

# Optimization and Analyzing Functions

## MATH 2511, BCIT

Technical Mathematics for Geomatics

February 5, 2018

# Relative Extrema

A function  $f$  has a **relative maximum** at  $x = c$  if there exists an open interval  $(a, b)$  containing  $c$  such that  $f(x) \leq f(c)$  for all  $x$  in  $(a, b)$ .

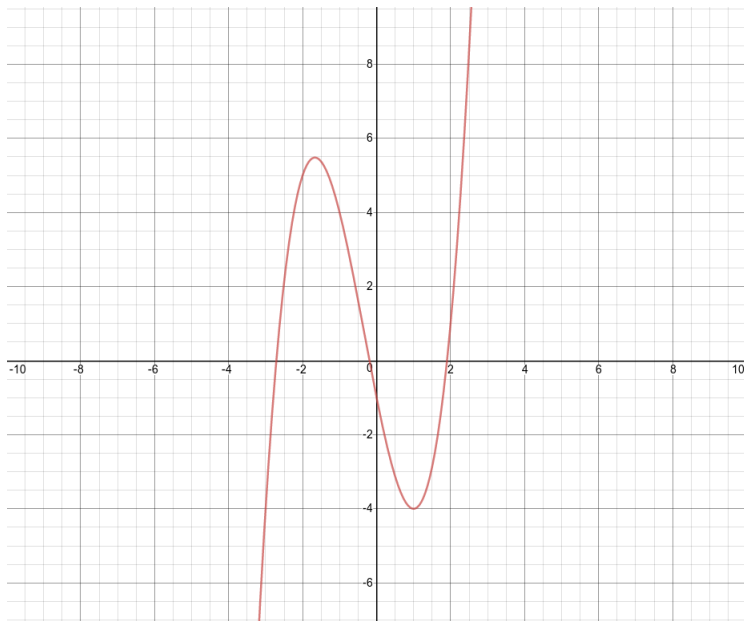
A function  $f$  has a **relative minimum** at  $x = c$  if there exists an open interval  $(a, b)$  containing  $c$  such that  $f(x) \geq f(c)$  for all  $x$  in  $(a, b)$ .

At any number  $c$  where a differentiable function  $f$  has a relative extremum,  $f'(c) = 0$ . The converse is not true. Consider the following two functions and their derivatives.

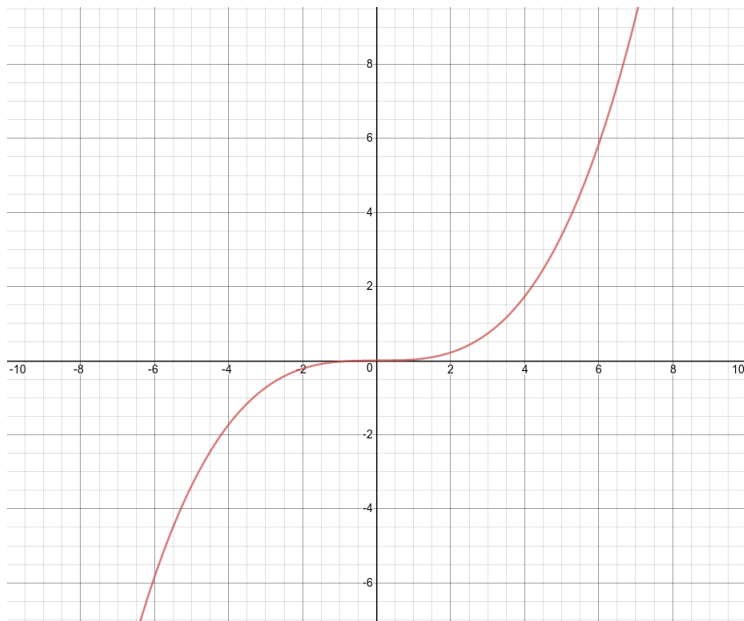
$$f_1(x) = x^3 + x^2 - 5x - 1 \quad (1)$$

