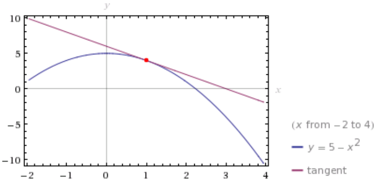
Fundamental Theorem of Calculus

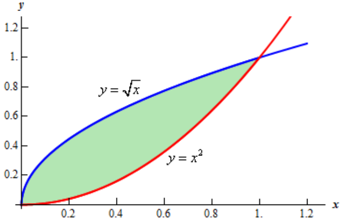
MATH 2511, BCIT

Technical Mathematics for Geomatics

March 9, 2018

Remember these two problems that we wanted to solve when we started with calculus:





We have solved the problem on the left. Now it is time to solve the problem on the right. For areas under a curve, we need antiderivatives. The antiderivative $F(x)$ of a function $f(x)$ is the function for which $F'(x) = f(x)$.

Differential Equations

Differential equations are like regular equations except that the unknown is a function, not a variable. Remember that

$$dy = f'(x)\,dx, \text{ therefore } f'(x) = \frac{dy}{dx} \tag{1}$$

Now consider this differential equation,

$$\frac{dy}{dx} = f(x) \tag{2}$$

This is an ODE, an ordinary differential equation.

Differential Equations

$$\frac{dy}{dx} = f(x) \tag{3}$$

This is an ODE, an ordinary differential equation. Any function

$$f(x) = e^x + C, C \in \mathbb{R} \tag{4}$$

would solve it. Often, an initial condition is provided to make the solution unique. Therefore, the solution to the differential equation

$$\frac{dy}{dx} = f(x) \tag{5}$$

with initial condition $f(0) = 1$ is $f(x) = e^x$.

Differential Equations

Antiderivatives are solutions to special differential equations. For example, the antiderivative of $f(x) = 6x$ is the solution to the differential equation

$$\frac{dy}{dx} = 6x \tag{6}$$

With an initial condition, the solution to this equation may be unique.

Rules for Finding Antiderivatives

Antiderivatives are not unique. If $F(x)$ is an antiderivative for $f(x)$, then $F(x)+c$ is an antiderivative as well, where c is any real number. In the following, we will use the notation $F(x)$ for one arbitrary antiderivative. There are many rules for finding antiderivatives called *table of integrals*. Here are a few.

Rule 1

If you find a function $g(x)$ for which $g'(x)=f(x)$, then $F(x)=g(x)+c$.

Exercise: show that the function $g(x)$ is an antiderivative of $f(x)=(x^3+3)^6(3x^2)$.

$$g(x)=\frac{(x^3+3)^7}{7} \tag{7}$$

More Rules for Finding Antiderivatives I

Rule 2

If $F(x)$ is an antiderivative for $f(x)$, then $aF(x)$ is an antiderivative for $af(x)$, where a is a constant.

More Rules for Finding Antiderivatives II

Rule 3

If $F_1(x)$ is an antiderivative for $f_1(x)$ and $F_2(x)$ is an antiderivative for $f_2(x)$, then $F_1(x) + F_2(x)$ is an antiderivative for $f_1(x) + f_2(x)$.

More Rules for Finding Antiderivatives III

Rule 4

If $f(x) = x^n$ and $n \neq -1$, then $F(x) = \frac{x^{n+1}}{n+1}$ is an antiderivative of $f(x)$.

Exercise: Find an antiderivative of $f(x) = 1/x$. The answer is not quite what you would expect (but very close).

Summary

Here is a table of antiderivatives, where F is an antiderivative of f and G is an antiderivative of g.

