R_2}{\overset{\frac{1}{2} \cdot R_2}{\sim}} \begin{bmatrix} 5 & -2 & 3 \\ 0 & 14 & -1 \\ 11 & 4 & 6 \end{bmatrix} \underset{\to R_3}{\overset{11 \cdot R_1 - 5 \cdot R_3}{\sim}}$$

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & 14 & -1 \\ 0 & -42 & 3 \end{bmatrix} \underset{\to R_3}{\overset{-\frac{1}{3} R_3}{\sim}} \begin{bmatrix} 5 & -2 & 3 \\ 0 & 14 & -1 \\ 0 & 14 & -1 \end{bmatrix} \underset{\to R_3}{\overset{R_2 - R_3}{\sim}} \begin{bmatrix} 5 & -2 & 3 \\ 0 & 14 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$14b = -1 \quad b = -\frac{1}{14} \quad 5a - \left(-\frac{1}{14}\right) \cdot 2 = 3 \Rightarrow a = \frac{4}{7}$$

$$\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} = \frac{4}{7} \cdot \begin{pmatrix} 5 \\ 1 \\ 11 \end{pmatrix} - \frac{1}{14} \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}$$

**(2)** Consider the vector space of lines in $\mathbb{R}^3$ going through the origin.

$$W = \{L | L : ax + by + cz = 0\} \tag{3}$$

Are the following three lines a basis for $W$?

$$\begin{aligned} L_1 &: \quad 2x + 5y + z = 0 \\ L_2 &: \quad x - y - 2z = 0 \\ L_3 &: \quad -3x + 4z = 0 \end{aligned} \tag{4}$$

- If yes, find the coordinates in terms of this basis for $L : 14x + 9y - 12z = 0$.
- If no, express one of the $L_i$ by the others, $i = 1, 2, 3$.

$$L_1 \sim \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \quad L_2 \sim \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad L_3 \sim \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$$

$$\det\left( \begin{bmatrix} 2 & 1 & -3 \\ 5 & -1 & 0 \\ 1 & -2 & 4 \end{bmatrix} \right) = -1$$

therefore, the three vectors are linearly independent, and because the vector space is three-dimensional they also span all of V; it follows that they form a basis

$$\begin{pmatrix} 14 \\ 9 \\ -12 \end{pmatrix} = a \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + c \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 1 & -3 \\ 5 & -1 & 0 \\ 1 & -2 & 4 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 14 \\ 9 \\ -12 \end{bmatrix}}_{v}$$

$$A \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = v \implies$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^{-1} \cdot v = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

$$\begin{pmatrix} 14 \\ 9 \\ -12 \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$$

**(3)** There are three points whose coordinates with measurement errors are

$$\begin{aligned} I &= (595.74, 537.76) \\ J &= (800.92, 658.44) \\ K &= (302.96, 168.88) \end{aligned} \tag{5}$$

From station $I$, you observe an angle of $158°49'21''$, again with the usual measurement error. In the next lesson, we will learn how to adjust the co-ordinates based on the observed angle. In the meantime, what is the angle between $\vec{IJ}$ and $\vec{IK}$ that you would have expected based on the coordinates? Use the inner product.

$$\vec{IJ} = \begin{pmatrix} 800.92 - 595.74 \\ 658.44 - 537.76 \end{pmatrix} = \begin{pmatrix} 205.18 \\ 120.68 \end{pmatrix}$$

$$\vec{IK} = \begin{pmatrix} 302.96 - 595.74 \\ 168.88 - 537.76 \end{pmatrix} = \begin{pmatrix} -292.78 \\ -368.88 \end{pmatrix}$$

$$\cos\vartheta = \frac{\vec{IJ} \cdot \vec{IK}}{\|\vec{IJ}\| \cdot \|\vec{IK}\|}$$

$$\vartheta = 158°54'6''$$

**(4)** Find an orthonormal basis for the following vector plane. Identify the normal vector to the plane.

$$-6x + 5y + 8z = 0 \longrightarrow \qquad (6)$$

orthonormal vector $\vec{n} = \begin{pmatrix} -6 \\ 5 \\ 8 \end{pmatrix}$

find a $\tilde{b}_1$ in the plane

$x = 1 \quad y = 0 \quad$ (arbitrary)

then $-6 \cdot 1 + 5 \cdot 0 + 8z = 0 \Rightarrow z = \frac{3}{4}$

$\tilde{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 3/4 \end{pmatrix} \Rightarrow$ also in the plane $\hat{b}_1 = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$

next: find $\tilde{b}_2 \perp \hat{b}_1$ with $\tilde{b}_2$ in the plane.

