Basic Rules of Differentiation

MATH 1441, BCIT

Technical Mathematics for Food Technology

October 25, 2018

Functions

A function is a rule that assigns to each element in a set A one and only one element in a set B.

Exercise: Find the maximum domain and range of the following functions on the real number line:

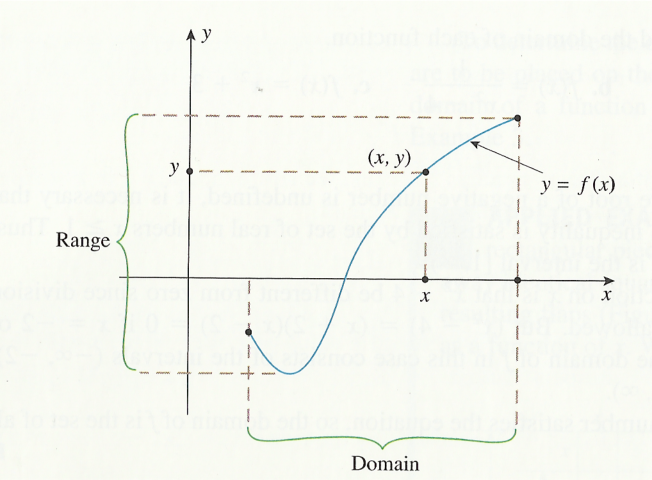
$$f(x) = \sqrt{x - 1} \tag{1}$$

$$f(x) = \frac{1}{x^2 - 4} \tag{2}$$

$$f(x) = x^2 + 3 \tag{3}$$

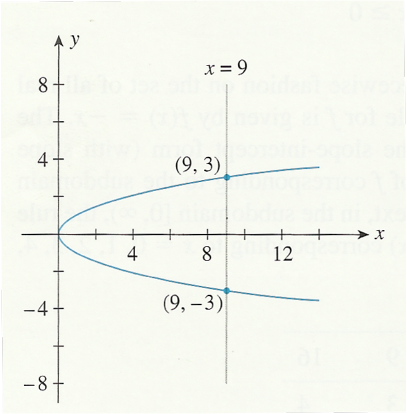
Function Graphs

The graph of a function f is the set of all points (x, y) in the xy-plane such that x is in the domain of f and $y = f(x)$.



Vertical Line Test

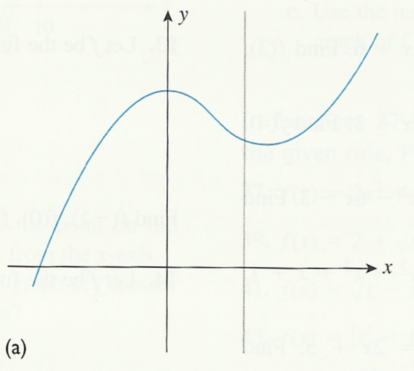
Every function f on a subset of the real numbers has a function graph, but not all graphs correspond to a function. Consider the graph $y^2 = x$. A curve in the xy-plane is the graph of a function $y = f(x)$ if and only if each vertical line intersects it in at most one point.



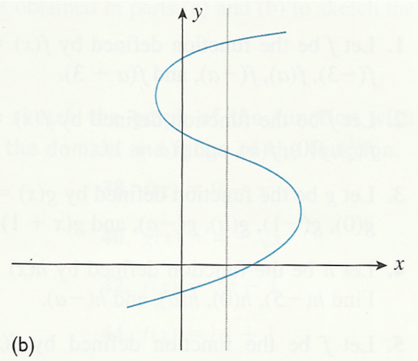
Vertical Line Test Exercise

In the next four slides, determine which graphs correspond to a function.

