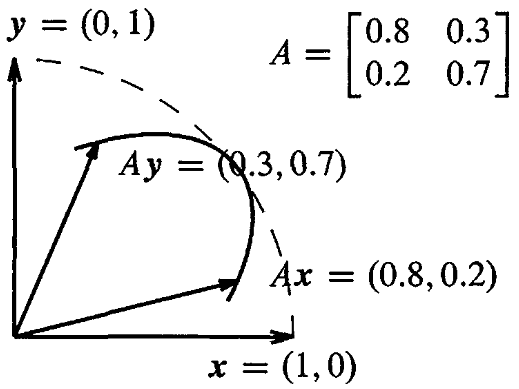
A symmetric matrix ($A^{\mathrm{T}} = A$) can be compared to a real number. A skew-symmetric matrix ($A^{\mathrm{T}} = -A$) can be compared to an imaginary number. An orthogonal matrix ($A^{\mathrm{T}}A = I$) can be compared to a complex number with $|\lambda| = 1$. For the eigenvalues those are more than analogies—they are theorems to be proved in Section 6.4.

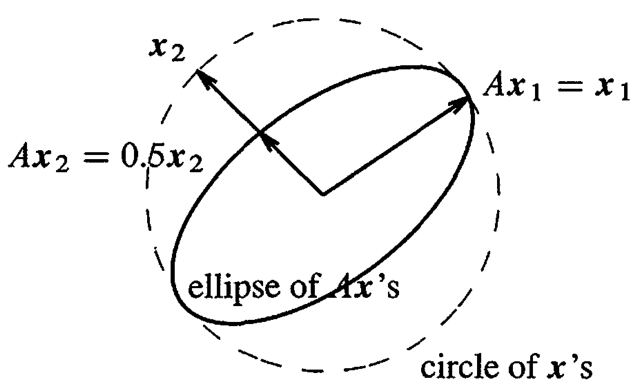
The eigenvectors for all these special matrices are perpendicular. Somehow $(i, 1)$ and $(1, i)$ are perpendicular (Chapter 10 explains the dot product of complex vectors).

Eigshow in MATLAB

There is a MATLAB demo (just type **eigshow**), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector $x = (1, 0)$. *The mouse makes this vector move around the unit circle.* At the same time the screen shows Ax, in color and also moving. Possibly Ax is ahead of x. Possibly Ax is behind x. *Sometimes Ax is parallel to x.* At that parallel moment, $Ax = \lambda x$ (at x_1 and x_2 in the second figure).



These are not eigenvectors



Ax lines up with x at eigenvectors

The eigenvalue λ is the length of Ax, when the unit eigenvector x lines up. The built-in choices for A illustrate three possibilities: 0, 1, or 2 directions where Ax crosses x.

0. There are *no real eigenvectors. Ax stays behind or ahead of x.* This means the eigenvalues and eigenvectors are complex, as they are for the rotation Q.

1. There is only *one* line of eigenvectors (unusual). The moving directions Ax and x touch but don't cross over. This happens for the last 2 by 2 matrix below.

2. There are eigenvectors in *two* independent directions. This is typical! Ax crosses x at the first eigenvector x_1, and it crosses back at the second eigenvector x_2. Then Ax and x cross again at $-x_1$ and $-x_2$.

You can mentally follow x and Ax for these five matrices. Under the matrices I will count their real eigenvectors. Can you see where Ax lines up with x?

$$A = \underset{\mathbf{2}}{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}} \quad \underset{\mathbf{2}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \quad \underset{\mathbf{0}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \quad \underset{\mathbf{1}}{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}} \quad \underset{\mathbf{1}}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$$

When A is singular (rank one), its column space is a line. The vector Ax goes up and down that line while x circles around. One eigenvector x is along the line. Another eigenvector appears when $Ax_2 = \mathbf{0}$. Zero is an eigenvalue of a singular matrix.

■ REVIEW OF THE KEY IDEAS ■

1. $Ax = \lambda x$ says that eigenvectors x keep the same direction when multiplied by A.
2. $Ax = \lambda x$ also says that $\det(A - \lambda I) = 0$. This determines n eigenvalues.
3. The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1}, with the same eigenvectors.
4. The sum of the λ's equals the sum down the main diagonal of A (*the trace*). The product of the λ's equals the determinant.
5. Projections P, reflections R, 90° rotations Q have special eigenvalues $1, 0, -1, i, -i$. Singular matrices have $\lambda = 0$. Triangular matrices have λ's on their diagonal.

■ WORKED EXAMPLES ■

6.1 A Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda - 1)(\lambda - 3) = 0$ so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. For the trace, the sum $2+2$ agrees with $1+3$. The determinant 3 agrees with the product $\lambda_1\lambda_2 = 3$. The eigenvectors come separately by solving $(A - \lambda I)x = \mathbf{0}$ which is $Ax = \lambda x$:

$$\lambda = 1: \quad (A - I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 3: \quad (A - 3I)x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors as* A. Their eigenvalues are λ^2 and λ^{-1} and $\lambda + 4$:

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \qquad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \qquad A + 4I \text{ has } \begin{matrix} 1+4=5 \\ 3+4=7 \end{matrix}$$

The trace of A^2 is $5 + 5$ which agrees with $1 + 9$. The determinant is $25 - 16 = 9$.

Notes for later sections: A has *orthogonal eigenvectors* (Section 6.4 on symmetric matrices). A can be *diagonalized* since $\lambda_1 \neq \lambda_2$ (Section 6.2). A is *similar* to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.6). A is a *positive definite matrix* (Section 6.5) since $A = A^{\mathrm{T}}$ and the λ's are positive.

6.1 B Find the eigenvalues and eigenvectors of this 3 by 3 matrix A:

Symmetric matrix
Singular matrix
Trace $1 + 2 + 1 = 4$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution Since all rows of A add to zero, the vector $x = (1, 1, 1)$ gives $Ax = \mathbf{0}$. This is an eigenvector for the eigenvalue $\lambda = 0$. To find λ_2 and λ_3 I will compute the 3 by 3 determinant:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \begin{aligned} &= (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 2] \\ &= (1-\lambda)(-\lambda)(3-\lambda). \end{aligned}$$

That factor $-\lambda$ confirms that $\lambda = 0$ is a root, and an eigenvalue of A. The other factors $(1 - \lambda)$ and $(3 - \lambda)$ give the other eigenvalues 1 and 3, adding to 4 (the trace). Each eigenvalue $0, 1, 3$ corresponds to an eigenvector:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad Ax_1 = \mathbf{0}x_1 \qquad x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad Ax_2 = \mathbf{1}x_2 \qquad x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad Ax_3 = \mathbf{3}x_3 .$$

I notice again that eigenvectors are perpendicular when A is symmetric.

The 3 by 3 matrix produced a third-degree (cubic) polynomial for $\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 3\lambda$. We were lucky to find simple roots $\lambda = 0, 1, 3$. Normally we would use a command like **eig**(A), and the computation will never even use determinants (Section 9.3 shows a better way for large matrices).

The full command $[S, D] =$ **eig**(A) will produce unit eigenvectors in the columns of the **eigenvector matrix** S. The first one happens to have three minus signs, reversed from $(1, 1, 1)$ and divided by $\sqrt{3}$. The eigenvalues of A will be on the diagonal of the ***eigenvalue matrix*** (typed as D but soon called Λ).

Problem Set 6.1

1 The example at the start of the chapter has powers of this matrix A:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors.

(a) Show from A how a row exchange can produce different eigenvalues.

(b) Why is a zero eigenvalue *not* changed by the steps of elimination?

2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the _____ eigenvectors as A. Its eigenvalues are _____ by 1.

3 Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the _____ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues _____.

4 Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same _____ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues _____. In this example, why is $\lambda_1^2 + \lambda_2^2 = 13$?

5 Find the eigenvalues of A and B (easy for triangular matrices) and $A + B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of $A + B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

6 Find the eigenvalues of A and B and AB and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

7 Elimination produces $A = LU$. The eigenvalues of U are on its diagonal; they are the _____. The eigenvalues of L are on its diagonal; they are all _____. The eigenvalues of A are not the same as _____.

8 (a) If you know that $\boldsymbol{x}$ is an eigenvector, the way to find λ is to _____.

(b) If you know that λ is an eigenvalue, the way to find $\boldsymbol{x}$ is to _____.

9 What do you do to the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$, in order to prove (a), (b), and (c)?

(a) λ^2 is an eigenvalue of A^2, as in Problem 4.

(b) λ^{-1} is an eigenvalue of A^{-1}, as in Problem 3.

(c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.

10 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain from those answers why A^{100} is close to A^∞:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

11 Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$: The columns of $A - \lambda_1 I$ are multiples of the eigenvector $\boldsymbol{x}_2$. Any idea why this should be?

12 Find three eigenvectors for this matrix P (projection matrices have $\lambda = 1$ and 0):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

13 From the unit vector $\boldsymbol{u} = \left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$ construct the rank one projection matrix $P = \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. This matrix has $P^2 = P$ because $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$.

(a) $P\boldsymbol{u} = \boldsymbol{u}$ comes from $(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}($_____$)$. Then $\boldsymbol{u}$ is an eigenvector with $\lambda = 1$.

(b) If $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ show that $P\boldsymbol{v} = \mathbf{0}$. Then $\lambda = 0$.

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

14 Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos\theta \pm i \sin\theta$:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{rotates the } xy \text{ plane by the angle } \theta. \text{ No real } \lambda\text{'s.}$$

Find the eigenvectors of Q by solving $(Q - \lambda I)\boldsymbol{x} = \mathbf{0}$. Use $i^2 = -1$.

15 Every permutation matrix leaves $x = (1, 1, \ldots, 1)$ unchanged. Then $\lambda = 1$. Find two more λ's (possibly complex) for these permutations, from $\det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

16 **The determinant of A equals the product $\lambda_1\lambda_2\cdots\lambda_n$.** Start with the polynomial $\det(A - \lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) \quad \text{so} \quad \det A = \underline{\qquad}.$$

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{2}$.

17 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues $\lambda = (a + d + \sqrt{})/2$ and $\lambda =$ _____. Their sum is _____. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) =$ _____.

18 If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$ and determinant 20 and $\lambda = 4, 5$.

19 A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these (give the answers where possible):

(a) the rank of B

(b) the determinant of $B^{\mathrm{T}}B$

(c) the eigenvalues of $B^{\mathrm{T}}B$

(d) the eigenvalues of $(B^2 + I)^{-1}$.

20 Choose the last rows of A and C to give eigenvalues $4, 7$ and $1, 2, 3$:

Companion matrices $$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$

21 ***The eigenvalues of A equal the eigenvalues of A^{T}.*** This is because $\det(A - \lambda I)$ equals $\det(A^{\mathrm{T}} - \lambda I)$. That is true because _____. Show by an example that the eigenvectors of A and A^{T} are *not* the same.

22 Construct any 3 by 3 Markov matrix M: positive entries down each column add to 1. Show that $M^{\mathrm{T}}(1, 1, 1) = (1, 1, 1)$. By Problem 21, $\lambda = 1$ is also an eigenvalue of M. Challenge: A 3 by 3 singular Markov matrix with trace $\frac{1}{2}$ has what λ's?

23 Find three 2 by 2 matrices that have $\lambda_1 = \lambda_2 = 0$. The trace is zero and the determinant is zero. A might not be the zero matrix but check that $A^2 = 0$.

24 This matrix is singular with rank one. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

25 Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Then $A = B$. *Reason*: Any vector $\boldsymbol{x}$ is a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. What is $A\boldsymbol{x}$? What is $B\boldsymbol{x}$?

26 The block B has eigenvalues 1, 2 and C has eigenvalues 3, 4 and D has eigenvalues 5, 7. Find the eigenvalues of the 4 by 4 matrix A:

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

27 Find the rank and the four eigenvalues of A and C:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

28 Subtract I from the previous A. Find the λ's and then the determinants of

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = I - A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

29 (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

30 When $a + b = c + d$ show that $(1, 1)$ is an eigenvector and find both eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

31 If we exchange rows 1 and 2 *and* columns 1 and 2, the eigenvalues don't change. Find eigenvectors of A and B for $\lambda = 11$. *Rank one gives* $\lambda_2 = \lambda_3 = 0$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad B = PAP^{\mathrm{T}} = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

32 Suppose A has eigenvalues $0, 3, 5$ with independent eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

(c) $A\boldsymbol{x} = \boldsymbol{u}$ has no solution. If it did then _____ would be in the column space.

33 Suppose $\boldsymbol{u}, \boldsymbol{v}$ are orthonormal vectors in $\mathbf{R}^2$, and $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Compute $A^2 = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ to discover the eigenvalues of A. Check that the trace of A agrees with $\lambda_1 + \lambda_2$.

34 Find the eigenvalues of this permutation matrix P from $\det(P - \lambda I) = 0$. Which vectors are not changed by the permutation? They are eigenvectors for $\lambda = 1$. Can you find three more eigenvectors?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Challenge Problems

35 There are six 3 by 3 permutation matrices P. What numbers can be the *determinants* of P? What numbers can be *pivots*? What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P, as in Problem 15?

36 *Is there a real* 2 *by* 2 *matrix* (other than I) *with* $A^3 = I$? Its eigenvalues must satisfy $\lambda^3 = 1$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What trace and determinant would this give? Construct a rotation matrix as A (which angle of rotation?).

37 (a) Find the eigenvalues and eigenvectors of A. They depend on c:

$$A = \begin{bmatrix} .4 & 1-c \\ .6 & c \end{bmatrix}.$$

(b) Show that A has just one line of eigenvectors when $c = 1.6$.

(c) This is a Markov matrix when $c = .8$. Then A^n will approach what matrix A^∞?

6.2 Diagonalizing a Matrix

When x is an eigenvector, multiplication by A is just multiplication by a number λ: $Ax = \lambda x$. All the difficulties of matrices are swept away. Instead of an interconnected system, we can follow the eigenvectors separately. It is like having a *diagonal matrix*, with no off-diagonal interconnections. The 100th power of a diagonal matrix is easy.

The point of this section is very direct. ***The matrix A turns into a diagonal matrix Λ when we use the eigenvectors properly.*** This is the matrix form of our key idea. We start right off with that one essential computation.

Diagonalization Suppose the n by n matrix A has n linearly independent eigenvectors $x_1, \ldots, x_n$. Put them into the columns of an ***eigenvector matrix*** S. Then $S^{-1}AS$ is the ***eigenvalue matrix*** Λ:

Eigenvector matrix S
Eigenvalue matrix Λ

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \tag{1}$$

The matrix A is "diagonalized." We use capital lambda for the eigenvalue matrix, because of the small λ's (the eigenvalues) on its diagonal.

Proof Multiply A times its eigenvectors, which are the columns of S. The first column of AS is Ax_1. That is $\lambda_1 x_1$. Each column of S is multiplied by its eigenvalue λ_i:

A times S

$$AS = A\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix}.$$

The trick is to split this matrix AS into S times Λ:

S times Λ

$$\begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = S\Lambda.$$

Keep those matrices in the right order! Then λ_1 multiplies the first column x_1, as shown. The diagonalization is complete, and we can write $AS = S\Lambda$ in two good ways:

$$AS = S\Lambda \quad \text{is} \quad S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}. \tag{2}$$

The matrix S has an inverse, because its columns (the eigenvectors of A) were assumed to be linearly independent. *Without n independent eigenvectors, we can't diagonalize.*

A and Λ have the same eigenvalues $\lambda_1, \ldots, \lambda_n$. The eigenvectors are different. The job of the original eigenvectors $x_1, \ldots, x_n$ was to diagonalize A. Those eigenvectors in S produce $A = S\Lambda S^{-1}$. You will soon see the simplicity and importance and meaning of the nth power $A^n = S\Lambda^n S^{-1}$.

Example 1 This A is triangular so the λ's are on the diagonal: $\lambda = 1$ and $\lambda = 6$.

$$\textbf{Eigenvectors} \quad \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \quad \underset{S^{-1}}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}} \underset{S}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} = \underset{\Lambda}{\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{6} \end{bmatrix}}$$

In other words $A = S\Lambda S^{-1}$. Then watch $A^2 = S\Lambda S^{-1} S\Lambda S^{-1}$. When you remove $S^{-1}S = I$, this becomes $S\Lambda^2 S^{-1}$. ***Same eigenvectors in S and squared eigenvalues in*** Λ^2.

The kth power will be $A^k = S\Lambda^k S^{-1}$ which is easy to compute:

$$\textbf{Powers of } A \qquad \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 6^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6^k - 1 \\ 0 & 6^k \end{bmatrix}.$$

With $k = 1$ *we get* A. *With* $k = 0$ *we get* $A^0 = I$ (and $\lambda^0 = 1$). *With* $k = -1$ *we get* A^{-1}. You can see how $A^2 = [1 \;\; 35; \;\; 0 \;\; 36]$ fits that formula when $k = 2$.

Here are four small remarks before we use Λ again.

Remark 1 Suppose the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all different. Then it is automatic that the eigenvectors $x_1, \ldots, x_n$ are independent. ***Any matrix that has no repeated eigenvalues can be diagonalized.***

Remark 2 ***We can multiply eigenvectors by any nonzero constants.*** $Ax = \lambda x$ will remain true. In Example 1, we can divide the eigenvector $(1, 1)$ by $\sqrt{2}$ to produce a unit vector.

Remark 3 The eigenvectors in S come in the same order as the eigenvalues in Λ. To reverse the order in Λ, put $(1, 1)$ before $(1, 0)$ in S:

$$\textbf{New order } 6, 1 \qquad \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = \Lambda_{\text{new}}$$

To diagonalize A we *must* use an eigenvector matrix. From $S^{-1}AS = \Lambda$ we know that $AS = S\Lambda$. Suppose the first column of S is x. Then the first columns of AS and $S\Lambda$ are Ax and $\lambda_1 x$. For those to be equal, x must be an eigenvector.

Remark 4 (repeated warning for repeated eigenvalues) Some matrices have too few eigenvectors. *Those matrices cannot be diagonalized.* Here are two examples:

$$\textbf{Not diagonalizable} \qquad A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \textbf{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Their eigenvalues happen to be 0 and 0. Nothing is special about $\lambda = 0$, it is the repetition of λ that counts. All eigenvectors of the first matrix are multiples of $(1, 1)$:

$$\begin{array}{l}\textbf{Only one line} \\ \textbf{of eigenvectors}\end{array} \qquad Ax = 0x \quad \text{means} \quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \; \\ x \\ \; \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

There is no second eigenvector, so the unusual matrix A cannot be diagonalized.

Those matrices are the best examples to test any statement about eigenvectors. In many true-false questions, non-diagonalizable matrices lead to *false*.

Remember that there is no connection between invertibility and diagonalizability:

- ***Invertibility*** is concerned with the ***eigenvalues*** ($\lambda = 0$ or $\lambda \neq 0$).
- ***Diagonalizability*** is concerned with the ***eigenvectors*** (too few or enough for S).

Each eigenvalue has at least one eigenvector! $A - \lambda I$ is singular. If $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ leads you to $\boldsymbol{x} = \mathbf{0}$, λ is *not* an eigenvalue. Look for a mistake in solving $\det(A - \lambda I) = 0$.

Eigenvectors for n different λ's are independent. Then we can diagonalize A.

Independent $\boldsymbol{x}$ from different λ Eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j$ that correspond to distinct (all different) eigenvalues are linearly independent. An n by n matrix that has n different eigenvalues (no repeated λ's) must be diagonalizable.

Proof Suppose $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \mathbf{0}$. Multiply by A to find $c_1\lambda_1\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 = \mathbf{0}$. Multiply by λ_2 to find $c_1\lambda_2\boldsymbol{x}_1 + c_2\lambda_2\boldsymbol{x}_2 = \mathbf{0}$. Now subtract one from the other:

$$\text{Subtraction leaves} \quad (\lambda_1 - \lambda_2)c_1\boldsymbol{x}_1 = \mathbf{0}. \quad \text{Therefore } c_1 = 0.$$

Since the λ's are different and $\boldsymbol{x}_1 \neq \mathbf{0}$, we are forced to this conclusion that $c_1 = 0$. Similarly $c_2 = 0$. No other combination gives $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \mathbf{0}$, so the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ must be independent.

This proof extends directly to j eigenvectors. Suppose $c_1\boldsymbol{x}_1 + \cdots + c_j\boldsymbol{x}_j = \mathbf{0}$. Multiply by A, multiply by λ_j, and subtract. This removes $\boldsymbol{x}_j$. Now multiply by A and by λ_{j-1} and subtract. This removes $\boldsymbol{x}_{j-1}$. Eventually only $\boldsymbol{x}_1$ is left:

$$(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_j)c_1\boldsymbol{x}_1 = \mathbf{0} \quad \text{which forces} \quad c_1 = 0. \tag{3}$$

Similarly every $c_i = 0$. When the λ's are all different, the eigenvectors are independent. A full set of eigenvectors can go into the columns of the eigenvector matrix S.

Example 2 **Powers of A** The Markov matrix $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ in the last section had $\lambda_1 = 1$ and $\lambda_2 = .5$. Here is $A = S\Lambda S^{-1}$ with those eigenvalues in the diagonal Λ:

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = S\Lambda S^{-1}.$$

The eigenvectors $(.6, .4)$ and $(1, -1)$ are in the columns of S. They are also the eigenvectors of A^2. Watch how A^2 has the same S, and ***the eigenvalue matrix of A^2 is Λ^2***:

$$\textbf{Same } S \textbf{ for } A^2 \qquad A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}. \tag{4}$$

Just keep going, and you see why the high powers A^k approach a "steady state":

$$\textbf{Powers of } A \qquad A^k = S\Lambda^k S^{-1} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & (.5)^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}.$$

As k gets larger, $(.5)^k$ gets smaller. In the limit it disappears completely. That limit is A^∞:

$$\textbf{Limit } k \to \infty \qquad A^\infty = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

The limit has the eigenvector $\boldsymbol{x}_1$ in both columns. We saw this A^∞ on the very first page of the chapter. Now we see it coming, from powers like $A^{100} = S\Lambda^{100}S^{-1}$.

Question ***When does $A^k \to$ zero matrix?*** **Answer** ***All*** $|\lambda| < 1$.

Fibonacci Numbers

We present a famous example, where eigenvalues tell how fast the Fibonacci numbers grow. ***Every new Fibonacci number is the sum of the two previous F's:***

The sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ ***comes from*** $F_{k+2} = F_{k+1} + F_k$.

These numbers turn up in a fantastic variety of applications. Plants and trees grow in a spiral pattern, and a pear tree has 8 growths for every 3 turns. For a willow those numbers can be 13 and 5. The champion is a sunflower of Daniel O'Connell, which had 233 seeds in 144 loops. Those are the Fibonacci numbers F_{13} and F_{12}. Our problem is more basic.

Problem: Find the Fibonacci number F_{100} The slow way is to apply the rule $F_{k+2} = F_{k+1} + F_k$ one step at a time. By adding $F_6 = 8$ to $F_7 = 13$ we reach $F_8 = 21$. Eventually we come to F_{100}. Linear algebra gives a better way.

The key is to begin with a matrix equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$. That is a *one-step* rule for vectors, while Fibonacci gave a two-step rule for scalars. We match those rules by putting two Fibonacci numbers into a vector. Then you will see the matrix A.

$$\text{Let } \boldsymbol{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}. \text{ The rule } \begin{array}{l} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{array} \text{ is } \boldsymbol{u}_{k+1} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \boldsymbol{u}_k. \tag{5}$$

Every step multiplies by $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. After 100 steps we reach $\boldsymbol{u}_{100} = A^{100}\boldsymbol{u}_0$:

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \ldots, \quad \boldsymbol{u}_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}.$$

This problem is just right for eigenvalues. Subtract λ from the diagonal of A:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \quad \text{leads to} \quad \det(A - \lambda I) = \lambda^2 - \lambda - 1.$$

The equation $\lambda^2 - \lambda - 1 = 0$ is solved by the quadratic formula $\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$:

$$\textbf{Eigenvalues} \quad \lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618.$$

These eigenvalues lead to eigenvectors $\boldsymbol{x}_1 = (\lambda_1, 1)$ and $\boldsymbol{x}_2 = (\lambda_2, 1)$. Step 2 finds the combination of those eigenvectors that gives $\boldsymbol{u}_0 = (1, 0)$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2}\left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}\right) \quad \text{or} \quad \boldsymbol{u}_0 = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{\lambda_1 - \lambda_2}. \tag{6}$$

Step 3 multiplies $\boldsymbol{u}_0$ by A^{100} to find $\boldsymbol{u}_{100}$. The eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ stay separate! They are multiplied by $(\lambda_1)^{100}$ and $(\lambda_2)^{100}$:

100 steps from $\boldsymbol{u}_0$
$$\boldsymbol{u}_{100} = \frac{(\lambda_1)^{100}\boldsymbol{x}_1 - (\lambda_2)^{100}\boldsymbol{x}_2}{\lambda_1 - \lambda_2}. \tag{7}$$

We want F_{100} = second component of $\boldsymbol{u}_{100}$. The second components of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are 1. The difference between $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ is $\lambda_1 - \lambda_2 = \sqrt{5}$. We have F_{100}:

$$F_{100} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1-\sqrt{5}}{2}\right)^{100}\right] \approx 3.54 \cdot 10^{20}. \tag{8}$$

Is this a whole number? *Yes.* The fractions and square roots must disappear, because Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ stays with integers. The second term in (8) is less than $\frac{1}{2}$, so it must move the first term to the nearest whole number:

$$k\text{th Fibonacci number} = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} = \text{nearest integer to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k. \tag{9}$$

The ratio of F_6 to F_5 is $8/5 = 1.6$. The ratio F_{101}/F_{100} must be very close to the limiting ratio $(1+\sqrt{5})/2$. The Greeks called this number the *"golden mean"*. For some reason a rectangle with sides 1.618 and 1 looks especially graceful.

Matrix Powers A^k

Fibonacci's example is a typical difference equation $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$. ***Each step multiplies by A.*** The solution is $\boldsymbol{u}_k = A^k\boldsymbol{u}_0$. We want to make clear how diagonalizing the matrix gives a quick way to compute A^k and find $\boldsymbol{u}_k$ in three steps.

The eigenvector matrix S produces $A = S\Lambda S^{-1}$. This is a factorization of the matrix, like $A = LU$ or $A = QR$. The new factorization is perfectly suited to computing powers, because ***every time S^{-1} multiplies S we get I***:

Powers of A
$$A^k\boldsymbol{u}_0 = (S\Lambda S^{-1})\cdots(S\Lambda S^{-1})\boldsymbol{u}_0 = S\Lambda^k S^{-1}\boldsymbol{u}_0$$

I will split $S\Lambda^k S^{-1}\boldsymbol{u}_0$ into three steps that show how eigenvalues work:

1. Write $\boldsymbol{u}_0$ as a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ of the eigenvectors. Then $\boldsymbol{c} = S^{-1}\boldsymbol{u}_0$.

2. Multiply each eigenvector $\boldsymbol{x}_i$ by $(\lambda_i)^k$. Now we have $\Lambda^k S^{-1}\boldsymbol{u}_0$.

3. Add up the pieces $c_i(\lambda_i)^k\boldsymbol{x}_i$ to find the solution $\boldsymbol{u}_k = A^k\boldsymbol{u}_0$. This is $S\Lambda^k S^{-1}\boldsymbol{u}_0$.

Solution for $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$
$$\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n. \tag{10}$$

In matrix language A^k equals $(S\Lambda S^{-1})^k$ which is S times Λ^k times S^{-1}. In Step 1, the eigenvectors in S lead to the c's in the combination $\boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$:

$$\textbf{Step 1} \qquad \boldsymbol{u}_0 = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \quad \text{This says that } \boldsymbol{u}_0 = S\boldsymbol{c}. \tag{11}$$

The coefficients in Step 1 are $\boldsymbol{c} = S^{-1}\boldsymbol{u}_0$. Then Step 2 multiplies by Λ^k. The final result $\boldsymbol{u}_k = \sum c_i(\lambda_i)^k\boldsymbol{x}_i$ in Step 3 is the product of S and Λ^k and $S^{-1}\boldsymbol{u}_0$:

$$A^k\boldsymbol{u}_0 = S\Lambda^kS^{-1}\boldsymbol{u}_0 = S\Lambda^k\boldsymbol{c} = \begin{bmatrix} \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} (\lambda_1)^k & & \\ & \ddots & \\ & & (\lambda_n)^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \tag{12}$$

This result is exactly $\boldsymbol{u}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n$. It solves $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$.

Example 3 Start from $\boldsymbol{u}_0 = (1, 0)$. Compute $A^k\boldsymbol{u}_0$ when S and Λ contain these eigenvectors and eigenvalues:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{has} \quad \lambda_1 = 2 \quad \text{and} \quad \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1 \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This matrix is like Fibonacci except the rule is changed to $F_{k+2} = F_{k+1} + 2F_k$. The new numbers start $0, 1, 1, 3$. They grow faster from $\lambda = 2$.

Solution in three steps Find $\boldsymbol{u}_0 = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ and then $\boldsymbol{u}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + c_2(\lambda_2)^k\boldsymbol{x}_2$

Step 1 $\qquad \boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{so} \quad c_1 = c_2 = \frac{1}{3}$

Step 2 $\qquad$ Multiply the two parts by $(\lambda_1)^k = 2^k$ and $(\lambda_2)^k = (-1)^k$

Step 3 $\qquad$ Combine eigenvectors $c_1(\lambda_1)^k\boldsymbol{x}_1$ and $c_2(\lambda_2)^k\boldsymbol{x}_2$ into $\boldsymbol{u}_k$:

$$\boldsymbol{u}_k = A^k\boldsymbol{u}_0 \qquad \boldsymbol{u}_k = \frac{1}{3}2^k\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}(-1)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{13}$$

The new number is $F_k = (2^k - (-1)^k)/3$. After $0, 1, 1, 3$ comes $F_4 = 15/3 = 5$.

Behind these numerical examples lies a fundamental idea: ***Follow the eigenvectors***. In Section 6.3 this is the crucial link from linear algebra to differential equations (powers λ^k will become $e^{\lambda t}$). Chapter 7 sees the same idea as "transforming to an eigenvector basis." The best example of all is a ***Fourier series***, built from the eigenvectors of d/dx.

Nondiagonalizable Matrices (Optional)

Suppose λ is an eigenvalue of A. We discover that fact in two ways:

1. Eigenvectors (geometric) There are nonzero solutions to $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

2. Eigenvalues (algebraic) The determinant of $A - \lambda I$ is zero.

The number λ may be a simple eigenvalue or a multiple eigenvalue, and we want to know its ***multiplicity***. Most eigenvalues have multiplicity $M = 1$ (simple eigenvalues). Then there is a single line of eigenvectors, and $\det(A - \lambda I)$ does not have a double factor.

For exceptional matrices, an eigenvalue can be ***repeated***. Then there are two different ways to count its multiplicity. Always GM $\leq$ AM for each λ:

1. (Geometric Multiplicity = GM) Count the **independent eigenvectors** for λ. This is the dimension of the nullspace of $A - \lambda I$.

2. (Algebraic Multiplicity = AM) Count the **repetitions of λ** among the eigenvalues. Look at the n roots of $\det(A - \lambda I) = 0$.

If A has $\lambda = 4, 4, 4$, that eigenvalue has AM $= 3$ and GM $= 1, 2,$ or 3.

The following matrix A is the standard example of trouble. Its eigenvalue $\lambda = 0$ is repeated. It is a double eigenvalue (AM $= 2$) with only one eigenvector (GM $= 1$).

AM = 2
GM = 1
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2.$$
$\lambda = 0, 0$ but
1 eigenvector

There "should" be two eigenvectors, because $\lambda^2 = 0$ has a double root. The double factor λ^2 makes AM $= 2$. But there is only one eigenvector $\boldsymbol{x} = (1, 0)$. ***This shortage of eigenvectors when* GM *is below* AM *means that A is not diagonalizable.***

The vector called "repeats" in the Teaching Code **eigval** gives the algebraic multiplicity AM for each eigenvalue. When repeats $= [1 \ 1 \ldots \ 1]$ we know that the n eigenvalues are all different and A is diagonalizable. The sum of all components in "repeats" is always n, because every nth degree equation $\det(A - \lambda I) = 0$ has n roots (counting repetitions).

The diagonal matrix **D** in the Teaching Code **eigvec** gives the geometric multiplicity GM for each eigenvalue. This counts the independent eigenvectors. The total number of independent eigenvectors might be less than n. Then A is not diagonalizable.

We emphasize again: $\lambda = 0$ makes for easy computations, but these three matrices also have the same shortage of eigenvectors. Their repeated eigenvalue is $\lambda = 5$. Traces are 10, determinants are 25:

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}.$$

Those all have $\det(A - \lambda I) = (\lambda - 5)^2$. The algebraic multiplicity is AM $= 2$. But each $A - 5I$ has rank $r = 1$. The geometric multiplicity is GM $= 1$. There is only one line of eigenvectors for $\lambda = 5$, and these matrices are not diagonalizable.

Eigenvalues of AB and $A+B$

The first guess about the eigenvalues of AB is not true. An eigenvalue λ of A times an eigenvalue β of B usually does *not* give an eigenvalue of AB:

False proof $$ABx = A\beta x = \beta Ax = \beta\lambda x. \tag{14}$$

It seems that β times λ is an eigenvalue. When x is an eigenvector for A and B, this proof is correct. ***The mistake is to expect that A and B automatically share the same eigenvector x.*** Usually they don't. Eigenvectors of A are not generally eigenvectors of B. A and B could have all zero eigenvalues while 1 is an eigenvalue of AB:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad \text{then} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For the same reason, the eigenvalues of $A + B$ are generally not $\lambda + \beta$. Here $\lambda + \beta = 0$ while $A + B$ has eigenvalues 1 and -1. (At least they add to zero.)

The false proof suggests what is true. Suppose x really is an eigenvector for both A and B. Then we do have $ABx = \lambda\beta x$ and $BAx = \lambda\beta x$. When all n eigenvectors are shared, we *can* multiply eigenvalues. The test $AB = BA$ for shared eigenvectors is important in quantum mechanics—time out to mention this application of linear algebra:

Commuting matrices share eigenvectors Suppose both A and B can be diagonalized. They share the same eigenvector matrix S if and only if $AB = BA$.

Heisenberg's uncertainty principle In quantum mechanics, the position matrix P and the momentum matrix Q do not commute. In fact $QP - PQ = I$ (these are infinite matrices). Then we cannot have $Px = \mathbf{0}$ at the same time as $Qx = \mathbf{0}$ (unless $x = \mathbf{0}$). If we knew the position exactly, we could not also know the momentum exactly. Problem 28 derives Heisenberg's uncertainty principle $\|Px\|\ \|Qx\| \geq \frac{1}{2}\|x\|^2$.

■ REVIEW OF THE KEY IDEAS ■

1. If A has n independent eigenvectors $x_1, \ldots, x_n$, they go into the columns of S.

   A is diagonalized by S $\qquad S^{-1}AS = \Lambda \quad \text{and} \quad A = S\Lambda S^{-1}.$

2. The powers of A are $A^k = S\Lambda^k S^{-1}$. The eigenvectors in S are unchanged.
3. The eigenvalues of A^k are $(\lambda_1)^k, \ldots, (\lambda_n)^k$ in the matrix Λ^k.
4. The solution to $u_{k+1} = Au_k$ starting from u_0 is $u_k = A^k u_0 = S\Lambda^k S^{-1}u_0$:

   $$u_k = c_1(\lambda_1)^k x_1 + \cdots + c_n(\lambda_n)^k x_n \quad \text{provided} \quad u_0 = c_1x_1 + \cdots + c_nx_n.$$

   That shows Steps 1, 2, 3 (c's from $S^{-1}u_0$, λ^k from Λ^k, and x's from S)

5. A is diagonalizable if every eigenvalue has enough eigenvectors (GM = AM).

■ WORKED EXAMPLES ■

6.2 A The **Lucas numbers** are like the Fibonacci numbers except they start with $L_1 = 1$ and $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$, the next Lucas numbers are $4, 7, 11, 18$. Show that the Lucas number L_{100} is $\lambda_1^{100} + \lambda_2^{100}$.