$$f_2(x) = \left(\frac{3}{10}x\right)^3 \quad (2)$$

# Derivatives and Extrema Graph I



# Derivatives and Extrema Graph II



# Derivatives and Extrema Caution

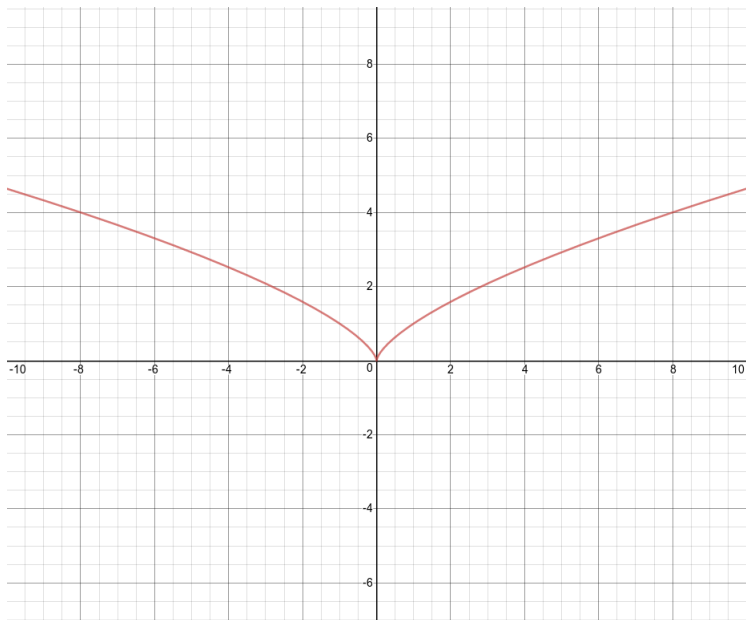
Note that a function may have an extremum at a point where the derivative is not 0 if at that point the function is not differentiable. Consider this function and its derivative.

$$f_3(x) = x^{\frac{2}{3}} \quad (3)$$

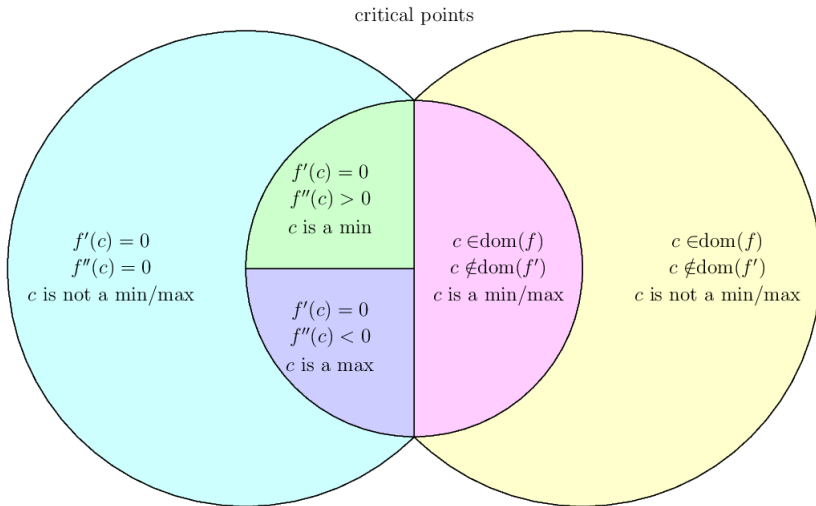
## Critical Number

A **critical number** of a function  $f$  is any number  $x$  in the domain of  $f$  such that  $f'(x) = 0$  or  $f'(x)$  does not exist.

