Newton's Method, L'Hôpital's Rule

MATH 2511, BCIT

Technical Mathematics for Geomatics

March 5, 2018

What are the x-intercepts of the following function?

$$f(x) = 2x^3 + 5x^2 - 11x + 3$$

We have not learned how to find x-intercepts for polynomials with degrees higher than 2. There are different methods. One method is called Newton's Method and approximates the x-intercept. I have created an instructional video for Newton's Method which you can watch here:

https://youtu.be/a28M5f0Dk_c

For Newton's Method, find a plausible x-value x_1 (near enough to the x-intercept that you are trying to find) and approximate the x-intercept using the following iterative procedure:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

Exercise 1: Approximate $\sqrt{7}$ to ten decimal places using Newton's method and the function $h(x) = x^2 - 7$.

Exercise

Exercise 2: Approximate the x-intercept of $f(x) = x^3 + 5x - 3$ using Newton's method.

Exercise

Exercise 3: Factor $g(t) = 24t^3 - 2t^2 - 9t + 2$. Remember that if x_1, x_2, x_3 are x-intercepts of the polynomial $ax^3 + bx^2 + cx + d$, then

$$ax^3 + bx^2 + cx + d = a(x - x_1)(x - x_2)(x - x_3) \tag{2}$$

Exercise 4: Find the x-intercepts for the following function:

$$f(x) = x^3 + 4x^2 + x - 6 \tag{3}$$

Exercise 5: Solve the equation

$$\cos x = x \tag{4}$$

using Newton's Method.

Exercise 6: Analyze the following function:

$$f(x) = \frac{2x^2 + 2}{x - 3} \tag{5}$$

Exercise

Exercise 7: Find one solution for the following equations using Newton's Method.

$$x^6 - x^5 - 6x^4 - x^2 + x + 10 = 0 \tag{6}$$

$$x^2(4 - x^2) = \frac{4}{x^2 + 1} \tag{7}$$

$$x^2\sqrt{2 - x - x^2} = 1 \tag{8}$$

$$4e^{-x^2}\sin x = x^2 - x + 1 \tag{9}$$

$$3\sin(x^2) = 2x \tag{10}$$

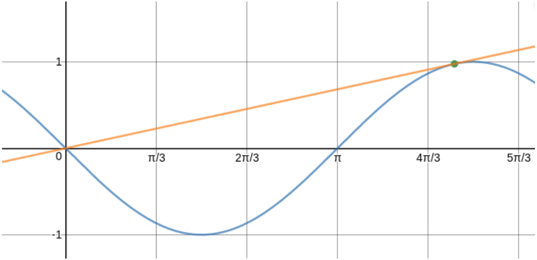
Exercise 8: Find the absolute minimum value of the following function correct to four decimal places.

$$f(x) = x^6 - x^4 + 3x^2 - 2x \tag{11}$$

Exercise 9: Of the infinitely many lines that are tangent to the curve

$$y = -\sin x \tag{12}$$

and pass through the origin, there is one that has the largest slope. Use Newton's Method to find the slope of that line.

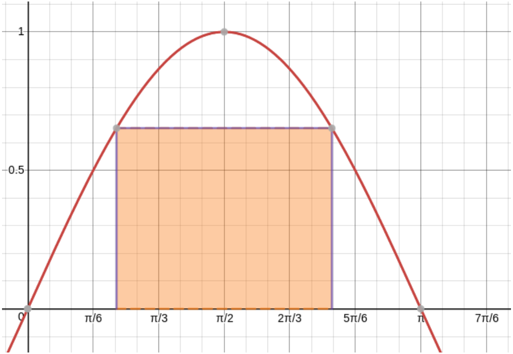


Exercise 10: Use Newton's Method to find the coordinates of the point on the parabola

$$y = (x - 1)^2 \tag{13}$$

that is closest to the origin.

