

# Area and Volume

## MATH 2511, BCIT

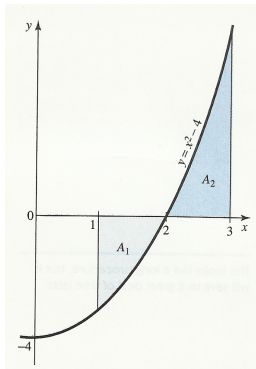
Technical Mathematics for Geomatics

March 21, 2018

# Negative Area I

Consider the following problem.

Find the area under the curve  $y = x^2 - 4$  between  $x = 1$  and  $x = 3$ .



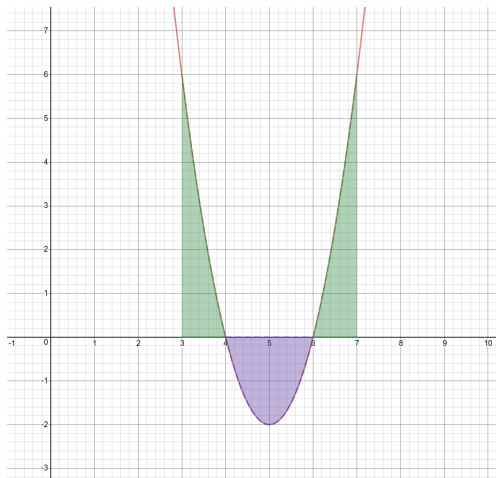
# Negative Area II

To solve this problem, find the  $x$ -intercept and treat the positive and negative area separately.

$$|A_1| + |A_2| = - \int_1^2 (x^2 - 4) dx + \int_1^2 (x^2 - 4) dx = - \left( -\frac{5}{3} \right) + \frac{7}{3} = 4$$

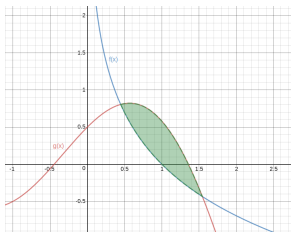
# Negative Area Exercise

Find the area between the curve  $y = 2(x - 5)^2 - 2$  and the  $x$ -axis between  $x = 3$  and  $x = 7$ .



# Area Between Curves

Find the area bounded by the curves  $f(x)$  and  $g(x)$ .



To find this area, solve for the two solutions  $x_1, x_2$  of

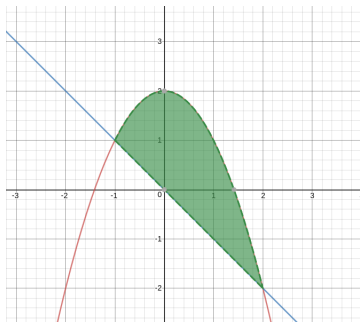
$$f(x) = g(x) \quad (1)$$

(you may have to use Newton's method) and then integrate

$$A = \int_{x_1}^{x_2} (g(x) - f(x)) \, dx \quad (2)$$

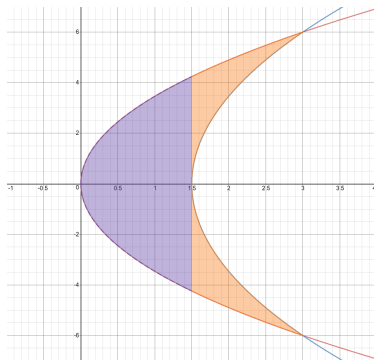
# Area Between Curves Exercise

**Exercise 1:** Find the area of the region enclosed by the parabola  $y = 2 - x^2$  and the line  $y = -x$ .



# Integrating Along the $y$ -Axis

Find the area bounded by the curves  $y^2 = 12x$  and  $y^2 = 24x - 36$ .



In this case, it is more efficient to integrate over  $y$ .

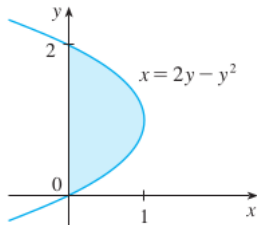
$$A = 2 \cdot \int_0^6 \left( \frac{y^2}{24} + \frac{36}{24} - \frac{y^2}{12} \right) dy \quad (3)$$

# Area Between Curves Exercise

**Exercise 2:** The area of the region that lies to the right of the  $y$ -axis and to the left of the parabola  $x = 2y - y^2$  (the shaded region in the figure) is given by the integral

$$\int_0^2 (2y - y^2) dy \quad (4)$$

Find the area of the region.