# Derivatives and Extrema Graph III



# Critical Points and Extrema





**Exercise 1:** Find the critical points of the following function,

$$f(x) = x^3 - 4x \quad (4)$$

**Exercise 2:** Find the critical points of the following function,

$$h(t) = -t^2 + 6t + 6 \quad (5)$$

**Exercise 3:** Find the critical points of the following function,

$$f(x) = \frac{1}{2}x^4 - x^2 \quad (6)$$

**Exercise 4:** Find the critical points of the following function,

$$g(x) = \frac{x+1}{x} \quad (7)$$

**Exercise 5:** Find the critical points of the following function,

$$f(x) = x\sqrt{x-4} \quad (8)$$

**Exercise 6:** Find the critical points of the following function,

$$f(x) = 2 \tan x - \tan^2 x \quad (9)$$

**Exercise 7:** Find the critical points of the following function,

$$h(s) = s^{\frac{5}{3}} \quad (10)$$

**Exercise 8:** Find local maxima and minima for the following function:

$$f(x) = 3x^3 - 12x + 5 \quad (11)$$



**Exercise 9:** Find local maxima and minima for the following function:

$$f(x) = \frac{x}{x^2 + 1} \quad (12)$$

**Exercise 10:** Find local maxima and minima for the following function:

$$f(t) = t\sqrt{4 - t^2} \quad (13)$$

**Exercise 11:** Find local maxima and minima for the following function:

$$g(t) = \sqrt[3]{t}(8 - t) \quad (14)$$

**Exercise 12:** Find local maxima and minima for the following function:

$$g(t) = \cos t + \sin t \quad (15)$$

**Exercise 13:** Find local maxima and minima for the following function:

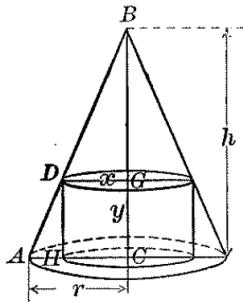
$$f(x) = \ln(x^2 + x + 1) \quad (16)$$

**Exercise 14:** Find local maxima and minima for the following function:

$$f(x) = \ln(\cos x) \quad (17)$$

# Optimization Word Problems

**Exercise 15:** Find the altitude of the cylinder of maximum volume that can be inscribed in a given right cone.



# Optimization Word Problem Hint

As a hint for the last exercise, consider the angle at  $B$  between the height of the cone and the generating line of the cone. The tangent of the angle can be expressed as two different ratios, which will help you to find a function  $V(x)$  and its derivative to maximize the volume.

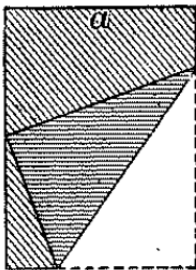


**Exercise 16:** A water tank is to be constructed with a square base and open top and is to hold 64 cubic yards. If the cost of the sides is \$1 a square yard, and of the bottom \$2 a square yard, what are the dimensions when the cost is a minimum? What is the minimum cost?

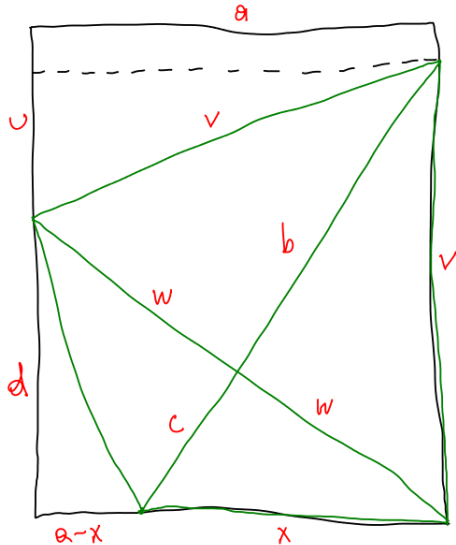
# Optimization Word Problems

**Exercise 17:** The lower corner of a leaf, whose width is  $a$ , is folded over so as just to reach the inner edge of the page.

- 1 Find the width of the part folded over when the length of the crease is a minimum.
- 2 Find the width when the area folded over is a minimum.



