Applications of Derivatives

MATH 2511, BCIT

Technical Mathematics for Geomatics

January 30, 2018

Rate of Change

If $f(t)$ expresses some type of position of something dependent on time t, then the rate of change over an interval is the slope of the secant line. The derivative $f'(t)$ is then the instantaneous rate of change or the slope of the tangent line at t.

Sometimes the rate of change is not with respect to a position. For example, acceleration is the rate of change with respect to a velocity. Another common example is the rate of change in numbers of a population. In economics, the rate of change in pricing may be of interest.

Exercises for Rate of Change (Position)

If a ball is given a push so that it has an initial velocity of 5m/s down a certain inclined plane, then the distance it has rolled after t seconds is $s = 5t + 3t^2$.

1. Find the velocity after 2 seconds.
2. How long does it take for the velocity to reach 35m/s?

Exercises for Rate of Change (Position)

Define $f(t) = 5t + 3t^2$. The derivative is $v(t) = f'(t) = 5 + 6t$. Locate the time at which $v(t) = 5$ by solving the equation $5 = 5 + 6t$. The initial time is $t = 0$. Then

$$v(2) = f'(2) = 5 + 6 \cdot 2 = 17 \tag{1}$$

The velocity after 2 seconds is 17m/s.

$$v(t) = 35 = 5 + 6t \qquad \Rightarrow \qquad t = 5 \tag{2}$$

The velocity after 5 seconds is 35m/s.

Exercises for Rate of Change (Geometry)

A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60cm/s. Find the rate at which the area within the circle is increasing after one second; three seconds; and five seconds. What can you conclude?

Exercises for Rate of Change (Geometry)

Define a function for the radius dependent on time.

$$r(t) = 60t \tag{3}$$

Define a function for the area of the circle dependent on time.

$$A(t) = \pi(r(t))^2 = \pi(60t)^2 = 3600\pi t^2 \tag{4}$$

Differentiate

$$A'(t) = 7200\pi t \tag{5}$$

The circle of the area increases by $7200\pi\text{cm}^2$ per second after one second; by $21600\pi\text{cm}^2$ per second after three seconds; by $36000\pi\text{cm}^2$ per second after five seconds.

Exercises for Rate of Change (Physics)

Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2} \tag{6}$$

where G is the gravitational constant and r is the distance between the bodies. Suppose it is known that the Earth attracts an object with a force that decreases at the rate of 2N/km when $r = 20,000$km (N=kg·m/s^2). What is the mass of the object? Assume the following as constants:

$$G = 6.67408 \cdot 10^{-11} m^3 kg^{-1} s^{-2} \tag{7}$$

$$m = 5.9722 \cdot 10^{24} kg \tag{8}$$

Exercises for Rate of Change (Physics)

Define the force as a function of time.

$$F(r) = \frac{GmM}{r^2} \tag{9}$$

Differentiate to find the rate of change.

$$F'(r) = -\frac{2GmM}{r^3} \tag{10}$$

Fill in the details from the word problem.

$$-2\frac{\mathrm{N}}{\mathrm{km}} = F'(20000) = (-2) \cdot \frac{GmM}{\left(2 \cdot 10^4\right)^3 \mathrm{km}^3} \tag{11}$$

M is the only unknown and equals approximately 20070.78kg.

Exercises for Rate of Change (Population)

The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

$$n = f(t) = \frac{a}{1 + be^{-0.7t}} \tag{12}$$

where t is measured in hours. At time $t = 0$ the population is 20 cells and is increasing at a rate of 12 cells per hour. Find the values of a and b. According to this model, what happens to the yeast population in the long run?

Exercises for Rate of Change (Population)

Find the rate of change for the population by differentiating,

$$f'(t) = \frac{0.7ab \cdot e^{-0.7t}}{(1 + be^{-0.7t})^2} \tag{13}$$

Find a in terms of b using the equation $f(0) = 20$.

$$a = 20 \cdot (1 + b) \tag{14}$$

Find b using $a = 20 \cdot (1 + b)$ and the equation $f'(0) = 12$

$$12 = \frac{0.7 \cdot 20 \cdot (1 + b)}{(1 + b)^2} \qquad \Rightarrow \qquad b = \frac{1}{6} \tag{15}$$

Use b and equation (14) to find $a = \frac{70}{3}$.

