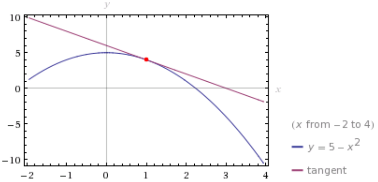
Basic Differentiation

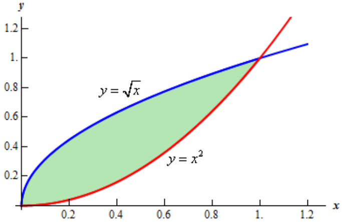
MATH 1511, BCIT

Technical Mathematics for Geomatics

January 15, 2018

Calculus solves many problems for which it was not originally designed. The initial motivation for calculus was to find the slope of a tangent on a curve and the area of a region bounded by a curve.





A Real-Life Example I

Consider a magnetic levitation train accelerating on a straight monorail track. The position of the train (in feet) from the origin at time t is given by

$$s = f(t) = 4t^2 \tag{1}$$

What is the velocity of the train at $t = 2$?

A Real-Life Example II

It appears to make sense only if we calculate the velocity given an interval of time rather than just one point in time. For example, the velocity between $t = 2$ and $t = 3$ is

$$v_{[2,3]} = \frac{f(3) - f(2)}{3 - 2} = 20 \tag{2}$$

More generally,

$$v_{[2,t]} = g(t) = \frac{f(t) - f(2)}{t - 2} = \frac{4(t^2 - 4)}{t - 2} \tag{3}$$

g is not defined at $t = 2$, but it is defined all around $t = 2$, so we can ask ourselves the question: what happens when $t \to 2$ from below; or when $t \to 2$ from above? It turns out that either way, the number approaches 16.

Velocity at a Point

If we found the limit as $t \to 2$ of

$$v_{[2,t]} = g(t) = \frac{f(t) - f(2)}{t - 2} = \frac{4(t^2 - 4)}{t - 2}$$

it would serve as an intuitive definition of what a velocity is at a point (instead of on an interval). Unfortunately, the limit has the indeterminate form

$$\lim_{t \to 2} \frac{4(t^2 - 4)}{t - 2} = \frac{0}{0} \tag{4}$$

However, notice that for $t \neq 2$,

$$g(t) = \frac{4(t^2 - 4)}{t - 2} = \frac{4(t + 2)\cancel{(t - 2)}}{\cancel{t - 2}} = 4(t + 2) \tag{5}$$

Tangent Lines I

Remember our magnetic levitation train. The distance-time function was

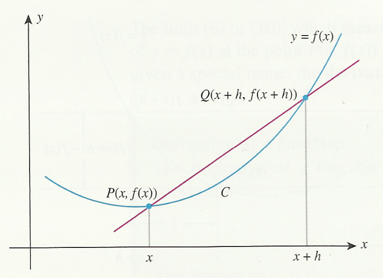
$$s = f(t) = 4t^2 \tag{6}$$

The velocity of the train over a given interval is

$$v_{[t_1,t_2]} = \frac{f(t_2) - f(t_1)}{t_2 - t_1} \tag{7}$$

This velocity is also the slope of the line going through the two function values $f(t_1)$ and $f(t_2)$. We call such a line a secant line.

Here is an example of a secant line.



Tangent Lines III

Now imagine t_1 and t_2 moving closer and closer together at a point a (for the train, we used $a = 2$). If both of these limits exist and agree with each other, we have a velocity at a point,

$$\lim_{t \to a} v_{[a,t]} \text{ for } t > a \tag{8}$$

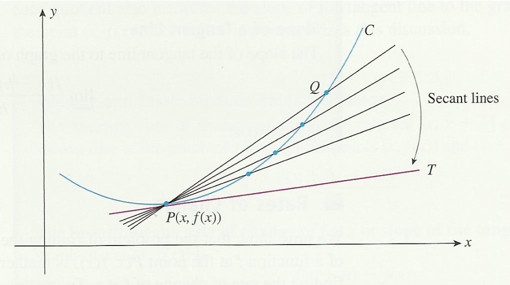
$$\lim_{t \to a} v_{[t,a]} \text{ for } t < a \tag{9}$$

This velocity at a point is also the slope of the line that just touches the function graph without crossing it. We call it a tangent line at $t = a$. The slope of the tangent line is sometimes also called the rate of change.

