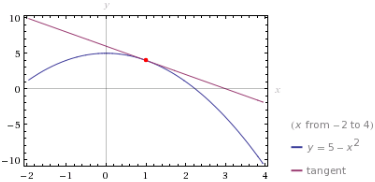
Fundamental Theorem of Calculus

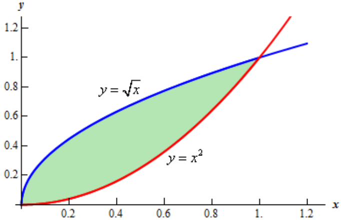
MATH 1441, BCIT

Technical Mathematics for Food Technology

November 26, 2018

Remember these two problems that we wanted to solve when we started with calculus:





We have solved the problem on the left. Now it is time to solve the problem on the right. For areas under a curve, we need antiderivatives. The antiderivative $F(x)$ of a function $f(x)$ is the function for which $F'(x) = f(x)$.

Differential Equations

Differential equations are like regular equations except that the unknown is a function, not a variable. Remember that

$$dy = f'(x)\,dx, \text{ therefore } f'(x) = \frac{dy}{dx} \tag{1}$$

Now consider this differential equation,

$$\frac{dy}{dx} = f(x) \tag{2}$$

This is an ODE, an ordinary differential equation.

Differential Equations

$$\frac{dy}{dx} = f(x) \tag{3}$$

This is an ODE, an ordinary differential equation. Any function

$$f(x) = e^x + C, C \in \mathbb{R} \tag{4}$$

would solve it. Often, an initial condition is provided to make the solution unique. Therefore, the solution to the differential equation

$$\frac{dy}{dx} = f(x) \tag{5}$$

with initial condition $f(0) = 1$ is $f(x) = e^x$.

Differential Equations

Antiderivatives are solutions to special differential equations. For example, the antiderivative of $f(x) = 6x$ is the solution to the differential equation

$$\frac{dy}{dx} = 6x \tag{6}$$

With an initial condition, the solution to this equation may be unique.

Rules for Finding Antiderivatives

Antiderivatives are not unique. If $F(x)$ is an antiderivative for $f(x)$, then $F(x) + c$ is an antiderivative as well, where c is any real number. In the following, we will use the notation $F(x)$ for one arbitrary antiderivative. There are many rules for finding antiderivatives called *table of integrals*. Here are a few.

Rule 1

If you find a function $g(x)$ for which $g'(x) = f(x)$, then $F(x) = g(x) + c$.

Exercise: show that the function $g(x)$ is an antiderivative of $f(x) = (x^3 + 3)^6(3x^2)$.

$$g(x) = \frac{(x^3 + 3)^7}{7} \tag{7}$$

More Rules for Finding Antiderivatives I

Rule 2

If $F(x)$ is an antiderivative for $f(x)$, then $aF(x)$ is an antiderivative for $af(x)$, where a is a constant.

More Rules for Finding Antiderivatives II

Rule 3

If $F_1(x)$ is an antiderivative for $f_1(x)$ and $F_2(x)$ is an antiderivative for $f_2(x)$, then $F_1(x) + F_2(x)$ is an antiderivative for $f_1(x) + f_2(x)$.

More Rules for Finding Antiderivatives III

Rule 4

If $f(x) = x^n$ and $n \neq -1$, then $F(x) = \frac{x^{n+1}}{n+1}$ is an antiderivative of $f(x)$.

Exercise: Find an antiderivative of $f(x) = 1/x$. The answer is not quite what you would expect (but very close).

Summary

Here is a table of antiderivatives, where F is an antiderivative of f and G is an antiderivative of g.

$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n with $n \neq -1$	$\dfrac{x^{n+1}}{n+1}$
$\dfrac{1}{x}$	$\ln \lvert x \rvert$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\dfrac{1}{1+x^2}$	$\arctan x$

The process of finding a derivative is called differentiation. The process of finding an antiderivative is called integration. Instead of the symbol 'prime' ($f'(x)$) for differentiation we use the sign $\int$ for integration. The symbol $\int$ stands for the word 'sum' because we take the limit of a sum of areas in order to find the area under a curve.

$$\int f(x)\,dx = F(x) + c \tag{8}$$

The differential helps to identify which letter is the variable for the function (there may be other letters that are just constants), for example

$$\int ax^2\,dx = \frac{ax^3}{3} + c \tag{9}$$

$$\int ax^2\,da = \frac{a^2x^2}{2} + c \tag{10}$$

Find the following indefinite integrals (another expression for antiderivatives).

