Optimization and Analyzing Functions

MATH 1441, BCIT

Technical Mathematics for Food Technology

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Relative Extrema

A function f has a relative maximum at $x = c$ if there exists an open interval (a, b) containing c such that $f(x) \leq f(c)$ for all x in (a, b).

A function f has a relative minimum at $x = c$ if there exists an open interval (a, b) containing c such that $f(x) \geq f(c)$ for all x in (a, b).

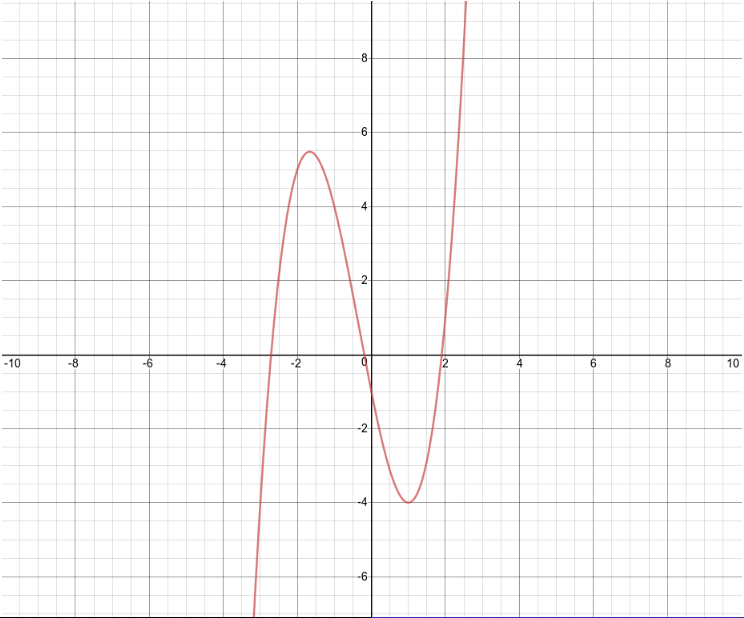
Derivatives and Extrema

At any number c where a differentiable function f has a relative extremum, $f'(c) = 0$. The converse is not true. Consider the following two functions and their derivatives.

