Area and Volume

MATH 2511, BCIT

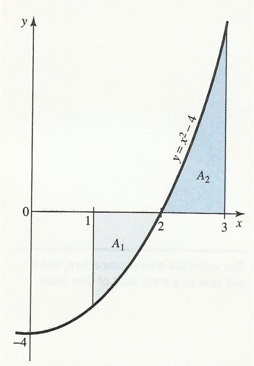
Technical Mathematics for Geomatics

March 21, 2018

Negative Area I

Consider the following problem.

Find the area under the curve $y = x^2 - 4$ between $x = 1$ and $x = 3$.



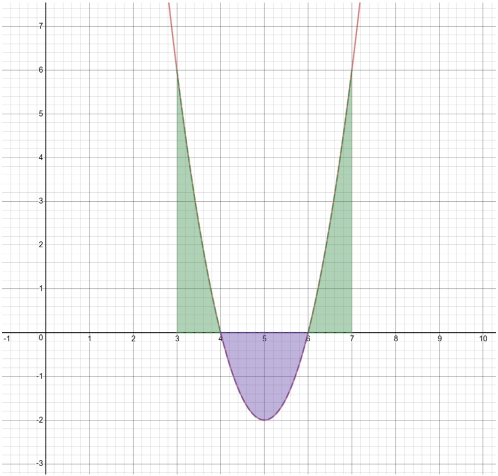
Negative Area II

To solve this problem, find the x-intercept and treat the positive and negative area separately.

$$|A_1|+|A_2| = -\int_1^2 (x^2-4)\,dx + \int_1^2 (x^2-4)\,dx = -\left(-\frac{5}{3}\right)+\frac{7}{3} = 4$$

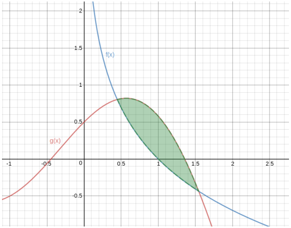
Negative Area Exercise

Find the area between the curve $y = 2(x-5)^2 - 2$ and the x-axis between $x = 3$ and $x = 7$.



Area Between Curves

Find the area bounded by the curves $f(x)$ and $g(x)$.



To find this area, solve for the two solutions x_1, x_2 of

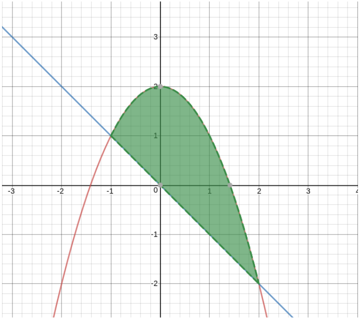
$$f(x) = g(x) \tag{1}$$

(you may have to use Newton's method) and then integrate

$$A = \int_{x_1}^{x_2} (g(x) - f(x))\ dx \tag{2}$$

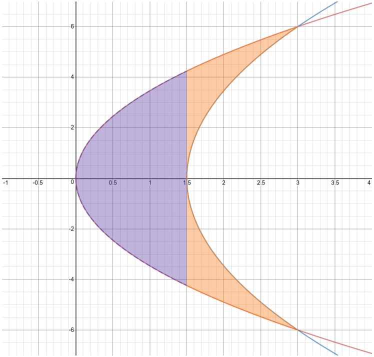
Area Between Curves Exercise

Exercise 1: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



Integrating Along the y-Axis

Find the area bounded by the curves $y^2 = 12x$ and $y^2 = 24x - 36$.



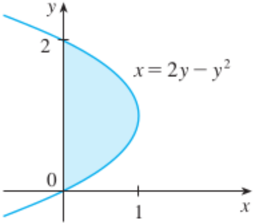
In this case, it is more efficient to integrate over y.

$$A = 2 \cdot \int_0^6 \left(\frac{y^2}{24} + \frac{36}{24} - \frac{y^2}{12} \right) \, dy \tag{3}$$

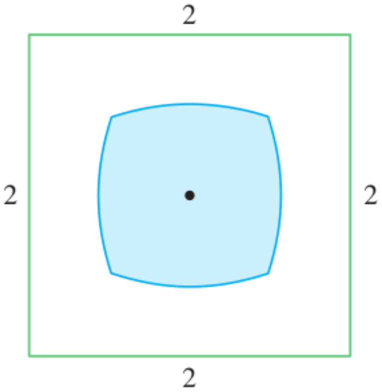
Exercise 2: The area of the region that lies to the right of the y-axis and to the left of the parabola $x = 2y - y^2$ (the shaded region in the figure) is given by the integral

$$\int_0^2 (2y - y^2)\, dy \tag{4}$$

Find the area of the region.



