# Limits

## MATH 2511, BCIT

Calculus for Geomatics

January 8, 2018

# Limits Introduction

Consider the function graph of the following function.

$$f(x) = \frac{x^2 - 1}{x - 1} \tag{1}$$

It looks like it is a linear equation! However, at $x = 1$, $f(x)$ is not defined. To fill the hole, we define the limit

$$\lim_{x \to a} f(x) = w \text{ if and only if } w = L = R \tag{2}$$

where $L$ is the number that the function $f$ approaches as $x$ gets closer to $a$ with $x < a$ (that means $x \neq a$!); and $R$ is the number that the function $f$ approaches as $x$ gets closer to $a$ with $x > a$. Note: for a mathematically rigorous definition of what "approaching" and "getting closer" means we would need to talk about sequences and series, which is a topic we won't cover here.

# Indeterminate Form I

Notice that

$$f(x) = \frac{x^2 - 1}{x - 1} \overset{x=1}{=} \frac{0}{0} \tag{3}$$

We call this an indeterminate form.

# Indeterminate Form

Notice that except at $x = 1$

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x + 1 = g(x) \tag{4}$$

$f$ and $g$ agree everywhere except on $x = 1$. Consider the following rule,

**One Disagreement Rule**

If $f = g$ except in one point, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ for all $a$, even the $a$ where $f$ and $g$ disagree.

Therefore

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1}(x + 1) = 2 \tag{5}$$

# Continuity

Consider a simple function like $f(x) = x^3$. What is $\lim_{x \to 4} f(x)$? The answer is almost trivial,

$$\lim_{x \to 4} f(x) = f(4) = 4^3 = 64 \tag{6}$$

Why is this true? Because $f$ is continuous at $x = 4$. There are no holes, jumps, gaps, or breaks of the function graph at $x = 4$. Constant functions, the identity function, linear functions, polynomial functions, exponential and logarithmic functions are all continuous. Rational functions, some trigonometric functions, and other functions are sometimes not continuous.

# Interesting Cases

A function is continuous if and only if $\lim_{x \to c} f(x) = f(c)$ for all $c$ in $\mathbb{R}$ (the logarithmic function is continuous only on $\mathbb{R}^+$). This means that (i) the function needs to be defined at $x = c$; (ii) the limit needs to be defined at $x = c$; and (iii) the function value and the limit need to be equal to each other.
Consider the following interesting cases:

1. A function that is continuous and well defined at $x = a$.
2. A function that is not continuous at $x = a$.
3. A function where the limit exists but $\lim_{x \to c} \neq f(c)$.
4. A function such as $f(x) = \sin(1/x)$.

# Interesting Cases

A function is continuous if and only if $\lim_{x \to c} f(x) = f(c)$ for all $c$ in $\mathbb{R}$ (the logarithmic function is continuous only on $\mathbb{R}^+$). This means that (i) the function needs to be defined at $x = c$; (ii) the limit needs to be defined at $x = c$; and (iii) the function value and the limit need to be equal to each other.

Consider the following interesting cases:

1. A function that is continuous and well defined at $x = a$.
2. A function that is not continuous at $x = a$.
3. A function where the limit exists but $\lim_{x \to c} \neq f(c)$.
4. A function such as $f(x) = \sin(1/x)$.

# Interesting Cases

A function is continuous if and only if $\lim_{x \to c} f(x) = f(c)$ for all $c$ in $\mathbb{R}$ (the logarithmic function is continuous only on $\mathbb{R}^+$). This means that (i) the function needs to be defined at $x = c$; (ii) the limit needs to be defined at $x = c$; and (iii) the function value and the limit need to be equal to each other.

Consider the following interesting cases:

1. A function that is continuous and well defined at $x = a$.
2. A function that is not continuous at $x = a$.
3. A function where the limit exists but $\lim_{x \to c} \neq f(c)$.
4. A function such as $f(x) = \sin(1/x)$.

