Optimization and Analyzing Functions

MATH 2511, BCIT

Technical Mathematics for Geomatics

February 5, 2018

A function f has a relative maximum at $x = c$ if there exists an open interval (a, b) containing c such that $f(x) \leq f(c)$ for all x in (a, b).

A function f has a relative minimum at $x = c$ if there exists an open interval (a, b) containing c such that $f(x) \geq f(c)$ for all x in (a, b).

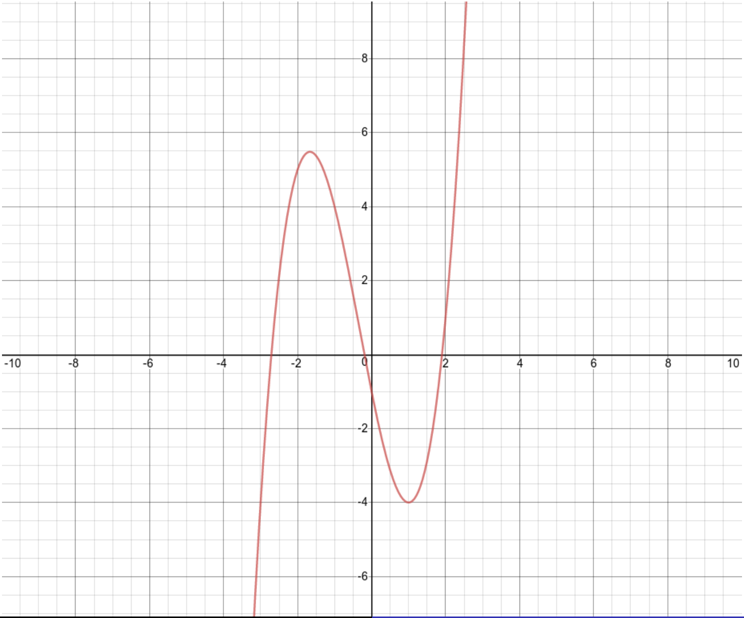
Derivatives and Extrema

At any number c where a differentiable function f has a relative extremum, $f'(c) = 0$. The converse is not true. Consider the following two functions and their derivatives.

