Notes on the Taylor Green Vortex

Comparing Domain $(0, 2\pi)^3$ to $(0, 1)^3$

B.A. Wingate

1 Introduction

The Taylor Green vortex solution is used to compare/debug and test DNS codes. The Sandia/LANL DNS code has a domain $(0, 1)^3$ while many DNS codes use the default domain $(0, 2\pi)^3$ and this will cause differences in the solution, such as the energy decay rate, to occur. This document dicusses some of the details of this computation. It has a companion sympy jupyter notebook titled `TaylorGreenExperiments.ipynb`.

2 In two-dimensions in $(0, 2\pi)^2$

The equations we solve are,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3}$$

On $\Omega \in (0, 2\pi)^3$ with initial condition,

$$\begin{aligned} u(\boldsymbol{x}, 0) &= \sin(x)\cos(y), \\ v(\boldsymbol{x}, 0) &= -\cos(x)\sin(y). \end{aligned}$$

This has the following exact solution,

$$\begin{aligned} u(\boldsymbol{x}, t) &= \sin(x)\cos(y)e^{-2\nu t}, \\ v(\boldsymbol{x}, t) &= -\cos(x)\sin(y)e^{-2\nu t}, \\ p(\boldsymbol{x}, t) &= -\left(\cos(2x) + \cos(2y)\right)e^{-4\nu t}. \end{aligned}$$

It is quite interesting to me to be reminded that the pressure has to decay like twice the others because the others are nonlinear (it is more like u.u). This therefore has the total energy per unit area decay like,

$$E = \frac{1}{\Omega}\frac{1}{2}\int_\Omega \left(u^2 + v^2\right) d\Omega \quad \Omega \in (0, 2\pi)^2$$

which gives,

$$\begin{aligned} E(t) &= \frac{\pi^2}{4\pi^2}e^{-4\nu t} \quad \Omega \in (0, 2\pi)^3 \\ &= \frac{1}{4}e^{-4\nu t} \end{aligned}$$

We can also compute the intitial energy, $E(0)$,

$$\begin{aligned} E(0) &= \frac{1}{4\pi^2}\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \sin^2(x)\cos^2(y) + \cos^2(x)\sin^2(y)\,dx\,dy \\ &= \frac{1}{4\pi^2}\frac{1}{2}(2\pi^2) = \frac{1}{4} \end{aligned}$$

3 In two-dimensions in $(0,1)^2$

The equations are the same as (1)-(3), but these are the following changes,
On $\Omega \in (0,1)$ with initial condition,

$$\begin{aligned} u(\boldsymbol{x},0) &= \sin(2\pi x)\cos(2\pi y), \\ v(\boldsymbol{x},0) &= -\cos(2\pi x)\sin(2\pi y). \end{aligned}$$

This has the following exact solution,

$$\begin{aligned} u(\boldsymbol{x},t) &= \sin(2\pi x)\cos(2\pi y)e^{-2\nu(2\pi)^2 t}, \\ v(\boldsymbol{x},t) &= -\cos(2\pi x)\sin(2\pi y)e^{-2\nu(2\pi)^2 t}, \\ p(\boldsymbol{x},t) &= -\left(\cos(2\,(2\pi x)) + \cos(2\,(2\pi y))\right) e^{-4\nu(2\pi)^2 t}. \end{aligned}$$

This therefore has the total energy decay like,

$$E = \frac{1}{\Omega}\frac{1}{2}\int_\Omega \left(u^2+v^2\right) d\Omega \quad \Omega \in (0,1)^2$$

which gives,

$$E(t) = \frac{1}{2}e^{-4\nu(2\pi)^2 t}.$$

Computing the energy at $t = 0$,

$$\begin{aligned} E(0) &= \frac{1}{1}\frac{1}{2}\int_0^1\int_0^1 \sin^2(2\pi x)\cos^2(2\pi y) + \cos^2(2\pi x)\sin^2(2\pi y)\,dx\,dy \\ &= \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}. \end{aligned}$$

4 In three-dimensions in $(0,2\pi)^3$

The equations we solve are,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \tag{5}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right), \tag{6}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{7}$$