Vertical Line Test Exercise I



Vertical Line Test Exercise II



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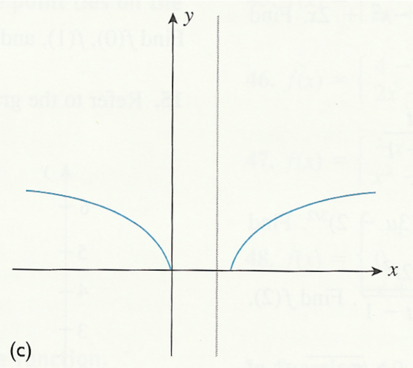
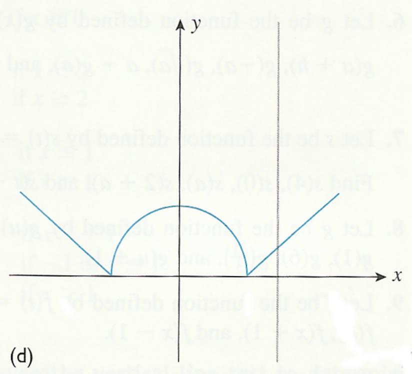


Vertical Line Test Exercise IV



Function Algebra

Let f and g be functions with domain A and B, respectively. Then the sum $f+g$, difference $f-g$, and product fg of f and g are functions with domain $A \cap B$ (the intersection of A and B) and rule given by

$$(f+g)(x) = f(x) + g(x) \tag{4}$$

$$(f-g)(x) = f(x) - g(x) \tag{5}$$

$$(fg)(x) = f(x) \cdot g(x) \tag{6}$$

The quotient f/g of f and g has domain $A \cap B$ excluding all points x such that $g(x) = 0$ and rule given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \tag{7}$$

Function Composition

Let f and g be functions. Then the composition of g and f is the function $g \circ f$ defined by

$$(g \circ f)(x) = g(f(x)) \tag{8}$$

The domain of $g \circ f$ is the set of all x in the domain of f such that $f(x)$ lies in the domain of g.

Consider the following two functions, $f(x) = \sqrt{x}$ and $g(y) = y - 2$. What are the maximal domains in the real numbers of $f \circ g$ and $g \circ f$?

Consider the function graph of the following function.

$$f(x) = \frac{x^2 - 1}{x - 1} \tag{9}$$

It looks like it is a linear equation! However, at $x = 1$, $f(x)$ is not defined. To fill the hole, we define the limit

$$\lim_{x \to a} f(x) = w \text{ if and only if } w = L = R \tag{10}$$

where L is the number that the function f approaches as x gets closer to a with $x < a$ (that means $x \neq a$!); and R is the number that the function f approaches as x gets closer to a with $x > a$.
Note: for a mathematically rigorous definition of what "approaching" and "getting closer" means we would need to talk about sequences and series, which is a topic we won't cover here.

Indeterminate Form I

Notice that

$$f(x) = \frac{x^2 - 1}{x - 1} \overset{x=1}{=} \frac{0}{0} \qquad (11)$$

We call this an indeterminate form.

Notice that except at $x = 1$

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x + 1 = g(x) \qquad (12)$$

f and g agree everywhere except on $x = 1$. Consider the following rule,

One Disagreement Rule

If $f = g$ except in one point, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ for all a, even the a where f and g disagree.

Therefore

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} (x + 1) = 2 \qquad (13)$$

Continuity

Consider a simple function like $f(x) = x^3$. What is $\lim_{x\to 4} f(x)$? The answer is almost trivial,

$$\lim_{x\to 4} f(x) = f(4) = 4^3 = 64 \tag{14}$$

Why is this true? Because f is continuous at $x = 4$. There are no holes, jumps, gaps, or breaks of the function graph at $x = 4$. Constant functions, the identity function, linear functions, polynomial functions, exponential and logarithmic functions are all continuous. Rational functions and other functions are sometimes not continuous.

Interesting Cases

A function is continuous if and only if $\lim_{x \to c} = f(c)$. This means that (i) the function needs to be defined at $x = c$; (ii) the limit needs to be defined at $x = c$; and (iii) the function value and the limit need to be equal to each other.

Consider the following interesting cases:

1. A function that is continuous and well defined at $x = a$.
2. A function that is not continuous at $x = a$.
3. A function where the limit exists but $\lim_{x \to c} \neq f(c)$.
4. A function such as $f(x) = \sin(1/x)$.

