

# Limits

MATH 2511, BCIT

Technical Mathematics for Geomatics

January 8, 2018

# Limits Introduction

Consider the function graph of the following function.

$$f(x) = \frac{x^2 - 1}{x - 1} \quad (1)$$

It looks like it is a linear equation! However, at  $x = 1$ ,  $f(x)$  is not defined. To fill the hole, we define the limit

$$\lim_{x \rightarrow a} f(x) = w \text{ if and only if } w = L = R \quad (2)$$

where  $L$  is the number that the function  $f$  approaches as  $x$  gets closer to  $a$  with  $x < a$  (that means  $x \neq a$ !); and  $R$  is the number that the function  $f$  approaches as  $x$  gets closer to  $a$  with  $x > a$ .

Note: for a mathematically rigorous definition of what “approaching” and “getting closer” means we would need to talk about sequences and series, which is a topic we won’t cover here.

# Indeterminate Form I

Notice that

$$f(x) = \frac{x^2 - 1}{x - 1} \stackrel{x=1}{=} \frac{0}{0} \quad (3)$$

We call this an **indeterminate form**.

# Indeterminate Form

Notice that except at  $x = 1$

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{\cancel{(x - 1)}(x + 1)}{\cancel{x - 1}} = x + 1 = g(x) \quad (4)$$

$f$  and  $g$  agree everywhere except on  $x = 1$ . Consider the following rule,

## One Disagreement Rule

If  $f = g$  except in one point, then  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$  for all  $a$ , even the  $a$  where  $f$  and  $g$  disagree.

Therefore

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 2 \quad (5)$$

Consider a simple function like  $f(x) = x^3$ . What is  $\lim_{x \rightarrow 4} f(x)$ ?  
The answer is almost trivial,

$$\lim_{x \rightarrow 4} f(x) = f(4) = 4^3 = 64 \quad (6)$$

Why is this true? Because  $f$  is continuous at  $x = 4$ . There are no holes, jumps, gaps, or breaks of the function graph at  $x = 4$ .

Constant functions, the identity function, linear functions, polynomial functions, exponential and logarithmic functions are all continuous. Rational functions, some trigonometric functions, and other functions are sometimes **not** continuous.

# Interesting Cases

A function is continuous if and only if  $\lim_{x \rightarrow c} f(x) = f(c)$  for all  $c$  in  $\mathbb{R}$  (the logarithmic function is continuous only on  $\mathbb{R}^+$ ). This means that (i) the function needs to be defined at  $x = c$ ; (ii) the limit needs to be defined at  $x = c$ ; and (iii) the function value and the limit need to be equal to each other.

Consider the following interesting cases:

- 1 A function that is continuous and well defined at  $x = a$ .
- 2 A function that is not continuous at  $x = a$ .
- 3 A function where the limit exists but  $\lim_{x \rightarrow c} f(x) \neq f(c)$ .
- 4 A function such as  $f(x) = \sin(1/x)$ .

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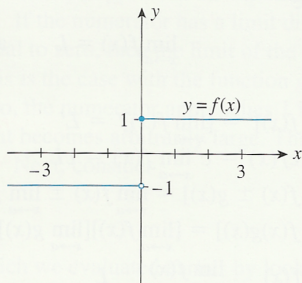
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# No Limit Examples I

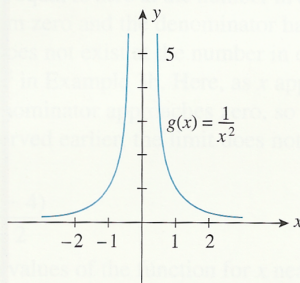
**EXAMPLE** Evaluate the limit of the following functions as  $x$  approaches the indicated point.

a.  $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}; x = 0$       b.  $g(x) = \frac{1}{x^2}; x = 0$