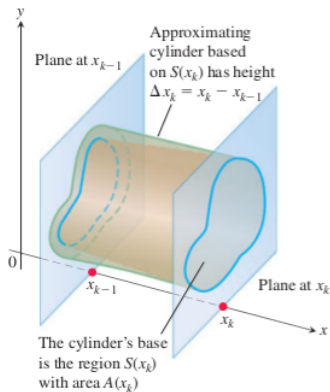




# Volume of Cross-Sections

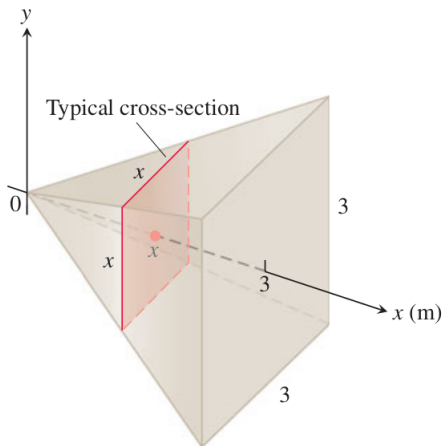
The volume of a solid integrable cross-sectional area  $A(x)$  from  $x = a$  to  $x = b$  is the integral of  $A$  from  $a$  to  $b$ ,

$$V = \int_a^b A(x) dx \quad (5)$$



# Volume of Cross-Sections Exercise

**Exercise 3:** A pyramid three metres high has a square base that is 3 metres on a side. The cross-section of the pyramid perpendicular to the altitude  $x$  metres down from the vertex is a square, whose side is  $x$  metres. Find the volume of the pyramid.



# Cavalieri's Principle

## Cavalieri's Principle

Suppose two regions in three-dimensional space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

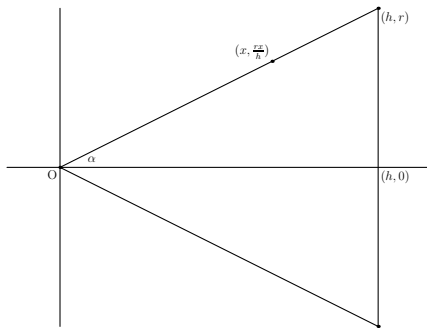


# Disk Method

Remember the formula for the volume of a cone:

$$V = \frac{1}{3}r^2\pi h \quad (6)$$

Let's see if we can give a reason for the formula using calculus. Let the height of a cone be  $h$  and the radius  $r$ .



# Disk Method

Using the volume of cross-sections formula,

$$A(x) = \left(\frac{rx}{h}\right)^2 \cdot \pi \quad (7)$$

and therefore

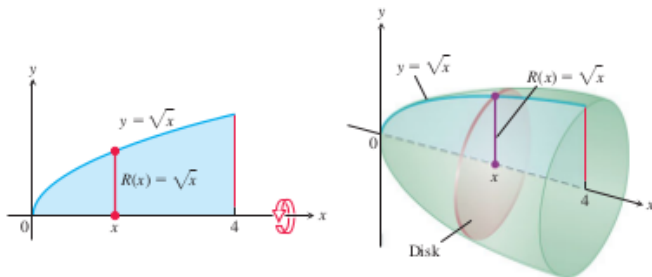
$$V = \int_0^h A(x) dx = \left[ \frac{r^2 \pi x^3}{3h^2} \right]_0^h = \frac{1}{3} r^2 \pi h \quad (8)$$

More generally, any integrable function rotated around the  $x$ -axis gives us the volume of a solid by the so-called **disk method** (see the diagram on the next slide),

$$V = \int_a^b \pi [R(x)]^2 dx \quad (9)$$

# Disk Method Exercise

**Exercise 4:** The region between the curve  $y = \sqrt{x}$ ,  $0 \leq x \leq 4$ , and the  $x$ -axis is revolved about the  $x$ -axis to generate a solid. Find its volume.

