Integration Methods

MATH 2511, BCIT

Technical Mathematics for Geomatics

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Integration Methods

We will learn about the following integration methods:

1. Using Integration Tables
2. Integration by Substitution
3. Integration by Parts
4. Trigonometric Integrals
5. Trigonometric Substitutions
6. Partial Fractions
7. Improper Integrals

Integration by Substitution

We know how to integrate the following functions

$$f_1(y) = y^3 \text{ and } f_2(x) = 2x + 5 \tag{1}$$

but how do you integrate $f = f_1 \circ f_2$, for example

$$f(x) = (2x + 5)^3 \tag{2}$$

We use the method of substitution. Write

$$u = 2x + 5 \tag{3}$$

Remember that the definition of differentials is as follows. If $u = f(x)$ and dx is some real number (usually small), then

$$du = f'(x)\,dx \tag{4}$$

The substitution changes the differential and the limits. For $u = 2x + 5$

$$du = 2dx \text{ and therefore } dx = \frac{1}{2}du \tag{5}$$

Consequently,

$$\int_a^b (2x+5)^3\,dx = \int_{2a+5}^{2b+5} u^3 \cdot \frac{1}{2}\,du \tag{6}$$

Integration by Substitution Four Steps

Step 1: Find Substitution replace $2x + 5$ by u (not all expressions involving x have to disappear yet)

Step 2: Find Substitution for Differential $du = 2 \cdot dx$, therefore $dx = \frac{1}{2} du$

Step 3: Perform Integration find $\frac{1}{2} \int u^3 \, du$

Step 4: Reverse the Substitution replace u by $2x + 5$ in the final result for the indefinite integral

Example 1: Integration by Substitution. Let's evaluate

$$\int_0^4 x\sqrt{9+x^2}\,dx \tag{7}$$

We will do this two ways.

method 1 Find the indefinite integral of $x\sqrt{9+x^2}$ and then use the limits $a = 0, b = 4$ to evaluate the definite integral.

method 2 Proceed as on the previous slide and change both differential and limits for the definite interval.

Integration by Substitution Example II

Here is method 1. Substitute $u = 9 + x^2$. Then, $du = 2x\,dx$, so

$$\frac{1}{2}\,du = x\,dx \tag{8}$$

Notice that we need the factor x on the right-hand side in order to make this integration work.

$$\int x\sqrt{9+x^2}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \tag{9}$$

Integration by Substitution Example III

Now reverse the substitution

$$\frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \frac{1}{3}(9 + x^2)^{\frac{3}{2}} \tag{10}$$

and evaluate the definite integral

$$\int_0^4 x\sqrt{9 + x^2}\, dx =$$

$$\frac{1}{3}(9 + x^2)^{\frac{3}{2}}\bigg|_{x=4} - \frac{1}{3}(9 + x^2)^{\frac{3}{2}}\bigg|_{x=0} = \frac{98}{3} \tag{11}$$

Integration by Substitution Example IV

Here is method 2.

$$\int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{2}\int_9^{25}\sqrt{u}\,du =$$

$$\frac{1}{3}\left(u^{\frac{3}{2}}\Big|_{u=25} - u^{\frac{3}{2}}\Big|_{u=9}\right) = \frac{1}{3}(125-27) = \frac{98}{3} \qquad (12)$$

Example 2: Integration by Substitution. Here is a more complicated example of substitution. Find

$$\int x^5 \sqrt{1+x^2}\, dx \tag{13}$$

Use the following trick: substitute $u = 1 + x^2$. The new differential is $du = 2x\, dx$. Take care of it by factoring $x^5 = x^4 \cdot x$. Now what to do with x^4? Notice that $x^4 = (u-1)^2$.

Integration by Substitution Example VI

Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 seconds. The maximum rate of air flow into the lungs is about 0.5 litres per second. This explains, in part, why the function

$$f(t) = \frac{1}{2} \sin\left(\frac{2\pi}{5} t\right) \tag{14}$$

has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t.