Exercises for Rate of Change (Economics)

The cost, in dollars, of producing x yards of a certain fabric is

$$C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3 \tag{16}$$

1. Find the marginal cost function.
2. Find $C'(200)$ and explain its meaning. What does it predict?
3. Compare $C'(200)$ with the cost of manufacturing the 201st yard of fabric.

Differentiate for the marginal cost function,

$$C'(x) = 12 - 0.2x + 0.0015x^2 \tag{17}$$

Therefore, $C'(200) = 32$. At 200 yards, the cost of producing yards is \$32 per yard. The cost of manufacturing the 201st yard of fabric is

$$C(201) - C(200) = 3632.2 - 3600 = 32.2 \tag{18}$$

Exponential Growth and Decay I

Growth often happens proportional to size. For example, a population of 500 will grow to 550 during a given period of time while at the same growth rate a population of 1000 will grow to 1100 during the same period of time. Interest on a bank account is another example. Radioactive decay is an example for decay, sometimes also called negative growth.

The growth rate $f'(t)$ should be proportional to the population $f(t)$, so $f'(t) = kf(t)$, where k is some constant. We already know that the function

$$f(t) = Ce^{kt} \tag{19}$$

fulfills this constraint. It turns out that the function in (19) is the only function fulfilling this constraint. Calculate C in terms of f.

Exponential Growth and Decay II

Many natural phenomena have been found to follow the law that an amount $f(t)$ varies with time t according to

$$f(t) = f(0)e^{kt} \tag{20}$$

If $k > 0$, there is growth. If $k < 0$, there is decay.

Uninhibited Growth and Decay Exercise

A colony of bacteria grows according to the law of uninhibited growth according to the function

$$N(t) = 100e^{0.045t} \tag{21}$$

where N is measured in grams and t is measured in days.

1. Determine the initial amount of bacteria.
2. What is the growth rate of the bacteria?
3. What is the population after five days?
4. How long will it take for the population to reach 140 grams?
5. What is the doubling time for the population?

Radiocarbon Dating

Radiocarbon dating is a method archeologists use to determine the age of ancient objects. The carbon dioxide in the atmosphere always contains a fixed fraction of radioactive carbon, carbon-14 (^{14}C), with a half-life of about 5730 years. Plants absorb carbon dioxide from the atmosphere, which then makes its way to animals through the food chain. Thus, all living creatures contain the same fixed proportions of ^{14}C to nonradioactive ^{12}C as the atmosphere.

After an organism dies, it stops assimilating ^{14}C, and the amount of ^{14}C in it begins to decay exponentially. We can then determine the time elapsed since the death of the organism by measuring the amount of ^{14}C left in it.

Radiocarbon Dating Example

If a donkey bone contains 73% as much ^{14}C as a living donkey, when did it die?

Look at the following table to notice the pattern,

$$\begin{array}{rcccc} 100\% & \dots & 2^{-0} & \dots & 5730 \cdot 0 \\ 50\% & \dots & 2^{-1} & \dots & 5730 \cdot 1 \\ 25\% & \dots & 2^{-2} & \dots & 5730 \cdot 2 \end{array} \tag{22}$$

Therefore (t being the number of years ago that the donkey died),

$$t = (-\log_2 0.73) \cdot 5730 = -\frac{\ln 0.73}{\ln 2} \cdot 5730 \approx 2600 \tag{23}$$

Radiocarbon Dating Exercise

^{14}C dating assumes that the carbon dioxid on earth today has the same radioactive content as it did centuries ago. If this is true, then amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a similar tree growing today. A piece of acient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? (The half-life of ^{14}C is 5730 years.)

Radioactive Decay

The half-life of radium-226 is 1590 years. A sample of radium-226 has a mass of 100mg.

1. Find a formula for the mass of the sample that remains after t years.
2. Find the mass ofter 1000 years correct to the nearest milligram.
3. When will the mass be reduced to 30 mg?

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Newton's Law of Cooling

The temperature u of a heated object at a given time t can be modeled by the following function,

$$u(t) = T + (u_0 - T)e^{kt} \tag{24}$$

where k is a negative constant, T is the constant temperature of the surrounding medium, and u_0 is the initial temperature of the heated object.

Newton's Law of Cooling Exercise 1

An object is heated to $100^\circ C$ (degrees Celsius) and is then allowed to cool in a room whose air temperature is $30^\circ C$.