Note The key point is that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1^2 + \lambda_2^2 = 3$, when the λ's are $(1 \pm \sqrt{5})/2$. The Lucas number L_k is $\boldsymbol{\lambda}_1^k + \boldsymbol{\lambda}_2^k$, since this is correct for L_1 and L_2.

Solution $\boldsymbol{u}_{k+1} = \left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]\boldsymbol{u}_k$ is the same as for Fibonacci, because $L_{k+2} = L_{k+1} + L_k$ is the same rule (with different starting values). The equation becomes a 2 by 2 system:

$$\text{Let } \boldsymbol{u}_k = \begin{bmatrix} L_{k+1} \\ L_k \end{bmatrix}. \quad \text{The rule } \begin{array}{l} L_{k+2} = L_{k+1} + L_k \\ L_{k+1} = L_{k+1} \end{array} \quad \text{is } \boldsymbol{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{u}_k.$$

The eigenvalues and eigenvectors of $A = \left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]$ still come from $\lambda^2 = \lambda + 1$:

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \boldsymbol{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}.$$

Now solve $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \boldsymbol{u}_1 = (3, 1)$. The solution is $c_1 = \lambda_1$ and $c_2 = \lambda_2$. Check:

$$\lambda_1\boldsymbol{x}_1 + \lambda_2\boldsymbol{x}_2 = \begin{bmatrix} \lambda_1^2 + \lambda_2^2 \\ \lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} \text{trace of } A^2 \\ \text{trace of } A \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \boldsymbol{u}_1$$

$\boldsymbol{u}_{100} = A^{99}\boldsymbol{u}_1$ tells us the Lucas numbers (L_{101}, L_{100}). The second components of the eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are 1, so the second component of $\boldsymbol{u}_{100}$ is the answer we want:

Lucas number $\qquad L_{100} = c_1\lambda_1^{99} + c_2\lambda_2^{99} = \lambda_1^{100} + \lambda_2^{100}.$

Lucas starts faster than Fibonacci, and ends up larger by a factor near $\sqrt{5}$.

6.2 B Find the inverse and the eigenvalues and the determinant of A:

$$A = 5 * \textbf{eye}(4) - \textbf{ones}(4) = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

Describe an eigenvector matrix S that gives $S^{-1}AS = \Lambda$.

Solution What are the eigenvalues of the all-ones matrix **ones**(4)? Its rank is certainly 1, so three eigenvalues are $\lambda = 0, 0, 0$. Its trace is 4, so the other eigenvalue is $\lambda = 4$. Subtract this all-ones matrix from $5I$ to get our matrix A:

Subtract the eigenvalues 4, 0, 0, 0 from 5, 5, 5, 5. The eigenvalues of A are 1, 5, 5, 5.

The determinant of A is 125, the product of those four eigenvalues. The eigenvector for $\lambda = 1$ is $\boldsymbol{x} = (1, 1, 1, 1)$ or (c, c, c, c). The other eigenvectors are perpendicular to $\boldsymbol{x}$ (since A is symmetric). The nicest eigenvector matrix S is the symmetric orthogonal Hadamard matrix H (normalized to unit column vectors):

$$\textbf{Orthonormal eigenvectors} \quad S = H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = H^{\mathrm{T}} = H^{-1}.$$

The eigenvalues of A^{-1} are $1, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$. The eigenvectors are not changed so $A^{-1} = H\Lambda^{-1}H^{-1}$. The inverse matrix is surprisingly neat:

$$A^{-1} = \frac{1}{5} * (\mathbf{eye}(4) + \mathbf{ones}(4)) = \frac{1}{5}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

A is a rank-one change from $5I$. So A^{-1} is a rank-one change $I/5 + \mathbf{ones}/5$.

The determinant 125 counts the "spanning trees" in a graph with 5 nodes (all edges included). *Trees have no loops* (graphs and trees are in Section 8.2).

With 6 nodes, the matrix $6 * \mathbf{eye}(5) - \mathbf{ones}(5)$ has the five eigenvalues $1, 6, 6, 6, 6$.

Problem Set 6.2

Questions 1–7 are about the eigenvalue and eigenvector matrices Λ and S.

1 (a) Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

(b) If $A = S\Lambda S^{-1}$ then $A^3 = (\quad)(\quad)(\quad)$ and $A^{-1} = (\quad)(\quad)(\quad)$.

2 If A has $\lambda_1 = 2$ with eigenvector $\boldsymbol{x}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\lambda_2 = 5$ with $\boldsymbol{x}_2 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and $\boldsymbol{x}$'s.

3 Suppose $A = S\Lambda S^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (\quad)(\quad)(\quad)^{-1}$.

4 True or false: If the columns of S (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) S is invertible (d) S is diagonalizable.

5 If the eigenvectors of A are the columns of I, then A is a _____ matrix. If the eigenvector matrix S is triangular, then S^{-1} is triangular. Prove that A is also triangular.

6 Describe all matrices S that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

7 Write down the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Questions 8–10 are about Fibonacci and Gibonacci numbers.

8 Diagonalize the Fibonacci matrix by completing S^{-1}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $S\Lambda^k S^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \\ G_{k+1} &= G_{k+1} \end{aligned} \qquad \text{is} \qquad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

10 Prove that every third Fibonacci number in $0, 1, 1, 2, 3, \ldots$ is even.

Questions 11–14 are about diagonalizability.

11 True or false: If the eigenvalues of A are $2, 2, 5$ then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

12 True or false: If the only eigenvectors of A are multiples of $(1, 4)$ then A has

(a) no inverse (b) a repeated eigenvalue (c) no diagonalization $S\Lambda S^{-1}$.

13 Complete these matrices so that $\det A = 25$. Then check that $\lambda = 5$ is repeated—the trace is 10 so the determinant of $A - \lambda I$ is $(\lambda - 5)^2$. Find an eigenvector with $Ax = 5x$. These matrices will not be diagonalizable because there is no second line of eigenvectors.

$$A = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

14 The matrix $A = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ is not diagonalizable because the rank of $A - 3I$ is _____. Change one entry to make A diagonalizable. Which entries could you change?

Questions 15–19 are about powers of matrices.

15 $A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than _____. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

16 (Recommended) Find Λ and S to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the _____.

17 Find Λ and S to diagonalize A_2 in Problem 15. What is $(A_2)^{10}\boldsymbol{u}_0$ for these $\boldsymbol{u}_0$?

$$\boldsymbol{u}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

18 Diagonalize A and compute $S\Lambda^k S^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{has} \quad A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

19 Diagonalize B and compute $S\Lambda^k S^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \quad \text{has} \quad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}.$$

20 Suppose $A = S\Lambda S^{-1}$. Take determinants to prove $\det A = \det \Lambda = \lambda_1\lambda_2 \cdots \lambda_n$. This quick proof only works when A can be _____.

21 Show that trace ST = trace TS, by adding the diagonal entries of ST and TS:

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} q & r \\ s & t \end{bmatrix}.$$

Choose T as ΛS^{-1}. Then $S\Lambda S^{-1}$ has the same trace as $\Lambda S^{-1}S = \Lambda$. The trace of A equals the trace of Λ = sum of the eigenvalues.

22 $AB - BA = I$ is impossible since the left side has trace $=$ _____. But find an elimination matrix so that $A = E$ and $B = E^{\mathrm{T}}$ give

$$AB - BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{which has trace zero.}$$

23 If $A = S\Lambda S^{-1}$, diagonalize the block matrix $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix}$. Find its eigenvalue and eigenvector (block) matrices.

24 Consider all 4 by 4 matrices A that are diagonalized by the same fixed eigenvector matrix S. Show that the A's form a subspace (cA and $A_1 + A_2$ have this same S). What is this subspace when $S = I$? What is its dimension?

25 Suppose $A^2 = A$. On the left side A multiplies each column of A. Which of our four subspaces contains eigenvectors with $\lambda = 1$? Which subspace contains eigenvectors with $\lambda = 0$? From the dimensions of those subspaces, A has a full set of independent eigenvectors. So a matrix with $A^2 = A$ can be diagonalized.

26 (Recommended) Suppose $Ax = \lambda x$. If $\lambda = 0$ then x is in the nullspace. If $\lambda \neq 0$ then x is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

27 The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = S\sqrt{\Lambda}\,S^{-1}$. Why is there no real matrix square root of B?

28 **(Heisenberg's Uncertainty Principle)** $AB - BA = I$ can happen for infinite matrices with $A = A^{\mathrm{T}}$ and $B = -B^{\mathrm{T}}$. Then

$$x^{\mathrm{T}}x = x^{\mathrm{T}}ABx - x^{\mathrm{T}}BAx \leq 2\|Ax\|\,\|Bx\|.$$

Explain that last step by using the Schwarz inequality. Then Heisenberg's inequality says that $\|Ax\|/\|x\|$ times $\|Bx\|/\|x\|$ is at least $\frac{1}{2}$. It is impossible to get the position error and momentum error both very small.

29 If A and B have the same λ's with the same independent eigenvectors, their factorizations into _____ are the same. So $A = B$.

30 Suppose the same S diagonalizes both A and B. They have the same eigenvectors in $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$. Prove that $AB = BA$.

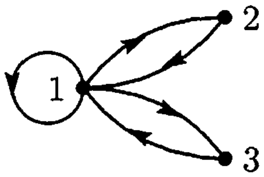
31 (a) If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ then the determinant of $A - \lambda I$ is $(\lambda - a)(\lambda - d)$. Check the "Cayley-Hamilton Theorem" that $(A - aI)(A - dI) = $ *zero matrix*.

(b) Test the Cayley-Hamilton Theorem on Fibonacci's $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.

32 Substitute $A = S\Lambda S^{-1}$ into the product $(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \det(A - \lambda I)$. The ***Cayley-Hamilton Theorem*** says that this product is always $p(A) =$ *zero matrix*, even if A is not diagonalizable.

33 Find the eigenvalues and eigenvectors and the kth power of A. For this "adjacency matrix" the i, j entry of A^k counts the k-step paths from i to j.

1's in A show edges between nodes
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



34 If $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ and $AB = BA$, show that $B = \left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ is also a diagonal matrix. B has the same eigen_____ as A but different eigen_____ . These diagonal matrices B form a two-dimensional subspace of matrix space. $AB - BA = 0$ gives four equations for the unknowns **a**, **b**, **c**, **d**—find the rank of the 4 by 4 matrix.

35 The powers A^k approach zero if all $|\lambda_i| < 1$ and they blow up if any $|\lambda_i| > 1$. Peter Lax gives these striking examples in his book *Linear Algebra*:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 6.9 \\ -3 & -4 \end{bmatrix}$$

$$\|A^{1024}\| > 10^{700} \quad B^{1024} = I \quad C^{1024} = -C \quad \|D^{1024}\| < 10^{-78}$$

Find the eigenvalues $\lambda = e^{i\theta}$ of B and C to show $B^4 = I$ and $C^3 = -I$.

Challenge Problems

36 The nth power of rotation through θ is rotation through $n\theta$:

$$A^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Prove that neat formula by diagonalizing $A = S\Lambda S^{-1}$. The eigenvectors (columns of S) are $(1, i)$ and $(i, 1)$. You need to know Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

37 The transpose of $A = S\Lambda S^{-1}$ is $A^{\mathrm{T}} = (S^{-1})^{\mathrm{T}}\Lambda S^{\mathrm{T}}$. The eigenvectors in $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{y}$ are the columns of that matrix $(S^{-1})^{\mathrm{T}}$. They are often called ***left eigenvectors***. How do you multiply matrices to find this formula for A?

$$\textbf{Sum of rank-1 matrices} \quad A = S\Lambda S^{-1} = \lambda_1 \boldsymbol{x}_1 \boldsymbol{y}_1^{\mathrm{T}} + \cdots + \lambda_n \boldsymbol{x}_n \boldsymbol{y}_n^{\mathrm{T}}.$$

38 The inverse of $A = \mathbf{eye}(n) + \mathbf{ones}(n)$ is $A^{-1} = \mathbf{eye}(n) + C * \mathbf{ones}(n)$. Multiply AA^{-1} to find that number C (depending on n).

6.3 Applications to Differential Equations

Eigenvalues and eigenvectors and $A = S\Lambda S^{-1}$ are perfect for matrix powers A^k. They are also perfect for differential equations $d\boldsymbol{u}/dt = A\boldsymbol{u}$. This section is mostly linear algebra, but to read it you need one fact from calculus: *The derivative of* $e^{\lambda t}$ *is* $\lambda e^{\lambda t}$. The whole point of the section is this: **To convert constant-coefficient differential equations into linear algebra.**

The ordinary scalar equation $du/dt = u$ is solved by $u = e^t$. The equation $du/dt = 4u$ is solved by $u = e^{4t}$. The solutions are exponentials!

$$\textbf{One equation} \qquad \frac{du}{dt} = \lambda u \quad \text{has the solutions} \quad u(t) = Ce^{\lambda t}. \tag{1}$$

The number C turns up on both sides of $du/dt = \lambda u$. At $t = 0$ the solution $Ce^{\lambda t}$ reduces to C (because $e^0 = 1$). By choosing $C = u(0)$, ***the solution that starts from*** $u(0)$ ***at*** $t = 0$ ***is*** $u(t) = u(0)e^{\lambda t}$.

We just solved a 1 by 1 problem. Linear algebra moves to n by n. The unknown is a vector $\boldsymbol{u}$ (now boldface). It starts from the initial vector $\boldsymbol{u}(0)$, which is given. The n equations contain a square matrix A. We expect n exponentials $e^{\lambda t}\boldsymbol{x}$ in $\boldsymbol{u}(t)$.

$$\boldsymbol{n}\ \textbf{equations} \qquad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \quad \text{starting from the vector } \boldsymbol{u}(0) \text{ at } t = 0. \tag{2}$$

These differential equations are *linear*. If $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ are solutions, so is $C\boldsymbol{u}(t) + D\boldsymbol{v}(t)$. We will need n constants like C and D to match the n components of $\boldsymbol{u}(0)$. Our first job is to find n "pure exponential solutions" $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ by using $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Notice that A is a *constant* matrix. In other linear equations, A changes as t changes. In nonlinear equations, A changes as $\boldsymbol{u}$ changes. We don't have those difficulties. $d\boldsymbol{u}/dt = A\boldsymbol{u}$ is "linear with constant coefficients". Those and only those are the differential equations that we will convert directly to linear algebra. The main point will be:

Solve linear constant coefficient equations by exponentials $e^{\lambda t}\boldsymbol{x}$, ***when*** $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Solution of $d\boldsymbol{u}/dt = A\boldsymbol{u}$

Our pure exponential solution will be $e^{\lambda t}$ times a fixed vector $\boldsymbol{x}$. You may guess that λ is an eigenvalue of A, and $\boldsymbol{x}$ ***is the eigenvector***. Substitute $\boldsymbol{u}(t) = e^{\lambda t}\boldsymbol{x}$ into the equation $d\boldsymbol{u}/dt = A\boldsymbol{u}$ to prove you are right (the factor $e^{\lambda t}$ will cancel):

$$\textbf{Use } \boldsymbol{u} = e^{\lambda t}\boldsymbol{x} \textbf{ when } A\boldsymbol{x} = \lambda\boldsymbol{x} \qquad \frac{d\boldsymbol{u}}{dt} = \lambda e^{\lambda t}\boldsymbol{x} \quad \text{agrees with} \quad A\boldsymbol{u} = Ae^{\lambda t}\boldsymbol{x} \tag{3}$$

All components of this special solution $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ share the same $e^{\lambda t}$. The solution grows when $\lambda > 0$. It decays when $\lambda < 0$. If λ is a complex number, its real part decides growth or decay. The imaginary part ω gives oscillation $e^{i\omega t}$ like a sine wave.

Example 1 Solve $du/dt = Au = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u$ starting from $u(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

This is a vector equation for $\boldsymbol{u}$. It contains two scalar equations for the components y and z. They are "coupled together" because the matrix is not diagonal:

$$\frac{du}{dt} = Au \qquad \frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} \text{ means that } \frac{dy}{dt} = z \text{ and } \frac{dz}{dt} = y.$$

The idea of eigenvectors is to combine those equations in a way that gets back to 1 by 1 problems. The combinations $y + z$ and $y - z$ will do it:

$$\frac{d}{dt}(y+z) = z + y \qquad \text{and} \qquad \frac{d}{dt}(y-z) = -(y-z).$$

The combination $y+z$ grows like e^t, because it has $\lambda = 1$. The combination $y-z$ decays like e^{-t}, because it has $\lambda = -1$. Here is the point: We don't have to juggle the original equations $du/dt = Au$, looking for these special combinations. The eigenvectors and eigenvalues of A will do it for us.

This matrix A has eigenvalues 1 and -1. The eigenvectors are $(1, 1)$ and $(1, -1)$. The pure exponential solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ take the form $e^{\lambda t}\boldsymbol{x}$ with $\lambda = 1$ and -1:

$$\boldsymbol{u}_1(t) = e^{\lambda_1 t}\boldsymbol{x}_1 = e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_2(t) = e^{\lambda_2 t}\boldsymbol{x}_2 = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{4}$$

Notice: These $\boldsymbol{u}$'s are eigenvectors. They satisfy $A\boldsymbol{u}_1 = \boldsymbol{u}_1$ and $A\boldsymbol{u}_2 = -\boldsymbol{u}_2$, just like $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The factors e^t and e^{-t} change with time. Those factors give $d\boldsymbol{u}_1/dt = \boldsymbol{u}_1 = A\boldsymbol{u}_1$ and $d\boldsymbol{u}_2/dt = -\boldsymbol{u}_2 = A\boldsymbol{u}_2$. We have two solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$. To find all other solutions, multiply those special solutions by any C and D and add:

$$\textbf{Complete solution} \qquad \boldsymbol{u}(t) = Ce^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} Ce^t + De^{-t} \\ Ce^t - De^{-t} \end{bmatrix}. \tag{5}$$

With these constants C and D, we can match any starting vector $\boldsymbol{u}(0)$. Set $t = 0$ and $e^0 = 1$. The problem asked for the initial value $\boldsymbol{u}(0) = (4, 2)$:

$$\boldsymbol{u}(0) \textbf{ gives } C, D \qquad C\begin{bmatrix} 1 \\ 1 \end{bmatrix} + D\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{yields} \quad C = 3 \quad \text{and} \quad D = 1.$$

With $C = 3$ and $D = 1$ in the solution (5), the initial value problem is solved.

The same three steps that solved $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ now solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

1. Write $\boldsymbol{u}(0)$ as a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ of the eigenvectors of A.
2. Multiply each eigenvector $\boldsymbol{x}_i$ by $e^{\lambda_i t}$.
3. The solution is the combination of pure solutions $e^{\lambda t}\boldsymbol{x}$:

$$\boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n. \tag{6}$$

Not included: If two λ's are equal, with only one eigenvector, another solution is needed. (It will be $te^{\lambda t}\boldsymbol{x}$). Step 1 needs $A = S\Lambda S^{-1}$ to be diagonalizable: a basis of eigenvectors.

Example 2 Solve $du/dt = Au$ knowing the eigenvalues $\lambda = 1, 2, 3$ of A:

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{u} \quad \text{starting from} \quad \boldsymbol{u}(0) = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix}.$$

The eigenvectors are $\boldsymbol{x}_1 = (1,0,0)$ and $\boldsymbol{x}_2 = (1,1,0)$ and $\boldsymbol{x}_3 = (1,1,1)$.

Step 1 The vector $\boldsymbol{u}(0) = (9,7,4)$ is $2\boldsymbol{x}_1 + 3\boldsymbol{x}_2 + 4\boldsymbol{x}_3$. Thus $(c_1, c_2, c_3) = (2,3,4)$.

Step 2 The pure exponential solutions are $e^t \boldsymbol{x}_1$ and $e^{2t}\boldsymbol{x}_2$ and $e^{3t}\boldsymbol{x}_3$.

Step 3 The combination that starts from $\boldsymbol{u}(0)$ is $\boldsymbol{u}(t) = 2e^t\boldsymbol{x}_1 + 3e^{2t}\boldsymbol{x}_2 + 4e^{3t}\boldsymbol{x}_3$.

The coefficients 2, 3, 4 came from solving the linear equation $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3 = \boldsymbol{u}(0)$:

$$\begin{bmatrix} & & \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ & & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix} \quad \text{which is} \quad Sc = \boldsymbol{u}(0). \tag{7}$$

You now have the basic idea—how to solve $du/dt = Au$. The rest of this section goes further. We solve equations that contain *second* derivatives, because they arise so often in applications. We also decide whether $\boldsymbol{u}(t)$ approaches zero or blows up or just oscillates.

At the end comes the ***matrix exponential*** e^{At}. Then $e^{At}\boldsymbol{u}(0)$ solves the equation $du/dt = Au$ in the same way that $A^k\boldsymbol{u}_0$ solves the equation $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$. In fact we ask whether $\boldsymbol{u}_k$ approaches $\boldsymbol{u}(t)$. Example 3 will show how "difference equations" help to solve differential equations. You will see real applications.

All these steps use the λ's and the $\boldsymbol{x}$'s. This section solves the constant coefficient problems that turn into linear algebra. It clarifies these simplest but most important differential equations—whose solution is completely based on $e^{\lambda t}$.

Second Order Equations

The most important equation in mechanics is $my'' + by' + ky = 0$. The first term is the mass m times the acceleration $a = y''$. This term ma balances the force F (*Newton's Law*!). The force includes the damping $-by'$ and the elastic restoring force $-ky$, proportional to distance moved. This is a second-order equation because it contains the second derivative $y'' = d^2y/dt^2$. It is still linear with constant coefficients m, b, k.

In a differential equations course, the method of solution is to substitute $y = e^{\lambda t}$. Each derivative brings down a factor λ. We want $y = e^{\lambda t}$ to solve the equation:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \quad \text{becomes} \quad (m\lambda^2 + b\lambda + k)e^{\lambda t} = 0. \tag{8}$$

Everything depends on $m\lambda^2 + b\lambda + k = 0$. This equation for λ has two roots λ_1 and λ_2. Then the equation for y has two pure solutions $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. Their combinations $c_1y_1 + c_2y_2$ give the complete solution unless $\lambda_1 = \lambda_2$.

In a linear algebra course we expect matrices and eigenvalues. Therefore we turn the scalar equation (with y'') into a vector equation (first derivative only). Suppose $m = 1$. The unknown vector $\boldsymbol{u}$ has components y and y'. The equation is $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

$$\begin{aligned} dy/dt &= y' \\ dy'/dt &= -ky - by' \end{aligned} \qquad \text{converts to} \qquad \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}. \tag{9}$$

The first equation $dy/dt = y'$ is trivial (but true). The second equation connects y'' to y' and y. Together the equations connect $\boldsymbol{u}'$ to $\boldsymbol{u}$. So we solve by eigenvalues of A:

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{bmatrix} \quad \text{has determinant} \quad \lambda^2 + b\lambda + k = 0.$$

The equation for the λ*'s is the same*! It is still $\lambda^2 + b\lambda + k = 0$, since $m = 1$. The roots λ_1 and λ_2 are now *eigenvalues of* A. The eigenvectors and the solution are

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

The first component of $\boldsymbol{u}(t)$ has $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$—the same solution as before. It can't be anything else. In the second component of $\boldsymbol{u}(t)$ you see the velocity dy/dt. The vector problem is completely consistent with the scalar problem.

Example 3 *Motion around a circle with* $y'' + y = 0$ *and* $y = \cos t$

This is our master equation with mass $m = 1$ and stiffness $k = 1$ and no damping dy'. Substitute $y = e^{\lambda t}$ into $y'' + y = 0$ to reach $\lambda^2 + 1 = 0$. *The roots are* $\lambda = i$ *and* $\lambda = -i$. Then half of $e^{it} + e^{-it}$ gives the solution $y = \cos t$.

As a first-order system, the initial values $y(0) = 1$, $y'(0) = 0$ go into $\boldsymbol{u}(0) = (1, 0)$:

$$\text{Use } y'' = -y \qquad \frac{d\boldsymbol{u}}{dt} = \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}. \tag{10}$$

The eigenvalues of A are again $\lambda = i$ and $\lambda = -i$ (no surprise). A is anti-symmetric with eigenvectors $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$. The combination that matches $\boldsymbol{u}(0) = (1, 0)$ is $\frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2)$. Step 2 multiplies $\frac{1}{2}$ by e^{it} and e^{-it}. Step 3 combines the pure oscillations into $\boldsymbol{u}(t)$ to find $y = \cos t$ as expected:

$$\boldsymbol{u}(t) = \frac{1}{2}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}. \qquad \text{This is } \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

All good. The vector $\boldsymbol{u} = (\cos t, -\sin t)$ goes around a circle (Figure 6.3). The radius is 1 because $\cos^2 t + \sin^2 t = 1$.

To display a circle on a screen, replace $y'' = -y$ by a ***finite difference equation***. Here are three choices using $\boldsymbol{Y}(t+\Delta t) - 2\boldsymbol{Y}(t) + \boldsymbol{Y}(t-\Delta t)$. Divide by $(\Delta t)^2$ to approximate y''.

Forward from $n-1$ **Centered at** n **Backward from** $n+1$	$\dfrac{Y_{n+1}-2Y_n+Y_{n-1}}{(\Delta t)^2} = \begin{matrix}-Y_{n-1}\\ -Y_n \\ -Y_{n+1}\end{matrix}$	(11)

Figure 6.3 shows the exact $y(t) = \cos t$ completing a circle at $t = 2\pi$. The three difference methods *don't* complete a perfect circle in 32 steps of length $\Delta t = 2\pi/32$. Those pictures will be explained by eigenvalues:

Forward $|\lambda| > 1$ **(spiral out)** **Centered** $|\lambda| = 1$ **(best)** **Backward** $|\lambda| < 1$ **(spiral in)**

The 2-step equations (11) reduce to 1-step systems. In the continuous case $\boldsymbol{u}$ was (y, y'). Now the discrete unknown is $\boldsymbol{U}_n = (Y_n, Z_n)$ after n time steps Δt from $\boldsymbol{U}_0$:

Forward	$\begin{aligned} Y_{n+1} &= Y_n + \Delta t\, Z_n \\ Z_{n+1} &= Z_n - \Delta t\, Y_n \end{aligned}$	becomes	$\boldsymbol{U}_{n+1} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix} = A\boldsymbol{U}_n.$	(12)

Those are like $Y' = Z$ and $Z' = -Y$. Eliminating Z will bring back equation (11). From the equation for Y_{n+1}, subtract the same equation for Y_n. That produces $Y_{n+1} - Y_n$ on the left side and $Y_n - Y_{n-1}$ on the right side. Also on the right is $\Delta t(Z_n - Z_{n-1})$, which is $-(\Delta t)^2 Y_{n-1}$ from the Z equation. This is the forward choice in equation (11).

My question is simple. Do the points (Y_n, Z_n) stay on the circle $Y^2 + Z^2 = 1$? They could grow to infinity, they could decay to $(0, 0)$. The answer must be found in the eigenvalues of A. $|\lambda|^2$ is $1 + (\Delta t)^2$, the determinant of A. Figure 6.3 shows growth!

We are taking powers A^n ***and not*** e^{At}***, so we test the magnitude*** $|\lambda|$ ***and not the real part of*** λ.

Eigenvalues of A	$\lambda = 1 \pm i\,\Delta t$	$|\lambda| > 1$ and (Y_n, Z_n) spirals out

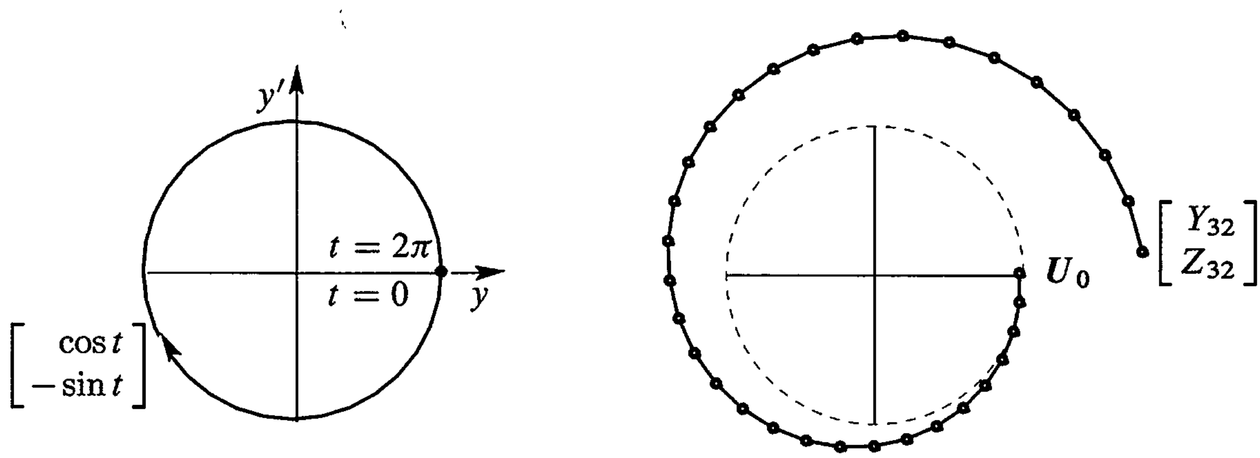


Figure 6.3: Exact $\boldsymbol{u} = (\cos t, -\sin t)$ on a circle. Forward Euler spirals out (32 steps).

The backward choice in (11) will do the opposite in Figure 6.4. Notice the difference:

$$\textbf{Backward}\quad \begin{array}{l} Y_{n+1} = Y_n + \Delta t\, Z_{n+1} \\ Z_{n+1} = Z_n - \Delta t\, Y_{n+1} \end{array} \text{ is } \begin{bmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} Y_n \\ Z_n \end{bmatrix} = U_n. \quad (13)$$

That matrix is A^{T}. It still has $\lambda = 1 \pm i\Delta t$. But now we *invert* it to reach U_{n+1}. When A^{T} has $|\lambda| > 1$, its inverse has $|\lambda| < 1$. That explains why the solution spirals in to $(0, 0)$ for backward differences.

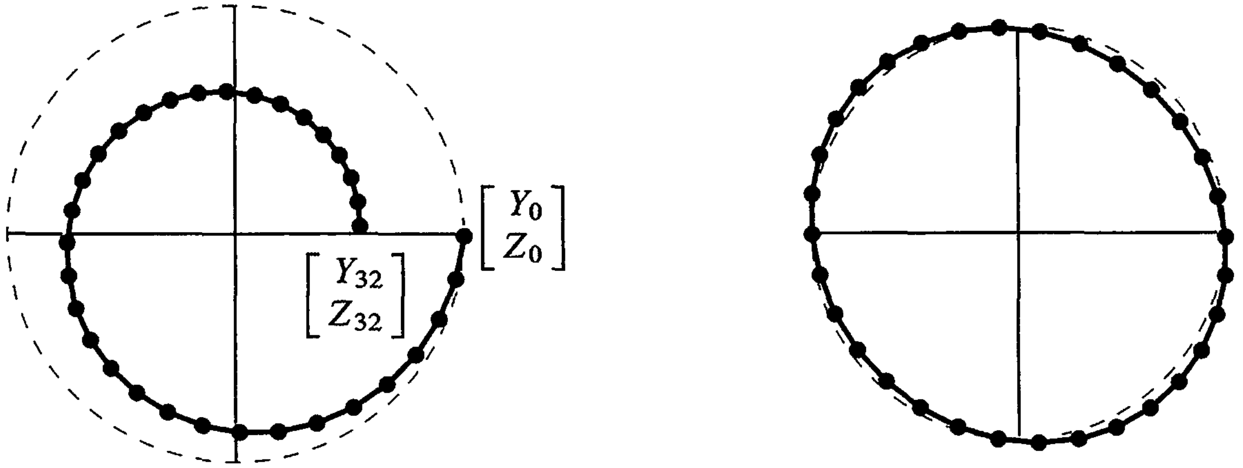


Figure 6.4: Backward differences spiral in. Leapfrog stays near the circle $Y_n^2 + Z_n^2 = 1$.

On the right side of Figure 6.4 you see 32 steps with the *centered* choice. The solution stays close to the circle (Problem 28) if $\Delta t < 2$. This is the **leapfrog method**. The second difference $Y_{n+1} - 2Y_n + Y_{n-1}$ "leaps over" the center value Y_n.

This is the way a chemist follows the motion of molecules (molecular dynamics leads to giant computations). Computational science is lively because one differential equation can be replaced by many difference equations—some unstable, some stable, some neutral. Problem 30 has a fourth (good) method that stays right on the circle.

Note Real engineering and real physics deal with systems (not just a single mass at one point). The unknown y is a vector. The coefficient of y'' is a *mass matrix* M, not a number m. The coefficient of y is a *stiffness matrix* K, not a number k. The coefficient of y' is a damping matrix which might be zero.

The equation $My'' + Ky = f$ is a major part of computational mechanics. It is controlled by the eigenvalues of $M^{-1}K$ in $Kx = \lambda Mx$.

Stability of 2 by 2 Matrices

For the solution of $du/dt = Au$, there is a fundamental question. *Does the solution approach* $u = 0$ *as* $t \to \infty$? Is the problem *stable*, by dissipating energy? The solutions in Examples 1 and 2 included e^t (unstable). Stability depends on the eigenvalues of A.

The complete solution $u(t)$ is built from pure solutions $e^{\lambda t}x$. If the eigenvalue λ is real, we know exactly when $e^{\lambda t}$ will approach zero: *The number* λ *must be negative.*

If the eigenvalue is a complex number $\lambda = r + is$, *the real part r must be negative.* When $e^{\lambda t}$ splits into $e^{rt}e^{ist}$, the factor e^{ist} has absolute value fixed at 1:

$$e^{ist} = \cos st + i \sin st \quad \text{has} \quad |e^{ist}|^2 = \cos^2 st + \sin^2 st = 1.$$

The factor e^{rt} controls growth ($r > 0$ is instability) or decay ($r < 0$ is stability).

The question is: ***Which matrices have negative eigenvalues?*** More accurately, when are the ***real parts*** of the λ's all negative? 2 by 2 matrices allow a clear answer.

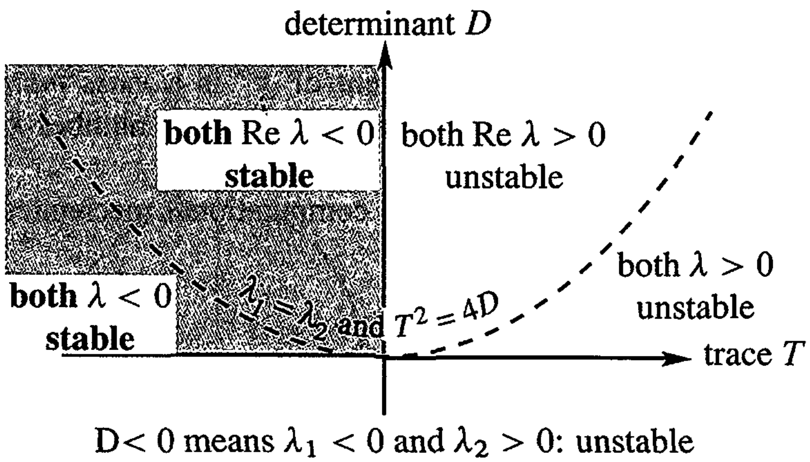
Stability A is ***stable*** and $\boldsymbol{u}(t) \to \boldsymbol{0}$ when all eigenvalues have ***negative real parts.*** The 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ must pass two tests:

$\lambda_1 + \lambda_2 < 0$	The trace $T = a + d$ must be negative.
$\lambda_1 \lambda_2 > 0$	The determinant $D = ad - bc$ must be positive.

Reason If the λ's are real and negative, their sum is negative. This is the trace T. Their product is positive. This is the determinant D. The argument also goes in the reverse direction. If $D = \lambda_1\lambda_2$ is positive, then λ_1 and λ_2 have the same sign. If $T = \lambda_1 + \lambda_2$ is negative, that sign will be negative. We can test T and D.

