Maclaurin and Taylor Series

MATH 2511, BCIT

Technical Mathematics for Geomatics

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Infinite Sequence

An infinite sequence is a function whose domain is the set of positive integers.

Here is an example: $a(n) = 2n$ for $n = 1, 2, 3, \ldots$. We usually wri a_n instead of $a(n)$. The infinite sequence is $2, 4, 6, \ldots$. The infini sequence itself is often called $(a_n)_{n\in\mathbb{N}}$.

Infinite Series

Given an infinite sequence a_n, the infinite series s_n is an infinite sequence defined as follows: $s_n = a_1 + a_2 + \ldots + a_n$.

s_n is called a partial sum of the sequence $(a_n)_{n\in\mathbb{N}}$. It is often written as

$$s_n = \sum_{i=1}^{n} a_i \qquad ($$

vergence and Divergence

converges $(a_n)_{n\in\mathbb{N}}$ converges to the real number L if for every positive real number ε there exists an integer N such that for all $n > N$ it is true that $|a_n - L| < \varepsilon$. L is the limit of this sequence.

diverges $(a_n)_{n\in\mathbb{N}}$ diverges if no limit exists. $(a_n)_{n\in\mathbb{N}}$ diverges to positive infinity ∞ if for every real number M there is an integer N such that for all n larger than N it is true that $a_n > M$. We say $\lim_{n\to\infty} a_n = \infty$ or $a_n \longrightarrow \infty$. $(a_n)_{n\in\mathbb{N}}$ diverges to negative infinity $-\infty$ if for every real number m there is an integer N such that for all n larger than N it is true that $a_n < m$. We say $\lim_{n\to\infty} a_n = -\infty$ or $a_n \longrightarrow -\infty$.

Geometric Series

Geometric series have the form

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1} \qquad ($$

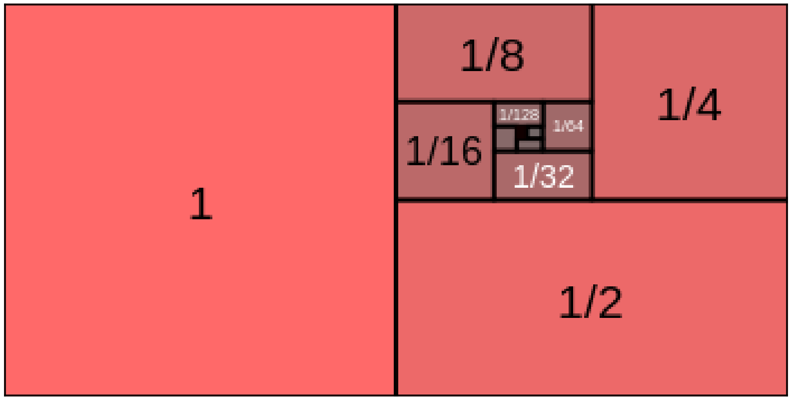
The notation for the limit is as follows,

$$\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} ar^{n-1} \qquad ($$

r is called the ratio of the geometric series. Subtract $s_n - rs_n$ to find out that

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1 \qquad ($$

If $|r| \geq 1$ then the limit does not exist.



Consider scenario 1,

$$\begin{aligned} a_n &= 2^n = \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots \\ s_n &= \textstyle\sum_{i=1}^{n} a_i \end{aligned}$$

Consider scenario 2,

$$\begin{aligned} a_n &= 2^n = 2, 4, 8, 16, \ldots \\ s_n &= \textstyle\sum_{i=1}^{n} a_i \end{aligned}$$

Now calculate the limits of these series. What goes wrong in scenario 2?

metric Series Example

ample 1: Limit of a Geometric Series. Find the limit of the llowing series.

$$\frac{7}{12} + \frac{7}{24} + \frac{7}{48} + \frac{7}{96} + \ldots \tag{7}$$

otice that 7 in the denominator and 12 in the numerator are mmon factors.

$$\frac{7}{12} + \frac{7}{24} + \frac{7}{48} + \frac{7}{96} + \ldots = \frac{7}{12} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) =$$

$$\frac{7}{12} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{7}{6} \tag{8}$$

Geometric Series Exercise

Exercise 1: Find the limit of the following series.

