

Maclaurin and Taylor Series

MATH 2511, BCIT

Technical Mathematics for Geomatics

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Sequences and Series

Infinite Sequence

An infinite sequence is a function whose domain is the set of positive integers.

Here is an example: $a(n) = 2n$ for $n = 1, 2, 3, \dots$. We usually write a_n instead of $a(n)$. The infinite sequence is $2, 4, 6, \dots$. The infinite sequence itself is often called $(a_n)_{n \in \mathbb{N}}$.

Infinite Series

Given an infinite sequence a_n , the infinite series s_n is an infinite sequence defined as follows: $s_n = a_1 + a_2 + \dots + a_n$.

s_n is called a partial sum of the sequence $(a_n)_{n \in \mathbb{N}}$. It is often written as

$$s_n = \sum_{i=1}^n a_i \quad (1)$$

Convergence and Divergence

converges $(a_n)_{n \in \mathbb{N}}$ converges to the real number L if for every positive real number ε there exists an integer N such that for all $n > N$ it is true that $|a_n - L| < \varepsilon$. L is the **limit** of this sequence.

diverges $(a_n)_{n \in \mathbb{N}}$ diverges if no limit exists. $(a_n)_{n \in \mathbb{N}}$ diverges to positive infinity ∞ if for every real number M there is an integer N such that for all n larger than N it is true that $a_n > M$. We say $\lim_{n \rightarrow \infty} a_n = \infty$ or $a_n \rightarrow \infty$. $(a_n)_{n \in \mathbb{N}}$ diverges to negative infinity $-\infty$ if for every real number m there is an integer N such that for all n larger than N it is true that $a_n < m$. We say $\lim_{n \rightarrow \infty} a_n = -\infty$ or $a_n \rightarrow -\infty$.

Geometric Series

Geometric series have the form

$$s_n = a + ar + ar^2 + \dots + ar^{n-1} \quad (2)$$

The notation for the limit is as follows,

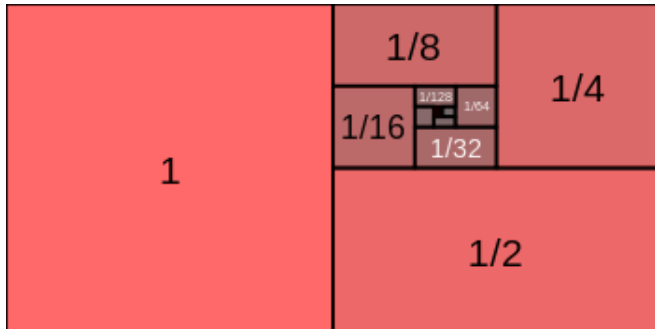
$$\lim_{n \rightarrow \infty} s_n = \sum_{n=1}^{\infty} ar^{n-1} \quad (3)$$

r is called the **ratio** of the geometric series. Subtract $s_n - rs_n$ to find out that

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1 \quad (4)$$

If $|r| \geq 1$ then the limit does not exist.

Geometric Series



Proof that $2 + 4 + 8 + 16 + \dots = -2$

Consider scenario 1,

$$\begin{aligned}a_n &= 2^n = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \\s_n &= \sum_{i=1}^n a_i\end{aligned}\tag{5}$$

Consider scenario 2,

$$\begin{aligned}a_n &= 2^n = 2, 4, 8, 16, \dots \\s_n &= \sum_{i=1}^n a_i\end{aligned}\tag{6}$$

Now calculate the limits of these series. What goes wrong in scenario 2?