# Optimization Word Problem Solution



# Optimization Word Problem Solution

Define a function  $f(x) = b + c$ . We want to minimize  $f$  on the interval  $(\frac{a}{2}, a]$ . Notice that  $b^2 = v^2 - w^2$  and  $c^2 = x^2 - w^2$ . We need to express  $v$  and  $w$  by  $x$  and the fixed number  $a$ . From

$$(2w)^2 = d^2 + a^2 \quad (18)$$

we get

$$w^2 = \frac{ax}{2} \quad (19)$$

From

$$v^2 = a^2 + u^2 \quad (20)$$

we get

$$v^2 = \frac{ax^2}{2x - a} \quad (21)$$

# Optimization Word Problem Solution

Consequently, the function to differentiate is

$$f(x) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{a^2 x}{2x - a}} + \sqrt{x(2x - a)} \right) \quad (22)$$

After differentiation, it turns out that  $f'(x) = 0$  if and only if

$$x = \frac{3}{4}a \quad (23)$$

This answers the first question. The answer for the second question is

$$x = \frac{2}{3}a \quad (24)$$

**Exercise 18:** A submarine telegraph cable consists of a core of copper wires with a covering made of nonconducting material. If  $x$  denotes the ratio of the radius of the core to the thickness of the covering, it is known that the speed of signaling varies as

$$x^2 \ln \frac{1}{x} \quad (25)$$

Show that the greatest speed is attained when  $x = \frac{1}{\sqrt{e}}$ .

**Exercise 19:** Shane is 2.2 miles offshore in a boat and wishes to reach a coastal village 7.5 miles down a straight shoreline from the point nearest the boat. He can row 1.9 miles per hour and can walk 5.1 miles per hour. Where should he land his boat to reach the village in the least amount of time?

# Optimization Word Problem Solution

Let the variable distance that Shane would row his boat be  $y$  and the variable distance that Shane would walk be  $x$ . Recall that time equals distance divided by velocity. Then

$$T(x) = \frac{y}{1.9} + \frac{x}{5.1} \quad (26)$$

for the time  $T(x)$  that it takes Shane to get from where he is to the coastal village. The domain of  $T$  is  $[0, 7.5]$ . Note that

$$y^2 = 2.2^2 + (7.5 - x)^2 \quad (27)$$



# Optimization Word Problem Solution

Therefore

$$T(x) = \frac{1}{1.9} \sqrt{x^2 - 15x + 61.09} + \frac{1}{5.1}x \quad (28)$$

and

$$T'(x) = \frac{2x - 15}{3.8\sqrt{x^2 - 15x + 61.09}} + \frac{1}{5.1} \quad (29)$$

$T'(x) = 0$  if and only if

$$89.6x^2 - 1344x + 4970.1 = 0 \quad (30)$$

with two solutions. The solution  $x = 8.3832$  falls outside the domain of  $T$ . Therefore, the time is optimized at  $x = 6.6168$ . One should check using the second derivative to make sure this is a minimum.

# Analyzing Functions I

To analyze a function, determine the following features:

- Domain and range of the function.
- Zeros (also called x-intercepts) of the function.
- Critical points, maxima, minima.
- Inflection points.
- Asymptotes.
- Is the function even ( $f_1(x) = x^2 + 1$ ) or odd ( $f_2(x) = x^3 - x$ )?

# Analyzing Functions Step-By-Step I

Here is a step-by-step guide to analyzing functions.

- 1 Determine the  $x$ -intercepts (also called zeros). Set  $f(x) = 0$  and find the solution set.
- 2 Determine the critical points. Find the derivative  $f'(x)$  and check whether there are points in the domain of  $f$  that are not in the domain of  $f'$ . Then set  $f'(x) = 0$  and find the solution set.
- 3 Determine whether the critical points are maxima or minima or neither. Find  $f''(x)$  and check whether  $f''$  at the critical points is positive, negative, or neither.

# Analyzing Functions Step-By-Step II

Here is a step-by-step guide to analyzing functions.

- 4 Determine the inflection points. Set  $f''(x) = 0$  and find the solution set.
- 5 Determine the asymptotes. See next slide.
- 6 Determine whether, for all  $x$  in the domain of  $f$ ,  
 $f(x) - f(-x) = 0$  (in which case  $f$  is even) or  
 $f(x) + f(-x) = 0$  (in which case  $f$  is odd).
- 7 Using the information you have, and possibly a table of function values, graph the function. Then determine the domain and range of  $f$ .

# Finding Asymptotes I

An asymptote is a linear function ( $y = kx + d$  with slope  $k$  and  $y$ -intercept  $d$ ) which the function graph of  $f$  approaches. There are three kinds of asymptotes.

