Multivariable Calculus

MATH 2511, BCIT

Technical Mathematics for Geomatics

May 8, 2018

Function of Multiple Variables

Let A be a domain and B a codomain (both of these objects are sets). Then f is a function from A to B if and only if for each $a \in A$ there is a $b \in B$ such that

$$(a, b) \in f \tag{1}$$

where f is a set of ordered pairs consisting of pairs with an element of A in the first position and an element of B in the second position. This is the set theory of functions. Usually, we think of f as a rule which assigns to each element of A an element of B.

$$f(a) = b \tag{2}$$

Functions with Multiple Arguments

We can use this definition for functions with multiple arguments. For example, let A be a subset of the xy plane (the domain) and B be a set of real numbers z. Then f is a function from A to B if and only if for each element of A there is a unique element of B which is assigned to it.

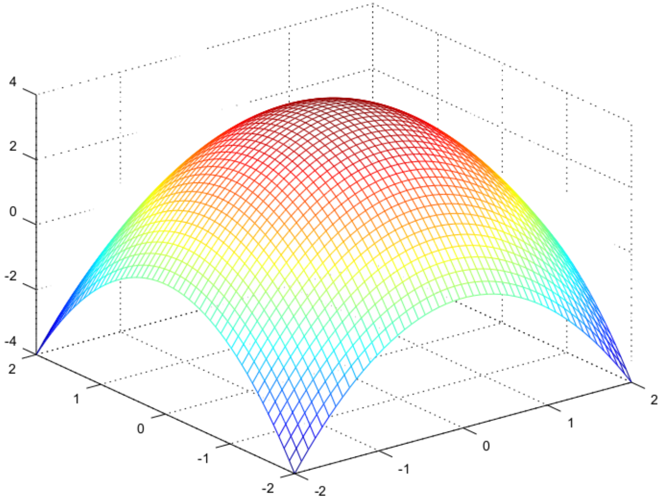
$$f(x, y) = z \tag{3}$$

Graphical Representations: 3D

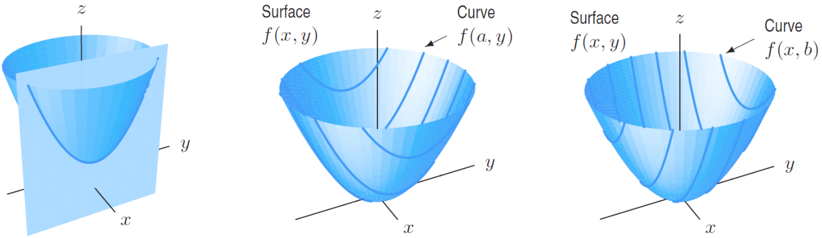
Here is some octave (matlab) code that simulates three dimensions in the plane using colours.

```
x=linspace(-2,2,50);

y=linspace(-2,2,50);

[xx,yy]=meshgrid(x,y);

mesh(xx,yy,4-(xx.^2+yy.^2))
```

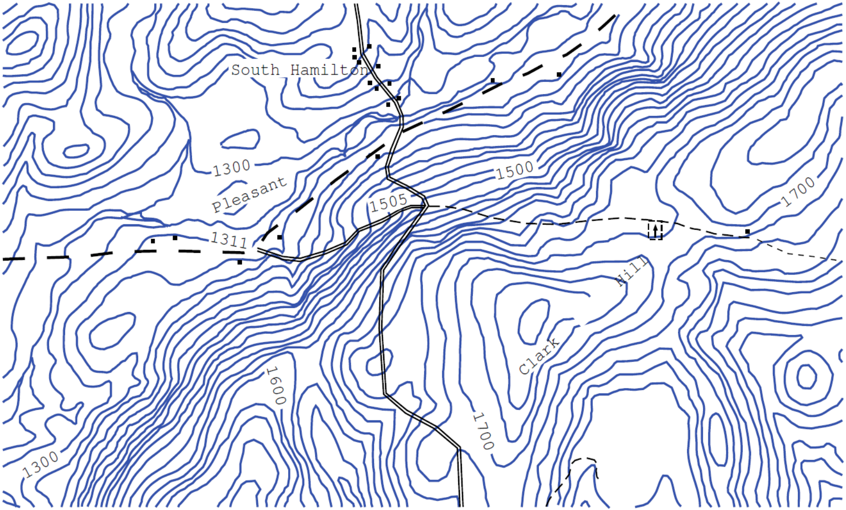
Graphical Representations: 3D



Graphical Representations: Cross-Sections



Graphical Representations: Contour Lines



Linear Functions

A linear function in three-dimensional space is characterized by three one-dimensional parameters, just as a linear function in two-dimensional space is characterized by two one-dimensional parameters (slope and y-intercept). There are many different ways, however, to conceive of these parameters. They may be three points that are not all on the same line; or a point and two slopes. We define a plane or a linear function in three-dimensional space to be the function