Geometric Series Example

Example 1: Limit of a Geometric Series. Find the limit of the following series.

$$\frac{7}{12} + \frac{7}{24} + \frac{7}{48} + \frac{7}{96} + \dots \quad (7)$$

Notice that 7 in the denominator and 12 in the numerator are common factors.

$$\begin{aligned} \frac{7}{12} + \frac{7}{24} + \frac{7}{48} + \frac{7}{96} + \dots &= \frac{7}{12} \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right) = \\ &= \frac{7}{12} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{7}{6} \end{aligned} \quad (8)$$

Exercise 1: Find the limit of the following series.

$$\sum_{n=2}^{\infty} \frac{3^n - 1}{6^n} \quad (9)$$

$$\sum_{n=0}^{\infty} \left(\frac{2n+1}{5^n} \right) \quad (10)$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n} \right) \quad (11)$$

Exercise 2: Find the following series limits using telescoping series.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \quad (12)$$

$$\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2} \right) \quad (13)$$

$$\sum_{n=1}^{\infty} \left(\sqrt{n+4} - \sqrt{n+3} \right) \quad (14)$$

Exercise 3: Find the following series limits using telescoping series.

$$\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2^{n+1})^2} \quad (15)$$

$$\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} \quad (16)$$

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} \quad (17)$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \quad (18)$$

Repeating Decimals

Express each of these numbers as the ratio of two integers.

$$0.\overline{23} = 0.23232323\ldots \quad (19)$$

$$0.0\overline{6} = 0.06666\ldots \quad (20)$$

$$1.24\overline{123} = 1.24123123123\ldots \quad (21)$$

Integral Test

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N is any positive integer).

Then the series

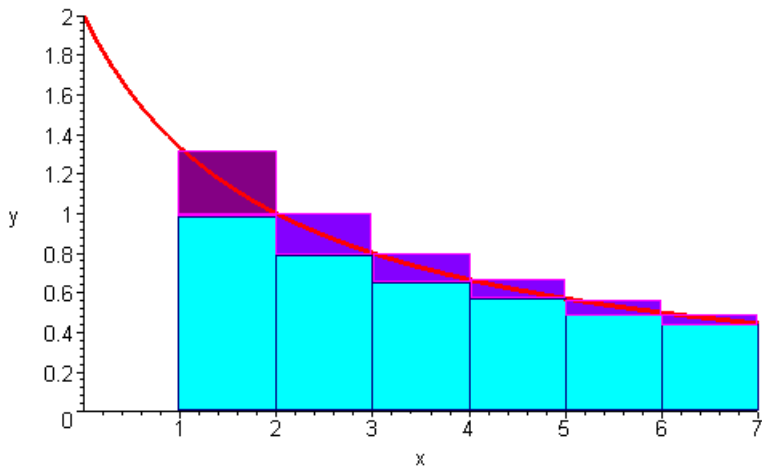
$$\sum_{n=N}^{\infty} a_n \quad (22)$$

and the integral

$$\int_N^{\infty} f(x) dx \quad (23)$$

both converge or both diverge.

Integral Test



Show that the famous harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \quad (24)$$

diverges. Then show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \quad (25)$$

converges.

Now show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \quad (26)$$

exists. Remember that the antiderivative of

$$f(x) = \frac{1}{x^2 + 1} \quad (27)$$

is $F(x) = \arctan x$. Showing that (26) exists does not mean that we know its value.

Integral Test Exercises

Give reasons why the following sums exist or do not exist.

$$\sum_{n=1}^{\infty} e^{-n}$$

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

$$\sum_{n=1}^{\infty} n \sin \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

$$\sum_{n=2}^{\infty} \frac{\ln n}{n}$$

$$\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3}$$

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$$

Integral Test Answers

$$\sum_{n=1}^{\infty} e^{-n} \quad (28)$$

is convergent because it is a geometric series with $0 \leq r = \frac{1}{e} < 1$.

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \quad (29)$$

is divergent because $\frac{n}{n+1} \rightarrow 1$, and $\frac{n}{n+1} \not\rightarrow 0$ implies divergence according to the n -th term test.

$$\sum_{n=1}^{\infty} n \sin \frac{1}{n} \quad (30)$$

is divergent because according to L'Hôpital's rule, $n \sin \frac{1}{n} \rightarrow 1$, and $n \sin \frac{1}{n} \not\rightarrow 0$ implies divergence according to the n -th term test.