On $\Omega \in (0,2\pi)^3$ with initial condition,

$$\begin{aligned} u(\boldsymbol{x},0) &= \sin(x)\cos(y)\cos(z), \\ v(\boldsymbol{x},0) &= -\cos(x)\sin(y)\cos(z). \\ w(\boldsymbol{x},0) &= 0. \end{aligned}$$

I think this has exact solution,

$$\begin{aligned} u(\boldsymbol{x},t) &= \sin(x)\cos(y)\cos(z)e^{-3\nu t}, \\ v(\boldsymbol{x},t) &= -\cos(x)\sin(y)\cos(z)e^{-3\nu t}, \\ w(\boldsymbol{x},t) &= ? \\ p(\boldsymbol{x},t) &= -\left(\cos(2x)+\cos(2y)\right)\left(\cos(z)\right)^2 e^{-6\nu t}. \end{aligned}$$

and therefor this has the following energy per unit volume,

$$
\begin{aligned}
E &= \frac{1}{8\pi^3}\frac{1}{2}\int_\Omega \left(u^2+v^2+w^2\right) d\Omega \quad \Omega \in (0,2\pi)^3 \\
&\approx \frac{1}{8\pi^3}\frac{1}{2}e^{-6\nu t}\int_\Omega \left(u^2+v^2+w^2\right) d\Omega \quad \Omega \in (0,2\pi)^3
\end{aligned}
$$

Computing the initial energy,

$$
\begin{aligned}
E(0) &= \frac{1}{8\pi^3}\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \sin^2(x)\cos^2(y)\cos^2(z)+\cos^2(x)\sin^2(y)\cos^2(z)\,dx\,dy\,dz \\
&= \frac{1}{8\pi^3}\frac{1}{2}\left(2\pi^3\right) = \frac{1}{8}
\end{aligned}
$$

5 In three-dimensions in $(0,1)^3$

The equations we solve are,

$$
\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), && (8)\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), && (9)\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right), && (10)\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0. && (11)
\end{aligned}
$$

On $\Omega \in (0,2\pi)^3$ with initial condition,

$$
\begin{aligned}
u(\boldsymbol{x},0) &= \sin(2\pi x)\cos(2\pi y)\cos(2\pi z), \\
v(\boldsymbol{x},0) &= -\cos(2\pi x)\sin(2\pi y)\cos(2\pi z). \\
w(\boldsymbol{x},0) &= 0.
\end{aligned}
$$

I think this has exact solution,

$$
\begin{aligned}
u(\boldsymbol{x},t) &= \sin(2\pi x)\cos(2\pi y)\cos(2\pi z)e^{-3(2\pi)^2\nu t}, \\
v(\boldsymbol{x},t) &= -\cos(2\pi x)\sin(2\pi y)\cos(2\pi z)e^{-3(2\pi)^2\nu t}, \\
w(\boldsymbol{x},t) &= ? \\
p(\boldsymbol{x},t) &= -\left(\cos(2x2\pi)+\cos(2y2\pi)\right)\left(\cos(2\pi z)\right)^2 e^{-6(2\pi)^2\nu t}.
\end{aligned}
$$

and therefor this has the following energy,

$$
\begin{aligned}
E &= \frac{1}{2}\int_\Omega \left(u^2+v^2+w^2\right) d\Omega \quad \Omega \in (0,2\pi)^3 \\
&\approx \frac{1}{2}e^{-6(2\pi)^2\nu t}\int_\Omega \left(u^2+v^2+w^2\right) d\Omega \quad \Omega \in (0,2\pi)^3
\end{aligned}
$$

Computing the energy at $t=0$,

$$
\begin{aligned}
E(0) &= \frac{1}{1}\frac{1}{2}\int_0^{1}\int_0^{1}\int_0^{1} \sin^2(2\pi x)\cos^2(2\pi y)\cos^2(2\pi z)+\cos^2(2\pi x)\sin^2(2\pi y)\cos^2(2\pi z)\,dx\,dy\,dz \\
&= \frac{1}{1}\frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}
\end{aligned}
$$