**Solution** The graphs of the functions  $f$  and  $g$  are shown in Figure 29.

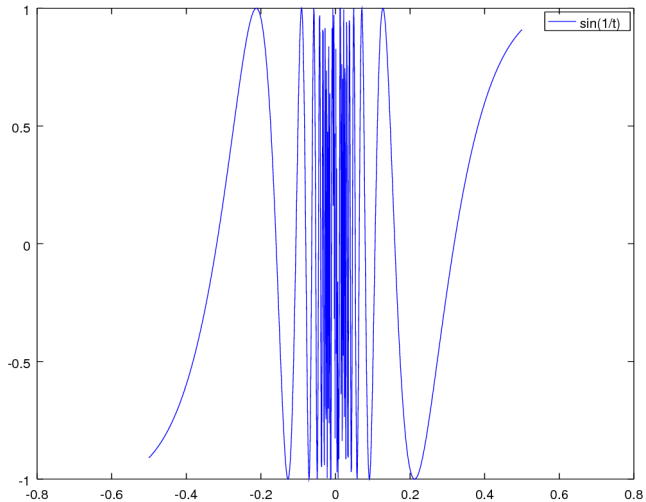


(a)  $\lim_{x \rightarrow 0} f(x)$  does not exist.



(b)  $\lim_{x \rightarrow 0} g(x)$  does not exist.

# No Limit Example II



# Properties of Limits

Suppose  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$ . Then

$$\lim_{x \rightarrow a} [f(x)]^r = L^r, r \text{ a real number} \quad (7)$$

$$\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot L, c \text{ a real number} \quad (8)$$

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = L \pm M \quad (9)$$

$$\lim_{x \rightarrow a} [f(x)g(x)] = LM \quad (10)$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}, \text{ provided that } M \neq 0 \quad (11)$$

# Properties of Limits Exercises

Use the properties of limits to evaluate the following,

$$\lim_{x \rightarrow 2} x^3 \quad (12)$$

$$\lim_{x \rightarrow 4} 5x^{3/2} \quad (13)$$

$$\lim_{x \rightarrow 1} (5x^4 - 2) \quad (14)$$

$$\lim_{x \rightarrow 3} 2x^3 \sqrt{x^2 + 7} \quad (15)$$

$$\lim_{x \rightarrow 2} \frac{2x^2 + 1}{x + 1} \quad (16)$$

## Another Indeterminate Form Example I

Here is an example where by skillful manipulation we can determine the limit even though at first the limit is in indeterminate form. Let

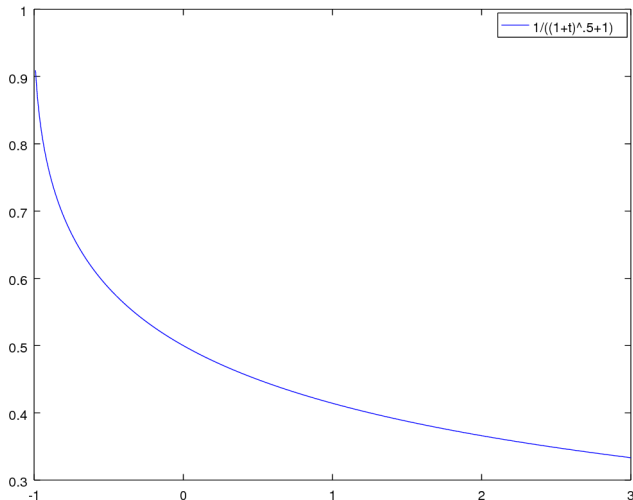
$$f(x) = \frac{\sqrt{1+x} - 1}{x} \quad (17)$$

What is  $\lim_{x \rightarrow 0} f(x)$ ? If we leave the fraction unchanged, it will give us an indeterminate form. However, if we multiply both numerator and denominator by  $(\sqrt{1+x} + 1)$ , we avoid the indeterminate form!

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2} \quad (18)$$

Look at the function graph of  $f(x)$  to verify that this is the correct limit.

## Another Indeterminate Form Example II



Sometimes, we want to know what happens to a function graph when either  $x$  or  $-x$  get very large. We use the infinity sign  $\infty$  for notation, but note that we do NOT use infinity to define these limits.

$$\lim_{x \rightarrow \infty} f(x) = w \text{ if and only if } w = S \quad (19)$$

where  $S$  is a number such that for any tiny number  $\varepsilon$  there is a real number  $x_0$  and  $|f(x) - S| < \varepsilon$  for all  $x > x_0$ .