Exercise 3: The figure shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region. (This is a difficult problem. Only try it if you are looking for a challenge.)



Finding an Area Example

Hint 1 Think of the curve to integrate in terms of the diagram on the next slide.

Hint 2 The definite integral is

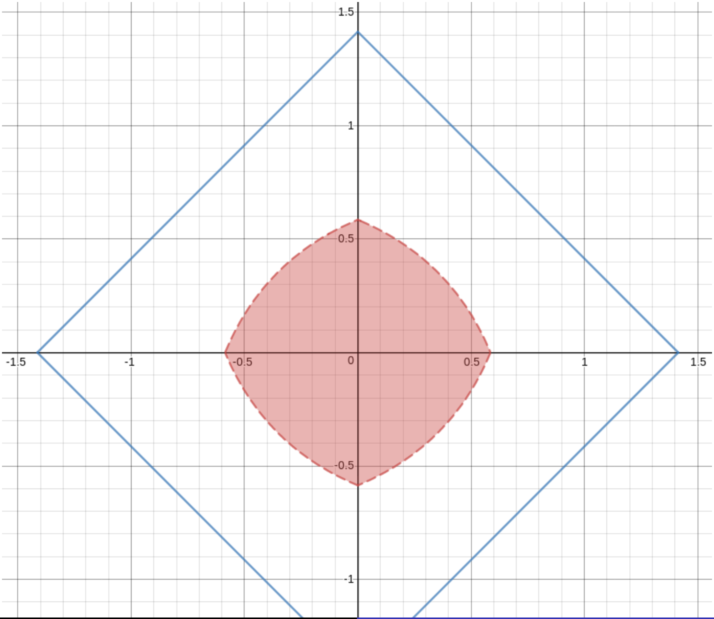
$$A = 4\int_0^{2-\sqrt{2}} \left(x - \sqrt{2} + 2\sqrt{1-\sqrt{2}x}\right)\, dx \qquad (5)$$

Hint 3 Use (from an integral table)

$$\int \sqrt{ax+b}\, dx = \frac{2\,(ax+b)^{\frac{3}{2}}}{3a} + C \qquad (6)$$

The solution is approximately $A = 0.87581$.

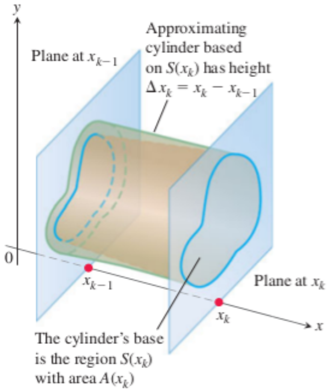
Finding an Area Example



Volume of Cross-Sections

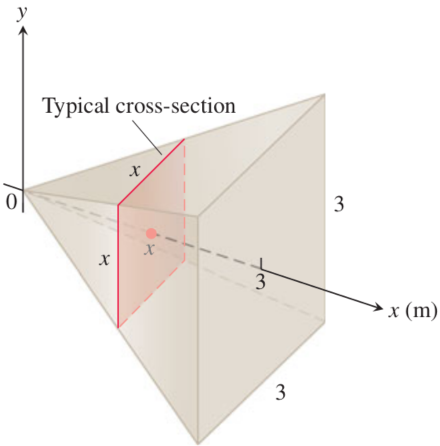
The volume of a solid integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx \tag{7}$$



Volume of Cross-Sections Exercise

Exercise 4: A pyramid three metres high has a square base that is 3 metres on a side. The cross-section of the pyramid perpendicular to the altitude x metres down from the vertex is a square, whose side is x metres. Find the volume of the pyramid.