If the λ's are complex numbers, they must have the form $r + is$ and $r - is$. Otherwise T and D will not be real. The determinant D is automatically positive, since $(r + is)(r - is) = r^2 + s^2$. The trace T is $r + is + r - is = 2r$. So a negative trace means that the real part r is negative and the matrix is stable. Q.E.D.

Figure 6.5 shows the parabola $T^2 = 4D$ which separates real from complex eigenvalues. Solving $\lambda^2 - T\lambda + D = 0$ leads to $\sqrt{T^2 - 4D}$. This is real below the parabola and imaginary above it. The stable region is the *upper left quarter* of the figure—where the trace T is negative and the determinant D is positive.



$$\begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} \ \textit{stable} \qquad \begin{bmatrix} 0 & 4 \\ 5 & -6 \end{bmatrix} \ \textit{unstable} \qquad \begin{bmatrix} 0 & -7 \\ 7 & 0 \end{bmatrix} \ \textit{neutral}$$

Figure 6.5: A 2 by 2 matrix is stable ($\boldsymbol{u}(t) \to \boldsymbol{0}$) when trace $<$ 0 and det $>$ 0.

The Exponential of a Matrix

We want to write the solution $u(t)$ in a new form $e^{At}u(0)$. This gives a perfect parallel with $A^k u_0$ in the previous section. First we have to say what e^{At} means, with a matrix in the exponent. To define e^{At} for matrices, we copy e^x for numbers.

The direct definition of e^x is by the infinite series $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$. When you substitute any square matrix At for x, this series defines the matrix exponential e^{At}:

Matrix exponential e^{At}	$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots$	(14)
Its t derivative is Ae^{At}	$A + A^2t + \frac{1}{2}A^3t^2 + \cdots = Ae^{At}$	
Its eigenvalues are $e^{\lambda t}$	$(I + At + \frac{1}{2}(At)^2 + \cdots)\boldsymbol{x} = (1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \cdots)\boldsymbol{x}$	

The number that divides $(At)^n$ is "n factorial". This is $n! = (1)(2)\cdots(n-1)(n)$. The factorials after $1, 2, 6$ are $4! = 24$ and $5! = 120$. They grow quickly. The series always converges and its derivative is always Ae^{At}. Therefore $e^{At}u(0)$ solves the differential equation with one quick formula—*even if there is a shortage of eigenvectors.*

I will use this series in Example 4, to see it work with a missing eigenvector. It will produce $te^{\lambda t}$. First let me reach $Se^{\Lambda t}S^{-1}$ in the good (diagonalizable) case.

This chapter emphasizes how to find $u(t) = e^{At}u(0)$ by diagonalization. Assume A does have n independent eigenvectors, so it is diagonalizable. Substitute $A = S\Lambda S^{-1}$ into the series for e^{At}. Whenever $S\Lambda S^{-1}S\Lambda S^{-1}$ appears, cancel $S^{-1}S$ in the middle:

Use the series $\qquad e^{At} = I + S\Lambda S^{-1}t + \frac{1}{2}(S\Lambda S^{-1}t)(S\Lambda S^{-1}t) + \cdots$

Factor out S and S^{-1} $\qquad = S\,[I + \Lambda t + \frac{1}{2}(\Lambda t)^2 + \cdots]\,S^{-1}$

Diagonalize e^{At} $\qquad = Se^{\Lambda t}S^{-1}$. $\qquad$ (15)

That equation says: e^{At} *equals* $Se^{\Lambda t}S^{-1}$. Then Λ is a diagonal matrix and so is $e^{\Lambda t}$. The numbers $e^{\lambda_i t}$ are on its diagonal. Multiply $Se^{\Lambda t}S^{-1}u(0)$ to recognize $u(t)$:

$$e^{At}u(0) = Se^{\Lambda t}S^{-1}u(0) = \begin{bmatrix} & & \\ x_1 & \cdots & x_n \\ & & \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \tag{16}$$

This solution $e^{At}u(0)$ is the same answer that came in equation (6) from three steps:

1. Write $u(0) = c_1x_1 + \cdots + c_nx_n$. Here we need n independent eigenvectors.
2. Multiply each x_i by $e^{\lambda_i t}$ to follow it forward in time.
3. The best form of $e^{At}u(0)$ is $\quad u(t) = c_1e^{\lambda_1 t}x_1 + \cdots + c_ne^{\lambda_n t}x_n$. $\qquad$ (17)

Example 4 When you substitute $y = e^{\lambda t}$ into $y'' - 2y' + y = 0$, you get an equation with **repeated roots:** $\lambda^2 - 2\lambda + 1 = 0 = (\lambda - 1)^2$. A differential equations course would propose e^t and te^t as two independent solutions. Here we discover why.

Linear algebra reduces $y'' - 2y' + y = 0$ to a vector equation for $\boldsymbol{u} = (y, y')$:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ 2y' - y \end{bmatrix} \quad \text{is} \quad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\boldsymbol{u}. \tag{18}$$

The eigenvalues of A are again $\lambda = 1, 1$ (with trace $= 2$ and $\det A = 1$). The only eigenvectors are multiples of $\boldsymbol{x} = (1, 1)$. Diagonalization is not possible, A has only one line of eigenvectors. So we compute e^{At} from its definition as a series:

$$\textbf{Short series} \qquad e^{At} = e^{It}\, e^{(A-I)t} = e^t\,[I + (A - I)t]. \tag{19}$$

The "infinite" series ends quickly because $(A - I)^2$ is the zero matrix! You can see te^t appearing in equation (19). The first component of $\boldsymbol{u}(t) = e^{At}\,\boldsymbol{u}(0)$ is our answer $y(t)$:

$$\boldsymbol{u}(t) = e^t\left[I + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t\right]\boldsymbol{u}(0) \qquad y(t) = e^t y(0) - te^t y(0) + te^t y'(0).$$

Example 5 Use the infinite series to find e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Notice that $A^4 = I$:

$$A = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \quad A^3 = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \quad A^4 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}.$$

A^5, A^6, A^7, A^8 will repeat these four matrices. The top right corner has $1, 0, -1, 0$ repeating over and over. The infinite series for e^{At} contains $t/1!$, 0, $-t^3/3!$, 0. Then $t - \frac{1}{6}t^3$ starts that top right corner, and $1 - \frac{1}{2}t^2$ starts the top left:

$$I + At + \tfrac{1}{2}(At)^2 + \tfrac{1}{6}(At)^3 + \cdots = \begin{bmatrix} 1 - \frac{1}{2}t^2 + \cdots & t - \frac{1}{6}t^3 + \cdots \\ -t + \frac{1}{6}t^3 - \cdots & 1 - \frac{1}{2}t^2 + \cdots \end{bmatrix}.$$

On the left side is e^{At}. The top row of that matrix shows the series for $\cos t$ and $\sin t$.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}. \tag{20}$$

A is a skew-symmetric matrix ($A^{\mathrm{T}} = -A$). Its exponential e^{At} is an orthogonal matrix. The eigenvalues of A are i and $-i$. The eigenvalues of e^{At} are e^{it} and e^{-it}. Three rules:

1 *e^{At} always has the inverse e^{-At}.*

2 *The eigenvalues of e^{At} are always $e^{\lambda t}$.*

3 *When A is skew-symmetric, e^{At} is orthogonal. Inverse = transpose = e^{-At}.*

Skew-symmetric matrices have pure imaginary eigenvalues like $\lambda = i\theta$. Then e^{At} has eigenvalues $e^{i\theta t}$. Their absolute value is 1 (neutral stability, pure oscillation, energy is conserved).

Our final example has a triangular matrix A. Then the eigenvector matrix S is triangular. So are S^{-1} and e^{At}. You will see the two forms of the solution: a combination of eigenvectors and the short form $e^{At}\boldsymbol{u}(0)$.

Example 6 Solve $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\boldsymbol{u}$ starting from $\boldsymbol{u}(0) = \begin{bmatrix}2\\ 1\end{bmatrix}$ at $t = 0$.

Solution The eigenvalues 1 and 2 are on the diagonal of A (since A is triangular). The eigenvectors are $(1,0)$ and $(1,1)$. The starting $\boldsymbol{u}(0)$ is $\boldsymbol{x}_1 + \boldsymbol{x}_2$ so $c_1 = c_2 = 1$. Then $\boldsymbol{u}(t)$ is the same combination of pure exponentials (*no* $te^{\lambda t}$ *when* $\lambda = 1, 2$):

$$\textbf{Solution to } \boldsymbol{u}' = A\boldsymbol{u} \qquad \boldsymbol{u}(t) = e^t\begin{bmatrix}1\\ 0\end{bmatrix} + e^{2t}\begin{bmatrix}1\\ 1\end{bmatrix}.$$

That is the clearest form. But the matrix form produces $\boldsymbol{u}(t)$ for every $\boldsymbol{u}(0)$:

$$\boldsymbol{u}(t) = Se^{\Lambda t}S^{-1}\boldsymbol{u}(0) \text{ is } \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}e^t & \\ & e^{2t}\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\boldsymbol{u}(0) = \begin{bmatrix}e^t & e^{2t} - e^t\\ 0 & e^{2t}\end{bmatrix}\boldsymbol{u}(0).$$

That last matrix is e^{At}. It's not bad to see what a matrix exponential looks like (this is a particularly nice one). The situation is the same as for $A\boldsymbol{x} = \boldsymbol{b}$ and inverses. We don't really need A^{-1} to find $\boldsymbol{x}$, and we don't need e^{At} to solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$. But as quick formulas for the answers, $A^{-1}\boldsymbol{b}$ and $e^{At}\boldsymbol{u}(0)$ are unbeatable.

■ REVIEW OF THE KEY IDEAS ■

1. The equation $\boldsymbol{u}' = A\boldsymbol{u}$ is linear with constant coefficients, starting from $\boldsymbol{u}(0)$.
2. Its solution is usually a combination of exponentials, involving each λ and $\boldsymbol{x}$:

$$\textbf{Independent eigenvectors} \qquad \boldsymbol{u}(t) = c_1e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_ne^{\lambda_n t}\boldsymbol{x}_n.$$

3. The constants $c_1, \ldots, c_n$ are determined by $\boldsymbol{u}(0) = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n = S\boldsymbol{c}$.
4. $\boldsymbol{u}(t)$ approaches zero (stability) if every λ has negative real part.
5. The solution is always $\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0)$, with the matrix exponential e^{At}.
6. Equations with y'' reduce to $\boldsymbol{u}' = A\boldsymbol{u}$ by combining y' and y into $\boldsymbol{u} = (y, y')$.

■ WORKED EXAMPLES ■

6.3 A Solve $y'' + 4y' + 3y = 0$ by substituting $e^{\lambda t}$ and also by linear algebra.

Solution Substituting $y = e^{\lambda t}$ yields $(\lambda^2 + 4\lambda + 3)e^{\lambda t} = 0$. That quadratic factors into $\lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0$. Therefore $\boldsymbol{\lambda_1 = -1}$ and $\boldsymbol{\lambda_2 = -3}$. The pure solutions are $y_1 = e^{-t}$ and $y_2 = e^{-3t}$. The complete solution $c_1 y_1 + c_2 y_2$ approaches zero.

To use linear algebra we set $\boldsymbol{u} = (y, y')$. Then the vector equation is $\boldsymbol{u}' = A\boldsymbol{u}$:

$$\begin{aligned} dy/dt &= y' \\ dy'/dt &= -3y - 4y' \end{aligned} \quad \text{converts to} \quad \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \boldsymbol{u}.$$

This A is called a "companion matrix" and its eigenvalues are again 1 and 3 :

$$\textbf{Same quadratic} \qquad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0.$$

The eigenvectors of A are $(1, \lambda_1)$ and $(1, \lambda_2)$. Either way, the decay in $y(t)$ comes from e^{-t} and e^{-3t}. With constant coefficients, calculus goes back to algebra $A\boldsymbol{x} = \lambda \boldsymbol{x}$.

Note In linear algebra the serious danger is a shortage of eigenvectors. Our eigenvectors $(1, \lambda_1)$ and $(1, \lambda_2)$ are the same if $\lambda_1 = \lambda_2$. Then we can't diagonalize A. In this case we don't yet have two independent solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$.

In differential equations the danger is also a repeated λ. After $y = e^{\lambda t}$, a second solution has to be found. It turns out to be $y = te^{\lambda t}$. This "impure" solution (with an extra t) appears in the matrix exponential e^{At}. Example 4 showed how.

6.3 B Find the eigenvalues and eigenvectors of A and write $\boldsymbol{u}(0) = (0, 2\sqrt{2}, 0)$ as a combination of the eigenvectors. Solve both equations $\boldsymbol{u}' = A\boldsymbol{u}$ and $\boldsymbol{u}'' = A\boldsymbol{u}$:

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \boldsymbol{u} \quad \text{and} \quad \frac{d^2\boldsymbol{u}}{dt^2} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \boldsymbol{u} \quad \text{with} \quad \frac{d\boldsymbol{u}}{dt}(0) = \boldsymbol{0}.$$

The $1, -2, 1$ diagonals make A into a *second difference matrix* (like a second derivative).

$\boldsymbol{u}' = A\boldsymbol{u}$ is like the heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$.

Its solution $u(t)$ will decay (negative eigenvalues).

$\boldsymbol{u}'' = A\boldsymbol{u}$ is like the wave equation $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$.

Its solution will oscillate (imaginary eigenvalues).

Solution The eigenvalues and eigenvectors come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)[(-2-\lambda)^2 - 2] = 0.$$

One eigenvalue is $\lambda = -2$, when $-2 - \lambda$ is zero. The other factor is $\lambda^2 + 4\lambda + 2$, so the other eigenvalues (also real and negative) are $\lambda = -2 \pm \sqrt{2}$. Find the eigenvectors:

$$\lambda = -2 \qquad (A + 2I)x = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda = -2 - \sqrt{2} \quad (A - \lambda I)x = \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

$$\lambda = -2 + \sqrt{2} \quad (A - \lambda I)x = \begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

The eigenvectors are *orthogonal* (proved in Section 6.4 for all symmetric matrices). All three λ_i are negative. This A is *negative definite* and e^{At} decays to zero (stability).

The starting $\boldsymbol{u}(0) = (0, 2\sqrt{2}, 0)$ is $\boldsymbol{x}_3 - \boldsymbol{x}_2$. The solution is $\boldsymbol{u}(t) = e^{\lambda_3 t}\boldsymbol{x}_3 - e^{\lambda_2 t}\boldsymbol{x}_2$.

Heat equation In Figure 6.6a, the temperature at the center starts at $2\sqrt{2}$. Heat diffuses into the neighboring boxes and then to the outside boxes (frozen at 0°). The rate of heat flow between boxes is the temperature difference. From box 2, heat flows left and right at the rate $u_1 - u_2$ and $u_3 - u_2$. So the flow out is $u_1 - 2u_2 + u_3$ in the second row of $A\boldsymbol{u}$.

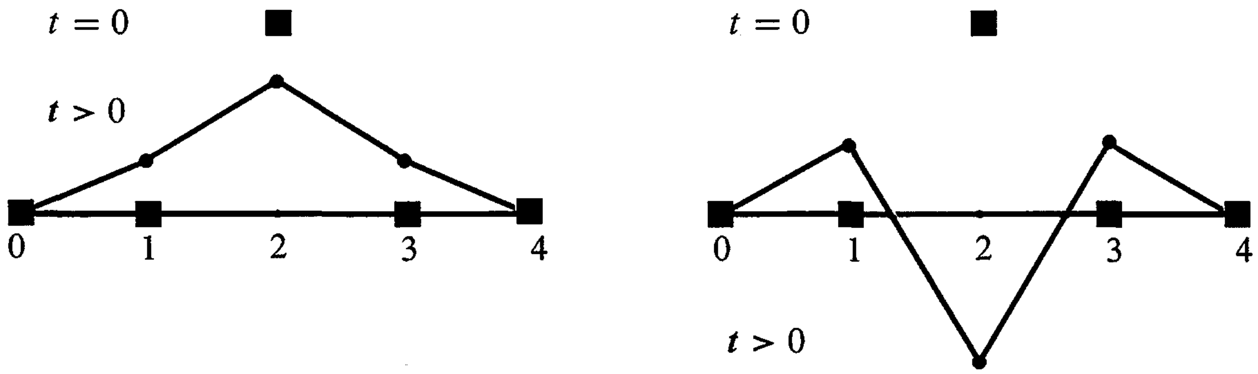


Figure 6.6: Heat diffuses away from box 2 (left). Wave travels from box 2 (right).

Wave equation $d^2\boldsymbol{u}/dt^2 = A\boldsymbol{u}$ has the same eigenvectors $\boldsymbol{x}$. But now the eigenvalues λ lead to **oscillations** $e^{i\omega t}\boldsymbol{x}$ and $e^{-i\omega t}\boldsymbol{x}$. The frequencies come from $\omega^2 = -\lambda$:

$$\frac{d^2}{dt^2}(e^{i\omega t}\boldsymbol{x}) = A(e^{i\omega t}\boldsymbol{x}) \qquad \text{becomes} \qquad (i\omega)^2 e^{i\omega t}\boldsymbol{x} = \lambda e^{i\omega t}\boldsymbol{x} \quad \text{and} \quad \omega^2 = -\lambda.$$

There are two square roots of $-\lambda$, so we have $e^{i\omega t}\boldsymbol{x}$ and $e^{-i\omega t}\boldsymbol{x}$. With three eigenvectors this makes *six* solutions to $\boldsymbol{u}'' = A\boldsymbol{u}$. A combination will match the six components of $\boldsymbol{u}(0)$ and $\boldsymbol{u}'(0)$. Since $\boldsymbol{u}' = \boldsymbol{0}$ in this problem, $e^{i\omega t}\boldsymbol{x}$ combines with $e^{-i\omega t}\boldsymbol{x}$ into $2\cos\omega t\,\boldsymbol{x}$.

6.3 C Solve the four equations $da/dt = 0, db/dt = a, dc/dt = 2b, dz/dt = 3c$ in that order starting from $\boldsymbol{u}(0) = (a(0), b(0), c(0), z(0))$. Solve the same equations by the matrix exponential in $\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0)$.

Four equations
$\boldsymbol{\lambda = 0, 0, 0, 0}$
Eigenvalues on the diagonal

$$\frac{d}{dt}\begin{bmatrix} a \\ b \\ c \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & \mathbf{3} & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ z \end{bmatrix} \quad \text{is} \quad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}.$$

First find A^2, A^3, A^4 and $e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3$. Why does the series stop? Why is it always true that $(e^A)(e^A) = (e^{2A})$? *Always* e^{As} *times* e^{At} *is* $e^{A(s+t)}$.

Solution 1 Integrate $da/dt = 0$, then $db/dt = a$, then $dc/dt = 2b$ and $dz/dt = 3c$:

$$\begin{aligned} a(t) &= a(0) \\ b(t) &= ta(0) + b(0) \\ c(t) &= t^2a(0) + 2tb(0) + c(0) \\ z(t) &= t^3a(0) + 3t^2b(0) + 3tc(0) + z(0) \end{aligned}$$

The 4 by 4 matrix which is multiplying $a(0), b(0), c(0), d(0)$ to produce $a(t), b(t), c(t), d(t)$ must be the same e^{At} as below

Solution 2 *The powers of* A *(strictly triangular) are all zero after* A^3.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{2} & 0 & 0 \\ 0 & 0 & \mathbf{3} & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{2} & 0 & 0 & 0 \\ 0 & \mathbf{6} & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{6} & 0 & 0 & 0 \end{bmatrix} \quad A^4 = \mathbf{0}$$

The diagonals move down at each step. So the series for e^{At} stops after four terms:

Same e^{At}

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} = \begin{bmatrix} 1 & & & \\ t & 1 & & \\ t^2 & 2t & 1 & \\ t^3 & 3t^2 & 3t & 1 \end{bmatrix}$$

The square of e^A is always e^{2A} for many reasons:

1. Solving with e^A from $t = 0$ to 1 and then from 1 to 2 agrees with e^{2A} from 0 to 2.

2. The squared series $(I + A + \frac{A^2}{2} + \cdots)^2$ matches $I + 2A + \frac{(2A)^2}{2} + \cdots = e^{2A}$.

3. If A can be diagonalized (this A can't!) then $(Se^{\Lambda}S^{-1})(Se^{\Lambda}S^{-1}) = Se^{2\Lambda}S^{-1}$.

But notice in Problem 23 that e^Ae^B and e^Be^A and e^{A+B} are all different.

Problem Set 6.3

1 Find two λ's and $\boldsymbol{x}$'s so that $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ solves

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}.$$

What combination $\boldsymbol{u} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{u}(0) = (5, -2)$?

2 Solve Problem 1 for $\boldsymbol{u} = (y, z)$ by back substitution, z before y:

$$\text{Solve } \frac{dz}{dt} = z \text{ from } z(0) = -2. \qquad \text{Then solve } \frac{dy}{dt} = 4y + 3z \text{ from } y(0) = 5.$$

The solution for y will be a combination of e^{4t} and e^{t}. The λ's are 4 and 1.

3 (a) If every column of A adds to zero, why is $\lambda = 0$ an eigenvalue?

(b) With negative diagonal and positive off-diagonal adding to zero, $\boldsymbol{u}' = A\boldsymbol{u}$ will be a "continuous" Markov equation. Find the eigenvalues and eigenvectors, and the *steady state* as $t \to \infty$

$$\text{Solve } \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \boldsymbol{u} \text{ with } \boldsymbol{u}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \text{ What is } \boldsymbol{u}(\infty)?$$

4 A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \qquad \text{and} \qquad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $d\boldsymbol{u}/dt = A\boldsymbol{u}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

5 Reverse the diffusion of people in Problem 4 to $d\boldsymbol{u}/dt = -A\boldsymbol{u}$:

$$\frac{dv}{dt} = v - w \qquad \text{and} \qquad \frac{dw}{dt} = w - v.$$

The total $v + w$ still remains constant. How are the λ's changed now that A is changed to $-A$? But show that $v(t)$ grows to infinity from $v(0) = 30$.

6 A has real eigenvalues but B has complex eigenvalues:

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \quad B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix} \quad (a \text{ and } b \text{ are real})$$

Find the conditions on a and b so that all solutions of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ and $d\boldsymbol{v}/dt = B\boldsymbol{v}$ approach zero as $t \to \infty$.

7 Suppose P is the projection matrix onto the 45° line $y = x$ in $\mathbf{R}^2$. What are its eigenvalues? If $d\boldsymbol{u}/dt = -P\boldsymbol{u}$ (notice minus sign) can you find the limit of $\boldsymbol{u}(t)$ at $t = \infty$ starting from $\boldsymbol{u}(0) = (3, 1)$?

8 The rabbit population shows fast growth (from $6r$) but loss to wolves (from $-2w$). The wolf population always grows in this model ($-w^2$ would control wolves):

$$\frac{dr}{dt} = 6r - 2w \qquad \text{and} \qquad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectors. If $r(0) = w(0) = 30$ what are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

9 (a) Write $(4, 0)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ of these two eigenvectors of A:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = i\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = -i\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

(b) The solution to $d\boldsymbol{u}/dt = A\boldsymbol{u}$ starting from $(4, 0)$ is $c_1e^{it}\boldsymbol{x}_1 + c_2e^{-it}\boldsymbol{x}_2$. Substitute $e^{it} = \cos t + i\sin t$ and $e^{-it} = \cos t - i\sin t$ to find $\boldsymbol{u}(t)$.

Questions 10–13 reduce second-order equations to first-order systems for (y, y').

10 Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $\boldsymbol{u} = (y, y')$:

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}.$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

11 The solution to $y'' = 0$ is a straight line $y = C + Dt$. Convert to a matrix equation:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} \text{ has the solution } \begin{bmatrix} y \\ y' \end{bmatrix} = e^{At}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}.$$

This matrix A has $\lambda = 0, 0$ and it cannot be diagonalized. Find A^2 and compute $e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots$. Multiply your e^{At} times $\big(y(0), y'(0)\big)$ to check the straight line $y(t) = y(0) + y'(0)t$.

12 Substitute $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ to show that $\lambda = 3$ is a repeated root. This is trouble; we need a second solution after e^{3t}. The matrix equation is

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}.$$

Show that this matrix has $\lambda = 3, 3$ and only one line of eigenvectors. *Trouble here too.* Show that the second solution to $y'' = 6y' - 9y$ is $y = te^{3t}$.

13 (a) Write down two familiar functions that solve the equation $d^2y/dt^2 = -9y$. Which one starts with $y(0) = 3$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -9y$ produces a vector equation $\boldsymbol{u}' = A\boldsymbol{u}$:

$$\boldsymbol{u} = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}.$$

Find $\boldsymbol{u}(t)$ by using the eigenvalues and eigenvectors of A: $\boldsymbol{u}(0) = (3, 0)$.

14 The matrix in this question is skew-symmetric ($A^{\mathrm{T}} = -A$):

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \boldsymbol{u} \qquad \text{or} \qquad \begin{aligned} u_1' &= cu_2 - bu_3 \\ u_2' &= au_3 - cu_1 \\ u_3' &= bu_1 - au_2. \end{aligned}$$

(a) The derivative of $\|\boldsymbol{u}(t)\|^2 = u_1^2 + u_2^2 + u_3^2$ is $2u_1u_1' + 2u_2u_2' + 2u_3u_3'$. Substitute u_1', u_2', u_3' to get *zero*. Then $\|\boldsymbol{u}(t)\|^2$ stays equal to $\|\boldsymbol{u}(0)\|^2$.

(b) *When A is skew-symmetric, $Q = e^{At}$ is orthogonal.* Prove $Q^{\mathrm{T}} = e^{-At}$ from the series for $Q = e^{At}$. Then $Q^{\mathrm{T}}Q = I$.

15 A particular solution to $d\boldsymbol{u}/dt = A\boldsymbol{u} - \boldsymbol{b}$ is $\boldsymbol{u}_p = A^{-1}\boldsymbol{b}$, if A is invertible. The usual solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$ give $\boldsymbol{u}_n$. Find the complete solution $\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_n$:

(a) $\dfrac{du}{dt} = u - 4$ (b) $\dfrac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \boldsymbol{u} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

16 If c is not an eigenvalue of A, substitute $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ and find a particular solution to $d\boldsymbol{u}/dt = A\boldsymbol{u} - e^{ct}\boldsymbol{b}$. How does it break down when c is an eigenvalue of A? The "nullspace" of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ contains the usual solutions $e^{\lambda_i t}\boldsymbol{x}_i$.

17 Find a matrix A to illustrate each of the unstable regions in Figure 6.5:

(a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$.

Questions 18–27 are about the matrix exponential e^{At}.

18 Write five terms of the infinite series for e^{At}. Take the t derivative of each term. Show that you have four terms of Ae^{At}. Conclusion: $e^{At}\boldsymbol{u}_0$ solves $\boldsymbol{u}' = A\boldsymbol{u}$.

19 The matrix $B = \left[\begin{smallmatrix} 0 & -4 \\ 0 & 0 \end{smallmatrix}\right]$ has $B^2 = 0$. Find e^{Bt} from a (short) infinite series. Check that the derivative of e^{Bt} is Be^{Bt}.

20 Starting from $\boldsymbol{u}(0)$ the solution at time T is $e^{AT}\boldsymbol{u}(0)$. Go an additional time t to reach $e^{At}\,e^{AT}\boldsymbol{u}(0)$. This solution at time $t + T$ can also be written as _____. Conclusion: e^{At} times e^{AT} equals _____.

21 Write $A = \left[\begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix}\right]$ in the form $S\Lambda S^{-1}$. Find e^{At} from $Se^{\Lambda t}S^{-1}$.

22 If $A^2 = A$ show that the infinite series produces $e^{At} = I + (e^t - 1)A$. For $A = \left[\begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix}\right]$ in Problem 21 this gives $e^{At} =$ _____.

23 Generally $e^A e^B$ is different from $e^B e^A$. They are both different from e^{A+B}. Check this using Problems 21–22 and 19. (If $AB = BA$, all three are the same.)

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} \qquad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

24 Write $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ as $S\Lambda S^{-1}$. Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At}. Check e^{At} and the derivative of e^{At} when $t = 0$.

25 Put $A = \left[\begin{smallmatrix} 1 & 3 \\ 0 & 0 \end{smallmatrix}\right]$ into the infinite series to find e^{At}. First compute A^2 and A^s:

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 3t \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} & \\ & \end{bmatrix} + \cdots = \begin{bmatrix} e^t & \\ 0 & \end{bmatrix}.$$

26 Give two reasons why the matrix exponential e^{At} is never singular:

(a) Write down its inverse.

(b) Write down its eigenvalues. If $Ax = \lambda x$ then $e^{At}x =$ _____ x.

27 Find a solution $x(t), y(t)$ that gets large as $t \to \infty$. To avoid this instability a scientist exchanged the two equations:

$$\begin{aligned} dx/dt &= 0x - 4y \\ dy/dt &= -2x + 2y \end{aligned} \qquad \text{becomes} \qquad \begin{aligned} dy/dt &= -2x + 2y \\ dx/dt &= 0x - 4y. \end{aligned}$$

Now the matrix $\left[\begin{smallmatrix} -2 & 2 \\ 0 & -4 \end{smallmatrix}\right]$ is stable. It has negative eigenvalues. How can this be?

Challenge Problems

28 Centering $y'' = -y$ in Example 3 will produce $Y_{n+1} - 2Y_n + Y_{n-1} = -(\Delta t)^2 Y_n$. This can be written as a one-step difference equation for $U = (Y, Z)$:

$$\begin{aligned} Y_{n+1} &= Y_n + \Delta t\ Z_n \\ Z_{n+1} &= Z_n - \Delta t\ Y_{n+1} \end{aligned} \qquad \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}$$

Invert the matrix on the left side to write this as $U_{n+1} = AU_n$. Show that $\det A = 1$. Choose the large time step $\Delta t = 1$ and find the eigenvalues λ_1 and $\lambda_2 = \overline{\lambda}_1$ of A:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \textbf{ has } |\lambda_1| = |\lambda_2| = 1. \textbf{ Show that } A^6 \textbf{ is exactly } I.$$

After 6 steps to $t = 6$, U_6 equals U_0. The exact $y = \cos t$ returns to 1 at $t = 2\pi$.

29 That centered choice (*leapfrog method*) in Problem 28 is very successful for small time steps Δt. But find the eigenvalues of A for $\Delta t = \sqrt{2}$ and 2:

$$A = \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}.$$

Both matrices have $|\lambda| = 1$. Compute A^4 in both cases and find the eigenvectors of A. That value $\Delta t = 2$ is at the border of instability. Time steps $\Delta t > 2$ will lead to $|\lambda| > 1$, and the powers in $\boldsymbol{U}_n = A^n \boldsymbol{U}_0$ will explode.

Note You might say that nobody would compute with $\Delta t > 2$. But if an atom vibrates with $y'' = -1000000y$, then $\Delta t > .0002$ will give instability. Leapfrog has a very strict stability limit. $Y_{n+1} = Y_n + 3Z_n$ and $Z_{n+1} = Z_n - 3Y_{n+1}$ will explode because $\Delta t = 3$ is too large.

30 Another good idea for $y'' = -y$ is the trapezoidal method (half forward/half back): ***This may be the best way to keep (Y_n, Z_n) exactly on a circle.***

$$\textbf{Trapezoidal} \quad \begin{bmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}.$$

(a) Invert the left matrix to write this equation as $\boldsymbol{U}_{n+1} = A\boldsymbol{U}_n$. *Show that A is an orthogonal matrix*: $A^{\mathrm{T}}A = I$. **These points $\boldsymbol{U}_n$ never leave the circle.** $A = (I - B)^{-1}(I + B)$ is always an orthogonal matrix if $B^{\mathrm{T}} = -B$.

(b) (Optional MATLAB) Take 32 steps from $\boldsymbol{U}_0 = (1, 0)$ to $\boldsymbol{U}_{32}$ with $\Delta t = 2\pi/32$. Is $\boldsymbol{U}_{32} = \boldsymbol{U}_0$? I think there is a small error.

31 The ***cosine of a matrix*** is defined like e^A, by copying the series for $\cos t$:

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \cdots \qquad \cos A = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \cdots$$

(a) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, multiply each term times $\boldsymbol{x}$ to find the eigenvalue of $\cos A$.

(b) Find the eigenvalues of $A = \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix}$ with eigenvectors $(1, 1)$ and $(1, -1)$. From the eigenvalues and eigenvectors of $\cos A$, find that matrix $C = \cos A$.

(c) The second derivative of $\cos(At)$ is $-A^2\cos(At)$.

$$\boldsymbol{u}(t) = \cos(A t)\, \boldsymbol{u}(0) \text{ solves } \frac{d^2\boldsymbol{u}}{dt^2} = -A^2\boldsymbol{u} \text{ starting from } \boldsymbol{u}'(0) = 0.$$

Construct $\boldsymbol{u}(t) = \cos(At)\,\boldsymbol{u}(0)$ by the usual three steps for that specific A:

1. Expand $\boldsymbol{u}(0) = (4, 2) = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ in the eigenvectors.
2. Multiply those eigenvectors by _____ and _____ (instead of $e^{\lambda t}$).
3. Add up the solution $\boldsymbol{u}(t) = c_1$ _____ $\boldsymbol{x}_1 + c_2$ _____ $\boldsymbol{x}_2$.

6.4 Symmetric Matrices

For projection onto a plane in $\mathbf{R}^3$, the plane is full of eigenvectors (where $Px = x$). The other eigenvectors are *perpendicular* to the plane (where $Px = \mathbf{0}$). The eigenvalues $\lambda = 1, 1, 0$ are real. Three eigenvectors can be chosen perpendicular to each other. I have to write "can be chosen" because the two in the plane are not automatically perpendicular. This section makes that best possible choice for ***symmetric matrices***: ***The eigenvectors of*** $P = P^{\mathrm{T}}$ ***are perpendicular unit vectors.***

Now we open up to all symmetric matrices. It is no exaggeration to say that these are the most important matrices the world will ever see—in the theory of linear algebra and also in the applications. We come immediately to the key question about symmetry. Not only the question, but also the answer.

What is special about $Ax = \lambda x$ ***when A is symmetric***? We are looking for special properties of the eigenvalues λ and the eigenvectors x when $A = A^{\mathrm{T}}$.

The diagonalization $A = S\Lambda S^{-1}$ will reflect the symmetry of A. We get some hint by transposing to $A^{\mathrm{T}} = (S^{-1})^{\mathrm{T}}\Lambda S^{\mathrm{T}}$. Those are the same since $A = A^{\mathrm{T}}$. Possibly S^{-1} in the first form equals S^{T} in the second form. Then $S^{\mathrm{T}}S = I$. That makes each eigenvector in S orthogonal to the other eigenvectors. The key facts get first place in the Table at the end of this chapter, and here they are:

1. A symmetric matrix has only ***real eigenvalues***.
2. The ***eigenvectors*** can be chosen ***orthonormal.***

Those n orthonormal eigenvectors go into the columns of S. Every symmetric matrix can be diagonalized. ***Its eigenvector matrix S becomes an orthogonal matrix Q***. Orthogonal matrices have $Q^{-1} = Q^{\mathrm{T}}$—what we suspected about S is true. To remember it we write $S = Q$, when we choose orthonormal eigenvectors.

Why do we use the word "choose"? Because the eigenvectors do not *have* to be unit vectors. Their lengths are at our disposal. We will choose unit vectors—eigenvectors of length one, which are orthonormal and not just orthogonal. Then $S\Lambda S^{-1}$ is in its special and particular form $Q\Lambda Q^{\mathrm{T}}$ for symmetric matrices:

(Spectral Theorem) Every symmetric matrix has the factorization $A = Q\Lambda Q^{\mathrm{T}}$ with real eigenvalues in Λ and orthonormal eigenvectors in $S = Q$:

Symmetric diagonalization $\quad A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}} \quad \text{with} \quad Q^{-1} = Q^{\mathrm{T}}.$

It is easy to see that $Q\Lambda Q^{\mathrm{T}}$ is symmetric. Take its transpose. You get $(Q^{\mathrm{T}})^{\mathrm{T}}\Lambda^{\mathrm{T}}Q^{\mathrm{T}}$, which is $Q\Lambda Q^{\mathrm{T}}$ again. The harder part is to prove that every symmetric matrix has real λ's and orthonormal x's. This is the *"spectral theorem"* in mathematics and the *"principal axis* *theorem*" in geometry and physics. We have to prove it! No choice. I will approach the proof in three steps:

1. By an example, showing real λ's in Λ and orthonormal x's in Q.

2. By a proof of those facts when no eigenvalues are repeated.

3. By a proof that allows repeated eigenvalues (at the end of this section).

Example 1 Find the λ's and x's when $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$.

