Functions

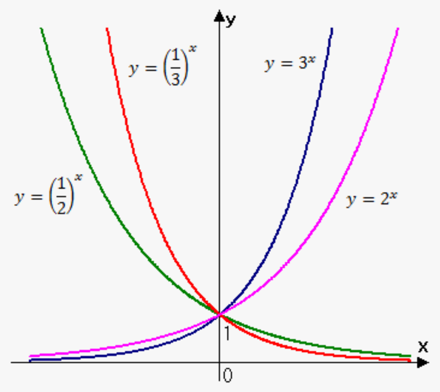
MATH 2511, BCIT

Calculus for Geomatics

January 5, 2018

The Exponential Function: Graph

Let's have a look at the graph for the exponential function.



The Exponential Function: Properties

Here are some properties for the following exponential function ($a > 0$),

$$f(x) = a^x \tag{1}$$

The Exponential Function: Properties

- if $a = 1$ then the exponential function is the constant function $f(x) = 1$
- $f(0) = 1$ and $f(1) = a$
- the domain of f is the real numbers, the range of f is all positive real numbers, and f is injective (one-to-one)
- if $a > 1$ then $f(x)$ tends to 0 as $x \to -\infty$, and $f(x)$ goes very fast to $+\infty$ as $x \to \infty$
- if $a < 1$ then $f(x)$ tends to 0 as $x \to \infty$, and $f(x)$ goes very fast to $+\infty$ as $x \to -\infty$
- how fast the graph rises to $+\infty$ on the left or the right depends on how large a is (if $a > 1$) or how small a is (if $a < 1$). The closer a is to 1, the flatter the graph. 'Flat,' of course, is a relative term here: no matter how close a is to 1, the function graph will still rise faster than any polynomial.

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Functions

Here are a few definitions,

function A function assigns a unique element of a set to each element of another (not necessarily distinct) set.

domain The domain is the set of elements to which the function assigns a unique element.

codomain The codomain is the set from which the function picks out elements to assign.

range The range is the subset of the codomain whose elements the function assigns to an element in the domain.

injective A function is injective if it does not assign the same element of the codomain to two distinct elements in the domain.

surjective A function is surjective if there are no elements in the codomain which are not assigned to an element in the domain.

Examples

What are possible domains and ranges for the following functions? Are the functions injective or surjective, given a particular domain and codomain?

$$f(x) = 2x + 3 \tag{2}$$

$$f(x) = x^2 - 1 \tag{3}$$

$$f(x) = \sqrt{x + 4} \tag{4}$$

$$f(x) = \frac{1}{x + 7} \tag{5}$$

$$f(x) = 10^{2x} \tag{6}$$

Inverse Functions

If a function f from a domain to a codomain is injective, then there is a function f^{-1} from the range of f to its domain which has the following property,

$$f^{-1}(y) = x \text{ if and only if } f(x) = y \tag{7}$$

We call f^{-1} the inverse function of f. Let, for example,

$$f(x) = 4x - 3 \tag{8}$$

Replace $f(x)$ by y for the equation $y = 4x - 3$ and manipulate the equation to isolate x. Then replace x by $f^{-1}(y)$ for the inverse function

$$f^{-1}(y) = \frac{y+3}{4} \tag{9}$$

Defining Logarithms

Let f be an exponential function with a base $a > 1$,

$$f(x) = a^x \tag{10}$$

Considering the function graph of this exponential function, it is apparent that f is an injective and surjective function for the domain $\mathbb{R}$ and the codomain $\mathbb{R}^+$. $\mathbb{R}^+$ is the set of all positive real numbers. There is therefore an inverse function from $\mathbb{R}^+$ to the real numbers, which we shall call $\log_a$,

$$\log_a(y) = x \text{ if and only if } a^x = y \tag{11}$$

Functions

A function is a rule that assigns to each element in a set A one and only one element in a set B.

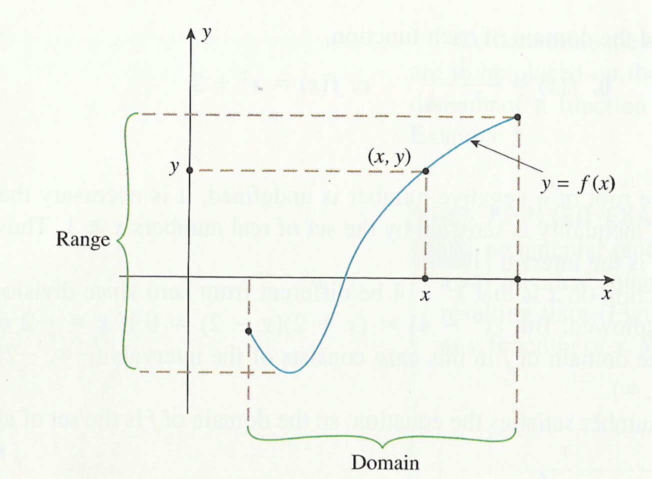
Exercise: Find the maximum domain and range of the following functions on the real number line:

$$f(x) = \sqrt{x-1} \tag{12}$$

$$f(x) = \frac{1}{x^2-4} \tag{13}$$

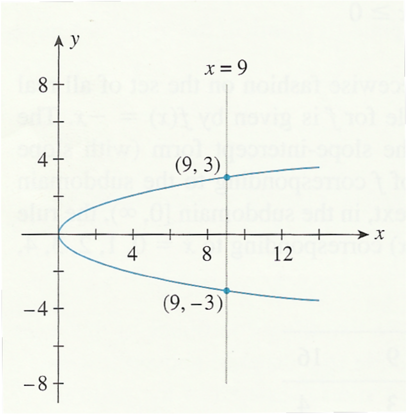
$$f(x) = x^2 + 3 \tag{14}$$

The graph of a function f is the set of all points (x, y) in the xy-plane such that x is in the domain of f and $y = f(x)$.



Vertical Line Test

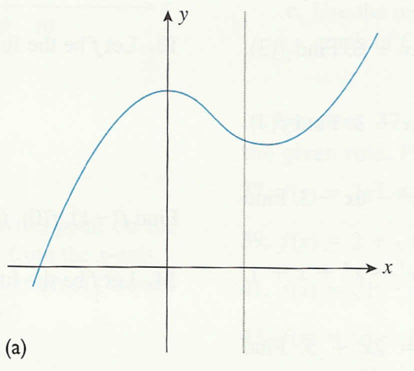
Every function f on a subset of the real numbers has a function graph, but not all graphs correspond to a function. Consider the graph $y^2 = x$. A curve in the xy-plane is the graph of a function $y = f(x)$ if and only if each vertical line intersects it in at most one point.



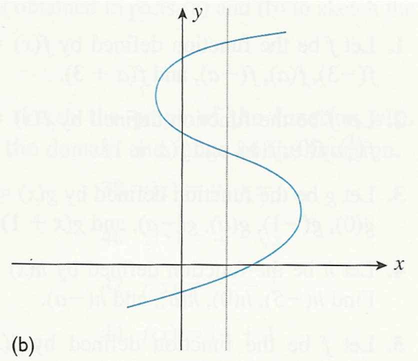
Vertical Line Test Exercise

In the next four slides, determine which graphs correspond to a function.

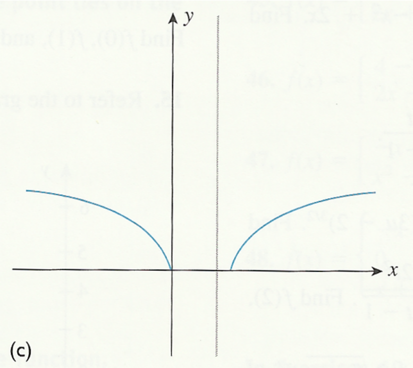
Vertical Line Test Exercise I



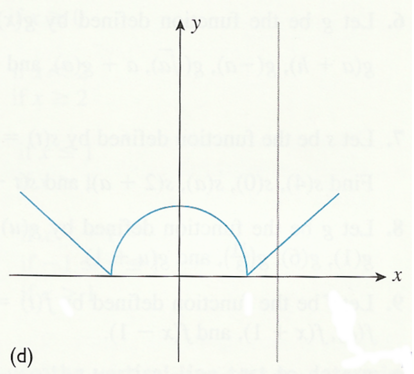
Vertical Line Test Exercise II



Vertical Line Test Exercise III



Vertical Line Test Exercise IV



Function Algebra

Let f and g be functions with domain A and B, respectively. Then the sum $f+g$, difference $f-g$, and product fg of f and g are functions with domain $A \cap B$ (the intersection of A and B) and rule given by

$$(f+g)(x) = f(x) + g(x) \tag{15}$$

$$(f-g)(x) = f(x) - g(x) \tag{16}$$

$$(fg)(x) = f(x) \cdot g(x) \tag{17}$$

The quotient f/g of f and g has domain $A \cap B$ excluding all points x such that $g(x) = 0$ and rule given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \tag{18}$$

Function Composition

Let f and g be functions. Then the composition of g and f is the function $g \circ f$ defined by

$$(g \circ f)(x) = g(f(x)) \tag{19}$$

The domain of $g \circ f$ is the set of all x in the domain of f such that $f(x)$ lies in the domain of g.

Consider the following two functions, $f(x) = \sqrt{x}$ and $g(y) = y - 2$. What are the maximal domains in the real numbers of $f \circ g$ and $g \circ f$?

End of Lesson

Next Lesson: Limits.