Differentiating Trigonometric Functions

MATH 2511, BCIT

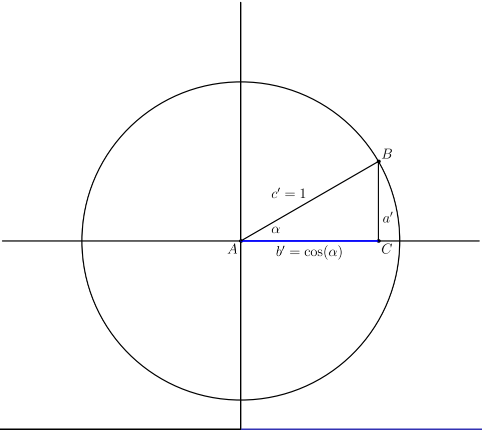
Technical Mathematics for Geomatics

January 22, 2018

Make sure to remember that the trigonometric functions (sine, cosine, tangent, cotangent, etc.) are functions from the real numbers into the real numbers. An angle is a real number in terms of its radian measure. If the angle is in degrees, it can be converted to radians as in the following example,

$$42^\circ = 42 \cdot \frac{\pi}{180} \approx 0.73304 \tag{1}$$

Any right triangle whose hypotenuse is of length $c' = 1$ can be inserted into the unit circle so that one of the two shorter sides rests on the x-axis and one of the vertices is at the origin (reference triangle). Then the vertex B in the diagram has the coordinates $(\cos\alpha, \sin\alpha)$.



Trigonometric Functions Review

The remaining trigonometric functions are defined as follows.

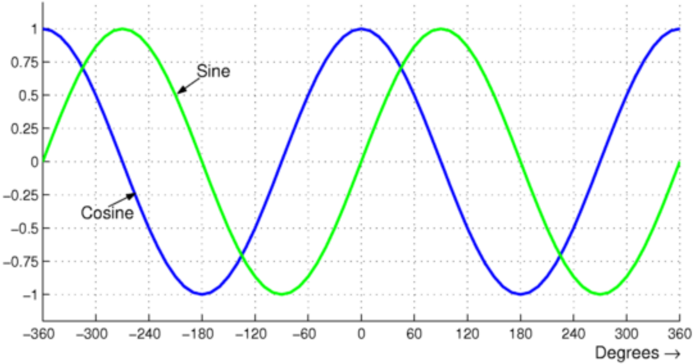
$$\tan x = \frac{\sin x}{\cos x} \tag{2}$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x} \tag{3}$$

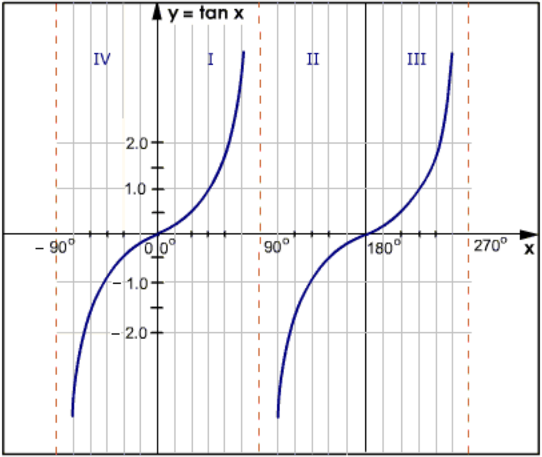
$$\csc x = \frac{1}{\sin x} \tag{4}$$

$$\sec x = \frac{1}{\cos x} \tag{5}$$

Here is a graph of the sine and cosine functions.



Here is a graph of the tangent function.



Trigonometric Functions Review

Consider the following table of well-known inverse functions commonly used in calculus:

function	**inverse**
e^x	$\ln x$
$\sin x$	$\arcsin x$ or $\sin^{-1}$
$\cos x$	$\arccos x$ or $\cos^{-1}$
$\tan x$	$\arctan x$ or $\tan^{-1}$