Solution The determinant of $A - \lambda I$ is $\lambda^2 - 5\lambda$. The eigenvalues are 0 and 5 (*both real*). We can see them directly: $\lambda = 0$ is an eigenvalue because A is singular, and $\lambda = 5$ matches the *trace* down the diagonal of A: $0 + 5$ agrees with $1 + 4$.

Two eigenvectors are $(2, -1)$ and $(1, 2)$—orthogonal but not yet orthonormal. The eigenvector for $\lambda = 0$ is in the *nullspace* of A. The eigenvector for $\lambda = 5$ is in the *column space*. We ask ourselves, why are the nullspace and column space perpendicular? The Fundamental Theorem says that the nullspace is perpendicular to the ***row space***—not the column space. But our matrix is *symmetric*! Its row and column spaces are the same. Its eigenvectors $(2, -1)$ and $(1, 2)$ must be (and are) perpendicular.

These eigenvectors have length $\sqrt{5}$. Divide them by $\sqrt{5}$ to get unit vectors. Put those into the columns of S (which is Q). Then $Q^{-1}AQ$ is Λ and $Q^{-1} = Q^{\mathrm{T}}$:

$$Q^{-1}AQ = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \Lambda.$$

Now comes the n by n case. The λ's are real when $A = A^{\mathrm{T}}$ and $Ax = \lambda x$.

Real Eigenvalues All the eigenvalues of a real symmetric matrix are real.

Proof Suppose that $Ax = \lambda x$. Until we know otherwise, λ might be a complex number $a + ib$ (a and b real). *Its complex conjugate is* $\overline{\lambda} = a - ib$. Similarly the components of x may be complex numbers, and switching the signs of their imaginary parts gives $\overline{x}$. The good thing is that $\overline{\lambda}$ times $\overline{x}$ is always the conjugate of λ times x. So we can take conjugates of $Ax = \lambda x$, remembering that A is real:

$$Ax = \lambda x \quad \text{leads to} \quad A\overline{x} = \overline{\lambda}\,\overline{x}. \qquad \text{Transpose to} \quad \overline{x}^{\mathrm{T}}A = \overline{x}^{\mathrm{T}}\,\overline{\lambda}. \tag{1}$$

Now take the dot product of the first equation with $\overline{x}$ and the last equation with x:

$$\overline{x}^{\mathrm{T}}Ax = \overline{x}^{\mathrm{T}}\lambda x \qquad \text{and also} \qquad \overline{x}^{\mathrm{T}}Ax = \overline{x}^{\mathrm{T}}\,\overline{\lambda}\,x. \tag{2}$$

The left sides are the same so the right sides are equal. One equation has λ, the other has $\overline{\lambda}$. They multiply $\overline{x}^{\mathrm{T}}x = |x_1|^2 + |x_2|^2 + \cdots =$ length squared which is not zero. *Therefore* λ *must equal* $\overline{\lambda}$, and $a + ib$ equals $a - ib$. The imaginary part is $b = 0$. Q.E.D.

The eigenvectors come from solving the real equation $(A - \lambda I)x = \mathbf{0}$. So the x's are also real. The important fact is that they are perpendicular.

Orthogonal Eigenvectors Eigenvectors of a real symmetric matrix (when they correspond to different λ's) are always perpendicular.

Proof Suppose $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$. We are assuming here that $\lambda_1 \neq \lambda_2$. Take dot products of the first equation with y and the second with x:

$$\textbf{Use } A^{\mathrm{T}} = A \qquad (\lambda_1 x)^{\mathrm{T}} y = (Ax)^{\mathrm{T}} y = x^{\mathrm{T}} A^{\mathrm{T}} y = x^{\mathrm{T}} A y = x^{\mathrm{T}} \lambda_2 y. \tag{3}$$

The left side is $x^{\mathrm{T}}\lambda_1 y$, the right side is $x^{\mathrm{T}}\lambda_2 y$. Since $\lambda_1 \neq \lambda_2$, this proves that $x^{\mathrm{T}} y = 0$. The eigenvector x (for λ_1) is perpendicular to the eigenvector y (for λ_2).

Example 2 The eigenvectors of a 2 by 2 symmetric matrix have a special form:

$$\textbf{Not widely known} \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \text{ has } x_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix}. \tag{4}$$

This is in the Problem Set. The point here is that x_1 is perpendicular to x_2:

$$x_1^{\mathrm{T}} x_2 = b(\lambda_2 - c) + (\lambda_1 - a)b = b(\lambda_1 + \lambda_2 - a - c) = 0.$$

This is zero because $\lambda_1 + \lambda_2$ equals the trace $a + c$. Thus $x_1^{\mathrm{T}} x_2 = 0$. Eagle eyes might notice the special case $a = c$, $b = 0$ when $x_1 = x_2 = \mathbf{0}$. This case has repeated eigenvalues, as in $A = I$. It still has perpendicular eigenvectors $(1, 0)$ and $(0, 1)$.

This example shows the main goal of this section—***to diagonalize symmetric matrices A by orthogonal eigenvector matrices $S = Q$***. Look again at the result:

$$\textbf{Symmetry} \qquad A = S\Lambda S^{-1} \quad \text{becomes} \quad A = Q\Lambda Q^{\mathrm{T}} \quad \text{with} \quad Q^{\mathrm{T}} Q = I.$$

This says that every 2 by 2 symmetric matrix looks like

$$A = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1^{\mathrm{T}} \\ x_2^{\mathrm{T}} \end{bmatrix}. \tag{5}$$

The columns x_1 and x_2 multiply the rows $\lambda_1 x_1^{\mathrm{T}}$ and $\lambda_2 x_2^{\mathrm{T}}$ to produce A:

$$\textbf{Sum of rank-one matrices} \qquad A = \lambda_1 x_1 x_1^{\mathrm{T}} + \lambda_2 x_2 x_2^{\mathrm{T}}. \tag{6}$$

This is the great factorization $Q\Lambda Q^{\mathrm{T}}$, written in terms of λ's and x's. When the symmetric matrix is n by n, there are n columns in Q multiplying n rows in Q^{T}. The n products $x_i x_i^{\mathrm{T}}$ are ***projection matrices***. Including the λ's, the spectral theorem $A = Q\Lambda Q^{\mathrm{T}}$ for symmetric matrices says that A is a combination of projection matrices:

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n \qquad \lambda_i = \text{eigenvalue}, \quad P_i = \text{projection onto eigenspace.}$$

Complex Eigenvalues of Real Matrices

Equation (1) went from $Ax = \lambda x$ to $A\overline{x} = \overline{\lambda}\,\overline{x}$. In the end, λ and x were real. Those two equations were the same. But a *non*symmetric matrix can easily produce λ and x that are complex. In this case, $A\overline{x} = \overline{\lambda}\,\overline{x}$ is different from $Ax = \lambda x$. It gives us a new eigenvalue (which is $\overline{\lambda}$) and a new eigenvector (which is $\overline{x}$):

For real matrices, complex λ's and x's come in "conjugate pairs."

$$\textit{If} \quad Ax = \lambda x \quad \textit{then} \quad A\overline{x} = \overline{\lambda}\,\overline{x}.$$

Example 3 $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has $\lambda_1 = \cos\theta + i\sin\theta$ and $\lambda_2 = \cos\theta - i\sin\theta$.

Those eigenvalues are conjugate to each other. They are λ and $\overline{\lambda}$. The eigenvectors must be x and $\overline{x}$, because A is real:

$$\begin{aligned} &\textbf{This is } \lambda x \qquad Ax = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos\theta + i\sin\theta)\begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &\textbf{This is } \overline{\lambda}\,\overline{x} \qquad A\overline{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos\theta - i\sin\theta)\begin{bmatrix} 1 \\ i \end{bmatrix}. \end{aligned} \tag{7}$$

Those eigenvectors $(1, -i)$ and $(1, i)$ are complex conjugates because A is real.

For this rotation matrix the absolute value is $|\lambda| = 1$, because $\cos^2\theta + \sin^2\theta = 1$. ***This fact $|\lambda| = 1$ holds for the eigenvalues of every orthogonal matrix.***

We apologize that a touch of complex numbers slipped in. They are unavoidable even when the matrix is real. Chapter 10 goes beyond complex numbers λ and complex vectors to complex matrices A. Then you have the whole picture.

We end with two optional discussions.

Eigenvalues versus Pivots

The eigenvalues of A are very different from the pivots. For eigenvalues, we solve $\det(A - \lambda I) = 0$. For pivots, we use elimination. The only connection so far is this:

product of pivots = determinant = product of eigenvalues.

We are assuming a full set of pivots $d_1, \ldots, d_n$. There are n real eigenvalues $\lambda_1, \ldots, \lambda_n$. The d's and λ's are not the same, but they come from the same matrix. This paragraph is about a hidden relation. ***For symmetric matrices the pivots and the eigenvalues have the same signs***:

The number of positive eigenvalues of $A = A^{\mathrm{T}}$ equals the number of positive pivots.
Special case: A has all $\lambda_i > 0$ if and only if all pivots are positive.

That special case is an all-important fact for **positive definite matrices** in Section 6.5.

Example 4 This symmetric matrix A has one positive eigenvalue and one positive pivot:

$$\textbf{Matching signs} \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad \begin{matrix}\text{has pivots 1 and } -8 \\ \text{eigenvalues 4 and } -2.\end{matrix}$$

The signs of the pivots match the signs of the eigenvalues, one plus and one minus. This could be false when the matrix is not symmetric:

$$\textbf{Opposite signs} \quad B = \begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix} \quad \begin{matrix}\text{has pivots 1 and 2} \\ \text{eigenvalues } -1 \text{ and } -2.\end{matrix}$$

The diagonal entries are a third set of numbers and we say nothing about them.

Here is a proof that the pivots and eigenvalues have matching signs, when $A = A^{\mathrm{T}}$.

You see it best when the pivots are divided out of the rows of U. Then A is LDL^{T}. The diagonal pivot matrix D goes between triangular matrices L and L^{T}:

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -8 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \textbf{This is } A = LDL^{\mathrm{T}}\textbf{. It is symmetric.}$$

Watch the eigenvalues when L and L^{T} move toward the identity matrix: $A \to D$.

The eigenvalues of LDL^{T} are 4 and -2. The eigenvalues of IDI^{T} are 1 and -8 (the pivots!). The eigenvalues are changing, as the "3" in L moves to zero. But to change *sign*, a real eigenvalue would have to cross zero. The matrix would at that moment be singular. Our changing matrix always has pivots 1 and -8, so it is *never* singular. The signs cannot change, as the λ's move to the d's.

We repeat the proof for any $A = LDL^{\mathrm{T}}$. Move L toward I, by moving the off-diagonal entries to zero. The pivots are not changing and not zero. The eigenvalues λ of LDL^{T} change to the eigenvalues d of IDI^{T}. Since these eigenvalues cannot cross zero as they move into the pivots, their signs cannot change. Q.E.D.

This connects the two halves of applied linear algebra—pivots and eigenvalues.

All Symmetric Matrices are Diagonalizable

When no eigenvalues of A are repeated, the eigenvectors are sure to be independent. Then A can be diagonalized. But a repeated eigenvalue can produce a shortage of eigenvectors. This *sometimes* happens for nonsymmetric matrices. It *never* happens for symmetric matrices. ***There are always enough eigenvectors to diagonalize $A = A^{\mathrm{T}}$.***

Here is one idea for a proof. Change A slightly by a diagonal matrix $\mathbf{diag}(c, 2c, \ldots, nc)$. If c is very small, the new symmetric matrix will have no repeated eigenvalues. Then we know it has a full set of orthonormal eigenvectors. As $c \to 0$ we obtain n orthonormal eigenvectors of the original A—even if some eigenvalues of that A are repeated.

Every mathematician knows that this argument is incomplete. How do we guarantee that the small diagonal matrix will separate the eigenvalues? (I am sure this is true.)

A different proof comes from a useful new factorization that applies to *all matrices*, symmetric or not. This new factorization immediately produces $A = Q\Lambda Q^{\mathrm{T}}$ with a full set of real orthonormal eigenvectors when A is any symmetric matrix.

Every square matrix factors into $A = QTQ^{-1}$ where T is upper triangular and $\overline{Q}^{\mathrm{T}} = Q^{-1}$. If A has real eigenvalues then Q and T can be chosen real: $Q^{\mathrm{T}}Q = I$.

This is Schur's Theorem. We are looking for $AQ = QT$. The first column $\boldsymbol{q}_1$ of Q must be a unit eigenvector of A. Then the first columns of AQ and QT are $A\boldsymbol{q}_1$ and $t_{11}\boldsymbol{q}_1$. But the other columns of Q need not be eigenvectors when T is only triangular (not diagonal). So use any $n-1$ columns that complete $\boldsymbol{q}_1$ to a matrix Q_1 with orthonormal columns. At this point only the first columns of Q and T are set, where $A\boldsymbol{q}_1 = t_{11}\boldsymbol{q}_1$:

$$\overline{Q}_1^{\mathrm{T}} A Q_1 = \begin{bmatrix} \overline{\boldsymbol{q}}_1^{\mathrm{T}} \\ \vdots \\ \overline{\boldsymbol{q}}_n^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} A\boldsymbol{q}_1 & \cdots & A\boldsymbol{q}_n \end{bmatrix} = \begin{bmatrix} t_{11} & \cdots \\ 0 & \boxed{A_2} \\ \vdots & \\ 0 & \end{bmatrix}. \tag{8}$$

Now I will argue by "induction". *Assume Schur's factorization $A_2 = Q_2 T_2 Q_2^{-1}$ is possible for that matrix A_2 of size $n-1$.* Put the orthogonal (or unitary) matrix Q_2 and the triangular T_2 into the final Q and T:

$$Q = Q_1 \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} t_{11} & \cdots \\ 0 & T_2 \end{bmatrix} \quad \text{and} \quad AQ = QT \quad \text{as desired.}$$

Note I had to allow $\boldsymbol{q}_1$ and Q_1 to be complex, in case A has complex eigenvalues. But if t_{11} is a real eigenvalue, then $\boldsymbol{q}_1$ and Q_1 can stay real. The induction step keeps everything real when A has real eigenvalues. Induction starts with 1 by 1, no problem.

Proof that T is the diagonal Λ when A is symmetric. Then we have $A = Q\Lambda Q^{\mathrm{T}}$.

Every symmetric A has real eigenvalues. Schur's $A = QTQ^{\mathrm{T}}$ with $Q^{\mathrm{T}}Q = I$ means that $T = Q^{\mathrm{T}}AQ$. This is a symmetric matrix (its transpose is $Q^{\mathrm{T}}AQ$). Now the key point: *If T is triangular and also symmetric, it must be diagonal*: $T = \Lambda$.

This proves $A = Q\Lambda Q^{\mathrm{T}}$. The matrix $A = A^{\mathrm{T}}$ has n orthonormal eigenvectors.

■ REVIEW OF THE KEY IDEAS ■

1. A symmetric matrix has *real eigenvalues* and *perpendicular eigenvectors*.
2. Diagonalization becomes $A = Q\Lambda Q^{\mathrm{T}}$ with an orthogonal matrix Q.
3. All symmetric matrices are diagonalizable, even with repeated eigenvalues.
4. The signs of the eigenvalues match the signs of the pivots, when $A = A^{\mathrm{T}}$.
5. Every square matrix can be "triangularized" by $A = QTQ^{-1}$.

■ WORKED EXAMPLES ■

6.4 A What matrix A has eigenvalues $\lambda = 1, -1$ and eigenvectors $\boldsymbol{x}_1 = (\cos\theta, \sin\theta)$ and $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$? Which of these properties can be predicted in advance?

$$A = A^{\mathrm{T}} \qquad A^2 = I \qquad \det A = -1 \qquad + \text{ and } - \text{ pivot} \qquad A^{-1} = A$$

Solution All those properties can be predicted! With real eigenvalues in Λ and orthonormal eigenvectors in Q, the matrix $A = Q\Lambda Q^{\mathrm{T}}$ must be symmetric. The eigenvalues 1 and -1 tell us that $A^2 = I$ (since $\lambda^2 = 1$) and $A^{-1} = A$ (same thing) and $\det A = -1$. The two pivots are positive and negative like the eigenvalues, since A is symmetric.

The matrix must be a reflection. Vectors in the direction of $\boldsymbol{x}_1$ are unchanged by A (since $\lambda = 1$). Vectors in the perpendicular direction are reversed (since $\lambda = -1$). The reflection $A = Q\Lambda Q^{\mathrm{T}}$ is across the "θ-line". Write c for $\cos\theta$, s for $\sin\theta$:

$$A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 - s^2 & 2cs \\ 2cs & s^2 - c^2 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Notice that $\boldsymbol{x} = (1, 0)$ goes to $A\boldsymbol{x} = (\cos 2\theta, \sin 2\theta)$ on the 2θ-line. And $(\cos 2\theta, \sin 2\theta)$ goes back across the θ-line to $\boldsymbol{x} = (1, 0)$.

6.4 B Find the eigenvalues of A_3 and B_4, and check the orthogonality of their first two eigenvectors. Graph these eigenvectors to see discrete sines and cosines:

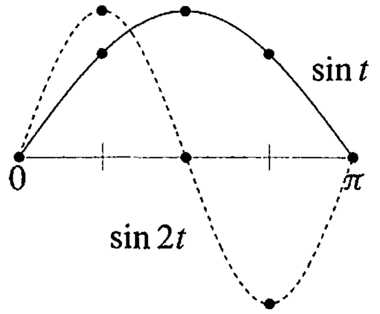
$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad B_4 = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$$

The $-1, 2, -1$ pattern in both matrices is a "second difference". Section 8.1 will explain how this is like a second derivative. Then $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $B\boldsymbol{x} = \lambda\boldsymbol{x}$ are like $d^2x/dt^2 = \lambda x$. This has eigenvectors $x = \sin kt$ and $x = \cos kt$ that are the bases for Fourier series. The matrices lead to "discrete sines" and "discrete cosines" that are the bases for the *Discrete Fourier Transform.* This DFT is absolutely central to all areas of digital signal processing. The favorite choice for JPEG in image processing has been B_8 of size 8.

Solution The eigenvalues of A_3 are $\lambda = 2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$. (see **6.3 B**). Their sum is 6 (the trace of A_3) and their product is 4 (the determinant). The eigenvector matrix S gives the "Discrete Sine Transform" and the graph shows how the first two eigenvectors fall onto sine curves. Please draw the third eigenvector onto a third sine curve!

$$S = \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$

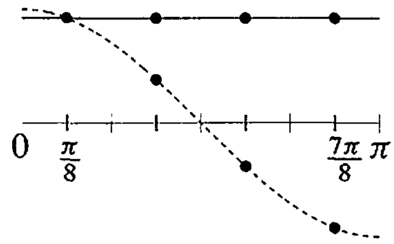
Eigenvector matrix for A_3



The eigenvalues of B_4 are $\lambda = 2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$ and 0 (the same as for A_3, plus the zero eigenvalue). The trace is still 6, but the determinant is now zero. The eigenvector matrix C gives the 4-point "Discrete Cosine Transform" and the graph shows how the first two eigenvectors fall onto cosine curves. (Please plot the third eigenvector.) These eigenvectors match cosines at the *halfway points* $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \sqrt{2}-1 & -1 & 1-\sqrt{2} \\ 1 & 1-\sqrt{2} & -1 & \sqrt{2}-1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Eigenvector matrix for B_4



S and C have orthogonal columns (eigenvectors of the symmetric A_3 and B_4). When we multiply a vector by S or C, that signal splits into pure frequencies—as a musical chord separates into pure notes. This is the most useful and insightful transform in all of signal processing. Here is a MATLAB code to create B_8 and its eigenvector matrix C:

```
n=8; e=ones(n−1,1); B=2∗eye(n)−diag(e,−1)−diag(e,1); B(1,1)=1; B(n,n)=1;
[C, Λ] = eig(B);
plot(C( : ,1:4), '−o')
```

Problem Set 6.4

1 Write A as $M + N$, symmetric matrix plus skew-symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = M + N \qquad (M^{\mathrm{T}} = M, N^{\mathrm{T}} = -N).$$

For any square matrix, $M = \frac{A+A^{\mathrm{T}}}{2}$ and $N =$ _____ add up to A.

2 If C is symmetric prove that $A^{\mathrm{T}}CA$ is also symmetric. (Transpose it.) When A is 6 by 3, what are the shapes of C and $A^{\mathrm{T}}CA$?

3 Find the eigenvalues and the unit eigenvectors of

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

4 Find an orthogonal matrix Q that diagonalizes $A = \left[\begin{smallmatrix} -2 & 6 \\ 6 & 7 \end{smallmatrix}\right]$. What is Λ?

5 Find an orthogonal matrix Q that diagonalizes this symmetric matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

6 Find *all* orthogonal matrices that diagonalize $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

7 (a) Find a symmetric matrix $\left[\begin{smallmatrix} 1 & b \\ b & 1 \end{smallmatrix}\right]$ that has a negative eigenvalue.

(b) How do you know it must have a negative pivot?

(c) How do you know it can't have two negative eigenvalues?

8 If $A^3 = 0$ then the eigenvalues of A must be ____. Give an example that has $A \neq 0$. But if A is symmetric, diagonalize it to prove that A must be zero.

9 If $\lambda = a + ib$ is an eigenvalue of a real matrix A, then its conjugate $\bar{\lambda} = a - ib$ is also an eigenvalue. (If $Ax = \lambda x$ then also $A\bar{x} = \bar{\lambda}\bar{x}$.) Prove that every real 3 by 3 matrix has at least one real eigenvalue.

10 Here is a quick "proof" that the eigenvalues of all real matrices are real:

$$\textbf{False proof} \quad Ax = \lambda x \quad \text{gives} \quad x^{\mathrm{T}}Ax = \lambda x^{\mathrm{T}}x \quad \text{so} \quad \lambda = \frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} \quad \text{is real.}$$

Find the flaw in this reasoning—a hidden assumption that is not justified. You could test those steps on the 90° rotation matrix $[0 \ \ -1; \ \ 1 \ \ 0]$ with $\lambda = i$ and $x = (i, 1)$.

11 Write A and B in the form $\lambda_1 x_1 x_1^{\mathrm{T}} + \lambda_2 x_2 x_2^{\mathrm{T}}$ of the spectral theorem $Q\Lambda Q^T$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \qquad (\text{keep } \|x_1\| = \|x_2\| = 1).$$

12 Every 2 by 2 symmetric matrix is $\lambda_1 x_1 x_1^{\mathrm{T}} + \lambda_2 x_2 x_2^{\mathrm{T}} = \lambda_1 P_1 + \lambda_2 P_2$. Explain $P_1 + P_2 = x_1 x_1^{\mathrm{T}} + x_2 x_2^{\mathrm{T}} = I$ from columns times rows of Q. Why is $P_1 P_2 = 0$?

13 What are the eigenvalues of $A = \left[\begin{smallmatrix} 0 & b \\ -b & 0 \end{smallmatrix}\right]$? Create a 4 by 4 skew-symmetric matrix $(A^{\mathrm{T}} = -A)$ and verify that all its eigenvalues are imaginary.

14 (Recommended) This matrix M is skew-symmetric and also _____. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. ($\|Mx\| = \|x\|$ for every x so $\|\lambda x\| = \|x\|$ for eigenvectors.) Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad \text{can only have eigenvalues } i \text{ or } -i.$$

15 Show that A (**symmetric but complex**) has only one line of eigenvectors:

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \text{ is not even diagonalizable: eigenvalues } \lambda = 0, 0.$$

$A^{\mathrm{T}} = A$ is not such a special property for complex matrices. The good property is $\overline{A}^{\mathrm{T}} = A$ (Section 10.2). Then all λ's are real and eigenvectors are orthogonal.

16 Even if A is rectangular, the block matrix $B = \begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}$ is symmetric:

$$Bx = \lambda x \quad \text{is} \quad \begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix} \quad \text{which is} \quad \begin{matrix} Az = \lambda y \\ A^{\mathrm{T}}y = \lambda z. \end{matrix}$$

(a) Show that $-\lambda$ is also an eigenvalue, with the eigenvector $(y, -z)$.

(b) Show that $A^{\mathrm{T}}Az = \lambda^2 z$, so that λ^2 is an eigenvalue of $A^{\mathrm{T}}A$.

(c) If $A = I$ (2 by 2) find all four eigenvalues and eigenvectors of B.

17 If $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in Problem 16, find all three eigenvalues and eigenvectors of B.

18 ***Another proof that eigenvectors are perpendicular when $A = A^{\mathrm{T}}$.*** Two steps:

1. Suppose $Ax = \lambda x$ and $Ay = 0y$ and $\lambda \neq 0$. Then y is in the nullspace and x is in the column space. They are perpendicular because _____. Go carefully—why are these subspaces orthogonal?
2. If $Ay = \beta y$, apply this argument to $A - \beta I$. The eigenvalue of $A - \beta I$ moves to zero and the eigenvectors stay the same—so they are perpendicular.

19 Find the eigenvector matrix S for A and for B. Show that S doesn't collapse at $d = 1$, even though $\lambda = 1$ is repeated. Are the eigenvectors perpendicular?

$$A = \begin{bmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -d & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix} \quad \text{have} \quad \lambda = 1, d, -d.$$

20 Write a 2 by 2 *complex* matrix with $\overline{A}^{\mathrm{T}} = A$ (a "Hermitian matrix"). Find λ_1 and λ_2 for your complex matrix. Adjust equations (1) and (2) to show that *the eigenvalues of a Hermitian matrix are real.*

21 ***True*** (with reason) or ***false*** (with example). "Orthonormal" is not assumed.

(a) A matrix with real eigenvalues and eigenvectors is symmetric.

(b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

(c) The inverse of a symmetric matrix is symmetric.

(d) The eigenvector matrix S of a symmetric matrix is symmetric.

22 (A paradox for instructors) If $AA^{\mathrm{T}} = A^{\mathrm{T}}A$ then A and A^{T} share the same eigenvectors (true). A and A^{T} always share the same eigenvalues. Find the flaw in this conclusion: They must have the same S and Λ. Therefore A equals A^{T}.

23 (Recommended) Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, $Q\Lambda Q^{\mathrm{T}}$?

24 What number b in $\left[\begin{smallmatrix} 2 & b \\ 1 & 0 \end{smallmatrix}\right]$ makes $A = Q\Lambda Q^{\mathrm{T}}$ possible? What number makes $A = S\Lambda S^{-1}$ impossible? What number makes A^{-1} impossible?

25 Find all 2 by 2 matrices that are orthogonal and also symmetric. Which two numbers can be eigenvalues?

26 This A is nearly symmetric. But its eigenvectors are far from orthogonal:

$$A = \begin{bmatrix} 1 & 10^{-15} \\ 0 & 1 + 10^{-15} \end{bmatrix} \quad \text{has eigenvectors} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [\,?\,]$$

What is the angle between the eigenvectors?

27 (MATLAB) Take two symmetric matrices with different eigenvectors, say $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 8 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Graph the eigenvalues $\lambda_1(A+tB)$ and $\lambda_2(A+tB)$ for $-8 < t < 8$. Peter Lax says on page 113 of *Linear Algebra* that λ_1 and λ_2 appear to be on a collision course at certain values of t. "Yet at the last minute they turn aside." How close do they come?

Challenge Problems

28 For complex matrices, the symmetry $A^{\mathrm{T}} = A$ that produces real eigenvalues changes to $\overline{A}^{\mathrm{T}} = A$. From $\det(A - \lambda I) = 0$, find the eigenvalues of the 2 by 2 "Hermitian" matrix $A = [4 \;\; 2+i; \;\; 2-i \;\; 0] = \overline{A}^{\mathrm{T}}$. To see why eigenvalues are real when $\overline{A}^{\mathrm{T}} = A$, adjust equation (1) of the text to $\overline{A}\,\overline{x} = \overline{\lambda}\,\overline{x}$.

Transpose to $\overline{x}^{\mathrm{T}}\overline{A}^{\mathrm{T}} = \overline{x}^{\mathrm{T}}\overline{\lambda}$. **With** $\overline{A}^{\mathrm{T}} = A$**, reach equation (2):** $\lambda = \overline{\lambda}$.

29 **Normal matrices** have $\overline{A}^{\mathrm{T}}A = A\overline{A}^{\mathrm{T}}$. For real matrices, $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ includes symmetric, skew-symmetric, and orthogonal. Those have real λ, imaginary λ, and $|\lambda| = 1$. Other normal matrices can have any complex eigenvalues λ.

Key point: ***Normal matrices have n orthonormal eigenvectors.*** Those vectors $\boldsymbol{x}_i$ probably will have complex components. In that complex case orthogonality means $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$ as Chapter 10 explains. Inner products (dot products) become $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{y}$.

The test for n orthonormal columns in Q becomes $\overline{Q}^{\mathrm{T}}Q = I$ instead of $Q^{\mathrm{T}}Q = I$.

A has n **orthonormal eigenvectors** ($A = Q\Lambda\overline{Q}^{\mathrm{T}}$) if and only if A is **normal**.

(a) Start from $A = Q\Lambda\overline{Q}^{\mathrm{T}}$ with $\overline{Q}^{\mathrm{T}}Q = I$. Show that $\overline{A}^{\mathrm{T}}A = A\overline{A}^{\mathrm{T}}$: A is normal.

(b) Now start from $\overline{A}^{\mathrm{T}}A = A\overline{A}^{\mathrm{T}}$. Schur found $A = QT\overline{Q}^{\mathrm{T}}$ for every matrix A, with a triangular T. For normal matrices we must show (in 3 steps) that this T will actually be diagonal. Then $T = \Lambda$.

Step 1. Put $A = QT\overline{Q}^{\mathrm{T}}$ into $\overline{A}^{\mathrm{T}}A = A\overline{A}^{\mathrm{T}}$ to find $\overline{T}^{\mathrm{T}}T = T\overline{T}^{\mathrm{T}}$.

Step 2. Suppose $T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\overline{T}^{\mathrm{T}}T = T\overline{T}^{\mathrm{T}}$. Prove that $b = 0$.

Step 3. Extend Step 2 to size n. A normal triangular T must be diagonal.

30 If $\lambda_{\max}$ is the largest eigenvalue of a symmetric matrix A, no diagonal entry can be larger than $\lambda_{\max}$. What is the first entry a_{11} of $A = Q\Lambda Q^{\mathrm{T}}$? Show why $a_{11} \le \lambda_{\max}$.

31 Suppose $A^{\mathrm{T}} = -A$ (real antisymmetric matrix). Explain these facts about A:

(a) $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 0$ for every real vector $\boldsymbol{x}$.

(b) The eigenvalues of A are pure imaginary.

(c) The determinant of A is positive or zero (not negative).

For (a), multiply out an example of $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$ and watch terms cancel. Or reverse $\boldsymbol{x}^{\mathrm{T}}(A\boldsymbol{x})$ to $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{x}$. For (b), $A\boldsymbol{z} = \lambda\boldsymbol{z}$ leads to $\overline{\boldsymbol{z}}^{\mathrm{T}}A\boldsymbol{z} = \lambda\overline{\boldsymbol{z}}^{\mathrm{T}}\boldsymbol{z} = \lambda\|\boldsymbol{z}\|^2$. Part (a) shows that $\overline{\boldsymbol{z}}^{\mathrm{T}}A\boldsymbol{z} = (\boldsymbol{x} - i\boldsymbol{y})^{\mathrm{T}}A(\boldsymbol{x} + i\boldsymbol{y})$ has zero real part. Then (b) helps with (c).

32 If A is symmetric and all its eigenvalues are $\lambda = 2$, how do you know that A must be $2I$? (Key point: Symmetry guarantees that A is diagonalizable. See "Proofs of the Spectral Theorem" on *web.mit.edu*/18.06.)

6.5 Positive Definite Matrices

This section concentrates on ***symmetric matrices that have positive eigenvalues***. If symmetry makes a matrix important, this extra property (*all* $\lambda > 0$) makes it truly special. When we say special, we don't mean rare. Symmetric matrices with positive eigenvalues are at the center of all kinds of applications. They are called ***positive definite***.

The first problem is to recognize these matrices. You may say, just find the eigenvalues and test $\lambda > 0$. That is exactly what we want to avoid. Calculating eigenvalues is work. When the λ's are needed, we can compute them. But if we just want to know that they are positive, there are faster ways. Here are two goals of this section:

- To find *quick tests* on a symmetric matrix that guarantee *positive eigenvalues*.
- To explain important applications of positive definiteness.

The λ's are automatically real because the matrix is symmetric.

Start with 2 by 2. When does $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ ***have*** $\lambda_1 > 0$ ***and*** $\lambda_2 > 0$?

The eigenvalues of A are positive if and only if $a > 0$ ***and*** $ac - b^2 > 0$.

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ is } \textit{not} \text{ positive definite because } ac - b^2 = 1 - 4 < 0$$

$$A_2 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix} \text{ is positive definite because } a = 1 \text{ and } ac - b^2 = 6 - 4 > 0$$

$$A_3 = \begin{bmatrix} -1 & 2 \\ 2 & -6 \end{bmatrix} \text{ is } \textit{not} \text{ positive definite (even with } \det A = +2\text{) because } a = -1$$

Notice that we didn't compute the eigenvalues 3 and -1 of A_1. Positive trace $3 - 1 = 2$, negative determinant $(3)(-1) = -3$. And $A_3 = -A_2$ is *negative* definite. The positive eigenvalues for A_2, two negative eigenvalues for A_3.

Proof that the 2 *by* 2 *test is passed when* $\lambda_1 > 0$ *and* $\lambda_2 > 0$. Their product $\lambda_1\lambda_2$ is the determinant so $ac - b^2 > 0$. Their sum is the trace so $a + c > 0$. Then a and c are both positive (if one of them is not positive, $ac - b^2 > 0$ will fail). Problem 1 reverses the reasoning to show that the tests guarantee $\lambda_1 > 0$ and $\lambda_2 > 0$.

This test uses the 1 by 1 determinant a and the 2 by 2 determinant $ac - b^2$. When A is 3 by 3, $\det A > 0$ is the third part of the test. The next test requires *positive pivots*.

The eigenvalues of $A = A^{\mathrm{T}}$ ***are positive if and only if the pivots are positive:***

$$a > 0 \quad \text{and} \quad \frac{ac - b^2}{a} > 0.$$

$a > 0$ is required in both tests. So $ac > b^2$ is also required, for the determinant test and now the pivot. The point is to recognize that ratio as the *second pivot* of A:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \xrightarrow[\text{The multiplier is } b/a]{\text{The first pivot is } a} \begin{bmatrix} a & b \\ 0 & c - \frac{b}{a}b \end{bmatrix} \qquad \textit{The second pivot is} \quad c - \frac{b^2}{a} = \frac{ac - b^2}{a}$$

This connects two big parts of linear algebra. ***Positive eigenvalues mean positive pivots and vice versa.*** We gave a proof for symmetric matrices of any size in the last section. The pivots give a quick test for $\lambda > 0$, and they are a lot faster to compute than the eigenvalues. It is very satisfying to see pivots and determinants and eigenvalues come together in this course.

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix} \qquad A_3 = \begin{bmatrix} -1 & 2 \\ 2 & -6 \end{bmatrix}$$

pivots 1 and -3 **(indefinite)** | pivots 1 and 2 **(positive definite)** | pivots -1 and -2 **(negative definite)**

Here is a different way to look at symmetric matrices with positive eigenvalues.