$$f(x, y) = z_0 + m(x - x_0) + n(y - y_0) \tag{4}$$

where $P = (x_0, y_0, z_0)$ is a point on the plane and m and n are the respective slopes along the x-axis and the y-axis.

Linear Functions

If

$$f(x, y) = z_0 + m(x - x_0) + n(y - y_0) \tag{5}$$

and $f(x, y) = z$, we can rewrite as a plane equation or linear combination of x, y, z

$$a_1x + a_2y + a_3z = 1 \tag{6}$$

Linear Function Exercise

Exercise 1: Find the linear function $f(x, y) = z$ for which the following three points are on the graph:

$$\begin{aligned} P_1 &= (1, 0, 1) \\ P_2 &= (1, -1, 3) \\ P_3 &= (3, 0, -1) \end{aligned} \tag{7}$$

Linear Function Exercise

Solution:

$$f(x, y) = 1 + (-1) \cdot (x - 1) + (-2) \cdot (y - 0) \tag{8}$$

For $z = f(x, y)$ we can rewrite as

$$\frac{1}{2}x + y + \frac{1}{2}z = 1 \tag{9}$$

The points P_1, P_2, P_3 were artificially chosen to make it easy to find the plane equation. The next few slides explain how to find a plane equation if three arbitrary, non-collinear points are provided.

Vectors

A vector is an ordered pair or triplet of real numbers. One way to interpret it is to make it refer to a point in the *xy*-plane or *xyz*-three-dimensional space. The usual interpretation, however, is as a displacement vector with a direction and a length. Here is an example:

$$\vec{v} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \tag{10}$$

Vector Algebra

Vectors can be added, subtracted, and multiplied by a scalar (a real number).

$$\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ \pi \\ -6 \end{pmatrix} = \begin{pmatrix} 5 \\ 5+\pi \\ -7 \end{pmatrix} \tag{11}$$

$$1.5 \cdot \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 4.5 \\ 7.5 \\ -1.5 \end{pmatrix} \tag{12}$$

Unit Vectors

All three-dimensional vectors can be expressed in components. For this expression we need unit vectors. Any three linearly-independent vectors would work, but it makes sense to use the following three:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{13}$$

Vector Decomposition

For any vector $\vec{v}$ (assuming from now on three dimensions),

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} \tag{14}$$

where $V = (v_x, v_y, v_z)$, and V is the point to which the origin $O = (0, 0, 0)$ would be displaced by vector

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \tag{15}$$

Vector Length and Distance Between Two Points

The length of vector $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{16}$$

The distance between two points P and Q is the length of a displacement vector between them. Let $\vec{OP}$ be the displacement vector from O to P and so on. Then

$$\vec{PQ} = \vec{PO} + \vec{OQ} = \vec{OQ} - \vec{OP} \tag{17}$$

and $\|\vec{PQ}\|$ is the distance between P and Q.

Dot Product

The following two definition of the dot product, or scalar product, $\vec{v} \cdot \vec{w}$ are equivalent:

geometric $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\vartheta$ where ϑ is the angle between $\vec{v}$ and $\vec{w}$, $0 \leq \vartheta \leq \pi$.

algebraic $\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z$

The dot product is a number, not a vector.

Dot Product

Now we need to show that the two definitions are equivalent. Consider a triangle PQR in three-dimensional space. Let $\vec{v} = \vec{PQ}, \vec{w} = \vec{PR}$. Then

$$\vec{QR} = \vec{QP} + \vec{PR} = -\vec{v} + \vec{w} = \vec{w} - \vec{v} \tag{18}$$

Here is the law of cosines for this triangle:

$$\|\vec{w} - \vec{v}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\| \cdot \|\vec{w}\| \cos\vartheta \tag{19}$$

It follows that the two definitions are equivalent.