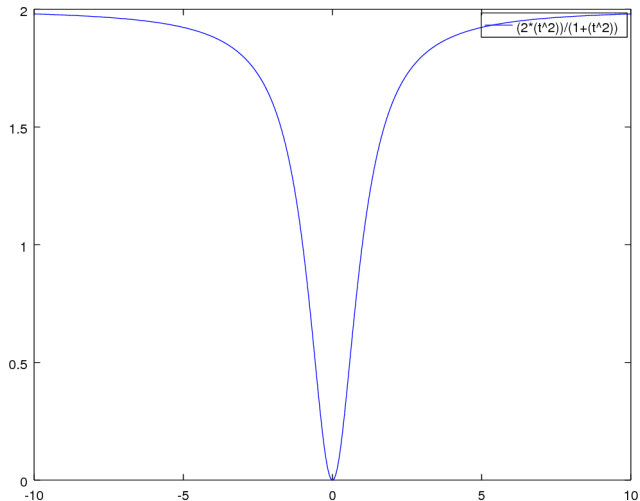
For example, what is

$$\lim_{x \rightarrow \infty} \frac{2x^2}{1 + x^2} \quad (20)$$

or

$$\lim_{x \rightarrow -\infty} \frac{2x^2}{1 + x^2} \quad (21)$$

# Limits at Infinity III



Here is another important property of limits. If  $1/x^n$  is defined and  $n > 0$ , then

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0 \text{ and } \lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0 \quad (22)$$

# Polynomial and Rational Functions

A polynomial function looks like this,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 \quad (23)$$

For example,  $p(x) = 7x^3 - 4.7x^2 + 6$ .  $n > 0$  is a natural number, and the  $a_i$  are called **coefficients**. They are real numbers. A rational function looks like this,

$$q(x) = \frac{p_1(x)}{p_2(x)} \quad (24)$$

where  $p_1(x)$  and  $p_2(x)$  are polynomial functions. For example,

$$q(x) = \frac{5x^2 - \pi x + 9000}{e^2 x + 2} \quad (25)$$

# Limits at Infinity V

When we are looking for the limit of rational functions as they go to negative or positive infinity, we often get an indeterminate form.

$$\lim_{x \rightarrow \infty} \frac{x^2 - x + 3}{2x^3 + 1} = \frac{\infty}{\infty} \quad (26)$$

Here is a technique that will almost always work. Divide both the numerator and the denominator by  $x^m$ , where  $m$  is the highest exponent you can find.

$$\lim_{x \rightarrow \infty} \frac{x^2 - x + 3}{2x^3 + 1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{1}{x^2} + \frac{3}{x^3}}{2 + \frac{1}{x^3}} = \frac{0}{2} = 0 \quad (27)$$

Here are two more examples.

$$\lim_{x \rightarrow -\infty} \frac{3x^2 + 8x - 4}{2x^2 + 4x - 5} = \lim_{x \rightarrow -\infty} \frac{3 - \frac{8}{x} - \frac{4}{x^2}}{2 + \frac{4}{x} - \frac{5}{x^2}} = \frac{3}{2} = 1.5 \quad (28)$$

$$\lim_{x \rightarrow \infty} \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4} = \lim_{x \rightarrow \infty} \frac{2 - \frac{3}{x} + \frac{1}{x^3}}{\frac{1}{x} + \frac{2}{x^2} + \frac{4}{x^3}} = \frac{2}{0} = \text{undefined} \quad (29)$$

In the second example, the limit does not exist. Sometimes, we write  $\lim_{x \rightarrow a} = \infty$  or  $\lim_{x \rightarrow a} = -\infty$ , depending on which way the function goes.

