Maclaurin and Taylor Series

MATH 2511, BCIT

Technical Mathematics for Geomatics

April 24, 2018

Sequences and Series

Infinite Sequence

An infinite sequence is a function whose domain is the set of positive integers.

Here is an example: $a(n) = 2n$ for $n = 1, 2, 3, \ldots$. We usually write a_n instead of $a(n)$. The infinite sequence is $2, 4, 6, \ldots$. The infinite sequence itself is often called $(a_n)_{n \in \mathbb{N}}$.

Infinite Series

Given an infinite sequence a_n, the infinite series s_n is an infinite sequence defined as follows: $s_n = a_1 + a_2 + \ldots + a_n$.

s_n is called a partial sum of the sequence $(a_n)_{n \in \mathbb{N}}$. It is often written as

$$s_n = \sum_{i=1}^{n} a_i \tag{1}$$

Convergence and Divergence

converges $(a_n)_{n\in\mathbb{N}}$ converges to the real number L if for every positive real number ε there exists an integer N such that for all $n > N$ it is true that $|a_n - L| < \varepsilon$. L is the limit of this sequence.

diverges $(a_n)_{n\in\mathbb{N}}$ diverges if no limit exists. $(a_n)_{n\in\mathbb{N}}$ diverges to positive infinity ∞ if for every real number M there is an integer N such that for all n larger than N it is true that $a_n > M$. We say $\lim_{n\to\infty} a_n = \infty$ or $a_n \longrightarrow \infty$. $(a_n)_{n\in\mathbb{N}}$ diverges to negative infinity $-\infty$ if for every real number m there is an integer N such that for all n larger than N it is true that $a_n < m$. We say $\lim_{n\to\infty} a_n = -\infty$ or $a_n \longrightarrow -\infty$.

Geometric Series

Geometric series have the form

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1} \tag{2}$$

The notation for the limit is as follows,

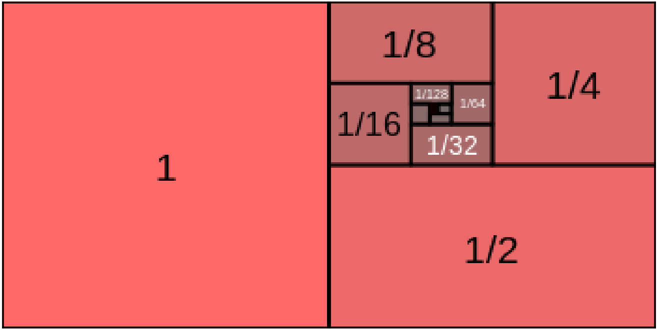
$$\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} ar^{n-1} \tag{3}$$

r is called the ratio of the geometric series. Subtract $s_n - rs_n$ to find out that

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1 \tag{4}$$

If $|r| \geq 1$ then the limit does not exist.

Geometric Series



Proof that $2 + 4 + 8 + 16 + \ldots = -2$

Consider scenario 1,

$$\begin{aligned} a_n &= 2^n = \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots \\ s_n &= \textstyle\sum_{i=1}^{n} a_i \end{aligned} \tag{5}$$

Consider scenario 2,

$$\begin{aligned} a_n &= 2^n = 2, 4, 8, 16, \ldots \\ s_n &= \textstyle\sum_{i=1}^{n} a_i \end{aligned} \tag{6}$$

Now calculate the limits of these series. What goes wrong in scenario 2?

Geometric Series Example

Example 1: Limit of a Geometric Series. Find the limit of the following series.

$$\frac{7}{12} + \frac{7}{24} + \frac{7}{48} + \frac{7}{96} + \dots \tag{7}$$

Notice that 7 in the denominator and 12 in the numerator are common factors.

$$\frac{7}{12} + \frac{7}{24} + \frac{7}{48} + \frac{7}{96} + \dots = \frac{7}{12} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) =$$

$$\frac{7}{12} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{7}{6} \tag{8}$$

Exercise 1: Find the limit of the following series.

$$\sum_{n=2}^{\infty} \frac{3^n - 1}{6^n} \tag{9}$$

$$\sum_{n=0}^{\infty} \left(\frac{2n+1}{5^n} \right) \tag{10}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n} \right) \tag{11}$$

Exercise 2: Find the following series limits using telescoping series.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) \tag{12}$$

$$\sum_{n=1}^{\infty}\left(\frac{3}{n^2}-\frac{3}{(n+1)^2}\right) \tag{13}$$

$$\sum_{n=1}^{\infty}\left(\sqrt{n+4}-\sqrt{n+3}\right) \tag{14}$$

Exercise 3: Find the following series limits using telescoping series.

$$\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2^{n+1})^2} \tag{15}$$

$$\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} \tag{16}$$

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} \tag{17}$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \tag{18}$$

Repeating Decimals

Express each of these numbers as the ratio of two integers.

$$0.\overline{23} = 0.23232323\ldots \tag{19}$$

$$0.0\overline{6} = 0.06666\ldots \tag{20}$$

$$1.24\overline{123} = 1.24123123123\ldots \tag{21}$$

Integral Test

Integral Test

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N is any positive integer).