let $\tilde{b}_2 = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}$ then $-6\tilde{x} + 5\tilde{y} + 8\tilde{z} = 0$

and $4\tilde{x} + 0\tilde{y} + 3\tilde{z} = 0$

$$\begin{bmatrix} -6 & 5 & 8 & 0 \\ 4 & 0 & 3 & 0 \end{bmatrix} \underset{\to R_2}{\overset{4 \cdot R_1 - 6 \cdot R_2}{\sim}} \begin{bmatrix} -6 & 5 & 8 & 0 \\ 0 & 20 & 50 & 0 \end{bmatrix}$$ (because $\hat{b}_1 \perp \tilde{b}_2$)

$\Rightarrow 2\tilde{y} + 5\tilde{z} = 0$, for example $\tilde{y} = 5 \quad \tilde{z} = -2$

$\Rightarrow \tilde{x} = \frac{3}{2} \Rightarrow \tilde{b}_2 = \begin{pmatrix} 3/2 \\ 5 \\ -2 \end{pmatrix} \Rightarrow \hat{b}_2 = \begin{pmatrix} 3 \\ 10 \\ -4 \end{pmatrix}$

$b_1 = (4/5, 0, 3/5)^T \quad b_2 = (3/\sqrt{125}, 10/\sqrt{125}, -4/\sqrt{125})^T$

**(5)** Consider the vector

$$\hat{b_1} = \begin{pmatrix} -5 \\ -7 \\ 4 \end{pmatrix} \tag{7}$$

Find $b_1$ such that $b_1 \| \hat{b_1}$ ($b_1$ and $\hat{b_1}$ are parallel) and $\|b_1\| = 1$. Then find vectors $b_2, b_3$ so that $B = \{b_1, b_2, b_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Combine $b_1, b_2, b_3$ in a matrix $M$ and find the inverse $M^{-1}$, using software. What do you notice?

$\tilde{b}_2 \perp \hat{b}_1$ let $\tilde{b}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ then

$-5x_2 - 7y_2 + 4z_2 = 0 \quad (\tilde{b}_2 \perp \hat{b}_1)$

let $x_2 = 1$ $y_2 = 0$ (arbitrary) then

$-5 + 4z_2 = 0 \Rightarrow z_2 = \frac{5}{4} \Rightarrow \tilde{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 5/4 \end{pmatrix}$

$\Rightarrow \hat{b}_2 = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$ now pick $\tilde{b}_3 \perp \hat{b}_1$ and $\tilde{b}_3 \perp \hat{b}_2$ let $\tilde{b}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$

$-5x_3 - 7y_3 + 4z_3 = 0$

$4x_3 \qquad + 5z_3 = 0$ choose $x_3 = -5$ then $z_3 = 4$

and $(-5)\cdot(-5) - 7y_3 + 4 \cdot 4 = 0 \Rightarrow y_3 = \frac{41}{7}$

$\Rightarrow \tilde{b}_3 = \begin{pmatrix} -5 \\ 41/7 \\ 4 \end{pmatrix} \Rightarrow \hat{b}_3 = \begin{pmatrix} -35 \\ 41 \\ 28 \end{pmatrix}$

