

Optimization and Analyzing Functions

MATH 2511, BCIT

Technical Mathematics for Geomatics

February 5, 2018

Relative Extrema

A function f has a **relative maximum** at $x = c$ if there exists an open interval (a, b) containing c such that $f(x) \leq f(c)$ for all x in (a, b) .

A function f has a **relative minimum** at $x = c$ if there exists an open interval (a, b) containing c such that $f(x) \geq f(c)$ for all x in (a, b) .

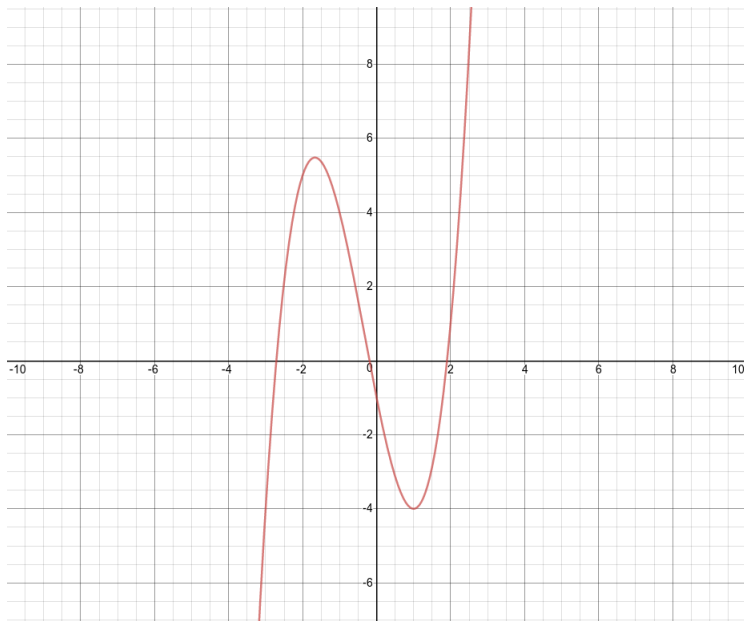
Derivatives and Extrema

At any number c where a differentiable function f has a relative extremum, $f'(c) = 0$. The converse is not true. Consider the following two functions and their derivatives.

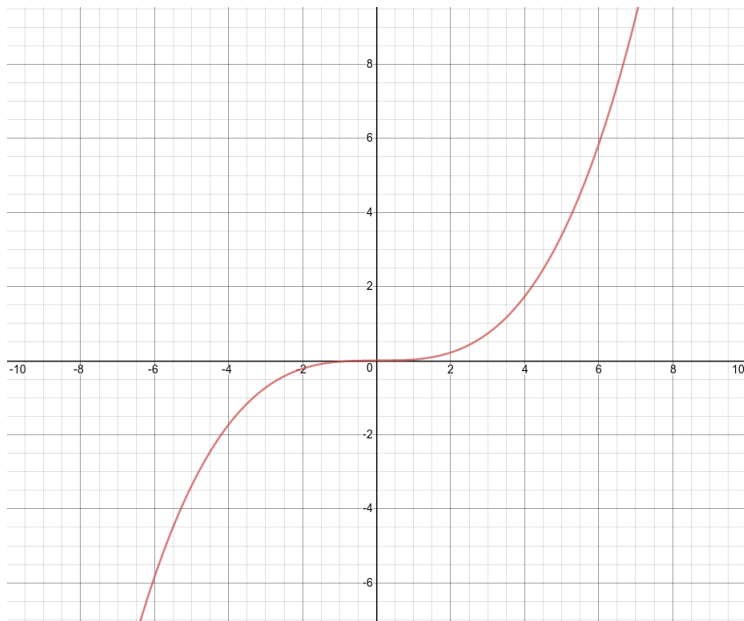
$$f_1(x) = x^3 + x^2 - 5x - 1 \quad (1)$$