# Interesting Cases

A function is continuous if and only if $\lim_{x \to c} f(x) = f(c)$ for all $c$ in $\mathbb{R}$ (the logarithmic function is continuous only on $\mathbb{R}^+$). This means that (i) the function needs to be defined at $x = c$; (ii) the limit needs to be defined at $x = c$; and (iii) the function value and the limit need to be equal to each other.

Consider the following interesting cases:

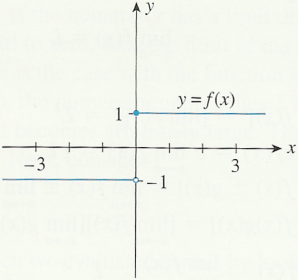
1. A function that is continuous and well defined at $x = a$.
2. A function that is not continuous at $x = a$.
3. A function where the limit exists but $\lim_{x \to c} \neq f(c)$.
4. A function such as $f(x) = \sin(1/x)$.

# No Limit Examples I

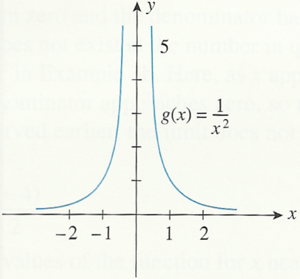
**EXAMPLE** Evaluate the limit of the following functions as $x$ approaches the indicated point.

**a.** $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}; \; x = 0$ **b.** $g(x) = \dfrac{1}{x^2}; \; x = 0$

**Solution** The graphs of the functions $f$ and $g$ are shown in Figure 29.

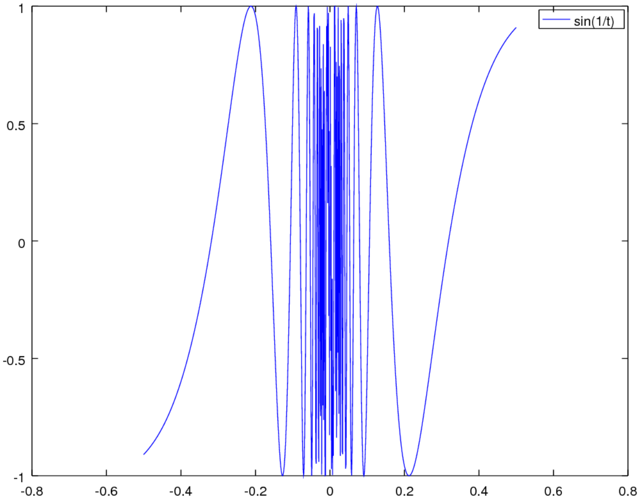



(a) $\lim_{x \to 0} f(x)$ does not exist.




(b) $\lim_{x \to 0} g(x)$ does not exist.

# No Limit Example II



# Properties of Limits

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

$$\lim_{x\to a}[f(x)]^r = L^r, r \text{ a real number} \tag{7}$$

$$\lim_{x\to a}[c \cdot f(x)] = c \cdot L, c \text{ a real number} \tag{8}$$

$$\lim_{x\to a}[f(x) \pm g(x)] = L \pm M \tag{9}$$

$$\lim_{x\to a}[f(x)g(x)] = LM \tag{10}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}, \text{provided that } M \neq 0 \tag{11}$$

# Properties of Limits Exercises

Use the properties of limits to evaluate the following,

$$\lim_{x \to 2} x^3 \tag{12}$$

$$\lim_{x \to 4} 5x^{3/2} \tag{13}$$

$$\lim_{x \to 1} \left(5x^4 - 2\right) \tag{14}$$

$$\lim_{x \to 3} 2x^3 \sqrt{x^2 + 7} \tag{15}$$

$$\lim_{x \to 2} \frac{2x^2 + 1}{x + 1} \tag{16}$$

# Another Indeterminate Form Example I

Here is an example where by skillful manipulation we can determine the limit even though at first the limit is in indeterminate form. Let