$b_1 = \frac{1}{3\sqrt{10}} \cdot (-5, -7, 4)^T$

$b_2 = \frac{1}{\sqrt{41}} \cdot (4, 0, 5)^T$

$b_3 = \frac{1}{3\sqrt{410}} \cdot (-35, 41, 28)^T$

First some calculus. We learned at the end of last year that for a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is differentiable at $(x_0, y_0)$ we can approximate

$$f(x,y) \approx L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$L(x,y) = z$ is a plane equation for the plane that is tangent to the point $P = (x_0, y_0)$. More generally, if $J$ is the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

then for a multivariable function $F : \mathbb{R}^n \to \mathbb{R}^m$ there is a linear approximation at point $\vec{a}$ such that

$$F(\vec{x}) \approx F(\vec{a}) + J(\vec{a})(\vec{x} - \vec{a})$$

All of this theory is a consequence of using Taylor polynomials and cutting them off after the first term (the remainder of the Taylor series expansion becomes an error term for the approximation).

Let $f(x,y) = \sqrt{1-x^2-y^2}$ (half-sphere above $x,y$-plane)

$$\frac{\partial f}{\partial x}(x,y) = -\frac{x}{\sqrt{1-x^2-y^2}} \qquad \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-0.2^2-0.3^2} - \frac{0.2}{\sqrt{1-0.2^2-0.3^2}}(x-0.2) - \frac{0.3}{\sqrt{1-0.2^2-0.3^2}}(y-0.3)$$

$$\Rightarrow \quad z = \sqrt{0.87} - \frac{1}{\sqrt{0.87}}(0.2x - 0.04) - \frac{1}{\sqrt{0.87}}(0.3y - 0.09)$$

$$\sqrt{0.87}\, z = 0.87 - 0.2x + 0.04 - 0.3y + 0.09$$

$$0.2x + 0.3y + \sqrt{0.87}\, z = 1$$

this is the tangent plane equation using calculus

for the linear algebra approach, consider that P is in the tangent plane and that the vector from O to P is a normal vector to the tangent plane

$$P = (0.2, 0.3, \sqrt{0.87}) \qquad \vec{OP} = \vec{n} = \begin{pmatrix} 0.2 \\ 0.3 \\ \sqrt{0.87} \end{pmatrix}$$

then the equation for the tangent plane is

$$0.2(x-0.2) + 0.3(y-0.3) + \sqrt{0.87}(z - \sqrt{0.87}) = 0$$

$$0.2x + 0.3y + \sqrt{0.87}\, z = 1$$

this is the tangent plane equation using linear algebra

**(7)** Solve the following system of linear equations. Use software.

$$\begin{array}{rcrcrcr} 4a & + & b & - & c & = & -12 \\ 3a & - & 2b & + & 4c & = & -5 \\ -a & + & 8b & - & 14c & = & -9 \end{array} \qquad (8)$$

If the system is consistent and dependent, provide your answer in the form

$$S = \{u \in \mathbb{R}^3 \mid u \text{ corresponds to } \vec{u} = \vec{v_0} + s_1\vec{v_1} + \ldots + s_n\vec{v_n}\} \qquad (9)$$

where $n$ is the dimension of the solution space and $s_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Note that $(-3, 2, 2)^\intercal$ solves the system.

Since $\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$ solves the equation, $\vec{v}_0 = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$

$$\det\left(\begin{bmatrix} 4 & 1 & -1 \\ 3 & -2 & 4 \\ -1 & 8 & -14 \end{bmatrix}\right) = 0$$

use elementary row operations for

$$\begin{bmatrix} 4 & 1 & -1 & -12 \\ 3 & -2 & 4 & -5 \\ -1 & 8 & -14 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & -7 \\ 0 & 11 & -19 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the last equation is redundant; the system is consistent with infinitely many solutions

find a solution that is <u>not</u> $\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$

$11b - 19c = -16$ choose $b = 0$ then $c = \frac{16}{19}$

and $a = -3 \cdot 0 + 5 \cdot \frac{16}{19} - 7 = -\frac{53}{19}$

then $\left(-\frac{53}{19}, 0, \frac{16}{19}\right)^\intercal$ is a solution and

$$\vec{v}_1 = \begin{pmatrix} -53/19 \\ 0 \\ 16/19 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4/19 \\ -2 \\ -22/19 \end{pmatrix}$$

we are only interested in the direction of v1, so we can multiply by 19 for

$$S = \left\{v \in \mathbb{R}^3 \mid v \sim \vec{v} = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} + s_1 \begin{pmatrix} 4 \\ -38 \\ -22 \end{pmatrix}, s_1 \in \mathbb{R}\right\}$$