Cavalieri's Principle

Cavalieri's Principle

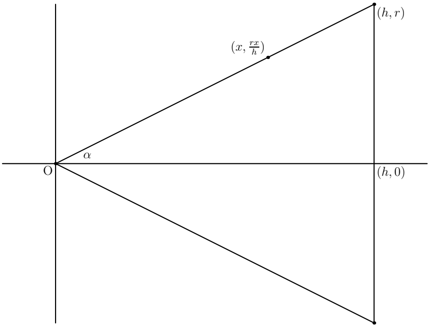
Suppose two regions in three-dimensional space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

Disk Method

Remember the formula for the volume of a cone:

$$V = \frac{1}{3} r^2 \pi h \tag{8}$$

Let's see if we can give a reason for the formula using calculus. Let the height of a cone be h and the radius r.



Disk Method

Using the volume of cross-sections formula,

$$A(x) = \left(\frac{rx}{h}\right)^2 \cdot \pi \tag{9}$$

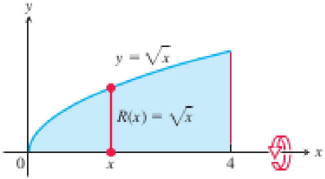
and therefore

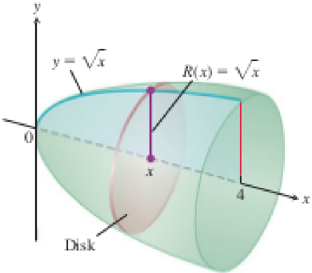
$$V = \int_0^h A(x)\,dx = \left[\frac{r^2 \pi x^3}{3h^2}\right]_0^h = \frac{1}{3} r^2 \pi h \tag{10}$$

More generally, any integrable function rotated around the x-axis gives us the volume of a solid by the so-called disk method (see the diagram on the next slide),

$$V = \int_a^b \pi [R(x)]^2\,dx \tag{11}$$

Exercise 5: The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

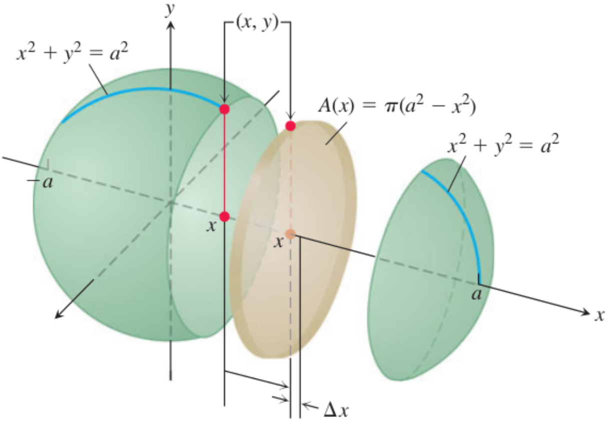




Exercise 6: The circle

$$x^2 + y^2 = r^2 \tag{12}$$

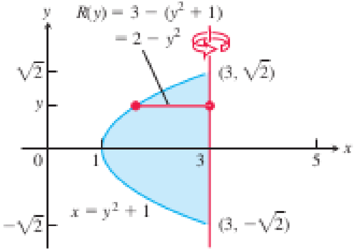
is rotated about the x-axis to generate a sphere. Find its volume.

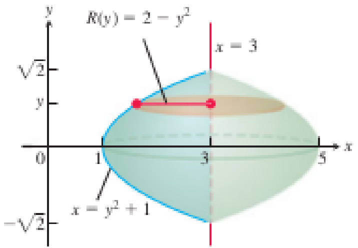


Exercise 7: Find the volume of the solid generated by revolving the region between the parabola

$$x = y^2 + 1 \tag{13}$$

and the line $x = 3$ about the line $x = 3$.

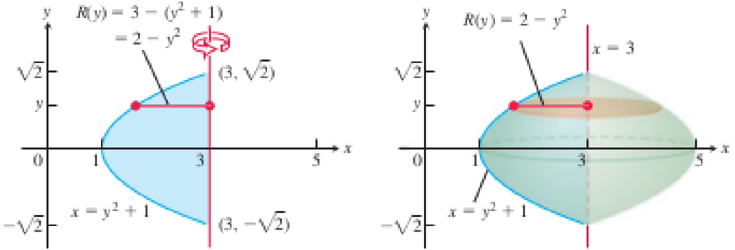




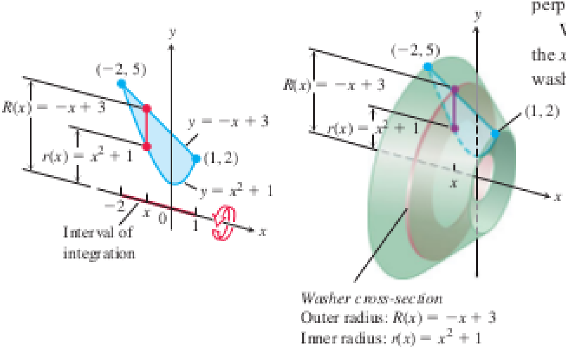
Exercise 7: Find the volume of the solid generated by revolving the region between the parabola