$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n with $n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\frac{1}{1+x^2}$	$\arctan x$

Integration

The process of finding a derivative is called differentiation. The process of finding an antiderivative is called integration. Instead of the symbol 'prime' ($f'(x)$) for differentiation we use the sign $\int$ for integration. The symbol $\int$ stands for the word 'sum' because we take the limit of a sum of areas in order to find the area under a curve.

$$\int f(x)\,dx = F(x) + c \tag{8}$$

The differential helps to identify which letter is the variable for the function (there may be other letters that are just constants), for example

$$\int ax^2\,dx = \frac{ax^3}{3} + c \tag{9}$$

$$\int ax^2\,da = \frac{a^2x^2}{2} + c \tag{10}$$

Integration Exercises I

Find the following indefinite integrals (another expression for antiderivatives).

$$\int 6\,dx \tag{11}$$

$$\int -2\,dx \tag{12}$$

$$\int 8x^4\,dx \tag{13}$$

$$\int \pi x^3\,dx \tag{14}$$

$$\int \left(x^3 + 7 - 2x^2\right)\,dx \tag{15}$$

$$\int \sqrt{x}\,dx \tag{16}$$

$$\int \frac{7}{2}x^{\frac{5}{2}}\,dx \tag{17}$$

Integration Exercises II

Find the following indefinite integrals (another expression for antiderivatives).

$$\int 9\sqrt[5]{2x}\, dx \tag{18}$$

$$\int \frac{3}{x^3}\, dx \tag{19}$$

$$\int \frac{7}{\sqrt[3]{x}}\, dx \tag{20}$$

$$\int \sqrt{x}\,(3x-2)\; dx \tag{21}$$

$$\int (x+1)^2\; dx \tag{22}$$

$$\int \frac{4x^2-2\sqrt{x}}{x}\, dx \tag{23}$$

$$\int \frac{x^3+2x^2-3x-6}{x+2}\, dx \tag{24}$$

Evaluating an integral at a point doesn't give us anything particularly meaningful.

$$\int x^2\,dx = \frac{x^3}{3} + c \tag{25}$$

$$\int x^2\,dx\bigg|_{x=6} = \frac{6^3}{3} + c = 72 + c \tag{26}$$

However, if we subtract one evaluated integral from another, we get a number.

$$\int x^2\,dx\bigg|_{x=6} - \int x^2\,dx\bigg|_{x=3} = \frac{6^3}{3} + c - \left(\frac{3^3}{3} + c\right) = 72 - 9 = 63$$

Definite Integrals II

We call this difference between evaluated integrals definite integral. The notation is

$$\int_3^6 x^2\,dx = \int x^2\,dx\bigg|_{x=6} - \int x^2\,dx\bigg|_{x=3} = 63$$

Evaluate each definite integral.

$$\int_1^2 x\,dx \qquad \int_{-2}^2 x^2\,dx \tag{27}$$

$$\int_1^3 7x^2\,dx \qquad \int_{-2}^2 3s^4\,ds \tag{28}$$

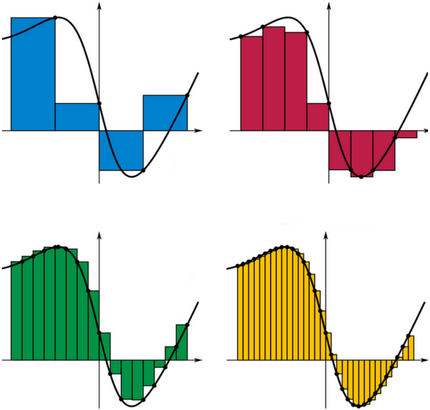
$$\int_0^4 (x^2+2x)\,dx \qquad \int_1^e \frac{1}{x}\,dx \tag{29}$$

$$\int_5^{10} \sqrt{x}\,dx \qquad \int_1^4 \frac{2+x^2}{\sqrt{x}}\,dx \tag{30}$$

$$\int_{-1}^2 (3u-2)(u+1)\,du \qquad \int_{\frac{\pi}{6}}^{\pi} \sin\vartheta\,d\vartheta \tag{31}$$

Fundamental Theorem of Calculus I

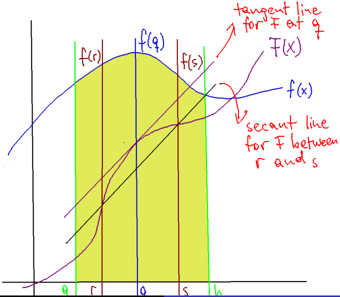
It turns out that the definite integral $\int_a^b f(x)\,dx$ gives you the area under the curve $y = f(x)$ between a and b. This area can be approximated by a series of rectangles.