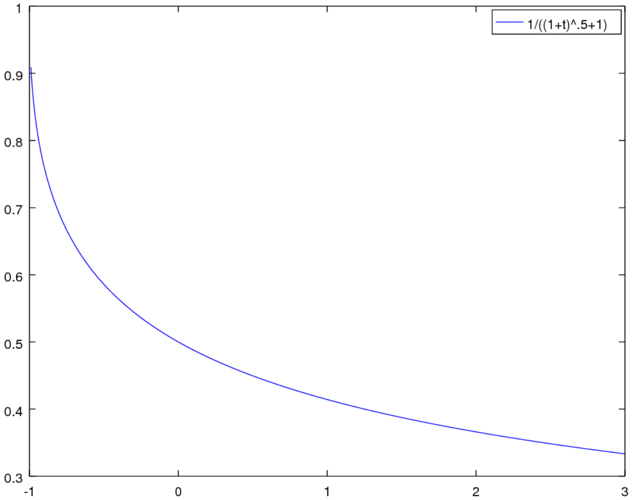
$$f(x) = \frac{\sqrt{1+x}-1}{x} \tag{17}$$

What is $\lim_{x\to 0} f(x)$? If we leave the fraction unchanged, it will give us an indeterminate form. However, if we multiply both numerator and denominator by $(\sqrt{1+x}+1)$, we avoid the indeterminate form!

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2} \tag{18}$$

Look at the function graph of $f(x)$ to verify that this is the correct limit.

# Another Indeterminate Form Example II



# Limits at Infinity I

Sometimes, we want to know what happens to a function graph when either $x$ or $-x$ get very large. We use the infinity sign $\infty$ for notation, but note that we do NOT use infinity to define these limits.

$$\lim_{x \to \infty} f(x) = w \text{ if and only if } w = S \tag{19}$$

where $S$ is a number such that for any tiny number $\varepsilon$ there is a real number $x_0$ and $|f(x) - S| < \varepsilon$ for all $x > x_0$.

# Limits at Infinity II

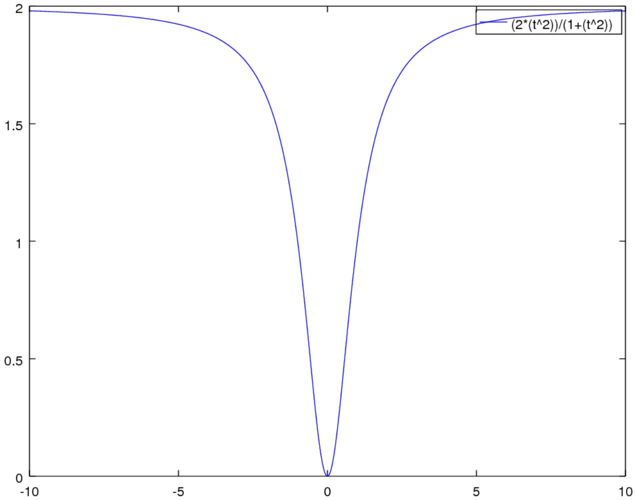
For example, what is

$$\lim_{x \to \infty} \frac{2x^2}{1+x^2} \tag{20}$$

or

$$\lim_{x \to -\infty} \frac{2x^2}{1+x^2} \tag{21}$$

# Limits at Infinity III



# Limits at Infinity IV

Here is another important property of limits. If $1/x^n$ is defined and $n > 0$, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \text{ and } \lim_{x\to-\infty} \frac{1}{x^n} = 0 \tag{22}$$

# Polynomial and Rational Functions

A polynomial function looks like this,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \qquad (23)$$

For example, $p(x) = 7x^3 - 4.7x^2 + 6$. $n > 0$ is a natural number, and the $a_i$ are called coefficients. They are real numbers. A rational function looks like this,

$$q(x) = \frac{p_1(x)}{p_2(x)} \qquad (24)$$

where $p_1(x)$ and $p_2(x)$ are polynomial functions. For example,

$$q(x) = \frac{5x^2 - \pi x + 9000}{e^2 x + 2} \qquad (25)$$

# Limits at Infinity V

When we are looking for the limit of rational functions as they go to negative or positive infinity, we often get an indeterminate form.

