Newton's Method, Optimization, L'Hôpital's Rule

MATH 2511, BCIT

Technical Mathematics for Geomatics

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Newton's Method

What are the x-intercepts of the following function?

$$f(x) = 2x^3 + 5x^2 - 11x + 3$$

We have not learned how to find x-intercepts for polynomials with degrees higher than 2. There are different methods. One method is called Newton's Method and approximates the x-intercept. I have created an instructional video for Newton's Method which you can watch here:

https://youtu.be/a28M5f0Dk_c

Newton's Method

For Newton's Method, find a plausible x-value x_1 (near enough to the x-intercept that you are trying to find) and approximate the x-intercept using the following iterative procedure:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

Exercise

Exercise 1: Approximate $\sqrt{7}$ to ten decimal places using Newton's method and the function $h(x) = x^2 - 7$.

Exercise

Exercise 2: Approximate the x-intercept of $f(x) = x^3 + 5x - 3$ using Newton's method.

Exercise 3: Factor $g(t) = 24t^3 - 2t^2 - 9t + 2$. Remember that if x_1, x_2, x_3 are x-intercepts of the polynomial $ax^3 + bx^2 + cx + d$, then

$$ax^3 + bx^2 + cx + d = a(x - x_1)(x - x_2)(x - x_3) \tag{2}$$

Exercise

Exercise 4: Find the x-intercepts for the following function:

$$f(x) = x^3 + 4x^2 + x - 6 \tag{3}$$

Exercise

Exercise 5: Solve the equation

$$\cos x = x \tag{4}$$

using Newton's Method.

Exercise

Exercise 6: Analyze the following function:

$$f(x) = \frac{2x^2 + 2}{x - 3} \tag{5}$$

Exercise 7: Solve the following equations using Newton's Method. Use a graphing calculator to get you started.

$$x^6 - x^5 - 6x^4 - x^2 + x + 10 = 0 \tag{6}$$

$$x^2(4 - x^2) = \frac{4}{x^2 + 1} \tag{7}$$

$$x^2\sqrt{2 - x - x^2} = 1 \tag{8}$$

$$4e^{-x^2} \sin x = x^2 - x + 1 \tag{9}$$

$$3\sin(x^2) = 2x \tag{10}$$

Exercise 8: Find the absolute minimum value of the following function correct to four decimal places.

$$f(x) = x^6 - x^4 + 3x^2 - 2x \tag{11}$$

Exercise

Exercise 9: Of the infinitely many lines that are tangent to the curve

$$y = -\sin x \tag{12}$$

and pass through the origin, there is one that has the largest slope. Use Newton's Method to find the slope of that line.

Exercise 10: Use Newton's Method to find the coordinates of the point on the parabola

$$y = (x - 1)^2 \tag{13}$$

that is closest to the origin.

Optimization

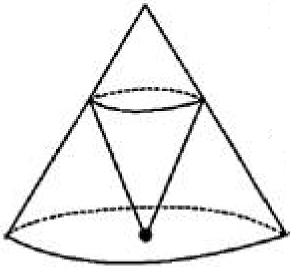
The last exercise gives us a nice segue to optimization. You already have all the tools for optimization. Optimization is often a matter of finding the solutions for $f'(x) = 0$ and then checking the second derivative to make sure the solution is what you were looking for. However, finding the function $f(x)$ can sometimes (as in the last exercise) be tricky! Here are some exercises.

Exercise 11: A farmer has 2400ft of fencing and wants to fence off a rectangular field that borders a straight river. She needs no fence along the river. What are the dimensions of the field that has the largest area?

Exercise 12: A cylindrical can is to be made to hold one litre of oil. Find the dimensions that wil minimize the cost of the metal to manufacture the can.

Exercise 13: Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Exercise 14: A cone with height h and radius r is inscribed in a larger cone with height H and radius R so that its vertex is at the centre of the base of the larger cone. Find h in terms of the dimensions of the larger cone that makes the volume of the smaller cone maximal.



Exercise 15: For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u ($u < v$), then the time required to swim a distance L is $L/(v-u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v-u} \tag{14}$$

where a is the proportionality constant. Determine the value of v that minimizes E.

Exercise

Exercise 16: How close does the semi-circle $y = \sqrt{16 - x^2}$ come to the point $P = (1, \sqrt{3})$?

