Exponential Functions

MATH 1441, BCIT

Technical Mathematics for Food Technology

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Functions

Here are a few definitions,

function
: A function assigns a unique element of a set to each element of another (not necessarily distinct) set.

domain
: The domain is the set of elements to which the function assigns a unique element.

codomain
: The codomain is the set from which the function picks out elements to assign.

range
: The range is the subset of the codomain whose elements the function assigns to an element in the domain.

injective
: A function is injective if it does not assign the same element of the codomain to two distinct elements in the domain.

surjective
: A function is surjective if there are no elements in the codomain which are not assigned to an element in the domain.

Examples

What are possible domains and ranges for the following functions? Are the functions injective or surjective, given a particular domain and codomain?

$$f(x) = 2x + 3 \tag{1}$$

$$f(x) = x^2 - 1 \tag{2}$$

$$f(x) = \sqrt{x + 4} \tag{3}$$

$$f(x) = \frac{1}{x + 7} \tag{4}$$

$$f(x) = 10^{2x} \tag{5}$$

Inverse Functions

If a function f from a domain to a codomain is injective, then there is a function f^{-1} from the range of f to its domain which has the following property,

$$f^{-1}(y) = x \text{ if and only if } f(x) = y \tag{6}$$

We call f^{-1} the inverse function of f. Let, for example,

$$f(x) = 4x - 3 \tag{7}$$

Replace $f(x)$ by y for the equation $y = 4x - 3$ and manipulate the equation to isolate x. Then replace x by $f^{-1}(y)$ for the inverse function

$$f^{-1}(y) = \frac{y+3}{4} \tag{8}$$

Defining Logarithms

Let f be an exponential function with a base $a > 1$,

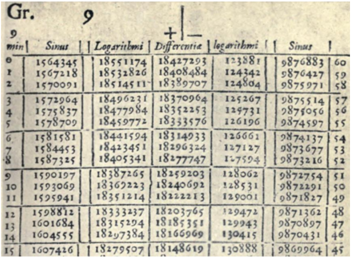
$$f(x) = a^x \tag{9}$$

Considering the function graph of this exponential function, it is apparent that f is an injective and surjective function for the domain $\mathbb{R}$ and the codomain $\mathbb{R}^+$. $\mathbb{R}^+$ is the set of all positive real numbers. There is therefore an inverse function from $\mathbb{R}^+$ to the real numbers, which we shall call $\log_a$,

$$\log_a(y) = x \text{ if and only if } a^x = y \tag{10}$$

Logarithm Charts

Here is an excerpt from John Napier's logarithm chart.



Gr. 9

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min	Sinus	Logarithmi	+ | − Differentiæ	logarithmi	Sinus	
0	1564345	18551174	18427293	123881	9876883	60
1	1567218	18532826	18408484	124342	9876427	59
2	1570091	18514511	18389707	124804	9875971	58
3	1572964	18496231	18370964	125267	9875514	57
4	1575837	18477984	18352253	125731	9875056	56
5	1578709	18459772	18333576	126196	9874597	55
6	1581581	18441594	18314933	126661	9874137	54
7	1584453	18423451	18296324	127127	9873677	53
8	1587325	18405341	18277747	127594	9873216	52
9	1590197	18387265	18259203	128062	9872754	51
10	1593069	18369223	18240692	128531	9872291	50
11	1595941	18351214	18222213	129001	9871827	49
12	1598812	18333237	18203765	129472	9871362	48
13	1601684	18315294	18185351	129943	9870897	47
14	1604555	18297384	18166969	130415	9870431	46
15	1607426	18279507	18148619	130888	9869964	45

Logarithm Charts I

A logarithm chart has a left-hand column for y and a right-hand column for x such that $a^x = y$. Here is an example, taking $a = 10$.

1	0.00000	$\vdots$	$\vdots$
2	0.30103	432	2.6355
3	0.47712	$\vdots$	$\vdots$
4	0.60206	703	2.8470
5	0.69897	$\vdots$	$\vdots$
6	0.77815	303696	5.4825
7	0.84510	$\vdots$	$\vdots$
8	0.90309		
9	0.95424		
10	1.00000		

Logarithm Charts II

Imagine you have no calculator and you need to multiply $432 \cdot 703$. It would take a while to do by hand! Alternatively, you could use a logarithm chart, look up the logarithms for 432 and 703, add them (as opposed to multiplying!), and then look up which number corresponds to the resulting logarithm.

$$432 \cdot 703 = 10^{2.6355} \cdot 10^{2.8470} =$$

$$10^{2.6355+2.8470} = 10^{5.4825} = 303696 \qquad (11)$$

Properties of the Logarithm

- The domain of $\log_a$ is all the positive real numbers.
- The range of $\log_a$ is the whole number line $\mathbb{R}$.
- The logarithm of $y = 1$ is always 0. *Reason:* $a^0 = 1$.
- The logarithm of the base $y = a$ is always 1. *Reason:* $a^1 = a$.
- The logarithm of $y = a^x$ is always x. *Reason:* $a^x = a^x$.
- a to the power of $\log_a y$ is y. *Reason:* that's precisely the definition of the logarithmic function.

Here are some laws, all of which make sense in terms of what we have learned so far.

$$\log_a(AB) = \log_a A + \log_a B \tag{12}$$

$$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B \tag{13}$$

$$\log_a(A^C) = C \cdot \log_a A \tag{14}$$

A and B must be positive real numbers for these laws to be generally valid. For example, $\log((-2)\cdot(-2)) \neq \log(-2) + \log(-2)$.

What Not to Do

Make sure that you **do not** apply the laws of logarithms the wrong way around!

$$\log_a(x+y) \neq \log_a x + \log_a y \tag{15}$$

$$\frac{\log_a x}{\log_a y} \neq \log_a x - \log_a y \tag{16}$$

$$(\log_a x)^3 \neq 3\log_a x \tag{17}$$

Notation: sometimes we write $\ln x$ instead of $\log_e(x)$. $\log_e$ is also called the natural logarithm.

Change of Base Law

Suppose you know what $\log_a x$ is but you want to know what $\log_b x$ is. Consider the following equivalent equations:

$$y = \log_b x \tag{18}$$

$$b^y = x \tag{19}$$

$$\log_a(b^y) = \log_a x \tag{20}$$

$$y \log_a b = \log_a x \tag{21}$$

$$y = \frac{\log_a x}{\log_a b} \tag{22}$$

This proves the Change of Base Formula:

$$\log_b x = \frac{\log_a x}{\log_a b} \tag{23}$$

Exercises I

(1) Express

$$3\ln x + \frac{1}{2}\ln(x+1) \tag{24}$$

as a single logarithm.
(2) Analyze the expression

$$\log_{12}\left(\frac{x^3}{y^{\frac{1}{2}}}\right) \tag{25}$$

(3) What is

$$\ln\,\ln e^{e^{\ln e^x}} \tag{26}$$

(4) Show that

$$-\ln\left(x-\sqrt{x^2-1}\right)=\ln\left(x+\sqrt{x^2-1}\right) \tag{27}$$

(5) Use the Change of Base Formula and the calculator to evaluate $\log_7 24$ and $\log_3 59049$.

End of Lesson

Next Lesson: Exponential Equations