

Limits

MATH 2511, BCIT

Technical Mathematics for Geomatics

January 8, 2018

Limits Introduction

Consider the function graph of the following function.

$$f(x) = \frac{x^2 - 1}{x - 1} \quad (1)$$

It looks like it is a linear equation! However, at $x = 1$, $f(x)$ is not defined. To fill the hole, we define the limit

$$\lim_{x \rightarrow a} f(x) = w \text{ if and only if } w = L = R \quad (2)$$

where L is the number that the function f approaches as x gets closer to a with $x < a$ (that means $x \neq a$!); and R is the number that the function f approaches as x gets closer to a with $x > a$.

Note: for a mathematically rigorous definition of what “approaching” and “getting closer” means we would need to talk about sequences and series, which is a topic we won’t cover here.

Indeterminate Form I

Notice that

$$f(x) = \frac{x^2 - 1}{x - 1} \stackrel{x=1}{=} \frac{0}{0} \quad (3)$$

We call this an **indeterminate form**.

Notice that except at $x = 1$

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{\cancel{(x - 1)}(x + 1)}{\cancel{x - 1}} = x + 1 = g(x) \quad (4)$$

f and g agree everywhere except on $x = 1$. Consider the following rule,

One Disagreement Rule

If $f = g$ except in one point, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ for all a , even the a where f and g disagree.

Therefore

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 2 \quad (5)$$

Consider a simple function like $f(x) = x^3$. What is $\lim_{x \rightarrow 4} f(x)$?
The answer is almost trivial,

$$\lim_{x \rightarrow 4} f(x) = f(4) = 4^3 = 64 \quad (6)$$

Why is this true? Because f is continuous at $x = 4$. There are no holes, jumps, gaps, or breaks of the function graph at $x = 4$.

Constant functions, the identity function, linear functions, polynomial functions, exponential and logarithmic functions are all continuous. Rational functions, some trigonometric functions, and other functions are sometimes **not** continuous.

Interesting Cases

A function is continuous if and only if $\lim_{x \rightarrow c} f(x) = f(c)$ for all c in \mathbb{R} (the logarithmic function is continuous only on \mathbb{R}^+). This means that (i) the function needs to be defined at $x = c$; (ii) the limit needs to be defined at $x = c$; and (iii) the function value and the limit need to be equal to each other.

Consider the following interesting cases:

- 1 A function that is continuous and well defined at $x = a$.
- 2 A function that is not continuous at $x = a$.
- 3 A function where the limit exists but $\lim_{x \rightarrow c} f(x) \neq f(c)$.
- 4 A function such as $f(x) = \sin(1/x)$.

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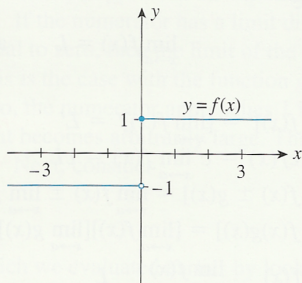
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No Limit Examples I

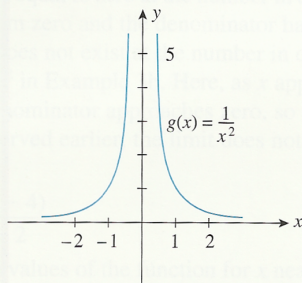
EXAMPLE Evaluate the limit of the following functions as x approaches the indicated point.

a. $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}; x = 0$ b. $g(x) = \frac{1}{x^2}; x = 0$

Solution The graphs of the functions f and g are shown in Figure 29.