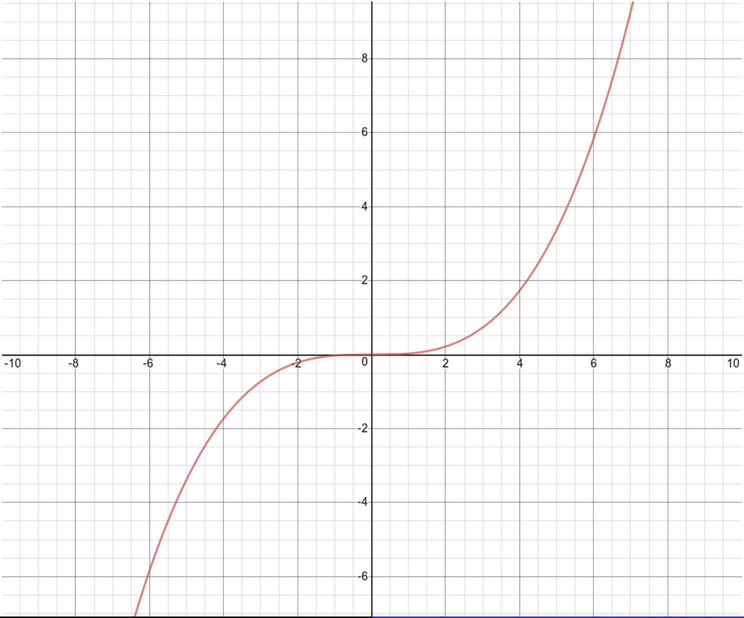
$$f_1(x) = x^3 + x^2 - 5x - 1 \tag{1}$$

$$f_2(x) = \left(\frac{3}{10}x\right)^3 \tag{2}$$

Derivatives and Extrema Graph I



Derivatives and Extrema Graph II



Derivatives and Extrema Caution

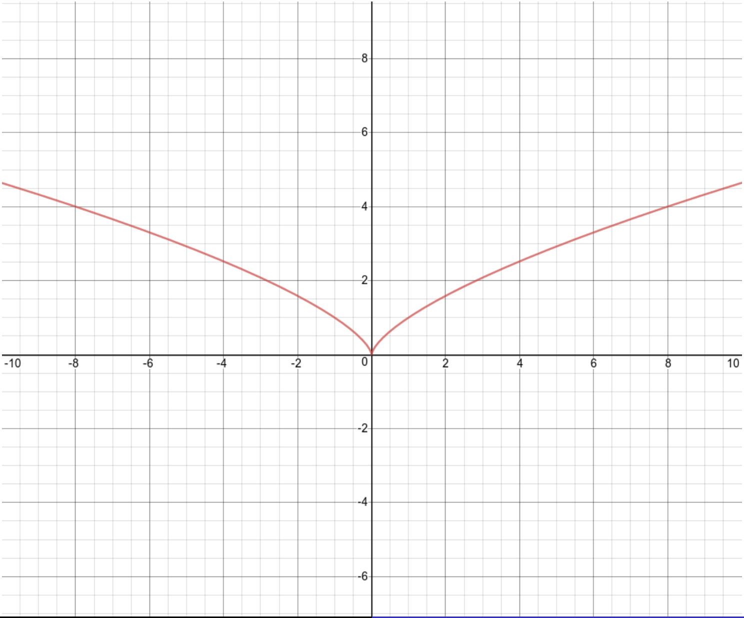
Note that a function may have an extremum at a point where the derivative is not 0 if at that point the function is not differentiable. Consider this function and its derivative.

$$f_3(x) = x^{\frac{2}{3}} \tag{3}$$

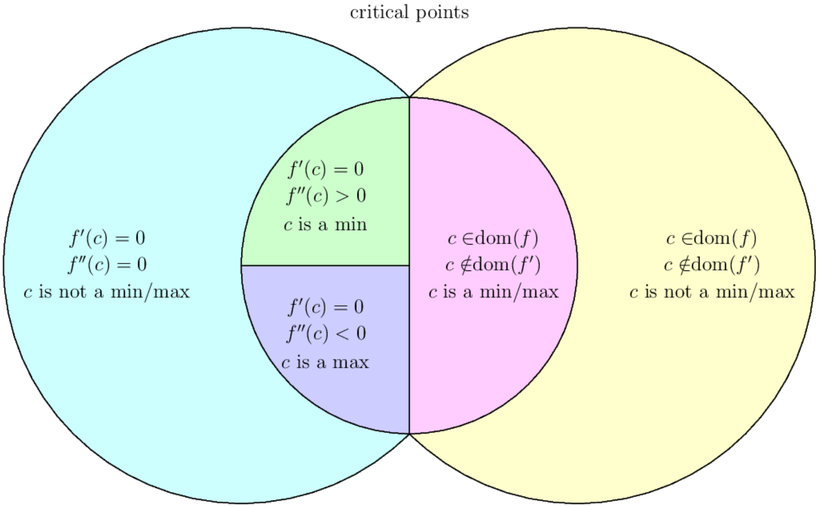
Critical Number

A critical number of a function f is any number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist.

Derivatives and Extrema Graph III



Critical Points and Extrema



Critical Points Exercises

Exercise 1: Find the critical points of the following function,

$$f(x) = x^3 - 4x \tag{4}$$

Exercise 2: Find the critical points of the following function,

$$h(t) = -t^2 + 6t + 6 \tag{5}$$

Exercise 3: Find the critical points of the following function,

$$f(x) = \frac{1}{2}x^4 - x^2 \tag{6}$$

Critical Points Exercises

Exercise 4: Find the critical points of the following function,

$$g(x) = \frac{x+1}{x} \tag{7}$$

Exercise 5: Find the critical points of the following function,

$$f(x) = x\sqrt{x-4} \tag{8}$$

Exercise 6: Find the critical points of the following function,

$$f(x) = 2\tan x - \tan^2 x \tag{9}$$

Exercise 7: Find the critical points of the following function,

$$h(s) = s^{\frac{5}{3}} \tag{10}$$

Exercise 8: Find local maxima and minima for the following function:

$$f(x) = 3x^3 - 12x + 5 \tag{11}$$

Extrema Exercises

Exercise 9: Find local maxima and minima for the following function:

$$f(x) = \frac{x}{x^2 + 1} \tag{12}$$

Exercise 10: Find local maxima and minima for the following function:

$$f(t) = t\sqrt{4 - t^2} \tag{13}$$

Exercise 11: Find local maxima and minima for the following function:

$$g(t) = \sqrt[3]{t}(8 - t) \tag{14}$$

Exercise 12: Find local maxima and minima for the following function:

$$g(t) = \cos t + \sin t \tag{15}$$

Exercise 13: Find local maxima and minima for the following function:

$$f(x) = \ln(x^2 + x + 1) \tag{16}$$

Exercise 14: Find local maxima and minima for the following function:

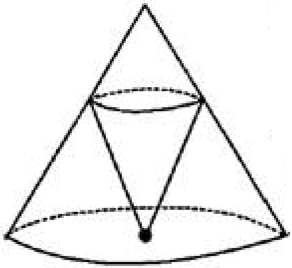
$$f(x) = \ln(\cos x) \tag{17}$$

Exercise 15: A farmer has 2400ft of fencing and wants to fence off a rectangular field that borders a straight river. She needs no fence along the river. What are the dimensions of the field that has the largest area?

Exercise 16: A cylindrical can is to be made to hold one litre of oil. Find the dimensions that wil minimize the cost of the metal to manufacture the can.

Exercise 17: Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Exercise 18: A cone with height h and radius r is inscribed in a larger cone with height H and radius R so that its vertex is at the centre of the base of the larger cone. Find h in terms of the dimensions of the larger cone that makes the volume of the smaller cone maximal.



Exercise 19: For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u ($u < v$), then the time required to swim a distance L is $L/(v - u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u} \tag{18}$$

where a is the proportionality constant. Determine the value of v that minimizes E.

Exercise 20: How close does the semi-circle $y = \sqrt{16 - x^2}$ come to the point $P = (1, \sqrt{3})$?

Exercise Solution

Note that the semi-circle $y = \sqrt{16 - x^2}$ is part of a circle with a centre of $M = (0, 0)$ and radius $r = 4$. If $Q = (x, y)$ is the point on the semi-circle closest to P, then the distance between P and Q is

$$f(x) = \sqrt{(x-1)^2 + (y - \sqrt{3})^2} \tag{19}$$

Since Q is on the semi-circle, we can replace $y = \sqrt{16 - x^2}$ to get

$$f(x) = \sqrt{(x-1)^2 + (\sqrt{16 - x^2} - \sqrt{3})^2} \tag{20}$$

Exercise Solution

The distance between P and Q is

$$f(x) = \sqrt{(x-1)^2 + (\sqrt{16-x^2} - \sqrt{3})^2} = \sqrt{g(x)} \tag{21}$$

Call the expression under the square root sign $g(x)$. Then

$$f'(x) = \frac{1}{2} \cdot (g(x))^{-\frac{1}{2}} \cdot g'(x) \tag{22}$$

Of these three factors, only $g'(x)$ can be zero. Setting $f'(x) = 0$ is therefore equivalent to $g'(x) = 0$. Note that

$$g'(x) = 2(x-1) + 2\left(\sqrt{16-x^2} - \sqrt{3}\right) \cdot \left(\frac{1}{2}(16-x^2)^{-\frac{1}{2}} \cdot (-2x)\right)$$