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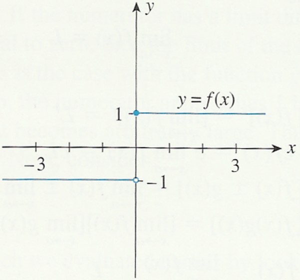
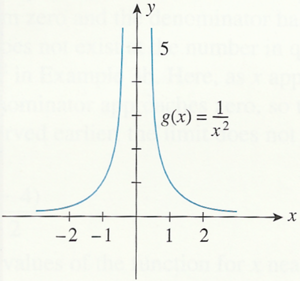
EXAMPLE Evaluate the limit of the following functions as x approaches the indicated point.

a. $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$; $x = 0$ **b.** $g(x) = \dfrac{1}{x^2}$; $x = 0$

Solution The graphs of the functions f and g are shown in Figure 29.

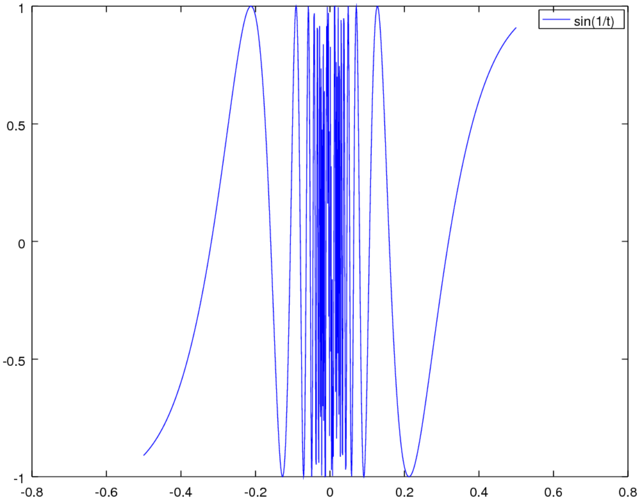


(a) $\lim_{x \to 0} f(x)$ does not exist.



(b) $\lim_{x \to 0} g(x)$ does not exist.

No Limit Example II



Properties of Limits

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then (call this Theorem 1),

$$\lim_{x\to a}[f(x)]^r = L^r, r \text{ a real number} \tag{15}$$

$$\lim_{x\to a}[c \cdot f(x)] = c \cdot L, c \text{ a real number} \tag{16}$$

$$\lim_{x\to a}[f(x) \pm g(x)] = L \pm M \tag{17}$$

$$\lim_{x\to a}[f(x)g(x)] = LM \tag{18}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}, \text{provided that } M \neq 0 \tag{19}$$

Use the properties of limits to evaluate the following,

$$\lim_{x \to 2} x^3 \tag{20}$$

$$\lim_{x \to 4} 5x^{3/2} \tag{21}$$

$$\lim_{x \to 1} \left(5x^4 - 2\right) \tag{22}$$

$$\lim_{x \to 3} 2x^3 \sqrt{x^2 + 7} \tag{23}$$

$$\lim_{x \to 2} \frac{2x^2 + 1}{x + 1} \tag{24}$$

Another Indeterminate Form Example I

Here is an example where by skillful manipulation we can determine the limit even though at first the limit is in indeterminate form. Let

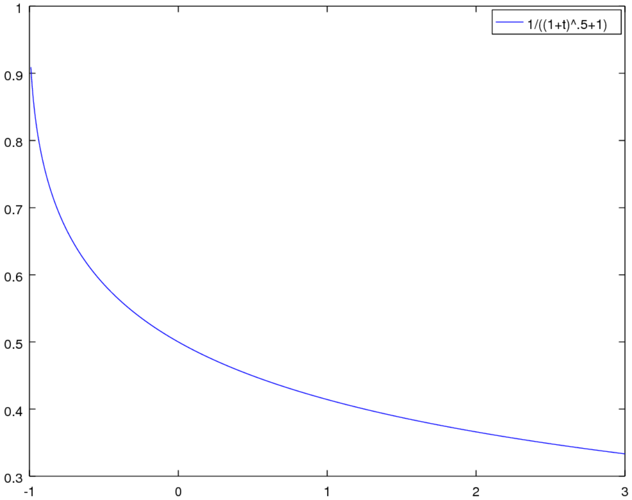
$$f(x) = \frac{\sqrt{1+x} - 1}{x} \tag{25}$$

What is $\lim_{x\to 0} f(x)$? If we leave the fraction unchanged, it will give us an indeterminate form. However, if we multiply both numerator and denominator by $(\sqrt{1+x} + 1)$, we avoid the indeterminate form!