**(8)** Solve the following system of linear equations. Use software.

$$\begin{array}{rcrcrcrcr} 6x & & & - & 4z & + & 5w & = & 23 \\ -3x & + & 8y & + & 3z & - & w & = & -14 \\ 9x & + & 8y & - & 5z & + & 9w & = & 32 \\ & & 16y & + & 2z & + & 3w & = & -5 \end{array} \tag{10}$$

If the system is consistent and dependent, provide your answer in the form

$$S = \{u \in \mathbb{R}^4 \mid u \text{ corresponds to } \vec{u} = \vec{v_0} + s_1\vec{v_1} + \ldots + s_n\vec{v_n}\} \tag{11}$$

where $n$ is the dimension of the solution space and $s_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Note that $(2, -1, 1, 3)^\intercal$ solves the system.

solution #1

$\vec{V}_0 = (2, -1, 1, 3)^T$

use software!

$$\begin{bmatrix} 6 & 0 & -4 & 5 & 23 \\ -3 & 8 & 3 & -1 & -14 \\ 9 & 8 & -5 & 9 & 32 \\ 0 & 16 & 2 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & 8 & -1 & 4 & 9 \\ 0 & 16 & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

the system is consistent with infinitely many solutions, the solution space has dimension 2

$16y + 2z + 3w = -5$ choose $y = 0$ $z = 0$

then $w = -\frac{5}{3}$ and $x = \frac{47}{9}$ solution #2

$$\hat{\vec{V}}_1 = \begin{pmatrix} 47/9 \\ 0 \\ 0 \\ -5/3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 29/9 \\ 1 \\ -1 \\ -14/3 \end{pmatrix}$$

I am only interested in the direction of v1, so multiply by 9

$$\vec{V}_1 = \begin{pmatrix} 29 \\ 9 \\ -9 \\ -42 \end{pmatrix}$$

$16y + 2z + 3w = -5$ choose $z = 0$ $w = 0$ solution #3

then $16y = -5 \Rightarrow y = -\frac{5}{16}$ and $x = \frac{23}{6}$

$$\hat{\vec{V}}_2 = \begin{pmatrix} 23/6 \\ -5/16 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 11/6 \\ 11/16 \\ -1 \\ -3 \end{pmatrix} \quad \vec{V}_2 = \begin{pmatrix} 88 \\ 33 \\ -48 \\ -144 \end{pmatrix}$$

use v0, v1, and v2 to construct the solution set

**(9)** Find all interior angles for and the plane equation containing the triangle with points

$$P = (-6, -2, -7), Q = (-2, 1, 6), R = (-8, 3, -5) \tag{12}$$

Hint: Use the dot product to find the interior angles. Use the cross product to find a normal vector to the plane. Remember that the cross product of two vectors is

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{13}$$

and is perpendicular to both $\vec{v}$ and $\vec{w}$. If $P = (p_x, p_y, p_z)$ is a point on a plane and $\vec{n} = (n_x, n_y, n_z)^\intercal$ is a normal vector to the plane, then the plane equation is (why?)

$$n_x(p_x - x) + n_y(p_y - y) + n_z(p_z - z) = 0 \tag{14}$$

$$\vec{PQ} = \begin{pmatrix} 4 \\ 3 \\ 13 \end{pmatrix} \quad \vec{PR} = \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix}$$

$$\vec{PQ} \times \vec{PR} = (3 \cdot 2 - 13 \cdot 5)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - (4 \cdot 2 - (-2) \cdot 13) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (4 \cdot 5 - (-2) \cdot 3)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -59 \\ -34 \\ 26 \end{pmatrix} = \vec{n}$$

plane equation:

$$-59(x+6) - 34(y+2) + 26(z+7) = 0$$

$$-59x - 34y + 26z = -240$$