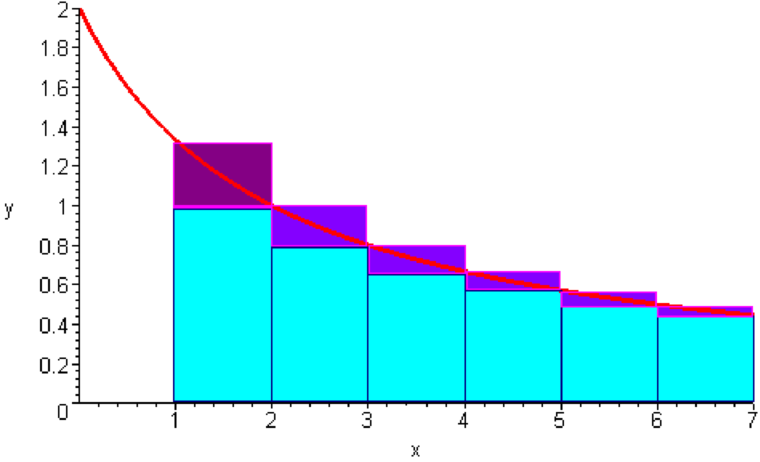
Then the series

$$\sum_{n=N}^{\infty} a_n \tag{22}$$

and the integral

$$\int_N^{\infty} f(x)\,dx \tag{23}$$

both converge or both diverge.


2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
y
1
2
3
4
5
6
7
x

Integral Test Exercises

Show that the famous harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \tag{24}$$

diverges. Then show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \tag{25}$$

converges.

Integral Test Exercises

Now show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \tag{26}$$

exists. Remember that the antiderivative of

$$f(x) = \frac{1}{x^2+1} \tag{27}$$

is $F(x) = \arctan x$. Showing that (26) exists does not mean that we know its value.

Integral Test Exercises

Give reasons why the following sums exist or do not exist.

$$\sum_{n=1}^{\infty} e^{-n} \qquad \sum_{n=1}^{\infty} \frac{n}{n+1} \qquad \sum_{n=1}^{\infty} n \sin \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} \qquad \sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$$\sum_{n=2}^{\infty} \frac{\ln n}{n} \qquad \sum_{n=1}^{\infty} \frac{5^n}{4^n+3} \qquad \sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$$

$$\sum_{n=1}^{\infty} e^{-n} \tag{28}$$

is convergent because it is a geometric series with $0 \leq r = \frac{1}{e} < 1$.

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \tag{29}$$

is divergent because $\frac{n}{n+1} \longrightarrow 1$, and $\frac{n}{n+1} \not\longrightarrow 0$ implies divergence according to the n-the term test.

$$\sum_{n=1}^{\infty} n \sin \frac{1}{n} \tag{30}$$

is divergent because according to L'Hôpital's rule, $n \sin \frac{1}{n} \longrightarrow 1$, and $n \sin \frac{1}{n} \not\longrightarrow 0$ implies divergence according to the n-the term test.

Integral Test Answers

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} \tag{31}$$

is divergent according to the integral test.

$$\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} \tag{32}$$

is convergent according to the integral test.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \tag{33}$$

is divergent because $a_n/b_n \longrightarrow 1$ for $a_n = n/(n^2+1)$ and $b_n = \frac{1}{n}$, using part 1 of the limit comparison test.

Integral Test Answers

$$\sum_{n=2}^{\infty} \frac{\ln n}{n} \tag{34}$$

is divergent because $\ln n/n > 1/n$ for $n > 2$ and the harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3} \tag{35}$$

is divergent because $a_n/b_n \longrightarrow 1$ for $a_n = 5^n/(4^n + 3)$ and $b_n = \frac{5^n}{4^n}$, using part 1 of the limit comparison test. $\sum b_n$ diverges because it is a geometric series with $r > 1$.