Tangent Lines IV

Think of a tangent line as the limit of secant lines. The slope of a tangent line at a point $P = (x, f(x))$, if it exists, is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{10}$$

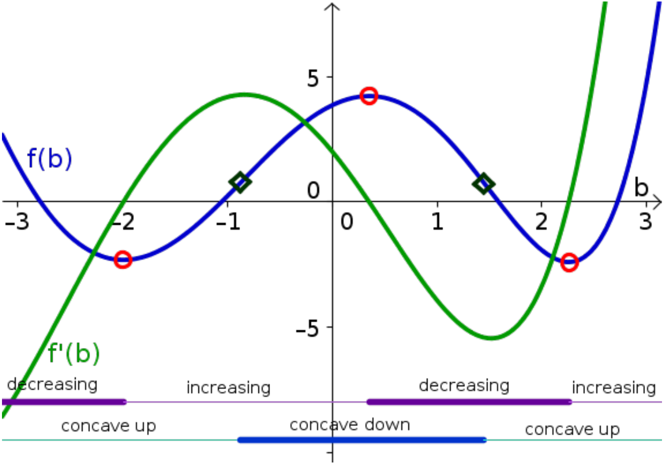


Derivatives

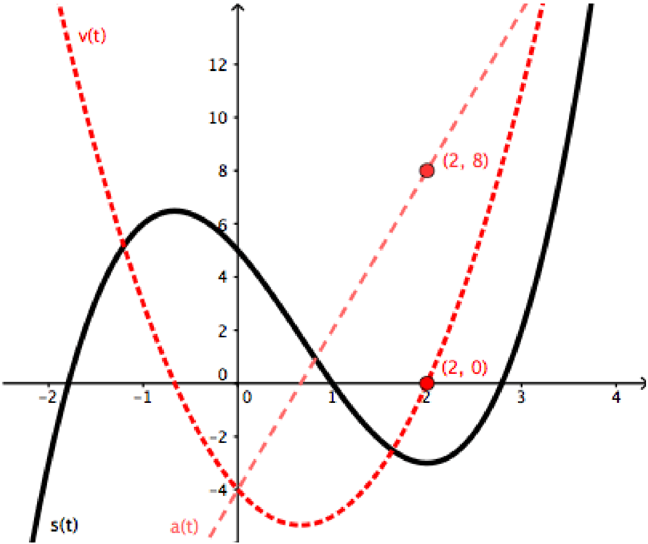
The derivative of a function f with respect to x is the function f' (read "f prime"),

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{11}$$

The domain of f' is the set of all x where the limit exists.


f(b)
f'(b)
b
5
0
-5
-3
-2
-1
0
1
2
3
decreasing
increasing
decreasing
increasing
concave up
concave down
concave up

Derivatives Diagram II



Basic Rules of Differentiation

Rule 1

Derivative of a Constant

$$f'(x) = 0 \text{ for } f(x) = c \tag{12}$$

Reason:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0 \tag{13}$$

Rule 2

The Power Rule

$$f'(x) = nx^{n-1} \text{ for } f(x) = x^n \tag{14}$$

This rule applies for any $n \in \mathbb{R}$. The proof is messy. However, we can show how the rule is justified for $n = 2$ and $n = \frac{1}{2}$.

Basic Rules of Differentiation

Case 1: $n = 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$\lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x \qquad (15)$$

Basic Rules of Differentiation

Case 2: $n = \frac{1}{2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$$

$$\lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x}) \cdot (\sqrt{x+h} + \sqrt{x})}{h \cdot (\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{x+h-x}{h \cdot (\sqrt{x+h} + \sqrt{x})} \tag{16}$$

Using the One Disagreement Rule, (16) equals

$$\lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}} \tag{17}$$

Rule 3

Derivative of a Constant Multiple of a Function

$$g'(x) = c \cdot f'(x) \text{ for } g(x) = c \cdot f(x) \tag{18}$$

Reason:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} =$$

$$\lim_{h\to 0} c \cdot \frac{f(x+h) - f(x)}{h} = c \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = c \cdot f'(x) \tag{19}$$

Rule 4

The Sum Rule

$$g'(x) = f_1'(x) + f_2'(x) \text{ for } g(x) = f_1(x) + f_2(x) \tag{20}$$

Reason:

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{f_1(x+h) + f_2(x+h) - f_1(x) - f_2(x)}{h} = \lim_{h \to 0} \left(\frac{f_1(x+h) - f_1(x)}{h} + \frac{f_2(x+h) - f_2(x)}{h} \right) = f_1'(x) + f_2'(x) \tag{21}$$

Exercise 1: Find the derivatives for the following functions.

1. $f(x) = 4x^5 + 3x^4 - 8x^2 + x + 3$
2. $f(t) = \frac{t^2}{5} + \frac{5}{t^3}$
3. $g(z) = 2z - 5\sqrt{z}$

Exercise 2: Find the slope and an equation of the tangent line to the graph of $f(x) = 2x + (1/\sqrt{x})$ at the point $(1, 3)$.

Exercise 3: Find the derivatives for the following functions.

$$f(x) = 5x^{\frac{4}{3}} - \frac{2}{3}x^{\frac{3}{2}} + x^2 - 3x + 1 \tag{22}$$

$$f(x) = 2t^2 + \sqrt{t^3} \tag{23}$$

$$f(x) = \frac{2}{x^2} - \frac{3}{x^{\frac{1}{3}}} \tag{24}$$

$$f(x) = \frac{3}{x^3} + \frac{4}{\sqrt{x}} + 1 \tag{25}$$

End of Lesson

Next Lesson: Product and Quotient Rule