Transcendental Functions and Differentials

MATH 2511, BCIT

Technical Mathematics for Geomatics

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Linear Approximation I

It would be tricky to calculate $\sqrt{3.92}$ by hand. Here is a way to approximate this number. Let

$$f(x) = \sqrt{x} \tag{1}$$

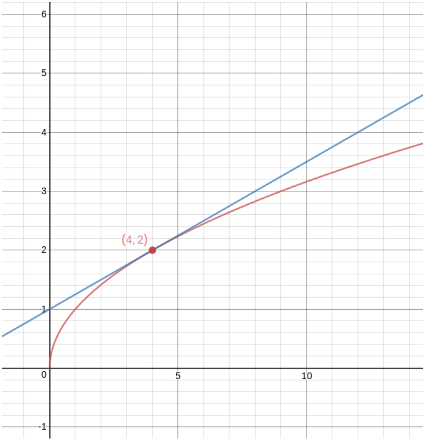
Then

$$f'(x) = \frac{1}{2\sqrt{x}} \tag{2}$$

and the tangent line at $P = (4, 2)$ is

$$y = \frac{1}{4}x + 1 \tag{3}$$

Linear Approximation II



Let's use how close the function is to the tangent line around the point $P = (4, 2)$ to approximate $\sqrt{3.92}$.

$$1.9799 = \sqrt{3.92} = f(3.92) \approx \frac{1}{4} \cdot 3.92 + 1 = 1.98 \qquad (4)$$

Differentials

It will be useful to define an independent variable dx called a differential. The variable dy is dependent on dx and x. It is determined by

$$dy = f'(x)dx \tag{5}$$

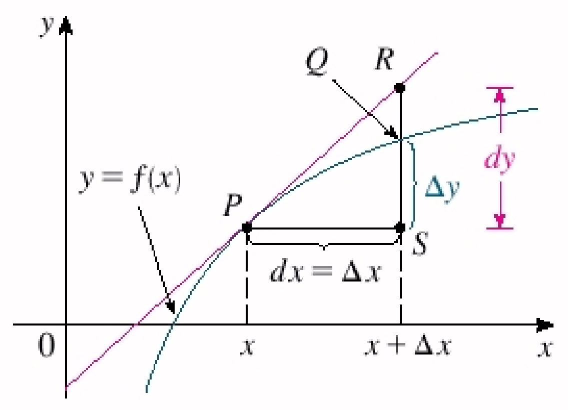
Now have a look at

$$f(x+dx) \approx f'(x)(x+dx) + f(x) - f'(x)x =$$

$$f(x) + f'(x)dx = f(x) + dy \tag{6}$$

This is the linear approximation we were talking about.

Differentials



Differentials are often used to approximate measurement errors.

Example 1: Volume of a Sphere. The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

If $V(r) = (4/3)r^3\pi$, then the answer to this question can be approximated using (where $dr = \Delta r$ is the measurement error)

$$dV = V'(r)dr \tag{7}$$

The precise answer is often called ΔV, which is approximated by dV.

Exercise 1: Find the linearization $L(x)$ of the function at a.

1. $f(x) = x^4 + 3x^2, a = -1$
2. $f(x) = \ln x, a = 1$
3. $f(x) = \cos x, a = \pi/2$
4. $f(x) = x^{\frac{3}{4}}, a = 16$

Differentials Exercises II

Exercise 2: Find the linear approximation of the function

$$g(x) = \sqrt[3]{1+x} \tag{8}$$

at $a = 0$ and use it to approximate the numbers

$$\sqrt[3]{0.95} \tag{9}$$

and

$$\sqrt[3]{1.1} \tag{10}$$

Differentials Exercises III

Exercise 3: Find the differential of each function.

1. $y = x^2 \sin 2x$
2. $y = \ln \sqrt{1 + t^2}$
3. $y = e^{-u} \cos u$
4. $y = e^{\tan \pi t}$

Differentials Exercises IV

Exercise 4: The edge of a cube was found to be 30cm with a possible error in measurement of 0.1cm. Use differentials to estimate the error in computing the volume of the cube and the surface area of the cube.

Exercise 5: Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05cm thick to a hemisphere dome with diameter 50m.

Transcendental Functions

Let's say that (among others) there are two extended families of functions. On the one hand, there are polynomials and their relatives. Constant functions, the identity function, linear functions, and polynomial functions with degrees greater or equal to two are polynomial functions. Rational functions are not polynomial functions, but they belong to the extended family. Functions with root signs also are not polynomial functions, but they belong to the extended family.

On the other hand, there are exponential functions and their relatives. Logarithmic functions are not exponential functions, but they belong to the extended family as the inverse of the exponential function. Where do trigonometric functions belong?

Complex Algebra

In order to answer this question, it is useful to go on a field trip to the complex numbers. Complex numbers are numbers of the form $a + b \cdot i$, where a and b are real numbers and i has the interesting property of $i^2 = -1$, so i is not a real number.

Once you have done some complex algebra, it turns out that (Euler's formula)

$$e^{ix} = \cos x + i \cdot \sin x \tag{11}$$

That means that

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \tag{12}$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \tag{13}$$

To show that this is true add or subtract Euler's formula for x and $-x$.