$$x = y^2 + 1 \tag{13}$$

and the line $x = 3$ about the line $x = 3$. The solution is $(1/15) \cdot 64\pi\sqrt{2}$.



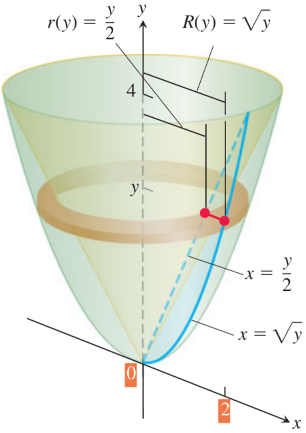
Exercise 8: The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved around the x-axis to generate a solid. Find the volume of the solid.



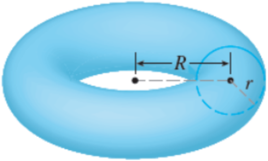
The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. When the region is revolved about the x-axis, the line segment generates a washer.

Washer Method Exercise

Exercise 9: The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.



Exercise 10: Find the volume of the solid torus with major radius R and minor radius r.



Solid Torus Volume

Define the following two functions:

$$f_1(x) = R + \sqrt{r^2 - x^2} \tag{14}$$

$$f_2(x) = R - \sqrt{r^2 - x^2} \tag{15}$$

Use the washer method for the following volume calculation,

$$V = \pi \int_{-r}^{r} \left(\left(R + \sqrt{r^2 - x^2} \right)^2 - \left(R - \sqrt{r^2 - x^2} \right)^2 \right) dx \tag{16}$$

Solid Torus Volume

Simplify the last equation to

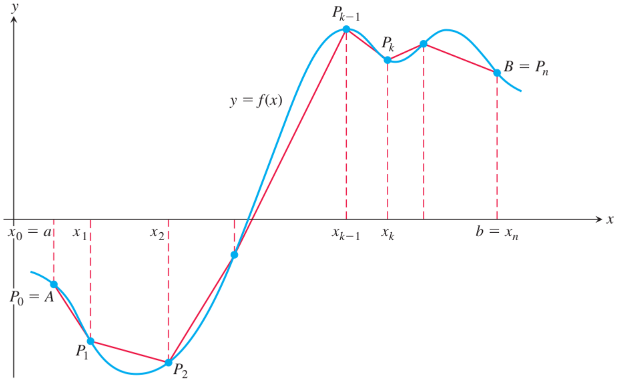
$$V = 4\pi R \int_{-r}^{r} \left(\sqrt{r^2 - x^2} \right) \, dx \tag{17}$$

The integral is just the area of a semicircle! Therefore,

$$V = 2\pi R r^2 \tag{18}$$

Arc Length

A curve $y = f(x)$ is called smooth on an interval $[a, b]$ if $f(x)$ has a continuous derivative at every point of $[a, b]$. The length of the polygonal path $P_0 P_1 P_2 \cdots P_n$ approximates the length of the curve $y = f(x)$ from point A to point B.



Arc Length

As we did for the fundamental theorem of calculus, divide up the interval $[a, b]$ into intervals of equal length $[x_i, x_{i+1}]$, where $i = 0, \ldots, n-1$ and $a = x_0, b = x_n$. Then the length of the curve $y = f(x)$ from a to b is

$$L = \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{(f(x_{i+1}) - f(x_i))^2 + (x_{i+1} - x_i)^2} \tag{19}$$