$$f_2(x) = \left(\frac{3}{10}x\right)^3 \quad (2)$$

Derivatives and Extrema Graph I



Derivatives and Extrema Graph II



Derivatives and Extrema Caution

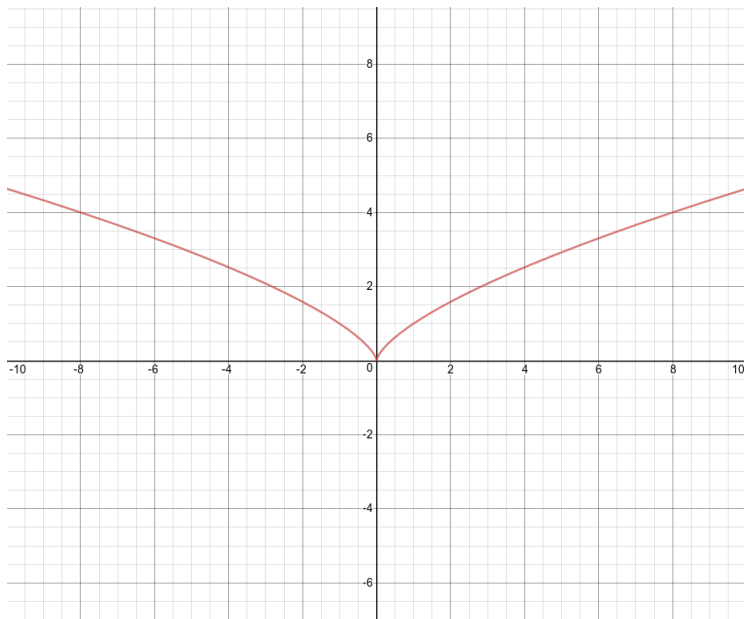
Note that a function may have an extremum at a point where the derivative is not 0 if at that point the function is not differentiable. Consider this function and its derivative.

$$f_3(x) = x^{\frac{2}{3}} \quad (3)$$

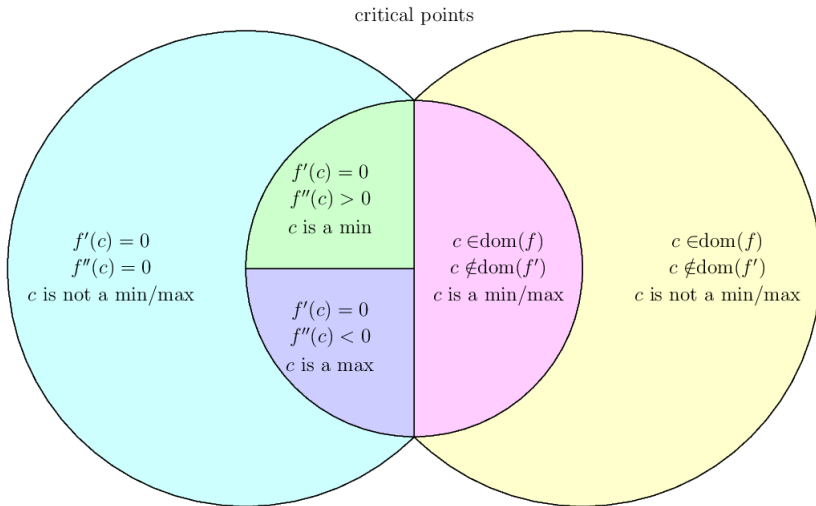
Critical Number

A **critical number** of a function f is any number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist.

Derivatives and Extrema Graph III



Critical Points and Extrema



Exercise 1: Find the critical points of the following function,

$$f(x) = x^3 - 4x \quad (4)$$

Exercise 2: Find the critical points of the following function,

$$h(t) = -t^2 + 6t + 6 \quad (5)$$

Exercise 3: Find the critical points of the following function,

$$f(x) = \frac{1}{2}x^4 - x^2 \quad (6)$$

Exercise 4: Find the critical points of the following function,

$$g(x) = \frac{x+1}{x} \quad (7)$$

Exercise 5: Find the critical points of the following function,

$$f(x) = x\sqrt{x-4} \quad (8)$$

Exercise 6: Find the critical points of the following function,

$$f(x) = 2 \tan x - \tan^2 x \quad (9)$$

Exercise 7: Find the critical points of the following function,

$$h(s) = s^{\frac{5}{3}} \quad (10)$$

Exercise 8: Find local maxima and minima for the following function:

$$f(x) = 3x^3 - 12x + 5 \quad (11)$$

Exercise 9: Find local maxima and minima for the following function:

$$f(x) = \frac{x}{x^2 + 1} \quad (12)$$

Exercise 10: Find local maxima and minima for the following function:

$$f(t) = t\sqrt{4 - t^2} \quad (13)$$

Exercise 11: Find local maxima and minima for the following function:

$$g(t) = \sqrt[3]{t}(8 - t) \quad (14)$$

Exercise 12: Find local maxima and minima for the following function:

$$g(t) = \cos t + \sin t \quad (15)$$

Exercise 13: Find local maxima and minima for the following function:

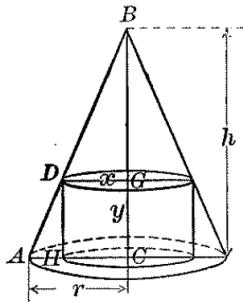
$$f(x) = \ln(x^2 + x + 1) \quad (16)$$

Exercise 14: Find local maxima and minima for the following function:

$$f(x) = \ln(\cos x) \quad (17)$$

Optimization Word Problems

Exercise 15: Find the altitude of the cylinder of maximum volume that can be inscribed in a given right cone.

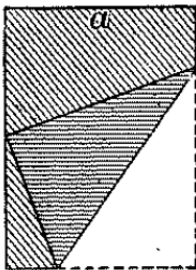


Exercise 16: A water tank is to be constructed with a square base and open top and is to hold 64 cubic yards. If the cost of the sides is \$1 a square yard, and of the bottom \$2 a square yard, what are the dimensions when the cost is a minimum? What is the minimum cost?

Optimization Word Problems

Exercise 17: The lower corner of a leaf, whose width is a , is folded over so as just to reach the inner edge of the page.

- 1 Find the width of the part folded over when the length of the crease is a minimum.
- 2 Find the width when the area folded over is a minimum.



Exercise 18: A submarine telegraph cable consists of a core of copper wires with a covering made of nonconducting material. If x denotes the ratio of the radius of the core to the thickness of the covering, it is known that the speed of signaling varies as

$$x^2 \ln \frac{1}{x} \tag{18}$$

Show that the greatest speed is attained when $x = \frac{1}{\sqrt{e}}$.

Analyzing Functions I

To analyze a function, determine the following features:

- Domain and range of the function.
- Zeros (also called x-intercepts) of the function.
- Critical points, maxima, minima.
- Inflection points.
- Asymptotes.
- Is the function even ($f_1(x) = x^2 + 1$) or odd ($f_2(x) = x^3 - x$)?

Analyzing Functions Step-By-Step I

Here is a step-by-step guide to analyzing functions.

- 1 Determine the x -intercepts (also called zeros). Set $f(x) = 0$ and find the solution set.
- 2 Determine the critical points. Find the derivative $f'(x)$ and check whether there are points in the domain of f that are not in the domain of f' . Then set $f'(x) = 0$ and find the solution set.
- 3 Determine whether the critical points are maxima or minima or neither. Find $f''(x)$ and check whether f'' at the critical points is positive, negative, or neither.

Analyzing Functions Step-By-Step II

Here is a step-by-step guide to analyzing functions.

- 4 Determine the inflection points. Set $f''(x) = 0$ and find the solution set.
- 5 Determine the asymptotes. See next slide.
- 6 Determine whether, for all x in the domain of f ,
 $f(x) - f(-x) = 0$ (in which case f is even) or
 $f(x) + f(-x) = 0$ (in which case f is odd).
- 7 Using the information you have, and possibly a table of function values, graph the function. Then determine the domain and range of f .

Finding Asymptotes I

An asymptote is a linear function ($y = kx + d$ with slope k and y -intercept d) which the function graph of f approaches. There are three kinds of asymptotes.