Energy-based Definition

From $Ax = \lambda x$, multiply by x^{T} to get $x^{\mathrm{T}}Ax = \lambda x^{\mathrm{T}}x$. The right side is a positive λ times a positive number $x^{\mathrm{T}}x = \|x\|^2$. So $x^{\mathrm{T}}Ax$ is positive for any eigenvector.

The new idea is that $x^{\mathrm{T}}Ax$ ***is positive for all nonzero vectors*** x, not just the eigenvectors. In many applications this number $x^{\mathrm{T}}Ax$ (or $\frac{1}{2}x^{\mathrm{T}}Ax$) is the **energy** in the system. The requirement of positive energy gives *another definition* of a positive definite matrix. I think this energy-based definition is the fundamental one.

Eigenvalues and pivots are two equivalent ways to test the new requirement $x^{\mathrm{T}}Ax > 0$.

Definition *A is positive definite if $x^{\mathrm{T}}Ax > 0$ for every nonzero vector x:*

$$x^{\mathrm{T}}Ax = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 > 0. \qquad (1)$$

The four entries a, b, b, c give the four parts of $x^{\mathrm{T}}Ax$. From a and c come the pure squares ax^2 and cy^2. From b and b off the diagonal come the cross terms bxy and byx (the same). Adding those four parts gives $x^{\mathrm{T}}Ax$. This energy-based definition leads to a basic fact:

If A and B are symmetric positive definite, so is $A + B$.

Reason: $x^{\mathrm{T}}(A + B)x$ is simply $x^{\mathrm{T}}Ax + x^{\mathrm{T}}Bx$. Those two terms are positive (for $x \neq \mathbf{0}$) so $A + B$ is also positive definite. The pivots and eigenvalues are not easy to follow when matrices are added, but the energies just add.

$x^T A x$ also connects with our final way to recognize a positive definite matrix. Start with any matrix R, possibly rectangular. We know that $A = R^T R$ is square and symmetric. More than that, A will be positive definite when R has independent columns:

If the columns of R are independent, then $A = R^T R$ is positive definite.

Again eigenvalues and pivots are not easy. But the number $x^T A x$ is the same as $x^T R^T R x$. That is exactly $(Rx)^T(Rx)$—another important proof by parenthesis! That vector Rx is not zero when $x \neq 0$ (this is the meaning of independent columns). Then $x^T A x$ is the positive number $\|Rx\|^2$ and the matrix A is positive definite.

Let me collect this theory together, into five equivalent statements of positive definiteness. You will see how that key idea connects the whole subject of linear algebra: pivots, determinants, eigenvalues, and least squares (from $R^T R$). Then come the applications.

When a symmetric matrix has one of these five properties, it has them all:

1. All n ***pivots*** are positive.
2. All n ***upper left determinants*** are positive.
3. All n ***eigenvalues*** are positive.
4. $x^T A x$ is positive except at $x = 0$. This is the ***energy-based*** definition.
5. A equals $R^T R$ for a matrix R with ***independent columns***.

The "upper left determinants" are 1 by 1, 2 by 2, ..., n by n. The last one is the determinant of the complete matrix A. This remarkable theorem ties together the whole linear algebra course—at least for symmetric matrices. We believe that two examples are more helpful than a detailed proof (we nearly have a proof already).

Example 1 Test these matrices A and B for positive definiteness:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}.$$

Solution The pivots of A are 2 and $\frac{3}{2}$ and $\frac{4}{3}$, all positive. Its upper left determinants are 2 and 3 and 4, all positive. The eigenvalues of A are $2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$, all positive. That completes tests **1**, **2**, and **3**.

We can write $x^T A x$ as a sum of three squares. The pivots 2, $\frac{3}{2}$, $\frac{4}{3}$ appear outside the squares. The multipliers $-\frac{1}{2}$ and $-\frac{2}{3}$ from elimination are inside the squares:

$$\begin{aligned} x^T A x &= 2\left(x_1^2 - x_1 x_2 + x_2^2 - x_2 x_3 + x_3^2\right) && \textbf{Rewrite with squares} \\ &= 2\left(x_1 - \tfrac{1}{2}x_2\right)^2 + \tfrac{3}{2}\left(x_2 - \tfrac{2}{3}x_3\right)^2 + \tfrac{4}{3}(x_3)^2. && \textbf{This sum is positive.} \end{aligned}$$

I have two candidates to suggest for R. Either one will show that $A = R^{\mathrm{T}}R$ is positive definite. R can be a rectangular first difference matrix, 4 by 3, to produce those second differences $-1, 2, -1$ in A:

$$\boldsymbol{A = R^{\mathrm{T}}R} \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

The three columns of this R are independent. A is positive definite.

Another R comes from $A = LDL^{\mathrm{T}}$ (the symmetric version of $A = LU$). Elimination gives the pivots $2, \frac{3}{2}, \frac{4}{3}$ in D and the multipliers $-\frac{1}{2}, 0, -\frac{2}{3}$ in L. **Just put $\sqrt{D}$ with L.**

$$LDL^{\mathrm{T}} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = \begin{array}{l} (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}} = R^{\mathrm{T}}R. \\ \textbf{\textit{R is the Cholesky factor}} \end{array} \qquad (2)$$

This choice of R has square roots (not so beautiful). But it is the only R that is 3 by 3 and upper triangular. It is the "Cholesky factor" of A and it is computed by MATLAB's command $R = \mathsf{chol}(A)$. In applications, the rectangular R is how we build A and this Cholesky R is how we break it apart.

Eigenvalues give the symmetric choice $R = Q\sqrt{\Lambda}Q^{\mathrm{T}}$. This is also successful with $R^{\mathrm{T}}R = Q\Lambda Q^{\mathrm{T}} = A$. All these tests show that the $-1, 2, -1$ matrix A is positive definite.

Now turn to B, where the $(1,3)$ and $(3,1)$ entries move away from 0 to b. This b must not be too large! *The determinant test is easiest.* The 1 by 1 determinant is 2, the 2 by 2 determinant is still 3. The 3 by 3 determinant involves b:

$$\det B = 4 + 2b - 2b^2 = (1+b)(4-2b) \quad \text{must be positive.}$$

At $b = -1$ and $b = 2$ we get $\det B = 0$. *Between $b = -1$ and $b = 2$ the matrix is positive definite.* The corner entry $b = 0$ in the first matrix A was safely between.

Positive Semidefinite Matrices

Often we are at the edge of positive definiteness. The determinant is zero. The smallest eigenvalue is zero. The energy in its eigenvector is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}0\boldsymbol{x} = 0$. These matrices on the edge are called *positive semidefinite*. Here are two examples (not invertible):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ are positive semidefinite.}$$

A has eigenvalues 5 and 0. Its upper left determinants are 1 and 0. Its rank is only 1. This matrix A factors into $R^{\mathrm{T}}R$ with **dependent columns** in R:

$$\begin{array}{l} \textbf{Dependent columns} \\ \textbf{Positive semidefinite} \end{array} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = R^{\mathrm{T}}R.$$

If 4 is increased by any small number, the matrix will become positive definite.

The cyclic B also has zero determinant (computed above when $b = -1$). It is singular. The eigenvector $\boldsymbol{x} = (1, 1, 1)$ has $B\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{x}^{\mathrm{T}}B\boldsymbol{x} = 0$. Vectors $\boldsymbol{x}$ in all other directions do give positive energy. This B can be written as $R^{\mathrm{T}}R$ in many ways, but R will always have *dependent* columns, with $(1, 1, 1)$ in its nullspace:

Second differences A
from first differences $R^{\mathrm{T}}R$
Cyclic A **from cyclic** R

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Positive semidefinite matrices have all $\lambda \geq 0$ and all $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} \geq 0$. Those weak inequalities ($\geq$ instead of $>$) include positive definite matrices and the singular matrices at the edge.

First Application: The Ellipse $ax^2 + 2bxy + cy^2 = 1$

Think of a tilted ellipse $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$. Its center is $(0, 0)$, as in Figure 6.7a. Turn it to line up with the coordinate axes (X and Y axes). That is Figure 6.7b. These two pictures show the geometry behind the factorization $A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$:

1. The tilted ellipse is associated with A. Its equation is $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$.

2. The lined-up ellipse is associated with Λ. Its equation is $\boldsymbol{X}^{\mathrm{T}}\Lambda\boldsymbol{X} = 1$.

3. The rotation matrix that lines up the ellipse is the eigenvector matrix Q.

Example 2 Find the axes of this tilted ellipse $5x^2 + 8xy + 5y^2 = 1$.

Solution Start with the positive definite matrix that matches this equation:

The equation is $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.$ The matrix is $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$

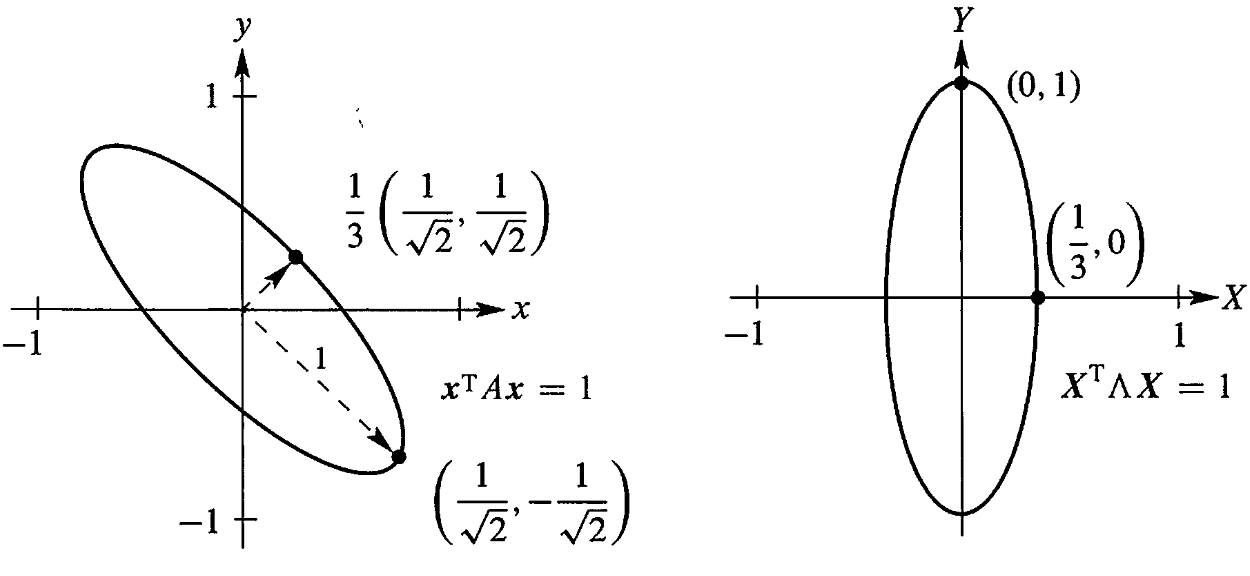


Figure 6.7: The tilted ellipse $5x^2 + 8xy + 5y^2 = 1$. Lined up it is $9X^2 + Y^2 = 1$.

The eigenvectors are $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$. Divide by $\sqrt{2}$ for unit vectors. Then $A = Q\Lambda Q^{\mathrm{T}}$:

$$\begin{array}{l}\textbf{Eigenvectors in } Q\\ \textbf{Eigenvalues 9 and 1}\end{array} \qquad \begin{bmatrix}5 & 4\\4 & 5\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}9 & 0\\0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}.$$

Now multiply by $\begin{bmatrix}x & y\end{bmatrix}$ on the left and $\begin{bmatrix}x\\y\end{bmatrix}$ on the right to get back to $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$:

$$x^{\mathrm{T}}Ax = \textbf{ sum of squares } \quad 5x^2 + 8xy + 5y^2 = 9\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1\left(\frac{x-y}{\sqrt{2}}\right)^2. \tag{3}$$

The coefficients are not the pivots 5 and 9/5 from D, they are the eigenvalues 9 and 1 from Λ. Inside *these* squares are the eigenvectors $(1,1)/\sqrt{2}$ and $(1,-1)/\sqrt{2}$.

The axes of the tilted ellipse point along the eigenvectors. This explains why $A = Q\Lambda Q^{\mathrm{T}}$ is called the "principal axis theorem"—it displays the axes. Not only the axis directions (from the eigenvectors) but also the axis lengths (from the eigenvalues). To see it all, use capital letters for the new coordinates that line up the ellipse:

$$\textbf{Lined up} \qquad \frac{x+y}{\sqrt{2}} = X \qquad \text{and} \qquad \frac{x-y}{\sqrt{2}} = Y \qquad \text{and} \qquad 9X^2 + Y^2 = 1.$$

The largest value of X^2 is $1/9$. The endpoint of the shorter axis has $X = 1/3$ and $Y = 0$. Notice: The *bigger* eigenvalue λ_1 gives the *shorter* axis, of half-length $1/\sqrt{\lambda_1} = 1/3$. The smaller eigenvalue $\lambda_2 = 1$ gives the greater length $1/\sqrt{\lambda_2} = 1$.

In the xy system, the axes are along the eigenvectors of A. In the XY system, the axes are along the eigenvectors of Λ—the coordinate axes. All comes from $A = Q\Lambda Q^{\mathrm{T}}$.

Suppose $A = Q\Lambda Q^{\mathrm{T}}$ is positive definite, so $\lambda_i > 0$. The graph of $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$ is an ellipse:

$$\begin{bmatrix}x & y\end{bmatrix} Q\Lambda Q^{\mathrm{T}}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}X & Y\end{bmatrix}\Lambda\begin{bmatrix}X\\Y\end{bmatrix} = \lambda_1 X^2 + \lambda_2 Y^2 = 1.$$

The axes point along eigenvectors. The half-lengths are $1/\sqrt{\lambda_1}$ and $1/\sqrt{\lambda_2}$.

$A = I$ gives the circle $x^2 + y^2 = 1$. If one eigenvalue is negative (exchange 4's and 5's in A), we don't have an ellipse. The sum of squares becomes a *difference of squares*: $9X^2 - Y^2 = 1$. This indefinite matrix gives a *hyperbola*. For a negative definite matrix like $A = -I$, with both λ's negative, the graph of $-x^2 - y^2 = 1$ has no points at all.

■ REVIEW OF THE KEY IDEAS ■

1. Positive definite matrices have positive eigenvalues and positive pivots.

2. A quick test is given by the upper left determinants: $a > 0$ and $ac - b^2 > 0$.

3. The graph of $x^{\mathrm{T}}Ax$ is then a "bowl" going up from $x = 0$:

$$x^{\mathrm{T}}Ax = ax^2 + 2bxy + cy^2 \text{ is positive except at } (x, y) = (0, 0).$$

4. $A = R^{\mathrm{T}}R$ is automatically positive definite if R has independent columns.

5. The ellipse $x^{\mathrm{T}}Ax = 1$ has its axes along the eigenvectors of A. Lengths $1/\sqrt{\lambda}$.

■ WORKED EXAMPLES ■

6.5 A The great factorizations of a symmetric matrix are $A = LDL^{\mathrm{T}}$ from pivots and multipliers, and $A = Q\Lambda Q^{\mathrm{T}}$ from eigenvalues and eigenvectors. Show that $x^{\mathrm{T}}Ax > 0$ for all nonzero x exactly when the pivots and eigenvalues are positive. Try these n by n tests on pascal(6) and ones(6) and hilb(6) and other matrices in MATLAB's gallery.

Solution To prove $x^{\mathrm{T}}Ax > 0$, put parentheses into $x^{\mathrm{T}}LDL^{\mathrm{T}}x$ and $x^{\mathrm{T}}Q\Lambda Q^{\mathrm{T}}x$:

$$x^{\mathrm{T}}Ax = (L^{\mathrm{T}}x)^{\mathrm{T}}D(L^{\mathrm{T}}x) \qquad \text{and} \qquad x^{\mathrm{T}}Ax = (Q^{\mathrm{T}}x)^{\mathrm{T}}\Lambda(Q^{\mathrm{T}}x).$$

If x is nonzero, then $y = L^{\mathrm{T}}x$ and $z = Q^{\mathrm{T}}x$ are nonzero (those matrices are invertible). So $x^{\mathrm{T}}Ax = y^{\mathrm{T}}Dy = z^{\mathrm{T}}\Lambda z$ becomes a sum of squares and A is shown as positive definite:

$$\begin{array}{lccccccc} \textbf{Pivots} & x^{\mathrm{T}}Ax & = & y^{\mathrm{T}}Dy & = & d_1y_1^2 + \cdots + d_ny_n^2 & > & 0 \\ \textbf{Eigenvalues} & x^{\mathrm{T}}Ax & = & z^{\mathrm{T}}\Lambda z & = & \lambda_1z_1^2 + \cdots + \lambda_nz_n^2 & > & 0 \end{array}$$

MATLAB has a gallery of unusual matrices (type help gallery) and here are four:

pascal(6) is positive definite because all its pivots are 1 (Worked Example **2.6 A**).

ones(6) is positive *semidefinite* because its eigenvalues are $0, 0, 0, 0, 0, 6$.

H=hilb(6) is positive definite even though eig(H) shows two eigenvalues very near zero.

Hilbert matrix $x^{\mathrm{T}}Hx = \int_0^1 (x_1 + x_2s + \cdots + x_6s^5)^2\, ds > 0$, $H_{ij} = 1/(i + j + 1)$.

rand(6)+rand(6)′ can be positive definite or not. Experiments gave only 2 in 20000.

$n = 20000$; $p = 0$; for $k = 1{:}n$, $A =$ rand(6); $p = p +$ all(eig($A + A'$) > 0); end, $p\,/\,n$

6.5 B **When is the symmetric block matrix** $M = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix}$ **positive definite?**

Solution Multiply the first row of M by $B^{\mathrm{T}}A^{-1}$ and subtract from the second row, to get a block of zeros. The *Schur complement* $S = C - B^{\mathrm{T}}A^{-1}B$ appears in the corner:

$$\begin{bmatrix} I & 0 \\ -B^{\mathrm{T}}A^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & C - B^{\mathrm{T}}A^{-1}B \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix} \tag{4}$$

Those two blocks A and S must be positive definite. Their pivots are the pivots of M.

6.5 C ***Second application: Test for a minimum.*** Does $F(x, y)$ have a minimum if $\partial F/\partial x = 0$ and $\partial F/\partial y = 0$ at the point $(x, y) = (0, 0)$?

Solution For $f(x)$, the test for a minimum comes from calculus: $df/dx = 0$ and $d^2 f/dx^2 > 0$. Moving to two variables x and y produces a symmetric matrix H. It contains the four second derivatives of $F(x, y)$. Positive f'' changes to positive definite H:

Second derivative matrix $$H = \begin{bmatrix} \partial^2 F/\partial x^2 & \partial^2 F/\partial x \partial y \\ \partial^2 F/\partial y \partial x & \partial^2 F/\partial y^2 \end{bmatrix}$$

$F(x, y)$ has a minimum if H is positive definite. Reason: H reveals the important terms $ax^2 + 2bxy + cy^2$ near $(x, y) = (0, 0)$. The second derivatives of F are $2a, 2b, 2b, 2c$!

6.5 D Find the eigenvalues of the $-1, 2, -1$ tridiagonal n by n matrix K (my favorite).

Solution The best way is to guess λ and $\boldsymbol{x}$. Then check $K\boldsymbol{x} = \lambda \boldsymbol{x}$. Guessing could not work for most matrices, but special cases are a big part of mathematics (pure and applied).

The key is hidden in a differential equation. The second difference matrix K is like a *second derivative*, and those eigenvalues are much easier to see:

Eigenvalues $\lambda_1, \lambda_2, \ldots$
Eigenfunctions $y_1, y_2, \ldots$ $$\frac{d^2 y}{dx^2} = \lambda y(x) \quad \textbf{with} \quad \begin{matrix} y(0) = 0 \\ y(1) = 0 \end{matrix} \tag{5}$$

Try $y = \sin cx$. Its second derivative is $y'' = -c^2 \sin cx$. So the eigenvalue will be $\lambda = -c^2$, provided $y(x)$ satisfies the end point conditions $y(0) = 0 = y(1)$.

Certainly $\sin 0 = 0$ (this is where cosines are eliminated by $\cos 0 = 1$). At $x = 1$, we need $y(1) = \sin c = 0$. The number c must be $k\pi$, a multiple of π, and λ is $-c^2$:

Eigenvalues $\lambda = -k^2\pi^2$
Eigenfunctions $y = \sin k\pi x$ $$\frac{d^2}{dx^2} \sin k\pi x = -k^2\pi^2 \sin k\pi x. \tag{6}$$

Now we go back to the matrix K and guess its eigenvectors. They come from $\sin k\pi x$ at n points $x = h, 2h, \ldots, nh$, equally spaced between 0 and 1. The spacing Δx is $h = 1/(n + 1)$, so the $(n + 1)$st point comes out at $(n + 1)h = 1$. Multiply that sine vector s by K:

Eigenvector of K = sine vector s $$\begin{aligned} Ks &= \lambda s = (2 - 2\cos k\pi h)\, s \\ s &= (\sin k\pi h, \ldots, \sin nk\pi h). \end{aligned} \tag{7}$$

I will leave that multiplication $Ks = \lambda s$ as a challenge problem. Notice what is important:

1. All eigenvalues $2 - 2\cos k\pi h$ are positive and K is positive definite.

2. The **sine matrix** S has orthogonal columns = eigenvectors $s_1, \ldots, s_n$ of K.

Discrete Sine Transform
The j,k entry is $\sin jk\pi h$

$$S = \begin{bmatrix} \sin \pi h & & \sin k\pi h & \\ \vdots & \cdots & \vdots & \cdots \\ \sin n\pi h & & \sin nk\pi h & \end{bmatrix}$$

Those eigenvectors are orthogonal just like the eigenfunctions: $\int_0^1 \sin j\pi x \sin k\pi x \, dx = 0$.

Problem Set 6.5

Problems 1–13 are about tests for positive definiteness.

1 Suppose the 2 by 2 tests $a > 0$ and $ac - b^2 > 0$ are passed. Then $c > b^2/a$ is also positive.

(i) λ_1 and λ_2 have the *same sign* because their product $\lambda_1\lambda_2$ equals _____.

(i) That sign is positive because $\lambda_1 + \lambda_2$ equals _____.

Conclusion: The tests $a > 0, ac - b^2 > 0$ guarantee positive eigenvalues λ_1, λ_2.

2 Which of A_1, A_2, A_3, A_4 has two positive eigenvalues? Use the test, don't compute the λ's. Find an x so that $x^{\mathrm{T}}A_1x < 0$, so A_1 fails the test.

$$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

3 For which numbers b and c are these matrices positive definite?

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \qquad A = \begin{bmatrix} c & b \\ b & c \end{bmatrix}.$$

With the pivots in D and multiplier in L, factor each A into LDL^{T}.

4 What is the quadratic $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write f as a sum of one or two squares $d_1(\quad)^2 + d_2(\quad)^2$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

5 Write $f(x,y) = x^2 + 4xy + 3y^2$ as a *difference* of squares and find a point (x,y) where f is negative. The minimum is not at $(0,0)$ even though f has positive coefficients.

6 The function $f(x,y) = 2xy$ certainly has a saddle point and not a minimum at $(0,0)$. What symmetric matrix A produces this f? What are its eigenvalues?

7 Test to see if $R^{\mathrm{T}}R$ is positive definite in each case:

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

8 The function $f(x, y) = 3(x + 2y)^2 + 4y^2$ is positive except at $(0, 0)$. What is the matrix in $f = [\,x \quad y\,]A[\,x \quad y\,]^{\mathrm{T}}$? Check that the pivots of A are 3 and 4.

9 Find the 3 by 3 matrix A and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

10 Which 3 by 3 symmetric matrices A and B produce these quadratics?

$$\begin{aligned} \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} &= 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3\right). \quad \text{Why is } A \text{ positive definite?} \\ \boldsymbol{x}^{\mathrm{T}}B\boldsymbol{x} &= 2\left(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3\right). \quad \text{Why is } B \text{ semidefinite?} \end{aligned}$$

11 Compute the three upper left determinants of A to establish positive definiteness. Verify that their ratios give the second and third pivots.

$$\textbf{Pivots = ratios of determinants} \qquad A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}.$$

12 For what numbers c and d are A and B positive definite? Test the 3 determinants:

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

13 Find a matrix with $a > 0$ and $c > 0$ and $a + c > 2b$ that has a negative eigenvalue.

Problems 14–20 are about applications of the tests.

14 *If A is positive definite then A^{-1} is positive definite.* Best proof: The eigenvalues of A^{-1} are positive because _____. *Second proof* (only for 2 by 2):

$$\text{The entries of } A^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \quad \text{pass the determinant tests} \quad \_\_\_\_\_.$$

15 *If A and B are positive definite, their sum $A + B$ is positive definite.* Pivots and eigenvalues are not convenient for $A + B$. Better to prove $\boldsymbol{x}^{\mathrm{T}}(A + B)\boldsymbol{x} > 0$. Or if $A = R^{\mathrm{T}}R$ and $B = S^{\mathrm{T}}S$, show that $A + B = [\,\mathrm{R} \;\; \mathrm{S}\,]^{\mathrm{T}} \left[\begin{smallmatrix} \mathrm{R} \\ \mathrm{S} \end{smallmatrix}\right]$ with independent columns.

16 A positive definite matrix cannot have a zero (or even worse, a negative number) on its diagonal. Show that this matrix fails to have $x^{\mathrm{T}}Ax > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad,\quad,\quad).$$

17 A diagonal entry a_{jj} of a symmetric matrix cannot be smaller than all the λ's. If it were, then $A - a_{jj}I$ would have _____ eigenvalues and would be positive definite. But $A - a_{jj}I$ has a _____ on the main diagonal.

18 If $Ax = \lambda x$ then $x^{\mathrm{T}}Ax =$ _____. If $x^{\mathrm{T}}Ax > 0$, prove that $\lambda > 0$.

19 Reverse Problem 18 to show that *if all* $\lambda > 0$ *then* $x^{\mathrm{T}}Ax > 0$. We must do this for *every* nonzero x, not just the eigenvectors. So write x as a combination of the eigenvectors and *explain why all "cross terms" are* $x_i^{\mathrm{T}}x_j = 0$. Then $x^{\mathrm{T}}Ax$ is

$$(c_1x_1 + \cdots + c_nx_n)^{\mathrm{T}}(c_1\lambda_1x_1 + \cdots + c_n\lambda_nx_n) = c_1^2\lambda_1x_1^{\mathrm{T}}x_1 + \cdots + c_n^2\lambda_nx_n^{\mathrm{T}}x_n > 0.$$

20 Give a quick reason why each of these statements is true:

(a) Every positive definite matrix is invertible.

(b) The only positive definite projection matrix is $P = I$.

(c) A diagonal matrix with positive diagonal entries is positive definite.

(d) A symmetric matrix with a positive determinant might not be positive definite!

Problems 21–24 use the eigenvalues; Problems 25–27 are based on pivots.

21 For which s and t do A and B have all $\lambda > 0$ (therefore positive definite)?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}.$$

22 From $A = Q\Lambda Q^{\mathrm{T}}$ compute the positive definite symmetric square root $Q\Lambda^{1/2}Q^{\mathrm{T}}$ of each matrix. Check that this square root gives $R^2 = A$:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}.$$

23 You may have seen the equation for an ellipse as $x^2/a^2 + y^2/b^2 = 1$. What are a and b when the equation is written $\lambda_1x^2 + \lambda_2y^2 = 1$? The ellipse $9x^2 + 4y^2 = 1$ has axes with half-lengths $a =$ _____ and $b =$ _____.

24 Draw the tilted ellipse $x^2 + xy + y^2 = 1$ and find the half-lengths of its axes from the eigenvalues of the corresponding matrix A.

25 With positive pivots in D, the factorization $A = LDL^{\mathrm{T}}$ becomes $L\sqrt{D}\sqrt{D}L^{\mathrm{T}}$. (Square roots of the pivots give $D = \sqrt{D}\sqrt{D}$.) Then $C = \sqrt{D}L^{\mathrm{T}}$ yields the ***Cholesky factorization*** $A = C^{\mathrm{T}}C$ which is "symmetrized $L\,U$":

$$\text{From} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{find } A. \qquad \text{From} \quad A = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} \quad \text{find } C = \mathbf{chol}(A).$$

26 In the Cholesky factorization $A = C^{\mathrm{T}}C$, with $C^{\mathrm{T}} = L\sqrt{D}$, the square roots of the pivots are on the diagonal of C. Find C (upper triangular) for

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}.$$

27 The symmetric factorization $A = LDL^{\mathrm{T}}$ means that $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}LDL^{\mathrm{T}}\boldsymbol{x}$:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & (ac-b^2)/a \end{bmatrix}\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

The left side is $ax^2 + 2bxy + cy^2$. The right side is $a\left(x + \frac{b}{a}y\right)^2 +$ _____ y^2. The second pivot completes the square! Test with $a = 2, b = 4, c = 10$.

28 Without multiplying $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find

(a) the determinant of A (b) the eigenvalues of A
(c) the eigenvectors of A (d) a reason why A is symmetric positive definite.

29 For $F_1(x,y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x,y) = x^3 + xy - x$ find the second derivative matrices H_1 and H_2:

$$\textbf{Test for minimum} \quad H = \begin{bmatrix} \partial^2F/\partial x^2 & \partial^2F/\partial x\partial y \\ \partial^2F/\partial y\partial x & \partial^2F/\partial y^2 \end{bmatrix} \text{ is positive definite}$$

H_1 is positive definite so F_1 is concave up (= convex). Find the minimum point of F_1 and the saddle point of F_2 (look only where first derivatives are zero).

30 The graph of $z = x^2 + y^2$ is a bowl opening upward. *The graph* of $z = x^2 - y^2$ *is a saddle*. The graph of $z = -x^2 - y^2$ is a bowl opening downward. What is a test on a, b, c for $z = ax^2 + 2bxy + cy^2$ to have a saddle point at $(0,0)$?

31 Which values of c give a bowl and which c give a saddle point for the graph of $z = 4x^2 + 12xy + cy^2$? Describe this graph at the borderline value of c.

Challenge Problems

32 A *group* of nonsingular matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these are groups (as in 2.7.37)?

Invent a "subgroup" of two of these groups (not I by itself $=$ the smallest group).

(a) Positive definite symmetric matrices A.

(b) Orthogonal matrices Q.

(c) All exponentials e^{tA} of a fixed matrix A.

(d) Matrices P with positive eigenvalues.

(e) Matrices D with determinant 1.

33 When A and B are symmetric positive definite, AB might not even be symmetric. But its eigenvalues are still positive. Start from $AB\boldsymbol{x} = \lambda\boldsymbol{x}$ and take dot products with $B\boldsymbol{x}$. Then prove $\lambda > 0$.

34 Write down the 5 by 5 sine matrix S from Worked Example **6.5 D**, containing the eigenvectors of K when $n = 5$ and $h = 1/6$. Multiply K times S to see the five positive eigenvalues.

Their sum should equal the trace 10. Their product should be $\det K = 6$.

35 Suppose C is positive definite (so $\boldsymbol{y}^{\mathrm{T}}C\boldsymbol{y} > 0$ whenever $\boldsymbol{y} \neq \boldsymbol{0}$) and A has independent columns (so $A\boldsymbol{x} \neq \boldsymbol{0}$ whenever $\boldsymbol{x} \neq \boldsymbol{0}$). Apply the energy test to $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}CA\boldsymbol{x}$ to show that $A^{\mathrm{T}}CA$ is positive definite: *the crucial matrix in engineering*.

6.6 Similar Matrices

The key step in this chapter is to diagonalize a matrix by using its eigenvectors. When S is the eigenvector matrix, the diagonal matrix $S^{-1}AS$ is Λ—the eigenvalue matrix. But diagonalization is not possible for every A. Some matrices have too few eigenvectors—we had to leave them alone. In this new section, the eigenvector matrix S remains the best choice when we can find it, ***but now we allow any invertible matrix M.***

Starting from A we go to $M^{-1}AM$. This matrix may be diagonal—probably not. It still shares important properties of A. No matter which M we choose, ***the eigenvalues stay the same.*** The matrices A and $M^{-1}AM$ are called "similar". A typical matrix A is similar to a whole family of other matrices because there are so many choices of M.

DEFINITION Let M be any invertible matrix. Then $B = M^{-1}AM$ is ***similar*** to A.

If $B = M^{-1}AM$ then immediately $A = MBM^{-1}$. That means: If B is similar to A then A is similar to B. The matrix in this reverse direction is M^{-1}—just as good as M.

A diagonalizable matrix is similar to Λ. In that special case M is S. We have $A = S\Lambda S^{-1}$ and $\Lambda = S^{-1}AS$. They certainly have the same eigenvalues! This section is opening up to other similar matrices $B = M^{-1}AM$, by allowing all invertible M.

The combination $M^{-1}AM$ appears when we change variables in a differential equation. Start with an equation for $\boldsymbol{u}$ and set $\boldsymbol{u} = M\boldsymbol{v}$:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \quad \text{becomes} \quad M\frac{d\boldsymbol{v}}{dt} = AM\boldsymbol{v} \quad \text{which is} \quad \frac{d\boldsymbol{v}}{dt} = M^{-1}AM\boldsymbol{v}.$$

The original coefficient matrix was A, the new one at the right is $M^{-1}AM$. Changing $\boldsymbol{u}$ to $\boldsymbol{v}$ leads to a similar matrix. When $M = S$ the new system is diagonal—the maximum in simplicity. Other choices of M could make the new system triangular and easier to solve. Since we can always go back to $\boldsymbol{u}$, similar matrices must give the same growth or decay. More precisely, *the eigenvalues of A and B are the same.*

(No change in λ's) Similar matrices A and $M^{-1}AM$ have the **same eigenvalues**. If $\boldsymbol{x}$ is an eigenvector of A, then $M^{-1}\boldsymbol{x}$ is an eigenvector of $B = M^{-1}AM$.

The proof is quick, since $B = M^{-1}AM$ gives $A = MBM^{-1}$. Suppose $A\boldsymbol{x} = \lambda\boldsymbol{x}$:

$$MBM^{-1}\boldsymbol{x} = \lambda\boldsymbol{x} \quad \textbf{\textit{means that}} \quad B(M^{-1}\boldsymbol{x}) = \lambda(M^{-1}\boldsymbol{x}).$$

The eigenvalue of B is the same λ. The eigenvector has changed to $M^{-1}\boldsymbol{x}$.

Two matrices can have the same *repeated* λ, and fail to be similar—as we will see.

Example 1 These matrices $M^{-1}AM$ all have the same eigenvalues 1 and 0.

$$\text{The projection } A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \text{ is similar to } \Lambda = S^{-1}AS = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\text{Now choose } M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}. \text{ The similar matrix } M^{-1}AM \text{ is } \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

$$\text{Also choose } M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \text{ The similar matrix } M^{-1}AM \text{ is } \begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix}.$$

All 2 by 2 matrices with those eigenvalues 1 and 0 are similar to each other. The eigenvectors change with M, the eigenvalues don't change.

The eigenvalues in that example are *not repeated*. This makes life easy. Repeated eigenvalues are harder. The next example has eigenvalues 0 and 0. The zero matrix shares those eigenvalues, but it is similar only to itself: $M^{-1}0M = 0$.

Example 2 A family of similar matrices with $\lambda = 0, 0$ (repeated eigenvalue)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ is similar to } \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \text{ and all } B = \begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix} \text{ except } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

These matrices B all have zero determinant (like A). They all have rank one (like A). One eigenvalue is zero and the trace is $cd - dc = 0$, so the other must be zero. I chose any $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $ad - bc = 1$, and $B = M^{-1}AM$.

These matrices B can't be diagonalized. In fact A is as close to diagonal as possible. It is the "***Jordan form***" for the family of matrices B. This is the outstanding member (my class says "Godfather") of the family. The Jordan form $J = A$ is as near as we can come to diagonalizing these matrices, when there is only one eigenvector. In going from A to $B = M^{-1}AM$, some things change and some don't. Here is a table to show this.

