Growth and Decay

MATH 1441, BCIT

Technical Mathematics for Food Technology

October 10, 2017

Radiocarbon Dating

Radiocarbon dating is a method archeologists use to determine the age of ancient objects. The carbon dioxide in the atmosphere always contains a fixed fraction of radioactive carbon, carbon-14 (^{14}C), with a half-life of about 5730 years. Plants absorb carbon dioxide from the atmosphere, which then makes its way to animals through the food chain. Thus, all living creatures contain the same fixed proportions of ^{14}C to nonradioactive ^{12}C as the atmosphere.

After an organism dies, it stops assimilating ^{14}C, and the amount of ^{14}C in it begins to decay exponentially. We can then determine the time elapsed since the death of the organism by measuring the amount of ^{14}C left in it.

Radiocarbon Dating Example

If a donkey bone contains 73% as much ^{14}C as a living donkey, when did it die?

Look at the following table to notice the pattern,

$$\begin{array}{rcccl} 100\% & \ldots & 2^{-0} & \ldots & 5730 \cdot 0 \\ 50\% & \ldots & 2^{-1} & \ldots & 5730 \cdot 1 \\ 25\% & \ldots & 2^{-2} & \ldots & 5730 \cdot 2 \end{array} \tag{1}$$

Therefore (t being the number of years ago that the donkey died),

$$t = (-\log_2 0.73) \cdot 5730 = -\frac{\ln 0.73}{\ln 2} \cdot 5730 \approx 2600 \tag{2}$$

Compound Interest 1

Now use a similar strategy calculating compound interest.
Consider a $\$5,000$ loan at 6% per annum (p.a.).

$$\begin{array}{lclcl}
\text{after 0 year(s)} & \ldots & \$5,000 \cdot 1.06^0 & \ldots & \$5,000.00 \\
\text{after 1 year(s)} & \ldots & \$5,000 \cdot 1.06^1 & \ldots & \$5,300.00 \\
\text{after 2 year(s)} & \ldots & \$5,000 \cdot 1.06^2 & \ldots & \$5,618.00 \\
\text{after 3 year(s)} & \ldots & \$5,000 \cdot 1.06^3 & \ldots & \$5,955.08
\end{array} \tag{3}$$

Compound Interest 2

Derive the compound interest formula,

$$A = P\left(1 + \frac{r}{m}\right)^{mt} \tag{4}$$

A	...	accumulated amount at the end of t years
P	...	principal
r	...	interest rate p.a.
m	...	number of conversion periods per year
t	...	term (number of years)

Compound Interest 3

Exercise 1: Find the accumulated amount after 3 years if $1,000 is invested at 8% per year compounded (a) annually, (b) semi-annually, (c) quarterly, (d) monthly, and (e) daily.

Look at the pattern and notice how it leads to the continuous compound interest formula,

$$A = Pe^{rt} \tag{5}$$

Compound Interest Exercises

Exercise 2: Find the amount that results from each investment; that is, find the future value. All percentages are per annum.

1. \$100 invested at 4% compounded quarterly after a period of 2 years.
2. \$500 invested at 8% compounded quarterly after a period of 2.5 years.
3. \$3,000 invested at 5% compounded annually after a period of 20 years.
4. \$1,000 invested at 10% compounded continuously after a period of $2\frac{1}{4}$ years.

Compound Interest Exercises

Exercise 3: Find the principal needed now to get each amount; that is, find the present value. All percentages are per annum.

1. To get \$300 after four years at 3% compounded daily.
2. To get \$75,000 after three years at 8% compounded quarterly.
3. To get \$400 after one year at 10% compounded continuously.
4. To get \$1,000,000 after two years at 6% compounded semi-annually.

Compound Interest Exercises

Exercise 4: Answer the following questions.

1. What rate of interest compounded annually is required to double an investment in three years?
2. What rate of inflation doubles prices every 14 years? (Inflation is like interest compounded annually.)
3. John will require $3,000 in 6 months to pay off a loan that has no prepayment privileges. If he has the $3,000 now, how much of it should he save in an account paying 3% compounded monthly so that in six months he will have exactly $3,000?
4. A business purchased for $650,000 in 1994 is sold in 1997 for $850,000. What is the annual rate of return for this investment?

Uninhibited Growth and Decay Formula

Many natural phenomena have been found to follow the law that an amount A varies with time t according to

$$A(t) = A_0 e^{kt} \tag{6}$$

where $A_0 = A(0)$ is the original amount at $t = 0$ and $k \neq 0$ is a constant. If $k > 0$, there is growth. If $k < 0$, there is decay.