Trigonometric Functions Review

Consider the following most important trigonometric identities:

$$\sin^2 x + \cos^2 x = 1 \tag{6}$$

$$\sin(-x) = -\sin x \tag{7}$$

$$\cos(-x) = \cos x \tag{8}$$

$$\tan(-x) = -\tan x \tag{9}$$

Trigonometric Functions Review

Consider the following most important trigonometric identities:

$$\sin(90^\circ - x) = \cos x \tag{10}$$

$$\cos(90^\circ - x) = \sin x \tag{11}$$

$$\tan(90^\circ - x) = \cot x \tag{12}$$

$$\sin(x + 180^\circ) = -\sin x \tag{13}$$

$$\cos(x + 180^\circ) = -\cos x \tag{14}$$

$$\tan(x + 180^\circ) = \tan x \tag{15}$$

$$\cot(x + 180^\circ) = \cot x \tag{16}$$

Trigonometric Functions Review

Here are the angle sum identities,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \tag{17}$$

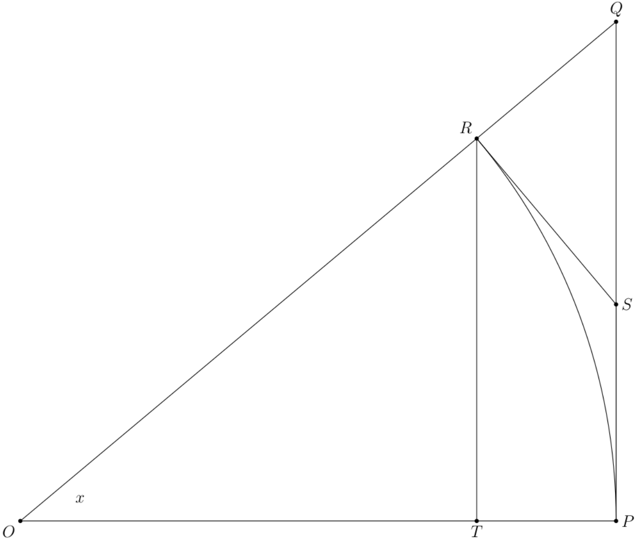
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{18}$$

from which we have, immediately following, the double angle identities,

$$\sin(2x) = 2\cos x \sin x \tag{19}$$

$$\cos(2x) = \cos^2 x - \sin^2 x \tag{20}$$

Limit of sin(x)/x



Limit of sin(x)/x

Let's find $\lim_{x\to 0} \frac{\sin x}{x}$. On the last slide, consider the unit circle with $\|\vec{OP}\| = \|\vec{OR}\| = 1$ and the angle x at O. For simplicity let's assume that $0 < x < \pi/2$. The angle x is also the length of the arc between P and R. Consequently

$$\|\vec{RT}\| = \sin x \leq x \tag{21}$$

and therefore

$$\frac{\sin x}{x} \leq 1 \tag{22}$$

$$x \leq \|\vec{PS}\| + \|\vec{SR}\| \leq \|\vec{PS}\| + \|\vec{SQ}\| = \|\vec{PQ}\| = \tan x \tag{23}$$

$\|\vec{SR}\| \leq \|\vec{SQ}\|$ because the angle QRS is a right angle. (23) means that

$$\cos x \leq \frac{\sin x}{x} \tag{24}$$

Since $\lim_{x\to 0}\cos x = 1$ and $\lim_{x\to 0} 1 = 1$, we can use the squeeze theorem, (22), and (24) for

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \tag{25}$$

Limit of (cos(x)-1)/x

Consider

$$\lim_{x\to 0}\frac{\cos x-1}{x}=\lim_{x\to 0}\left[\frac{\cos x-1}{x}\cdot\frac{\cos x+1}{\cos x+1}\right]=$$

$$\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}=-\lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{\cos x+1}\right]=$$

$$-1\cdot\left(\frac{0}{1+1}\right)=0 \tag{26}$$

Derivative of Sine

The derivative of $f(x) = \sin x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} =$$

$$\lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} =$$

$$\lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] = \cos x \qquad (27)$$

Exercise 1: Differentiate $f(x) = x^2 \sin x$.