Exercise 11: Inscribe a rectangle based on the x-axis under the sine curve between $x = 0$ and $x = \pi$. Maximize the area. Provide the dimensions of the rectangle. Use Newton's Method.



Evaluate the following limits:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} \tag{14}$$

$$\lim_{x \to 0} \frac{\sin x}{x} \text{ (it equals 1 based on geometry)} \tag{15}$$

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} \tag{16}$$

These limits have in common that they are of indeterminate form when you plug in the a towards which the x goes. Sometimes the tricks we have found don't work, for example for

$$\lim_{x \to 1} \frac{\ln x}{x - 1} \tag{17}$$

or for

$$\lim_{x \to \infty} \frac{\ln x}{x - 1} \tag{18}$$

L'Hôpital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Exercise

Exercise 12: Find

$$\lim_{x \to 1} \frac{\ln x}{x - 1} \qquad (19)$$

Exercise

Exercise 13: Find

$$\lim_{x\to\infty} \frac{e^x}{x^2} \tag{20}$$

Exercise 14: Find

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} \tag{21}$$

Exercise 15: Find

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} \tag{22}$$

Exercise Solution

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} \overset{\text{LHR}}{=} \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \frac{0}{0} \qquad (23)$$

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \overset{\text{LHR}}{=} \lim_{x\to 0} \frac{2\tan x \sec^2 x}{6x} = \frac{0}{0} \qquad (24)$$

$$\lim_{x\to 0} \frac{\tan x \sec^2 x}{3x} = \lim_{x\to 0} \frac{\sec^4 x + 2\tan^2 x \sec^2 x}{3} = \frac{1}{3} \qquad (25)$$

Exercise 16: Fynd

$$\lim_{x \to \pi} \frac{\pi - \pi \cos x + \sin x}{1 - \cos x} \tag{26}$$

Exercise

Exercise 17: Find

$$\lim_{x \to 0^+} x \ln x \tag{27}$$

Exercise 18: Find

$$\lim_{x \to \frac{\pi}{2}^-} (\sec x - \tan x) \tag{28}$$

Exercise 19: Find

$$\lim_{x \to 0^+} x^x \tag{29}$$

Exercise 20: Find

$$\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x \tag{30}$$

Note that this is a question-begging exercise. e is defined to be the limit in (30); this exercise only shows that l'Hôpital's rule is consistent with this definition.

Exercise 21: Find

$$\lim_{x\to 0^+} (1 + \sin 4x)^{\cot x} \tag{31}$$

Exercise

Exercise 22: Find

$$\lim_{x\to\infty}(\sqrt{x^2+x}-x)$$

Note: this is an example where l'Hôpital's rule does not work and another method has to be used. This is not unusual for radicals. Further examples include

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} \tag{32}$$

$$\lim_{x\to\infty}(\sqrt{x^2+1}-x) \tag{33}$$

Exercise Solution

$$\lim_{x\to\infty}(\sqrt{x^2+x}-x)=\lim_{x\to\infty}(\sqrt{x(x+1)}-x)=\lim_{x\to\infty}(\sqrt{x}(\sqrt{x+1}-\sqrt{x})) \tag{34}$$

Multiply by the conjugate for

$$\lim_{x\to\infty}(\sqrt{x}(\sqrt{x+1}-\sqrt{x}))=\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} \tag{35}$$

Take the reciprocal for

$$\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}=\left(\lim_{x\to\infty}\left(1+\sqrt{\frac{x+1}{x}}\right)\right)^{-1}=\frac{1}{2} \tag{36}$$

Exercise 23: Find

$$\lim_{x\to 1} \frac{1 - x + \ln x}{1 + \cos \pi x} \tag{37}$$

Exercise 24: Find

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \tag{38}$$

Exercise

Exercise 25: Find

$$\lim_{x \to 0^+} \sin x \ln x \tag{39}$$

Exercise 26: Find

$$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) \tag{40}$$

End of Lesson

Next Lesson: Fundamental Theorem of Calculus