# Example I

Consider the function,

$$f(x) = \frac{x - 4}{\sqrt{x} - 2} \quad (30)$$

Let's find

$$\lim_{x \rightarrow 4} f(x) \quad (31)$$

## Example II

First, fill out the table:

$x = 3$	$f(x) = 3.7321$	$x = 5$	$f(x) = 4.2361$
$x = 3.5$		$x = 4.5$	
$x = 3.75$		$x = 4.25$	
$x = 3.9$		$x = 4.1$	
$x = 3.95$		$x = 4.05$	
$x = 3.99$		$x = 4.01$	



## Example III

Next, let's assume that  $x \neq 4$  and expand both the numerator and denominator by  $\sqrt{x} + 2$ . Simplify

$$f(x) = \frac{(x - 4) \cdot (\sqrt{x} + 2)}{(\sqrt{x} - 2) \cdot (\sqrt{x} + 2)} \text{ on domain } \mathbb{R} \setminus \{4\} \quad (32)$$

$$g(x) = \sqrt{x} + 2 \text{ on domain } \mathbb{R} \quad (33)$$

Except on  $x = 4$ ,  $g$  agrees with  $f$ . Determine  $\lim_{x \rightarrow 4} g(x)$ .

Evaluate the following two limits.

$$\lim_{x \rightarrow 3} = \frac{\sqrt{x^2 + 7} + \sqrt{3x - 5}}{x + 2} \quad (34)$$

$$\lim_{x \rightarrow -1} \frac{x^2 - x - 2}{2x^2 - x - 3} \quad (35)$$

Evaluate the following three limits.

$$\lim_{x \rightarrow 2} 3 \quad (36)$$

$$\lim_{x \rightarrow \infty} \frac{3x + 2}{x - 5} \quad (37)$$

$$\lim_{x \rightarrow \infty} \frac{x^5 - x^3 + x - 1}{x^6 + 2x^2 + 1} \quad (38)$$

# Finding Limits Exercises

Find the following limits,

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \quad (39)$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 8x}}{2x + 1} \quad (40)$$

$$\lim_{x \rightarrow -1} \frac{x^2 - x - 2}{2x^2 - x - 3} \quad (41)$$

$$\lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x+4}}{3 - \frac{1}{x^2}} \quad (42)$$

$$\lim_{x \rightarrow \infty} \frac{x - 2x^3}{(1 + x)^3} \quad (43)$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 3}}{x + 5} \quad (44)$$

In Einstein's theory of relativity, the length  $L$  of an object moving at a velocity  $v$  is

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} \quad (45)$$

where  $c$  is the speed of light and  $L_0$  is the length of the object at rest. What is the one-sided limit of  $L$  as  $v$  gets faster and faster?

Sometimes you need some ingenuity to find a limit. Consider

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \quad (46)$$

# Squeeze Theorem

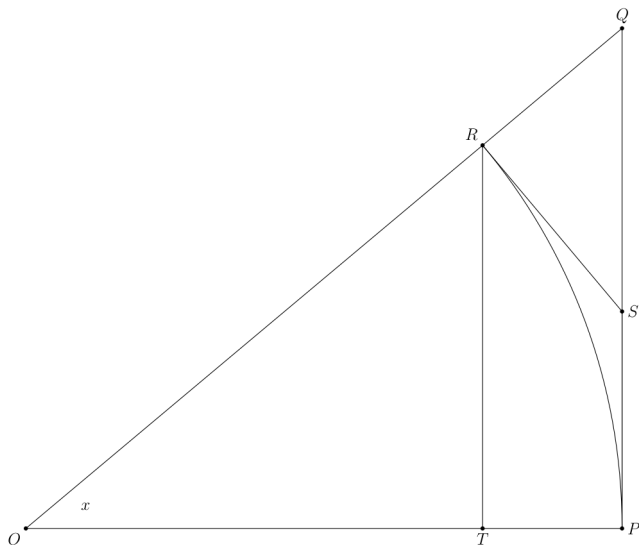
If  $f(x) \leq g(x) \leq h(x)$  when  $x$  is near  $a$  (except possibly at  $x = a$ )  
and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \quad (47)$$

then

$$\lim_{x \rightarrow a} g(x) = L \quad (48)$$

# Squeeze Theorem





# Squeeze Theorem

In the previous slide, consider the unit circle with  $\|\vec{OP}\| = \|\vec{OR}\| = 1$  and the angle  $x$  at  $O$ . For simplicity let's assume that  $0 < x < \pi/2$ . The angle  $x$  is also the length of the arc between  $P$  and  $R$ . Consequently

$$\|\vec{RT}\| = \sin x \leq x \quad (49)$$

and therefore

$$\frac{\sin x}{x} \leq 1 \quad (50)$$

# Squeeze Theorem

Now consider

$$x \leq \|\vec{PS}\| + \|\vec{SR}\| \leq \|\vec{PS}\| + \|\vec{SQ}\| = \|\vec{PQ}\| = \tan x \quad (51)$$