## Vertical Asymptote

A vertical asymptote, strictly speaking, is not a linear function. It is a curve defined by  $x = c$ , where  $c$  is a real number (we call real numbers like  $c$  **constants**). You can often find vertical asymptotes at points where  $f$  is undefined.

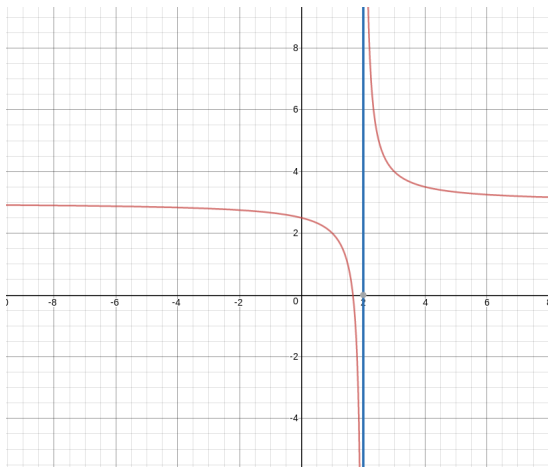
Find vertical asymptotes by checking points which are not in the domain of the function  $f$ .

$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2 \quad (31)$$

# Finding Asymptotes I

Example:

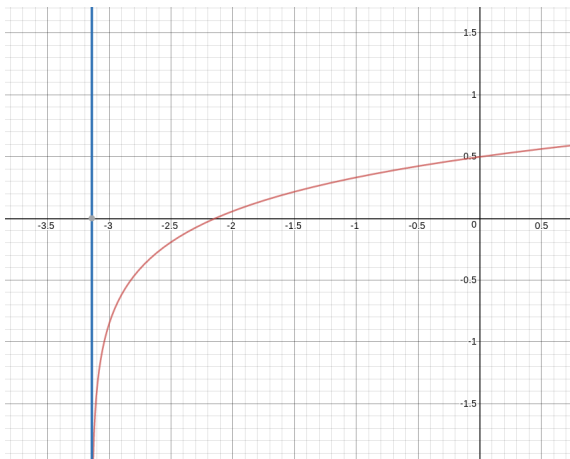
$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2 \quad (32)$$



# Finding Asymptotes I

Example:

$$f(x) = \ln(x + \pi) \text{ has an asymptote at } x = -\pi \quad (33)$$



# Finding Asymptotes II

## Horizontal Asymptote

A horizontal asymptote is a linear function with slope  $k = 0$ . Its equation is  $y = c$ , where  $c$  is a constant. There are horizontal asymptotes for functions whose limits is a constant and for rational functions whose numerator and denominator polynomials share the same degree.

Find horizontal asymptotes by checking

$$\lim_{x \rightarrow \infty} f'(x) \text{ and } \lim_{x \rightarrow -\infty} f'(x) \quad (34)$$

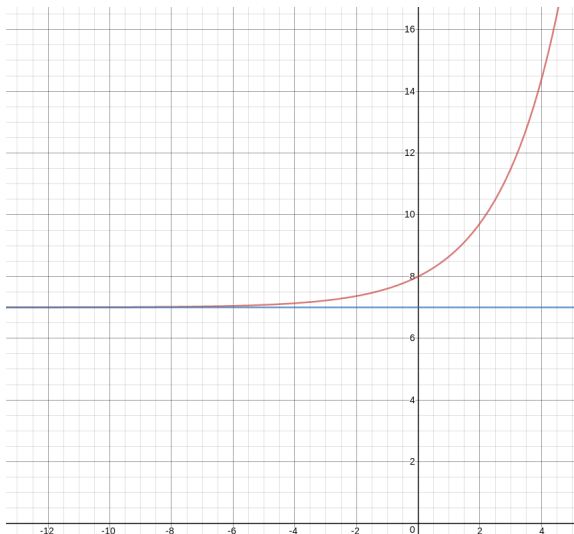
If the limit is  $k = 0$ , then that is also the slope of the asymptote.