Not changed by *M*	**Changed by *M***
Eigenvalues	**Eigenvectors**
Trace and determinant	**Nullspace**
Rank	**Column space**
Number of independent eigenvectors	**Row space**
	Left nullspace
Jordan form	**Singular values**

The eigenvalues don't change for similar matrices; the eigenvectors do. The trace is the sum of the λ's (unchanged). The determinant is the product of the same λ's.[1] The nullspace consists of the eigenvectors for $\lambda = 0$ (if any), so it can change. Its dimension $n - r$ does not change! The *number* of eigenvectors stays the same for each λ, while the vectors themselves are multiplied by M^{-1}. The *singular values* depend on $A^{\mathrm{T}}A$, which definitely changes. They come in the next section.

[1] The determinant is unchanged because $\det B = (\det M^{-1})(\det A)(\det M) = \det A$.

Examples of the Jordan Form

The *Jordan form* is the serious new idea here. We lead up to it with one more example of similar matrices: *triple eigenvalue, one eigenvector.*

Example 3 This **Jordan matrix** J has $\lambda = 5, 5, 5$ on its diagonal. Its only eigenvectors are multiples of $\boldsymbol{x} = (1, 0, 0)$. Algebraic multiplicity is 3, geometric multiplicity is 1:

$$\text{If} \quad J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{then} \quad J - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{has rank 2.}$$

Every similar matrix $B = M^{-1}JM$ has the same triple eigenvalue $5, 5, 5$. Also $B - 5I$ must have the same rank 2. Its nullspace has dimension 1. So every B that is similar to this "Jordan block" J has only one independent eigenvector $M^{-1}\boldsymbol{x}$.

The transpose matrix J^{T} has the same eigenvalues $5, 5, 5$, and $J^{\mathrm{T}} - 5I$ has the same rank 2. ***Jordan's theorem says that J^{T} is similar to J.*** The matrix M that produces the similarity happens to be the *reverse identity*:

$$J^{\mathrm{T}} = M^{-1}JM \quad \text{is} \quad \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}.$$

All blank entries are zero. An eigenvector of J^{T} is $M^{-1}(1, 0, 0) = (0, 0, 1)$. There is one line of eigenvectors $(x_1, 0, 0)$ for J and another line $(0, 0, x_3)$ for J^{T}.

The key fact is that this matrix J is similar to *every* matrix A with eigenvalues $5, 5, 5$ and *one line* of eigenvectors. There is an M with $M^{-1}AM = J$.

Example 4 Since J is as close to diagonal as we can get, the equation $d\boldsymbol{u}/dt = J\boldsymbol{u}$ cannot be simplified by changing variables. We must solve it as it stands:

$$\frac{d\boldsymbol{u}}{dt} = J\boldsymbol{u} = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{is} \quad \begin{aligned} dx/dt &= 5x + y \\ dy/dt &= 5y + z \\ dz/dt &= 5z. \end{aligned}$$

The system is triangular. We think naturally of back substitution. Solve the last equation and work upwards. Main point: *All solutions contain e^{5t}* since $\lambda = 5$:

$$\begin{array}{llll} \textbf{Last equation} & \dfrac{dz}{dt} = 5z & \text{yields} & z = z(0)e^{5t} \\ \textbf{Notice } te^{5t} & \dfrac{dy}{dt} = 5y + z & \text{yields} & y = \big(y(0) + tz(0)\big)e^{5t} \\ \textbf{Notice } t^2e^{5t} & \dfrac{dx}{dt} = 5x + y & \text{yields} & x = \big(x(0) + ty(0) + \tfrac{1}{2}t^2z(0)\big)e^{5t}. \end{array}$$

The two missing eigenvectors are responsible for the te^{5t} and t^2e^{5t} terms in y and x. The factors t and t^2 enter because $\lambda = 5$ is a triple eigenvalue with one eigenvector.

Note Chapter 7 will explain another approach to similar matrices. Instead of changing variables by $\boldsymbol{u} = M\boldsymbol{v}$, we "***change the basis***". In this approach, similar matrices will represent the same transformation of n-dimensional space. When we choose a basis for $\mathbf{R}^n$, we get a matrix. The standard basis vectors ($M = I$) lead to $I^{-1}AI$ which is A. Other bases lead to similar matrices $B = M^{-1}AM$.

The Jordan Form

For every A, we want to choose M so that $M^{-1}AM$ is as ***nearly diagonal as possible***. When A has a full set of n eigenvectors, they go into the columns of M. Then $M = S$. The matrix $S^{-1}AS$ is diagonal, period. This matrix Λ is the Jordan form of A—when A can be diagonalized. In the general case, eigenvectors are missing and Λ can't be reached.

Suppose A has s independent eigenvectors. Then it is similar to a matrix with s blocks. Each block is like J in Example 3. *The eigenvalue is on the diagonal with* 1's *just above it.* This block accounts for one eigenvector of A. When there are n eigenvectors and n blocks, they are all 1 by 1. In that case J is Λ.

(Jordan form) If A has s independent eigenvectors, it is similar to a matrix J that has s Jordan blocks on its diagonal: Some matrix M puts A into Jordan form:

Jordan form
$$M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J. \tag{1}$$

Each block in J has one eigenvalue λ_i, one eigenvector, and 1's above the diagonal:

Jordan block
$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & 1 \\ & & & \lambda_i \end{bmatrix}. \tag{2}$$

A is similar to B if they share the same Jordan form J—not otherwise.

The Jordan form J has an off-diagonal 1 for each missing eigenvector (and the 1's are next to the eigenvalues). This is the big theorem about matrix similarity. In every family of similar matrices, we are picking one outstanding member called J. It is nearly diagonal (or if possible completely diagonal). For that J, we can solve $d\boldsymbol{u}/dt = J\boldsymbol{u}$ as in Example 4. We can take powers J^k as in Problems 9–10. Every other matrix in the family has the form $A = MJM^{-1}$. The connection through M solves $d\boldsymbol{u}/dt = A\boldsymbol{u}$.

The point you must see is that $MJM^{-1}MJM^{-1} = MJ^2M^{-1}$. That cancellation of $M^{-1}M$ in the middle has been used through this chapter (when M was S). We found A^{100} from $S\Lambda^{100}S^{-1}$—by diagonalizing the matrix. Now we can't quite diagonalize A. So we use $MJ^{100}M^{-1}$ instead.

Jordan's Theorem is proved in my textbook *Linear Algebra and Its Applications*. Please refer to that book (or more advanced books) for the proof. The reasoning is rather intricate and in actual computations the Jordan form is not at all popular—its calculation is not stable. A slight change in A will separate the repeated eigenvalues and remove the off-diagonal 1's—switching to a diagonal Λ.

Proved or not, you have caught the central idea of similarity—to make A as simple as possible while preserving its essential properties.

■ REVIEW OF THE KEY IDEAS ■

1. B is similar to A if $B = M^{-1}AM$, for some invertible matrix M.
2. Similar matrices have the same eigenvalues. Eigenvectors are multiplied by M^{-1}.
3. If A has n independent eigenvectors then A is similar to Λ (take $M = S$).
4. Every matrix is similar to a Jordan matrix J (which has Λ as its diagonal part). J has a block for each eigenvector, and 1's for missing eigenvectors.

■ WORKED EXAMPLES ■

6.6 A The 4 by 4 triangular Pascal matrix A and its inverse (alternating diagonals) are

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Check that A and A^{-1} have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives $A^{-1} = D^{-1}AD$. This A is similar to A^{-1}, which is unusual.

These similar matrices must have the same Jordan form J. This J has only one block because the Pascal matrix has only one line of eigenvectors.

Solution The triangular matrices A and A^{-1} both have $\lambda = 1, 1, 1, 1$ on their main diagonals. Choose D with alternating 1 and -1 on its diagonal. *D equals D^{-1}*:

$$D^{-1}AD = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} = A^{-1}.$$

Check: Changing signs in rows 1 and 3 of A, and columns 1 and 3, produces the four negative entries in A^{-1}. We are multiplying row i by $(-1)^i$ and column j by $(-1)^j$, which gives the alternating diagonals in A^{-1}. Then AD has *columns with alternating signs.*

6.6 B ***The best way to compute eigenvalues of a large matrix is not from solving*** $\det(A - \lambda I) = 0$. That high degree polynomial is a numerical disaster.

Instead we compute similar matrices $A_1, A_2, \ldots$ that approach a triangular matrix. Then the eigenvalues of A (unchanged) are almost sitting on the main diagonal.

One way is to factor $A = QR$ by "Gram-Schmidt". Reverse the order to $A_1 = RQ$. This matrix is similar to A because $RQ = Q^{-1}(QR)Q$. An example with $c = \cos\theta$ and $s = \sin\theta$ shows how a small off-diagonal s can be *cubed* in A_1:

$$A = \begin{bmatrix} c & s \\ s & 0 \end{bmatrix} \quad \text{factors into} \quad \begin{bmatrix} c & s \\ s & -c \end{bmatrix}\begin{bmatrix} 1 & cs \\ 0 & s^2 \end{bmatrix} = QR.$$

$$A_1 = RQ = \begin{bmatrix} c + cs^2 & s^3 \\ s^3 & -cs^2 \end{bmatrix} \quad \text{has } s^3 \text{ below the diagonal}$$

Another step can factor $A_1 = Q_1R_1$ and reverse to $A_2 = R_1Q_1$. This ***QR method*** is in Section 9.3 with a further improvement for A_1. Add cs^2 to its diagonal (to get zero in the corner) and then subtract back from A_2:

Shift and factor $A_1 + cs^2I = Q_1R_1$ **Reverse and shift back** $A_2 = R_1Q_1 - cs^2I$

Shifted QR is an amazing success—just about the best way to compute eigenvalues.

Problem Set 6.6

1 If $C = F^{-1}AF$ and also $C = G^{-1}BG$, what matrix M gives $B = M^{-1}AM$? Conclusion: If C is similar to A and also to B then _____.

2 If $A = \text{diag}(1, 3)$ and $B = \text{diag}(3, 1)$ show that A and B are similar (find an M).

3 Show that A and B are similar by finding M so that $B = M^{-1}AM$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.$$

4 If a 2 by 2 matrix A has eigenvalues 0 and 1, why is it similar to $\Lambda = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$? Deduce from Problem 1 that all 2 by 2 matrices with those eigenvalues are similar.

5 Which of these six matrices are similar? Check their eigenvalues.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

6 There are sixteen 2 by 2 matrices whose entries are 0's and 1's. Similar matrices go into the same family. How many families? How many matrices (total 16) in each family?

7 (a) If $\boldsymbol{x}$ is in the nullspace of A show that $M^{-1}\boldsymbol{x}$ is in the nullspace of $M^{-1}AM$.

(b) The nullspaces of A and $M^{-1}AM$ have the same (vectors)(basis)(dimension).

8 Suppose $Ax = \lambda x$ and $Bx = \lambda x$ with the same λ's and x's. With n independent eigenvectors we have $A = B$: *Why*? Find $A \neq B$ when both have eigenvalues $0, 0$ but only one line of eigenvectors $(x_1, 0)$.

9 By direct multiplication find A^2 and A^3 and A^5 when

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Guess the form of A^k. Set $k = 0$ to find A^0 and $k = -1$ to find A^{-1}.

Questions 10–14 are about the Jordan form.

10 By direct multiplication, find J^2 and J^3 when

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Guess the form of J^k. Set $k = 0$ to find J^0. Set $k = -1$ to find J^{-1}.

11 Solve $du/dt = J\boldsymbol{u}$ for J in Problem 10, starting from $\boldsymbol{u}(0) = (5, 2)$. Remember $te^{\lambda t}$.

12 These Jordan matrices have eigenvalues $0, 0, 0, 0$. They have two eigenvectors (one from each block). But the block sizes don't match and they are *not similar*:

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad K = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

For any matrix M, compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is impossible: *J is not similar to K*.

13 Based on Problem 12, what are the five Jordan forms when $\lambda = 0, 0, 0, 0$?

14 Prove that A^{T} *is always similar to* A (we know the λ's are the same):

1. For one Jordan block J_i: Find M_i so that $M_i^{-1}J_iM_i = J_i^{\mathrm{T}}$ (see Example 3).
2. For any J with blocks J_i: Build M_0 from blocks so that $M_0^{-1}JM_0 = J^{\mathrm{T}}$.
3. For any $A = MJM^{-1}$: Show that A^{T} is similar to J^{T} and so to J and to A.

15 Prove that $\det(A - \lambda I) = \det(M^{-1}AM - \lambda I)$. (You could write $I = M^{-1}M$ and factor out $\det M^{-1}$ and $\det M$.) Since these *characteristic polynomials* of A and $M^{-1}AM$ are the same, the eigenvalues are the same (with the same multiplicities).

16 Which pairs are similar? Choose a, b, c, d to prove that the other pairs aren't:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} b & a \\ d & c \end{bmatrix} \quad \begin{bmatrix} c & d \\ a & b \end{bmatrix} \quad \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

17 True or false, with a good reason:

(a) A symmetric matrix can't be similar to a nonsymmetric matrix.

(b) An invertible matrix can't be similar to a singular matrix.

(c) A can't be similar to $-A$ unless $A = 0$.

(d) A can't be similar to $A + I$.

18 If B is invertible, prove that AB is similar to BA. *They have the same eigenvalues.*

19 If A is 6 by 4 and B is 4 by 6, AB and BA have different sizes. But with blocks

$$M^{-1}FM = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} = G.$$

(a) What sizes are the four blocks (the same four sizes in each matrix)?

(b) This equation is $M^{-1}FM = G$, so F and G have the same 10 eigenvalues. F has the 6 eigenvalues of AB plus 4 zeros; G has the 4 eigenvalues of BA plus 6 zeros. ***AB has the same eigenvalues as BA plus _____ zeros.***

20 Why are these statements all true?

(a) If A is similar to B then A^2 is similar to B^2.

(b) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$).

(c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$.

(d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is not similar to $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

(e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2, **the eigenvalues stay the same.** In this case $M =$ _____ .

21 If J is the 5 by 5 Jordan block with $\lambda = 0$, find J^2 and count its eigenvectors and find its Jordan form (there will be two blocks).

Challenge Problems

22 If an n by n matrix A has all eigenvalues $\lambda = 0$, prove that $A^n =$ zero matrix. (Maybe prove first that $J^n =$ zero matrix, by direct multiplication. Or use the Cayley-Hamilton Theorem?)

23 For the shifted QR method in the Worked Example **6.6 B**, show that A_2 is similar to A_1. No change in eigenvalues, and the A's quickly approach a diagonal matrix.

24 If A is similar to A^{-1}, must all the eigenvalues equal 1 or -1?

6.7 Singular Value Decomposition (SVD)

The Singular Value Decomposition is a highlight of linear algebra. A is any m by n matrix, square or rectangular. Its rank is r. We will diagonalize this A, but not by $S^{-1}AS$. The eigenvectors in S have three big problems: They are usually not orthogonal, there are not always enough eigenvectors, and $Ax = \lambda x$ requires A to be square. The ***singular vectors*** of A solve all those problems in a perfect way.

The price we pay is to have two sets of singular vectors, $\boldsymbol{u}$'s and $\boldsymbol{v}$'s. The $\boldsymbol{u}$'s are eigenvectors of AA^{T} and the $\boldsymbol{v}$'s are eigenvectors of $A^{\mathrm{T}}A$. Since those matrices are both symmetric, their eigenvectors can be chosen orthonormal. In equation (13) below, the simple fact that A times $A^{\mathrm{T}}A$ is the same as AA^{T} times A will lead to a remarkable property of these $\boldsymbol{u}$'s and $\boldsymbol{v}$'s:

$$\textbf{“}\boldsymbol{A}\textbf{ is diagonalized”} \qquad A\boldsymbol{v}_1 = \sigma_1\boldsymbol{u}_1 \quad A\boldsymbol{v}_2 = \sigma_2\boldsymbol{u}_2 \quad \ldots \quad A\boldsymbol{v}_r = \sigma_r\boldsymbol{u}_r \tag{1}$$

The singular vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are in the *row space* of A. The outputs $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ are in the *column space* of A. The *singular values* $\sigma_1, \ldots, \sigma_r$ are all positive numbers. When the $\boldsymbol{v}$'s and $\boldsymbol{u}$'s go into the columns of V and U, orthogonality gives $V^{\mathrm{T}}V = I$ and $U^{\mathrm{T}}U = I$. The σ's go into a diagonal matrix Σ.

Just as $A\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i$ led to the diagonalization $AS = S\Lambda$, the equations $A\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i$ tell us column by column that $\boldsymbol{AV = U\Sigma}$:

$$\begin{array}{c}(m \text{ by } n)(n \text{ by } r)\\ \text{equals}\\ (m \text{ by } r)(r \text{ by } r)\end{array} \qquad A\begin{bmatrix} \\ \boldsymbol{v}_1 \cdots \boldsymbol{v}_r \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \boldsymbol{u}_1 \cdots \boldsymbol{u}_r \\ \\ \end{bmatrix}\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}. \tag{2}$$

This is the heart of the SVD, but there is more. Those $\boldsymbol{v}$'s and $\boldsymbol{u}$'s account for the row space and column space of A. We need $n - r$ more $\boldsymbol{v}$'s and $m - r$ more $\boldsymbol{u}$'s, from the nullspace $\boldsymbol{N}(A)$ and the left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$. They can be orthonormal bases for those two nullspaces (and then automatically orthogonal to the first r $\boldsymbol{v}$'s and $\boldsymbol{u}$'s). Include all the $\boldsymbol{v}$'s and $\boldsymbol{u}$'s in V and U, so these matrices become *square*. ***We still have*** $\boldsymbol{AV = U\Sigma}$.

$$\begin{array}{c}(m \text{ by } n)(n \text{ by } n)\\ \text{equals}\\ (m \text{ by } m)(m \text{ by } n)\end{array} \qquad A\begin{bmatrix} \\ \boldsymbol{v}_1 \cdots \boldsymbol{v}_r \cdots \boldsymbol{v}_n \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \boldsymbol{u}_1 \cdots \boldsymbol{u}_r \cdots \boldsymbol{u}_m \\ \\ \end{bmatrix}\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \\ & & & \end{bmatrix} \tag{3}$$

The new Σ is m by n. It is just the old r by r matrix (call that Σ_r) with $m - r$ new zero rows and $n - r$ new zero columns. The real change is in the shapes of U and V and Σ. Still $V^{\mathrm{T}}V = I$ and $U^{\mathrm{T}}U = I$, with sizes n and m.

V is now a square orthogonal matrix, with inverse $V^{-1} = V^{\mathrm{T}}$. So $AV = U\Sigma$ can become $\boldsymbol{A = U\Sigma V^{\mathrm{T}}}$. This is the ***Singular Value Decomposition***:

$$\textbf{SVD} \qquad A = U\Sigma V^{\mathrm{T}} = \boldsymbol{u}_1\sigma_1\boldsymbol{v}_1^{\mathrm{T}} + \cdots + \boldsymbol{u}_r\sigma_r\boldsymbol{v}_r^{\mathrm{T}}. \tag{4}$$

I would write the earlier "reduced SVD" from equation (2) as $A = U_r \Sigma_r V_r^{\mathrm{T}}$. That is equally true, without the extra zeros in Σ. This reduced SVD gives the same splitting of A into a sum of r matrices, each of rank one.

We will see that $\sigma_i^2 = \lambda_i$ is an eigenvalue of $A^{\mathrm{T}}A$ and also AA^{T}. When we put the singular values in descending order, $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > 0$, the splitting in equation (4) gives the r rank-one pieces of A ***in order of importance***.

Example 1 When is $U\Sigma V^{\mathrm{T}}$ (singular values) the same as $S\Lambda S^{-1}$ (eigenvalues)?

Solution We need orthonormal eigenvectors in $S = U$. We need nonnegative eigenvalues in $\Lambda = \Sigma$. So A must be a ***positive semidefinite (or definite) symmetric matrix*** $Q\Lambda Q^{\mathrm{T}}$.

Example 2 If $A = \boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$ with unit vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, what is the SVD of A?

Solution The reduced SVD in (2) is exactly $\boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$, with rank $r = 1$. It has $\boldsymbol{u}_1 = \boldsymbol{x}$ and $\boldsymbol{v}_1 = \boldsymbol{y}$ and $\sigma_1 = 1$. For the full SVD, complete $\boldsymbol{u}_1 = \boldsymbol{x}$ to an orthonormal basis of $\boldsymbol{u}$'s, and complete $\boldsymbol{v}_1 = \boldsymbol{y}$ to an orthonormal basis of $\boldsymbol{v}$'s. No new σ's.

I will describe an application before proving that $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$. This key equation gave the diagonalizations (2) and (3) and (4) of the SVD: $A = U\Sigma V^{\mathrm{T}}$.

Image Compression

Unusually, I am going to stop the theory and describe applications. This is the century of data, and often that data is stored in a matrix. *A digital image is really a matrix of pixel values.* Each little picture element or "pixel" has a gray scale number between black and white (it has three numbers for a color picture). The picture might have $512 = 2^9$ pixels in each row and $256 = 2^8$ pixels down each column. We have a 256 by 512 pixel matrix with 2^{17} entries! To store one picture, the computer has no problem. But a CT or MR scan produces an image at every cross section—a ton of data. If the pictures are frames in a movie, 30 frames a second means 108,000 images per hour. Compression is especially needed for high definition digital TV, or the equipment could not keep up in real time.

What is compression? We want to replace those 2^{17} matrix entries by a smaller number, *without losing picture quality*. A simple way would be to use larger pixels—replace groups of four pixels by their average value. This is $4:1$ compression. But if we carry it further, like $16:1$, our image becomes "blocky". We want to replace the mn entries by a smaller number, in a way that the human visual system won't notice.

Compression is a billion dollar problem and everyone has ideas. Later in this book I will describe Fourier transforms (used in **jpeg**) and wavelets (now in **JPEG2000**). Here we try an SVD approach: *Replace the* 256 *by* 512 *pixel matrix by* ***a matrix of rank one****: a column times a row.* If this is successful, the storage requirement becomes $256 + 512$ (add instead of multiply). The compression ratio $(256)(512)/(256 + 512)$ is better than 170 to 1. This is more than we hope for. We may actually use five matrices of rank one (so a matrix approximation of rank 5). The compression is still $34 : 1$ and the crucial question is the picture quality.

Where does the SVD come in? *The best rank one approximation to A is the matrix* $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}}$. It uses the largest singular value σ_1. The best rank 5 approximation includes also $\sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}} + \cdots + \sigma_5 \boldsymbol{u}_5 \boldsymbol{v}_5^{\mathrm{T}}$. **The SVD puts the pieces of A in descending order.**

A library compresses a different matrix. The rows correspond to key words. Columns correspond to titles in the library. The entry in this *word-title matrix* is $a_{ij} = 1$ if word i is in title j (otherwise $a_{ij} = 0$). We normalize the columns so long titles don't get an advantage. We might use a table of contents or an abstract. (Other books might share the title "*Introduction to Linear Algebra*".) Instead of $a_{ij} = 1$, the entries of A can include the *frequency* of the search words. See Section 8.6 for the SVD in statistics.

Once the indexing matrix is created, the search is a linear algebra problem. This giant matrix has to be compressed. The SVD approach gives an optimal low rank approximation, better for library matrices than for natural images. There is an ever-present tradeoff in the cost to compute the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s. We still need a better way (with sparse matrices).

The Bases and the SVD

Start with a 2 by 2 matrix. Let its rank be $r = 2$, so A is invertible. We want $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ to be perpendicular unit vectors. ***We also want $A\boldsymbol{v}_1$ and $A\boldsymbol{v}_2$ to be perpendicular.*** (This is the tricky part. It is what makes the bases special.) Then the unit vectors $\boldsymbol{u}_1 = A\boldsymbol{v}_1/\|A\boldsymbol{v}_1\|$ and $\boldsymbol{u}_2 = A\boldsymbol{v}_2/\|A\boldsymbol{v}_2\|$ will be orthonormal. Here is a specific example:

$$\textbf{Unsymmetric matrix} \qquad A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}. \tag{5}$$

No orthogonal matrix Q will make $Q^{-1}AQ$ diagonal. We need $U^{-1}AV$. The two bases will be different—one basis cannot do it. The output is $A\boldsymbol{v}_1 = \sigma_1 \boldsymbol{u}_1$ when the input is $\boldsymbol{v}_1$. The "singular values" σ_1 and σ_2 are the lengths $\|A\boldsymbol{v}_1\|$ and $\|A\boldsymbol{v}_2\|$.

$$\begin{matrix} \boldsymbol{AV = U\Sigma} \\ \boldsymbol{A = U\Sigma V^{\mathrm{T}}} \end{matrix} \qquad A\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 \boldsymbol{u}_1 & \sigma_2 \boldsymbol{u}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}. \tag{6}$$

There is a neat way to remove U and see V by itself. *Multiply A^{T} times A.*

$$A^{\mathrm{T}}A = (U\Sigma V^{\mathrm{T}})^{\mathrm{T}}(U\Sigma V^{\mathrm{T}}) = V\Sigma^{\mathrm{T}}\Sigma V^{\mathrm{T}}. \tag{7}$$

$U^{\mathrm{T}}U$ disappears because it equals I. (We require $\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{u}_1 = 1 = \boldsymbol{u}_2^{\mathrm{T}}\boldsymbol{u}_2$ and $\boldsymbol{u}_1^{\mathrm{T}}\boldsymbol{u}_2 = 0$.) Multiplying those diagonal Σ^{T} and Σ gives σ_1^2 and σ_2^2. That leaves an ordinary diagonalization of the crucial symmetric matrix $A^{\mathrm{T}}A$, whose eigenvalues are σ_1^2 and σ_2^2:

$$\begin{matrix} \textbf{Eigenvalues } \sigma_1^2, \sigma_2^2 \\ \textbf{Eigenvectors } \boldsymbol{v}_1, \boldsymbol{v}_2 \end{matrix} \qquad A^{\mathrm{T}}A = V\begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} V^{\mathrm{T}}. \tag{8}$$

This is exactly like $A = Q\Lambda Q^{\mathrm{T}}$. But the symmetric matrix is not A itself. Now the symmetric matrix is $A^{\mathrm{T}}A$! *And the columns of V are the eigenvectors of $A^{\mathrm{T}}A$.* Last is U:

Compute the eigenvectors v and eigenvalues σ^2 of $A^{\mathrm{T}}A$. Then each $u = Av/\sigma$.

For large matrices LAPACK finds a special way to avoid multiplying $A^{\mathrm{T}}A$ in **svd** (A).

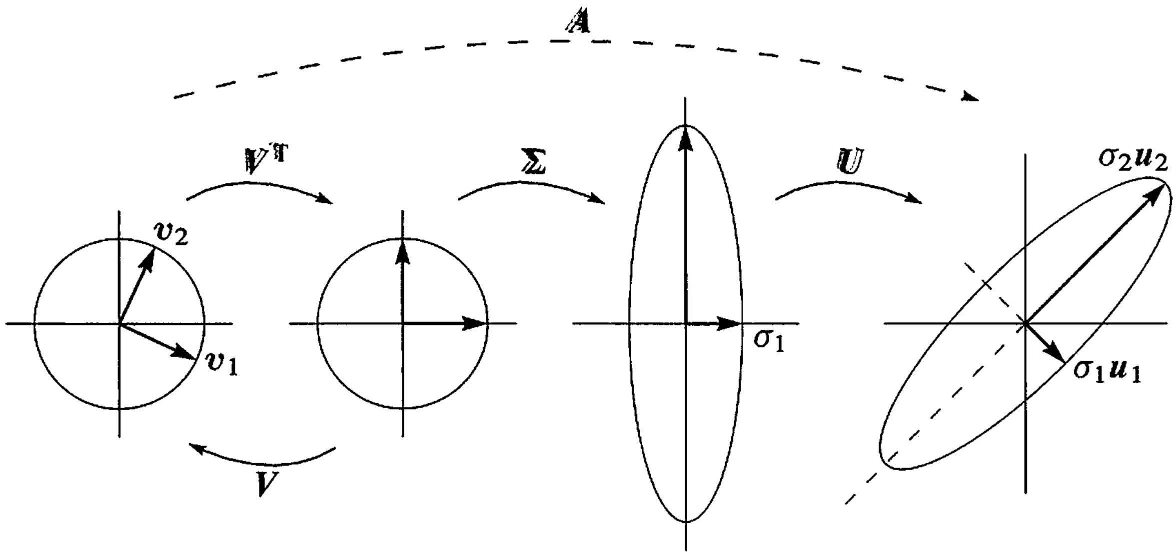


Figure 6.8: U and V are rotations and reflections. Σ stretches circle to ellipse.

Example 3 Find the singular value decomposition of that matrix $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

Solution Compute $A^{\mathrm{T}}A$ and its eigenvectors. Then make them unit vectors:

$$A^{\mathrm{T}}A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \quad \text{has unit eigenvectors} \quad v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

The eigenvalues of $A^{\mathrm{T}}A$ are 8 and 2. The v's are perpendicular, because eigenvectors of every symmetric matrix are perpendicular—and $A^{\mathrm{T}}A$ is automatically symmetric.

Now the u's are quick to find, because Av_1 is going to be in the direction of u_1:

$$Av_1 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}. \quad \text{The unit vector is} \quad u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Clearly Av_1 is the same as $2\sqrt{2}\,u_1$. The first singular value is $\sigma_1 = 2\sqrt{2}$. Then $\sigma_1^2 = 8$.

$$Av_2 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}. \quad \text{The unit vector is} \quad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now Av_2 is $\sqrt{2}\,u_2$ and $\sigma_2 = \sqrt{2}$. Thus σ_2^2 agrees with the other eigenvalue 2 of $A^{\mathrm{T}}A$.

$$A = U\Sigma V^{\mathrm{T}} \quad \text{is} \quad \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & \\ & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}. \tag{9}$$

This matrix, and every invertible 2 by 2 matrix, ***transforms the unit circle to an ellipse.*** You can see that in the figure, which was created by Cliff Long and Tom Hern.

One final point about that example. We found the $\boldsymbol{u}$'s from the $\boldsymbol{v}$'s. Could we find the $\boldsymbol{u}$'s directly? *Yes*, by multiplying AA^{T} instead of $A^{\mathrm{T}}A$:

$$\text{Use } V^{\mathrm{T}}V = I \qquad AA^{\mathrm{T}} = (U\Sigma V^{\mathrm{T}})(V\Sigma^{\mathrm{T}}U^{\mathrm{T}}) = U\Sigma\Sigma^{\mathrm{T}}U^{\mathrm{T}}. \tag{10}$$

Multiplying $\Sigma\Sigma^{\mathrm{T}}$ gives σ_1^2 and σ_2^2 as before. *The $\boldsymbol{u}$'s are eigenvectors of AA^{T}*:

$$\textbf{Diagonal in this example} \qquad AA^{\mathrm{T}} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}.$$

The eigenvectors $(1,0)$ and $(0,1)$ agree with $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ found earlier. Why take the first eigenvector to be $(1,0)$ instead of $(-1,0)$ or $(0,1)$? Because we have to follow $A\boldsymbol{v}_1$ (I missed that in my video lecture ...). Notice that AA^{T} has the same eigenvalues (8 and 2) as $A^{\mathrm{T}}A$. The singular values are $\sqrt{8}$ and $\sqrt{2}$.

Example 4 Find the SVD of the singular matrix $A = \left[\begin{smallmatrix} 2 & 2 \\ 1 & 1 \end{smallmatrix}\right]$. The rank is $r = 1$.

Solution The row space has only one basis vector $\boldsymbol{v}_1 = (1,1)/\sqrt{2}$. The column space has only one basis vector $\boldsymbol{u}_1 = (2,1)/\sqrt{5}$. Then $A\boldsymbol{v}_1 = (4,2)/\sqrt{2}$ must equal $\sigma_1\boldsymbol{u}_1$. It does, with $\sigma_1 = \sqrt{10}$.

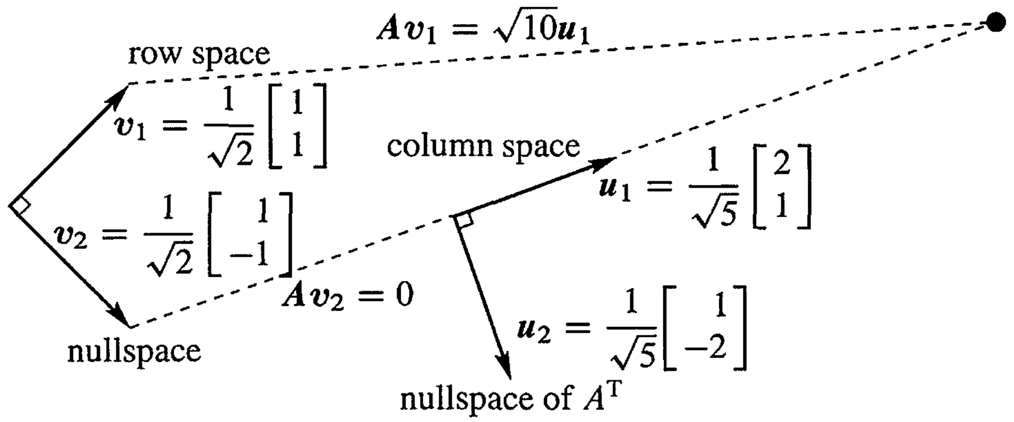


Figure 6.9: The SVD chooses orthonormal bases for 4 subspaces so that $A\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i$.

The SVD could stop after the row space and column space (it usually doesn't). It is customary for U and V to be square. The matrices need a second column. ***The vector $\boldsymbol{v}_2$ is in the nullspace.*** It is perpendicular to $\boldsymbol{v}_1$ in the row space. Multiply by A to get $A\boldsymbol{v}_2 = \mathbf{0}$. We could say that the second singular value is $\sigma_2 = 0$, but singular values are like pivots—only the r nonzeros are counted.

$$\begin{array}{l} A = U\Sigma V^{\mathrm{T}} \\ \textbf{Full size} \end{array} \qquad \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{11}$$

The matrices U and V contain orthonormal bases for all four subspaces:

first	r	columns of V :	*row space* of A
last	$n-r$	columns of V :	*nullspace* of A
first	r	columns of U :	*column space* of A
last	$m-r$	columns of U :	*nullspace* of A^{T}

The first columns $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ are eigenvectors of $A^{\mathrm{T}}A$ and AA^{T}. We now explain why $A\boldsymbol{v}_i$ falls in the direction of $\boldsymbol{u}_i$. The last $\boldsymbol{v}$'s and $\boldsymbol{u}$'s (in the nullspaces) are easier. As long as those are orthonormal, the SVD will be correct.

Proof of the SVD: Start from $A^{\mathrm{T}}A\boldsymbol{v}_i = \sigma_i^2\boldsymbol{v}_i$, which gives the $\boldsymbol{v}$'s and σ's. Multiplying by $\boldsymbol{v}_i^{\mathrm{T}}$ leads to $\|A\boldsymbol{v}_i\|^2$. To prove that $A\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i$, the key step is to multiply by A:

$$\boldsymbol{v}_i^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{v}_i = \sigma_i^2\boldsymbol{v}_i^{\mathrm{T}}\boldsymbol{v}_i \quad \textbf{gives} \quad \|A\boldsymbol{v}_i\|^2 = \sigma_i^2 \quad \textbf{so that} \quad \|A\boldsymbol{v}_i\| = \sigma_i \tag{12}$$

$$AA^{\mathrm{T}}A\boldsymbol{v}_i = \sigma_i^2 A\boldsymbol{v}_i \quad \textbf{gives} \quad \boldsymbol{u}_i = A\boldsymbol{v}_i/\sigma_i \quad \textbf{as a unit eigenvector of } AA^{\mathrm{T}}. \tag{13}$$

Equation (12) used the small trick of placing parentheses in $(\boldsymbol{v}_i^{\mathrm{T}}A^{\mathrm{T}})(A\boldsymbol{v}_i) = \|A\boldsymbol{v}_i\|^2$. Equation (13) placed the all-important parentheses in $(AA^{\mathrm{T}})(A\boldsymbol{v}_i)$. This shows that $A\boldsymbol{v}_i$ is an eigenvector of AA^{T}. Divide by its length σ_i to get the unit vector $\boldsymbol{u}_i = A\boldsymbol{v}_i/\sigma_i$. These $\boldsymbol{u}$'s are orthogonal because $(A\boldsymbol{v}_i)^{\mathrm{T}}(A\boldsymbol{v}_j) = \boldsymbol{v}_i^{\mathrm{T}}(A^{\mathrm{T}}A\boldsymbol{v}_j) = \boldsymbol{v}_i^{\mathrm{T}}(\sigma_j^2\boldsymbol{v}_j) = 0$.

