Product and Quotient Rule

MATH 1441, BCIT

Technical Mathematics for Food Technology

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Product Rule

Rule 5

The Product Rule

$$g'(x) = f_1(x)f_2'(x) + f_1'(x)f_2(x) \text{ for } g(x) = f_1(x)f_2(x) \tag{1}$$

Product Rule Reason

Reason:

$$g'(x) =$$

$$\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{f_1(x+h)f_2(x+h) - f_1(x)f_2(x)}{h} =$$

$$\lim_{h\to 0} \frac{f_1(x+h)f_2(x+h) - f_1(x)f_2(x+h) + f_1(x)f_2(x+h) - f_1(x)f_2(x)}{h} =$$

$$\lim_{h\to 0} \frac{(f_1(x+h) - f_1(x))f_2(x+h) + f_1(x)(f_2(x+h) - f_2(x))}{h} =$$

$$f_1(x)f_2'(x) + f_1'(x)f_2(x) \tag{2}$$

Product Rule Exercises

Exercise 1: Differentiate the following functions.

$$f(x) = (2x^2 - 1)(x^3 + 3) \tag{3}$$

$$g(t) = t^3 \left(\sqrt{t} + 1\right) \tag{4}$$

Quotient Rule

Rule 6

The Quotient Rule

$$g'(x) = \frac{f_1'(x) f_2(x) - f_1(x) f_2'(x)}{\left(f_2(x)\right)^2} \text{ for } g(x) = \frac{f_1(x)}{f_2(x)} \qquad (5)$$

Quotient Rule Reason

Reason:

$$g(x) = \frac{f_1(x)}{f_2(x)} \tag{6}$$

$$f_1(x) = g(x) f_2(x) \tag{7}$$

$$f_1'(x) = g'(x) f_2(x) + g(x) f_2'(x) \text{ now isolate } g'(x) \tag{8}$$

$$g'(x) = \frac{f_1'(x) - g(x) f_2'(x)}{f_2(x)} \text{ now substitute } g(x) = \frac{f_1(x)}{f_2(x)} \tag{9}$$

$$g'(x) = \frac{\frac{f_1'(x) f_2(x)}{f_2(x)} - \frac{f_1(x) f_2'(x)}{f_2(x)}}{f_2(x)} \tag{10}$$

$$g'(x) = \frac{f_1'(x) f_2(x) - f_1(x) f_2'(x)}{(f_2(x))^2} \tag{11}$$

Quotient Rule Exercises

Exercise 2: Differentiate the following functions.

$$f(z) = \frac{3z^2 + 5z - 2}{3z - 1} \tag{12}$$

$$h(x) = \frac{\sqrt{x}}{x^2 + 1} \tag{13}$$

Quotient Rule Exercise Solution

$$g(x) = \frac{3x^2+5x-2}{3x-1}$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

[1] $$g(x) = \frac{3x^2+5x-2}{3x-1} = \frac{\cancel{(3x-1)}(x+2)}{\cancel{3x-1}} = x+2 \quad \text{except at } x = \frac{1}{3}$$

$$g'(x) = \frac{d}{dx}(x+2) = 1$$

[2] $$g'(x) = \frac{(6x+5)(3x-1) - (3x^2+5x-2)3}{(3x-1)^2} =$$

$$\frac{18x^2+15x-6x-5-[9x^2+15x-6]}{9x^2-6x+1} = \frac{18x^2+15x-6x-5-9x^2-15x+6}{9x^2-6x+1} =$$

$$\frac{9x^2-6x+1}{9x^2-6x+1} = 1$$

Euler's Number

The number e is defined as follows,

$$e = \lim_{t \to \infty} \left(1 + \frac{1}{t}\right)^t \tag{14}$$

Lemma

Consider two functions f_1 and f_2. They are related in so far as

$$f_1(x) = f_2\left(\frac{1}{x}\right) \tag{15}$$

For example,

$$f_1(x) = \frac{2x+1}{5x-7} \text{ and } f_2(x) = -\frac{x+2}{7x-5} \tag{16}$$

Then

$$\text{If } \lim_{x\to\infty} f_1(x) = a \text{ then } \lim_{x\to 0} f_2(x) = a \tag{17}$$

The Derivative of the Logarithmic Function

Now consider the function $f(x) = \ln x$ and the definition of the derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h} \ln \frac{x+h}{x} =$$

$$\lim_{h\to 0} \frac{1}{x} \cdot \frac{x}{h} \ln \left(1 + \frac{h}{x}\right) = \lim_{h\to 0} \frac{1}{x} \ln \left(1 + \frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}} \qquad (18)$$

Use the lemma of the last slide and the definition of Euler's number to see that

$$f'(x) = \frac{1}{x} \qquad (19)$$

Exercise 3: Differentiate the following functions.

$$f(x) = x^2 \ln x \tag{20}$$

$$g(y) = \ln y^{(5y-2)} \tag{21}$$

$$h(z) = (\ln z)^2 \tag{22}$$

$$f(x) = \log_2 x \tag{23}$$

Exercises

Exercise 4: Differentiate the following functions.

$$f(x) = \frac{21 - 5x - 6x^2}{3 - 2x} \tag{24}$$

$$g(t) = (t^2 + 1)(\pi t - 6) \tag{25}$$

$$h(w) = \left(\frac{1}{\sqrt{x}} - x^3\right) \ln x^2 \tag{26}$$

$$f(x) = (2x + 1)\left(\sqrt[3]{x^2} + \frac{1}{x^3}\right) \tag{27}$$

Exercise 5: Find the equation of the tangent line at P for the function f.

$$f(x) = \frac{x}{2x+3}, P = \left(1, \frac{1}{5}\right) \tag{28}$$

$$f(x) = 2x^3(3x^4 + x), P = (-2, -736) \tag{29}$$

$$f(x) = (3x+2)(2x-5), P = \left(\frac{1}{2}, -14\right) \tag{30}$$

$$f(x) = \sqrt{x}\ln x^3, P = (4, 12\ln 2) \tag{31}$$

End of Lesson

Next Lesson: Chain Rule