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} \quad (31)$$

is divergent according to the integral test.

$$\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} \quad (32)$$

is convergent according to the integral test.

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \quad (33)$$

is divergent because $a_n/b_n \rightarrow 1$ for $a_n = n/(n^2 + 1)$ and $b_n = \frac{1}{n}$, using part 1 of the limit comparison test.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n} \quad (34)$$

is divergent because $\ln n/n > 1/n$ for $n > 2$ and the harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3} \quad (35)$$

is divergent because $a_n/b_n \rightarrow 1$ for $a_n = 5^n/(4^n + 3)$ and $b_n = \frac{5^n}{4^n}$, using part 1 of the limit comparison test. $\sum b_n$ diverges because it is a geometric series with $r > 1$.

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n} \quad (36)$$

is divergent because according to L'Hôpital's rule, $\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln x}$ does not exist.

The n -th Term Test

We could prove this theorem, but it is also accessible to intuition:

$$\text{If } \sum_{i=1}^n a_i \text{ converges, then } a_n \longrightarrow 0 \quad (37)$$

Test for Divergence

$\sum_{i=1}^n a_i$ diverges if $\lim_{n \rightarrow \infty} a_n$ fails to exist or is different from 0.

The converse of the n -th term test is not true. For the following sequence, the corresponding series diverges even though the sequence goes to 0.

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{2 \text{ terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4 \text{ terms}} + \frac{1}{8} + \dots \quad (38)$$

Comparison Tests

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with no negative terms. Then

- ① $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N .
- ② $\sum a_n$ diverges if there is a divergent series $\sum d_n$ with $a_n \geq d_n \geq 0$ for all $n > N$, for some integer N .

Example 2: Comparison Test Example. The series

$$\sum_{n=1}^{\infty} \frac{5}{5n-1} \quad (39)$$

diverges because

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n} \quad (40)$$

for all $n \in \mathbb{N}$.

Limit Comparison Tests

1. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0 \quad (41)$$

then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0 \quad (42)$$

and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty \quad (43)$$

and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence with positive terms and suppose that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \varrho \quad (44)$$

Then

- ① the series $\sum a_n$ converges if $\varrho < 1$
- ② the series $\sum a_n$ diverges if $\varrho > 1$ or ϱ is infinite
- ③ the test is inconclusive if $\varrho = 1$

Exercise 4: Use the ratio test to find out if the following exist:

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \quad \sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} \quad \sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$$

Leibniz's Theorem

Let $(u_n)_{n \in \mathbb{N}}$ be a sequence with $u_n > 0$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots \quad (45)$$

is an **alternating series**. It converges if the following two conditions are satisfied:

- ① $u_n > u_{n+1}$ for all $n > N$, for some integer N
- ② $u_n \longrightarrow 0$

It immediately follows that the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \quad (46)$$

converges. It equals $\ln 2$.

Harmonic Series

An ant starts to crawl along a taut rubber rope 1 km long at a speed of 1 cm per second (relative to the rubber it is crawling on). At the same time, the rope starts to stretch uniformly by 1 km per second, so that after 1 second it is 2 km long, after 2 seconds it is 3 km long, etc. Will the ant ever reach the end of the rope? Counterintuitively, yes. This is a consequence of the divergent harmonic series.

Another example is the block-stacking problem: given a collection of identical dominoes, it is clearly possible to stack them at the edge of a table so that they hang over the edge of the table without falling. The counterintuitive result is that one can stack them in such a way as to make the overhang arbitrarily large, provided there are enough dominoes.

Harmonic Series



Absolute Convergence

A series $\sum a_n$ **converges absolutely** if the corresponding series of absolute values $\sum |a_n|$ converges. Are the following two series absolutely convergent?

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots \quad (47)$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \quad (48)$$

A series that converges but does not converge absolutely is said to **converge conditionally**. If $\sum |a_n|$ converges, then $\sum a_n$ must converge. Absolutely (and ONLY absolutely) convergent series can be rearranged. The alternating harmonic series can be rearranged to diverge or to reach any preassigned infinite sum.