The mean value theorem tells us that there is always a point x_i^* between x_i and x_{i+1} such that

$$f'(x_i^*) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \tag{20}$$

Arc Length

Consequently,

$$L = \lim_{n \to \infty} \sum_{i=0}^{n-1} \sqrt{\left(f'(x_i^*)\,(x_{i+1} - x_i)\right)^2 + (x_{i+1} - x_i)^2} \tag{21}$$

which is equivalent to

$$L = \lim_{n \to \infty} \sum_{i=0}^{n-1} (x_{i+1} - x_i)\,\sqrt{\left(1 + f'(x_i^*)\right)^2} \tag{22}$$

Now let g be the function

$$g(x) = \sqrt{1 + (f'(x))^2} \tag{23}$$

Arc Length

Then

$$L = \lim_{n\to\infty} \sum_{i=0}^{n-1} (x_{i+1} - x_i)\, g(x_i^*) \tag{24}$$

We already know from the fundamental theorem of calculus that this is

$$L = \int_a^b g(x)\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx \tag{25}$$

This is our formula for arc length.

Arc Length Exercise

Exercise 11: Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1, 0 \leq x \leq 1 \tag{26}$$

Check the plausibility of your result by approximating the curve length calculating the straight-line distance between the two end points.

Exercise 12: Find the length of the curve

$$y = \frac{x^3}{12} + \frac{1}{x}, 1 \leq x \leq 4 \tag{27}$$

Check the plausibility of your result by approximating the curve length calculating the straight-line distance between the two end points.

Arc Length Exercise

Exercise 13: A telephone line hangs between two poles 14 metres apart in the shape of a catenary

$$y = 20\cosh\left(\frac{x}{20}\right) - 15 \tag{28}$$

where x and y are measured in metres. Find the length of telephone wire needed between the two poles.

Exercise 13: A telephone line hangs between two poles 14 metres apart in the shape of a catenary

$$y = 20\cosh\left(\frac{x}{20}\right) - 15 \tag{28}$$

where x and y are measured in metres. Find the length of telephone wire needed between the two poles. The answer is $20(\sinh(7/20) - \sinh(-7/20)) = 14.288$.

Surface Area

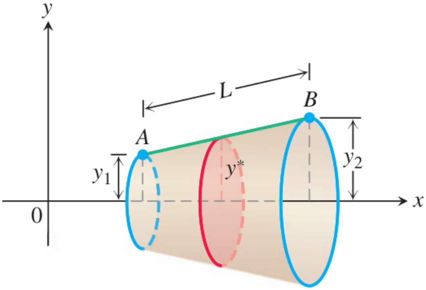
If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx \tag{29}$$

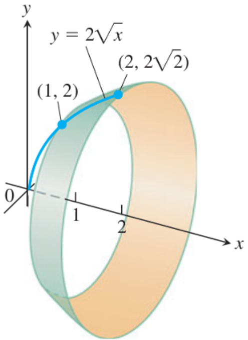
Exercise 14: Show that the lateral surface area of a frustum (without base and top) is

$$S = 2\pi y^* L \tag{30}$$

where y^* is the average height of AB above the x-axis and L is the length of AB.

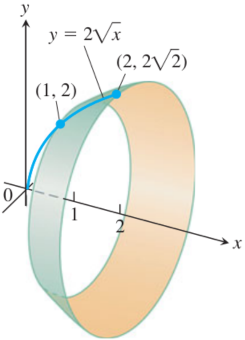


Exercise 15: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis.

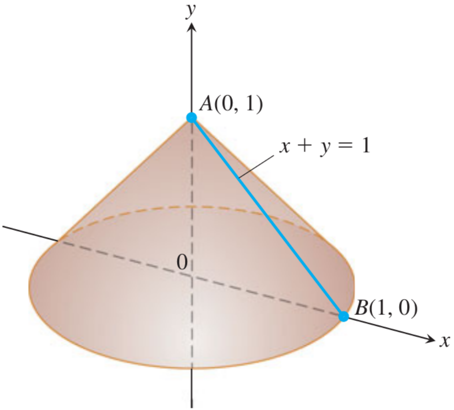


Surface Area Exercise

Exercise 15: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis. The solution is $(8\pi/3) \cdot (\sqrt{27} - \sqrt{8}) = 19.836$.



Exercise 16: The line segment $x = 1 - y, 0 \leq y \leq 1$, is revolved around the y-axis to generate a cone. Find its lateral surface area (which excludes the base area).



End of Lesson

Next Lesson: Integration Methods