Dot Product

Perpendicularity and Dot Product

Two non-zero vectors $\vec{v}$ and $\vec{w}$ are perpendicular, or orthogonal, if and only if $\vec{v} \cdot \vec{w} = 0$.

Magnitude and Dot Product

Magnitude and dot product are related as follows: $\vec{v} \cdot \vec{v} = \|\vec{v}\|$.

Dot Product Exercise

Exercise 2: Find the angle between

$$\vec{v} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \tag{20}$$

Consider the dot product

$$4 \cdot (-2) + 0 \cdot 1 + 7 \cdot 3 = 13 \tag{21}$$

According to the two equivalent definitions of the dot product, this is equal to

$$\|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\vartheta = \sqrt{4^2 + 7^2} \cdot \sqrt{(-2)^2 + 1^2 + 3^2} \cdot \cos\vartheta \tag{22}$$

Therefore,

$$\vartheta = \arccos \frac{13}{\sqrt{4^2 + 7^2} \cdot \sqrt{(-2)^2 + 1^2 + 3^2}} = 64.47^\circ \tag{23}$$

Planes Again

The equation of the plane with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ and containing the point $P = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{24}$$

Alternatively, for $d = ax_0 + by_0 + cz_0$

$$ax + by + cz = d \tag{25}$$

Cross Product

The following two definitions of the **cross product** or **vector product** $\vec{v} \times \vec{w}$ are equivalent:

- **Geometric definition**

  If $\vec{v}$ and $\vec{w}$ are not parallel, then

  $$\vec{v} \times \vec{w} = \begin{pmatrix} \text{Area of parallelogram} \\ \text{with edges } \vec{v} \text{ and } \vec{w} \end{pmatrix} \vec{n} = (\|\vec{v}\| \|\vec{w}\| \sin\theta)\vec{n},$$

  where $0 \le \theta \le \pi$ is the angle between $\vec{v}$ and $\vec{w}$ and $\vec{n}$ is the unit vector perpendicular to $\vec{v}$ and $\vec{w}$ pointing in the direction given by the right-hand rule. If $\vec{v}$ and $\vec{w}$ are parallel, then $\vec{v} \times \vec{w} = \vec{0}$.

- **Algebraic definition**

  $$\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}$$

  where $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

Cross Product

If you know what a determinant is, you can remember the algebraic definition as follows.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{26}$$

Note that $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$.

Exercise 3: Use the cross product to find the linear equation containing the three points

$$\begin{aligned} P &= (1,3,0) \\ Q &= (3,4,-3) \\ R &= (3,6,2) \end{aligned} \tag{27}$$

Cross Product Exercise Answer

One way to find the answer to the last exercise (without using the cross product) is to solve the following system of linear equations for the plane $x + ay + bz = c$,

$$\begin{aligned} 1 + 3a + 0b &= c \\ 3 + 4a - 3b &= c \\ 3 + 6a + 2b &= c \end{aligned} \tag{28}$$

Change this to

$$\begin{aligned} 3a + 0b - c &= -1 \\ 4a - 3b - c &= -3 \\ 6a + 2b - c &= -3 \end{aligned} \tag{29}$$

Using matrices,

$$\begin{pmatrix} 3 & 0 & -1 \\ 4 & -3 & -1 \\ 6 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -3 \end{pmatrix} \tag{30}$$

Cross Product Exercise Answer

Equation (30) yields the solution

$$x - \frac{10}{11}y + \frac{4}{11}z = -\frac{19}{11} \tag{31}$$

Now let's use the cross product instead, avoiding the matrices. Note that

$$\begin{aligned} \vec{PQ} &= 2\vec{i} + \vec{j} - 3\vec{k} \\ \vec{PR} &= 2\vec{i} + 3\vec{j} + 2\vec{k} \end{aligned} \tag{32}$$

The cross product, using the algebraic definition, is $\vec{u} = \vec{PQ} \times \vec{PR} = 11\vec{i} - 10\vec{j} + 4\vec{k}$.

