Optimization and Analyzing Functions

MATH 2511, BCIT

Technical Mathematics for Geomatics

February 5, 2018

Relative Extrema

A function f has a relative maximum at $x = c$ if there exists an open interval (a, b) containing c such that $f(x) \leq f(c)$ for all x in (a, b).

A function f has a relative minimum at $x = c$ if there exists an open interval (a, b) containing c such that $f(x) \geq f(c)$ for all x in (a, b).

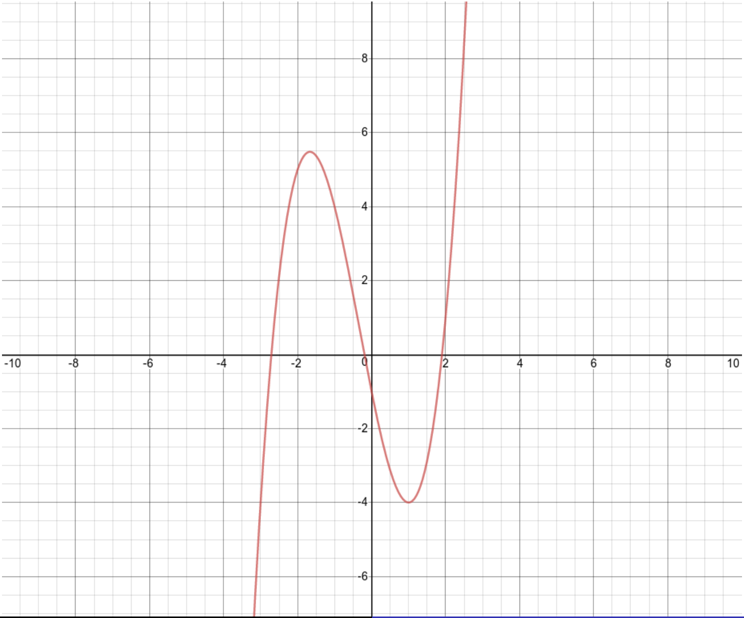
Derivatives and Extrema

At any number c where a differentiable function f has a relative extremum, $f'(c) = 0$. The converse is not true. Consider the following two functions and their derivatives.