I will give my opinion directly. The SVD is the climax of this linear algebra course. I think of it as the final step in the Fundamental Theorem. First come the *dimensions* of the four subspaces. Then their *orthogonality*. Then *the orthonormal bases diagonalize A*. It is all in the formula $A = U\Sigma V^{\mathrm{T}}$. You have made it to the top.

Eigshow (Part 2)

Section 6.1 described the MATLAB demo called **eigshow**. The first option is *eig*, when $\boldsymbol{x}$ moves in a circle and $A\boldsymbol{x}$ follows on an ellipse. The second option is *svd*, when two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ stay perpendicular as they travel around a circle. Then $A\boldsymbol{x}$ and $A\boldsymbol{y}$ move too (not usually perpendicular). The four vectors are in the Java demo on **web.mit.edu/18.06**.

The SVD is seen graphically when $A\boldsymbol{x}$ is perpendicular to $A\boldsymbol{y}$. Their directions at that moment give an orthonormal basis $\boldsymbol{u}_1, \boldsymbol{u}_2$. Their lengths give the singular values σ_1, σ_2. The vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ at that same moment are the orthonormal basis $\boldsymbol{v}_1, \boldsymbol{v}_2$.

Searching the Web

I will end with an application of the SVD to web search engines. When you google a word, you get a list of web sites in order of importance. You could try "four subspaces".

The HITS algorithm that we describe is one way to produce that ranked list. It begins with about 200 sites found from an index of key words, and after that we look only at links between pages. Search engines are link-based more than content-based.

Start with the 200 sites and all sites that link to them and all sites they link to. That is our list, to be put in order. Importance can be measured by links out and links in.

1. The site is an ***authority***: *links come in from many sites.* Especially from hubs.

2. The site is a ***hub***: *links go out to many sites in the list.* Especially to authorities.

We want numbers $x_1, \ldots, x_N$ to rank the authorities and $y_1, \ldots, y_N$ to rank the hubs. Start with a simple count: x_i^0 and y_i^0 count the links into and out of site i.

Here is the point: *A good authority has links from important sites* (like hubs). Links from universities count more heavily than links from friends. *A good hub is linked to important sites* (like authorities). A link to **amazon.com** unfortunately means more than a link to **wellesleycambridge.com**. The rankings x^0 and y^0 from counting links are updated to x^1 and y^1 by taking account of *good* links (measuring their quality by x^0 and y^0):

$$\textbf{Authority values} \quad x_i^1 = \sum_{j \text{ links to } i} y_j^0 \qquad \textbf{Hub values} \quad y_i^1 = \sum_{i \text{ links to } j} x_j^0 \tag{14}$$

In matrix language those are $x^1 = A^{\mathrm{T}}y^0$ and $y^1 = Ax^0$. The matrix A contains 1's and 0's, with $a_{ij} = 1$ when i links to j. In the language of graphs, A is an "adjacency matrix" for the World Wide Web (an enormous matrix). The new x^1 and y^1 give better rankings, but not the best. Take another step like (14), to reach x^2 and y^2:

$$\textbf{\textit{A}}^{\mathrm{T}}\textbf{\textit{A}} \textbf{ and } \textbf{\textit{AA}}^{\mathrm{T}} \textbf{ appear} \quad x^2 = A^{\mathrm{T}}y^1 = A^{\mathrm{T}}Ax^0 \quad \text{and} \quad y^2 = A^{\mathrm{T}}x^1 = AA^{\mathrm{T}}y^0. \tag{15}$$

In two steps we are multiplying by $A^{\mathrm{T}}A$ and AA^{T}. Twenty steps will multiply by $(A^{\mathrm{T}}A)^{10}$ and $(AA^{\mathrm{T}})^{10}$. When we take powers, the largest eigenvalue σ_1^2 begins to dominate. And the vectors x and y line up with the leading eigenvectors v_1 and u_1 of $A^{\mathrm{T}}A$ and AA^{T}. We are computing the top terms in the SVD, by the *power method* that is discussed in Section 9.3. It is wonderful that linear algebra helps to understand the Web.

Google actually creates rankings by a random walk that follows web links. The more often this random walk goes to a site, the higher the ranking. The frequency of visits gives the leading eigenvector ($\lambda = 1$) of the normalized adjacency matrix for the Web. ***That Markov matrix has 2.7 billion rows and columns, from 2.7 billion web sites.***

This is the largest eigenvalue problem ever solved. The excellent book by Langville and Meyer, *Google's PageRank and Beyond*, explains in detail the science of search engines. See `mathworks.com/company/newsletter/clevescorner/oct02_cleve.shtml`

But many of the important techniques are well-kept secrets of Google. Probably Google starts with last month's eigenvector as a first approximation, and runs the random walk very fast. To get a high ranking, you want a lot of links from important sites. The HITS algorithm is described in the 1999 *Scientific American* (June 16). But I don't think the SVD is mentioned there...

■ REVIEW OF THE KEY IDEAS ■

1. The SVD factors A into $U\Sigma V^{\mathrm{T}}$, with r singular values $\sigma_1 \geq \ldots \geq \sigma_r > 0$.

2. The numbers $\sigma_1^2, \ldots, \sigma_r^2$ are the nonzero eigenvalues of AA^{T} and $A^{\mathrm{T}}A$.
3. The orthonormal columns of U and V are eigenvectors of AA^{T} and $A^{\mathrm{T}}A$.
4. Those columns hold orthonormal bases for the four fundamental subspaces of A.
5. Those bases diagonalize the matrix: $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ for $i \leq r$. This is $AV = U\Sigma$.

■ WORKED EXAMPLES ■

6.7 A Identify by name these decompositions $A = \boldsymbol{c}_1\boldsymbol{b}_1 + \cdots + \boldsymbol{c}_r\boldsymbol{b}_r$ of an m by n matrix. Each term is a rank one matrix (column $\boldsymbol{c}$ times row $\boldsymbol{b}$). The rank of A is r.

1. *Orthogonal* columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r$ and *orthogonal* rows $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_r$.
2. *Orthogonal* columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r$ and *triangular* rows $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_r$.
3. *Triangular* columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r$ and *triangular* rows $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_r$.

$A = CB$ is (m by r)(r by n). Triangular vectors $\boldsymbol{c}_i$ and $\boldsymbol{b}_i$ have zeros up to component i. The matrix C with columns $\boldsymbol{c}_i$ is lower triangular, the matrix B with rows $\boldsymbol{b}_i$ is upper triangular. Where do the rank and the pivots and singular values come into this picture?

Solution These three splittings $A = CB$ are basic to linear algebra, pure or applied:

1. **Singular Value Decomposition** $A = U\Sigma V^{\mathrm{T}}$ (*orthogonal* U*, orthogonal* ΣV^{T})
2. **Gram-Schmidt Orthogonalization** $A = QR$ (*orthogonal* Q*, triangular* R)
3. **Gaussian Elimination** $A = LU$ (*triangular* L*, triangular* U)

You might prefer to separate out the σ_i and pivots d_i and heights h_i:

1. $A = U\Sigma V^{\mathrm{T}}$ with unit vectors in U and V. The singular values are in Σ.
2. $A = QHR$ with unit vectors in Q and diagonal 1's in R. The heights h_i are in H.
3. $A = LDU$ with diagonal 1's in L and U. The pivots are in D.

Each h_i tells the height of column i above the base from earlier columns. The volume of the full n-dimensional box ($r = m = n$) comes from $A = U\Sigma V^{\mathrm{T}} = LDU = QHR$:

$$|\,\mathbf{det}\,A\,| = |\,\textbf{\textit{product of}}\ \boldsymbol{\sigma}\text{'s}\,| = |\,\textbf{\textit{product of}}\ \boldsymbol{d}\text{'s}\,| = |\,\textbf{\textit{product of}}\ \boldsymbol{h}\text{'s}\,|.$$

6.7.B For $A = \boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$ of rank one (2 by 2), compare $A = U\Sigma V^{\mathrm{T}}$ with $A = S\Lambda S^{-1}$.

Comment This started as an exam problem in 2007. It led further and became interesting. Now there is an essay called "The Four Fundamental Subspaces: 4 Lines" on **web.mit.edu/18.06**. The Jordan form enters when $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} = 0$ and $\lambda = 0$ is repeated.

6.7.C Show that $\sigma_1 \geq |\lambda|_{\max}$. The largest singular value dominates all eigenvalues.
Show that $\sigma_1 \geq |a_{ij}|_{\max}$. The largest singular value dominates all entries of A.

Solution Start from $A = U\Sigma V^{\mathrm{T}}$. Remember that multiplying by an orthogonal matrix *does not change length*: $\|Qx\| = \|x\|$ because $\|Qx\|^2 = x^{\mathrm{T}}Q^{\mathrm{T}}Qx = x^{\mathrm{T}}x = \|x\|^2$. This applies to $Q = U$ and $Q = V^{\mathrm{T}}$. In between is the diagonal matrix Σ.

$$\|Ax\| = \|U\Sigma V^{\mathrm{T}}x\| = \|\Sigma V^{\mathrm{T}}x\| \leq \sigma_1 \|V^{\mathrm{T}}x\| = \sigma_1 \|x\|. \tag{16}$$

An eigenvector has $\|Ax\| = |\lambda|\|x\|$. So (16) says that $|\lambda|\|x\| \leq \sigma_1\|x\|$. Then $|\lambda| \leq \sigma_1$.

Apply also to the unit vector $x = (1, 0, \ldots, 0)$. Now Ax is the first column of A. Then by inequality (16), this column has length $\leq \sigma_1$. Every entry must have magnitude $\leq \sigma_1$.

Example 5 Estimate the singular values σ_1 and σ_2 of A and A^{-1}:

$$\textbf{Eigenvalues} = \mathbf{1} \qquad A = \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 1 & 0 \\ -C & 1 \end{bmatrix}. \tag{17}$$

Solution The length of the first column is $\sqrt{1 + C^2} \leq \sigma_1$, from the reasoning above. This confirms that $\sigma_1 \geq 1$ and $\sigma_1 \geq C$. Then σ_1 dominates the eigenvalues $1, 1$ and the entry C. If C is very large then σ_1 is much bigger than the eigenvalues.

This matrix A has determinant $= 1$. $A^{\mathrm{T}}A$ also has determinant $= 1$ and then $\sigma_1\sigma_2 = 1$. For this matrix, $\sigma_1 \geq 1$ and $\sigma_1 \geq C$ lead to $\sigma_2 \leq 1$ and $\sigma_2 \leq 1/C$.

Conclusion: If $C = 1000$ then $\sigma_1 \geq 1000$ and $\sigma_2 \leq 1/1000$. A ***is ill-conditioned***, slightly sick. Inverting A is easy by algebra, but solving $Ax = b$ by elimination could be dangerous. *A is close to a singular matrix even though both eigenvalues are $\lambda = 1$.* By slightly changing the $1, 2$ entry from zero to $1/C = 1/1000$, the matrix becomes singular.

Section 9.2 will explain how the ratio $\sigma_{\max}/\sigma_{\min}$ governs the roundoff error in elimination. MATLAB warns you if this "*condition number*" is large. Here $\sigma_1/\sigma_2 \geq C^2$.

Problem Set 6.7

Problems 1–3 compute the SVD of a square singular matrix A.

1 Find the eigenvalues and unit eigenvectors v_1, v_2 of $A^{\mathrm{T}}A$. Then find $u_1 = Av_1/\sigma_1$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} \quad \text{and} \quad AA^{\mathrm{T}} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}.$$

Verify that u_1 is a unit eigenvector of AA^{T}. Complete the matrices U, Σ, V.

$$\textbf{SVD} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}}.$$

2 Write down orthonormal bases for the four fundamental subspaces of this A.

3 (a) Why is the trace of $A^{\mathrm{T}}A$ equal to the sum of all a_{ij}^2 ?

(b) For every rank-one matrix, why is σ_1^2 = sum of all a_{ij}^2?

Problems 4–7 ask for the SVD of matrices of rank 2.

4 Find the eigenvalues and unit eigenvectors of $A^{\mathrm{T}}A$ and AA^{T}. Keep each $A\boldsymbol{v} = \sigma\boldsymbol{u}$:

$$\textbf{Fibonacci matrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Construct the singular value decomposition and verify that A equals $U\Sigma V^{\mathrm{T}}$.

5 Use the **svd** part of the MATLAB demo **eigshow** to find those $\boldsymbol{v}$'s graphically.

6 Compute $A^{\mathrm{T}}A$ and AA^{T} and their eigenvalues and unit eigenvectors for V and U.

$$\textbf{Rectangular matrix} \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check $AV = U\Sigma$ (this will decide $\pm$ signs in U). Σ has the same shape as A.

7 What is the closest rank-one approximation to that 2 by 3 matrix?

8 A square invertible matrix has $A^{-1} = V\Sigma^{-1}U^{\mathrm{T}}$. This says that the *singular values of A^{-1} are $1/\sigma(A)$*. Show that $\sigma_{\max}(A^{-1})\,\sigma_{\max}(A) \geq 1$.

9 Suppose $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are orthonormal bases for $\mathbf{R}^n$. Construct the matrix A that transforms each $\boldsymbol{v}_j$ into $\boldsymbol{u}_j$ to give $A\boldsymbol{v}_1 = \boldsymbol{u}_1, \ldots, A\boldsymbol{v}_n = \boldsymbol{u}_n$.

10 Construct the matrix with rank one that has $A\boldsymbol{v} = 12\boldsymbol{u}$ for $\boldsymbol{v} = \frac{1}{2}(1,1,1,1)$ and $\boldsymbol{u} = \frac{1}{3}(2,2,1)$. Its only singular value is $\sigma_1 =$ _____.

11 Suppose A has orthogonal columns $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ of lengths $\sigma_1, \sigma_2, \ldots, \sigma_n$. What are U, Σ, and V in the SVD?

12 Suppose A is a 2 by 2 symmetric matrix with unit eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. If its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$, what are the matrices U, Σ, V^{T} in its SVD?

13 If $A = QR$ with an orthogonal matrix Q, the SVD of A is almost the same as the SVD of R. Which of the three matrices U, Σ, V is changed because of Q?

14 Suppose A is invertible (with $\sigma_1 > \sigma_2 > 0$). Change A by ***as small a matrix as possible*** to produce a singular matrix A_0. Hint: U and V do not change:

$$\text{From} \qquad A = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}} \qquad \text{find the nearest } A_0.$$

15 Why doesn't the SVD for $A+I$ just use $\Sigma+I$?

Challenge Problems

16 (Search engine) Run a **random walk** $x(2),\ldots,x(n)$ starting from web site $x(1)=1$. Count the visits to each site. At each step the code chooses the next website $x(k)$ with probabilities given by column $x(k-1)$ of A. At the end, $\boldsymbol{p}$ gives the fraction of time at each site from a histogram: count visits. ***The rankings are based on $\boldsymbol{p}$.***

Please compare $\boldsymbol{p}$ to the steady state eigenvector of the Markov matrix A:

$$A=[0\ \ .1\ \ .2\ \ .7;\ \ .05\ \ 0\ \ .15\ \ .8;\ \ .15\ \ .25\ \ 0\ \ .6;\ \ .1\ \ .3\ \ .6\ \ 0]'$$

$n=100$; $x=\text{zeros}(1,n)$; $x(1)=1$;
for $k=2:n$ $\;x(k)=\text{min(find(rand<cumsum}(A(:,x(k-1)))))$; end
$p=\text{hist}(x,1:4)/n$

17 The $1,-1$ first difference matrix A has $A^{\mathrm{T}}A$ = second difference matrix. The singular vectors of A are *sine vectors $\boldsymbol{v}$* and *cosine vectors $\boldsymbol{u}$*. Then $A\boldsymbol{v}=\sigma\boldsymbol{u}$ is the discrete form of $d/dx(\sin cx)=c(\cos cx)$. This is the best SVD I have seen.

$$\textbf{SVD of } A \qquad A=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\\0&0&-1\end{bmatrix} \qquad A^{\mathrm{T}}A=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$$

$$\textit{Orthogonal sine matrix} \qquad V=\frac{1}{\sqrt{2}}\begin{bmatrix}\sin\pi/4&\sin2\pi/4&\sin3\pi/4\\\sin2\pi/4&\sin4\pi/4&\sin6\pi/4\\\sin3\pi/4&\sin6\pi/4&\sin9\pi/4\end{bmatrix}$$

(a) Put numbers in V: The unit eigenvectors of $A^{\mathrm{T}}A$ are singular vectors of A. Show that the columns of V have $A^{\mathrm{T}}A\boldsymbol{v}=\lambda\boldsymbol{v}$ with $\lambda=2-\sqrt{2},\ 2,\ 2+\sqrt{2}$.

(b) Multiply AV and verify that its columns are orthogonal. They are $\sigma_1\boldsymbol{u}_1$ and $\sigma_2\boldsymbol{u}_2$ and $\sigma_3\boldsymbol{u}_3$. The first columns of the cosine matrix U are $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3$.

(c) Since A is 4 by 3, we need a fourth orthogonal vector $\boldsymbol{u}_4$. It comes from the nullspace of A^{T}. What is $\boldsymbol{u}_4$?

The cosine vectors in U are eigenvectors of AA^{T}. The fourth cosine is $(1,1,1,1)/2$.

$$AA^{\mathrm{T}}=\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix} \qquad U=\frac{1}{\sqrt{2}}\begin{bmatrix}\cos\pi/8&\cos2\pi/8&\cos3\pi/8\\\cos3\pi/8&\cos6\pi/8&\cos9\pi/8\\\cos5\pi/8&\cos10\pi/8&\cos15\pi/8\\\cos7\pi/8&\cos14\pi/8&\cos21\pi/8\end{bmatrix}$$

Those angles $\pi/8, 3\pi/8, 5\pi/8, 7\pi/8$ fit 4 points with spacing $\pi/4$ between 0 and π. The sine transform has three points $\pi/4, 2\pi/4, 3\pi/4$. The full cosine transform includes $\boldsymbol{u}_4$ from the "zero frequency" or *direct current* eigenvector $(1,1,1,1)$.

The 8 by 8 cosine transform in 2D is the workhorse of **jpeg compression**. Linear algebra (circulant, Toeplitz, orthogonal matrices) is at the heart of signal processing.

Table of Eigenvalues and Eigenvectors

How are the properties of a matrix reflected in its eigenvalues and eigenvectors? This question is fundamental throughout Chapter 6. A table that organizes the key facts may be helpful. Here are the special properties of the eigenvalues λ_i and the eigenvectors x_i.

Symmetric: $A^{\mathrm{T}} = A$	real λ's	orthogonal $x_i^{\mathrm{T}} x_j = 0$
Orthogonal: $Q^{\mathrm{T}} = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Skew-symmetric: $A^{\mathrm{T}} = -A$	imaginary λ's	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Complex Hermitian: $\overline{A}^{\mathrm{T}} = A$	real λ's	orthogonal $\overline{x}_i^{\mathrm{T}} x_j = 0$
Positive Definite: $x^{\mathrm{T}} A x > 0$	all $\lambda > 0$	orthogonal since $A^{\mathrm{T}} = A$
Markov: $m_{ij} > 0, \sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $x > 0$
Similar: $B = M^{-1}AM$	$\lambda(B) = \lambda(A)$	$x(B) = M^{-1}x(A)$
Projection: $P = P^2 = P^{\mathrm{T}}$	$\lambda = 1;\ 0$	column space; nullspace
Plane Rotation	$e^{i\theta}$ and $e^{-i\theta}$	$x = (1, i)$ and $(1, -i)$
Reflection: $I - 2uu^{\mathrm{T}}$	$\lambda = -1;\ 1, .., 1$	u; whole plane $u^{\perp}$
Rank One: uv^{T}	$\lambda = v^{\mathrm{T}}u;\ 0, .., 0$	u; whole plane $v^{\perp}$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Stable Exponential: $e^{At} \to 0$	all Re $\lambda < 0$	any eigenvectors
Cyclic Permutation: row 1 of I last	$\lambda_k = e^{2\pi i k/n}$	$x_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Tridiagonal: $-1, 2, -1$ on diagonals	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$x_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots\right)$
Diagonalizable: $A = S\Lambda S^{-1}$	diagonal of Λ	columns of S are independent
Symmetric: $A = Q\Lambda Q^{\mathrm{T}}$	diagonal of Λ (real)	columns of Q are orthonormal
Schur: $A = QTQ^{-1}$	diagonal of T	columns of Q *if* $A^{\mathrm{T}}A = AA^{\mathrm{T}}$
Jordan: $J = M^{-1}AM$	diagonal of J	each block gives $x = (0, .., 1, .., 0)$
Rectangular: $A = U\Sigma V^{\mathrm{T}}$	$\mathrm{rank}(A) = \mathrm{rank}(\Sigma)$	eigenvectors of $A^{\mathrm{T}}A, AA^{\mathrm{T}}$ in V, U

Chapter 7

Linear Transformations

7.1 The Idea of a Linear Transformation

When a matrix A multiplies a vector $\boldsymbol{v}$, it "transforms" $\boldsymbol{v}$ into another vector $A\boldsymbol{v}$. ***In goes*** $\boldsymbol{v}$, ***out comes*** $T(\boldsymbol{v}) = A\boldsymbol{v}$. A transformation T follows the same idea as a function. In goes a number x, out comes $f(x)$. For one vector $\boldsymbol{v}$ or one number x, we multiply by the matrix or we evaluate the function. The deeper goal is to see all $\boldsymbol{v}$'s at once. We are transforming the whole space $\mathbf{V}$ when we multiply every $\boldsymbol{v}$ by A.

Start again with a matrix A. It transforms $\boldsymbol{v}$ to $A\boldsymbol{v}$. It transforms $\boldsymbol{w}$ to $A\boldsymbol{w}$. Then we *know* what happens to $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$. There is no doubt about $A\boldsymbol{u}$, it has to equal $A\boldsymbol{v} + A\boldsymbol{w}$. Matrix multiplication $T(\boldsymbol{v}) = A\boldsymbol{v}$ gives a ***linear transformation***:

A ***transformation*** T assigns an output $T(\boldsymbol{v})$ to each input vector $\boldsymbol{v}$ in $\mathbf{V}$. The transformation is ***linear*** if it meets these requirements for all $\boldsymbol{v}$ and $\boldsymbol{w}$:

(a) $T(\boldsymbol{v} + \boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ (b) $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ for all c.

If the input is $\boldsymbol{v} = \mathbf{0}$, the output must be $T(\boldsymbol{v}) = \mathbf{0}$. We combine (a) and (b) into one:

$$\textbf{Linear transformation} \qquad T(c\boldsymbol{v} + d\boldsymbol{w}) \quad \textbf{\textit{must equal}} \quad cT(\boldsymbol{v}) + dT(\boldsymbol{w}).$$

Again I can test matrix multiplication for linearity: $A(c\boldsymbol{v} + d\boldsymbol{w}) = cA\boldsymbol{v} + dA\boldsymbol{w}$ is *true*.

A linear transformation is highly restricted. Suppose T adds $\boldsymbol{u}_0$ to every vector. Then $T(\boldsymbol{v}) = \boldsymbol{v} + \boldsymbol{u}_0$ and $T(\boldsymbol{w}) = \boldsymbol{w} + \boldsymbol{u}_0$. This isn't good, or at least *it isn't linear*. Applying T to $\boldsymbol{v} + \boldsymbol{w}$ produces $\boldsymbol{v} + \boldsymbol{w} + \boldsymbol{u}_0$. That is not the same as $T(\boldsymbol{v}) + T(\boldsymbol{w})$:

$$\textbf{Shift is not linear} \qquad \boldsymbol{v} + \boldsymbol{w} + \boldsymbol{u}_0 \quad \text{is not} \quad T(\boldsymbol{v}) + T(\boldsymbol{w}) = \boldsymbol{v} + \boldsymbol{u}_0 + \boldsymbol{w} + \boldsymbol{u}_0.$$

The exception is when $\boldsymbol{u}_0 = \mathbf{0}$. The transformation reduces to $T(\boldsymbol{v}) = \boldsymbol{v}$. This is the ***identity transformation*** (nothing moves, as in multiplication by the identity matrix). That is certainly linear. In this case the input space $\mathbf{V}$ is the same as the output space $\mathbf{W}$.

The linear-plus-shift transformation $T(\boldsymbol{v}) = A\boldsymbol{v} + \boldsymbol{u}_0$ is called *"affine"*. Straight lines stay straight although T is not linear. Computer graphics works with affine transformations in Section 8.6, because we must be able to move images.

Example 1 Choose a fixed vector $\boldsymbol{a} = (1,3,4)$, and let $T(\boldsymbol{v})$ be the dot product $\boldsymbol{a} \cdot \boldsymbol{v}$:

$$\text{The input is} \quad \boldsymbol{v} = (v_1, v_2, v_3). \qquad \text{The output is} \quad T(\boldsymbol{v}) = \boldsymbol{a} \cdot \boldsymbol{v} = v_1 + 3v_2 + 4v_3.$$

This is linear. The inputs $\boldsymbol{v}$ come from three-dimensional space, so $\mathbf{V} = \mathbf{R}^3$. The outputs are just numbers, so the output space is $\mathbf{W} = \mathbf{R}^1$. We are multiplying by the row matrix $A = [1 \;\; 3 \;\; 4]$. Then $T(\boldsymbol{v}) = A\boldsymbol{v}$.

You will get good at recognizing which transformations are linear. If the output involves squares or products or lengths, v_1^2 or $v_1 v_2$ or $\|\boldsymbol{v}\|$, then T is not linear.

Example 2 The length $T(\boldsymbol{v}) = \|\boldsymbol{v}\|$ is not linear. Requirement (a) for linearity would be $\|\boldsymbol{v} + \boldsymbol{w}\| = \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$. Requirement (b) would be $\|c\boldsymbol{v}\| = c\|\boldsymbol{v}\|$. Both are false!

Not (a): The sides of a triangle satisfy an *inequality* $\|\boldsymbol{v} + \boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

Not (b): The length $\| - \boldsymbol{v}\|$ is not $-\|\boldsymbol{v}\|$. For negative c, we fail.

Example 3 (Important) T is the transformation that *rotates every vector by* 30°. The "*domain*" is the xy plane (all input vectors $\boldsymbol{v}$). The "*range*" is also the xy plane (all rotated vectors $T(\boldsymbol{v})$). We described T without a matrix: rotate by 30°.

Is rotation linear? *Yes it is.* We can rotate two vectors and add the results. The sum of rotations $T(\boldsymbol{v}) + T(\boldsymbol{w})$ is the same as the rotation $T(\boldsymbol{v} + \boldsymbol{w})$ of the sum. The whole plane is turning together, in this linear transformation.

Lines to Lines, Triangles to Triangles

Figure 7.1 shows the line from $\boldsymbol{v}$ to $\boldsymbol{w}$ in the input space. It also shows the line from $T(\boldsymbol{v})$ to $T(\boldsymbol{w})$ in the output space. Linearity tells us: Every point on the input line goes onto the output line. And more than that: ***Equally spaced points go to equally spaced points.*** The middle point $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$ goes to the middle point $T(\boldsymbol{u}) = \frac{1}{2}T(\boldsymbol{v}) + \frac{1}{2}T(\boldsymbol{w})$.

The second figure moves up a dimension. Now we have three corners $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$. Those inputs have three outputs $T(\boldsymbol{v}_1)$, $T(\boldsymbol{v}_2)$, $T(\boldsymbol{v}_3)$. *The input triangle goes onto the output triangle.* Equally spaced points stay equally spaced (along the edges, and then between the edges). The middle point $\boldsymbol{u} = \frac{1}{3}(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3)$ goes to the middle point $T(\boldsymbol{u}) = \frac{1}{3}(T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2) + T(\boldsymbol{v}_3))$.

The rule of linearity extends to combinations of three vectors or n vectors:

$$\textbf{Linearity} \qquad \begin{array}{c} \boldsymbol{u} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n \quad \textbf{transforms to} \\ T(\boldsymbol{u}) = c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + \cdots + c_nT(\boldsymbol{v}_n) \end{array} \qquad (1)$$

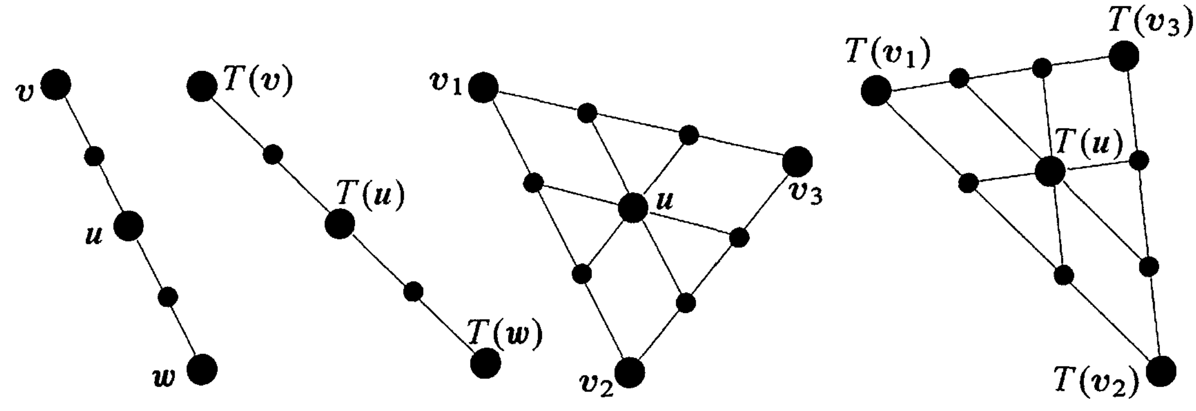


Figure 7.1: Lines to lines, equal spacing to equal spacing, $\boldsymbol{u} = \mathbf{0}$ to $T(\boldsymbol{u}) = \mathbf{0}$.

Note Transformations have a language of their own. Where there is no matrix, we can't talk about a column space. But the idea can be rescued. The column space consisted of all outputs $A\boldsymbol{v}$. The nullspace consisted of all inputs for which $A\boldsymbol{v} = \mathbf{0}$. Translate those into "range" and "kernel":

Range of T = set of all outputs $T(\boldsymbol{v})$: range corresponds to column space

Kernel of T = set of all inputs for which $T(\boldsymbol{v}) = \mathbf{0}$: kernel corresponds to nullspace.

The range is in the output space $\mathbf{W}$. The kernel is in the input space $\mathbf{V}$. When T is multiplication by a matrix, $T(\boldsymbol{v}) = A\boldsymbol{v}$, you can translate to column space and nullspace.

Examples of Transformations (mostly linear)

Example 4 Project every 3-dimensional vector straight down onto the xy plane. Then $T(x, y, z) = (x, y, 0)$. The range is that plane, which contains every $T(\boldsymbol{v})$. The kernel is the z axis (which projects down to zero). This projection is linear.

Example 5 Project every 3-dimensional vector onto the horizontal plane $z = 1$. The vector $\boldsymbol{v} = (x, y, z)$ is transformed to $T(\boldsymbol{v}) = (x, y, 1)$. This transformation is not linear. Why not? It doesn't even transform $\boldsymbol{v} = \mathbf{0}$ into $T(\boldsymbol{v}) = \mathbf{0}$.

Multiply every 3-dimensional vector by a 3 by 3 matrix A. This $T(\boldsymbol{v}) = A\boldsymbol{v}$ is linear.

$$T(\boldsymbol{v} + \boldsymbol{w}) = A(\boldsymbol{v} + \boldsymbol{w}) \quad \textbf{does equal} \quad A\boldsymbol{v} + A\boldsymbol{w} = T(\boldsymbol{v}) + T(\boldsymbol{w}).$$

Example 6 Suppose A is an *invertible matrix*. The kernel of T is the zero vector; the range $\mathbf{W}$ equals the domain $\mathbf{V}$. Another linear transformation is multiplication by A^{-1}. This is the ***inverse transformation*** T^{-1}, which brings every vector $T(\boldsymbol{v})$ back to $\boldsymbol{v}$:

$$T^{-1}(T(\boldsymbol{v})) = \boldsymbol{v} \quad \text{matches the matrix multiplication} \quad A^{-1}(A\boldsymbol{v}) = \boldsymbol{v}.$$

We are reaching an unavoidable question. ***Are all linear transformations from*** $\mathbf{V} = \mathbf{R}^n$ ***to*** $\mathbf{W} = \mathbf{R}^m$ ***produced by matrices?*** When a linear T is described as a "rotation" or "projection" or "...", is there always a matrix hiding behind T?

The answer is *yes*. This is an approach to linear algebra that doesn't start with matrices. The next section shows that we still end up with matrices.

Linear Transformations of the Plane

It is more interesting to *see* a transformation than to define it. When a 2 by 2 matrix A multiplies all vectors in $\mathbf{R}^2$, we can watch how it acts. Start with a "house" that has eleven endpoints. Those eleven vectors v are transformed into eleven vectors Av. Straight lines between v's become straight lines between the transformed vectors Av. (The transformation from house to house is linear!) Applying A to a standard house produces a new house—possibly stretched or rotated or otherwise unlivable.

This part of the book is visual, not theoretical. We will show four houses and the matrices that produce them. The columns of H are the eleven corners of the first house. (H is 2 by 12, so **plot2d** will connect the 11th corner to the first.) The 11 points in the house matrix H are multiplied by A to produce the corners AH of the other houses.

House matrix $$H = \begin{bmatrix} -6 & -6 & -7 & 0 & 7 & 6 & 6 & -3 & -3 & 0 & 0 & -6 \\ -7 & 2 & 1 & 8 & 1 & 2 & -7 & -7 & -2 & -2 & -7 & -7 \end{bmatrix}.$$

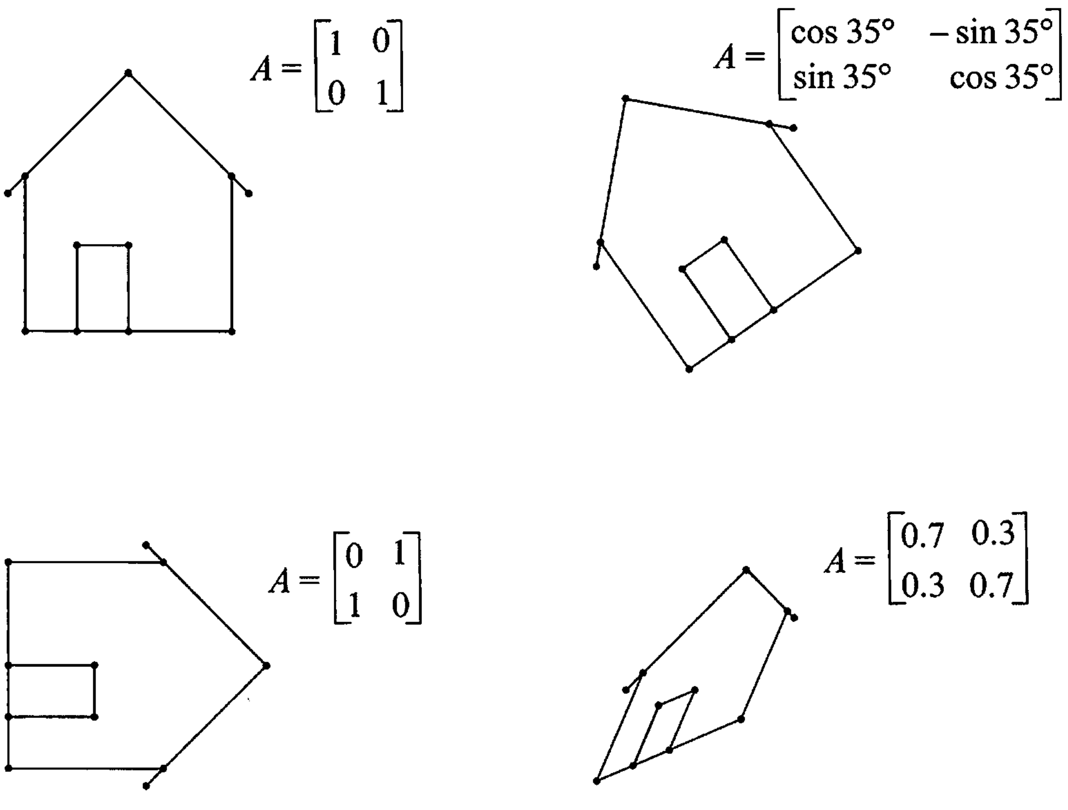


Figure 7.2: Linear transformations of a house drawn by **plot2d**$(A * H)$.