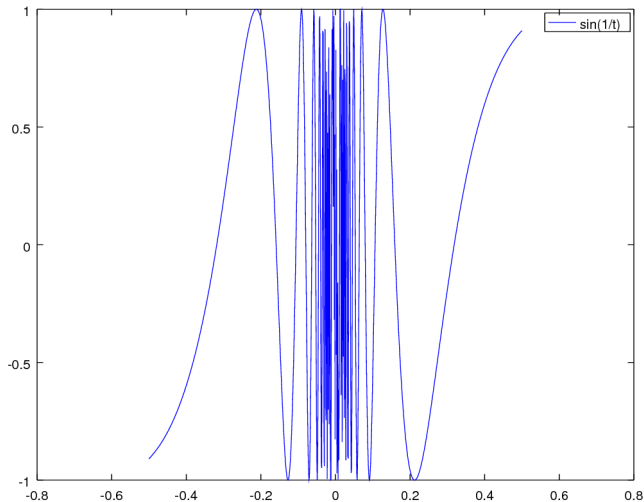


(a) $\lim_{x \rightarrow 0} f(x)$ does not exist.



(b) $\lim_{x \rightarrow 0} g(x)$ does not exist.

No Limit Example II



Properties of Limits

Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Then

$$\lim_{x \rightarrow a} [f(x)]^r = L^r, r \text{ a real number} \quad (7)$$

$$\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot L, c \text{ a real number} \quad (8)$$

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = L \pm M \quad (9)$$

$$\lim_{x \rightarrow a} [f(x)g(x)] = LM \quad (10)$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}, \text{ provided that } M \neq 0 \quad (11)$$

Properties of Limits Exercises

Use the properties of limits to evaluate the following,

$$\lim_{x \rightarrow 2} x^3 \quad (12)$$

$$\lim_{x \rightarrow 4} 5x^{3/2} \quad (13)$$

$$\lim_{x \rightarrow 1} (5x^4 - 2) \quad (14)$$

$$\lim_{x \rightarrow 3} 2x^3 \sqrt{x^2 + 7} \quad (15)$$

$$\lim_{x \rightarrow 2} \frac{2x^2 + 1}{x + 1} \quad (16)$$

Another Indeterminate Form Example I

Here is an example where by skillful manipulation we can determine the limit even though at first the limit is in indeterminate form. Let

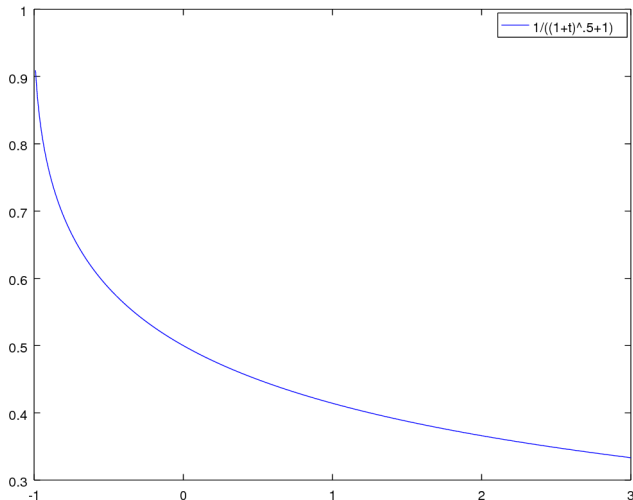
$$f(x) = \frac{\sqrt{1+x} - 1}{x} \quad (17)$$

What is $\lim_{x \rightarrow 0} f(x)$? If we leave the fraction unchanged, it will give us an indeterminate form. However, if we multiply both numerator and denominator by $(\sqrt{1+x} + 1)$, we avoid the indeterminate form!

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2} \quad (18)$$

Look at the function graph of $f(x)$ to verify that this is the correct limit.

Another Indeterminate Form Example II



Sometimes, we want to know what happens to a function graph when either x or $-x$ get very large. We use the infinity sign ∞ for notation, but note that we do NOT use infinity to define these limits.

$$\lim_{x \rightarrow \infty} f(x) = w \text{ if and only if } w = S \quad (19)$$

where S is a number such that for any tiny number ε there is a real number x_0 and $|f(x) - S| < \varepsilon$ for all $x > x_0$.

