Optimization and Analyzing Functions MATH 2511, BCIT

Technical Mathematics for Geomatics

February 5, 2017

Relative Extrema

A function f has a relative maximum at x = c if there exists an open interval (a, b) containing c such that $f(x) \le f(c)$ for all x in (a, b).

A function f has a relative minimum at x = c if there exists an open interval (a, b) containing c such that $f(x) \ge f(c)$ for all x in (a, b).

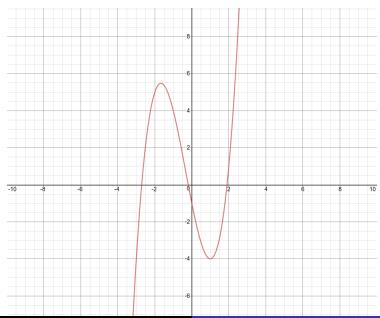
Derivatives and Extrema

At any number c where a differentiable function f has a relative extremum, f'(c) = 0. The converse is not true. Consider the following two functions and their derivatives.

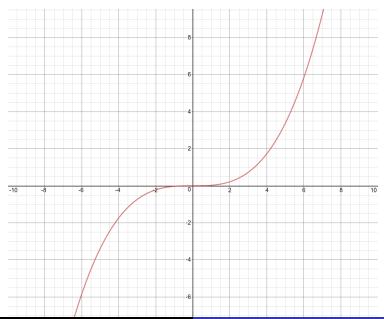
$$f_1(x) = x^3 + x^2 - 5x - 1 \tag{1}$$

$$f_2(x) = \left(\frac{3}{10}x\right)^3\tag{2}$$

Derivatives and Extrema Graph I



Derivatives and Extrema Graph II



Derivatives and Extrema Caution

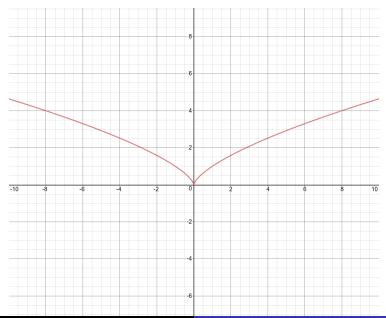
Note that a function may have an extremum at a point where the derivative is not 0 if at that point the function is not differentiable. Consider this function and its derivative.

$$f_3(x) = x^{\frac{2}{3}} \tag{3}$$

Critical Number

A critical number of a function f is any number x in the domain of f such that f'(x) = 0 or f'(x) does not exist.

Derivatives and Extrema Graph III



Extrema Exercises

Find the relative maxima and relative minima, if any, of each function.

$$f(x) = x^3 - 4x \tag{4}$$

$$h(t) = -t^2 + 6t + 6 (5)$$

$$f(x) = \frac{1}{2}x^4 - x^2 \tag{6}$$

$$g(x) = \frac{x+1}{x} \tag{7}$$

$$f(x) = x\sqrt{x-4} \tag{8}$$

$$h(s) = s^{\frac{5}{3}} \tag{9}$$

Analyzing Functions I

To analyze a function, determine the following features:

- Domain and range of the function.
- Zeros (also called *x*-intercepts) of the function.
- Critical points, maxima, minima.
- Inflection points.
- Asymptotes.
- Is the function even $(f_1(x) = x^2 + 1)$ or odd $(f_2(x) = x^3 x)$?

Analyzing Functions Step-By-Step I

Here is a step-by-step guide to analyzing functions.

- ① Determine the x-intercepts (also called zeros). Set f(x) = 0 and find the solution set.
- ② Determine the critical points. Find the derivative f'(x) and check whether there are points in the domain of f that are not in the domain of f'. Then set f'(x) = 0 and find the solution set.
- **3** Determine whether the critical points are maxima or minima or neither. Find f''(x) and check whether f'' at the critical points is positive, negative, or neither.

Analyzing Functions Step-By-Step II

Here is a step-by-step guide to analyzing functions.

- **3** Determine the inflection points. Set f''(x) = 0 and find the solution set.
- Determine the asymptotes. See next slide.
- Determine whether, for all x in the domain of f, f(x) f(-x) = 0 (in which case f is even) or f(x) + f(-x) = 0 (in which case f is odd).
- Using the information you have, and possibly a table of function values, graph the function. Then determine the domain and range of f.

Finding Asymptotes I

An asymptote is a linear function (y = kx + d with slope k and y-intercept d) which the function graph of f approaches. There are three kinds of asymptotes.

Vertical Asymptote

A vertical asymptote, strictly speaking, is not a linear function. It is a curve defined by x=c, where c is a real number (we call real numbers like c constants). You can often find vertical asymptotes at points where f is undefined.

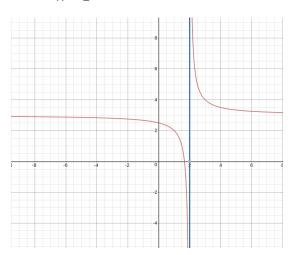
Find vertical asymptotes by checking points which are not in the domain of the function f.