$$\lim_{x\to\infty} \frac{x^2 - x + 3}{2x^3 + 1} = \frac{\infty}{\infty} \tag{26}$$

Here is a technique that will almost always work. Divide both the numerator and the denominator by $x^m$, where $m$ is the highest exponent you can find.

$$\lim_{x\to\infty} \frac{x^2 - x + 3}{2x^3 + 1} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x^2} + \frac{3}{x^3}}{2 + \frac{1}{x^3}} = \frac{0}{2} = 0 \tag{27}$$

# Limits at Infinity VI

Here are two more examples.

$$\lim_{x\to-\infty}\frac{3x^2+8x-4}{2x^2+4x-5}=\lim_{x\to-\infty}\frac{3-\frac{8}{x}-\frac{4}{x^2}}{2+\frac{4}{x}-\frac{5}{x^2}}=\frac{3}{2}=1.5 \tag{28}$$

$$\lim_{x\to\infty}\frac{2x^3-3x^2+1}{x^2+2x+4}=\lim_{x\to\infty}\frac{2-\frac{3}{x}+\frac{1}{x^3}}{\frac{1}{x}+\frac{2}{x^2}+\frac{4}{x^3}}=\frac{2}{0}=\text{undefined} \tag{29}$$

In the second example, the limit does not exist. Sometimes, we write $\lim_{x\to a}=\infty$ or $\lim_{x\to a}=-\infty$, depending on which way the function goes.

# Example I

Consider the function,

$$f(x) = \frac{x-4}{\sqrt{x}-2} \tag{30}$$

Let's find

$$\lim_{x \to 4} f(x) \tag{31}$$

# Example II

First, fill out the table:

| | | | |
|---|---|---|---|
| $x = 3$ | $f(x) = 3.7321$ | $x = 5$ | $f(x) = 4.2361$ |
| $x = 3.5$ | | $x = 4.5$ | |
| $x = 3.75$ | | $x = 4.25$ | |
| $x = 3.9$ | | $x = 4.1$ | |
| $x = 3.95$ | | $x = 4.05$ | |
| $x = 3.99$ | | $x = 4.01$ | |

# Example III

Next, let's assume that $x \neq 4$ and expand both the numerator and denominator by $\sqrt{x} + 2$. Simplify

$$g(x) = \frac{(x-4) \cdot (\sqrt{x} + 2)}{(\sqrt{x} - 2) \cdot (\sqrt{x} + 2)} \text{ on domain } \mathbb{R} \setminus \{4\} \qquad (32)$$

Except on $x = 4$, $g$ agrees with $f$. Determine $\lim_{x \to 4} g(x)$.

# Exercises I

Evaluate the following two limits.

$$\lim_{x\to 3} = \frac{\sqrt{x^2+7}+\sqrt{3x-5}}{x+2} \tag{33}$$

$$\lim_{x\to -1} \frac{x^2-x-2}{2x^2-x-3} \tag{34}$$

# Exercises II

Evaluate the following three limits.

$$\lim_{x \to 2} 3 \tag{35}$$

$$\lim_{x \to \infty} \frac{3x + 2}{x - 5} \tag{36}$$

$$\lim_{x \to \infty} \frac{x^5 - x^3 + x - 1}{x^6 + 2x^2 + 1} \tag{37}$$

Find the following limits,

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \tag{38}$$

$$\lim_{x \to \infty} \frac{\sqrt{x^2 - 8x}}{2x + 1} \tag{39}$$

$$\lim_{x \to -1} \frac{x^2 - x - 2}{2x^2 - x - 3} \tag{40}$$

$$\lim_{x \to \infty} \frac{2 + \frac{1}{x+4}}{3 - \frac{1}{x^2}} \tag{41}$$

$$\lim_{x \to \infty} \frac{x - 2x^3}{(1 + x)^3} \tag{42}$$

$$\lim_{x \to \infty} \frac{\sqrt{4x^2 + 3}}{x + 5} \tag{43}$$

In Einstein's theory of relativity, the length $L$ of an object moving at a velocity $v$ is

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} \tag{44}$$

where $c$ is the speed of light and $L_0$ is the length of the object at rest. What is the one-sided limit of $L$ as $v$ gets faster and faster?