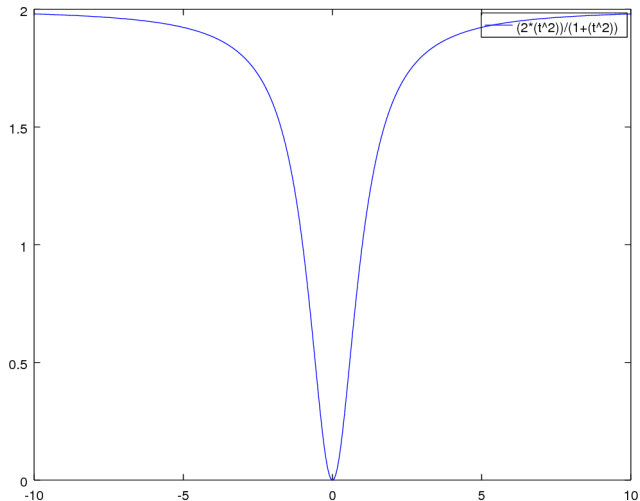
For example, what is

$$\lim_{x \rightarrow \infty} \frac{2x^2}{1 + x^2} \quad (20)$$

or

$$\lim_{x \rightarrow -\infty} \frac{2x^2}{1 + x^2} \quad (21)$$

Limits at Infinity III



Here is another important property of limits. If $1/x^n$ is defined and $n > 0$, then

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0 \text{ and } \lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0 \quad (22)$$

Polynomial and Rational Functions

A polynomial function looks like this,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 \quad (23)$$

For example, $p(x) = 7x^3 - 4.7x^2 + 6$. $n > 0$ is a natural number, and the a_i are called **coefficients**. They are real numbers. A rational function looks like this,

$$q(x) = \frac{p_1(x)}{p_2(x)} \quad (24)$$

where $p_1(x)$ and $p_2(x)$ are polynomial functions. For example,

$$q(x) = \frac{5x^2 - \pi x + 9000}{e^2 x + 2} \quad (25)$$

Limits at Infinity V

When we are looking for the limit of rational functions as they go to negative or positive infinity, we often get an indeterminate form.

$$\lim_{x \rightarrow \infty} \frac{x^2 - x + 3}{2x^3 + 1} = \frac{\infty}{\infty} \quad (26)$$

Here is a technique that will almost always work. Divide both the numerator and the denominator by x^m , where m is the highest exponent you can find.

$$\lim_{x \rightarrow \infty} \frac{x^2 - x + 3}{2x^3 + 1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{1}{x^2} + \frac{3}{x^3}}{2 + \frac{1}{x^3}} = \frac{0}{2} = 0 \quad (27)$$

Here are two more examples.

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 8x - 4}{2x^2 + 4x - 5} = \lim_{x \rightarrow -\infty} \frac{3 - \frac{8}{x} - \frac{4}{x^2}}{2 + \frac{4}{x} - \frac{5}{x^2}} = \frac{3}{2} = 1.5 \quad (28)$$

$$\lim_{x \rightarrow \infty} \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4} = \lim_{x \rightarrow \infty} \frac{2 - \frac{3}{x} + \frac{1}{x^3}}{\frac{1}{x} + \frac{2}{x^2} + \frac{4}{x^3}} = \frac{2}{0} = \text{undefined} \quad (29)$$

In the second example, the limit does not exist. Sometimes, we write $\lim_{x \rightarrow a} = \infty$ or $\lim_{x \rightarrow a} = -\infty$, depending on which way the function goes.