Exercise Solution

Simplify and expand to

$$\frac{1}{2}g'(x) = (x-1) - x + \frac{\sqrt{3}x}{\sqrt{16-x^2}} \tag{23}$$

$g'(x) = 0$ just when

$$1 = \frac{\sqrt{3}x}{\sqrt{16-x^2}} \tag{24}$$

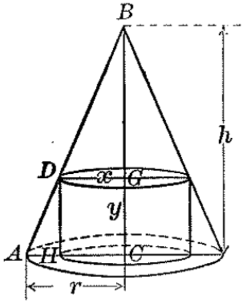
Square both sides for the polynomial equation

$$4x^2 - 16 = 0 \tag{25}$$

and the two solutions $x_1 = -2$ and $x_2 = 2$. The first solution is where the distance between P and Q is at a maximum. The second solution is where the distance is at a minimum. Therefore, the point $Q = (2, 2\sqrt{3})$ is the answer to the question in this exercise.

Exercise 21: Find the area of the largest rectangle that can be inscribed in a semicircle of radius r (two vertices of the rectangle must be on a diameter of the circle).

Exercise 22: Find the altitude of the cylinder of maximum volume that can be inscribed in a given right cone.



Optimization Word Problem Hint

As a hint for the last exercise, consider the angle at B between the height of the cone and the generating line of the cone. The tangent of the angle can be expressed as two different ratios, which will help you to find a function $V(x)$ and its derivative to maximize the volume.

Exercise 23: A water tank is to be constructed with a square base and open top and is to hold 64 cubic yards. If the cost of the sides is \$1 a square yard, and of the bottom \$2 a square yard, what are the dimensions when the cost is a minimum? What is the minimum cost?

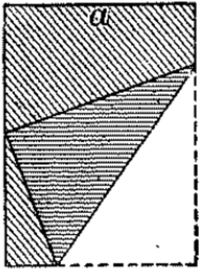
Between 0°C and 30°C, the volume V (in cubic centimetres) of one kilogram of water at a temperature T is given approximately by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \qquad (26)$$

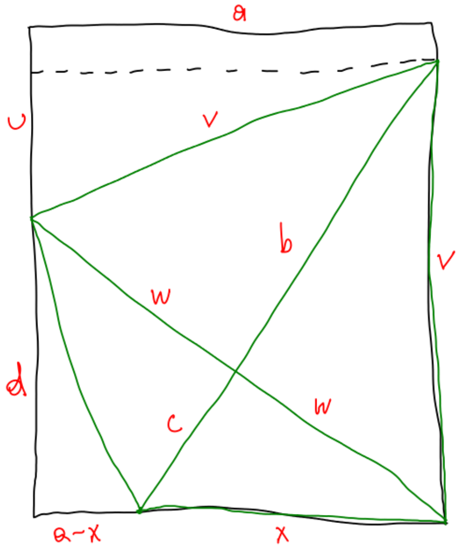
Find the temperature at which water has its maximum density.

Exercise 24: The lower corner of a leaf, whose width is a, is folded over so as just to reach the inner edge of the page.

1. Find the width of the part folded over when the length of the crease is a minimum.
2. Find the width when the area folded over is a minimum.



Optimization Word Problem Solution



Optimization Word Problem Solution

Define a function $f(x) = b + c$. We want to minimize f on the interval $(\frac{a}{2}, a]$. Notice that $b^2 = v^2 - w^2$ and $c^2 = x^2 - w^2$. We need to express v and w by x and the fixed number a. From

$$(2w)^2 = d^2 + a^2 \tag{27}$$

we get

$$w^2 = \frac{ax}{2} \tag{28}$$

From

$$v^2 = a^2 + u^2 \tag{29}$$

we get

$$v^2 = \frac{ax^2}{2x - a} \tag{30}$$

Optimization Word Problem Solution

Consequently, the function to differentiate is

$$f(x) = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{a^2 x}{2x - a}} + \sqrt{x(2x - a)} \right) \tag{31}$$

After differentiation, it turns out that $f'(x) = 0$ if and only if

$$x = \frac{3}{4} a \tag{32}$$

This answers the first question. The answer for the second question is

$$x = \frac{2}{3} a \tag{33}$$

Exercise 25: A submarine telegraph cable consists of a core of copper wires with a covering made of nonconducting material. If x denotes the ratio of the radius of the core to the thickness of the covering, it is known that the speed of signaling varies as

$$x^2 \ln \frac{1}{x} \tag{34}$$

Show that the greatest speed is attained when $x = \frac{1}{\sqrt{e}}$.