Antiderivative of tan and cot

We can use the substitution method to find the antiderivative of the tangent and the cotangent. For $u = \cos x$, note that $du = -\sin x\, dx$. Then,

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int u^{-1} du = -\ln|u| + c =$$

$$-\ln|cos x| + c = \ln|\sec x| + c \tag{15}$$

Now try a similar idea for the $\cot x$, which yields

$$\int \cot x\, dx = \ln|\sin x| + c \tag{16}$$

Evaluate the following definite integrals.

$$\int_0^2 x(x^2-1)^3\,dx \qquad \int_0^1 x^2(2x^3-1)^4\,dx \tag{17}$$

$$\int_0^1 x\sqrt{5x^2+4}\,dx \qquad \int_1^3 x\sqrt{3x^2-2}\,dx \tag{18}$$

$$\int_0^2 x^2(x^3+1)^{\frac{3}{2}}\,dx \qquad \int_1^5 (2x-1)^{\frac{5}{2}}\,dx \tag{19}$$

$$\int_0^1 \frac{1}{\sqrt{2x+1}}\,dx \qquad \int_0^2 \frac{x}{\sqrt{x^2+5}}\,dx \tag{20}$$

Evaluate the following definite integrals.

$$\int_1^2 (2x+4)(x^2+4x-8)^3\,dx \qquad \int_{-1}^1 x^2(x^3+1)^4\,dx$$

$$\int_0^2 xe^{x^2}\,dx \qquad \int_0^1 e^{-1}\,dx$$

$$\int_3^6 \frac{2}{x-2}\,dx \qquad \int_0^1 \frac{e^x}{1+e^x}\,dx$$

$$\int_0^1 \frac{x}{1+2x^2}\,dx \qquad \int_1^2 \frac{\ln x}{x}\,dx$$

Here are integration tables and tables of derivatives to last you for a while:

`http://www.ambrsoft.com/Equations/Derivation/Derivation.htm`

Example 3: Using an Integration Table. Evaluate the indefinite integral

$$\int -7\sqrt{\cot x}\,\csc^2 x\,dx \tag{21}$$

The derivative of $f(x) = \cot x$ is $f'(x) = -\csc^2 x$. Using the substitution $u = \cot x$ and $du = -\csc^2 x\,dx$ yields

$$\int -7\sqrt{\cot x}\,\csc^2 x\,dx = 7\int u^{\frac{1}{2}}\,du = \frac{14}{3}\sqrt{\cot^3 x} + C \tag{22}$$

Example 4: Using an Integration Table. Evaluate the indefinite integral

$$\int \frac{9-9x}{1+x^2}\,dx \tag{23}$$

Gather from an integration table (or a table of derivatives) that if $f(x) = \arctan x$ then $f'(x) = 1/(1+x^2)$. Therefore, using the substitution $u = 1 + x^2$ with $du = 2x\,dx$,

$$\int \frac{9-9x}{1+x^2}\,dx = 9 \cdot \left(\int \frac{1}{1+x^2}\,dx - \int \frac{x}{1+x^2}\,dx \right) =$$

$$9 \arctan x - \frac{9}{2} \ln |1+x^2| + C \tag{24}$$

Example 5: Using an Integration Table. Evaluate the indefinite integral

$$\int \frac{5x}{\sqrt{3-x^4}}\,dx \tag{25}$$

Notice in the integration table that

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C \tag{26}$$

Thus, substituting $u = x^2$ and $du = 2x\,dx$,

$$\int \frac{5x}{\sqrt{3-x^4}}\,dx = \frac{5}{2}\arcsin\left(\frac{x^2}{\sqrt{3}}\right) + C \tag{27}$$

Find the following integrals.

$$\int \frac{1}{\sqrt{8x - x^2}}\,dx^{*} \tag{28}$$

$$\int_0^{\frac{\pi}{4}} \frac{1}{1 - \sin x}\,dx^{\dagger} \tag{29}$$

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx \tag{30}$$

$$\int_4^{(e+1)^2} \frac{1}{x - \sqrt{x}}\,dx \tag{31}$$

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*Hint: complete the square to find out that $8x - x^2 = 16 - (x - 4)^2$.

†Hint: expand the fraction by $1 + \sin x$.

For the integral in (31), there are two ways to solve this problem.

1. Multiply by the conjugate, simplify, and substitute $u = x - 1$. Then use formula 29a from Thomas' table of integrals,

$$\int \frac{1}{x\sqrt{ax+b}}\,dx = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}} \right| + C \qquad (32)$$

2. Substitute $u^2 = x$. It is generally a good idea to try substituting expressions under a square root by u^2.

The solution for the definite integral is 2.

There is no product rule for integration, so integrals of the form

$$\int f(x) \cdot g(x)\, dx \tag{33}$$

are a problem. Notice, however, that

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \tag{34}$$

and therefore

$$\int f'(x)g(x)\, dx + \int f(x)g'(x)\, dx = f(x)g(x) + C \tag{35}$$

Consequently,

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx + C \tag{36}$$

If we happen to know everything on the right-hand side (RHS), then we have an integral for the left-hand side (LHS).