Example I

Consider the function,

$$f(x) = \frac{x - 4}{\sqrt{x} - 2} \quad (30)$$

Let's find

$$\lim_{x \rightarrow 4} f(x) \quad (31)$$

Example II

First, fill out the table:

$x = 3$	$f(x) = 3.7321$	$x = 5$	$f(x) = 4.2361$
$x = 3.5$		$x = 4.5$	
$x = 3.75$		$x = 4.25$	
$x = 3.9$		$x = 4.1$	
$x = 3.95$		$x = 4.05$	
$x = 3.99$		$x = 4.01$	

Example III

Next, let's assume that $x \neq 4$ and expand both the numerator and denominator by $\sqrt{x} + 2$. Simplify

$$f(x) = \frac{(x - 4) \cdot (\sqrt{x} + 2)}{(\sqrt{x} - 2) \cdot (\sqrt{x} + 2)} \text{ on domain } \mathbb{R} \setminus \{4\} \quad (32)$$

$$g(x) = \sqrt{x} + 2 \text{ on domain } \mathbb{R} \quad (33)$$

Except on $x = 4$, g agrees with f . Determine $\lim_{x \rightarrow 4} g(x)$.

Evaluate the following two limits.

$$\lim_{x \rightarrow 3} = \frac{\sqrt{x^2 + 7} + \sqrt{3x - 5}}{x + 2} \quad (34)$$

$$\lim_{x \rightarrow -1} \frac{x^2 - x - 2}{2x^2 - x - 3} \quad (35)$$

Evaluate the following three limits.

$$\lim_{x \rightarrow 2} 3 \quad (36)$$

$$\lim_{x \rightarrow \infty} \frac{3x + 2}{x - 5} \quad (37)$$

$$\lim_{x \rightarrow \infty} \frac{x^5 - x^3 + x - 1}{x^6 + 2x^2 + 1} \quad (38)$$

Finding Limits Exercises

Find the following limits,

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \quad (39)$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 8x}}{2x + 1} \quad (40)$$

$$\lim_{x \rightarrow -1} \frac{x^2 - x - 2}{2x^2 - x - 3} \quad (41)$$

$$\lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x+4}}{3 - \frac{1}{x^2}} \quad (42)$$

$$\lim_{x \rightarrow \infty} \frac{x - 2x^3}{(1 + x)^3} \quad (43)$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 3}}{x + 5} \quad (44)$$

In Einstein's theory of relativity, the length L of an object moving at a velocity v is

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} \quad (45)$$

where c is the speed of light and L_0 is the length of the object at rest. What is the one-sided limit of L as v gets faster and faster?

Sometimes you need some ingenuity to find a limit. Consider

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \quad (46)$$

Squeeze Theorem

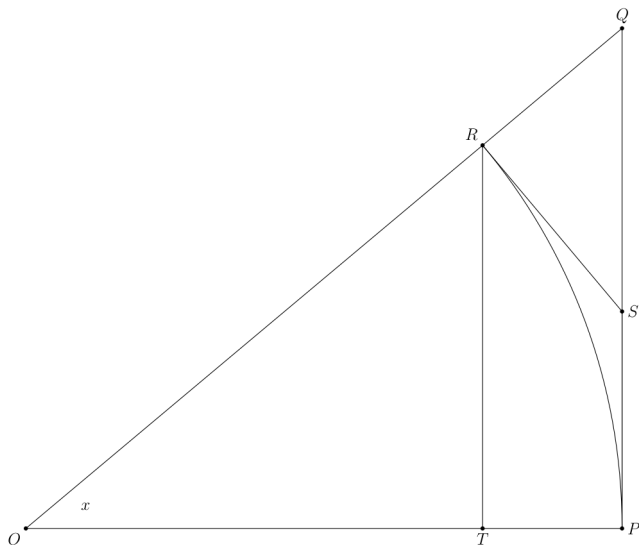
If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at $x = a$)
and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \quad (47)$$

then

$$\lim_{x \rightarrow a} g(x) = L \quad (48)$$

Squeeze Theorem



Squeeze Theorem

In the previous slide, consider the unit circle with $\|\vec{OP}\| = \|\vec{OR}\| = 1$ and the angle x at O . For simplicity let's assume that $0 < x < \pi/2$. The angle x is also the length of the arc between P and R . Consequently

$$\|\vec{RT}\| = \sin x \leq x \quad (49)$$

and therefore

$$\frac{\sin x}{x} \leq 1 \quad (50)$$

Squeeze Theorem

Now consider

$$x \leq \|\vec{PS}\| + \|\vec{SR}\| \leq \|\vec{PS}\| + \|\vec{SQ}\| = \|\vec{PQ}\| = \tan x \quad (51)$$