$$\int 6\,dx \tag{11}$$

$$\int -2\,dx \tag{12}$$

$$\int 8x^4\,dx \tag{13}$$

$$\int \pi x^3\,dx \tag{14}$$

$$\int \left(x^3 + 7 - 2x^2\right)\,dx \tag{15}$$

$$\int \sqrt{x}\,dx \tag{16}$$

$$\int \frac{7}{2}x^{\frac{5}{2}}\,dx \tag{17}$$

Integration Exercises II

Find the following indefinite integrals (another expression for antiderivatives).

$$\int 9\sqrt[5]{2x}\,dx \tag{18}$$

$$\int \frac{3}{x^3}\,dx \tag{19}$$

$$\int \frac{7}{\sqrt[3]{x}}\,dx \tag{20}$$

$$\int \sqrt{x}\,(3x-2)\;dx \tag{21}$$

$$\int (x+1)^2\;dx \tag{22}$$

$$\int \frac{4x^2-2\sqrt{x}}{x}\,dx \tag{23}$$

$$\int \frac{x^3+2x^2-3x-6}{x+2}\,dx \tag{24}$$

Definite Integrals I

Evaluating an integral at a point doesn't give us anything particularly meaningful.

$$\int x^2\,dx = \frac{x^3}{3} + c \tag{25}$$

$$\left.\int x^2\,dx\right|_{x=6} = \frac{6^3}{3} + c = 72 + c \tag{26}$$

However, if we subtract one evaluated integral from another, we get a number.

$$\left.\int x^2\,dx\right|_{x=6} - \left.\int x^2\,dx\right|_{x=3} = \frac{6^3}{3} + c - \left(\frac{3^3}{3} + c\right) = 72 - 9 = 63$$

Definite Integrals II

We call this difference between evaluated integrals definite integral. The notation is

$$\int_3^6 x^2\,dx = \int x^2\,dx\Big|_{x=6} - \int x^2\,dx\Big|_{x=3} = 63$$

Definite Integrals Exercises

Evaluate each definite integral.

$$\int_1^2 x\,dx \qquad \int_{-2}^{2} x^2\,dx \tag{27}$$

$$\int_1^3 7x^2\,dx \qquad \int_{-2}^{2} 3s^4\,ds \tag{28}$$

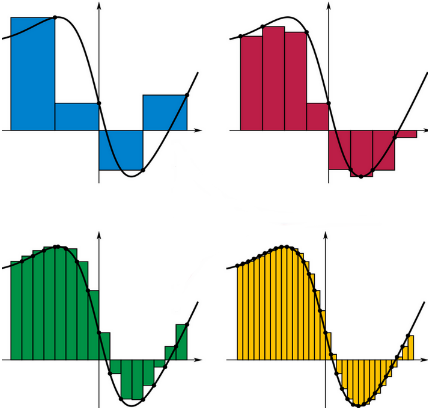
$$\int_0^4 (x^2+2x)\,dx \qquad \int_1^e \frac{1}{x}\,dx \tag{29}$$

$$\int_5^{10} \sqrt{x}\,dx \qquad \int_1^4 \frac{2+x^2}{\sqrt{x}}\,dx \tag{30}$$

$$\int_{-1}^{2} (3u-2)(u+1)\,du \qquad \int_{\frac{\pi}{6}}^{\pi} \sin\vartheta\,d\vartheta \tag{31}$$

Fundamental Theorem of Calculus I

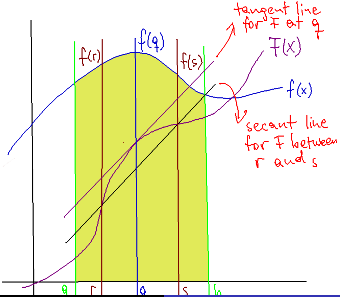
It turns out that the definite integral $\int_a^b f(x)\,dx$ gives you the area under the curve $y = f(x)$ between a and b. This area can be approximated by a series of rectangles.