Cross Product Exercise Answer

Let $P = (x_0, y_0, z_0)$ be a fixed point on the plane with known coordinates. Since any point $S = (x, y, z)$ on the plane fulfills

$$\vec{PS} \cdot \vec{u} = 0 \tag{33}$$

this can be turned into the plane equation

$$u_x(x - x_0) + u_y(y - y_0) + u_z(z - z_0) = 0 \tag{34}$$

Therefore, using $P = (1, 3, 0)$, this translates into

$$11x - 10y + 4z = 19 \tag{35}$$

which is equivalent to (31). Notice how easy it is to find a linear equation when you have a point $P = (x_0, y_0, z_0)$ on the plane and a normal vector $\vec{u}$ to the plane $u_x\vec{i} + u_y\vec{j} + u_z\vec{k}$:

$$u_x x + u_y y + u_z z = u_x x_0 + u_y y_0 + u_z z_0 \tag{36}$$

Partial Derivatives

The derivative of a one-variable function measures its rate of change. We will see how a two-variable function has two rates of change: one as x changes (with y held constant) and one as y changes (with x held constant).

For all points at which the limits exist, we define the partial derivatives at the point (a, b) by

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} \tag{37}$$

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h} \tag{38}$$

If we let a and b vary, we have the partial derivative functions $f_x(x, y)$ and $f_y(x, y)$.

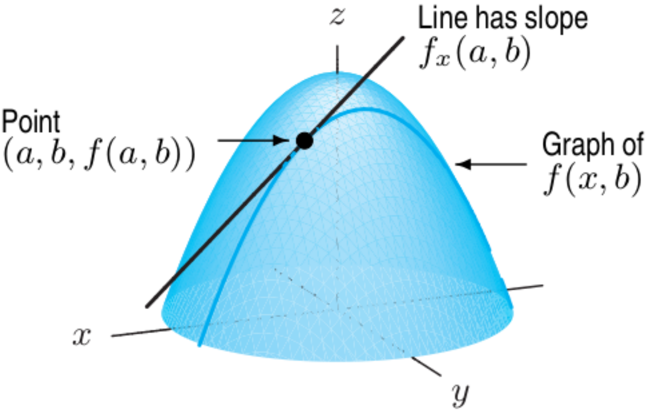
Partial Derivatives Notation

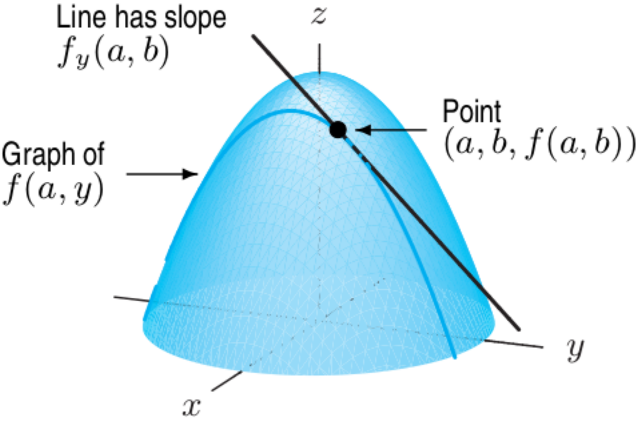
There is some terrifying notation for partial derivatives. If $z = f(x, y)$, we can write

$$f_x(x, y) = \frac{\partial z}{\partial x} \text{ and } f_y(x, y) = \frac{\partial z}{\partial y} \tag{39}$$

$$f_x(a, b) = \left.\frac{\partial z}{\partial x}\right|_{(a,b)} \text{ and } f_y(a, b) = \left.\frac{\partial z}{\partial y}\right|_{(a,b)} \tag{40}$$

Partial Derivatives




Line has slope
$f_y(a, b)$
z
Point
$(a, b, f(a, b))$
Graph of
$f(a, y)$
y
x

Tangent Planes

Assuming f is differentiable at (a, b), the equation of the tangent plane is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{41}$$

$f(x, y)$ can be approximated around (a, b) by the tangent plane,

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \text{ near } (a, b) \tag{42}$$

Partial Derivatives Exercise

Exercise 4: Let

$$f(x, y) = \frac{x^2}{y + 1} \tag{43}$$

Find $f_x(3, 2)$.