1. If the temperature of the object is $80^\circ C$ after five minutes, when will its temperature be $50^\circ C$?
2. Determine the elapsed time before the temperature of the object is $35^\circ C$.
3. What do you notice about $u(t)$, the temperature, as t, time, passes?

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A frozen steak has a temperature of $28^\circ F$. It is placed in a room with a constant temperature of $70^\circ F$. After 10 minutes, the temperature of the steak has risen to $35^\circ F$.

1. What will the temperature of the steak be after 30 minutes?
2. How long will it take the steak to thaw to a temperature of $45^\circ F$?

Newton's Law of Cooling Exercise 2

A frozen steak has a temperature of $28^\circ F$. It is placed in a room with a constant temperature of $70^\circ F$. After 10 minutes, the temperature of the steak has risen to $35^\circ F$.

1. What will the temperature of the steak be after 30 minutes?
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Newton's Law of Cooling Exercise 3

The hotel Bora-Bora is having a pig roast. At noon, the chef put the pig in a large earthen oven. The pig's original temperature was $75^\circ F$. At 2:00PM the chef checked the pig's temperature and was upset because it had reached only $100^\circ F$.

1. If the oven's temperature remains a constant $325^\circ F$, at what time may the hotel serve its guests, assuming that pork is done when it reaches $175^\circ F$?

Suppose that a corpse was discovered in a motel room at midnight and its temperature was $80^\circ F$. The temperature of the room is kept constant at $60^\circ F$. Two hours later the temperature of the corpse dropped to 75°. Find the time of death (i.e. the time when the temperature of the body was $98.6^\circ F$). Round the time to the nearest minute.

Newton's Law of Cooling Exercise 5

Suppose we are preparing a lovely Canard à l'Orange (roast duck with orange sauce). We first take our duck out of a 36°F refrigerator and place it in a 350°F oven to roast. After 10 minutes the internal temperature is 53°F. If we want to roast the duck until just under well-done (about 170°F internally), when will it be ready?

Mixed Exercises

The total cost (in dollars) of producing x coffee machines is

$$C(x) = 2100 + 30x - 0.3x^2 \tag{25}$$

1. Find the exact cost of producing the 21st machine.
2. Use marginal cost to approximate the cost of producing the 21st machine.

The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by

$$Q(t) = t^3 2t^2 + 6t + 4 \tag{26}$$

Find the current at $t = .5$sec and $t = 1$sec. The unit of current is 1 ampere, which equals 1 coulombs per second.

Mixed Exercises

If a tank holds 4000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 4000\left(1\frac{t}{40}\right)^2 \text{ for } 0 \leq t \leq 40 \tag{27}$$

Find the rate at which the water is draining out of the tank after

1. 5 min
2. 10 min
3. 20 min
4. 40 min

Mixed Exercises

A Cephid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximal brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by ± 0.35. In view of these data, the brightness of Delta Cephei at time t, where t is measured in days, has been modeled by the function

$$B(t) = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right) \tag{28}$$

1. Find the rate of change of the brightness after t days.
2. Find the rate of increase after one day.

Suppose that a bacteria population starts with 500 bacteria and triples every hour.

1. What is the population after t hours?
2. When will the population reach ten thousand?

A coal-burning electrical generating plant emits sulfur dioxide into the surrounding air. The concentration $C(x)$, in parts per million, is approximately given by the function

$$C(x) = \frac{0.7}{x^2} \tag{29}$$

where x is the distance away from the plant in miles. Calculate the rate of change for the sulfur dioxide concentration one mile from the plant and eight miles from the plant.

Mixed Exercises

Find the first and second derivatives of the following function:

$$h(x) = \arctan(x^2) \tag{30}$$

If $g(x) = \sec x$, find $g'''(\pi/4)$.

Let

$$f(x) = \frac{9x^2}{2 - 3x} \tag{31}$$

Find $f'(x)$ and $f''(x)$.

Mixed Exercises

Suppose that the equation of motion for a particle (where s is in meters and t in seconds) is

$$s = \sin 6\pi t \tag{32}$$

1. Find the velocity and acceleration as functions of t.
2. Find the acceleration after 1 second.
3. Find the acceleration (in absolute value) at the instant when the velocity is 0.

End of Lesson

Next Lesson: Optimization and Analyzing Functions