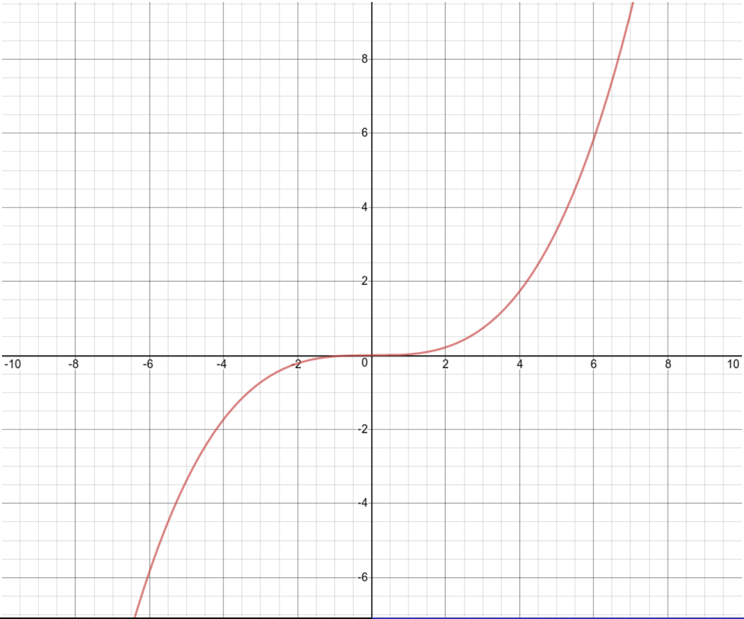
$$f_1(x) = x^3 + x^2 - 5x - 1 \tag{1}$$

$$f_2(x) = \left(\frac{3}{10}x\right)^3 \tag{2}$$

Derivatives and Extrema Graph I



Derivatives and Extrema Graph II



Derivatives and Extrema Caution

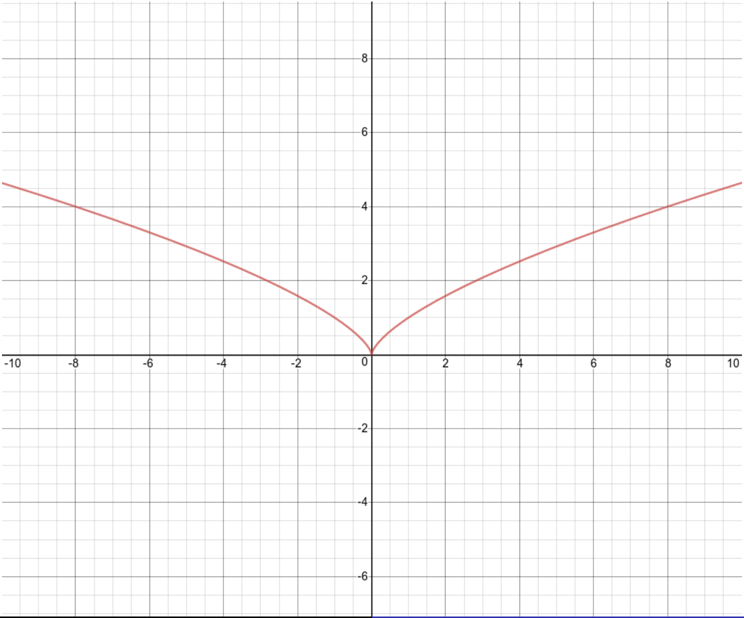
Note that a function may have an extremum at a point where the derivative is not 0 if at that point the function is not differentiable. Consider this function and its derivative.

$$f_3(x) = x^{\frac{2}{3}} \tag{3}$$

Critical Number

A critical number of a function f is any number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist.

Derivatives and Extrema Graph III



Extrema Exercises

Find the relative maxima and relative minima, if any, of each function.

$$f(x) = x^3 - 4x \tag{4}$$

$$h(t) = -t^2 + 6t + 6 \tag{5}$$

$$f(x) = \frac{1}{2}x^4 - x^2 \tag{6}$$

$$g(x) = \frac{x+1}{x} \tag{7}$$

$$f(x) = x\sqrt{x-4} \tag{8}$$

$$h(s) = s^{\frac{5}{3}} \tag{9}$$

Analyzing Functions

To analyze a function, determine the following features:

- Domain and range of the function.
- Zeros (also called x-intercepts) of the function.
- Critical points, maxima, minima.
- Inflection points.
- Asymptotes.
- Is the function even $(f_1(x) = x^2 + 1)$ or odd $(f_2(x) = x^3 - x)$?

Analyzing Functions Step-By-Step I

Here is a step-by-step guide to analyzing functions.

1. Determine the x-intercepts (also called zeros). Set $f(x) = 0$ and find the solution set.
2. Determine the critical points. Find the derivative $f'(x)$ and check whether there are points in the domain of f that are not in the domain of f'. Then set $f'(x) = 0$ and find the solution set.
3. Determine whether the critical points are maxima or minima or neither. Find $f''(x)$ and check whether f'' at the critical points is positive, negative, or neither.

Analyzing Functions Step-By-Step II

Here is a step-by-step guide to analyzing functions.

4. Determine the inflection points. Set $f''(x) = 0$ and find the solution set.
5. Determine the asymptotes. See next slide.
6. Determine whether, for all x in the domain of f,
$f(x) - f(-x) = 0$ (in which case f is even) or
$f(x) + f(-x) = 0$ (in which case f is odd).
7. Using the information you have, and possibly a table of function values, graph the function. Then determine the domain and range of f.

Finding Asymptotes I

An asymptote is a linear function ($y = kx + d$ with slope k and y-intercept d) which the function graph of f approaches. There are three kinds of asymptotes.

Vertical Asymptote

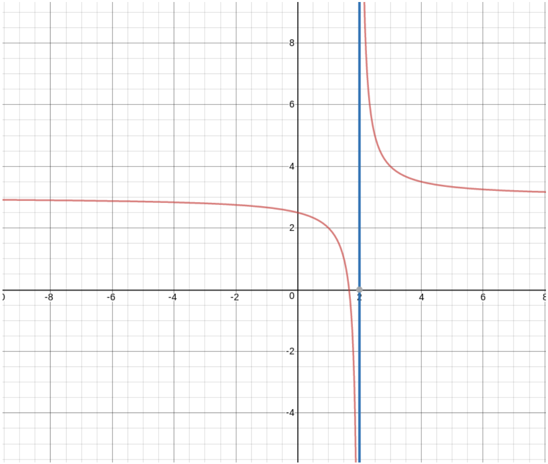
A vertical asymptote, strictly speaking, is not a linear function. It is a curve defined by $x = c$, where c is a real number (we call real numbers like c constants). You can often find vertical asymptotes at points where f is undefined.