Fundamental Theorem of Calculus II

Let's assume our function is positive between a and b, so $f(x) \geq 0$ for $a \leq x \leq b$. Let F be an antiderivative of f. Here is the mean value theorem, a theorem we need to assume without proof: between two arguments r and s we can always find a point q such that the slope of the secant line between $F(r)$ and $F(s)$ equals the slope of the tangent line at $F(q)$, so

$$F'(q) = \frac{F(s) - F(r)}{s - r} \text{ (MVT)}$$



Fundamental Theorem of Calculus III

Now divide the interval from a to b (the notation for this interval is $[a, b]$) into n intervals that are of equal length. For this, we need intermediate points $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$. The approximate area under the curve between a and b is

$$A \approx \frac{x_1 - a}{n} f(x_1^*) + \frac{x_2 - x_1}{n} f(x_2^*) + \ldots + \frac{b - x_{n-1}}{n} f(x_n^*) \qquad (32)$$

where x_1^* is some point in the first interval and so on. Notice that the fractions all equal $(b - a)/n$ because the intervals are all of equal length. Therefore

$$A = \lim_{n \to \infty} \frac{b - a}{n} \left(f(x_1^*) + \ldots + f(x_n^*) \right) \qquad (33)$$

Fundamental Theorem of Calculus IV

Now choose x_1^* such that

$$f(x_1^*) = F'(x_1^*) = \frac{F(x_1) - F(x_0)}{x_1 - x_0} \tag{34}$$

and so on with $x_2^*, x_3^*, \ldots, x_n^*$. Then

$$A = \lim_{n\to\infty} \frac{b-a}{n} \left(\frac{F(x_1) - F(x_0)}{x_1 - x_0} + \ldots + \frac{F(x_n) - F(x_{n-1})}{x_n - x_{n-1}} \right) \tag{35}$$

Note that $x_i - x_{i-1}$ (where i is any number between 1 and n) is again just the length of the intervals $(b-a)/n$. After appropriate simplification,

$$A = F(b) - F(a) = \int_a^b f(x)\,dx \tag{36}$$

Fundamental Theorem of Calculus V

Here are two different ways to express the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Note that we need not require $a \leq b$. If the limits of integration are unintuitively placed, you can rectify the situation by using

$$\int_b^a f(x)\,dx = F(a) - F(b) = -(F(b) - F(a)) = -\int_a^b f(x)\,dx$$

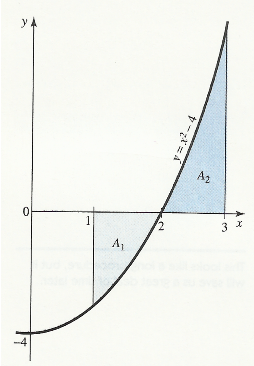
Exercise 1: Find the area under the parabola

$$y = x^2 \tag{37}$$

from 0 to 1.

Consider the following problem.

Find the area under the curve $y = x^2 - 4$ between $x = 1$ and $x = 3$.



Negative Area

To solve this problem, find the x-intercept and treat the positive and negative area separately.

$$|A_1|+|A_2| = -\int_1^2 (x^2-4)\,dx + \int_1^2 (x^2-4)\,dx = -\left(-\frac{5}{3}\right)+\frac{7}{3} = 4$$

We know how to integrate the following functions

$$f_1(y) = y^3 \text{ and } f_2(x) = 2x + 5 \quad (38)$$

but how do you integrate $f = f_1 \circ f_2$, so

$$f(x) = (2x + 5)^3 \quad (39)$$

We use the method of substitution. Write

$$y = 2x + 5 \quad (40)$$

The important part here is that the substitution changes the differential and the limits.

$$dy = 2dx \text{ and therefore } dx = \frac{1}{2}dy \quad (41)$$

Therefore,

$$\int_a^b (2x + 5)^3 dx = \int_{2a+5}^{2b+5} y^3 \cdot \frac{1}{2} dy \quad (42)$$

Integration by Substitution Example

Let's evaluate $\int_0^4 x\sqrt{9+x^2}dx$. We will do this two ways. For method 1, we find the indefinite integral of $x\sqrt{9+x^2}$ and then use the limits $a=0, b=4$ to evaluate the definite integral. For method 2, we proceed as on the previous slide and change both differential and limits for the definite interval. Here is method 1. Substitute $y=9+x^2$. Then, $dy=2xdx$, so

$$\frac{1}{2}dy = xdx \tag{43}$$

Notice that we need the factor x on the right-hand side in order to make this integration work.

$$\int x\sqrt{9+x^2}dx = \frac{1}{2}\int \sqrt{y}dy = \frac{1}{2}\cdot\frac{y^{\frac{3}{2}}}{\frac{3}{2}} \tag{44}$$

