Chain Rule

MATH 2511, BCIT

Technical Mathematics for Geomatics

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Euler's Number

The number e is defined as follows,

$$e = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t \tag{1}$$

Lemma

Consider two functions f_1 and f_2. They are related in so far as

$$f_1(x) = f_2\left(\frac{1}{x}\right) \tag{2}$$

For example,

$$f_1(x) = \frac{2x+1}{5x-7} \text{ and } f_2(x) = -\frac{x+2}{7x-5} \tag{3}$$

Then

$$\text{If } \lim_{x\to\infty} f_1(x) = a \text{ then } \lim_{x\to 0} f_2(x) = a \tag{4}$$

The Derivative of the Logarithmic Function

Now consider the function $f(x) = \ln x$ and the definition of the derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{h}\ln\frac{x+h}{x} =$$

$$\lim_{h\to 0}\frac{1}{x}\cdot\frac{x}{h}\ln\left(1+\frac{h}{x}\right) = \lim_{h\to 0}\frac{1}{x}\ln\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}} \tag{5}$$

Use the lemma of the last slide and the definition of Euler's number to see that

$$f'(x) = \frac{1}{x} \tag{6}$$

Problematic Functions

Here are some functions that we either don't know how to differentiate or whose differentiation would take an inordinate amount of time.

$$f(x) = 2^x \tag{7}$$

$$f(x) = \sqrt{x^2+1} \tag{8}$$

$$f(x) = (x^2+x+1)^{100} \tag{9}$$

$$f(x) = \sin(1+\sqrt{x-7}) \tag{10}$$

$$f(x) = \log_{10} x \tag{11}$$

$$f(x) = \ln(x^2+1) \tag{12}$$

The Chain Rule

Rule 7

The Chain Rule

$$g'(x) = f_1'(f_2(x))f_2'(x) \text{ for } g(x) = (f_1 \circ f_2)(x) \tag{13}$$

Chain Rule Reason

Consider

$$(f\circ g)'(x) = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h} =$$

$$\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \tag{14}$$

$$f'(g(x))g'(x) \tag{15}$$

This is only a hint, not a rigorous proof, since we have replaced $g(x+h)$ by $g(x)+h$, which isn't covered by our rules and is, in fact, false in some situations.

Exercises

1. Diffentiate: $f(x) = 2^x$
2. Diffentiate: $f(x) = \sqrt{x^2+1}$
3. Diffentiate: $f(x) = (x^2+x+1)^{100}$
4. Diffentiate: $f(x) = \log_{10} x$
5. Diffentiate: $f(x) = \ln(x^2+1)$

Inverse and Identity Function

Remember how we defined the logarithmic function,

$$\ln y = x \text{ if and only if } e^x = y \tag{16}$$

so the logarithmic function is the inverse of the exponential function. Consequently, if $f(x) = e^x$ and $g(y) = \ln y$

$$(f \circ g)(y) = y \text{ and } (g \circ f)(x) = x \tag{17}$$

When (17) is true we call f the inverse function of g and vice versa. The function $\mathrm{id}(x) = x$ is called the identity function.

The Derivative of the Exponential Function

We know the derivative of the identity function.

$$\mathrm{id}'(x) = 1 \tag{18}$$

Consequently,

$$\frac{d}{dx}\ln\left(e^x\right) = 1 \tag{19}$$

We also know that according to the chain rule

$$\frac{d}{dx}\ln\left(e^x\right) = \frac{1}{e^x}\exp'(x) \tag{20}$$

where $\exp(x) = e^x$. Therefore,

$$\exp'(x) = e^x \tag{21}$$

The exponential function is its own derivative!

Derivative of the Exponential Function: Exercises

Exercise 1: Differentiate the following functions:

$$f(x) = e^{\sin x} \tag{22}$$

$$g(t) = \frac{1}{e^t} \tag{23}$$

Derivative of the Exponential Function: Exercises

Exercise 2: Differentiate the following functions:

$$v(w) = w^2 e^w \tag{24}$$

$$g(z) = \frac{e^z - 1}{e^z + 1} \tag{25}$$

Exercises for Chain Rule

Exercise 3: Differentiate

$$f(\vartheta) = \tan(\sin\vartheta) \tag{26}$$

$$F(x) = \sqrt[4]{1 + 2x + x^3} \tag{27}$$

$$g(t) = \frac{\pi}{(t^4 + 1)^3} \tag{28}$$

Exercises for Chain Rule

Exercise 4: Differentiate

$$f(s) = \sqrt[3]{1 + \tan s} \tag{29}$$

$$y = (x^2 + 1)\sqrt[3]{x^2 + 2} \tag{30}$$

$$y = e^{x\cos x} \tag{31}$$

$$y = x \sin\frac{1}{x} \tag{32}$$

Exercises for Chain Rule

Exercise 5: Differentiate the following functions or find dy/dx for the following curves:

$$y = \frac{1}{\arcsin x} \tag{33}$$

$$y = xe^{-kx} \tag{34}$$

$$f(x) = x \arctan\sqrt{x} \tag{35}$$

Exercises for Chain Rule

Exercise 6: Differentiate the following functions or find dy/dx for the following curves:

$$y = 3\cot(nx) \tag{36}$$

$$y = \arccos\left(e^{2x}\right) \tag{37}$$

$$h(t) = (t^4 - 1)^3(t^3 + 1)^4 \tag{38}$$

Exercises for Chain Rule

Exercise 7: Differentiate the following functions or find dy/dx for the following curves:

$$y = (x^2+1)\sqrt{x^2+2} \tag{39}$$

$$G(y) = \left(\frac{y^2}{y+1}\right)^5 \tag{40}$$

$$y = \tan^2(3\vartheta) \tag{41}$$

$$y = \arctan\left(x - \sqrt{1+x^2}\right) \tag{42}$$

Exercises for Differentiation III

Exercise 8: Find an equation of the tangent line to the curve

$$y = \frac{2}{1+e^{-x}} \tag{43}$$

at $x = 0$.

Exercise 9: Here is a model for the length of daylight (in hours) in Toronto on the t-th day of the year

$$L(t) = 12 + 2.8\sin\left(\frac{2\pi}{365}(t-80)\right) \tag{44}$$

Compare how the number of hours of daylight is increasing in Toronto on March 21 and May 21.

End of Lesson

Next Lesson: Higher Order Derivatives