Vertical Asymptote

A vertical asymptote, strictly speaking, is not a linear function. It is a curve defined by $x = c$, where c is a real number (we call real numbers like c **constants**). You can often find vertical asymptotes at points where f is undefined.

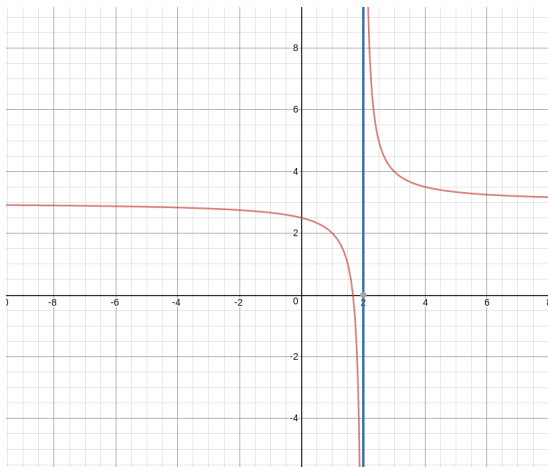
Find vertical asymptotes by checking points which are not in the domain of the function f .

$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2 \quad (19)$$

Finding Asymptotes I

Example:

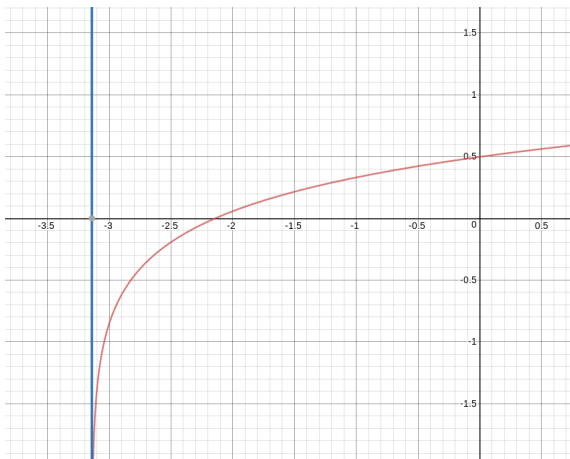
$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2 \quad (20)$$



Finding Asymptotes I

Example:

$$f(x) = \ln(x + \pi) \text{ has an asymptote at } x = -\pi \quad (21)$$



Finding Asymptotes II

Horizontal Asymptote

A horizontal asymptote is a linear function with slope $k = 0$. Its equation is $y = c$, where c is a constant. There are horizontal asymptotes for functions whose limits is a constant and for rational functions whose numerator and denominator polynomials share the same degree.

Find horizontal asymptotes by checking

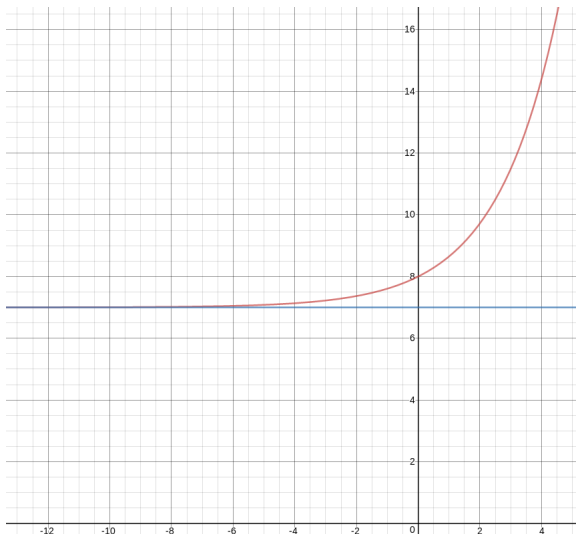
$$\lim_{x \rightarrow \infty} f'(x) \text{ and } \lim_{x \rightarrow -\infty} f'(x) \quad (22)$$

If the limit is $k = 0$, then that is also the slope of the asymptote.