$$\sum_{n=2}^{\infty} \frac{3^n - 1}{6^n}$$

$$\sum_{n=0}^{\infty} \left(\frac{2n+1}{5^n}\right)$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$$

ercise 2: Find the following series limits using telescoping ries.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) \tag{12}$$

$$\sum_{n=1}^{\infty}\left(\frac{3}{n^2}-\frac{3}{(n+1)^2}\right) \tag{13}$$

$$\sum_{n=1}^{\infty}\left(\sqrt{n+4}-\sqrt{n+3}\right) \tag{14}$$

Exercise 3: Find the following series limits using telescoping series.

$$\sum_{n=1}^{\infty}\frac{40n}{(2n-1)^2(2^{n+1})^2}$$

$$\sum_{n=1}^{\infty}\frac{4}{(4n-3)(4n+1)}$$

$$\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$$

$$\sum_{n=1}^{\infty}\frac{n}{2^n}$$

eating Decimals

xpress each of these numbers as the ratio of two integers.

$$0.\overline{23} = 0.23232323\ldots \tag{19}$$

$$0.0\overline{6} = 0.06666\ldots \tag{20}$$

$$1.24\overline{123} = 1.24123123123\ldots \tag{21}$$

Integral Test

Integral Test

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function x for all $x \geq N$ (N is any positive integer).

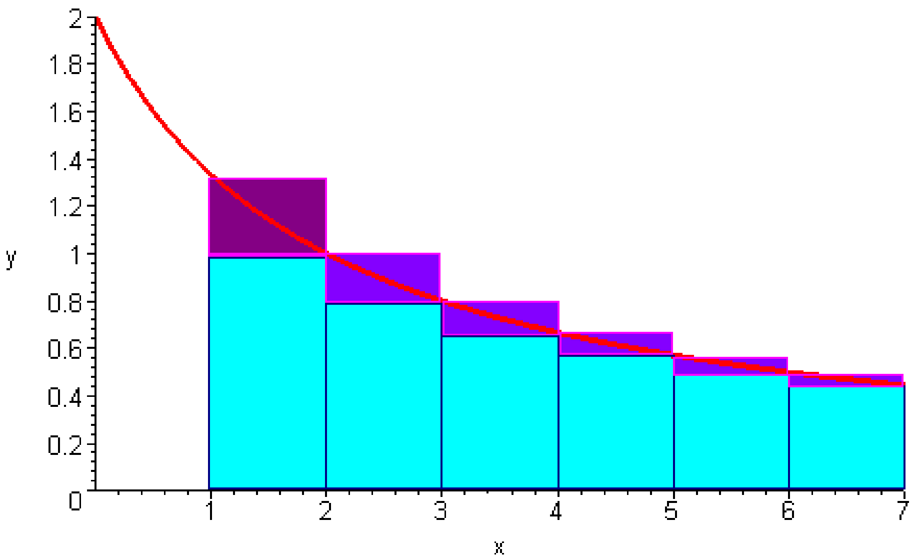
Then the series

$$\sum_{n=N}^{\infty} a_n$$

and the integral

$$\int_N^{\infty} f(x)\,dx$$

both converge or both diverge.



Show that the famous harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots \qquad (2$$

diverges. Then show that

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots \qquad (2$$

converges.

gral Test Exercises

ow show that

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1} \qquad (26)$$

ists. Remember that the antiderivative of

$$f(x)=\frac{1}{x^2+1} \qquad (27)$$

$F(x)=\arctan x$. Showing that (26) exists does not mean that e know its value.

Integral Test Exercises

Give reasons why the following sums exist or do not exist.

$$\sum_{n=1}^{\infty}e^{-n} \qquad \sum_{n=1}^{\infty}\frac{n}{n+1} \qquad \sum_{n=1}^{\infty}n\sin\frac{1}{n}$$

$$\sum_{n=1}^{\infty}\frac{3}{\sqrt{n}} \qquad \sum_{n=1}^{\infty}\frac{-2}{n\sqrt{n}} \qquad \sum_{n=1}^{\infty}\frac{n}{n^2+1}$$

$$\sum_{n=2}^{\infty}\frac{\ln n}{n} \qquad \sum_{n=1}^{\infty}\frac{5^n}{4^n+3} \qquad \sum_{n=2}^{\infty}\frac{\sqrt{n}}{\ln n}$$

$$\sum_{n=1}^{\infty} e^{-n} \quad (28)$$

convergent because it is a geometric series with $0 \leq r = \frac{1}{e} < 1$.