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2} \tag{26}$$

Look at the function graph of $f(x)$ to verify that this is the correct limit.

Another Indeterminate Form Example II



Limits at Infinity I

Sometimes, we want to know what happens to a function graph when either x or $-x$ get very large. We use the infinity sign ∞ for notation, but note that we do NOT use infinity to define these limits.

$$\lim_{x \to \infty} f(x) = w \text{ if and only if } w = S \tag{27}$$

where S is a number such that for any tiny number ε there is a real number x_0 and $|f(x) - S| < \varepsilon$ for all $x > x_0$.

Limits at Infinity II

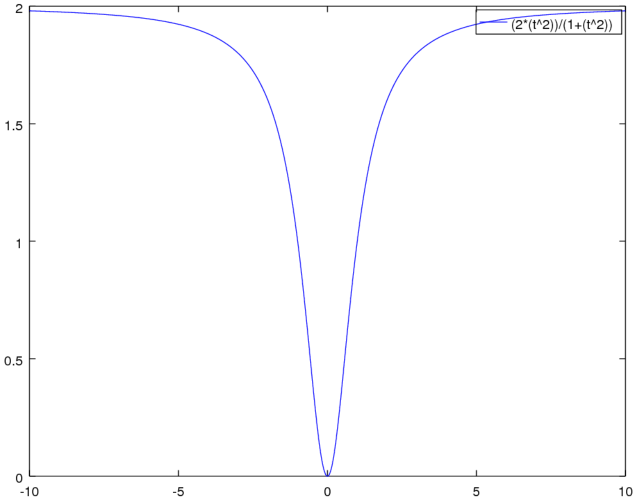
For example, what is

$$\lim_{x\to\infty} \frac{2x^2}{1+x^2} \tag{28}$$

or

$$\lim_{x\to-\infty} \frac{2x^2}{1+x^2} \tag{29}$$

Limits at Infinity III



Limits at Infinity IV

Here is another important property of limits (call this Theorem 2).
If $1/x^n$ is defined and $n > 0$, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \text{ and } \lim_{x\to-\infty} \frac{1}{x^n} = 0 \tag{30}$$

Polynomial and Rational Functions

A polynomial function looks like this,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \qquad (31)$$

For example, $p(x) = 7x^3 - 4.7x^2 + 6$. $n > 0$ is a natural number, and the a_i are called coefficients. They are real numbers. A rational function looks like this,

$$q(x) = \frac{p_1(x)}{p_2(x)} \qquad (32)$$

where $p_1(x)$ and $p_2(x)$ are polynomial functions. For example,

$$q(x) = \frac{5x^2 - \pi x + 9000}{e^2 x + 2} \qquad (33)$$

Limits at Infinity V

When we are looking for the limit of rational functions as they go to negative or positive infinity, we often get an indeterminate form.

$$\lim_{x\to\infty} \frac{x^2 - x + 3}{2x^3 + 1} = \frac{\infty}{\infty} \tag{34}$$

Here is a technique that will almost always work. Divide both the numerator and the denominator by x^m, where m is the highest exponent you can find.

$$\lim_{x\to\infty} \frac{x^2 - x + 3}{2x^3 + 1} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x^2} + \frac{3}{x^3}}{2 + \frac{1}{x^3}} = \frac{0}{2} = 0 \tag{35}$$

Limits at Infinity VI

Here are two more examples.

$$\lim_{x \to -\infty} \frac{3x^2 + 8x - 4}{2x^2 + 4x - 5} = \lim_{x \to -\infty} \frac{3 - \frac{8}{x} - \frac{4}{x^2}}{2 + \frac{4}{x} - \frac{5}{x^2}} = \frac{3}{2} = 1.5 \qquad (36)$$

$$\lim_{x \to \infty} \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4} = \lim_{x \to \infty} \frac{2 - \frac{3}{x} + \frac{1}{x^3}}{\frac{1}{x} + \frac{2}{x^2} + \frac{4}{x^3}} = \frac{2}{0} = \text{undefined} \qquad (37)$$

In the second example, the limit does not exist. Sometimes, we write $\lim_{x \to a} = \infty$ or $\lim_{x \to a} = -\infty$, depending on which way the function goes.

