Chain Rule

MATH 1441, BCIT

Technical Mathematics for Food Technology

November 9, 2017

Problematic Functions

Here are some functions that we either don't know how to differentiate or whose differentiation would take an inordinate amount of time.

$$f(x) = 2^x \tag{1}$$

$$f(x) = \sqrt{x^2 + 1} \tag{2}$$

$$f(x) = (x^2 + x + 1)^{100} \tag{3}$$

$$f(x) = \sin(1 + \sqrt{x - 7}) \tag{4}$$

$$f(x) = \log_{10} x \tag{5}$$

$$f(x) = \ln(x^2 + 1) \tag{6}$$

The Chain Rule

Rule 7

The Chain Rule

$$g'(x) = f_1'(f_2(x))f_2'(x) \text{ for } g(x) = (f_1 \circ f_2)(x) \tag{7}$$

Chain Rule Reason

Consider

$$
\begin{aligned}
(f \circ g)'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} &= \\
\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} &= \qquad (8) \\
f'(g(x))g'(x) & \qquad (9)
\end{aligned}
$$

This is only a hint, not a rigorous proof, since we have replaced $g(x+h)$ by $g(x)+h$, which isn't covered by our rules and is, in fact, false in some situations.

Exercises

1. Diffentiate: $f(x) = 2^x$
2. Diffentiate: $f(x) = \sqrt{x^2 + 1}$
3. Diffentiate: $f(x) = (x^2 + x + 1)^{100}$
4. Diffentiate: $f(x) = \log_{10} x$
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Inverse and Identity Function

Remember how we defined the logarithmic function,

$$\ln y = x \text{ if and only if } e^x = y \tag{10}$$

so the logarithmic function is the inverse of the exponential function. Consequently, if $f(x) = e^x$ and $g(y) = \ln y$

$$(f \circ g)(y) = y \text{ and } (g \circ f)(x) = x \tag{11}$$

When (11) is true we call f the inverse function of g and vice versa. The function $\mathrm{id}(x) = x$ is called the identity function.

The Derivative of the Exponential Function

We know the derivative of the identity function.

$$\mathrm{id}'(x) = 1 \tag{12}$$

Consequently,

$$\frac{d}{dx} \ln \left(e^x\right) = 1 \tag{13}$$

We also know that according to the chain rule

$$\frac{d}{dx} \ln \left(e^x\right) = \frac{1}{e^x} \exp'(x) \tag{14}$$

where $\exp(x) = e^x$. Therefore,

$$\exp'(x) = e^x \tag{15}$$

The exponential function is its own derivative!

Derivative of the Exponential Function: Exercises

Differentiate the following functions:

$$f(x) = e^{\sin x} \tag{16}$$

$$g(t) = \frac{1}{e^t} \tag{17}$$

$$v(w) = w^2 e^w \tag{18}$$

$$g(z) = \frac{e^z - 1}{e^z + 1} \tag{19}$$

Exercises for Differentiation I

Differentiate the following functions or find dy/dx for the following curves:

$$f(\vartheta) = \tan(\sin\vartheta) \tag{20}$$

$$F(x) = \sqrt[4]{1 + 2x + x^3} \tag{21}$$

$$g(t) = \frac{\pi}{(t^4 + 1)^3} \tag{22}$$

$$f(s) = \sqrt[3]{1 + \tan s} \tag{23}$$

$$y = (x^2 + 1)\sqrt[3]{x^2 + 2} \tag{24}$$

$$y = e^{x\cos x} \tag{25}$$

$$y = x\sin\frac{1}{x} \tag{26}$$

Exercises for Differentiation II

Differentiate the following functions or find dy/dx for the following curves:

$$y = 3\cot(nx) \tag{27}$$

$$y = xe^{-kx} \tag{28}$$

$$h(t) = (t^4 - 1)^3(t^3 + 1)^4 \tag{29}$$

$$y = (x^2 + 1)\sqrt{x^2 + 2} \tag{30}$$

$$G(y) = \left(\frac{y^2}{y + 1}\right)^5 \tag{31}$$

$$y = \tan^2(3\vartheta) \tag{32}$$

Exercises for Differentiation III

Find an equation of the tangent line to the curve

$$y = \frac{2}{1 + e^{-x}} \tag{33}$$

at $x = 0$.

Here is a model for the length of daylight (in hours) in Toronto on the t-th day of the year

$$L(t) = 12 + 2.8 \sin\left(\frac{2\pi}{365}(t - 80)\right) \tag{34}$$

Compare how the number of hours of daylight is increasing in Toronto on March 21 and May 21.

End of Lesson

Next Lesson: Optimization