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \quad (29)$$

divergent because $\frac{n}{n+1} \longrightarrow 1$, and $\frac{n}{n+1} \not\longrightarrow 0$ implies divergence
cording to the n-the term test.

$$\sum_{n=1}^{\infty} n \sin \frac{1}{n} \quad (30)$$

divergent because according to L'Hôpital's rule, $n \sin \frac{1}{n} \longrightarrow 1$,
d $n \sin \frac{1}{n} \not\longrightarrow 0$ implies divergence according to the n-the term
st.

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} \quad (3$$

is divergent according to the integral test.

$$\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} \quad (3$$

is convergent according to the integral test.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \quad (3$$

is divergent because $a_n/b_n \longrightarrow 1$ for $a_n = n/(n^2+1)$ and $b_n =$
using part 1 of the limit comparison test.

gral Test Answers

$$\sum_{n=2}^{\infty} \frac{\ln n}{n} \quad (34)$$

divergent because $\ln n/n > 1/n$ for $n > 2$ and the harmonic
ries diverges.

$$\sum_{n=1}^{\infty} \frac{5^n}{4^n+3} \quad (35)$$

divergent because $a_n/b_n \longrightarrow 1$ for $a_n = 5^n/(4^n+3)$ and
$= \frac{5^n}{4^n}$, using part 1 of the limit comparison test. $\sum b_n$ diverges
cause it is a geometric series with $r > 1$.

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n} \quad (36)$$

divergent because according to L'Hôpital's rule, $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x}$
es not exist

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The n-th Term Test

We could prove this theorem, but it is also accessible to intuition

$$\text{If } \sum_{i=1}^{n} a_i \text{ converges, then } a_n \longrightarrow 0 \quad (3$$

Test for Divergence

$\sum_{i=1}^{n} a_i$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from 0.

The converse of the n-th term test is not true. For the following sequence, the corresponding series diverges even though the sequence goes to 0.

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{\text{2 terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{\text{4 terms}} + \frac{1}{8} + \ldots \quad (3$$

t $(a_n)_{n\in\mathbb{N}}$ be a sequence with no negative terms. Then

1. $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.
2. $\sum a_n$ diverges if there is a divergent series $\sum d_n$ with $a_n \geq d_n \geq 0$ for all $n > N$, for some integer N.

Example 2: Comparison Test Example. The series

$$\sum_{n=1}^{\infty} \frac{5}{5n-1} \tag{3}$$

diverges because

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n} \tag{4}$$

for all $n \in \mathbb{N}$.

t Comparison Tests

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0 \tag{41}$$

en $\sum a_n$ and $\sum b_n$ both converge or both diverge.

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \tag{42}$$

d $\sum b_n$ converges, them $\sum a_n$ converges.

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \tag{43}$$

d $\sum b_n$ diverges, them $\sum a_n$ diverges.

Ratio Test

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \varrho \tag{4}$$

Then

1. the series $\sum a_n$ converges if $\varrho < 1$
2. the series $\sum a_n$ diverges if $\varrho > 1$ or ϱ is infinite
3. the test is inconclusive if $\varrho = 1$

xercise 4: Use the ratio test to find out if the following exist:

$$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n} \qquad \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} \qquad \sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$$

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence with $u_n > 0$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots \qquad (4$$

is an alternating series. It converges if the following two conditio are satisfied:

1. $u_n > u_{n+1}$ for all $n > N$, for some integer N
2. $u_n \longrightarrow 0$

It immediately follows that the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \qquad (4$$

converges. It equals $\ln 2$.

monic Series

n ant starts to crawl along a taut rubber rope 1 km long at a eed of 1 cm per second (relative to the rubber it is crawling on). the same time, the rope starts to stretch uniformly by 1 km per cond, so that after 1 second it is 2 km long, after 2 seconds it is km long, etc. Will the ant ever reach the end of the rope? ounterintuitively, yes. This is a consequence of the divergent rmonic series.

nother example is the block-stacking problem: given a collection identical dominoes, it is clearly possible to stack them at the ge of a table so that they hang over the edge of the table thout falling. The counterintuitive result is that one can stack em in such a way as to make the overhang arbitrarily large, ovided there are enough dominoes.