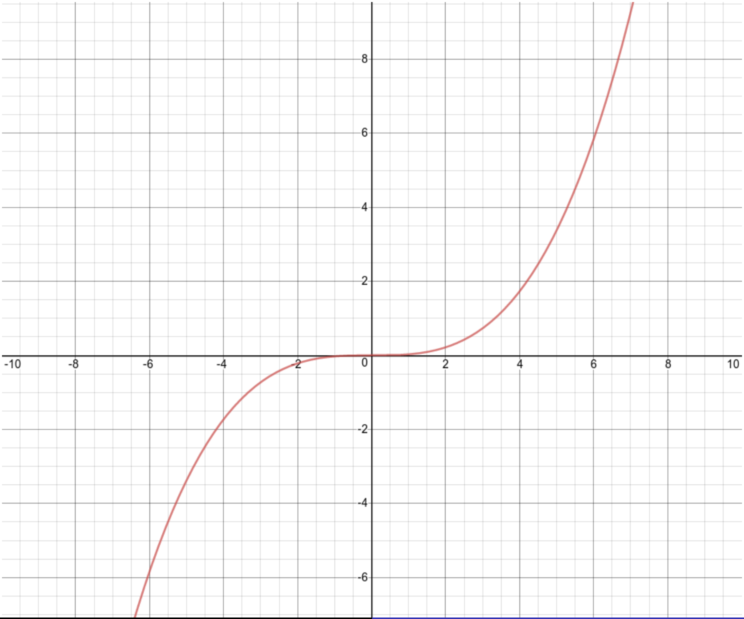
$$f_1(x) = x^3 + x^2 - 5x - 1 \tag{1}$$

$$f_2(x) = \left(\frac{3}{10}x\right)^3 \tag{2}$$

Derivatives and Extrema Graph I



Derivatives and Extrema Graph II



Derivatives and Extrema Caution

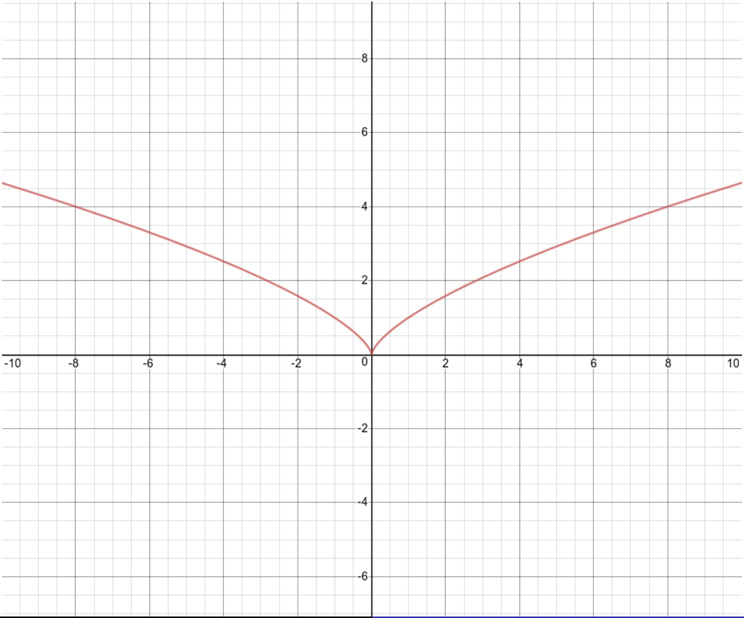
Note that a function may have an extremum at a point where the derivative is not 0 if at that point the function is not differentiable. Consider this function and its derivative.

$$f_3(x) = x^{\frac{2}{3}} \tag{3}$$

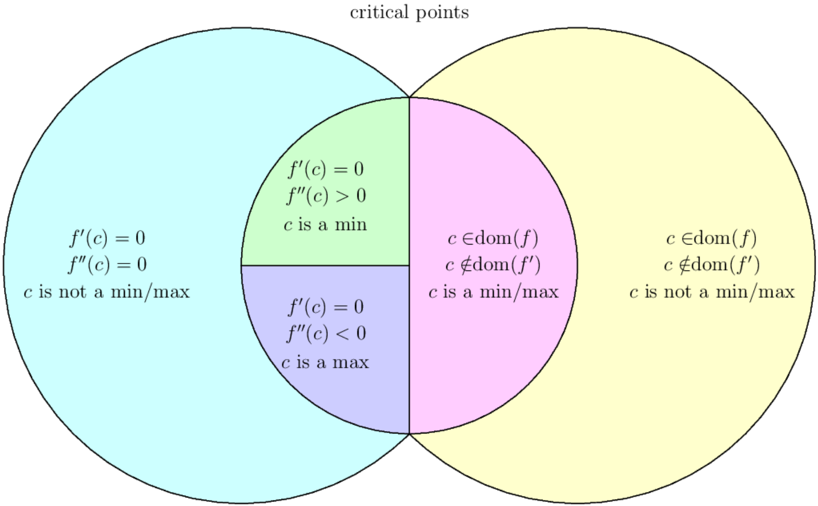
Critical Number

A critical number of a function f is any number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist.

Derivatives and Extrema Graph III



Critical Points and Extrema



Critical Points Exercises

Exercise 1: Find the critical points of the following function,

$$f(x) = x^3 - 4x \tag{4}$$

Exercise 2: Find the critical points of the following function,

$$h(t) = -t^2 + 6t + 6 \tag{5}$$

Critical Points Exercises

Exercise 3: Find the critical points of the following function,

$$f(x) = \frac{1}{2}x^4 - x^2 \tag{6}$$

Critical Points Exercises

Exercise 4: Find the critical points of the following function,

$$g(x) = \frac{x+1}{x} \tag{7}$$

Exercise 5: Find the critical points of the following function,

$$f(x) = x\sqrt{x-4} \tag{8}$$

Exercise 6: Find the critical points of the following function,

$$f(x) = 2\tan x - \tan^2 x \tag{9}$$

Exercise 7: Find the critical points of the following function,

$$h(s) = s^{\frac{5}{3}} \tag{10}$$

Exercise 8: Find local maxima and minima for the following function:

$$f(x) = 3x^3 - 12x + 5 \tag{11}$$

Exercise 9: Find local maxima and minima for the following function:

$$f(x) = \frac{x}{x^2+1} \tag{12}$$

Exercise 10: Find local maxima and minima for the following function:

$$f(t) = t\sqrt{4 - t^2} \tag{13}$$

Exercise 11: Find local maxima and minima for the following function:

$$g(t) = \sqrt[3]{t}(8 - t) \tag{14}$$

Exercise 12: Find local maxima and minima for the following function:

$$g(t) = \cos t + \sin t \tag{15}$$

Extrema Exercises

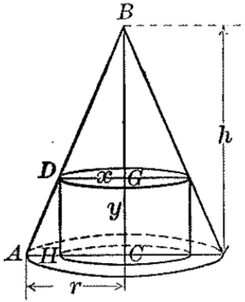
Exercise 13: Find local maxima and minima for the following function:

$$f(x) = \ln(x^2 + x + 1) \tag{16}$$

Exercise 14: Find local maxima and minima for the following function:

$$f(x) = \ln(\cos x) \tag{17}$$

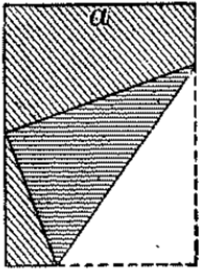
Exercise 15: Find the altitude of the cylinder of maximum volume that can be inscribed in a given right cone.