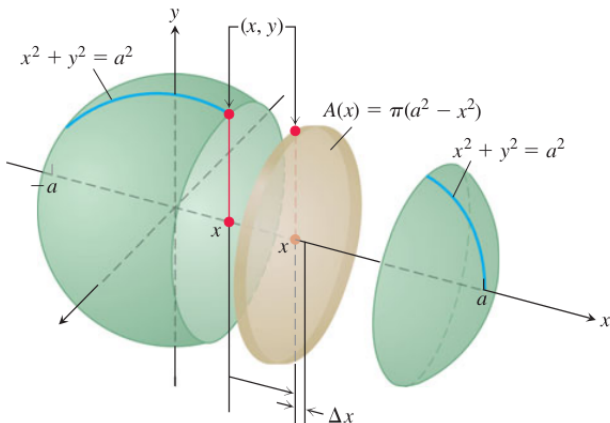


# Disk Method Exercise

## Exercise 5: The circle

$$x^2 + y^2 = r^2 \quad (10)$$

is rotated about the  $x$ -axis to generate a sphere. Find its volume.

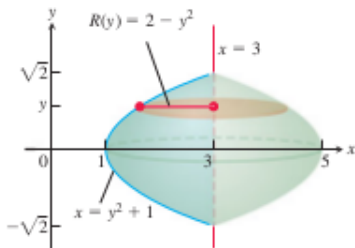
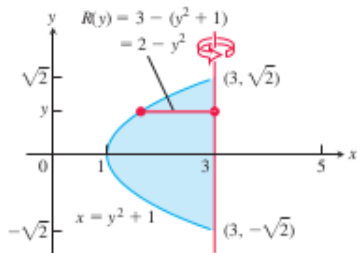


# Disk Method Exercise

**Exercise 6:** Find the volume of the solid generated by revolving the region between the parabola

$$x = y^2 + 1 \quad (11)$$

and the line  $x = 3$  about the line  $x = 3$ .



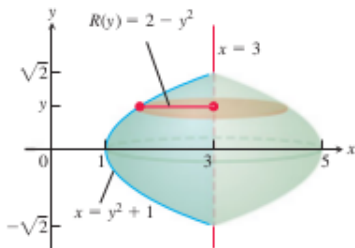
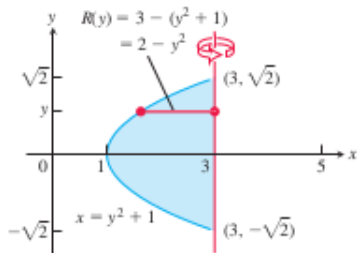


# Disk Method Exercise

**Exercise 6:** Find the volume of the solid generated by revolving the region between the parabola

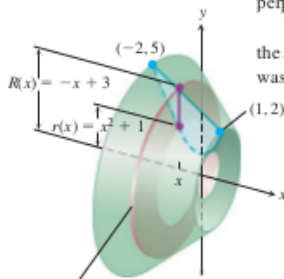
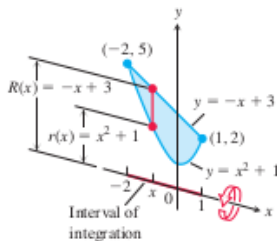
$$x = y^2 + 1 \quad (11)$$

and the line  $x = 3$  about the line  $x = 3$ . The solution is  $(1/15) \cdot 64\pi\sqrt{2}$ .



# Washer Method Exercise

**Exercise 7:** The region bounded by the curve  $y = x^2 + 1$  and the line  $y = -x + 3$  is revolved around the  $x$ -axis to generate a solid. Find the volume of the solid.



*Washer cross-section*

Outer radius:  $R(x) = -x + 3$

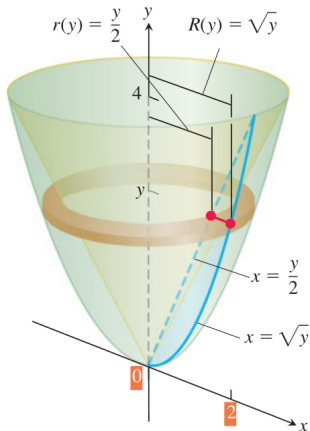
Inner radius:  $r(x) = x^2 + 1$

The region in Example 9 spanned by a line segment perpendicular to the axis of revolution.