Harmonic Series

series $\sum a_n$ converges absolutely if the corresponding series of
solute values $\sum |a_n|$ converges. Are the following two series
solutely convergent?

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots \qquad (47)$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \qquad (48)$$

series that converges but does not converge absolutely is said to
nverge conditionally. If $\sum |a_n|$ converges, then $\sum a_n$ must
nverge. Absolutely (and ONLY absolutely) convergent series can
rearranged. The alternating harmonic series can be rearranged
diverge or to reach any preassigned infinite sum.

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots \qquad (4$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots \qquad (5$$

in which the centre a and the coefficients c_0, c_1, c_2 are real numbers.

Term-by-Term Differentiation Theorem

$\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $a - R < x < a + R$ for some
> 0, it defines a function f

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ on the domain } a - R < x < a + R \qquad (51)$$

ch a function f has derivatives of all orders inside the interval of
nvergence. We can obtain the derivatives by differentiating the
iginal series term by term.

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \qquad (52)$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2} \qquad (53)$$

d so on. Each of these derived series converges at every interior
int of the interval of convergence of the original series

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The Term-by-Term Integration Theorem

If $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $a - R < x < a + R$ for some $R > 0$, it defines a function f

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ on the domain } a - R < x < a + R \qquad (5$$

Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \qquad (5$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C \qquad (5$$

for $a - R < x < a + R$.

se these two theorems to find power series expansions for $x) = \arctan x$ and $g(x) = \ln(1+x)$ on the domain $-1 < x < 1$.

se the following two functions to succeed in this endeavour.

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad (57)$$

$$g(x) = 1 - x + x^2 - x^3 + \dots \quad (58)$$

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutel for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k} \quad (5$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$,

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n \quad (6$$

Use term-by-term differentiation and the series multiplication theorem for power series independently to show that for $|x| < 1$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots \quad (6$$

or and Mclaurin Series

ow think about it the other way around. If a power series gives us continuous function with derivatives of all orders, will a ntinuous function with derivatives of all orders give us a power ries? What would be the coefficients? Let's assume that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \quad (62)$$

th a positive radius of convergence.

Taylor and Mclaurin Series

Then

$$f^{(n)}(x) = n! a_n + \text{ a sum of terms with } x - a \text{ as a factor} \quad (6$$

Since these equations all hold at $x = a$, we have

$$\begin{aligned} f'(a) &= 1 \cdot a_1 \\ f''(a) &= 1 \cdot 2 \cdot a_2 \\ f'''(a) &= 1 \cdot 2 \cdot 3 \cdot a_3 \end{aligned} \quad (6$$

and in general $f^{(n)} = n! a_n$.

, if (and that's a significant "if") a function f has a series oresentation, then the series must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots +$$

$$\frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots \qquad (65)$$

f is infinitely differentiable, then this series is determined, but it not always true that the series has a positive radius of nvergence. All kinds of things can go wrong. For example, the nction

$$f(x) = e^{-\frac{1}{x^2}} \qquad (66)$$

s a Mclaurin series which converges everywhere but only at $= 0$ does the limit equal $f(x)$!

Let f be a function with derivatives of all orders throughout som interval containing a as an interior point. Then the Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) +$$

$$\frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots \qquad (6$$

The Mclaurin series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x)^k = f(0) + f'(0)(x) +$$

$$\frac{f''(0)}{2!}(x)^2 + \ldots + \frac{f^{(n)}(0)}{n!}(x)^n + \ldots \qquad (6$$

or Series Exercises

nd the Taylor polynomials of orders 0, 1, 2, 3 generated by f at a.

$$f(x) = \ln x, \ a = 1 \qquad (69)$$

$$f(x) = \frac{1}{x}, \ a = 2 \qquad (70)$$

$$f(x) = \sin x, \ a = \frac{\pi}{4} \qquad (71)$$

$$f(x) = \sqrt{x}, \ a = 4 \qquad (72)$$

$$f(x) = \cos x, \ a = \frac{\pi}{4} \qquad (73)$$

Mclaurin Series Exercises

Find the Mclaurin series for the following functions.

$$f(x) = e^{-x} \qquad (7$$

$$f(x) = e^{\frac{x}{2}} \qquad (7$$

$$f(x) = \frac{1}{1+x} \qquad (7$$

$$f(x) = \cosh x \qquad (7$$

$$f(x) = \sinh x \qquad (7$$