Exercise 16: A water tank is to be constructed with a square base and open top and is to hold 64 cubic yards. If the cost of the sides is \$1 a square yard, and of the bottom \$2 a square yard, what are the dimensions when the cost is a minimum? What is the minimum cost?

Exercise 17: The lower corner of a leaf, whose width is a, is folded over so as just to reach the inner edge of the page.

1. Find the width of the part folded over when the length of the crease is a minimum.
2. Find the width when the area folded over is a minimum.



Exercise 18: A submarine telegraph cable consists of a core of copper wires with a covering made of nonconducting material. If x denotes the ratio of the radius of the core to the thickness of the covering, it is known that the speed of signaling varies as

$$x^2 \ln \frac{1}{x} \tag{18}$$

Show that the greatest speed is attained when $x = \frac{1}{\sqrt{e}}$.

Analyzing Functions I

To analyze a function, determine the following features:

- Domain and range of the function.
- Zeros (also called x-intercepts) of the function.
- Critical points, maxima, minima.
- Inflection points.
- Asymptotes.
- Is the function even ($f_1(x) = x^2 + 1$) or odd ($f_2(x) = x^3 - x$)?

Analyzing Functions Step-By-Step I

Here is a step-by-step guide to analyzing functions.

1. Determine the x-intercepts (also called zeros). Set $f(x) = 0$ and find the solution set.
2. Determine the critical points. Find the derivative $f'(x)$ and check whether there are points in the domain of f that are not in the domain of f'. Then set $f'(x) = 0$ and find the solution set.
3. Determine whether the critical points are maxima or minima or neither. Find $f''(x)$ and check whether f'' at the critical points is positive, negative, or neither.

Analyzing Functions Step-By-Step II

Here is a step-by-step guide to analyzing functions.

4. Determine the inflection points. Set $f''(x) = 0$ and find the solution set.
5. Determine the asymptotes. See next slide.
6. Determine whether, for all x in the domain of f, $f(x) - f(-x) = 0$ (in which case f is even) or $f(x) + f(-x) = 0$ (in which case f is odd).
7. Using the information you have, and possibly a table of function values, graph the function. Then determine the domain and range of f.

Finding Asymptotes I

An asymptote is a linear function ($y = kx + d$ with slope k and y-intercept d) which the function graph of f approaches. There are three kinds of asymptotes.

Vertical Asymptote

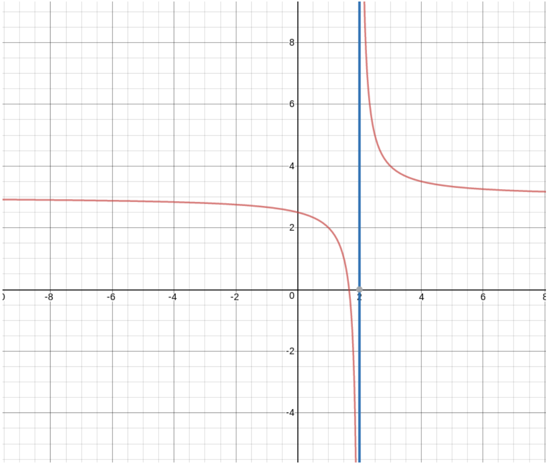
A vertical asymptote, strictly speaking, is not a linear function. It is a curve defined by $x = c$, where c is a real number (we call real numbers like c constants). You can often find vertical asymptotes at points where f is undefined.

Find vertical asymptotes by checking points which are not in the domain of the function f.