When the region is revolved about the  $x$ -axis, the line segment generates a washer.

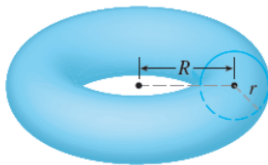
# Washer Method Exercise

**Exercise 8:** The region bounded by the parabola  $y = x^2$  and the line  $y = 2x$  in the first quadrant is revolved about the  $y$ -axis to generate a solid. Find the volume of the solid.



# Washer Method Exercise

**Exercise 9:** Find the volume of the solid torus with major radius  $R$  and minor radius  $r$ .



# Solid Torus Volume

Define the following two functions:

$$f_1(x) = R + \sqrt{r^2 - x^2} \quad (12)$$

$$f_2(x) = R - \sqrt{r^2 - x^2} \quad (13)$$

Use the washer method for the following volume calculation,

$$V = \pi \int_{-r}^r \left( \left( R + \sqrt{r^2 - x^2} \right)^2 - \left( R - \sqrt{r^2 - x^2} \right)^2 \right) dx \quad (14)$$

# Solid Torus Volume

Simplify the last equation to

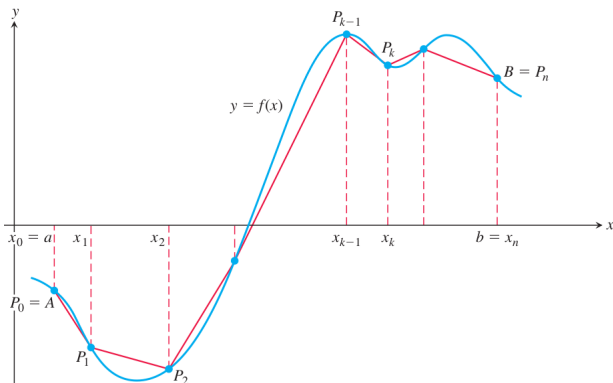
$$V = 4\pi R \int_{-r}^r \left( \sqrt{r^2 - x^2} \right) dx \quad (15)$$

The integral is just the area of a semicircle! Therefore,

$$V = 2\pi R r^2 \quad (16)$$

# Arc Length

A curve  $y = f(x)$  is called **smooth** on an interval  $[a, b]$  if  $f(x)$  has a continuous derivative at every point of  $[a, b]$ . The length of the polygonal path  $P_0P_1P_2 \cdots P_n$  approximates the length of the curve  $y = f(x)$  from point  $A$  to point  $B$ .



# Arc Length

As we did for the fundamental theorem of calculus, divide up the interval  $[a, b]$  into intervals of equal length  $[x_i, x_{i+1}]$ , where  $i = 0, \dots, n-1$  and  $a = x_0, b = x_n$ . Then the length of the curve  $y = f(x)$  from  $a$  to  $b$  is

$$L = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \sqrt{(f(x_{i+1}) - f(x_i))^2 + (x_{i+1} - x_i)^2} \quad (17)$$

The mean value theorem tells us that there is always a point  $x_i^*$  between  $x_i$  and  $x_{i+1}$  such that

$$f'(x_i^*) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \quad (18)$$



Consequently,

$$L = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \sqrt{(f'(x_i^*)(x_{i+1} - x_i))^2 + (x_{i+1} - x_i)^2} \quad (19)$$

which is equivalent to

$$L = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sqrt{(1 + f'(x_i^*))^2} \quad (20)$$

Now let  $g$  be the function

$$g(x) = \sqrt{1 + (f'(x))^2} \quad (21)$$

# Arc Length

Then

$$L = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (x_{i+1} - x_i) g(x_i^*) \quad (22)$$

We already know from the fundamental theorem of calculus that this is

$$L = \int_a^b g(x) dx = \int_a^b \sqrt{1 + (f'(x))^2} dx \quad (23)$$

This is our **formula for arc length**.

**Exercise 10:** Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1, 0 \leq x \leq 1 \quad (24)$$

Check the plausibility of your result by approximating the curve length calculating the straight-line distance between the two end points.