Example 6: Integration by Parts. Find

$$\int x \cos x \, dx \tag{37}$$

If we choose $f(x) = \cos x$ and $g'(x) = x$, then integration by parts yields

$$\int x \cos x \, dx = \frac{1}{2}x^2 \cos x + \frac{1}{2}\int x^2 \sin x \, dx \tag{38}$$

We have not helped our cause. Let's try this the other way around with $f(x) = x$ and $g(x) = \sin x$. Then

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx = x \sin x + \cos x + C \tag{39}$$

Success!

Integration by Parts

When we learned integration by substitution, we were able to find the antiderivative of $\tan x$ and $\cot x$. Now it is time to find the antiderivative of $\ln x$. Use integration by parts for

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx \tag{40}$$

and find out that

$$\int \ln x \, dx = x \ln x - x + C \tag{41}$$

Add this integral to your personal list.

Exercises. Evaluate the following integrals.

$$\int x^2 e^x \, dx \tag{42}$$

$$\int e^x \cos x \, dx \tag{43}$$

$$\int \cos^n x \, dx \tag{44}$$

For (43), you will need to use a trick. For (44), all we want is a reduction formula to decrease the exponent n to express the integral in terms of $\int \cos^{n-1} x \, dx$.

Trigonometric Integrals

There are several tricks for integrals with trigonometric functions. It is best to consult a textbook when you have to solve a particular integral. Sometimes we can solve an integral that doesn't involve trigonometric functions by substituting trigonometric functions: this is called trigonometric substitution. Here is our first challenge: solve integrals of the form

$$\int \sin^m x \cos^n x \, dx \tag{45}$$

Trigonometric Integrals Case (1)

Distinguish two cases: (1) one of the exponents is odd; (2) both exponents are even. In case (1), notice that for some natural number k

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x \qquad (46)$$

I have assumed here that the sine has the odd exponent. If the sine's exponent is even then use the cosine's exponent instead.

Trigonometric Integrals Case (1)

Let's demonstrate the rest of the procedure by example, using the substitution $u = \cos x$ and $du = -\sin x\, dx$

$$\int \sin^3 x \cos^2 x\, dx = \int (1 - \cos^2 x) \cos^2 x \sin x\, dx = -\int (1 - u^2) u^2\, du =$$

$$-\int u^2\, du + \int u^4\, du = -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

Here is what happens when the sine's exponent is even. Substitute $u = \sin x$ and $du = \cos x \, dx$.

$$\int \cos^3 x \sin^2 x \, dx = \int (1 - \sin^2 x) \sin^2 x \cos x \, dx = \int (1 - u^2) u^2 \, du =$$

$$\int u^2 \, du - \int u^4 \, du = \frac{1}{3} u^3 - \frac{1}{5} u^5 + C = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$$

Trigonometric Integrals Case (2)

If both exponents are even, in case (2), remember that

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ 1 &= \cos^2 x + \sin^2 x \end{aligned} \tag{47}$$

Add and subtract these two equations for

$$\begin{aligned} \sin^2 x &= \tfrac{1}{2} - \tfrac{1}{2}\cos 2x \\ \cos^2 x &= \tfrac{1}{2} + \tfrac{1}{2}\cos 2x \end{aligned} \tag{48}$$

Trigonometric Integrals Case (2)

Substitute (48), as in the following example.

$$\int \cos^4 x \sin^2 x \, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^2 \cdot \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) \, dx$$

$$\int \cos^4 x \sin^2 x \, dx = \frac{1}{8} \int \left(1 + \cos 2x - \cos^2 2x - \cos^3 2x\right) \, dx$$

You can use conventional methods and reducing the term involving $\cos^2 2x$ again to provide the solution

$$\int \cos^4 x \sin^2 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) \tag{49}$$

Trigonometric Integrals Case (2)

You can solve case (2) by using trigonometric identities, as on the last slide; you can also solve it by using integration by parts. Consider the example on the next slide.

On the next slide, make sure that the solutions in (50) and (51) agree (use the double angle formula for $\sin 2x$).

Trigonometric Integrals Case (2)

Using trigonometric identities,

$$\int \sin^2 x\, dx = \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right)\, dx = \frac{1}{2}x - \frac{1}{4} sin2x + C \quad (50)$$

Using integration by parts,

$$\int \sin^2 x\, dx = \int \sin x \sin x\, dx = -\sin x \cos x + \int \cos^2 x\, dx =$$

$$-\sin x \cos x + \int \left(1 - \sin^2 x\right)\, dx = -\sin x \cos x + x - \int \sin^2 x\, dx$$

Note that for $A = \int \sin^2 x\, dx$, this means that

$$2A = -\sin x \cos x + x + C, \text{ so } A = -\frac{1}{2} \sin x \cos x + \frac{1}{2}x + C \quad (51)$$

Exercises

$$\int \cos^3 x \, dx \tag{52}$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x \, dx \tag{53}$$

$$\int_0^{\frac{\pi}{6}} 3\cos^5 3x \, dx \tag{54}$$

$$\int_0^{\pi} \sqrt{1 - \cos 2x} \, dx^{\ddagger} \tag{55}$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 16 \sin^2 x \cos^2 x \, dx \tag{56}$$

$$\int_0^{\frac{\pi}{2}} 35 \sin^4 x \cos^3 x \, dx \tag{57}$$

‡Hint: Use the double-angle formula to get rid of the square root sign.

