

# Newton's Method, L'Hôpital's Rule

## MATH 2511, BCIT

Technical Mathematics for Geomatics

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What are the  $x$ -intercepts of the following function?

$$f(x) = 2x^3 + 5x^2 - 11x + 3$$

We have not learned how to find  $x$ -intercepts for polynomials with degrees higher than 2. There are different methods. One method is called **Newton's Method** and approximates the  $x$ -intercept. I have created an instructional video for Newton's Method which you can watch here:

[https://youtu.be/a28M5f0Dk\\_c](https://youtu.be/a28M5f0Dk_c)

# Newton's Method

For Newton's Method, find a plausible  $x$ -value  $x_1$  (near enough to the  $x$ -intercept that you are trying to find) and approximate the  $x$ -intercept using the following iterative procedure:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1)$$

**Exercise 1:** Approximate  $\sqrt{7}$  to ten decimal places using Newton's method and the function  $h(x) = x^2 - 7$ .

**Exercise 2:** Approximate the  $x$ -intercept of  $f(x) = x^3 + 5x - 3$  using Newton's method.

**Exercise 3:** Factor  $g(t) = 24t^3 - 2t^2 - 9t + 2$ . Remember that if  $x_1, x_2, x_3$  are  $x$ -intercepts of the polynomial  $ax^3 + bx^2 + cx + d$ , then

$$ax^3 + bx^2 + cx + d = a(x - x_1)(x - x_2)(x - x_3) \quad (2)$$

**Exercise 4:** Find the  $x$ -intercepts for the following function:

$$f(x) = x^3 + 4x^2 + x - 6 \quad (3)$$

**Exercise 5:** Solve the equation

$$\cos x = x \quad (4)$$

using Newton's Method.



**Exercise 6:** Analyze the following function:

$$f(x) = \frac{2x^2 + 2}{x - 3} \quad (5)$$

**Exercise 7:** Solve the following equations using Newton's Method. Use a graphing calculator to get you started.

$$x^6 - x^5 - 6x^4 - x^2 + x + 10 = 0 \quad (6)$$

$$x^2(4 - x^2) = \frac{4}{x^2 + 1} \quad (7)$$

$$x^2\sqrt{2 - x - x^2} = 1 \quad (8)$$

$$4e^{-x^2} \sin x = x^2 - x + 1 \quad (9)$$

$$3 \sin(x^2) = 2x \quad (10)$$

**Exercise 8:** Find the absolute minimum value of the following function correct to four decimal places.

$$f(x) = x^6 - x^4 + 3x^2 - 2x \quad (11)$$

**Exercise 9:** Of the infinitely many lines that are tangent to the curve

$$y = -\sin x \quad (12)$$

and pass through the origin, there is one that has the largest slope. Use Newton's Method to find the slope of that line.

**Exercise 10:** Use Newton's Method to find the coordinates of the point on the parabola

$$y = (x - 1)^2 \quad (13)$$

that is closest to the origin.

# L'Hôpital's Rule

Evaluate the following limits:

$$\lim_{x \rightarrow 1} \frac{x^2 - x}{x^2 - 1} \quad (14)$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \text{ (it equals 1 based on geometry)} \quad (15)$$

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{2x^2 + 1} \quad (16)$$

These limits have in common that they are of **indeterminate form** when you plug in the  $a$  towards which the  $x$  goes. Sometimes the tricks we have found don't work, for example for

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} \quad (17)$$

or for

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x - 1} \quad (18)$$

# L'Hôpital's Rule

Suppose  $f$  and  $g$  are differentiable and  $g'(x) \neq 0$  on an open interval  $I$  that contains  $a$  (except possibly at  $a$ ). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

or that

$$\lim_{x \rightarrow a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type  $\frac{0}{0}$  or  $\infty/\infty$ .) Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is  $\infty$  or  $-\infty$ ).

**Exercise 11:** Find

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} \quad (19)$$



**Exercise 12:** Find

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} \quad (20)$$

**Exercise 13:** Find

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt[3]{x}} \quad (21)$$

**Exercise 14:** Find

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} \quad (22)$$

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} \stackrel{\text{LHR}}{=} \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} = \frac{0}{0} \quad (23)$$

$$\lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} \stackrel{\text{LHR}}{=} \lim_{x \rightarrow 0} \frac{2 \tan x \sec^2 x}{6x} = \frac{0}{0} \quad (24)$$

$$\lim_{x \rightarrow 0} \frac{\tan x \sec^2 x}{3x} = \lim_{x \rightarrow 0} \frac{\sec^4 x + 2 \tan^2 x \sec^2 x}{3} = \frac{1}{3} \quad (25)$$

**Exercise 15:** Find

$$\lim_{x \rightarrow \pi} \frac{\pi - \pi \cos x + \sin x}{1 - \cos x} \quad (26)$$

**Exercise 16:** Find

$$\lim_{x \rightarrow 0^+} x \ln x \quad (27)$$

**Exercise 17:** Find

$$\lim_{x \rightarrow \frac{\pi}{2}^-} (\sec x - \tan x) \quad (28)$$

**Exercise 18:** Find

$$\lim_{x \rightarrow 0^+} (1 + \sin 4x)^{\cot x} \quad (29)$$



**Exercise 19:** Find

$$\lim_{x \rightarrow 0^+} x^x \quad (30)$$

**Exercise 20:** Find

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x) \quad (31)$$

Note: this is an example where l'Hôpital's rule does not work and another method has to be used. This is not unusual for radicals. Further examples include

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} \text{ and } \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) \quad (32)$$

# Exercise Solution

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x) = \lim_{x \rightarrow \infty} (\sqrt{x(x+1)} - x) = \lim_{x \rightarrow \infty} (\sqrt{x}(\sqrt{x+1} - \sqrt{x})) \quad (33)$$

Multiply by the conjugate for

$$\lim_{x \rightarrow \infty} (\sqrt{x}(\sqrt{x+1} - \sqrt{x})) = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{x+1} + \sqrt{x}} \quad (34)$$

Take the reciprocal for

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{x+1} + \sqrt{x}} = \left( \lim_{x \rightarrow \infty} \left( 1 + \sqrt{\frac{x+1}{x}} \right) \right)^{-1} = \frac{1}{2} \quad (35)$$

**Exercise 21:** Find

$$\lim_{x \rightarrow 1} \frac{1 - x + \ln x}{1 + \cos \pi x} \quad (36)$$

**Exercise 22:** Find

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \quad (37)$$

**Exercise 23:** Find

$$\lim_{x \rightarrow 0^+} \sin x \ln x \quad (38)$$

**Exercise 24:** Find

$$\lim_{x \rightarrow 1} \left( \frac{x}{x-1} - \frac{1}{\ln x} \right) \quad (39)$$

# End of Lesson

Next Lesson: Fundamental Theorem of Calculus