Set Theory, Functions, Numbers

MATH 1441, BCIT

Technical Mathematics for Food Technology

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Sets

A set is a rule: for each object, the rule tells you unambiguously whether the object belongs to the set or not. We say

$$x \in A \tag{1}$$

when an object x belongs to a set A. When it does not belong to set A we say

$$x \notin A \tag{2}$$

Statements like (1) or (2) are called propositions. They are either true or false. Logic is "under the hood" for the vehicle that is set theory. Set theory is "under the hood" for the vehicle that is mathematics.

Sets

There are many different ways to write out the rules whether an object belongs to a set or not. One is to just list the objects in curly braces, such as

$$S = \{\clubsuit, \spadesuit, \diamondsuit, \heartsuit\} \tag{3}$$

Rule (3) tells us that no object belongs to set B except these four: $\clubsuit, \spadesuit, \diamondsuit, \heartsuit$. They are the elements of S. Order does not matter. There is one very special set: the empty set $\emptyset$. The rule for the empty set is that no object belongs to it. It is sometimes written $\{\}$.

Set Operators

Sets can relate to each other in certain ways. For example,

$$\{\clubsuit, \diamondsuit\} \subset \{\clubsuit, \spadesuit, \diamondsuit, \heartsuit\} \tag{4}$$

$A \subset B$ (A is a subset of B) if and only if all elements of A are also elements of B. Two sets are equal if and only if $A \subset B$ and $B \subset A$. We say $A = B$. If they are not equal, we say $A \neq B$. $B \supset A$ means the same as $A \subset B$, but we usually say that B is a superset of A.

Set Operators

If $A \subset X$, then the complement of A with respect to X is the set of objects that belong to X but not to A. We say A^C or $\bar{A}$, and often X as the universal set is not mentioned explicitly.
For two sets A and B, set subtraction gives us $A - B$ or $A \setminus B$, which contains all elements in A which are not in B. If $A \subset X$ then

$$X \setminus A = A^C \tag{5}$$

Unions and Intersections

The union of two sets A and B is the group of objects x for which it is true that $x \in A$ or $x \in B$. We say

$$A \cup B \tag{6}$$

The intersection of two sets A and B is the group of objects x for which it is true that $x \in A$ and $x \in B$. We say

$$A \cap B \tag{7}$$

Two sets A and B are disjoint if they have no elements in common, i.e. $A \cap B = \emptyset$.

Russell's Paradox

Sets can be elements of sets. The following set has three elements.

$$A = \{\{\clubsuit, \heartsuit\}, \{\}, \{\heartsuit, \spadesuit, \diamondsuit\}\} \tag{8}$$

How many elements does this set have?

$$B = \{\{\{\{\{\{\{\}\}\}\}\}\}\} \tag{9}$$

The father of set theory, Gottlob Frege, thought that all sets make sense as long as they are well-defined. Another famous mathematician, Bertrand Russell, figured out that that was not quite right. Think of the set R that contains all sets which do not contain themselves (some sets do contain themselves, for example the set of all non-lemons). Does R contain itself?

Natural Numbers

On the previous slide, I said that a set had three elements. The point of set theory is to give a foundation for numbers, not the other way around. So I jumped the gun. We don't know yet what "three" means. To avoid Russell's paradox, let's restrict ourselves to sets that do not contain themselves. Then we define the successor A^+ of a set A to be the following

$$A^+ = A \cup \{A\} \tag{10}$$

Note that A does not contain itself and A^+ does not contain itself.

Natural Numbers

Now we give names to the following sets:

$\emptyset$	"zero"	0
$\emptyset^+$	"one"	1
$\left(\emptyset^+\right)^+$	"two"	2
$\left(\left(\emptyset^+\right)^+\right)^+$	"three"	3

and so on. The set of all these sets is $\mathbb{N}$, the natural numbers.
Exercise 1: Write the set 3 using curly braces.

Natural Numbers

Another way to write a rule for a set is to use pattern matching

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\} \tag{11}$$

For example, the set of odd numbers is

$$\mathbb{N}_{\text{odd}} = \{1, 3, 5, 7, \ldots\} \tag{12}$$

Ordered Pairs

An ordered pair is a set which determines an order in which the components of an ordered pair can be read. Let $G = \{a, b\}$. Then the collection of ordered pairs in G (we call this collection the set $G \times G$)

$$G \times G = \{(a, a), (a, b), (b, a), (b, b)\} \tag{13}$$

(a, b) and the other elements of this set are ordered pairs. They are defined as follows:

$$(a, b) = \{\{a\}, \{a, b\}\} \tag{14}$$

Notice that even though $\{a, b\} = \{b, a\}$ (order does not matter for sets), $(a, b) \neq (b, a)$ (order matters for ordered pairs). Coordinates are an example of ordered pairs (or ordered triplets, and so on).