Partial Fractions

Find the integral

$$\int \frac{5x-3}{x^2-2x-3} \tag{58}$$

We have no quotient rule for integration, so this integral presents a challenge. If we could express the rational function as a sum of simpler fractions, called partial fractions, we may be able to solve this. First, factor the denominator

$$x^2-2x-3=(x+1)(x-3) \tag{59}$$

Then find A and B for

$$\frac{5x-3}{x^2-2x-3}=\frac{A}{x+1}+\frac{B}{x-3} \tag{60}$$

Getting rid of all the fractions, (60) is equivalent to

$$5x+(-3)=(A+B)x+(B-3A) \tag{61}$$

(61) is true only when

$$\begin{array}{rcrcr} A & + & B & = & 5 \\ -3A & + & B & = & -3 \end{array} \tag{62}$$

This system of linear equations has the solution $A = 2$ and $B = 3$. Therefore,

$$\int \frac{5x-3}{x^2-2x-3} = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx =$$

$$2\ln|x+1| + 3\ln|x-3| + C \tag{63}$$

Exercise 1: Use partial fractions to evaluate the following integrals.

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx \tag{64}$$

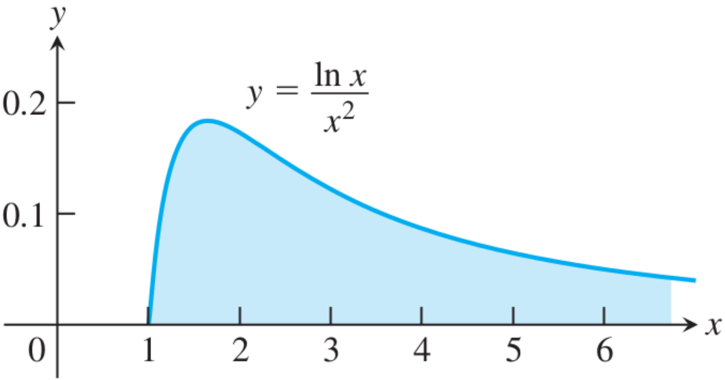
$$\int \frac{6x+7}{(x+2)^2}\,dx^{\S} \tag{65}$$

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx^{\P} \tag{66}$$

§Hint: In this case, the denominator for A is $x+2$ and the denominator for B is $(x+2)^2$.

¶Hint: Use polynomial division for $2x^3-4x^2-x-3 = 2x(x^2-2x-3)+(5x-3)$.

There are sometimes infinite curves with finite areas under them. Consider the following two examples.



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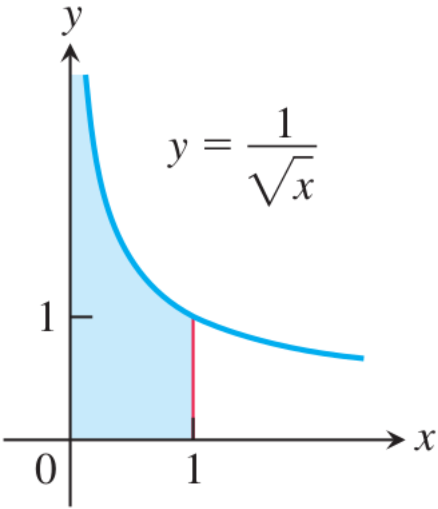

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Improper Integrals

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Improper Integrals Exercises

$$\int_0^{\infty} e^{-\frac{x}{2}} \tag{67}$$

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx \tag{68}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx \tag{69}$$

$$\int_0^{1} \frac{1}{1-x}\, dx \tag{70}$$

$$\int_0^{3} \frac{1}{(x-1)^{\frac{2}{3}}}\, dx \tag{71}$$

$$\int_{-\infty}^{0} e^{-|x|}\, dx \tag{72}$$

$$\int_0^{1} x \ln x\, dx \tag{73}$$

End of Lesson

Next Lesson: Maclaurin and Taylor Series Expansion