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots \quad (49)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \dots \quad (50)$$

in which the centre a and the coefficients c_0, c_1, c_2 are real numbers.

The Term-by-Term Differentiation Theorem

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $a-R < x < a+R$ for some $R > 0$, it defines a function f

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ on the domain } a-R < x < a+R \quad (51)$$

Such a function f has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term.

$$f'(x) = \sum_{n=1}^{\infty} n c_n(x-a)^{n-1} \quad (52)$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n(x-a)^{n-2} \quad (53)$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

The Term-by-Term Integration Theorem

If $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $a-R < x < a+R$ for some $R > 0$, it defines a function f

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ on the domain } a-R < x < a+R \quad (54)$$

Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \quad (55)$$

converges for $a-R < x < a+R$ and

$$\int f(x) dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C \quad (56)$$

for $a-R < x < a+R$.

First Power Series Expansions

Use these two theorems to find power series expansions for $f(x) = \arctan x$ and $g(x) = \ln(1+x)$ on the domain $-1 < x < 1$.

Use the following two functions to succeed in this endeavour.

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad (57)$$

$$g(x) = 1 - x + x^2 - x^3 + \dots \quad (58)$$

The Series Multiplication Theorem for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^n a_k b_{n-k} \quad (59)$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$,

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n \quad (60)$$

Use term-by-term differentiation and the series multiplication theorem for power series independently to show that for $|x| < 1$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots \quad (61)$$

Now think about it the other way around. If a power series gives us a continuous function with derivatives of all orders, will a continuous function with derivatives of all orders give us a power series? What would be the coefficients? Let's assume that

$$f(x) = \sum_{n=0}^{\infty} a_n(x - a)^n \quad (62)$$

with a positive radius of convergence.

Then

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with } x - a \text{ as a factor} \quad (63)$$

Since these equations all hold at $x = a$, we have

$$\begin{aligned} f'(a) &= 1 \cdot a_1 \\ f''(a) &= 1 \cdot 2 \cdot a_2 \\ f'''(a) &= 1 \cdot 2 \cdot 3 \cdot a_3 \end{aligned} \quad (64)$$

and in general $f^{(n)} = n!a_n$.

Taylor and Mclaurin Series

So, if (and that's a significant "if") a function f has a series representation, then the series must be

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \dots \quad (65)$$

If f is infinitely differentiable, then this series is determined, but it is not always true that the series has a positive radius of convergence. All kinds of things can go wrong. For example, the function

$$f(x) = e^{-\frac{1}{x^2}} \quad (66)$$

has a Mclaurin series which converges everywhere but only at $x = 0$ does the limit equal $f(x)$!

Taylor and Mclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series** generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \dots \quad (67)$$

The **Mclaurin series** generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (x)^k = f(0) + f'(0)(x) + \frac{f''(0)}{2!} (x)^2 + \dots + \frac{f^{(n)}(0)}{n!} (x)^n + \dots \quad (68)$$

Taylor Series Exercises

Find the Taylor polynomials of orders 0, 1, 2, 3 generated by f at a .

$$f(x) = \ln x, \quad a = 1 \quad (69)$$

$$f(x) = \frac{1}{x}, \quad a = 2 \quad (70)$$

$$f(x) = \sin x, \quad a = \frac{\pi}{4} \quad (71)$$

$$f(x) = \sqrt{x}, \quad a = 4 \quad (72)$$

$$f(x) = \cos x, \quad a = \frac{\pi}{4} \quad (73)$$

Maclaurin Series Exercises

Find the Maclaurin series for the following functions.

$$f(x) = e^{-x} \quad (74)$$

$$f(x) = e^{\frac{x}{2}} \quad (75)$$

$$f(x) = \frac{1}{1+x} \quad (76)$$

$$f(x) = \cosh x \quad (77)$$

$$f(x) = \sinh x \quad (78)$$

End of Lesson

Next Lesson: Spiral Curve