Uninhibited Growth and Decay Exercise

Exercise 5: A colony of bacteria grows according to the law of uninhibited growth according to the function

$$N(t) = 100e^{0.045t} \tag{7}$$

where N is measured in grams and t is measured in days.

1. Determine the initial amount of bacteria.
2. What is the growth rate of the bacteria?
3. What is the population after five days?
4. How long will it take for the population to reach 140 grams?
5. What is the doubling time for the population?

Newton's Law of Cooling

The temperature u of a heated object at a given time t can be modeled by the following function,

$$u(t) = T + (u_0 - T)e^{kt} \tag{8}$$

where k is a negative constant, T is the constant temperature of the surrounding medium, and u_0 is the initial temperature of the heated object.

Newton's Law of Cooling Exercise

Exercise 6: An object is heated to $100^\circ C$ (degrees Celsius) and is then allowed to cool in a room whose air temperature is $30^\circ C$.

1. If the temperature of the object is $80^\circ C$ after five minutes, when will its temperature be $50^\circ C$?
2. Determine the elapsed time before the temperature of the object is $35^\circ C$.
3. What do you notice about $u(t)$, the temperature, as t, time, passes?

Exercise 7: A frozen steak has a temperature of $28^\circ F$. It is placed in a room with a constant temperature of $70^\circ F$. After 10 minutes, the temperature of the steak has risen to $35^\circ F$.

1. What will the temperature of the steak be after 30 minutes?
2. How long will it take the steak to thaw to a temperature of $45^\circ F$?

Newton's Law of Cooling Exercise

Exercise 8: A frozen steak has a temperature of $28^\circ F$. It is placed in a room with a constant temperature of $70^\circ F$. After 10 minutes, the temperature of the steak has risen to $35^\circ F$.

1. What will the temperature of the steak be after 30 minutes?
2. How long will it take the steak to thaw to a temperature of $45^\circ F$?

Newton's Law of Cooling Exercise

Exercise 9: The hotel Bora-Bora is having a pig roast. At noon, the chef put the pig in a large earthen oven. The pig's original temperature was $75^\circ F$. At 2:00PM the chef checked the pig's temperature and was upset because it had reached only $100^\circ F$.

1. If the oven's temperature remains a constant $325^\circ F$, at what time may the hotel serve its guests, assuming that pork is done when it reaches $175^\circ F$?

Exercise 10: A small lake is stocked with a certain species of fish. The fish population is modeled by the function

$$P = \frac{10}{1 + 4e^{-0.8t}} \tag{9}$$

where P is the number of fish in thousands and t is measured in years since the lake was stocked.

1. Find the fish population after 3 years.
2. After how many years will the fish population reach 5000 fish?

Exercise 11: A culture starts with 8600 bacteria. After one hour the count is $10,000$.

1. Find a function that models the number of bacteria $n(t)$ after t hours.
2. Find the number of bacteria after 2 hours.
3. After how many hours will the number of bacteria double?

Exercise 12: Atmospheric pressure decreases exponentially as you go higher above sea level. It decreases about 12% for every 1000 metres. The pressure at sea level is 1013 hectopascal (hPa). The formula is

$$y(t) = y(0) \cdot e^{ks} \tag{10}$$

where s is the distance above sea level in metres and k is a positive constant. What is your prediction for the air pressure on Mount Everest (8848 metres above sea level)?

Exercise 13: The number of people living in a country is increasing each year exponentially. The number of people 5 years ago was 4 million. The number of people in five years is projected to be 6.25 million. What is the present population of the country?

Exercises

Exercise 14: "Loudness" is measured in decibels. The formula for the loudness of a sound is given by

$$L = 10 \log \frac{I}{I_0} \tag{11}$$

where I_0 is the intensity of "threshold sound," or sound that can barely be perceived. The logarithm here is to base 10, i.e. the common logarithm. Other sounds are defined in terms of how many times more intense they are than threshold sound. For instance, a cat's purr is about 316 times as intense as threshold sound, for a decibel rating of:

$$10 \log \frac{I}{I_0} = 10 \log \frac{316 \cdot I_0}{I_0} = 10 \log 316 \approx 25 \text{ decibels} \tag{12}$$

Exercises

1. An airplane jet take-off heard from 100 metres away measures 100 decibels. How much more intense than a normal conversation (50 decibels) is the sound of an airplane jet take-off? (Do not use exponents in your answer.)
2. Prolonged exposure to sounds above 85 decibels can cause hearing damage or loss. What is the approximate loudness of a gunshot from a .22 rimfire rifle with an intensity of about $I = (2.5 \cdot 10^{13})I_0$? Should you follow the rules and wear ear protection when visiting the rifle range?

End of Lesson

Next Lesson: Basic Rules of Differentiation