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n} \tag{36}$$

is divergent because according to L'Hôpital's rule, $\lim_{x \to \infty} \frac{\sqrt{x}}{\ln x}$ does not exist.

The n-th Term Test

We could prove this theorem, but it is also accessible to intuition:

$$\text{If } \sum_{i=1}^{n} a_i \text{ converges, then } a_n \longrightarrow 0 \tag{37}$$

Test for Divergence

$\sum_{i=1}^{n} a_i$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from 0.

The converse of the n-th term test is not true. For the following sequence, the corresponding series diverges even though the sequence goes to 0.

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{\text{2 terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{\text{4 terms}} + \frac{1}{8} + \dots \tag{38}$$

Comparison Tests

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with no negative terms. Then

1. $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.
2. $\sum a_n$ diverges if there is a divergent series $\sum d_n$ with $a_n \geq d_n \geq 0$ for all $n > N$, for some integer N.

Comparison Test Example

Example 2: Comparison Test Example. The series

$$\sum_{n=1}^{\infty} \frac{5}{5n-1} \tag{39}$$

diverges because

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n} \tag{40}$$

for all $n \in \mathbb{N}$.

Limit Comparison Tests

1. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0 \tag{41}$$

then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \tag{42}$$

and $\sum b_n$ converges, them $\sum a_n$ converges.

3. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \tag{43}$$

and $\sum b_n$ diverges, them $\sum a_n$ diverges.

Ratio Test

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \varrho \tag{44}$$

Then

1. the series $\sum a_n$ converges if $\varrho < 1$
2. the series $\sum a_n$ diverges if $\varrho > 1$ or ϱ is infinite
3. the test is inconclusive if $\varrho = 1$

Exercise 4: Use the ratio test to find out if the following exist:

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} \qquad \sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$$

Leibniz's Theorem

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence with $u_n > 0$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \dots \tag{45}$$

is an alternating series. It converges if the following two conditions are satisfied:

1. $u_n > u_{n+1}$ for all $n > N$, for some integer N
2. $u_n \longrightarrow 0$

It immediately follows that the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \tag{46}$$

converges. It equals $\ln 2$.

Harmonic Series

An ant starts to crawl along a taut rubber rope 1 km long at a speed of 1 cm per second (relative to the rubber it is crawling on). At the same time, the rope starts to stretch uniformly by 1 km per second, so that after 1 second it is 2 km long, after 2 seconds it is 3 km long, etc. Will the ant ever reach the end of the rope? Counterintuitively, yes. This is a consequence of the divergent harmonic series.

Another example is the block-stacking problem: given a collection of identical dominoes, it is clearly possible to stack them at the edge of a table so that they hang over the edge of the table without falling. The counterintuitive result is that one can stack them in such a way as to make the overhang arbitrarily large, provided there are enough dominoes.

Harmonic Series

Absolute Convergence

A series $\sum a_n$ converges absolutely if the corresponding series of absolute values $\sum |a_n|$ converges. Are the following two series absolutely convergent?

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots \tag{47}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \tag{48}$$

A series that converges but does not converge absolutely is said to converge conditionally. If $\sum |a_n|$ converges, then $\sum a_n$ must converge. Absolutely (and ONLY absolutely) convergent series can be rearranged. The alternating harmonic series can be rearranged to diverge or to reach any preassigned infinite sum.

Power Series

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots \tag{49}$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots \tag{50}$$

in which the centre a and the coefficients c_0, c_1, c_2 are real numbers.

The Term-by-Term Differentiation Theorem

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $a-R<x<a+R$ for some $R>0$, it defines a function f

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n \text{ on the domain } a-R<x<a+R \tag{51}$$

Such a function f has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term.

$$f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \tag{52}$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2} \tag{53}$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

The Term-by-Term Integration Theorem

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $a-R<x<a+R$ for some $R>0$, it defines a function f

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n \text{ on the domain } a-R<x<a+R \tag{54}$$

Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \tag{55}$$

converges for $a-R<x<a+R$ and

$$\int f(x)\,dx=\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}+C \tag{56}$$

for $a-R<x<a+R$.

First Power Series Expansions

Use these two theorems to find power series expansions for $f(x) = \arctan x$ and $g(x) = \ln(1+x)$ on the domain $-1 < x < 1$.

Use the following two functions to succeed in this endeavour.