Note that the semi-circle $y = \sqrt{16 - x^2}$ is part of a circle with a centre of $M = (0, 0)$ and radius $r = 4$. If $Q = (x, y)$ is the point on the semi-circle closest to P, then the distance between P and Q is

$$f(x) = \sqrt{(x-1)^2 + (y - \sqrt{3})^2} \tag{15}$$

Since Q is on the semi-circle, we can replace $y = \sqrt{16 - x^2}$ to get

$$f(x) = \sqrt{(x-1)^2 + (\sqrt{16 - x^2} - \sqrt{3})^2} \tag{16}$$

Exercise Solution

The distance between P and Q is

$$f(x) = \sqrt{(x-1)^2 + (\sqrt{16-x^2} - \sqrt{3})^2} = \sqrt{g(x)} \tag{17}$$

Call the expression under the square root sign $g(x)$. Then

$$f'(x) = \frac{1}{2} \cdot (g(x))^{-\frac{1}{2}} \cdot g'(x) \tag{18}$$

Of these three factors, only $g'(x)$ can be zero. Setting $f'(x) = 0$ is therefore equivalent to $g'(x) = 0$. Note that

$$g'(x) = 2(x-1) + 2\left(\sqrt{16-x^2} - \sqrt{3}\right) \cdot \left(\frac{1}{2}(16-x^2)^{-\frac{1}{2}} \cdot (-2x)\right)$$

Exercise Solution

Simplify and expand to

$$\frac{1}{2}g'(x) = (x-1) - x + \frac{\sqrt{3}x}{\sqrt{16-x^2}} \tag{19}$$

$g'(x) = 0$ just when

$$1 = \frac{\sqrt{3}x}{\sqrt{16-x^2}} \tag{20}$$

Square both sides for the polynomial equation

$$4x^2 - 16 = 0 \tag{21}$$

and the two solutions $x_1 = -2$ and $x_2 = 2$. The first solution is where the distance between P and Q is at a maximum. The second solution is where the distance is at a minimum. Therefore, the point $Q = (2, 2\sqrt{3})$ is the answer to the question in this exercise.

L'Hôpital's Rule

Evaluate the following limits:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} \tag{22}$$

$$\lim_{x \to 0} \frac{\sin x}{x} \text{ (it equals 1 based on geometry)} \tag{23}$$

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} \tag{24}$$

These limits have in common that they are of indeterminate form when you plug in the a towards which the x goes. Sometimes the tricks we have found don't work, for example for

$$\lim_{x \to 1} \frac{\ln x}{x - 1} \tag{25}$$

or for

$$\lim_{x \to \infty} \frac{\ln x}{x - 1} \tag{26}$$

L'Hôpital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Exercise 17: Find

$$\lim_{x \to 1} \frac{\ln x}{x - 1} \tag{27}$$

Exercise 18: Find

$$\lim_{x \to \infty} \frac{e^x}{x^2} \tag{28}$$

Exercise

Exercise 19: Find

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} \tag{29}$$

Exercise 20: Find

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} \tag{30}$$

Exercise 21: Fynd

$$\lim_{x \to \pi} \frac{\pi - \pi \cos x + \sin x}{1 - \cos x} \tag{31}$$

Exercise

Exercise 22: Find

$$\lim_{x \to 0^+} x \ln x \tag{32}$$

Exercise 23: Find

$$\lim_{x \to \frac{\pi}{2}^-} (\sec x - \tan x) \tag{33}$$

Exercise

Exercise 24: Find

$$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} \tag{34}$$

Exercise 25: Find

$$\lim_{x \to 0^+} x^x \tag{35}$$

Exercise 26: Find

$$\lim_{x\to\infty}(\sqrt{x^2+x}-x) \tag{36}$$

Exercise

Exercise 27: Find

$$\lim_{x \to 1} \frac{1 - x + \ln x}{1 + \cos \pi x} \tag{37}$$

Exercise 28: Find

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \tag{38}$$

Exercise 29: Find

$$\lim_{x \to 0^+} \sin x \ln x \tag{39}$$

Exercise 30: Find

$$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) \tag{40}$$

End of Lesson

Next Lesson: Fundamental Theorem of Calculus