Higher-Order Derivatives

MATH 2511, BCIT

Technical Mathematics for Geomatics

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Expanding the Logarithm

The logarithm is only defined on the positive real numbers. It turns out that

$$\frac{d}{dx} \ln|x| = \frac{1}{x} \text{ for all real numbers except } x = 0 \tag{1}$$

Remember that the absolute value $|x|$ is defined as

$$|x| = \begin{cases} x \text{ for } x \geq 0 \\ -x \text{ for } x < 0 \end{cases} \tag{2}$$

Differentiating the Absolute Value

Treat functions with an absolute value as you would treat piecewise defined functions. A piecewise defined function looks like this:

$$f(x) = \begin{cases} x^2 - 7 \text{ for } x > 7 \\ e^{1/x} \text{ for } x \leq 7 \end{cases} \tag{3}$$

Now differentiate

$$g(t) = |t^2 + t| \tag{4}$$

Absolute Value Example

Differentiate

$$f(x) = |x - 1| \tag{5}$$

You can do this two ways: (a) use the chain rule and

$$f(x) = \sqrt{(x-1)^2} \tag{6}$$

(b) use the piecewise definition

$$f(x) = \begin{cases} x - 1 \text{ for } x \geq 1 \\ 1 - x \text{ for } x < 1 \end{cases} \tag{7}$$

Exercise 1: Differentiate

$$f(x) = -x + 2 + |-x + 2| \tag{8}$$

Absolute Value Exercises

Exercise 2: Differentiate

$$f(x) = |2x - 5| \tag{9}$$

Absolute Value Exercises

Exercise 3: Differentiate

$$f(x) = (x-2)^2 + |x-2| \tag{10}$$

Absolute Value Exercises

Exercise 4: Differentiate

$$f(x) = -3 \cdot |x + 2| - 1 \tag{11}$$

Proving the Power Rule

We have shown the power rule to be true for $n = 2$ and $n = 0.5$ $(x \geq 0)$. Here is a proof that it is true for all real numbers n. Let

$$f(x) = \ln(x^n) \tag{12}$$

On the one hand,

$$f'(x) = \frac{1}{x^n} \frac{d}{dx} x^n \tag{13}$$

On the other hand, using $f(x) = n \cdot \ln x$,

$$f'(x) = \frac{n}{x} \tag{14}$$

Consequently,

$$\frac{d}{dx} x^n = nx^{n-1} \tag{15}$$

Logarithmic Differentiation

Using this method, we can differentiate a function such as

$$f(x) = x^{\sqrt{x}}, \text{ using the helper function } g(y) = \ln x^{\sqrt{x}} \tag{16}$$

Now use the chain rule for

$$g'(y) = \frac{1}{x^{\sqrt{x}}} \frac{d}{dx} x^{\sqrt{x}} \tag{17}$$

and the properties of logarithms for

$$g'(y) = \frac{d}{dx}\left(\sqrt{x} \cdot \ln x\right) \tag{18}$$

Compare the results and isolate $d/dx(x^{\sqrt{x}})$. Alternatively, use

$$x^{\sqrt{x}} = e^{\ln x^{\sqrt{x}}} \tag{19}$$

Logarithmic Differentiation Exercises

Exercise 5: Differentiate

$$f_1(x) = x^{\pi} \text{ and } f_2(x) = \pi^{x} \tag{20}$$

Logarithmic Differentiation Exercises

Exercise 6: Differentiate

$$g(s) = 7^{2s^2 - s} \tag{21}$$

Higher-Order Derivatives

Derivatives often have derivatives themselves, and so on. If f is a function of time determining the position of an object, then f' is sometimes called the velocity of the object as a function of time, f'' is called the acceleration of the object as a function of time, and f''' is called the jerk of the object as a function of time. For higher-order derivatives than that, we write $f^{(n)}$, for example

$$f(x) = x^4 + 3x^3 + 7x^2 - \pi x + 1 \text{ and } f^{(4)} = 24 \tag{22}$$

Exercise 7: If $g(\vartheta) = \vartheta \sin \vartheta$, find $g''(\pi/6)$.

Exercise 8: Find $f^{(n)}(x)$ if $f(x) = 1/(2 - x)$.

There is a differentiation worksheet with solutions on D2L.

End of Lesson

Next Lesson: Applications