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \tag{57}$$

$$g(x) = 1 - x + x^2 - x^3 + \dots \tag{58}$$

The Series Multiplication Theorem for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k} \qquad (59)$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$,

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n \qquad (60)$$

Use term-by-term differentiation and the series multiplication theorem for power series independently to show that for $|x| < 1$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \qquad (61)$$

Now think about it the other way around. If a power series gives us a continuous function with derivatives of all orders, will a continuous function with derivatives of all orders give us a power series? What would be the coefficients? Let's assume that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \tag{62}$$

with a positive radius of convergence.

Then

$$f^{(n)}(x) = n!a_n + \text{ a sum of terms with } x - a \text{ as a factor} \qquad (63)$$

Since these equations all hold at $x = a$, we have

$$\begin{aligned} f'(a) &= 1 \cdot a_1 \\ f''(a) &= 1 \cdot 2 \cdot a_2 \\ f'''(a) &= 1 \cdot 2 \cdot 3 \cdot a_3 \end{aligned} \qquad (64)$$

and in general $f^{(n)} = n!a_n$.

So, if (and that's a significant "if") a function f has a series representation, then the series must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots +$$

$$\frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots \tag{65}$$

If f is infinitely differentiable, then this series is determined, but it is not always true that the series has a positive radius of convergence. All kinds of things can go wrong. For example, the function

$$f(x) = e^{-\frac{1}{x^2}} \tag{66}$$

has a Mclaurin series which converges everywhere but only at $x = 0$ does the limit equal $f(x)$!

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) +$$

$$\frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots \qquad (67)$$

The Mclaurin series generated by f is

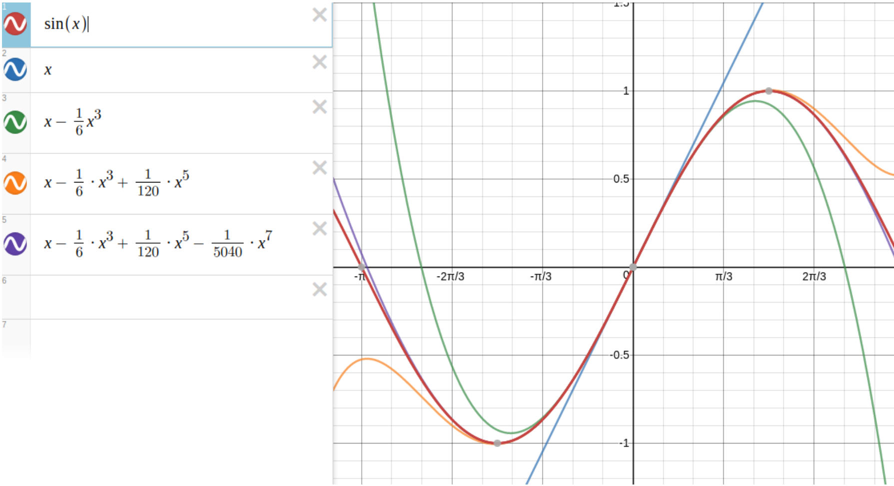
$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x)^k = f(0) + f'(0)(x) +$$

$$\frac{f''(0)}{2!}(x)^2 + \ldots + \frac{f^{(n)}(0)}{n!}(x)^n + \ldots \qquad (68)$$

Taylor and Mclaurin Polynomials

When we learned differentiation, we learned about the linear approximation of a function $f(x)$ at $x = a$. This linear approximation turns out to be the Taylor polynomial of order 1. A Taylor polynomial (or Mclaurin polynomial) is a Taylor series with the tail cut off. If a function has a Taylor series expansion, you can approximate it arbitrarily well with a polynomial as seen on the next slide for $\sin(x)$.

Taylor and Mclaurin Polynomials



Taylor Series Exercises

Find the Taylor polynomials of orders 0, 1, 2, 3 generated by f at a.

$$f(x) = \ln x,\ a = 1 \tag{69}$$

$$f(x) = \frac{1}{x},\ a = 2 \tag{70}$$

$$f(x) = \sin x,\ a = \frac{\pi}{4} \tag{71}$$

$$f(x) = \sqrt{x},\ a = 4 \tag{72}$$

$$f(x) = \cos x,\ a = \frac{\pi}{4} \tag{73}$$

Mclaurin Series Exercises

Find the Mclaurin series for the following functions.

$$f(x) = e^{-x} \tag{74}$$

$$f(x) = e^{\frac{x}{2}} \tag{75}$$

$$f(x) = \frac{1}{1+x} \tag{76}$$

$$f(x) = \cosh x \tag{77}$$

$$f(x) = \sinh x \tag{78}$$

End of Lesson

Next Lesson: Multivariable Calculus