Euler's Formula

We haven't defined the sine and cosine function on the imaginary numbers yet. Using Euler's formula, it makes sense to define

$$\cos(iy) = \frac{1}{2}\left(e^{-y} + e^{y}\right) \tag{14}$$

$$\sin(iy) = \frac{1}{2i}\left(e^{-y} - e^{y}\right) \tag{15}$$

These look (almost) like plain old real-valued functions, and they should have many of the special features of trigonometric functions (think of all those identities!).

Hyperbolic Functions Definition

Let's return to the safe ground of real-valued functions on the domain of the real numbers. We define

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right) \tag{16}$$

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right) \tag{17}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \tag{18}$$

$$\coth x = \frac{\cosh x}{\sinh x} \tag{19}$$

Identities and Derivatives

The following can be checked easily:

$$\cosh x + \sinh x = e^x \tag{20}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{21}$$

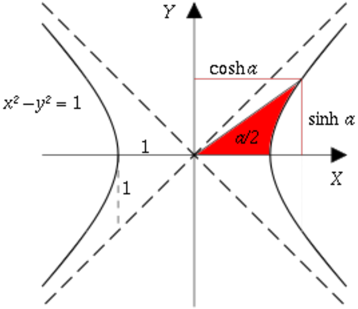
$$(\sinh x)' = \cosh x \tag{22}$$

$$(\cosh x)' = \sinh x \tag{23}$$

There is lots more, but this is enough theory for now. The take-home lesson: once you expand the real numbers to the complex numbers, trigonometric functions turn out to be exponential functions in disguise.

Unit Circles and Hyperbolas

$\cosh(x)$ and $\sinh(x)$ are called hyperbolic functions because they parametrize the unit hyperbola $x^2 - y^2 = 1$ the way $\cos(x)$ and $\sin(x)$ parametrize the unit circle $x^2 + y^2 = 1$.



Catenaries and Ocean Waves

Example 2: Catenaries. Have you ever wondered what function graph models a hanging wire? Galileo thought it was a parabola. It turns out to be $\cosh(x)$. The function

$$f(x) = c + a\cosh\left(\frac{x}{a}\right) \tag{24}$$

is called a catenary.

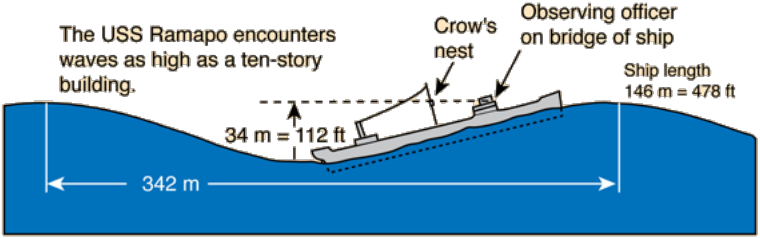
Example 3: Ocean Waves. Another application occurs in the description of ocean waves. The velocity of a water wave with length L moving across a body of water with depth d is modeled by the function ($g \approx 9.8m/s^2$)

$$v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)} \tag{25}$$

Example 4: USS Ramapo. In February, 1933, the USS Ramapo, a 146 meter (478 ft) Navy oiler found itself in an extraordinary storm on its way from Manila to San Diego. The storm lasted 7 days and stretched from the coast of Asia to New York, producing strong winds over thousands of miles of unobstructed ocean. Driven from behind by winds on the order of 60 knots, the crew had time to carefully observe the nearly sinusoidal mountainous waves. An officer on the deck observed the crest of the wave approaching from behind just over the level of the crow's nest while the stern of the ship was at the trough of the wave. Subsequent scaling yielded the height of 34 meters for the wave.

The USS Ramapo in Mountainous Waves

How fast did the waves travel across the ocean if the depth of the ocean was 3000 metres?



Notice that if you could measure the velocity of the waves you could calculate the depth of the ocean using inverse hyperbolic functions.

Exercises

Exercise 6: Find the numerical value of $\tanh 2$ and $\cosh(\ln 3)$.

Exercise 7: Show that $\sinh(x)$ is an odd function and that $\cosh(x)$ is an even function.

Exercise 8: Prove the double angle formula

$$\cosh 2x = \cosh^2 x + \sinh^2 x \tag{26}$$

Exercise 9: Differentiate

$$f(x) = \tanh(1 + e^{2x}) \tag{27}$$

$$g(y) = \cosh(\ln x) \tag{28}$$

$$h(z) = \sinh(\cosh z) \tag{29}$$

Exercise 10: At what point of the curve

$$y = \cosh x \tag{30}$$

does the tangent have slope 1?

Exercises

Exercise 11: If

$$x = \ln(\sec\vartheta + \tan\vartheta) \tag{31}$$

show that

$$\sec\vartheta = \cosh x \tag{32}$$

Exercise 12: Evaluate

$$\lim_{x \to \infty} \frac{\sinh x}{e^x} \tag{33}$$

Exercise 13: A telephone line hangs between two poles 14 metres apart in the shape of a catenary

$$y = 20 \cosh\left(\frac{x}{20}\right) - 15 \tag{34}$$

where x and y are measured in metres. Find the slope of this curve where it meets the right pole in order to determine the angle between the line and the pole.

Exercise 14: If you know that the depth of the ocean is 1500 metres and you measure the length L of a wave to be 17 metres with a 50cm margin of error, then what is your margin of error for the calculated velocity of the wave? Use equation (25).

End of Lesson

Next Lesson: Newton's Method and Optimization