Least Squares Adjustments with Non-Linear Equations

(1) Find a matrix C such that

$$C^2 = A \text{ where } A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

We will learn how to solve this problem using eigenvalues. For now, we are faced with a system of non-linear equations given

$$C = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

and

$$\begin{array}{ccccc} x_1^2 & + & x_2x_3 & = & 3 \\ x_1x_2 & + & x_2x_4 & = & -1 \\ x_1x_3 & + & x_3x_4 & = & 2 \\ x_2x_3 & + & x_4^2 & = & 0 \end{array}$$

(2) You are trying to measure the coordinates of stations A and B. Your provisional estimate is $(8.3995, 3.0161)$ and $(-2.872, 1.4937)$. Then you observe the length between A and B to be 11.391. How would you report your least squares adjusted coordinates for A and B, given that you weigh equally the errors for A and B's coordinates as well as the distance between them?

Setting up the first four yave equations is simple. The fifth one, however, is non-linear.

$$\begin{aligned} x &= 8.3995 + \epsilon_1 \\ y &= 3.0161 + \epsilon_2 \\ z &= -2.872 + \epsilon_3 \\ w &= 1.4937 + \epsilon_4 \\ \sqrt{(z-x)^2 + (w-y)^2} &= 11.391 + \epsilon_5 \end{aligned}$$

Linearize the fifth equation using the Taylor polynomial expansion of the function $G(x, y, z, w) = \sqrt{(z-x)^2 + (w-y)^2}$.

(3) Solve the following system of nonlinear equations numerically. Use $(x_0, y_0) = (0.6, 0.8)$ as your first approximation.

$$\begin{array}{ccccc} \cos x & - & y & = & 0 \\ x & - & y^2 & = & 0 \end{array}$$

The solution set is

$$S = \{(x, y) \in \mathbb{R}^2 | x \approx 0.64171, y \approx 0.80107\}$$

(4) There are three points whose coordinates with measurement errors are

$$\begin{aligned} I &= (595.74, 537.76) \\ J &= (800.92, 658.44) \\ K &= (302.96, 168.88) \end{aligned}$$

From station I, you observe an angle of $158°49'21''$ instead of the expected $158°54'5.9107''$ between $\vec{IJ}$ and $\vec{IK}$. How should you least squares adjust the coordinates of I, J, K in light of your angle measurement? (Note that it is unnatural to give equal weight to the errors in coordinate measurements and angle measurements: this can be addressed by weight factors, but let us skip this step here for simplicity.)

(5) For the following matrices, find the eigenvalues and one corresponding eigenvector for each eigenvalue.

$$A_1 = \begin{bmatrix} 22 & 20 \\ -25 & -23 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad A_5 = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad A_6 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ -1 & 1 & 3 & 6 \end{bmatrix}$$