Exercise 26: Shane is 2.2 miles offshore in a boat and wishes to reach a coastal village 7.5 miles down a straight shoreline from the point nearest the boat. He can row 1.9 miles per hour and can walk 5.1 miles per hour. Where should he land his boat to reach the village in the least amount of time?

Optimization Word Problem Solution

Let the variable distance that Shane would row his boat be y and the variable distance that Shane would walk be x. Recall that time equals distance divided by velocity. Then

$$T(x) = \frac{y}{1.9} + \frac{x}{5.1} \tag{35}$$

for the time $T(x)$ that it takes Shane to get from where he is to the coastal village. The domain of T is $[0, 7.5]$. Note that

$$y^2 = 2.2^2 + (7.5 - x)^2 \tag{36}$$

Optimization Word Problem Solution

Therefore

$$T(x) = \frac{1}{1.9}\sqrt{x^2 - 15x + 61.09} + \frac{1}{5.1}x \tag{37}$$

and

$$T'(x) = \frac{2x - 15}{3.8\sqrt{x^2 - 15x + 61.09}} + \frac{1}{5.1} \tag{38}$$

$T'(x) = 0$ if and only if

$$89.6x^2 - 1344x + 4970.1 = 0 \tag{39}$$

with two solutions. The solution $x = 8.3832$ falls outside the domain of T. Therefore, the time is optimized at $x = 6.6168$. One should check using the second derivative to make sure this is a minimum.

Analyzing Functions I

To analyze a function, determine the following features:

- Domain and range of the function.
- Zeros (also called x-intercepts) of the function.
- Critical points, maxima, minima.
- Inflection points.
- Asymptotes.
- Is the function even ($f_1(x) = x^2 + 1$) or odd ($f_2(x) = x^3 - x$)?

Analyzing Functions Step-By-Step I

Here is a step-by-step guide to analyzing functions.

1. Determine the x-intercepts (also called zeros). Set $f(x) = 0$ and find the solution set.
2. Determine the critical points. Find the derivative $f'(x)$ and check whether there are points in the domain of f that are not in the domain of f'. Then set $f'(x) = 0$ and find the solution set.
3. Determine whether the critical points are maxima or minima or neither. Find $f''(x)$ and check whether f'' at the critical points is positive, negative, or neither.

Analyzing Functions Step-By-Step II

Here is a step-by-step guide to analyzing functions.

4. Determine the inflection points. Set $f''(x) = 0$ and find the solution set.
5. Determine the asymptotes. See next slide.
6. Determine whether, for all x in the domain of f, $f(x) - f(-x) = 0$ (in which case f is even) or $f(x) + f(-x) = 0$ (in which case f is odd).
7. Using the information you have, and possibly a table of function values, graph the function. Then determine the domain and range of f.

Finding Asymptotes I

An asymptote is a linear function ($y = kx + d$ with slope k and y-intercept d) which the function graph of f approaches. There are three kinds of asymptotes.

