Differentiating Trigonometric Functions

MATH 2511, BCIT

Technical Mathematics for Geomatics

January 22, 2018

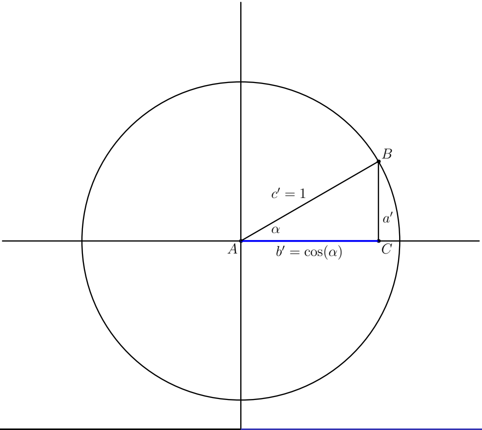
Trigonometric Functions Review

Make sure to remember that the trigonometric functions (sine, cosine, tangent, cotangent, etc.) are functions from the real numbers into the real numbers. An angle is a real number in terms of its radian measure. If the angle is in degrees, it can be converted to radians as in the following example,

$$42^\circ = 42 \cdot \frac{\pi}{180} \approx 0.73304 \tag{1}$$

Trigonometric Functions Review

Any right triangle whose hypotenuse is of length $c' = 1$ can be inserted into the unit circle so that one of the two shorter sides rests on the x-axis and one of the vertices is at the origin (reference triangle). Then the vertex B in the diagram has the coordinates $(\cos\alpha, \sin\alpha)$.



Trigonometric Functions Review

The remaining trigonometric functions are defined as follows.

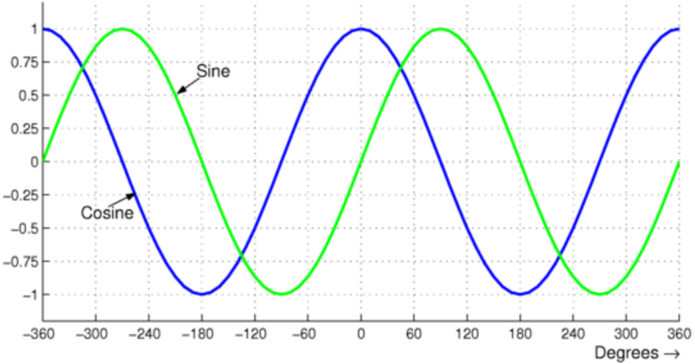
$$\tan x = \frac{\sin x}{\cos x} \tag{2}$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x} \tag{3}$$

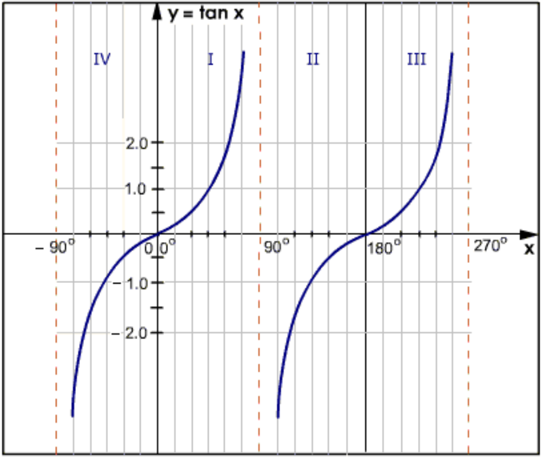
$$\csc x = \frac{1}{\sin x} \tag{4}$$

$$\sec x = \frac{1}{\cos x} \tag{5}$$

Here is a graph of the sine and cosine functions.



Here is a graph of the tangent function.



Trigonometric Functions Review

Consider the following table of well-known inverse functions commonly used in calculus:

function	**inverse**
e^x	$\ln x$
$\sin x$	$\arcsin x$ or $\sin^{-1}$
$\cos x$	$\arccos x$ or $\cos^{-1}$
$\tan x$	$\arctan x$ or $\tan^{-1}$

Trigonometric Functions Review

Consider the following most important trigonometric identities:

$$\sin^2 x + \cos^2 x = 1 \tag{6}$$

$$\sin(-x) = -\sin x \tag{7}$$

$$\cos(-x) = \cos x \tag{8}$$

$$\tan(-x) = -\tan x \tag{9}$$

Trigonometric Functions Review

Consider the following most important trigonometric identities:

$$\sin(90^\circ - x) = \cos x \tag{10}$$

$$\cos(90^\circ - x) = \sin x \tag{11}$$

$$\tan(90^\circ - x) = \cot x \tag{12}$$

$$\sin(x + 180^\circ) = -\sin x \tag{13}$$

$$\cos(x + 180^\circ) = -\cos x \tag{14}$$

$$\tan(x + 180^\circ) = \tan x \tag{15}$$

$$\cot(x + 180^\circ) = \cot x \tag{16}$$

Trigonometric Functions Review

Here are the angle sum identities,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{17}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \tag{18}$$