Exercise 5: Compute the partial derivatives with respect to x and with respect to y for the following functions.

$$f(x, y) = y^2 e^{3x} \tag{44}$$

$$z = (3xy + 2x)^5 \tag{45}$$

$$g(x, y) = e^{x+3y} \sin(xy) \tag{46}$$

Second Order Partial Derivatives

The second order partial derivatives are

$$\frac{\partial^2 z}{\partial x^2} = f_{xx} = (f_x)_x \qquad \frac{\partial^2 z}{\partial x \partial y} = f_{yx} = (f_y)_x \tag{47}$$

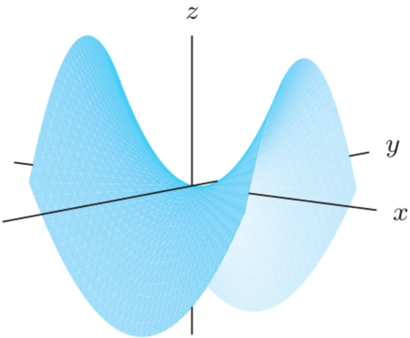
$$\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = (f_x)_y \qquad \frac{\partial^2 z}{\partial y^2} = f_{yy} = (f_y)_y \tag{48}$$

It turns out that if f_{xy} and f_{yx} are continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b) \tag{49}$$

Saddle Points

consider the following function graph:



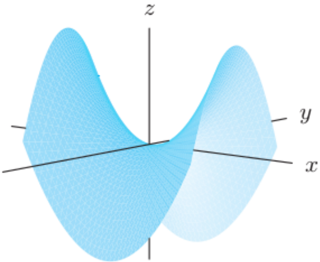
Is $f_{xx}(0,0)$ positive or negative? Is $f_{yy}(0,0)$ positive or negative?

Critical Points

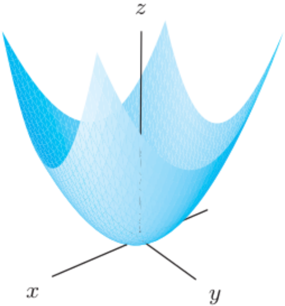
f has a local maximum at the point P_0 if $f(P_0) \geq f(P)$ for all points P near P_0.

f has a local minimum at the point P_0 if $f(P_0) \leq f(P)$ for all points P near P_0.

Critical Points



Graph of $g(x, y) = x^2 - y^2$, showing saddle shape at the origin



Graph of $h(x, y) = x^2 + y^2$, showing minimum at the origin

Suppose (x_0, y_0) is a point where the partial derivatives are both 0. Let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 \qquad (50)$$

Then

- If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
- If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
- If $D < 0$, then f has a saddle point at (x_0, y_0).
- If $D = 0$, then anything can happen: f can have a local maximum, or a local minimum, or a saddle point, or none of these, at (x_0, y_0).

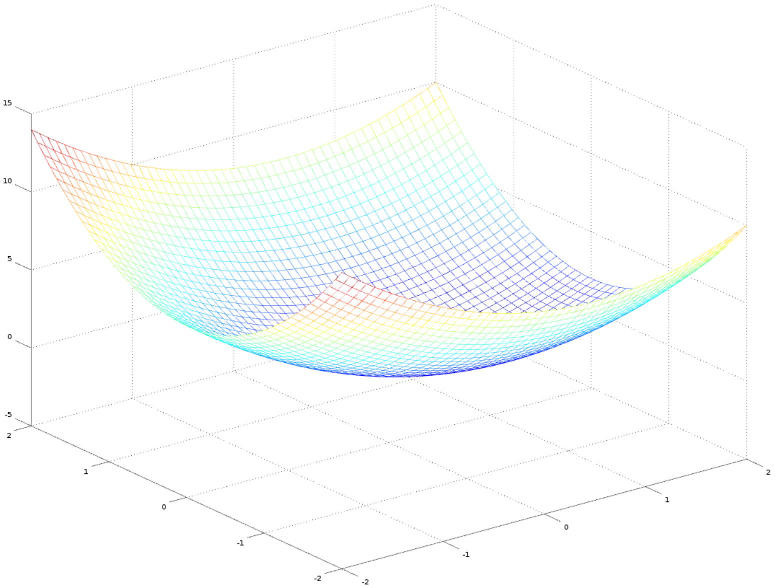
Exercise 6: Find the local maxima, minima, and saddle points of