Finding Asymptotes II

Example

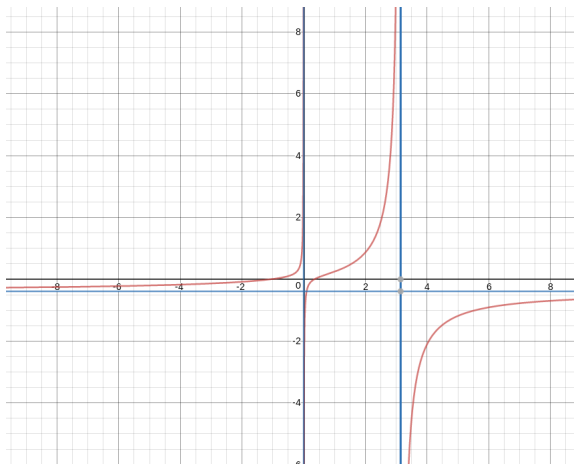
$f(x) = e^{\frac{x}{2}} + 7$ has the asymptote $y = 7$ (23)



Finding Asymptotes II

Example (this example additionally has two vertical asymptotes):

$$f(x) = \frac{\pi x^2 + 2x - 1}{-7x^2 + 3x} \text{ has asymptotes } y = -\frac{e}{7}, x = \frac{22}{7}, x = 0 \quad (24)$$



Finding Asymptotes III

Sloped Asymptote

A sloped asymptote is a linear function with a positive or a negative slope, $y = kx + d$ with $k \neq 0$. There are sloped asymptotes for rational functions where the numerator polynomial's degree exceeds the denominator polynomial's degree by 1.

Find sloped asymptotes by checking

$$\lim_{x \rightarrow \infty} f'(x) \text{ and } \lim_{x \rightarrow -\infty} f'(x) \quad (25)$$

If the limit is $k \neq 0$, then that is also the slope of the asymptote. Hyperbolas also sometimes have sloped asymptotes.

Finding Asymptotes III

Example:

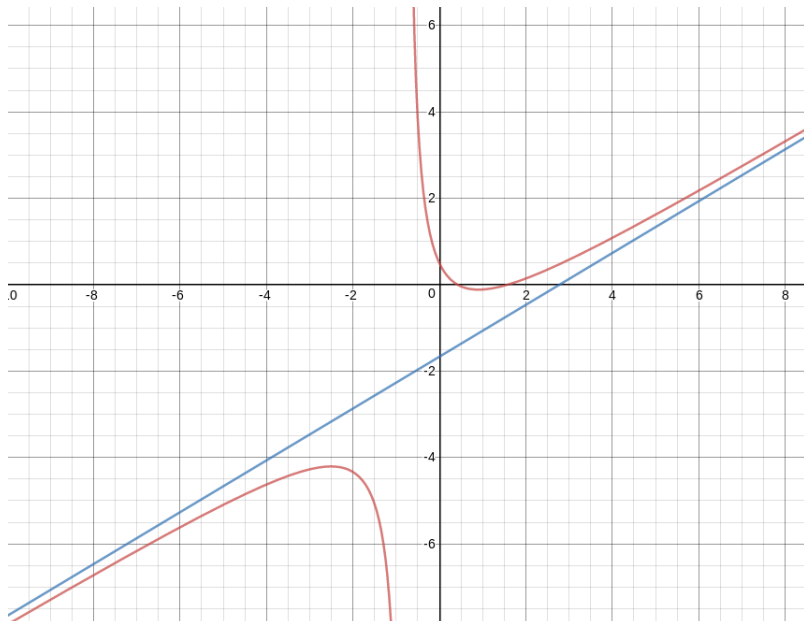
$$f(x) = \frac{3x^2 - 6x + 2}{5x + 4} \text{ has the asymptote } y = \frac{3}{5}x - \frac{5}{3} \quad (26)$$

Find the y -intercept by making sure that

$$\lim_{x \rightarrow \infty} (f(x) - g(x)) = 0 \quad (27)$$

where $y = g(x) = kx + d$ for the sloped asymptote. This results in an equation where d is the only unknown.

