

# Chain Rule

MATH 2511, BCIT

Technical Mathematics for Geomatics

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# Euler's Number

The number  $e$  is defined as follows,

$$e = \lim_{t \rightarrow \infty} \left(1 + \frac{1}{t}\right)^t \quad (1)$$

Consider two functions  $f_1$  and  $f_2$ . They are related in so far as

$$f_1(x) = f_2\left(\frac{1}{x}\right) \quad (2)$$

For example,

$$f_1(x) = \frac{2x+1}{5x-7} \text{ and } f_2(x) = -\frac{x+2}{7x-5} \quad (3)$$

Then

$$\text{If } \lim_{x \rightarrow \infty} f_1(x) = a \text{ then } \lim_{x \rightarrow 0} f_2(x) = a \quad (4)$$

# The Derivative of the Logarithmic Function

Now consider the function  $f(x) = \ln x$  and the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \ln \frac{x+h}{x} = \\ &= \lim_{h \rightarrow 0} \frac{1}{x} \cdot \frac{x}{h} \ln \left( 1 + \frac{h}{x} \right) = \lim_{h \rightarrow 0} \frac{1}{x} \ln \left( 1 + \frac{1}{\frac{x}{h}} \right)^{\frac{x}{h}} \end{aligned} \quad (5)$$

Use the lemma of the last slide and the definition of Euler's number to see that

$$f'(x) = \frac{1}{x} \quad (6)$$

# Problematic Functions

Here are some functions that we either don't know how to differentiate or whose differentiation would take an inordinate amount of time.

$$f(x) = 2^x \quad (7)$$

$$f(x) = \sqrt{x^2 + 1} \quad (8)$$

$$f(x) = (x^2 + x + 1)^{100} \quad (9)$$

$$f(x) = \sin(1 + \sqrt{x - 7}) \quad (10)$$

$$f(x) = \log_{10} x \quad (11)$$

$$f(x) = \ln(x^2 + 1) \quad (12)$$

## Rule 7

### The Chain Rule

$$g'(x) = f_1'(f_2(x))f_2'(x) \text{ for } g(x) = (f_1 \circ f_2)(x) \quad (13)$$

Consider

$$\begin{aligned}(f \circ g)'(x) &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} = \\ \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} &= \quad (14)\end{aligned}$$

$$f'(g(x))g'(x) \quad (15)$$

This is only a hint, not a rigorous proof, since we have replaced  $g(x+h)$  by  $g(x) + h$ , which isn't covered by our rules and is, in fact, false in some situations.

- 1 Diffentiate:  $f(x) = 2^x$
- 2 Diffentiate:  $f(x) = \sqrt{x^2 + 1}$
- 3 Diffentiate:  $f(x) = (x^2 + x + 1)^{100}$
- 4 Diffentiate:  $f(x) = \log_{10} x$
- 5 Diffentiate:  $f(x) = \ln(x^2 + 1)$



# Inverse and Identity Function

Remember how we defined the logarithmic function,

$$\ln y = x \text{ if and only if } e^x = y \quad (16)$$

so the logarithmic function is the inverse of the exponential function. Consequently, if  $f(x) = e^x$  and  $g(y) = \ln y$

$$(f \circ g)(y) = y \text{ and } (g \circ f)(x) = x \quad (17)$$

When (17) is true we call  $f$  the **inverse function** of  $g$  and vice versa. The function  $\text{id}(x) = x$  is called the **identity function**.

# The Derivative of the Exponential Function

We know the derivative of the identity function.

$$\text{id}'(x) = 1 \quad (18)$$

Consequently,

$$\frac{d}{dx} \ln(e^x) = 1 \quad (19)$$

We also know that according to the chain rule

$$\frac{d}{dx} \ln(e^x) = \frac{1}{e^x} \exp'(x) \quad (20)$$

where  $\exp(x) = e^x$ . Therefore,

$$\exp'(x) = e^x \quad (21)$$

The exponential function is its own derivative!

**Exercise 1:** Differentiate the following functions:

$$f(x) = e^{\sin x} \quad (22)$$

$$g(t) = \frac{1}{e^t} \quad (23)$$

**Exercise 2:** Differentiate the following functions:

$$v(w) = w^2 e^w \quad (24)$$

$$g(z) = \frac{e^z - 1}{e^z + 1} \quad (25)$$

**Exercise 3:** Differentiate

$$f(\vartheta) = \tan(\sin \vartheta) \quad (26)$$

$$F(x) = \sqrt[4]{1 + 2x + x^3} \quad (27)$$

$$g(t) = \frac{\pi}{(t^4 + 1)^3} \quad (28)$$

**Exercise 4:** Differentiate

$$f(s) = \sqrt[3]{1 + \tan s} \quad (29)$$

$$y = (x^2 + 1)\sqrt[3]{x^2 + 2} \quad (30)$$

$$y = e^{x \cos x} \quad (31)$$

$$y = x \sin \frac{1}{x} \quad (32)$$

**Exercise 5:** Differentiate the following functions or find  $dy/dx$  for the following curves:

$$y = \frac{1}{\arcsin x} \quad (33)$$

$$y = xe^{-kx} \quad (34)$$

$$f(x) = x \arctan \sqrt{x} \quad (35)$$

**Exercise 6:** Differentiate the following functions or find  $dy/dx$  for the following curves:

$$y = 3 \cot(nx) \quad (36)$$

$$y = \arccos(e^{2x}) \quad (37)$$

$$h(t) = (t^4 - 1)^3(t^3 + 1)^4 \quad (38)$$



**Exercise 7:** Differentiate the following functions or find  $dy/dx$  for the following curves:

$$y = (x^2 + 1)\sqrt{x^2 + 2} \quad (39)$$

$$G(y) = \left( \frac{y^2}{y+1} \right)^5 \quad (40)$$

$$y = \tan^2(3\vartheta) \quad (41)$$

$$y = \arctan \left( x - \sqrt{1 + x^2} \right) \quad (42)$$

# Exercises for Differentiation III

**Exercise 8:** Find an equation of the tangent line to the curve

$$y = \frac{2}{1 + e^{-x}} \quad (43)$$

at  $x = 0$ .

**Exercise 9:** Here is a model for the length of daylight (in hours) in Toronto on the  $t$ -th day of the year

$$L(t) = 12 + 2.8 \sin \left( \frac{2\pi}{365}(t - 80) \right) \quad (44)$$

Compare how the number of hours of daylight is increasing in Toronto on March 21 and May 21.

# End of Lesson

Next Lesson: Higher Order Derivatives