$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2 \tag{19}$$

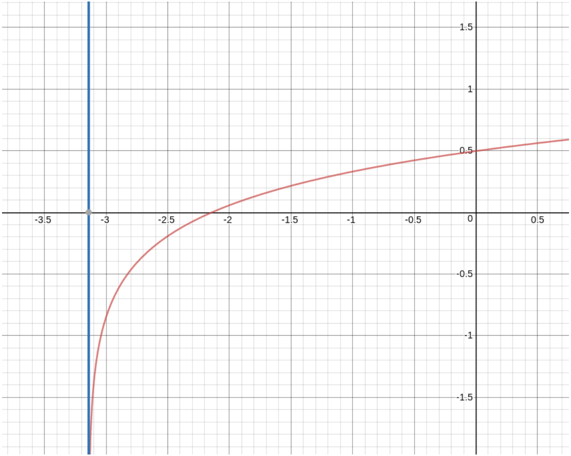
Example:

$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2 \tag{20}$$



Example:

$$f(x) = \ln(x + \pi) \text{ has an asymptote at } x = -\pi \tag{21}$$



Finding Asymptotes II

Horizontal Asymptote

A horizontal asymptote is a linear function with slope $k = 0$. Its equation is $y = c$, where c is a constant. There are horizontal asymptotes for functions whose limits is a constant and for rational functions whose numerator and denominator polynomials share the same degree.

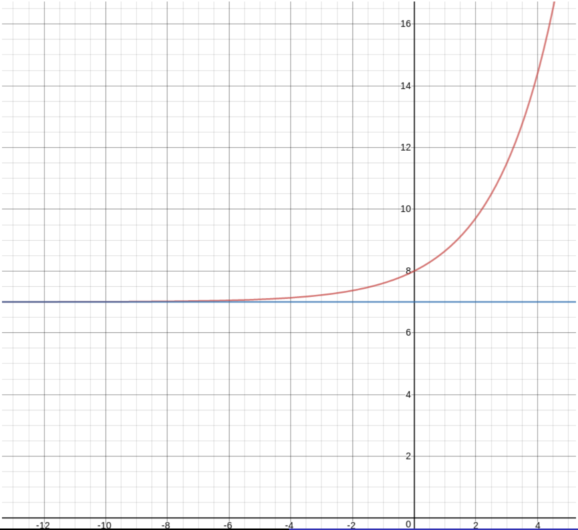
Find horizontal asymptotes by checking

$$\lim_{x \to \infty} f'(x) \text{ and } \lim_{x \to -\infty} f'(x) \tag{22}$$

If the limit is $k = 0$, then that is also the slope of the asymptote.

Example

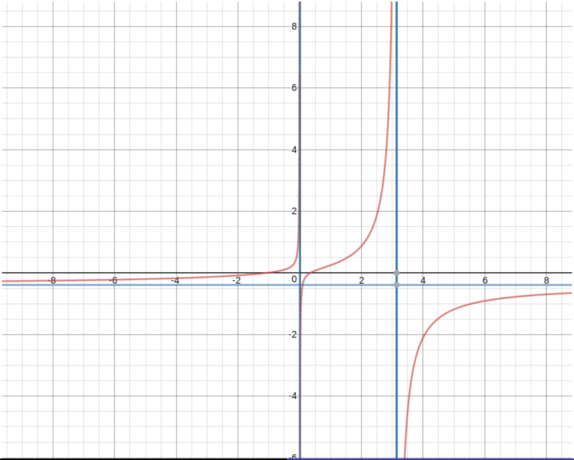
$$f(x) = e^{\frac{x}{2}} + 7 \text{ has the asymptote } y = 7 \tag{23}$$



Finding Asymptotes II

Example (this example additionally has two vertical asymptotes):

$$f(x) = \frac{\pi x^2 + 2x - 1}{-7x^2 + 3x} \text{ has asymptotes } y = -\frac{e}{7}, x = \frac{22}{7}, x = 0 \tag{24}$$



Finding Asymptotes III

Sloped Asymptote

A sloped asymptote is a linear function with a positive or a negative slope, $y = kx + d$ with $k \neq 0$. There are sloped asymptotes for rational functions where the numerator polynomial's degree exceeds the denominator polynomial's degree by 1.

Find sloped asymptotes by checking

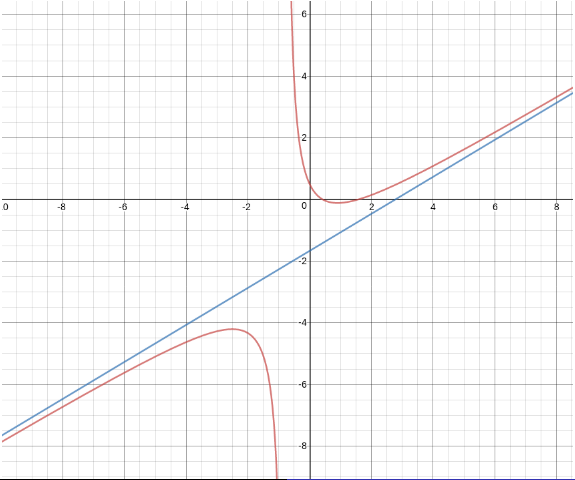
$$\lim_{x \to \infty} f'(x) \text{ and } \lim_{x \to -\infty} f'(x) \tag{25}$$

If the limit is $k \neq 0$, then that is also the slope of the asymptote. Hyperbolas also sometimes have sloped asymptotes.