Integration by Substitution Example

Now reverse the substitution

$$\frac{1}{2} \cdot \frac{y^{\frac{3}{2}}}{\frac{3}{2}} = \frac{1}{3}(9 + x^2)^{\frac{3}{2}} \tag{45}$$

and evaluate the definite integral

$$\int_0^4 x\sqrt{9 + x^2}dx =$$

$$\frac{1}{3}(9 + x^2)^{\frac{3}{2}}\bigg|_{x=4} - \frac{1}{3}(9 + x^2)^{\frac{3}{2}}\bigg|_{x=0} = \frac{98}{3} \tag{46}$$

Integration by Substitution Example

Here is method 2.

$$\int_0^4 x\sqrt{9+x^2}dx = \frac{1}{2}\int_9^{25}\sqrt{y}dy =$$

$$\frac{1}{3}\left(y^{\frac{3}{2}}\Big|_{y=25} - y^{\frac{3}{2}}\Big|_{y=9}\right) = \frac{1}{3}(125-27) = \frac{98}{3} \tag{47}$$

Exercise 2: Evaluate the following definite integrals.

$$\int_0^2 x(x^2-1)^3 dx \qquad \int_0^1 x^2(2x^3-1)^4 dx \tag{48}$$

$$\int_0^1 x\sqrt{5x^2+4} dx \qquad \int_1^3 x\sqrt{3x^2-2} dx \tag{49}$$

$$\int_0^2 x^2(x^3+1)^{\frac{3}{2}} dx \qquad \int_1^5 (2x-1)^{\frac{5}{2}} dx \tag{50}$$

$$\int_0^1 \frac{1}{\sqrt{2x+1}} dx \qquad \int_0^2 \frac{x}{\sqrt{x^2+5}} dx \tag{51}$$

Exercises

Exercise 3: Evaluate the following definite integrals.

$$\int_1^2 (2x+4)(x^2+4x-8)^3 dx \qquad \int_{-1}^1 x^2(x^3+1)^4 dx \quad (52)$$

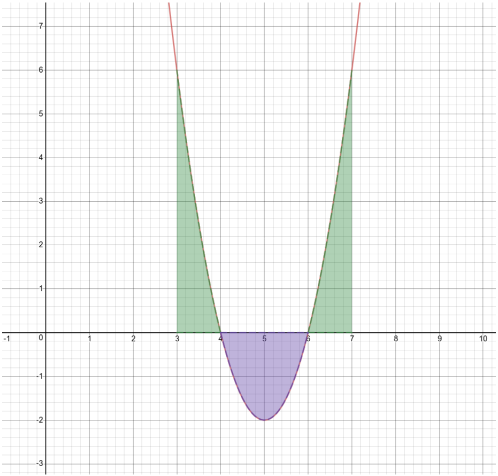
$$\int_0^2 xe^{x^2} dx \qquad \int_0^1 e^{-1} dx \quad (53)$$

$$\int_3^6 \frac{2}{x-2} dx \qquad \int_0^1 \frac{e^x}{1+e^x} dx \quad (54)$$

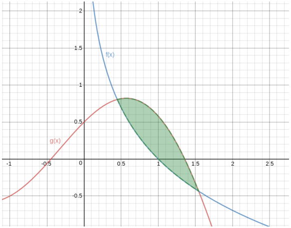
$$\int_0^1 \frac{x}{1+2x^2} dx \qquad \int_1^2 \frac{\ln x}{x} dx \quad (55)$$

Negative Area Exercise

Exercise 4: Find the area between the curve $y = 2(x-5)^2 - 2$ and the x-axis between $x = 3$ and $x = 7$.



Exercise 5: Find the area bounded by the curves $f(x)$ and $g(x)$.



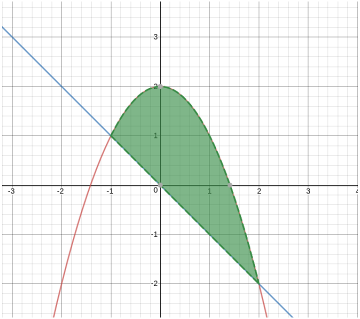
To find this area, solve for the two solutions x_1, x_2 of

$$f(x) = g(x) \tag{56}$$

(you may have to use Newton's method) and then integrate

$$A = \int_{x_1}^{x_2} (g(x) - f(x))\, dx \tag{57}$$

Exercise 6: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



End of Lesson

Next Lesson: That's all, folks! See you next year!