$\|\vec{SR}\| \leq \|\vec{SQ}\|$  because the angle  $QRS$  is a right angle. (51) means that

$$\cos x \leq \frac{\sin x}{x} \quad (52)$$

Since  $\lim_{x \rightarrow 0} \cos x = 1$  and  $\lim_{x \rightarrow 0} 1 = 1$ , we can use the squeeze theorem, (50), and (52) for

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (53)$$

Here is a summary of methods to use to find limits.

- 1 If a function  $f$  is continuous, then  $\lim_{x \rightarrow a} f(x) = f(a)$ .
- 2 If a function is composed of continuous functions, use the properties of limits (Theorem 1) to find the limit.
- 3 If the last step gives you an indeterminate form, try to factor either the numerator or the denominator and use the One Disagreement Rule. Example:  
$$\lim_{x \rightarrow -2} [(x^2 - x - 6)/(x + 2)] = -5.$$
- 4 If there is a square root in a fraction, another thing to try is to multiply both numerator and denominator by the conjugate. Example:  $\lim_{x \rightarrow -2} [(x - 4)/(\sqrt{x} - 2)] = 4.$
- 5 For rational functions, use the property of limits involving  $x^{-n}$ . Example:  $\lim_{x \rightarrow \infty} (5x^3 - x^2 - x + 3)/(2x^3 + 1) = 5/2.$

**Exercise 1:** Find the following limit.

$$\lim_{x \rightarrow 4} (\sqrt{x} - 2 - (x - 4)) \quad (54)$$

**Exercise 2:** Find the following limit.

$$\lim_{x \rightarrow 0} (\ln x) \quad (55)$$

**Exercise 3:** Find the following limit.

$$\lim_{x \rightarrow 3} \frac{x^3 - 5}{(x - 3)^2} \quad (56)$$

**Exercise 4:** Find the following limit.

$$\lim_{x \rightarrow 0} \frac{x^2}{x^2 - 6x + 9} \quad (57)$$

**Exercise 5:** Find the following limit.

$$\lim_{x \rightarrow 2} \frac{3x - 4}{(x - 2)^2} \quad (58)$$



**Exercise 6:** Find the following limit.

$$\lim_{x \rightarrow \pi} \frac{(\pi - x)^2}{\cos 2x} \quad (59)$$

**Exercise 7:** Find the following limit.

$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{1 - \cos x}} \quad (60)$$

**Exercise 8:** Find the following limit.

$$\lim_{x \rightarrow 5} \frac{x}{x^2 - 25} \quad (61)$$

**Exercise 9:** Find the following limit.

$$\lim_{x \rightarrow \infty} \arctan x \quad (62)$$

**Exercise 10:** Find the following limit.

$$\lim_{x \rightarrow -1} \frac{2x^2 - x - 3}{x + 1} \quad (63)$$

**Exercise 11:** Find the following limit.

$$\lim_{x \rightarrow -3} \frac{x^2 + 5x + 6}{x^2 - x - 12} \quad (64)$$

**Exercise 12:** Find the following limit.

$$\lim_{x \rightarrow -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4} \quad (65)$$

**Exercise 13:** Find the following limit.

$$\lim_{x \rightarrow 5} \frac{-2x^2 + 11x - 5}{(x - 5)(x + 3)} \quad (66)$$



**Exercise 14:** Find the following limit.

$$\lim_{x \rightarrow -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4} \quad (67)$$

**Exercise 15:** Find the following limit.

$$\lim_{x \rightarrow \infty} \frac{1 - 5x^3}{4x^2 + x^6} \quad (68)$$

**Exercise 16:** Find the following limit.

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} \quad (69)$$

**Exercise 17:** Find the following limit.

$$\lim_{x \rightarrow \infty} \frac{7x^2 - 9}{4x + 3} \quad (70)$$

# End of Lesson

Next Lesson: Derivatives