from which we have, immediately following, the double angle identities,

$$\sin(2x) = 2 \cos x \sin x \tag{19}$$

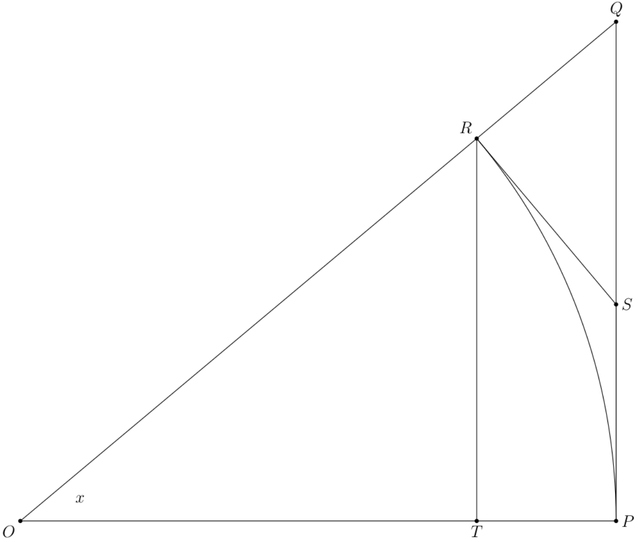
$$\cos(2x) = \cos^2 x - \sin^2 x \tag{20}$$

Limit of sin(x)/x

Consider

$$\lim_{x \to 0} \frac{\sin x}{x} \tag{21}$$

Limit of sin(x)/x



In the previous slide, consider the unit circle with $\|\vec{OP}\| = \|\vec{OR}\| = 1$ and the angle x at O. For simplicity let's assume that $0 < x < \pi/2$. The angle x is also the length of the arc between P and R. Consequently

$$\|\vec{RT}\| = \sin x \leq x \tag{22}$$

and therefore

$$\frac{\sin x}{x} \leq 1 \tag{23}$$

Now consider

$$x \leq \|\vec{PS}\| + \|\vec{SR}\| \leq \|\vec{PS}\| + \|\vec{SQ}\| = \|\vec{PQ}\| = \tan x \qquad (24)$$

$\|\vec{SR}\| \leq \|\vec{SQ}\|$ because the angle QRS is a right angle. (24) means that

$$\cos x \leq \frac{\sin x}{x} \qquad (25)$$

Since $\lim_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} 1 = 1$, we can use the squeeze theorem, (23), and (25) for

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad (26)$$

Limit of (cos(x)-1)/x

Consider

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = \lim_{x\to 0}\left[\frac{\cos x - 1}{x}\cdot\frac{\cos x + 1}{\cos x + 1}\right] =$$

$$\lim_{x\to 0}\frac{\cos^2 x - 1}{x(\cos x + 1)} = -\lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{\cos x + 1}\right] =$$

$$-1\cdot\left(\frac{0}{1+1}\right) = 0 \tag{27}$$

Derivative of Sine

The derivative of $f(x) = \sin x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} =$$

$$\lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} =$$

$$\lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] = \cos x \tag{28}$$

Exercise 1: Differentiate $f(x) = x^2 \sin x$.

The Derivative of Cosine

The derivative of $f(x) = \cos x$ is

$$f'(x) = -\sin x \tag{29}$$

The proof is analogous to the proof for $\sin x$.
Exercise 2: Differentiate

$$f(t) = \frac{1 + \sin t}{t + \cos t} \tag{30}$$

The Derivative of Tangent

The derivative of $f(x) = \tan x$ is

$$f'(x) = \sec^2 x \tag{31}$$

Use the quotient rule to prove this. Remember that

$$\sec x = \frac{1}{\cos x} \tag{32}$$

Derivatives of Trigonometric Functions

$$
\begin{aligned}
\tfrac{d}{dx}\sin x &= \cos x, & \tfrac{d}{dx}\arcsin x &= \tfrac{1}{\sqrt{1-x^2}} \\
\tfrac{d}{dx}\cos x &= -\sin x, & \tfrac{d}{dx}\arccos x &= \tfrac{-1}{\sqrt{1-x^2}} \\
\tfrac{d}{dx}\tan x &= \sec^2 x, & \tfrac{d}{dx}\arctan x &= \tfrac{1}{1+x^2} \\
\tfrac{d}{dx}\cot x &= -\csc^2 x, & \tfrac{d}{dx}\operatorname{arccot} x &= \tfrac{-1}{1+x^2} \\
\tfrac{d}{dx}\sec x &= \tan x \sec x, & \tfrac{d}{dx}\operatorname{arcsec} x &= \tfrac{1}{|x|\sqrt{x^2-1}} \\
\tfrac{d}{dx}\csc x &= -\csc x \cot x, & \tfrac{d}{dx}\operatorname{arccsc} x &= \tfrac{-1}{|x|\sqrt{x^2-1}}
\end{aligned}
$$

Exercises

Exercise 3: Differentiate the following function:

$$f(x) = 3x^2 - 2\cos x \tag{33}$$

Exercise 4: Find the equation of the tangent line at $(\pi/3, 2)$ for

$$y = \sec x \tag{34}$$

Exercise 5: Differentiate the following function:

$$f(x) = \sqrt{x} \sin x \tag{35}$$

Exercise 6: Find the equation of the tangent line at $(\pi/6, 4 + \frac{5}{2}\sqrt{3})$ for

$$f(x) = 2 \csc x + 5 \cos x \tag{36}$$

Exercises

Exercise 7: Differentiate the following functions:

$$g(t) = 4\sec t + \tan t \tag{37}$$

Exercise 8: Differentiate the following functions:

$$f(x) = \csc x(x + \cot x) \tag{38}$$

Exercise 9: Differentiate the following functions:

$$v(w) = \frac{\sin w}{w^2} \tag{39}$$

End of Lesson

Next Lesson: Chain Rule