**Exercise 11:** Find the length of the curve

$$y = \frac{x^3}{12} + \frac{1}{x}, 1 \leq x \leq 4 \quad (25)$$

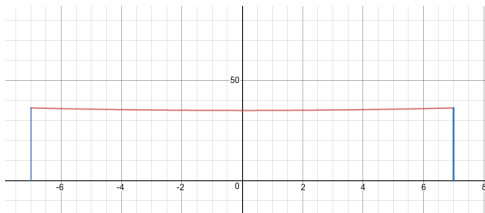
Check the plausibility of your result by approximating the curve length calculating the straight-line distance between the two end points.

# Arc Length Exercise

**Exercise 12:** A telephone line hangs between two poles 14 metres apart in the shape of a catenary

$$y = 20 \cosh\left(\frac{x}{20}\right) - 15 \quad (26)$$

where  $x$  and  $y$  are measured in metres. Find the length of telephone wire needed between the two poles.



# Arc Length Exercise

**Exercise 12:** A telephone line hangs between two poles 14 metres apart in the shape of a catenary

$$y = 20 \cosh\left(\frac{x}{20}\right) - 15 \quad (26)$$

where  $x$  and  $y$  are measured in metres. Find the length of telephone wire needed between the two poles. The answer is  $20(\sinh(7/20) - \sinh(-7/20)) = 14.288$ .



If the function  $f(x) \geq 0$  is continuously differentiable on  $[a, b]$ , the **area of the surface** generated by revolving the graph of  $y = f(x)$  about the  $x$ -axis is

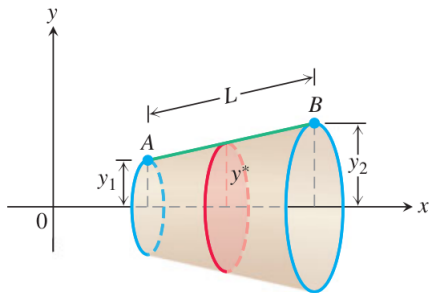
$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx \quad (27)$$

# Surface Area Exercise

**Exercise 13:** Show that the lateral surface area of a frustum (without base and top) is

$$S = 2\pi y^* L \quad (28)$$

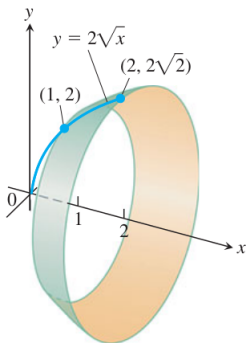
where  $y^*$  is the average height of  $AB$  above the  $x$ -axis and  $L$  is the length of  $AB$ .





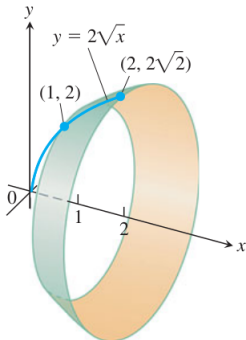
# Surface Area Exercise

**Exercise 14:** Find the area of the surface generated by revolving the curve  $y = 2\sqrt{x}$ ,  $1 \leq x \leq 2$ , about the  $x$ -axis.



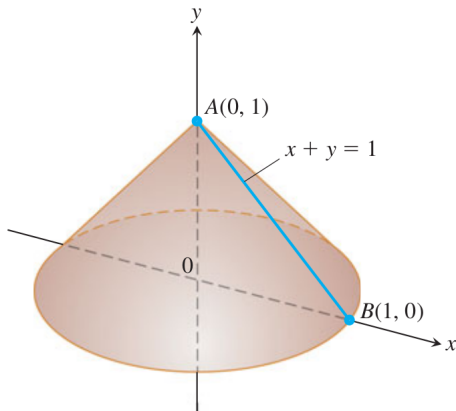
# Surface Area Exercise

**Exercise 14:** Find the area of the surface generated by revolving the curve  $y = 2\sqrt{x}$ ,  $1 \leq x \leq 2$ , about the  $x$ -axis. The solution is  $(8\pi/3) \cdot (\sqrt{27} - \sqrt{8}) = 19.836$ .



# Surface Area Exercise

**Exercise 15:** The line segment  $x = 1 - y$ ,  $0 \leq y \leq 1$ , is revolved around the  $y$ -axis to generate a cone. Find its lateral surface area (which excludes the base area).



# End of Lesson

Next Lesson: Integration Methods