$\|\vec{SR}\| \leq \|\vec{SQ}\|$ because the angle QRS is a right angle. (51) means that

$$\cos x \leq \frac{\sin x}{x} \quad (52)$$

Since $\lim_{x \rightarrow 0} \cos x = 1$ and $\lim_{x \rightarrow 0} 1 = 1$, we can use the squeeze theorem, (50), and (52) for

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (53)$$

Here is a summary of methods to use to find limits.

- 1 If a function f is continuous, then $\lim_{x \rightarrow a} f(x) = f(a)$.
- 2 If a function is composed of continuous functions, use the properties of limits (Theorem 1) to find the limit.
- 3 If the last step gives you an indeterminate form, try to factor either the numerator or the denominator and use the One Disagreement Rule. Example:
$$\lim_{x \rightarrow -2} [(x^2 - x - 6)/(x + 2)] = -5.$$
- 4 If there is a square root in a fraction, another thing to try is to multiply both numerator and denominator by the conjugate. Example: $\lim_{x \rightarrow -2} [(x - 4)/(\sqrt{x} - 2)] = 4.$
- 5 For rational functions, use the property of limits involving x^{-n} . Example: $\lim_{x \rightarrow \infty} (5x^3 - x^2 - x + 3)/(2x^3 + 1) = 5/2.$

Exercise 1: Find the following limit.

$$\lim_{x \rightarrow 4} (\sqrt{x} - 2 - (x - 4)) \quad (54)$$

Exercise 2: Find the following limit.

$$\lim_{x \rightarrow 0} (\ln x) \quad (55)$$

Exercise 3: Find the following limit.

$$\lim_{x \rightarrow 3} \frac{x^3 - 5}{(x - 3)^2} \quad (56)$$

Exercise 4: Find the following limit.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^2 - 6x + 9} \quad (57)$$

Exercise 5: Find the following limit.

$$\lim_{x \rightarrow 2} \frac{3x - 4}{(x - 2)^2} \quad (58)$$

Exercise 6: Find the following limit.

$$\lim_{x \rightarrow \pi} \frac{(\pi - x)^2}{\cos 2x} \quad (59)$$

Exercise 7: Find the following limit.

$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{1 - \cos x}} \quad (60)$$

Exercise 8: Find the following limit.

$$\lim_{x \rightarrow 5} \frac{x}{x^2 - 25} \quad (61)$$

Exercise 9: Find the following limit.

$$\lim_{x \rightarrow \infty} \arctan x \quad (62)$$

Exercise 10: Find the following limit.

$$\lim_{x \rightarrow -1} \frac{2x^2 - x - 3}{x + 1} \quad (63)$$

Exercise 11: Find the following limit.

$$\lim_{x \rightarrow -3} \frac{x^2 + 5x + 6}{x^2 - x - 12} \quad (64)$$

Exercise 12: Find the following limit.

$$\lim_{x \rightarrow -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4} \quad (65)$$

Exercise 13: Find the following limit.

$$\lim_{x \rightarrow 5} \frac{-2x^2 + 11x - 5}{(x - 5)(x + 3)} \quad (66)$$

Exercise 14: Find the following limit.

$$\lim_{x \rightarrow -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4} \quad (67)$$

Exercise 15: Find the following limit.

$$\lim_{x \rightarrow \infty} \frac{1 - 5x^3}{4x^2 + x^6} \quad (68)$$

Exercise 16: Find the following limit.

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} \quad (69)$$

Exercise 17: Find the following limit.

$$\lim_{x \rightarrow \infty} \frac{7x^2 - 9}{4x + 3} \quad (70)$$

End of Lesson

Next Lesson: Derivatives