Vertical Asymptote

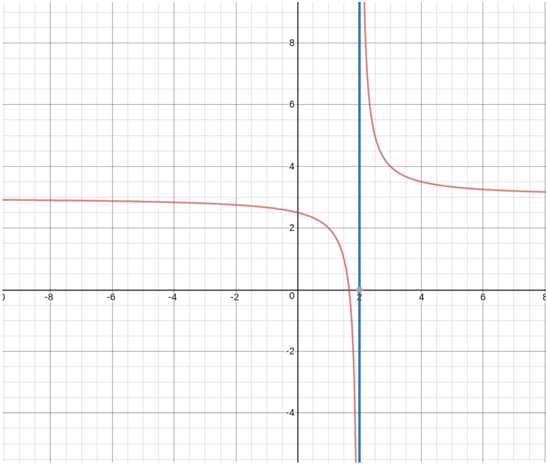
A vertical asymptote, strictly speaking, is not a linear function. It is a curve defined by $x = c$, where c is a real number (we call real numbers like c constants). You can often find vertical asymptotes at points where f is undefined.

Find vertical asymptotes by checking points which are not in the domain of the function f.

$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2 \tag{40}$$

Example:

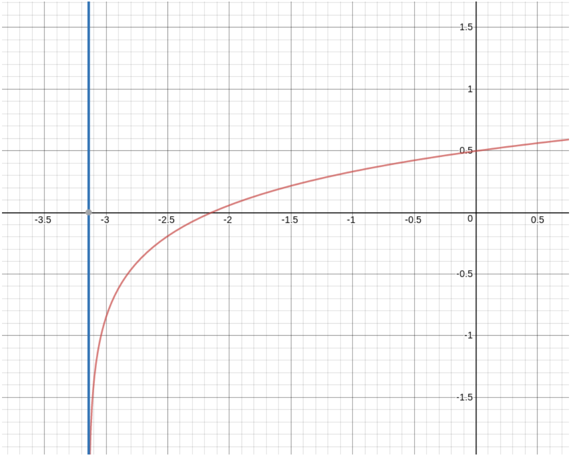
$$f(x) = \frac{1}{x-2} + 3 \text{ has an asymptote at } x = 2 \tag{41}$$



Finding Asymptotes I

Example:

$$f(x) = \ln(x+\pi) \text{ has an asymptote at } x = -\pi \tag{42}$$



Finding Asymptotes II

Horizontal Asymptote

A horizontal asymptote is a linear function with slope $k = 0$. Its equation is $y = c$, where c is a constant. There are horizontal asymptotes for functions whose limits is a constant and for rational functions whose numerator and denominator polynomials share the same degree.

Find horizontal asymptotes by checking

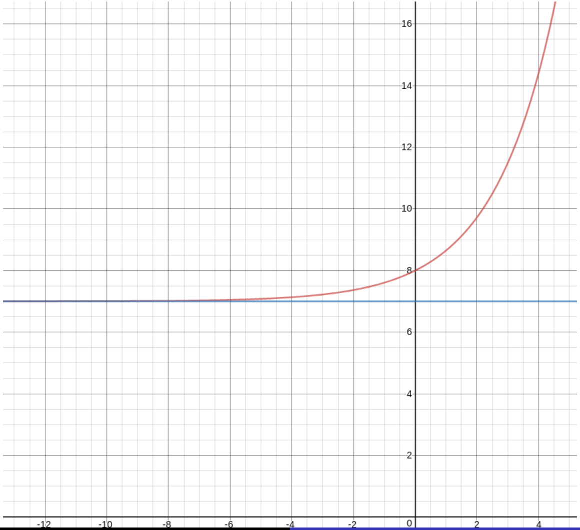
$$\lim_{x \to \infty} f'(x) \text{ and } \lim_{x \to -\infty} f'(x) \tag{43}$$

If the limit is $k = 0$, then that is also the slope of the asymptote.