# Squeeze Theorem

Sometimes you need some ingenuity to find a limit. Consider

$$\lim_{x \to 0} \frac{\sin x}{x} \tag{45}$$

# Squeeze Theorem

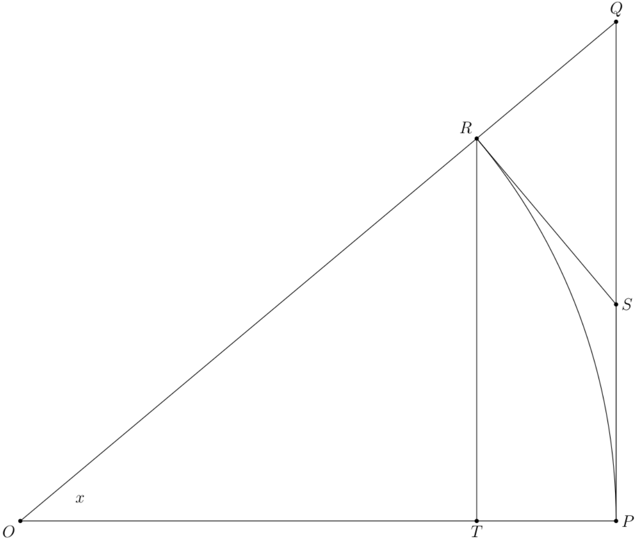
If $f(x) \leq g(x) \leq h(x)$ when $x$ is near $a$ (except possibly at $x = a$) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \tag{46}$$

then

$$\lim_{x \to a} g(x) = L \tag{47}$$

# Squeeze Theorem



# Squeeze Theorem

In the previous slide, consider the unit circle with $\|\vec{OP}\| = \|\vec{OR}\| = 1$ and the angle $x$ at $O$. For simplicity let's assume that $0 < x < \pi/2$. The angle $x$ is also the length of the arc between $P$ and $R$. Consequently

$$\|\vec{RT}\| = \sin x \leq x \tag{48}$$

and therefore

$$\frac{\sin x}{x} \leq 1 \tag{49}$$

# Squeeze Theorem

Now consider

$$x \leq \|\vec{PS}\| + \|\vec{SR}\| \leq \|\vec{PS}\| + \|\vec{SQ}\| = \|\vec{PQ}\| = \tan x \tag{50}$$

$\|\vec{SR}\| \leq \|\vec{SQ}\|$ because the angle $QRS$ is a right angle. (50) means that

$$\cos x \leq \frac{\sin x}{x} \tag{51}$$

Since $\lim_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} 1 = 1$, we can use the squeeze theorem, (49), and (51) for

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{52}$$

Here is a summary of methods to use to find limits.

1. If a function $f$ is continuous, then $\lim_{x\to a} f(x) = f(a)$.
2. If a function is composed of continuous functions, use the properties of limits (Theorem 1) to find the limit.
3. If the last step gives you an indeterminate form, try to factor either the numerator or the denominator and use the One Disagreement Rule. Example: $\lim_{x\to -2}[(x^2 - x - 6)/(x+2)] = -5$.
4. If there is a square root in a fraction, another thing to try is to multiply both numerator and denominator by the conjugate. Example: $\lim_{x\to -2}[(x-4)/(\sqrt{x}-2)] = 4$.
5. For rational functions, use the property of limits involving $x^{-n}$. Example: $\lim_{x\to\infty}(5x^3 - x^2 - x + 3)/(2x^3+1) = 5/2$.

# End of Lesson

Next Lesson: Derivatives