

# Product and Quotient Rule

## MATH 1441, BCIT

Technical Mathematics for Food Technology

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## Rule 5

### The Product Rule

$$g'(x) = f_1(x)f_2'(x) + f_1'(x)f_2(x) \text{ for } g(x) = f_1(x)f_2(x) \quad (1)$$

# Product Rule Reason

Reason:

$$\begin{aligned} g'(x) &= \\ \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} &= \lim_{h \rightarrow 0} \frac{f_1(x+h)f_2(x+h) - f_1(x)f_2(x)}{h} = \\ \lim_{h \rightarrow 0} \frac{f_1(x+h)f_2(x+h) - \cancel{f_1(x)f_2(x+h)} + \cancel{f_1(x)f_2(x+h)} - f_1(x)f_2(x)}{h} &= \\ \lim_{h \rightarrow 0} \frac{(f_1(x+h) - f_1(x))f_2(x+h) + f_1(x)(f_2(x+h) - f_2(x))}{h} &= \\ f_1(x)f_2'(x) + f_1'(x)f_2(x) &\quad (2) \end{aligned}$$

**Exercise 1:** Differentiate the following functions.

$$f(x) = (2x^2 - 1)(x^3 + 3) \quad (3)$$

$$g(t) = t^3 (\sqrt{t} + 1) \quad (4)$$

# Quotient Rule

## Rule 6

### The Quotient Rule

$$g'(x) = \frac{f_1'(x)f_2(x) - f_1(x)f_2'(x)}{(f_2(x))^2} \text{ for } g(x) = \frac{f_1(x)}{f_2(x)} \quad (5)$$

# Quotient Rule Reason

Reason:

$$g(x) = \frac{f_1(x)}{f_2(x)} \quad (6)$$

$$f_1(x) = g(x)f_2(x) \quad (7)$$

$$f_1'(x) = g'(x)f_2(x) + g(x)f_2'(x) \text{ now isolate } g'(x) \quad (8)$$

$$g'(x) = \frac{f_1'(x) - g(x)f_2'(x)}{f_2(x)} \text{ now substitute } g(x) = \frac{f_1(x)}{f_2(x)} \quad (9)$$

$$g'(x) = \frac{\frac{f_1'(x)f_2(x)}{f_2(x)} - \frac{f_1(x)f_2'(x)}{f_2(x)}}{f_2(x)} \quad (10)$$

$$g'(x) = \frac{f_1'(x)f_2(x) - f_1(x)f_2'(x)}{(f_2(x))^2} \quad (11)$$

**Exercise 2:** Differentiate the following functions.

$$f(z) = \frac{3z^2 + 5z - 2}{3z - 1} \quad (12)$$

$$h(x) = \frac{\sqrt{x}}{x^2 + 1} \quad (13)$$

# Quotient Rule Exercise Solution

$$g(x) = \frac{3x^2 + 5x - 2}{3x - 1}$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$[1] \quad g(x) = \frac{3x^2 + 5x - 2}{3x - 1} = \frac{\cancel{(3x-1)}(x+2)}{\cancel{3x-1}} = x+2 \quad \text{except at } x = \frac{1}{3}$$

$$g'(x) = \frac{d}{dx}(x+2) = 1$$

$$[2] \quad g'(x) = \frac{(6x+5)(3x-1) - (3x^2+5x-2)3}{(3x-1)^2} =$$

$$\frac{18x^2 + 15x - 6x - 5 - [9x^2 + 15x - 6]}{9x^2 - 6x + 1} = \frac{18x^2 + 15x - 6x - 5 - 9x^2 - 15x + 6}{9x^2 - 6x + 1} =$$

$$\frac{9x^2 - 6x + 1}{9x^2 - 6x + 1} = 1$$



# Euler's Number

The number  $e$  is defined as follows,

$$e = \lim_{t \rightarrow \infty} \left(1 + \frac{1}{t}\right)^t \quad (14)$$

Consider two functions  $f_1$  and  $f_2$ . They are related in so far as

$$f_1(x) = f_2\left(\frac{1}{x}\right) \quad (15)$$

For example,

$$f_1(x) = \frac{2x+1}{5x-7} \text{ and } f_2(x) = -\frac{x+2}{7x-5} \quad (16)$$

Then

$$\text{If } \lim_{x \rightarrow \infty} f_1(x) = a \text{ then } \lim_{x \rightarrow 0} f_2(x) = a \quad (17)$$

# The Derivative of the Logarithmic Function

Now consider the function  $f(x) = \ln x$  and the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \ln \frac{x+h}{x} = \\ &= \lim_{h \rightarrow 0} \frac{1}{x} \cdot \frac{x}{h} \ln \left( 1 + \frac{h}{x} \right) = \lim_{h \rightarrow 0} \frac{1}{x} \ln \left( 1 + \frac{1}{\frac{x}{h}} \right)^{\frac{x}{h}} \end{aligned} \quad (18)$$

Use the lemma of the last slide and the definition of Euler's number to see that

$$f'(x) = \frac{1}{x} \quad (19)$$

**Exercise 3:** Differentiate the following functions.

$$f(x) = x^2 \ln x \quad (20)$$

$$g(y) = \ln y^{(5y-2)} \quad (21)$$

$$h(z) = (\ln z)^2 \quad (22)$$

$$f(x) = \log_2 x \quad (23)$$

**Exercise 4:** Differentiate the following functions.

$$f(x) = \frac{21 - 5x - 6x^2}{3 - 2x} \quad (24)$$

$$g(t) = (t^2 + 1)(\pi t - 6) \quad (25)$$

$$h(w) = \left( \frac{1}{\sqrt{x}} - x^3 \right) \ln x^2 \quad (26)$$

$$f(x) = (2x + 1) \left( \sqrt[3]{x^2} + \frac{1}{x^3} \right) \quad (27)$$

**Exercise 5:** Find the equation of the tangent line at  $P$  for the function  $f$ .

$$f(x) = \frac{x}{2x+3}, P = \left(1, \frac{1}{5}\right) \quad (28)$$

$$f(x) = 2x^3(3x^4 + x), P = (-2, -736) \quad (29)$$

$$f(x) = (3x+2)(2x-5), P = \left(\frac{1}{2}, -14\right) \quad (30)$$

$$f(x) = \sqrt{x} \ln x^3, P = (4, 12 \ln 2) \quad (31)$$

# End of Lesson

Next Lesson: Chain Rule