Finding Asymptotes II

Example

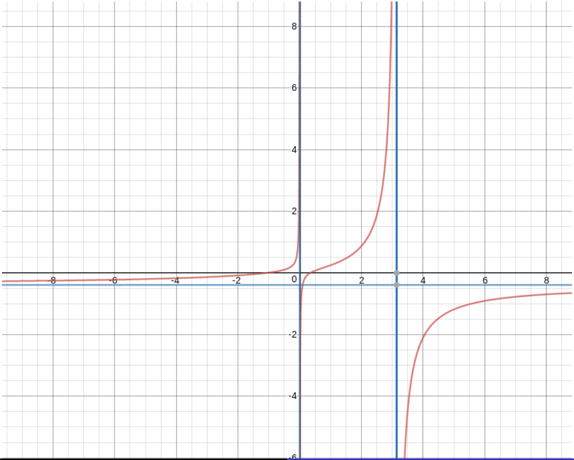
$$f(x) = e^{\frac{x}{2}} + 7 \text{ has the asymptote } y = 7 \qquad (44)$$



Finding Asymptotes II

Example (this example additionally has two vertical asymptotes):

$$f(x) = \frac{\pi x^2 + 2x - 1}{-7x^2 + 3x} \text{ has asymptotes } y = -\frac{e}{7}, x = \frac{22}{7}, x = 0 \tag{45}$$



Finding Asymptotes III

Sloped Asymptote

A sloped asymptote is a linear function with a positive or a negative slope, $y = kx + d$ with $k \neq 0$. There are sloped asymptotes for rational functions where the numerator polynomial's degree exceeds the denominator polynomial's degree by 1.

Find sloped asymptotes by checking

$$\lim_{x \to \infty} f'(x) \text{ and } \lim_{x \to -\infty} f'(x) \tag{46}$$

If the limit is $k \neq 0$, then that is also the slope of the asymptote. Hyperbolas also sometimes have sloped asymptotes.

Finding Asymptotes III

Example:

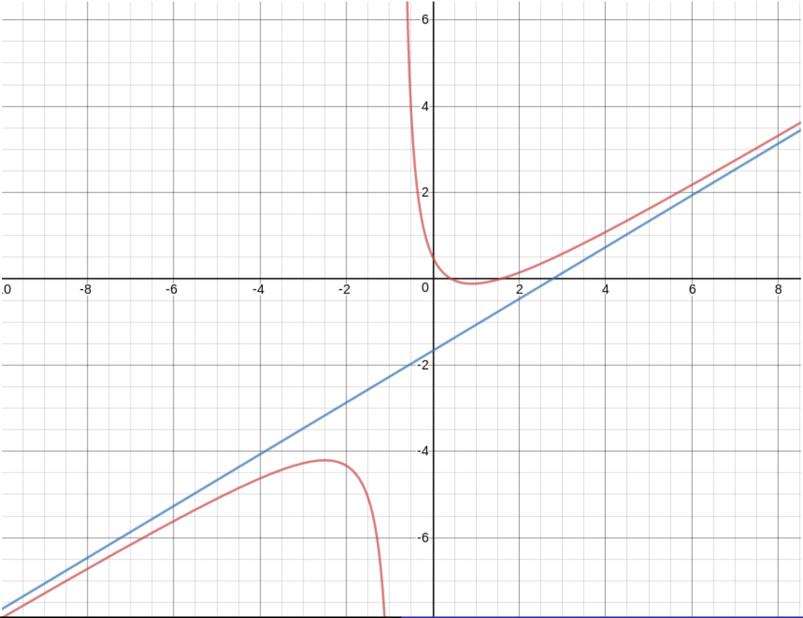
$$f(x) = \frac{3x^2 - 6x + 2}{5x + 4} \text{ has the asymptote } y = \frac{3}{5}x - \frac{5}{3} \qquad (47)$$