$$f(x, y) = x^2 + 2y^2 - x \tag{51}$$

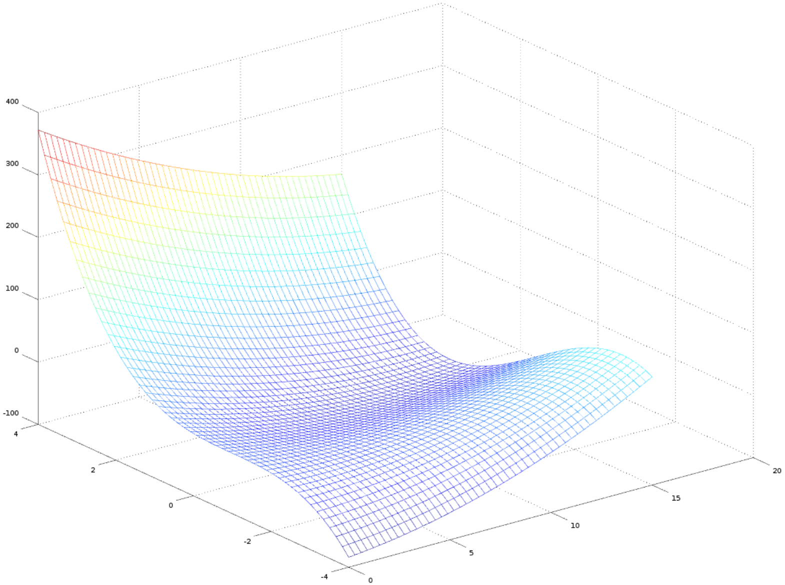
$$g(x, y) = \frac{1}{2}x^2 + 3y^3 + 9y^2 - 3xy + 9y - 9x \tag{52}$$

$$h(x, y) = e^{-x^2-y^2} \tag{53}$$

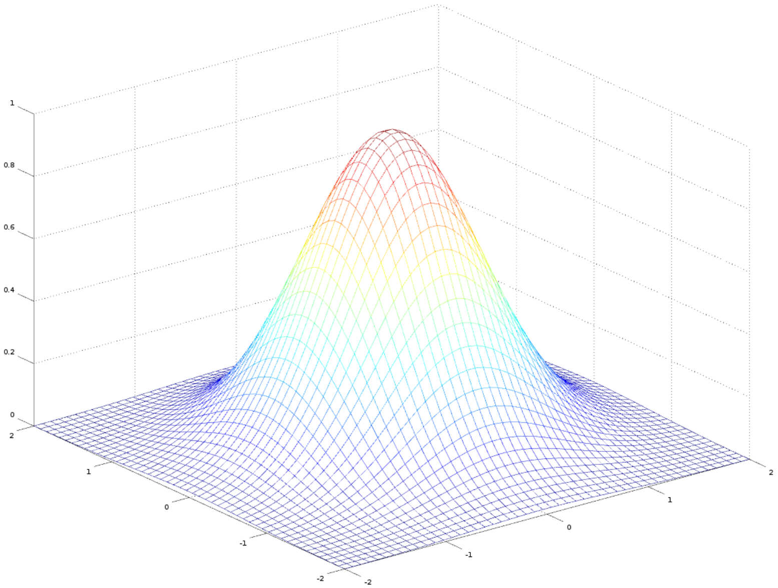
Optimization Visual



Optimization Visual



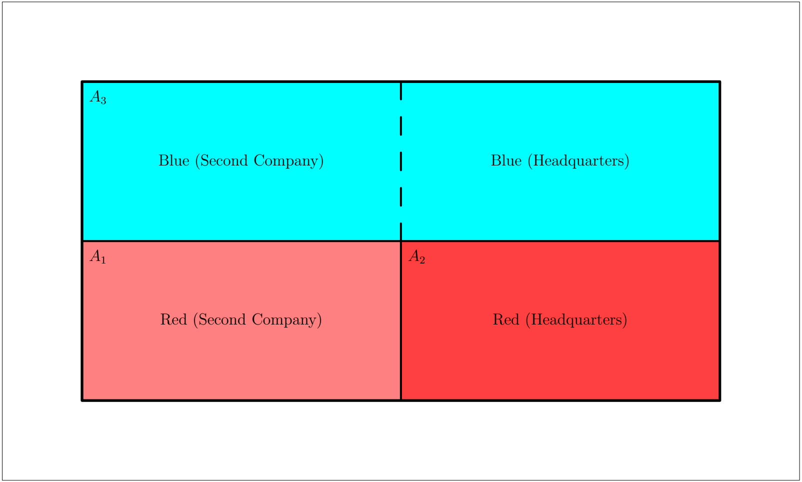
Optimization Visual



Optimization Exercise

Exercise 7: A delivery of 480 cubic meters of gravel is to be made to a landfill. The trucker plans to purchase an open-top box in which to transport the gravel in numerous trips. The total cost to the trucker is the cost of the box plus \$80 per trip. The box must have height 2 metres, but the trucker can choose the length and width. The cost of the box is $\$100/m^2$ for the ends, $\$50/m^2$ for the sides and $\$200/m^2$ for the bottom. Notice the tradeoff: A smaller box is cheaper to buy but requires more trips. What size box should the trucker buy to minimize the total cost?

