

# Notes on the Taylor Green Vortex

Comparing Domain  $(0, 2\pi)^3$  to  $(0, 1)^3$

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## 1 Introduction

The Taylor Green vortex solution is used to compare/debug and test DNS codes. The Sandia/LANL DNS code has a domain  $(0, 1)^3$  while many DNS codes use the default domain  $(0, 2\pi)^3$  and this will cause differences in the solution, such as the energy decay rate, to occur. This document dicusses some of the details of this computation. It has a companion sympy jupyter notebook titled *TaylorGreenExperiments.ipynb*.

## 2 In two-dimensions in $(0, 2\pi)^2$

The equations we solve are,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (3)$$

On  $\Omega \in (0, 2\pi)^3$  with initial condition,

$$\begin{aligned} u(\mathbf{x}, 0) &= \sin(x) \cos(y), \\ v(\mathbf{x}, 0) &= -\cos(x) \sin(y). \end{aligned}$$

This has the following exact solution,

$$\begin{aligned} u(\mathbf{x}, t) &= \sin(x) \cos(y) e^{-2\nu t}, \\ v(\mathbf{x}, t) &= -\cos(x) \sin(y) e^{-2\nu t}, \\ p(\mathbf{x}, t) &= -(\cos(2x) + \cos(2y)) e^{-4\nu t}. \end{aligned}$$

It is quite interesting to me to be reminded that the pressure has to decay like twice the others because the others are nonlinear (it is more like  $u \cdot u$ ). This therefore has the total energy per unit volume decay like,

$$E = \frac{1}{\Omega} \frac{1}{2} \int_{\Omega} (u^2 + v^2) d\Omega \quad \Omega \in (0, 2\pi)^2$$

which gives,

$$\begin{aligned} E(t) &= \frac{\pi^2}{4\pi^2} e^{-4\nu t} \quad \Omega \in (0, 2\pi)^3 \\ &= \frac{1}{4} e^{-4\nu t} \end{aligned}$$

## 3 In two-dimensions in $(0, 1)^2$

The equations are the same as (1)-(3), but these are the following changes,

On  $\Omega \in (0, 1)$  with initial condition,

$$\begin{aligned} u(\mathbf{x}, 0) &= \sin(2\pi x) \cos(2\pi y), \\ v(\mathbf{x}, 0) &= -\cos(2\pi x) \sin(2\pi y). \end{aligned}$$

This has the following exact solution,

$$\begin{aligned} u(\mathbf{x}, t) &= \sin(2\pi x) \cos(2\pi y) e^{-2\nu(2\pi)^2 t}, \\ v(\mathbf{x}, t) &= -\cos(2\pi x) \sin(2\pi y) e^{-2\nu(2\pi)^2 t}, \\ p(\mathbf{x}, t) &= -(\cos(2(2\pi x)) + \cos(2(2\pi y))) e^{-4\nu(2\pi)^2 t}. \end{aligned}$$

This therefore has the total energy decay like,

$$E = \frac{1}{\Omega} \frac{1}{2} \int_{\Omega} (u^2 + v^2) d\Omega \quad \Omega \in (0, 1)^2$$

which gives,

$$E(t) = \frac{1}{2} e^{-4\nu(2\pi)^2 t}$$

## 4 In three-dimensions in $(0, 2\pi)^3$

The equations we solve are,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad (4)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \quad (5)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \quad (6)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (7)$$

On  $\Omega \in (0, 2\pi)^3$  with initial condition,

$$\begin{aligned} u(\mathbf{x}, 0) &= \sin(x) \cos(y) \cos(z), \\ v(\mathbf{x}, 0) &= -\cos(x) \sin(y) \cos(z), \\ w(\mathbf{x}, 0) &= 0. \end{aligned}$$

I think this has exact solution,

$$\begin{aligned} u(\mathbf{x}, t) &= \sin(x) \cos(y) \cos(z) e^{-3\nu t}, \\ v(\mathbf{x}, t) &= -\cos(x) \sin(y) \cos(z) e^{-3\nu t}, \\ w(\mathbf{x}, t) &= ? \\ p(\mathbf{x}, t) &= -(\cos(2x) + \cos(2y)) (\cos(z))^2 e^{-6\nu t}. \end{aligned}$$

and therefor this has the following energy,

$$\begin{aligned} E &= \frac{1}{8\pi^3} \frac{1}{2} \int_{\Omega} (u^2 + v^2 + w^2) d\Omega \quad \Omega \in (0, 2\pi)^3 \\ &\approx \frac{1}{8\pi^3} \frac{1}{2} e^{-6\nu t} \int_{\Omega} (u^2 + v^2 + w^2) d\Omega \quad \Omega \in (0, 2\pi)^3 \end{aligned}$$

## 5 In three-dimensions in $(0, 1)^3$

The equations we solve are,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad (8)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \quad (9)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \quad (10)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (11)$$

On  $\Omega \in (0, 2\pi)^3$  with initial condition,

$$\begin{aligned} u(\mathbf{x}, 0) &= \sin(2\pi x) \cos(2\pi y) \cos(2\pi z), \\ v(\mathbf{x}, 0) &= -\cos(2\pi x) \sin(2\pi y) \cos(2\pi z). \\ w(\mathbf{x}, 0) &= 0. \end{aligned}$$

I think this has exact solution,

$$\begin{aligned} u(\mathbf{x}, t) &= \sin(2\pi x) \cos(2\pi y) \cos(2\pi z) e^{-3(2\pi)^2 \nu t}, \\ v(\mathbf{x}, t) &= -\cos(2\pi x) \sin(2\pi y) \cos(2\pi z) e^{-3(2\pi)^2 \nu t}, \\ w(\mathbf{x}, t) &= ? \\ p(\mathbf{x}, t) &= -(\cos(2x2\pi) + \cos(2y2\pi)) (\cos(2\pi z))^2 e^{-6(2\pi)^2 \nu t}. \end{aligned}$$

and therefor this has the following energy,

$$\begin{aligned} E &= \frac{1}{2} \int_{\Omega} (u^2 + v^2 + w^2) d\Omega \quad \Omega \in (0, 2\pi)^3 \\ &\approx \frac{1}{2} e^{-6(2\pi)^2 \nu t} \int_{\Omega} (u^2 + v^2 + w^2) d\Omega \quad \Omega \in (0, 2\pi)^3 \end{aligned}$$