Functions

The set product is best explained by example. Let $A = \{a_1, a_2, a_3\}$ and $W = \{w_1, w_2\}$. Then

$$A \times W = \{(a_1, w_1), (a_1, w_2), (a_2, w_1), (a_2, w_2), (a_3, w_1), (a_3, w_2)\}$$

The elements of $A \times W$ are ordered pairs. A function $f : A \to W$ is a subset of $A \times W$ with the following rule (we will write $f(a) = w$ instead of $(a, w) \in f$ to make it more readable):

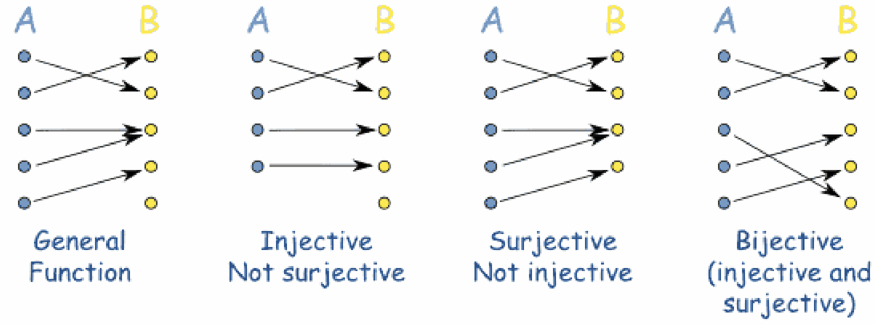
> *For each element in $a \in A$, there must be exactly one ordered pair in f such that $f(a) = w$ ($w \in W$).*

Functions

If $f : A \to W$ is a function, then A is the domain and W is the codomain. Those w in W for which $f(a) = w$ form a subset of W called the range of the function.

If $f(a) = w$ and $f(b) = w$ implies that $a = b$ ($a = b$ means that there is one and the same object with two labels a and b), then f is called one-to-one or injective. If the codomain is a subset of the range, then f is called onto or surjective. If a function is both injective and surjective, then it is called bijective.

Injective and Surjective Functions



Cardinality

If there is a bijective function $f : A \to B$ for two sets A and B, then A and B share the same cardinality $n(A) = n(B)$. If there is a bijective function from any set B to a set in $\mathbb{N}$, then that element of $\mathbb{N}$ is the cardinality of B. In other words, since there is a bijective function from 3 to

$$B = \{\clubsuit, \heartsuit, \diamondsuit\} \tag{15}$$

the cardinality of B is three. Not all cardinalities are natural numbers, for example $n(\mathbb{N})$.

Integers and Fractions

We use more set theory to define the integers:

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\} \tag{16}$$

Then we use a similar strategy to define the rational numbers:

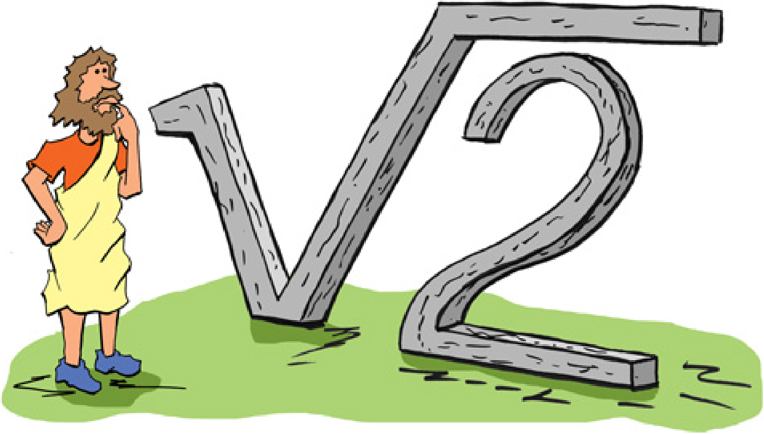
$$\mathbb{Q} = \left\{\dots, -\frac{14}{17}, \dots, -\frac{1}{2}, \dots, 0, \dots, \frac{1}{7}, \dots, 4, \dots, \frac{19}{4}, \dots\right\}$$

For the definition of $\mathbb{Z}$ and $\mathbb{Q}$ we need a concept of set theory called equivalence relation. For the definition of operators such as $+, -, \cdot, \div$ and exponents we need a concept of set theory called induction. We will not cover these two concepts in this brief introduction and rely on our intuition instead.