Find the y-intercept by making sure that

$$\lim_{x \to \infty} (f(x) - g(x)) = 0 \qquad (48)$$

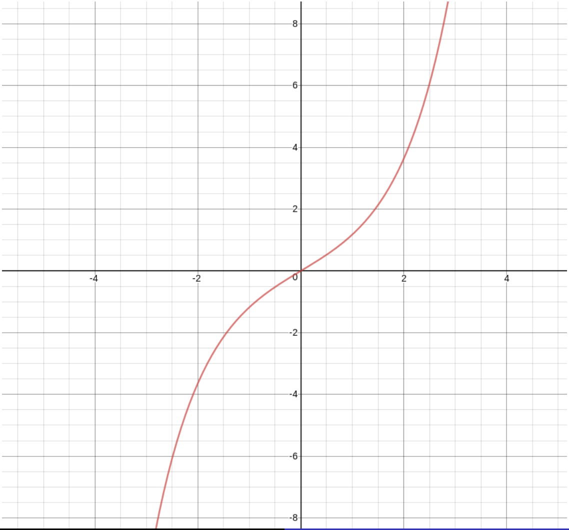
where $y = g(x) = kx + d$ for the sloped asymptote. This results in an equation where d is the only unknown.

Finding Asymptotes III



Example:

$$f(\vartheta) = \sinh\vartheta \tag{49}$$



Exercise 27: Analyze the following function:

$$g_1(x) = -x^2 + 3x \tag{50}$$

Analyzing Functions

Exercise 28: Analyze the following function:

$$g_2(x) = 3x^{\frac{2}{3}} - 2x \tag{51}$$

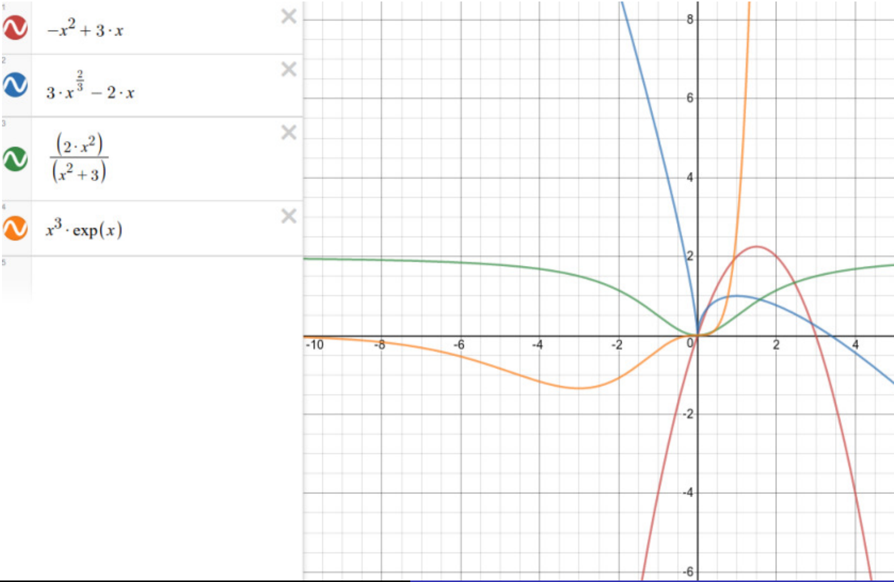
Exercise 29: Analyze the following function:

$$g_3(t) = \frac{2t^2}{t^2 + 3} \tag{52}$$

Exercise 30: Analyze the following function:

$$g_4(x) = x^3 e^x \tag{53}$$

Analyzing Functions Exercises Graph



Analyzing Functions Further Exercises

Exercise 31: Analyze the following function.

$$f(x) = x \cdot \ln x^2 \tag{54}$$

Exercise 32: Analyze the following function.

$$f(x) = \frac{2x^2 + 2}{x - 3} \text{ (do not look for inflection points)} \qquad (55)$$

Exercise 33: Analyze the following function.

$$f(x) = x^3 + 4x^2 + x - 6 \tag{56}$$

Note that $x = 1$ is an x-intercept so that $(x - 1)$ can be factored as in

$$x^3 + 4x^2 + x - 6 = (x - 1)(x^2 + 5x + 6) \tag{57}$$

Exercise 34: Analyze the following function.

$$f(x) = 4 - \frac{e^x + 1}{e^x} \tag{58}$$

Exercise 35: Analyze the following function.

$$f(x) = \frac{3x^2 - 5}{x - 2} \tag{59}$$

End of Lesson

Next Lesson: Transcendental Functions and Differentials