Find vertical asymptotes by checking points which are not in the domain of the function f.

$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2 \tag{10}$$

Example:

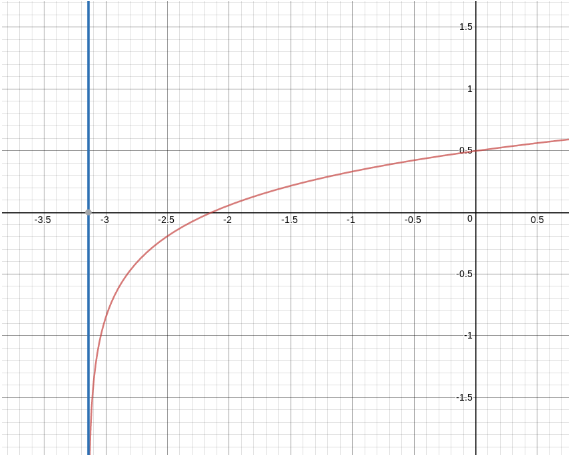
$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2 \tag{11}$$



Finding Asymptotes I

Example:

$$f(x) = \ln(x + \pi) \text{ has an asymptote at } x = -\pi \qquad (12)$$



Finding Asymptotes II

Horizontal Asymptote

A horizontal asymptote is a linear function with slope $k = 0$. Its equation is $y = c$, where c is a constant. There are horizontal asymptotes for functions whose limits is a constant and for rational functions whose numerator and denominator polynomials share the same degree.

Find horizontal asymptotes by checking

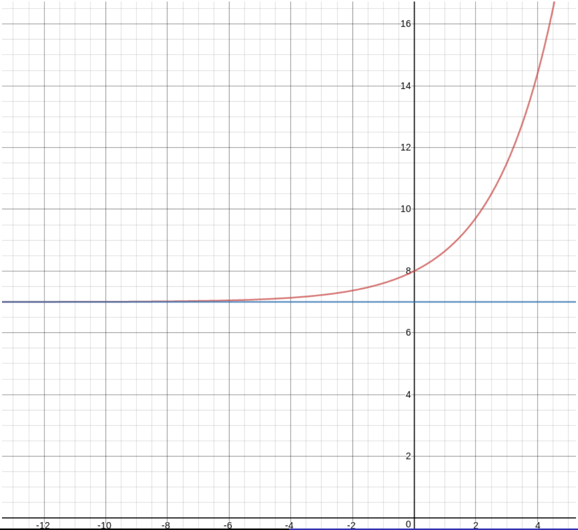
$$\lim_{x \to \infty} f'(x) \text{ and } \lim_{x \to -\infty} f'(x) \tag{13}$$

If the limit is $k = 0$, then that is also the slope of the asymptote.

Finding Asymptotes II

Example

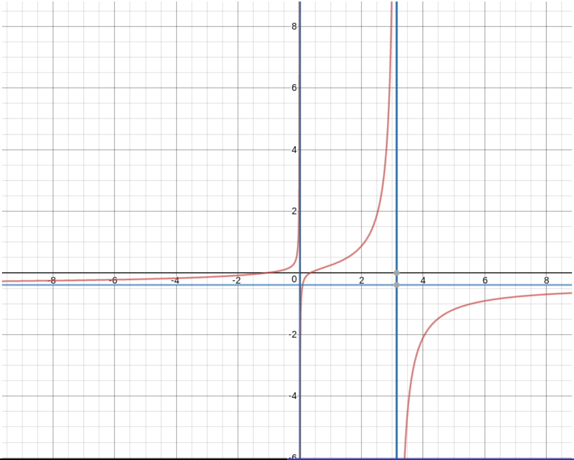
$$f(x) = e^{\frac{x}{2}} + 7 \text{ has the asymptote } y = 7 \qquad (14)$$



Finding Asymptotes II

Example (this example additionally has two vertical asymptotes):

$$f(x) = \frac{\pi x^2 + 2x - 1}{-7x^2 + 3x} \text{ has asymptotes } y = -\frac{e}{7}, x = \frac{22}{7}, x = 0 \tag{15}$$



Finding Asymptotes III

Sloped Asymptote

A sloped asymptote is a linear function with a positive or a negative slope, $y = kx + d$ with $k \neq 0$. There are sloped asymptotes for rational functions where the numerator polynomial's degree exceeds the denominator polynomial's degree by 1.