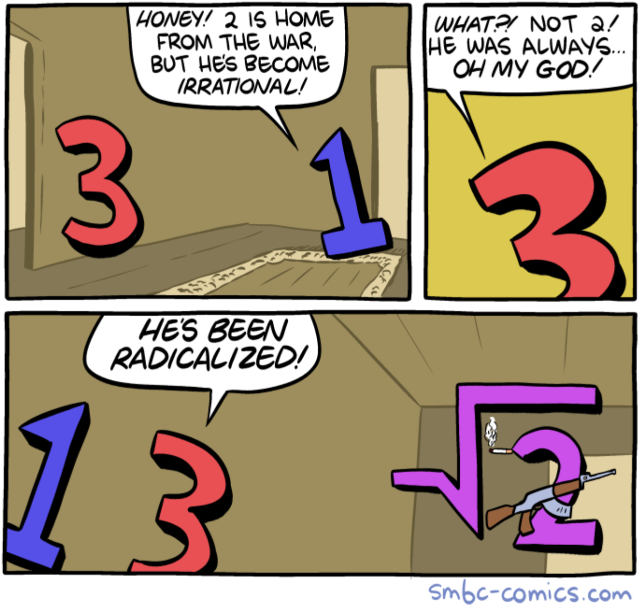
Hippasus of Metapontum

It turns out that $\mathbb{N}$,$\mathbb{Z}$, and $\mathbb{Q}$ have the same cardinality. A Greek mathematician called Hippasus figured out that the square root of 2 is not an element of any of these sets. You can duckduckgo the proof for this fact. It is not difficult to understand; it uses prime numbers.

Hippasus of Metapontum



Terrible Twos



The Real Numbers

Some more complicated math is necessary to construct the real number line, including such irrational numbers as $\sqrt{2}, \pi$, and e. We call the set of real numbers $\mathbb{R}$. Make sure to remember that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \tag{17}$$

However, the cardinality of $\mathbb{R}$ is not the same as the cardinality of $\mathbb{Q}$. Again, you can duckduckgo the proof, which is not difficult to understand; it uses Cantor's diagonal argument. It is a very difficult question (unresolved as far as I know) whether there is a set that has more elements than $\mathbb{N}$ and fewer elements than $\mathbb{R}$.

Power Sets and Interval Notation

The power set of a set A is the set which contains all the subsets of A (it always contains A and $\emptyset$). It is sometimes called $\mathfrak{p}(A)$.
The cardinality of $\mathfrak{p}(\mathbb{N})$ is the same as the cardinality of $\mathbb{R}$.
Another way to write a rule for a set is as follows:

$$[-7, 2) = \{x \in \mathbb{R} | -7 \leq x \text{ and } x < 2\} \tag{18}$$

The infinity sign ∞ is used when there is no boundary on the left or the right of the number line:

$$\left(-\infty, \frac{1}{2}\right] = \left\{x \in \mathbb{R} | x \leq \frac{1}{2}\right\} \tag{19}$$

Equations

Equations are rules for sets. The equation $2x = 6$ is a rule for the set

$$S = \{x \in \mathbb{R} | 2x = 6\} \tag{20}$$

The two rules $2x = 6$ and $5 - x = 2$ yield the same set. We say that these two equations are equivalent. Note that the solution set is assumed to be a subset of $\mathbb{R}$.

$$S_{\mathbb{R}} = \{x \in \mathbb{R} | x^2 = -1\} = \{\} \tag{21}$$

$$S_{\mathbb{C}} = \{x \in \mathbb{C} | x^2 = -1\} = \{i, -i\} \tag{22}$$

Manipulating Equations

Solving an equation often means listing the elements of its solution set in curly braces, for example

$$S = \{x \in \mathbb{R} | 2x = 6\} = \{3\} \tag{23}$$

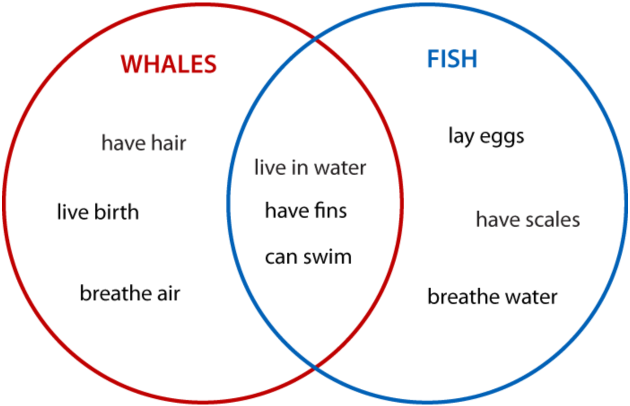
If we apply a bijective function to both sides of the equation, the original and the resulting equation are equivalent. Let F be a bijective function $F : \mathbb{R} \to \mathbb{R}$. We usually use linear functions F for this purpose. Then

$$\{x \in \mathbb{R} | 2x = 6\} = \{x \in \mathbb{R} | F(2x) = F(6)\} \tag{24}$$

Exercise 2: Use the function $F(X) = \frac{1}{2} \cdot X$ to find the equivalent equation to $2x = 6$.

Venn Diagrams

Sometimes we use Venn diagrams to define or clarify the composition of various sets.



End of Lesson

Next Lesson: Linear Equations