Finding Asymptotes III

Example:

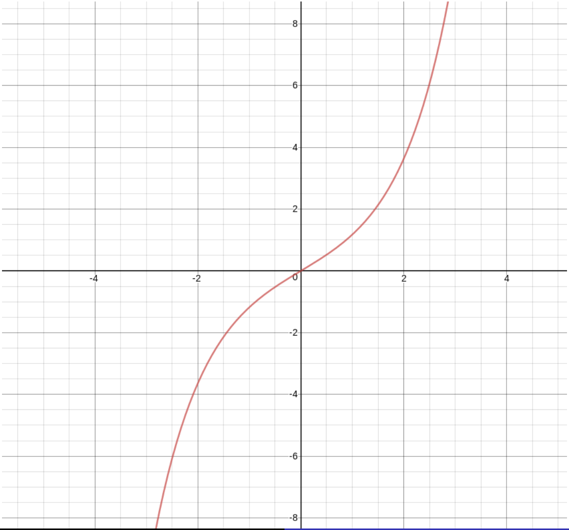
$$f(x) = \frac{3x^2 - 6x + 2}{5x + 4} \text{ has the asymptote } y = \frac{3}{5}x - \frac{5}{3} \tag{26}$$



Finding Asymptotes III

Example:

$$f(\vartheta) = \sinh \vartheta \tag{27}$$



Analyzing Functions

Analyze the following functions:

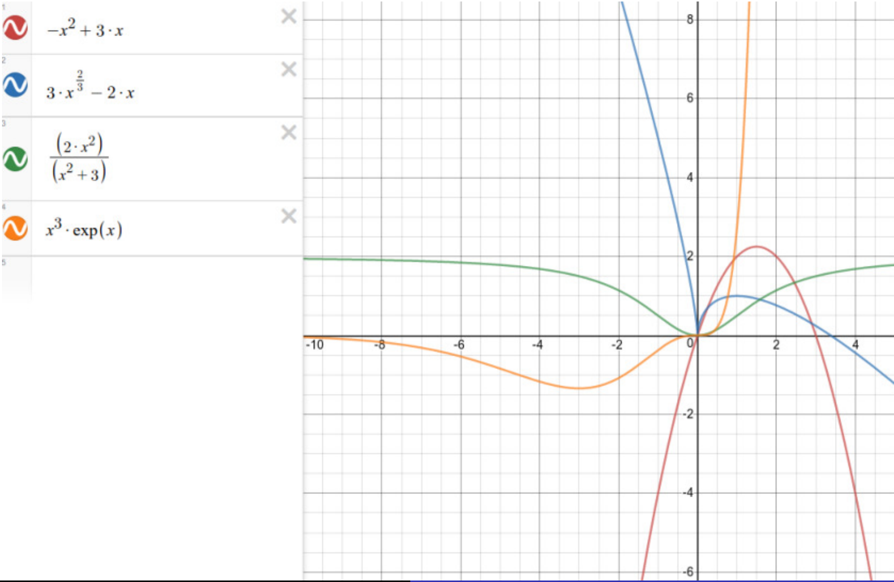
$$g_1(x) = -x^2 + 3x \tag{28}$$

$$g_2(x) = 3x^{\frac{2}{3}} - 2x \tag{29}$$

$$g_3(t) = \frac{2t^2}{t^2 + 3} \tag{30}$$

$$g_4(x) = x^3 e^x \tag{31}$$

Analyzing Functions Exercises Graph



Analyzing Functions Further Exercises

Analyze the following functions.

$$f(x) = x \cdot \ln x^2 \tag{32}$$

$$f(x) = \frac{2x^2 + 2}{x - 3} \text{ (do not look for inflection points)} \tag{33}$$

$$f(x) = x^3 + 4x^2 + x - 6 \tag{34}$$

For (34), note that $x = 1$ is an x-intercept so that $(x - 1)$ can be factored as in

$$x^3 + 4x^2 + x - 6 = (x - 1)(x^2 + 5x + 6) \tag{35}$$

End of Lesson

Next Lesson: Transcendental Functions and Differentials