Find sloped asymptotes by checking

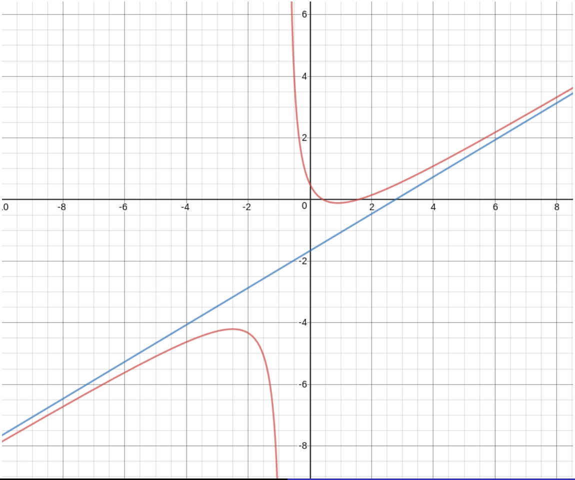
$$\lim_{x \to \infty} f'(x) \text{ and } \lim_{x \to -\infty} f'(x) \tag{16}$$

If the limit is $k \neq 0$, then that is also the slope of the asymptote. Hyperbolas also sometimes have sloped asymptotes.

Finding Asymptotes III

Example:

$$f(x) = \frac{3x^2 - 6x + 2}{5x + 4} \text{ has the asymptote } y = \frac{3}{5}x - \frac{5}{3} \qquad (17)$$



Analyze the following functions:

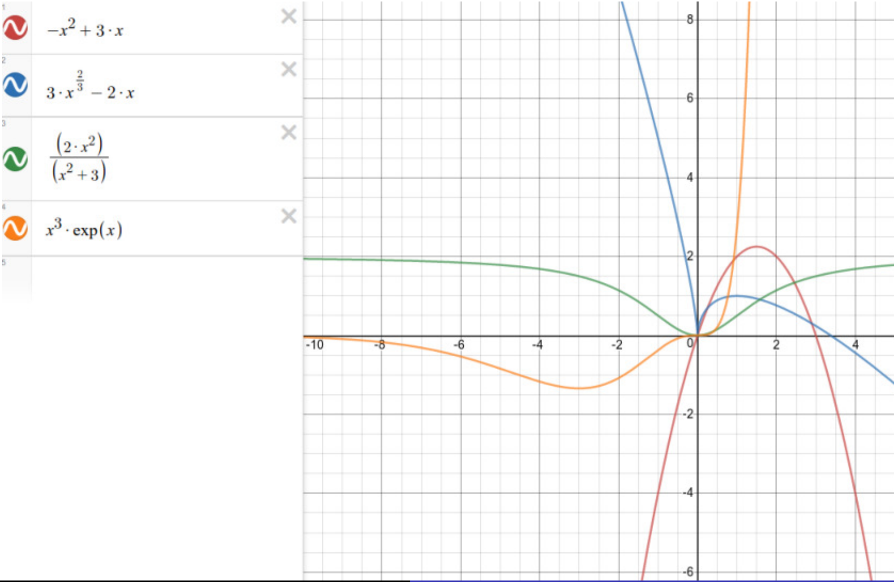
$$g_1(x) = -x^2 + 3x \tag{18}$$

$$g_2(x) = 3x^{\frac{2}{3}} - 2x \tag{19}$$

$$g_3(x) = \frac{2t^2}{t^2 + 3} \tag{20}$$

$$g_4(x) = x^3 e^x \tag{21}$$

Analyzing Functions Exercises Graph



End of Lesson

Next Lesson: Analyzing Functions