$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2$$
 (10)

Finding Asymptotes I

Example:

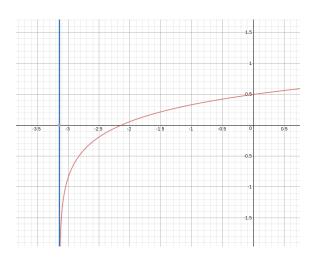
$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2$$
 (11)



Finding Asymptotes I

Example:

$$f(x) = \ln(x + \pi)$$
 has an asymptote at $x = -\pi$ (12)



Finding Asymptotes II

Horizontal Asymptote

A horizontal asymptote is a linear function with slope k=0. Its equation is y=c, where c is a constant. There are horizontal asymptotes for functions whose limits is a constant and for rational functions whose numerator and denominator polynomials share the same degree.

Find horizontal asymptotes by checking

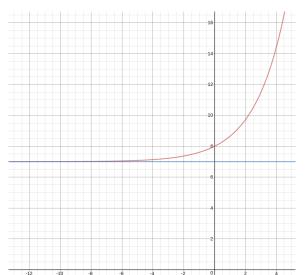
$$\lim_{x \to \infty} f'(x) \text{ and } \lim_{x \to -\infty} f'(x) \tag{13}$$

If the limit is k = 0, then that is also the slope of the asymptote.

Finding Asymptotes II

Example

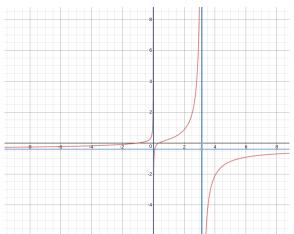
$$f(x) = e^{\frac{x}{2}} + 7 \text{ has the asymptote } y = 7$$
 (14)



Finding Asymptotes II

Example (this example additionally has two vertical asymptotes):

$$f(x) = \frac{\pi x^2 + 2x - 1}{-7x^2 + 3x} \text{ has asymptotes } y = -\frac{e}{7}, x = \frac{22}{7}, x = 0$$
 (15)



Finding Asymptotes III

Sloped Asymptote

A sloped asymptote is a linear function with a positive or a negative slope, y=kx+d with $k\neq 0$. There are sloped asymptotes for rational functions where the numerator polynomial's degree exceeds the denominator polynomial's degree by 1.

Find sloped asymptotes by checking

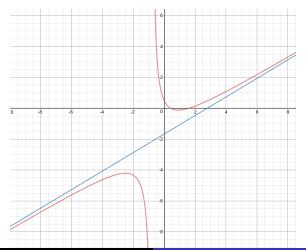
$$\lim_{x \to \infty} f'(x) \text{ and } \lim_{x \to -\infty} f'(x) \tag{16}$$

If the limit is $k \neq 0$, then that is also the slope of the asymptote. Hyperbolas also sometimes have sloped asymptotes.

Finding Asymptotes III

Example:

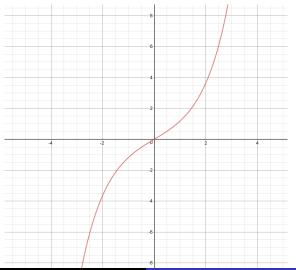
$$f(x) = \frac{3x^2 - 6x + 2}{5x + 4}$$
 has the asymptote $y = \frac{3}{5}x - \frac{5}{3}$ (17)



Finding Asymptotes III

Example:

$$f(\vartheta) = \sinh \vartheta \tag{18}$$



Analyzing Functions

Analyze the following functions:

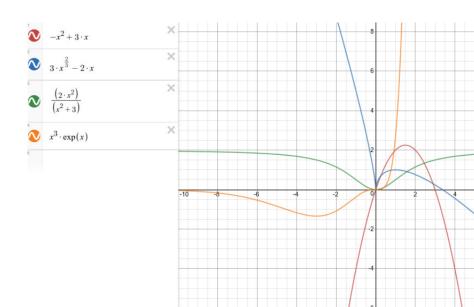
$$g_1(x) = -x^2 + 3x (19)$$

$$g_2(x) = 3x^{\frac{2}{3}} - 2x \tag{20}$$

$$g_3(t) = \frac{2t^2}{t^2 + 3} \tag{21}$$

$$g_4(x) = x^3 e^x \tag{22}$$

Analyzing Functions Exercises Graph



Analyzing Functions Further Exercises

Analyze the following functions.

$$f(x) = x \cdot \ln x^2 \tag{23}$$

$$f(x) = \frac{2x^2 + 2}{x - 3}$$
 (do not look for inflection points) (24)

$$f(x) = x^3 + 4x^2 + x - 6 (25)$$

For (25), note that x=1 is an x-intercept so that (x-1) can be factored as in

$$x^3 + 4x^2 + x - 6 = (x - 1)(x^2 + 5x + 6)$$
 (26)

More Exercises

Find local maxima and minima for the following function:

$$f(x) = 3x^3 - 12x + 5 (27)$$

Find the critical point(s) for the following function:

$$f(x) = \frac{x}{x^2 + 1} \tag{28}$$

Find the critical point(s) for the following function:

$$f(t) = t\sqrt{4 - t^2} \tag{29}$$

Find the critical point(s) for the following function:

$$g(t) = \sqrt[3]{t}(8-t) \tag{30}$$

Find local maxima and minima for the following function:

$$f(x) = \ln(x^2 + x + 1) \tag{31}$$



Next Lesson: Transcendental Functions and Differentials