Example I

Consider the function,

$$f(x) = \frac{x-4}{\sqrt{x}-2} \tag{38}$$

Let's find

$$\lim_{x \to 4} f(x) \tag{39}$$

Example II

First, fill out the table:

$x = 3$	$f(x) = 3.7321$	$x = 5$	$f(x) = 4.2361$
$x = 3.5$		$x = 4.5$	
$x = 3.75$		$x = 4.25$	
$x = 3.9$		$x = 4.1$	
$x = 3.95$		$x = 4.05$	
$x = 3.99$		$x = 4.01$	

Example III

Next, let's assume that $x \neq 4$ and expand both the numerator and denominator by $\sqrt{x}+2$. Simplify

$$g(x) = \frac{(x-4)\cdot(\sqrt{x}+2)}{(\sqrt{x}-2)\cdot(\sqrt{x}+2)} \text{ on domain } \mathbb{R}\setminus\{4\} \qquad (40)$$

Except on $x = 4$, g agrees with f. Determine $\lim_{x\to 4} g(x)$.

Exercises I

Evaluate the following two limits.

$$\lim_{x \to 3} = \frac{\sqrt{x^2 + 7} + \sqrt{3x - 5}}{x + 2} \tag{41}$$

$$\lim_{x \to -1} \frac{x^2 - x - 2}{2x^2 - x - 3} \tag{42}$$

Exercises II

Evaluate the following three limits.

$$\lim_{x \to 2} 3 \tag{43}$$

$$\lim_{x \to \infty} \frac{3x + 2}{x - 5} \tag{44}$$

$$\lim_{x \to \infty} \frac{x^5 - x^3 + x - 1}{x^6 + 2x^2 + 1} \tag{45}$$

Finding Limits Exercises

Find the following limits,

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \tag{46}$$

$$\lim_{x\to\infty} \frac{\sqrt{x^2-8x}}{2x+1} \tag{47}$$

$$\lim_{x\to -1} \frac{x^2-x-2}{2x^2-x-3} \tag{48}$$

$$\lim_{x\to\infty} \frac{2+\frac{1}{x+4}}{3-\frac{1}{x^2}} \tag{49}$$

$$\lim_{x\to\infty} \frac{x-2x^3}{(1+x)^3} \tag{50}$$

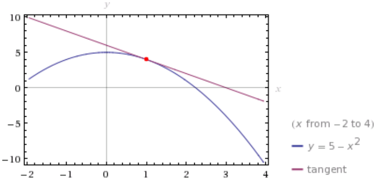
$$\lim_{x\to\infty} \frac{\sqrt{4x^2+3}}{x+5} \tag{51}$$

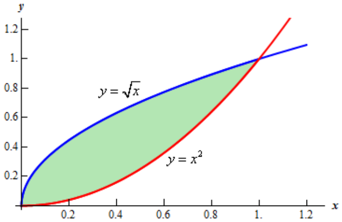
In Einstein's theory of relativity, the length L of an object moving at a velocity v is

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} \tag{52}$$

where c is the speed of light and L_0 is the length of the object at rest. What is the one-sided limit of L as v gets faster and faster?

Calculus solves many problems for which it was not originally designed. The initial motivation for calculus was to find the slope of a tangent on a curve and the area of a region bounded by a curve.





A Real-Life Example I

Consider a magnetic levitation train accelerating on a straight monorail track. The position of the train (in feet) from the origin at time t is given by

$$s = f(t) = 4t^2 \tag{53}$$

What is the velocity of the train at $t = 2$?