# Finding Asymptotes II

Example

$f(x) = e^{\frac{x}{2}} + 7$  has the asymptote  $y = 7$  (35)

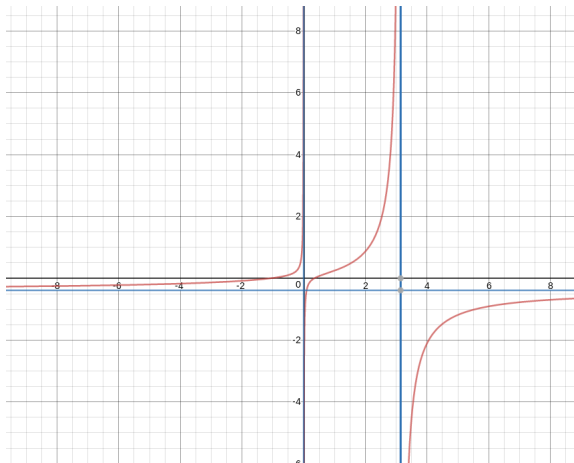


# Finding Asymptotes II

Example (this example additionally has two vertical asymptotes):

$$f(x) = \frac{\pi x^2 + 2x - 1}{-7x^2 + 3x} \text{ has asymptotes } y = -\frac{e}{7}, x = \frac{22}{7}, x = 0$$

(36)



# Finding Asymptotes III

## Sloped Asymptote

A sloped asymptote is a linear function with a positive or a negative slope,  $y = kx + d$  with  $k \neq 0$ . There are sloped asymptotes for rational functions where the numerator polynomial's degree exceeds the denominator polynomial's degree by 1.

Find sloped asymptotes by checking

$$\lim_{x \rightarrow \infty} f'(x) \text{ and } \lim_{x \rightarrow -\infty} f'(x) \quad (37)$$

If the limit is  $k \neq 0$ , then that is also the slope of the asymptote. Hyperbolas also sometimes have sloped asymptotes.

# Finding Asymptotes III

Example:

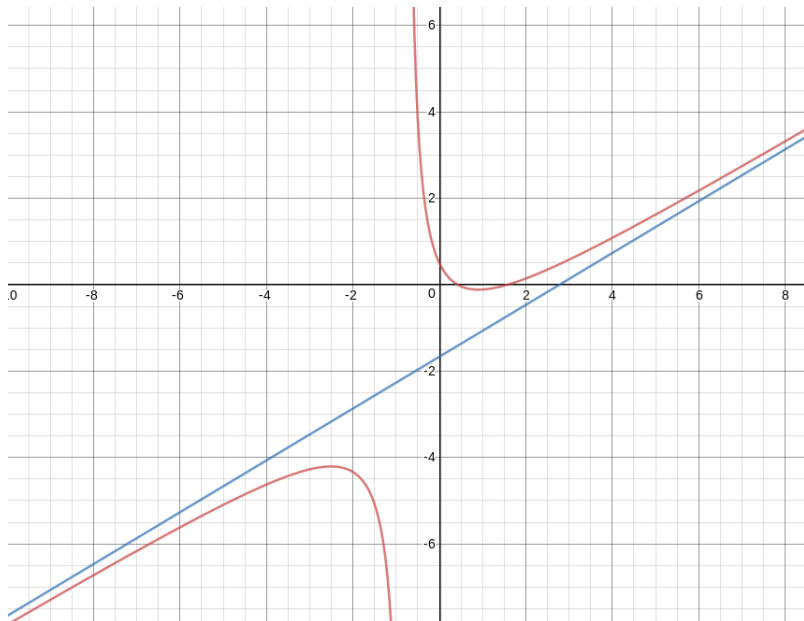
$$f(x) = \frac{3x^2 - 6x + 2}{5x + 4} \text{ has the asymptote } y = \frac{3}{5}x - \frac{5}{3} \quad (38)$$

Find the  $y$ -intercept by making sure that

$$\lim_{x \rightarrow \infty} (f(x) - g(x)) = 0 \quad (39)$$

where  $y = g(x) = kx + d$  for the sloped asymptote. This results in an equation where  $d$  is the only unknown.