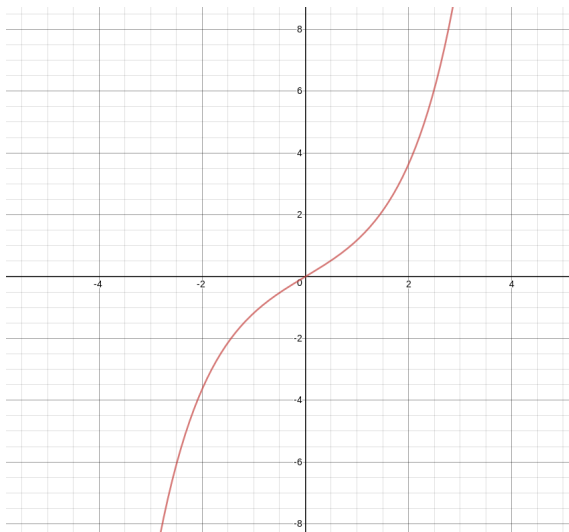
Finding Asymptotes III



Finding Asymptotes III

Example:

$$f(\vartheta) = \sinh \vartheta \quad (28)$$



Exercise 19: Analyze the following function:

$$g_1(x) = -x^2 + 3x \quad (29)$$

Exercise 20: Analyze the following function:

$$g_2(x) = 3x^{\frac{2}{3}} - 2x \quad (30)$$

Exercise 21: Analyze the following function:

$$g_3(t) = \frac{2t^2}{t^2 + 3} \quad (31)$$

Exercise 22: Analyze the following function:

$$g_4(x) = x^3 e^x \quad (32)$$

Analyzing Functions Exercises Graph



$$-x^2 + 3 \cdot x$$



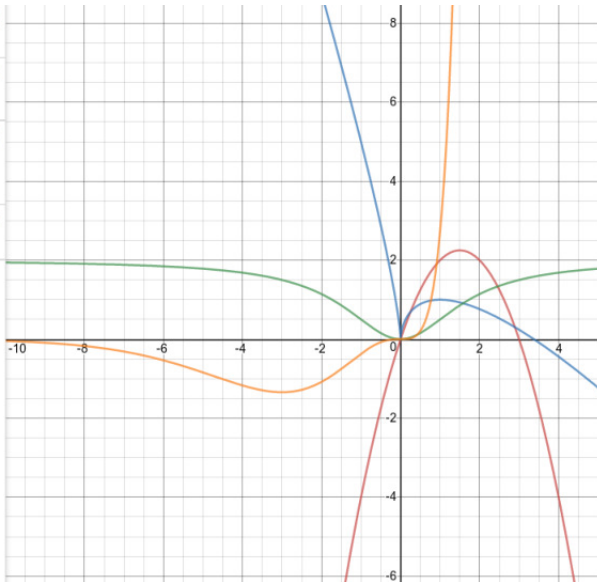
$$3 \cdot x^{\frac{2}{3}} - 2 \cdot x$$



$$\frac{(2 \cdot x^2)}{(x^2 + 3)}$$



$$x^3 \cdot \exp(x)$$



Exercise 23: Analyze the following function.

$$f(x) = x \cdot \ln x^2 \quad (33)$$

Exercise 24: Analyze the following function.

$$f(x) = \frac{2x^2 + 2}{x - 3} \text{ (do not look for inflection points)} \quad (34)$$

Exercise 25: Analyze the following function.

$$f(x) = x^3 + 4x^2 + x - 6 \quad (35)$$

Note that $x = 1$ is an x -intercept so that $(x - 1)$ can be factored as in

$$x^3 + 4x^2 + x - 6 = (x - 1)(x^2 + 5x + 6) \quad (36)$$

Exercise 26: Analyze the following function.

$$f(x) = 4 - \frac{e^x + 1}{e^x} \quad (37)$$

Exercise 27: Analyze the following function.

$$f(x) = \frac{3x^2 - 5}{x - 2} \quad (38)$$

Next Lesson: Transcendental Functions and Differentials