Judy Benjamin

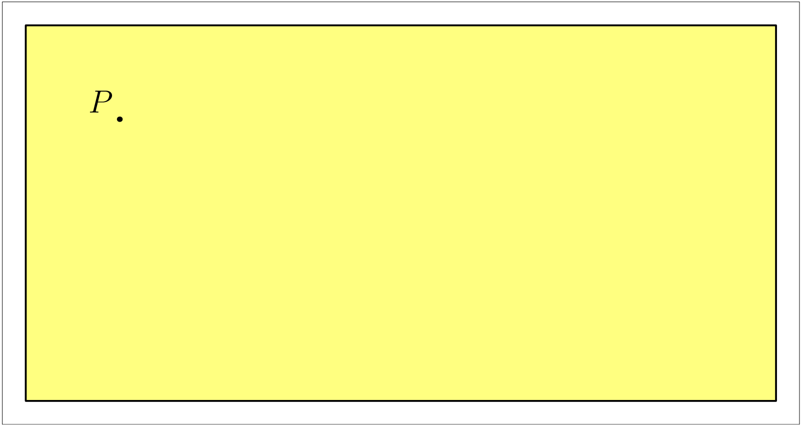


- (MAP) Judy has no idea where she is. She is on team Blue. Because of the map, her probability of being in Blue territory equals the probability of being in Red territory, and being on Red Second Company ground equals the probability of being on Red Headquarters ground.
- (HDQ) Headquarters inform Judy that in case she is in Red territory, her chance of being on their Headquarters ground is three times the chance of being on their Second Company ground.

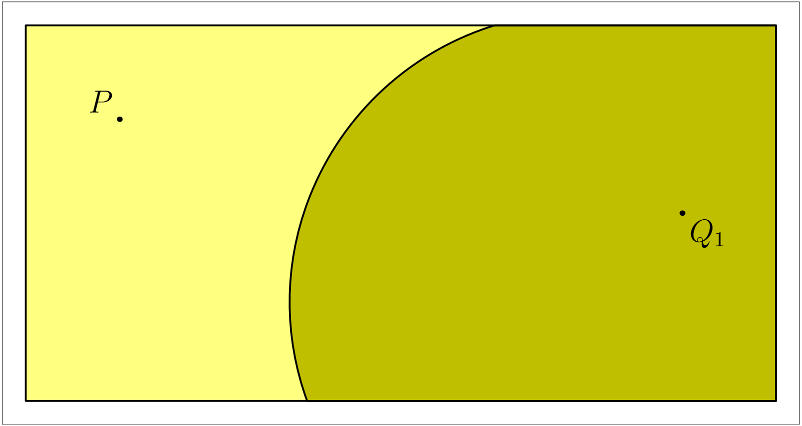
$$2 \cdot p_1 = 2 \cdot p_2 = p_3 \tag{MAP}$$