Fundamental Theorem of Calculus II

Let's assume our function is positive between a and b, so $f(x) \geq 0$ for $a \leq x \leq b$. Let F be an antiderivative of f. Here is the mean value theorem, a theorem we need to assume without proof: between two arguments r and s we can always find a point q such that the slope of the secant line between $F(r)$ and $F(s)$ equals the slope of the tangent line at $F(q)$, so

$$F'(q) = \frac{F(s) - F(r)}{s - r} \text{ (MVT)}$$



Fundamental Theorem of Calculus III

Now divide the interval from a to b (the notation for this interval is $[a, b]$) into n intervals that are of equal length. For this, we need intermediate points $a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$. The approximate area under the curve between a and b is

$$A \approx \frac{x_1 - a}{n} f(x_1^*) + \frac{x_2 - x_1}{n} f(x_2^*) + \ldots + \frac{b - x_{n-1}}{n} f(x_n^*) \qquad (32)$$

where x_1^* is some point in the first interval and so on. Notice that the fractions all equal $(b-a)/n$ because the intervals are all of equal length. Therefore

$$A = \lim_{n \to \infty} \frac{b-a}{n} \left(f(x_1^*) + \ldots + f(x_n^*) \right) \qquad (33)$$

Fundamental Theorem of Calculus IV

Now choose x_1^* such that

$$f(x_1^*) = F'(x_1^*) = \frac{F(x_1) - F(x_0)}{x_1 - x_0} \tag{34}$$

and so on with $x_2^*, x_3^*, \ldots, x_n^*$. Then

$$A = \lim_{n \to \infty} \frac{b-a}{n} \left(\frac{F(x_1) - F(x_0)}{x_1 - x_0} + \ldots + \frac{F(x_n) - F(x_{n-1})}{x_n - x_{n-1}} \right) \tag{35}$$

Note that $x_i - x_{i-1}$ (where i is any number between 1 and n) is again just the length of the intervals $(b-a)/n$. After appropriate simplification,

$$A = F(b) - F(a) = \int_a^b f(x)\, dx \tag{36}$$

Fundamental Theorem of Calculus V

Here are two different ways to express the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Note that we need not require $a \leq b$. If the limits of integration are unintuitively placed, you can rectify the situation by using

$$\int_b^a f(x)\,dx = F(a) - F(b) = -(F(b) - F(a)) = -\int_a^b f(x)\,dx$$

Fundamental Theorem of Calculus Exercises

Exercise 1: Find the area under the parabola

$$y = x^2 \tag{37}$$

from 0 to 1.

Exercise 2: Find the area under the cosine curve from 0 to b, where $0 \leq b \leq \frac{\pi}{2}$.

Example 1: Distance Problem. Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times. If the velocity remains constant, then the distance problem is easy to solve by means of the formula distance

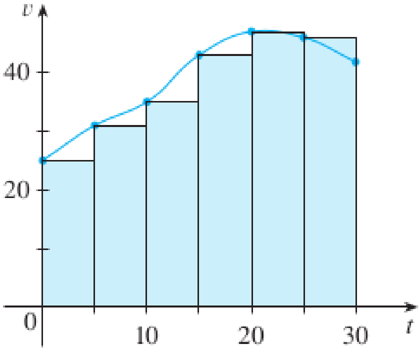
$$\text{distance} = \text{velocity} \times \text{time} \tag{38}$$

But if the velocity varies, it's not so easy to find the distance traveled. Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table.

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

Other Applications: Distance

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41



Other Applications: Net Change Theorem

Net Change Theorem

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a) \tag{39}$$

Example 2: Water. If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1) \tag{40}$$

is the change in the amount of water in the reservoir between time t_1 and t_2.

Example 3: Production Cost. If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$. So

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1) \tag{41}$$

is the increase in cost when production is increased from x_1 units to x_2 units.

Exercise 3: Water flows from the bottom of a storage tank at a rate of

$$r(t) = 200 - 4t \tag{42}$$

litres per minute, where $0 \leq t \leq 50$. Find the amount of water that flows from the tank during the first 10 minutes.

End of Lesson

Next Lesson: Area and Volume