A Real-Life Example II

It appears to make sense only if we calculate the velocity given an interval of time rather than just one point in time. For example, the velocity between $t = 2$ and $t = 3$ is

$$v_{[2,3]} = \frac{f(3) - f(2)}{3 - 2} = 20 \tag{54}$$

More generally,

$$v_{[2,t]} = g(t) = \frac{f(t) - f(2)}{t - 2} = \frac{4(t^2 - 4)}{t - 2} \tag{55}$$

g is not defined at $t = 2$, but it is defined all around $t = 2$, so we can ask ourselves the question: what happens when $t \to 2$ from below; or when $t \to 2$ from above? It turns out that either way, the number approaches 16.

Velocity at a Point

Now remember formula (55) from a few slides ago.

$$v_{[2,t]} = g(t) = \frac{f(t) - f(2)}{t-2} = \frac{4(t^2-4)}{t-2}$$

If we found the limit as $t \to 2$, it would serve as an intuitive definition of what a velocity is at a point (instead of on an interval). Unfortunately, the limit has the indeterminate form

$$\lim_{t \to 2} \frac{4(t^2-4)}{t-2} = \frac{0}{0} \tag{56}$$

However, notice that for $t \neq 2$,

$$g(t) = \frac{4(t^2-4)}{t-2} = \frac{4(t+2)\cancel{(t-2)}}{\cancel{t-2}} = 4(t+2) \tag{57}$$

Tangent Lines I

Remember our magnetic levitation train. The distance-time function was

$$s = f(t) = 4t^2 \tag{58}$$

The velocity of the train over a given interval is

$$v_{[t_1,t_2]} = \frac{f(t_2) - f(t_1)}{t_2 - t_1} \tag{59}$$

This velocity is also the slope of the line going through the two function values $f(t_1)$ and $f(t_2)$. We call such a line a secant line.

Tangent Lines II

Now imagine t_1 and t_2 moving closer and closer together at a point a (for the train, we used $a = 2$). If both of these limits exist and agree with each other, we have a velocity at a point,

$$\lim_{t \to a} v_{[a,t]} \text{ for } t > a \tag{60}$$

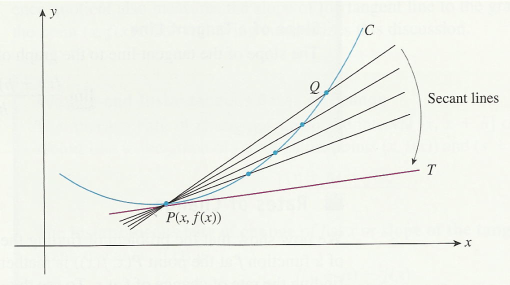
$$\lim_{t \to a} v_{[t,a]} \text{ for } t < a \tag{61}$$

This velocity at a point is also the slope of the line that just touches the function graph without crossing it. We call it a tangent line at $t = a$. The slope of the tangent line is sometimes also called the rate of change.

Tangent Lines III

Think of a tangent line as the limit of secant lines. The slope of a tangent line at a point $P = (x, f(x))$, if it exists, is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{62}$$



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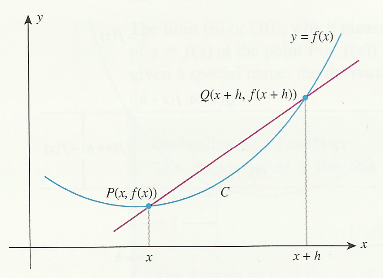


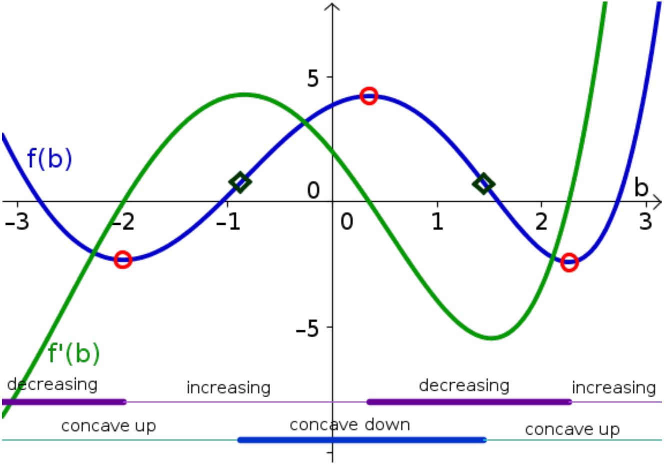
Derivatives

The derivative of a function f with respect to x is the function f' (read "f prime"),