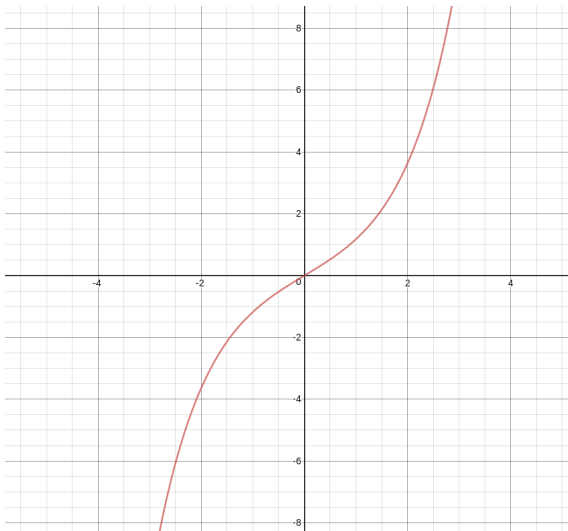
# Finding Asymptotes III



# Finding Asymptotes III

Example:

$$f(\vartheta) = \sinh \vartheta \quad (40)$$



**Exercise 20:** Analyze the following function:

$$g_1(x) = -x^2 + 3x \quad (41)$$

**Exercise 21:** Analyze the following function:

$$g_2(x) = 3x^{\frac{2}{3}} - 2x \quad (42)$$



**Exercise 22:** Analyze the following function:

$$g_3(t) = \frac{2t^2}{t^2 + 3} \quad (43)$$

**Exercise 23:** Analyze the following function:

$$g_4(x) = x^3 e^x \quad (44)$$

# Analyzing Functions Exercises Graph



$$-x^2 + 3 \cdot x$$



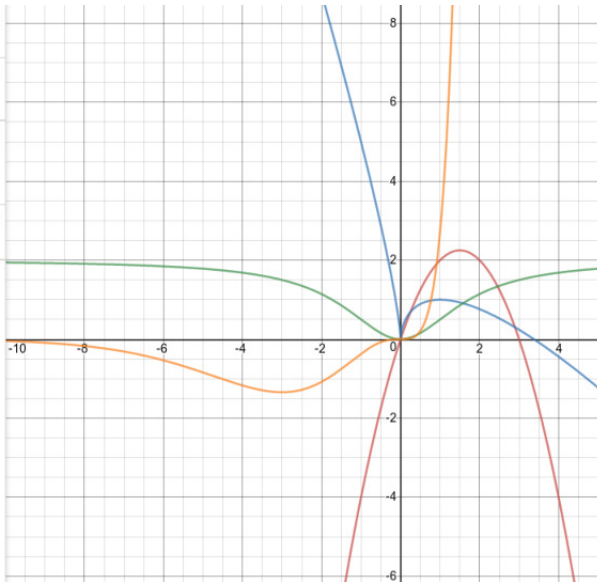
$$3 \cdot x^{\frac{2}{3}} - 2 \cdot x$$



$$\frac{(2 \cdot x^2)}{(x^2 + 3)}$$



$$x^3 \cdot \exp(x)$$



**Exercise 24:** Analyze the following function.

$$f(x) = x \cdot \ln x^2 \quad (45)$$

**Exercise 25:** Analyze the following function.

$$f(x) = \frac{2x^2 + 2}{x - 3} \text{ (do not look for inflection points)} \quad (46)$$

**Exercise 26:** Analyze the following function.

$$f(x) = x^3 + 4x^2 + x - 6 \quad (47)$$

Note that  $x = 1$  is an  $x$ -intercept so that  $(x - 1)$  can be factored as in

$$x^3 + 4x^2 + x - 6 = (x - 1)(x^2 + 5x + 6) \quad (48)$$

**Exercise 27:** Analyze the following function.

$$f(x) = 4 - \frac{e^x + 1}{e^x} \quad (49)$$

**Exercise 28:** Analyze the following function.

$$f(x) = \frac{3x^2 - 5}{x - 2} \quad (50)$$



Next Lesson: Transcendental Functions and Differentials