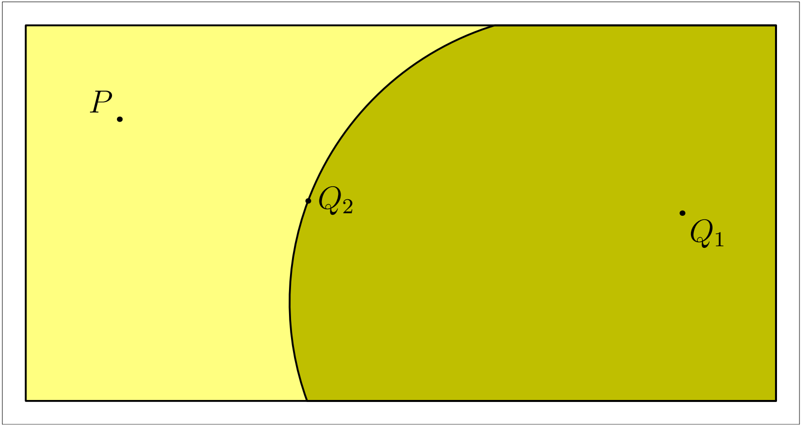
$$\frac{q_2}{q_3} = 3 \tag{HDQ}$$

The Principle of Maximum Entropy

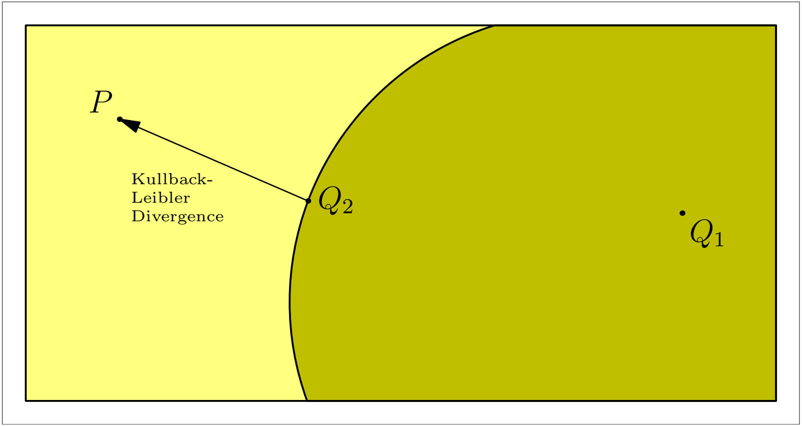


The Principle of Maximum Entropy




P
Q_2
Q_1

The Principle of Maximum Entropy



Kullback-Leibler Divergence

The Kullback-Leibler Divergence is

$$D(Q, P) = \sum_{i=1}^{m} q_i \log_2 \frac{q_i}{p_i}$$

Our point $P = (0.5, 0.25, 0.25)$ is fixed. Define the function

$$f(q_1, q_2, q_3) = q_1 \log \frac{q_1}{0.5} + q_2 \log \frac{q_2}{0.25} + q_3 \log \frac{q_3}{0.25} \tag{54}$$

and find the minimum with the constraint that $q_1 + q_2 + q_3 = 1$ and $q_2 = 3q_3$.

The function with Lagrange Multipliers is

$$LM(q_1, q_2, q_3, \lambda, \mu) = q_1 \log \frac{q_1}{0.5} + q_2 \log \frac{q_2}{0.25} + q_3 \log \frac{q_3}{0.25} + \lambda(q_1 + q_2 + q_3 - 1) + \mu(q_2 - 3q_3) \quad (55)$$

Differentiation with respect to λ and μ and setting to zero gives you the two constraints.

$$\frac{\partial LM}{\partial q_1} = \log \frac{q_1}{0.5} + 1 + \lambda \quad (56)$$

$$\frac{\partial LM}{\partial q_2} = \log \frac{q_2}{0.25} + 1 + \lambda + \mu \quad (57)$$

$$\frac{\partial LM}{\partial q_3} = \log \frac{q_3}{0.25} + 1 + \lambda - 3\mu \quad (58)$$

Setting (56)–(58) to zero with these two constraints gives us a system of five equations with five variables.

$$q_1 = \frac{1}{2}e^{1+\lambda} \tag{59}$$

$$q_2 = \frac{1}{4}e^{1+\lambda+\mu} \tag{60}$$

$$q_3 = \frac{1}{4}e^{1+\lambda-3\mu} \tag{61}$$

$$\frac{1}{2}e^{1+\lambda} + \frac{1}{4}e^{1+\lambda+\mu} + \frac{1}{4}e^{1+\lambda-3\mu} = 1 \tag{62}$$

$$\frac{1}{4}e^{1+\lambda+\mu} = \frac{3}{4}e^{1+\lambda-3\mu} \tag{63}$$

Equation (63) gives us $\mu = \frac{1}{4}\ln 3$. Substituting this in (62) gives us

$$\lambda = \ln \frac{4}{e\left(2 + 3^{\frac{1}{4}} + 3^{-\frac{3}{4}}\right)} \tag{64}$$

Substituting these values in (59), (60), and (61) yields

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \approx \begin{pmatrix} 0.533 \\ 0.350 \\ 0.117 \end{pmatrix} \tag{65}$$

Surprisingly, even though the radio call from headquarters was all about the enemy territory, Judy ought to increase her degree of belief that she has landed in friendly territory from 0.50 to 0.533.

End of Lesson

Next Lesson: End of Term! Have a Happy Holiday!