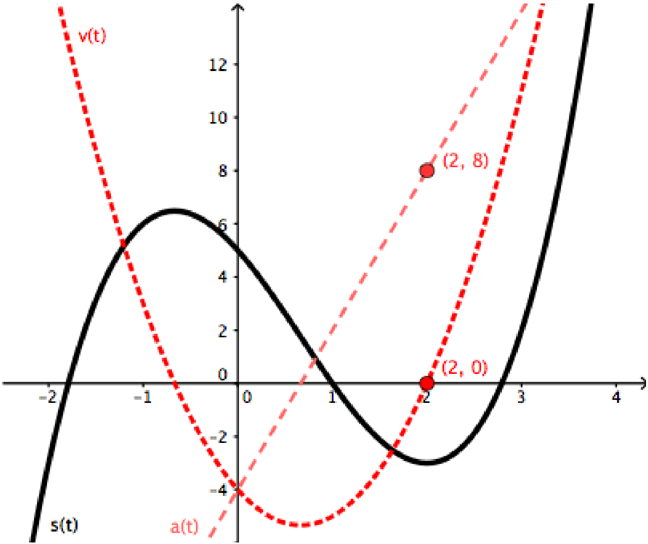
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{64}$$

The domain of f' is the set of all x where the limit exists.

Derivatives Diagram I



Derivatives Diagram II



Basic Rules of Differentiation I

Rule 1

Derivative of a Constant

$$f'(x) = 0 \text{ for } f(x) = c \tag{65}$$

Reason:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0 \tag{66}$$

Rule 2

The Power Rule

$$f'(x) = nx^{n-1} \text{ for } f(x) = x^n \tag{67}$$

Reason (the general case is messy, we will just show it for $f(x) = x^2$):

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$\lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0}(2x + h) = 2x \tag{68}$$

Rule 3

Derivative of a Constant Multiple of a Function

$$g'(x) = c \cdot f'(x) \text{ for } g(x) = c \cdot f(x) \tag{69}$$

Reason:

$$g'(x) = \lim_{h\to0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} =$$

$$\lim_{h\to0} c \cdot \frac{f(x+h) - f(x)}{h} = c \cdot \lim_{h\to0} \frac{f(x+h) - f(x)}{h} = c \cdot f'(x) \tag{70}$$

Rule 4

The Sum Rule

$$g'(x) = f_1'(x) + f_2'(x) \text{ for } g(x) = f_1(x) + f_2(x) \tag{71}$$

Reason:

$$g'(x) =$$
$$\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{f_1(x+h) + f_2(x+h) - f_1(x) - f_2(x)}{h} =$$
$$\lim_{h\to 0} \left(\frac{f_1(x+h) - f_1(x)}{h} + \frac{f_2(x+h) - f_2(x)}{h} \right) = f_1'(x) + f_2'(x) \tag{72}$$

Basic Differentiation Exercises I

Find the derivatives for the following functions.

1. $f(x) = 4x^5 + 3x^4 - 8x^2 + x + 3$
2. $f(t) = \frac{t^2}{5} + \frac{5}{t^3}$
3. $g(z) = 2z - 5\sqrt{z}$

Find the slope and an equation of the tangent line to the graph of $f(x) = 2x + (1/\sqrt{x})$ at the point $(1, 3)$.

Basic Differentiation Exercises II

Find the derivatives for the following functions.

$$f(x) = 5x^{\frac{4}{3}} - \frac{2}{3}x^{\frac{3}{2}} + x^2 - 3x + 1 \tag{73}$$

$$f(x) = 2t^2 + \sqrt{t^3} \tag{74}$$

$$f(x) = \frac{2}{x^2} - \frac{3}{x^{\frac{1}{3}}} \tag{75}$$

$$f(x) = \frac{3}{x^3} + \frac{4}{\sqrt{x}} + 1 \tag{76}$$

End of Lesson

Next Lesson: Product and Quotient Rule