The Derivative of Cosine

The derivative of $f(x) = \cos x$ is

$$f'(x) = -\sin x \tag{28}$$

The proof is analogous to the proof for $\sin x$.
Exercise 2: Differentiate

$$f(t) = \frac{1 + \sin t}{t + \cos t} \tag{29}$$

The Derivative of Tangent

The derivative of $f(x) = \tan x$ is

$$f'(x) = \sec^2 x \tag{30}$$

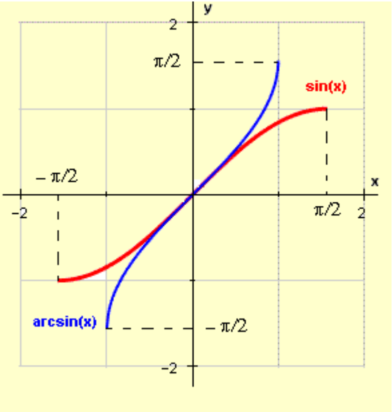
Use the quotient rule to prove this. Remember that

$$\sec x = \frac{1}{\cos x} \tag{31}$$

Exercise 3: Find the following arcsine values:

$$\arcsin(-1) \qquad \arcsin(1)$$

$$\arcsin\left(\frac{1}{2}\right) \qquad \arcsin\left(\frac{\sqrt{3}}{2}\right) \qquad \arcsin\left(-\frac{1}{\sqrt{2}}\right)$$



The Derivative of Arcsine

We can tell from the graph that

1. $\frac{d}{dx} \arcsin x$ will be positive
2. $\frac{d}{dx} \arcsin x$ will be defined on the interval $[-1, 1]$

Let's restrict our attention to the first quadrant so that we may say with confidence for $f(x) = \arcsin x$ that

$$sin(f(x)) = x \text{ and therefore } \cos f(x) = \sqrt{1 - \sin^2 f(x)} = \sqrt{1 - x^2} \tag{32}$$

The Derivative of Arcsine

Take the equation $\sin(f(x)) = x$ and differentiate with respect to x on both sides.

$$\sin(f(x)) = x \tag{33}$$

$$\frac{d}{dx}\sin(f(x)) = \frac{d}{dx}x \tag{34}$$

We know the right-hand side equals 1. For the left-hand side, we know that

$$\frac{d}{dx}\sin x = \cos x \tag{35}$$

but be very careful here

$$\frac{d}{dx}\sin(f(x)) \neq \cos f(x) \tag{36}$$

The Derivative of Arcsine

We need some magic here—exactly the kind of magic provided in the next lesson: it is called the chain rule.

$$\frac{d}{dx}\sin(f(x)) = \cos f(x) \cdot f'(x) \tag{37}$$

Now that we also have the left-hand side, (34) becomes

$$\cos f(x) \cdot f'(x) = 1 \tag{38}$$

and therefore, using (32)

$$\frac{d}{dx}\arcsin x = f'(x) = \frac{1}{\cos f(x)} = \frac{1}{\sqrt{1-x^2}} \tag{39}$$

Derivatives of Trigonometric Functions

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \cos x, & \frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\cos x &= -\sin x, & \frac{d}{dx}\arccos x &= \frac{-1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\tan x &= \sec^2 x, & \frac{d}{dx}\arctan x &= \frac{1}{1+x^2} \\
\frac{d}{dx}\cot x &= -\csc^2 x, & \frac{d}{dx}\operatorname{arccot} x &= \frac{-1}{1+x^2} \\
\frac{d}{dx}\sec x &= \tan x \sec x, & \frac{d}{dx}\operatorname{arcsec} x &= \frac{1}{|x|\sqrt{x^2-1}} \\
\frac{d}{dx}\csc x &= -\csc x \cot x, & \frac{d}{dx}\operatorname{arccsc} x &= \frac{-1}{|x|\sqrt{x^2-1}}
\end{aligned}
$$

Exercises

Exercise 4: Differentiate the following function:

$$f(x) = 3x^2 - 2\cos x \tag{40}$$

Exercise 5: Find the equation of the tangent line at $(\pi/3, 2)$ for

$$y = \sec x \tag{41}$$

Exercise 6: Differentiate the following function:

$$f(x) = \sqrt{x}\sin x \tag{42}$$

Exercise 7: Find the equation of the tangent line at $(\pi/6, 4 + \frac{5}{2}\sqrt{3})$ for

$$f(x) = 2\csc x + 5\cos x \tag{43}$$

Exercise 8: Differentiate the following functions:

$$g(t) = 4\sec t + \tan t \tag{44}$$

$$f(x) = \csc x(x + \cot x) \tag{45}$$

$$v(w) = \frac{\sin w}{w^2} \tag{46}$$

End of Lesson

Next Lesson: Chain Rule