■ REVIEW OF THE KEY IDEAS ■

1. A transformation T takes each v in the input space to $T(v)$ in the output space.
2. T is **linear** if $T(v + w) = T(v) + T(w)$ and $T(cv) = cT(v)$: lines to lines.

3. Combinations to combinations: $T\,(c_1v_1 + \cdots + c_nv_n) = c_1\,T(v_1) + \cdots + c_n\,T(v_n)$.

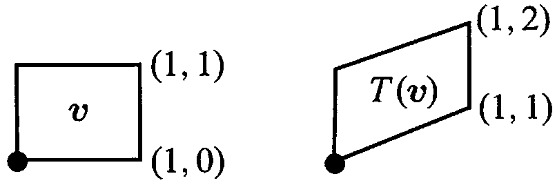
4. The transformation $T(v) = Av + v_0$ is linear only if $v_0 = \mathbf{0}$. Then $T(v) = Av$.

■ WORKED EXAMPLES ■

7.1 A The elimination matrix $\left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right]$ gives a ***shearing transformation*** from (x, y) to $T(x, y) = (x, x + y)$. Draw the xy plane and show what happens to $(1, 0)$ and $(1, 1)$. What happens to points on the vertical lines $x = 0$ and $x = a$? If the inputs fill the unit square $0 \le x \le 1,\ 0 \le y \le 1$, draw the outputs (the transformed square).

Solution The points $(1, 0)$ and $(2, 0)$ on the x axis transform by T to $(1, 1)$ and $(2, 2)$. The horizontal x axis transforms to the 45° line (going through $(0, 0)$ of course). The points on the y axis are *not moved* because $T(0, y) = (0, y)$. The y axis is the line of eigenvectors of T with $\lambda = 1$. Points with $x = a$ move up by a.

Vertical lines slide up
This is the shearing
Squares to parallelograms



7.1 B A **nonlinear transformation** T is invertible if every $\boldsymbol{b}$ in the output space comes from exactly one $\boldsymbol{x}$ in the input space: $T(\boldsymbol{x}) = \boldsymbol{b}$ always has exactly one solution. Which of these transformations (on real numbers x) is invertible and what is T^{-1}? ***None are linear, not even T_3.*** When you solve $T(\boldsymbol{x}) = \boldsymbol{b}$, you are inverting T:

$$T_1(x) = x^2 \quad T_2(x) = x^3 \quad T_3(x) = x + 9 \quad T_4(x) = e^x \quad T_5(x) = \frac{1}{x} \text{ for nonzero } x\text{'s}$$

Solution T_1 is not invertible: $x^2 = 1$ has *two* solutions and $x^2 = -1$ has *no* solution. T_4 is not invertible because $e^x = -1$ has no solution. (If the output space changes to *positive* b's then the inverse of $e^x = b$ is $x = \ln b$.)

Notice $T_5^2 =$ identity. But $T_3^2(x) = x + 18$. What are $T_2^2(x)$ and $T_4^2(x)$?

T_2, T_3, T_5 are invertible. The solutions to $x^3 = b$ and $x + 9 = b$ and $\frac{1}{x} = b$ are unique:

$$x = T_2^{-1}(b) = b^{1/3} \qquad x = T_3^{-1}(b) = b - 9 \qquad x = T_5^{-1}(b) = 1/b$$

Problem Set 7.1

1 A linear transformation must leave the zero vector fixed: $T(\mathbf{0}) = \mathbf{0}$. Prove this from $T(\boldsymbol{v}+\boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ by choosing $\boldsymbol{w} =$ _____ (and finish the proof). Prove it also from $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ by choosing $c =$ _____.

2 Requirement (b) gives $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ and also $T(d\boldsymbol{w}) = dT(\boldsymbol{w})$. Then by addition, requirement (a) gives $T(\quad) = (\quad)$. What is $T(c\boldsymbol{v} + d\boldsymbol{w} + e\boldsymbol{u})$?

3 Which of these transformations are not linear? The input is $\boldsymbol{v} = (v_1, v_2)$:

(a) $T(\boldsymbol{v}) = (v_2, v_1)$ (b) $T(\boldsymbol{v}) = (v_1, v_1)$ (c) $T(\boldsymbol{v}) = (0, v_1)$

(d) $T(\boldsymbol{v}) = (0, 1)$ (e) $T(\boldsymbol{v}) = v_1 - v_2$ (f) $T(\boldsymbol{v}) = v_1 v_2$.

4 If S and T are linear transformations, is $S(T(\boldsymbol{v}))$ linear or quadratic?

(a) (Special case) If $S(\boldsymbol{v}) = \boldsymbol{v}$ and $T(\boldsymbol{v}) = \boldsymbol{v}$, then $S(T(\boldsymbol{v})) = \boldsymbol{v}$ or $\boldsymbol{v}^2$?

(b) (General case) $S(\boldsymbol{w}_1+\boldsymbol{w}_2) = S(\boldsymbol{w}_1)+S(\boldsymbol{w}_2)$ and $T(\boldsymbol{v}_1+\boldsymbol{v}_2) = T(\boldsymbol{v}_1)+T(\boldsymbol{v}_2)$ combine into

$$S(T(\boldsymbol{v}_1 + \boldsymbol{v}_2)) = S(\_\_\_\_) = \_\_\_\_ + \_\_\_\_.$$

5 Suppose $T(\boldsymbol{v}) = \boldsymbol{v}$ except that $T(0, v_2) = (0, 0)$. Show that this transformation satisfies $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ but not $T(\boldsymbol{v}+\boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$.

6 Which of these transformations satisfy $T(\boldsymbol{v}+\boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ and which satisfy $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$?

(a) $T(\boldsymbol{v}) = \boldsymbol{v}/\|\boldsymbol{v}\|$ (b) $T(\boldsymbol{v}) = v_1+v_2+v_3$ (c) $T(\boldsymbol{v}) = (v_1, 2v_2, 3v_3)$

(d) $T(\boldsymbol{v}) =$ largest component of $\boldsymbol{v}$.

7 For these transformations of $\mathbf{V} = \mathbf{R}^2$ to $\mathbf{W} = \mathbf{R}^2$, find $T(T(\boldsymbol{v}))$. Is this transformation T^2 linear?

(a) $T(\boldsymbol{v}) = -\boldsymbol{v}$ (b) $T(\boldsymbol{v}) = \boldsymbol{v} + (1, 1)$

(c) $T(\boldsymbol{v}) = 90°$ rotation $= (-v_2, v_1)$

(d) $T(\boldsymbol{v}) =$ projection $= \left(\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2}\right)$.

8 Find the range and kernel (like the column space and nullspace) of T:

(a) $T(v_1, v_2) = (v_1 - v_2, 0)$ (b) $T(v_1, v_2, v_3) = (v_1, v_2)$

(c) $T(v_1, v_2) = (0, 0)$ (d) $T(v_1, v_2) = (v_1, v_1)$.

9 The "cyclic" transformation T is defined by $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$. What is $T(T(\boldsymbol{v}))$? What is $T^3(\boldsymbol{v})$? What is $T^{100}(\boldsymbol{v})$? Apply T a hundred times to $\boldsymbol{v}$.

10 A linear transformation from $\mathbf{V}$ to $\mathbf{W}$ has an *inverse* from $\mathbf{W}$ to $\mathbf{V}$ when the range is all of $\mathbf{W}$ and the kernel contains only $v = \mathbf{0}$. Then $T(v) = w$ has one solution v for each w in $\mathbf{W}$. Why are these T's not invertible?

(a) $T(v_1, v_2) = (v_2, v_2)$ $\quad \mathbf{W} = \mathbf{R}^2$

(b) $T(v_1, v_2) = (v_1, v_2, v_1 + v_2)$ $\quad \mathbf{W} = \mathbf{R}^3$

(c) $T(v_1, v_2) = v_1$ $\quad \mathbf{W} = \mathbf{R}^1$

11 If $T(v) = Av$ and A is m by n, then T is "multiplication by A."

(a) What are the input and output spaces $\mathbf{V}$ and $\mathbf{W}$?

(b) Why is range of T = column space of A?

(c) Why is kernel of T = nullspace of A?

12 Suppose a linear T transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$. Find $T(v)$:

(a) $v = (2, 2)$ (b) $v = (3, 1)$ (c) $v = (-1, 1)$ (d) $v = (a, b)$.

Problems 13-19 may be harder. The input space V contains all 2 by 2 matrices M.

13 M is any 2 by 2 matrix and $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. The transformation T is defined by $T(M) = AM$. What rules of matrix multiplication show that T is linear?

14 Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. Show that the range of T is the whole matrix space $\mathbf{V}$ and the kernel is the zero matrix:

(1) If $AM = 0$ prove that M must be the zero matrix.

(2) Find a solution to $AM = B$ for any 2 by 2 matrix B.

15 Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Show that the identity matrix I is not in the range of T. Find a nonzero matrix M such that $T(M) = AM$ is zero.

16 Suppose T transposes every matrix M. Try to find a matrix A which gives $AM = M^{\mathrm{T}}$ for every M. Show that no matrix A will do it. *To professors:* Is this a linear transformation that doesn't come from a matrix?

17 The transformation T that transposes every matrix is definitely linear. Which of these extra properties are true?

(a) T^2 = identity transformation.

(b) The kernel of T is the zero matrix.

(c) Every matrix is in the range of T.

(d) $T(M) = -M$ is impossible.

18 Suppose $T(M) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} M \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Find a matrix with $T(M) \neq 0$. Describe all matrices with $T(M) = 0$ (the kernel) and all output matrices $T(M)$ (the range).

19 If A and B are invertible and $T(M) = AMB$, find $T^{-1}(M)$ in the form $(\quad)M(\quad)$.

Questions 20–26 are about house transformations. The output is $T(H) = AH$.

20 How can you tell from the picture of T (house) that A is

(a) a diagonal matrix?

(b) a rank-one matrix?

(c) a lower triangular matrix?

21 Draw a picture of T (house) for these matrices:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} .7 & .7 \\ .3 & .3 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

22 What are the conditions on $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to ensure that T (house) will

(a) sit straight up?

(b) expand the house by 3 in all directions?

(c) rotate the house with no change in its shape?

23 Describe T (house) when $T(v) = -v + (1, 0)$. This T is "affine".

24 Change the house matrix H to add a chimney.

25 The standard house is drawn by **plot2d**(H). Circles from o and lines from –:

$$x = H(1,:)'; y = H(2,:)';$$
$$\text{axis}([-1010\text{–}1010]), \text{axis}(\text{'square'})$$
$$\text{plot}(x, y, 'o', x, y, '-');$$

Test **plot2d**(A′∗ H) and **plot2d**(A′∗ A ∗ H) with the matrices in Figure 7.1.

26 Without a computer sketch the houses $A * H$ for these matrices A:

$$\begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

27 This code creates a vector theta of 50 angles. It draws the unit circle and then T (circle) = ellipse. **$T(v) = Av$ takes circles to ellipses.**

```
A = [2 1;1 2]    % You can change A
theta = [0:2 * pi/50:2 * pi];
circle = [cos(theta); sin(theta)];
ellipse = A * circle;
axis([–4 4 –4 4]); axis('square')
plot(circle(1,:), circle(2,:), ellipse(1,:), ellipse(2,:))
```

28 Add two eyes and a smile to the circle in Problem 27. (If one eye is dark and the other is light, you can tell when the face is reflected across the y axis.) Multiply by matrices A to get new faces.

Challenge Problems

29 What conditions on $\det A = ad - bc$ ensure that the output house AH will

(a) be squashed onto a line?

(b) keep its endpoints in clockwise order (not reflected)?

(c) have the same area as the original house?

30 From $A = U\Sigma V^{\mathrm{T}}$ (Singular Value Decomposition) A takes **circles to ellipses.** $AV = U\Sigma$ says that the radius vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of the circle go to the semi-axes $\sigma_1\boldsymbol{u}_1$ and $\sigma_2\boldsymbol{u}_2$ of the ellipse. Draw the circle and the ellipse for $\theta = 30°$:

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

31 Why does every linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^2$ take squares to parallelograms? Rectangles also go to parallelograms (squashed if T is not invertible).

7.2 The Matrix of a Linear Transformation

The next pages assign a matrix to every linear transformation T. For ordinary column vectors, the input v is in $\mathbf{V} = \mathbf{R}^n$ and the output $T(v)$ is in $\mathbf{W} = \mathbf{R}^m$. The matrix A for this transformation T will be m by n. Our choice of bases in $\mathbf{V}$ and $\mathbf{W}$ will decide A.

The standard basis vectors for $\mathbf{R}^n$ and $\mathbf{R}^m$ are the columns of I. That choice leads to a standard matrix, and $T(v) = Av$ in the normal way. But these spaces also have other bases, so the same T is represented by other matrices. A main theme of linear algebra is to choose the bases that give the best matrix for T.

When $\mathbf{V}$ and $\mathbf{W}$ are not $\mathbf{R}^n$ and $\mathbf{R}^m$, they still have bases. Each choice of basis leads to a matrix for T. When the input basis is different from the output basis, the matrix for $T(v) = v$ will not be the identity I. It will be the "change of basis matrix".

Key idea of this section

> ***Suppose we know** $T(v_1), \ldots, T(v_n)$ **for the basis vectors** $v_1, \ldots, v_n$.*
> ***Then linearity produces** $T(v)$ **for every other input vector** v.*

Reason Every v is a unique combination $c_1v_1 + \cdots + c_nv_n$ of the basis vectors v_i. Since T is a linear transformation (here is the moment for linearity), $T(v)$ must be **the same combination** $c_1T(v_1) + \cdots + c_nT(v_n)$ **of the known outputs** $T(v_i)$.

Our first example gives the outputs $T(v)$ for the standard basis vectors $(1,0)$ and $(0,1)$.

Example 1 Suppose T transforms $v_1 = (1,0)$ to $T(v_1) = (2,3,4)$. Suppose the second basis vector $v_2 = (0,1)$ goes to $T(v_2) = (5,5,5)$. If T is linear from $\mathbf{R}^2$ to $\mathbf{R}^3$ then its "standard matrix" is 3 by 2. Those outputs $T(v_1)$ and $T(v_2)$ go into its columns:

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix}. \qquad \begin{matrix} T(v_1 + v_2) = T(v_1) + T(v_2) \\ \text{combines the columns} \end{matrix} \qquad \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Example 2 The derivatives of the functions $1, x, x^2, x^3$ are $0, 1, 2x, 3x^2$. Those are four facts about the transformation T that "***takes the derivative***". The inputs and the outputs are functions! Now add the crucial fact that the "derivative transformation" T is linear:

$$T(v) = \frac{dv}{dx} \quad \text{obeys the linearity rule} \quad \frac{d}{dx}(cv + dw) = c\frac{dv}{dx} + d\frac{dw}{dx}. \tag{1}$$

It is exactly this linearity that you use to find all other derivatives. From the derivative of each separate power $1, x, x^2, x^3$ (those are the basis vectors v_1, v_2, v_3, v_4) you find the derivative of any polynomial like $4 + x + x^2 + x^3$:

$$\frac{d}{dx}(4 + x + x^2 + x^3) = 1 + 2x + 3x^2 \qquad \text{(because of linearity!)}$$

This example applies T (the derivative d/dx) to the input $v = 4v_1 + v_2 + v_3 + v_4$. Here the input space $\mathbf{V}$ contains all combinations of $1, x, x^2, x^3$. I call them vectors, you might call them functions. Those four vectors are a basis for the space $\mathbf{V}$ of cubic polynomials (degree ≤ 3). Four derivatives tell us all derivatives in $\mathbf{V}$.

For the nullspace of A, we solve $Av = \mathbf{0}$. For the kernel of the derivative T, we solve $dv/dx = \mathbf{0}$. The solution is v = constant. The nullspace of T is one-dimensional, containing all constant functions (like the first basis function $v_1 = 1$).

To find the range (or column space), look at all outputs from $T(v) = dv/dx$. The inputs are cubic polynomials $a + bx + cx^2 + dx^3$, so the outputs are *quadratic polynomials* (degree ≤ 2). For the output space $\mathbf{W}$ we have a choice. If $\mathbf{W}$ = cubics, then the range of T (the quadratics) is a subspace. If $\mathbf{W}$ = quadratics, then the range is all of $\mathbf{W}$.

That second choice emphasizes the difference between the domain or input space ($\mathbf{V}$ = cubics) and the image or output space ($\mathbf{W}$ = quadratics). $\mathbf{V}$ has dimension $n = 4$ and $\mathbf{W}$ has dimension $m = 3$. The "derivative matrix" below will be 3 by 4.

The range of T is a three-dimensional subspace. The matrix will have rank $r = 3$. The kernel is one-dimensional. The sum $3 + 1 = 4$ is the dimension of the input space. This was $r + (n - r) = n$ in the Fundamental Theorem of Linear Algebra. Always ***(dimension of range) + (dimension of kernel) = dimension of input space.***

Example 3 The ***integral*** is the inverse of the derivative. That is the Fundamental Theorem of Calculus. We see it now in linear algebra. The transformation T^{-1} that "takes the integral from 0 to x" is linear! Apply T^{-1} to $1, x, x^2$, which are $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$:

$$\textbf{Integration is } T^{-1} \qquad \int_0^x 1\,dx = x, \qquad \int_0^x x\,dx = \tfrac{1}{2}x^2, \qquad \int_0^x x^2\,dx = \tfrac{1}{3}x^3.$$

By linearity, the integral of $\boldsymbol{w} = B + Cx + Dx^2$ is $T^{-1}(\boldsymbol{w}) = Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$. The integral of a quadratic is a cubic. The input space of T^{-1} is the quadratics, the output space is the cubics. ***Integration takes* $\mathbf{W}$ *back to* $\mathbf{V}$.** Its matrix will be 4 by 3.

Range of T^{-1} The outputs $Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$ are cubics with no constant term.

Kernel of T^{-1} The output is zero only if $B = C = D = 0$. The nullspace is $\mathbf{Z} = \{\mathbf{0}\}$.

Fundamental Theorem $3 + 0$ is the dimension of the input space $\mathbf{W}$ for T^{-1}.

Matrices for the Derivative and Integral

We will show how the matrices A and A^{-1} copy the derivative T and the integral T^{-1}. This is an excellent example from calculus. (I write A^{-1} but I don't quite mean it.) Then comes the general rule—how to represent any linear transformation T by a matrix A.

The derivative transforms the space $\mathbf{V}$ of cubics to the space $\mathbf{W}$ of quadratics. The basis for $\mathbf{V}$ is $1, x, x^2, x^3$. The basis for $\mathbf{W}$ is $1, x, x^2$. *The derivative matrix is* 3 *by* 4*:*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \textbf{matrix form of derivative } T. \tag{2}$$

Why is A the correct matrix? Because ***multiplying by A agrees with transforming by T.*** The derivative of $v = a + bx + cx^2 + dx^3$ is $T(v) = b + 2cx + 3dx^2$. The same numbers b and $2c$ and $3d$ appear when we multiply by the matrix A:

$$\textbf{Take the derivative} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}. \tag{3}$$

Look also at T^{-1}. The integration matrix is 4 by 3. Watch how the following matrix starts with $\boldsymbol{w} = B + Cx + Dx^2$ and produces its integral $0 + Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$:

$$\textbf{Take the integral} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ B \\ \frac{1}{2}C \\ \frac{1}{3}D \end{bmatrix}. \tag{4}$$

I want to call that matrix A^{-1}, and I will. But you realize that rectangular matrices don't have inverses. At least they don't have two-sided inverses. This rectangular A has a *one-sided inverse*. The integral is a one-sided inverse of the derivative!

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{but} \quad A^{-1}A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If you integrate a function and then differentiate, you get back to the start. So $AA^{-1} = I$. But if you differentiate before integrating, the constant term is lost. ***The integral of the derivative of* 1 *is zero:***

$$T^{-1}T(1) = \text{integral of zero function} = 0.$$

This matches $A^{-1}A$, whose first column is all zero. The derivative T has a kernel (the constant functions). Its matrix A has a nullspace. Main point again: $A\boldsymbol{v}$ copies $T(\boldsymbol{v})$.

Construction of the Matrix

Now we construct a matrix for any linear transformation. Suppose T transforms the space $\mathbf{V}$ (n-dimensional) to the space $\mathbf{W}$ (m-dimensional). We choose a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ for $\mathbf{V}$ and we choose a basis $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$ for $\mathbf{W}$. The matrix A will be m by n. To find the first column of A, apply T to the first basis vector $\boldsymbol{v}_1$. The output $T(\boldsymbol{v}_1)$ is in $\mathbf{W}$.

$$T(\boldsymbol{v}_1) \quad \textit{is a combination} \quad a_{11}\boldsymbol{w}_1 + \cdots + a_{m1}\boldsymbol{w}_m \quad \textit{of the output basis for } \mathbf{W}.$$

These numbers $a_{11}, \ldots, a_{m1}$ go into the first column of A. Transforming $\boldsymbol{v}_1$ to $T(\boldsymbol{v}_1)$ matches multiplying $(1, 0, \ldots, 0)$ by A. It yields that first column of the matrix.

When T is the derivative and the first basis vector is 1, its derivative is $T(\boldsymbol{v}_1) = \mathbf{0}$. So for the derivative matrix, the first column of A was all zero.

For the integral, the first basis function is again 1. Its integral is the second basis function x. So the first column of A^{-1} was $(0, 1, 0, 0)$. Here is the construction of A.

Key rule: The jth column of A is found by applying T to the jth basis vector $\boldsymbol{v}_j$

$$T(\boldsymbol{v}_j) = \text{combination of basis vectors of } \mathbf{W} = a_{1j}\boldsymbol{w}_1 + \cdots + a_{mj}\boldsymbol{w}_m. \tag{5}$$

These numbers $a_{1j}, \ldots, a_{mj}$ go into column j of A. *The matrix is constructed to get the basis vectors right.* ***Then linearity gets all other vectors right.*** Every $\boldsymbol{v}$ is a combination $c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$, and $T(\boldsymbol{v})$ is a combination of the $\boldsymbol{w}$'s. When A multiplies the coefficient vector $\boldsymbol{c} = (c_1, \ldots, c_n)$ in the $\boldsymbol{v}$ combination, $A\boldsymbol{c}$ produces the coefficients in the $T(\boldsymbol{v})$ combination. This is because matrix multiplication (combining columns) is linear like T.

The matrix A tells us what T does. Every linear transformation from $\mathbf{V}$ to $\mathbf{W}$ can be converted to a matrix. This matrix depends on the bases.

Example 4 ***If the bases change, T is the same but the matrix A is different.***

Suppose we reorder the basis to $x, x^2, x^3, 1$ for the cubics in $\mathbf{V}$. Keep the original basis $1, x, x^2$ for the quadratics in $\mathbf{W}$. The derivative of the first basis vector $\boldsymbol{v}_1 = x$ is the first basis vector $\boldsymbol{w}_1 = 1$. So the first column of A looks different:

$$A_{\text{new}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{matrix} \text{matrix for the derivative } T \\ \text{when the bases change to} \\ x, x^2, x^3, 1 \text{ and } 1, x, x^2. \end{matrix}$$

When we reorder the basis of $\mathbf{V}$, we reorder the columns of A. The input basis vector $\boldsymbol{v}_j$ is responsible for column j. The output basis vector $\boldsymbol{w}_i$ is responsible for row i. Soon the changes in the bases will be more than permutations.

Products AB Match Transformations TS

The examples of derivative and integral made three points. First, linear transformations T are everywhere—in calculus and differential equations and linear algebra. Second, spaces other than $\mathbf{R}^n$ are important—we had functions in $\mathbf{V}$ and $\mathbf{W}$. Third, T still boils down to a matrix A. Now we make sure that we can find this matrix.

The next examples have $\mathbf{V} = \mathbf{W}$. We choose the same basis for both spaces. Then we can compare the matrices A^2 and AB with the transformations T^2 and TS.

Example 5 T rotates every vector by the angle θ. Here $\mathbf{V} = \mathbf{W} = \mathbf{R}^2$. *Find A.*

Solution The standard basis is $\boldsymbol{v}_1 = (1, 0)$ and $\boldsymbol{v}_2 = (0, 1)$. To find A, apply T to those basis vectors. In Figure 7.3a, they are rotated by θ. *The first vector* $(1, 0)$ ***swings around to*** $(\cos\theta, \sin\theta)$. This equals $\cos\theta$ times $(1, 0)$ plus $\sin\theta$ times $(0, 1)$. Therefore those numbers $\cos\theta$ and $\sin\theta$ go into the *first column* of A:

$$\begin{bmatrix} \cos\theta & \\ \sin\theta & \end{bmatrix} \text{ shows column 1} \quad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ shows both columns.}$$

For the second column, transform the second vector $(0, 1)$. The figure shows it rotated to $(-\sin\theta, \cos\theta)$. *Those numbers go into the second column.* Multiplying A times $(0, 1)$ produces that column. A agrees with T on the basis, and on all $\boldsymbol{v}$.

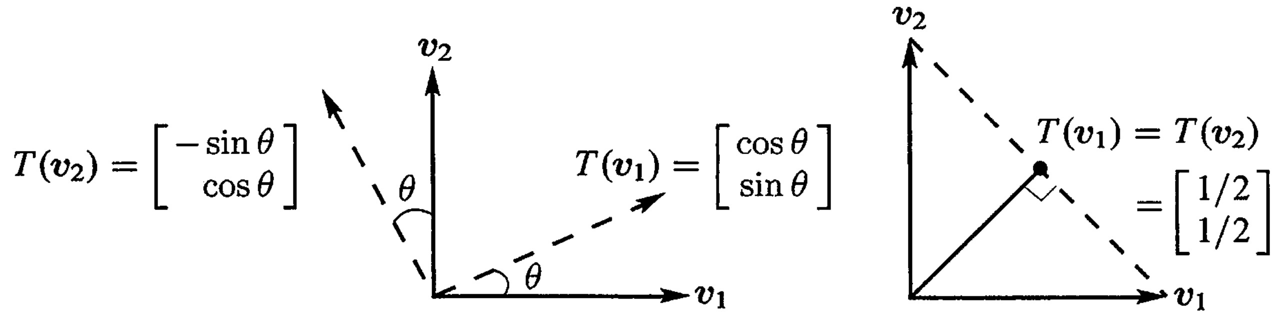


Figure 7.3: Two transformations: Rotation by θ and projection onto the 45° line.

Example 6 (*Projection*) Suppose T projects every plane vector onto the 45° line. Find its matrix for two different choices of the basis. We will find two matrices.

Solution Start with a specially chosen basis, not drawn in Figure 7.3. The basis vector $\boldsymbol{v}_1$ is along the 45° line. *It projects to itself*: $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$. So the first column of A contains 1 and 0. The second basis vector $\boldsymbol{v}_2$ is along the perpendicular line (135°). *This basis vector projects to zero.* So the second column of A contains 0 and 0:

$$\textbf{Projection} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{when } \mathbf{V} \text{ and } \mathbf{W} \text{ have the 45° and 135° basis.}$$

Now take the standard basis $(1, 0)$ and $(0, 1)$. Figure 7.3b shows how $(1, 0)$ projects to $(\frac{1}{2}, \frac{1}{2})$. That gives the first column of A. The other basis vector $(0, 1)$ also projects to $(\frac{1}{2}, \frac{1}{2})$. So the standard matrix for this projection is A:

$$\textbf{Same projection} \quad A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{for the standard basis.}$$

Both A's are projection matrices. If you square A it doesn't change. Projecting twice is the same as projecting once: $T^2 = T$ so $A^2 = A$. Notice what is hidden in that statement: ***The matrix for T^2 is A^2.***

We have come to something important—the real reason for the way matrices are multiplied. *At last we discover why*! Two transformations S and T are represented by two matrices B and A. When we apply T to the output from S, we get the "composition" TS. When we apply A after B, we get the matrix product AB. ***Matrix multiplication gives the correct matrix AB to represent TS.***

The transformation S is from a space $\mathbf{U}$ to $\mathbf{V}$. Its matrix B uses a basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p$ for $\mathbf{U}$ and a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ for $\mathbf{V}$. The matrix is n by p. The transformation T is from $\mathbf{V}$ to $\mathbf{W}$ as before. *Its matrix A must use the same basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ for* $\mathbf{V}$—this is the output space for S and the input space for T. ***Then the matrix AB matches TS:***

Multiplication The linear transformation TS starts with any vector $\boldsymbol{u}$ in $\mathbf{U}$, goes to $S(\boldsymbol{u})$ in $\mathbf{V}$ and then to $T(S(\boldsymbol{u}))$ in $\mathbf{W}$. The matrix AB starts with any $\boldsymbol{x}$ in $\mathbf{R}^p$, goes to $B\boldsymbol{x}$ in $\mathbf{R}^n$ and then to $AB\boldsymbol{x}$ in $\mathbf{R}^m$. The matrix AB correctly represents TS:

$$TS: \quad \mathbf{U} \to \mathbf{V} \to \mathbf{W} \qquad AB: \quad (m \text{ by } n)(n \text{ by } p) = (m \text{ by } p).$$

The input is $\boldsymbol{u} = x_1\boldsymbol{u}_1 + \cdots + x_p\boldsymbol{u}_p$. The output $T(S(\boldsymbol{u}))$ matches the output $AB\boldsymbol{x}$. ***Product of transformations matches product of matrices.***

The most important cases are when the spaces $\mathbf{U}, \mathbf{V}, \mathbf{W}$ are the same and their bases are the same. With $m = n = p$ we have square matrices.

Example 7 S rotates the plane by θ and T also rotates by θ. Then TS rotates by 2θ. This transformation T^2 corresponds to the rotation matrix A^2 through 2θ:

$$T = S \qquad A = B \qquad T^2 = \text{rotation by } 2\theta \qquad A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}. \tag{6}$$

By matching (transformation)^2 with (matrix)^2, we pick up the formulas for $\cos 2\theta$ and $\sin 2\theta$. Multiply A times A:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}. \tag{7}$$

Comparing (6) with (7) produces $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. Trigonometry (the double angle rule) comes from linear algebra.

Example 8 S rotates by θ and T rotates by $-\theta$. Then $TS = I$ matches $AB = I$.

In this case $T(S(\boldsymbol{u}))$ is $\boldsymbol{u}$. We rotate forward and back. For the matrices to match, $AB\boldsymbol{x}$ must be $\boldsymbol{x}$. The two matrices are inverses. Check this by putting $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ into the backward rotation matrix:

$$AB = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = I.$$

Earlier T took the derivative and S took the integral. The transformation TS is the identity but not ST. Therefore AB is the identity matrix but not BA:

$$AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = I \quad \text{but} \quad BA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Identity Transformation and the Change of Basis Matrix

We now find the matrix for the special and boring transformation $T(v) = v$. This ***identity transformation*** does nothing to v. The matrix for $T = I$ also does nothing, *provided* the output basis is the same as the input basis. The output $T(v_1)$ is v_1. When the bases are the same, this is w_1. So the first column of A is $(1, 0, \ldots, 0)$.

When each output $T(v_j) = v_j$ is the same as w_j, the matrix is just I.

This seems reasonable: The identity transformation is represented by the identity matrix. But suppose the bases are *different*. Then $T(v_1) = v_1$ is a combination of the w's. That combination $m_{11}w_1 + \cdots + m_{n1}w_n$ tells the first column of the matrix (call it M).

Identity transformation ***When the outputs $T(v_j) = v_j$ are combinations $\sum_{i=1}^{n} m_{ij} w_i$, the "change of basis matrix" is M.***

The basis is changing but the vectors themselves are not changing: $T(v) = v$. When the inputs have one basis and the outputs have another basis, the matrix is not I.

Example 9 The input basis is $v_1 = (3,7)$ and $v_2 = (2,5)$. The output basis is $w_1 = (1,0)$ and $w_2 = (0,1)$. Then the matrix M is easy to compute:

Change of basis The matrix for $T(v) = v$ is $M = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$.

Reason The first input is the basis vector $v_1 = (3,7)$. The output is also $(3,7)$ which we express as $3w_1 + 7w_2$. Then the first column of M contains 3 and 7.

This seems too simple to be important. It becomes trickier when the change of basis goes the other way. We get the inverse of the previous matrix M:

Example 10 The input basis is now $v_1 = (1,0)$ and $v_2 = (0,1)$. The outputs are just $T(v) = v$. But the output *basis* is now $w_1 = (3,7)$ and $w_2 = (2,5)$.

Reverse the bases
Invert the matrix The matrix for $T(v) = v$ is $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$.

Reason The first input is $v_1 = (1,0)$. The output is also v_1 but we express it as $5w_1 - 7w_2$. Check that $5(3,7) - 7(2,5)$ does produce $(1,0)$. We are combining the columns of the previous M to get the columns of I. The matrix to do that is M^{-1}.

$$\begin{matrix}\textbf{Change basis}\\ \textbf{Change back}\end{matrix} \quad \begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \text{ is } MM^{-1} = I.$$

A mathematician would say that the matrix MM^{-1} corresponds to the product of two identity transformations. We start and end with the same basis $(1,0)$ and $(0,1)$. Matrix multiplication must give I. So the two change of basis matrices are inverses.

One thing is sure. Multiplying A times $(1,0,\ldots,0)$ gives column 1 of the matrix. The novelty of this section is that $(1,0,\ldots,0)$ stands for the first vector v_1, *written in the basis of* v*'s*. Then column 1 of the matrix is that same vector v_1, *written in the standard basis.*

Wavelet Transform = Change to Wavelet Basis

Wavelets are little waves. They have different lengths and they are localized at different places. The first basis vector is not actually a wavelet, it is the very useful flat vector of all ones. This example shows "Haar wavelets":

$$\textbf{Haar basis} \quad w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad w_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad w_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad w_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}. \tag{8}$$

Those vectors are *orthogonal*, which is good. You see how w_3 is localized in the first half and w_4 is localized in the second half. The **wavelet transform** finds the coefficients c_1, c_2, c_3, c_4 when the input signal $v = (v_1, v_2, v_3, v_4)$ is expressed in the wavelet basis:

$$\textbf{Transform } v \textbf{ to } c \qquad v = c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4 = Wc \tag{9}$$

The coefficients c_3 and c_4 tell us about details in the first half and last half of v. The coefficient c_1 is the average.

Why do want to change the basis? I think of v_1, v_2, v_3, v_4 as the intensities of a signal. In audio they are volumes of sound. In images they are pixel values on a scale of black to white. An electrocardiogram is a medical signal. Of course $n = 4$ is very short, and $n = 10{,}000$ is more realistic. We may need to *compress* that long signal, by keeping only the largest 5% of the coefficients. This is 20 : 1 compression and (to give only two of its applications) it makes High Definition TV and video conferencing possible.

If we keep only 5% of the *standard* basis coefficients, we lose 95% of the signal. In image processing, 95% of the image disappears. In audio, 95% of the tape goes blank. But if we choose a better basis of w's, 5% of the basis vectors can combine to come very close to the original signal. In image processing and audio coding, you can't see or hear the difference. We don't need the other 95%!

One good basis vector is the flat $(1,1,1,1)$. That part alone can represent the constant background of our image. A short wave like $(0,0,1,-1)$ or in higher dimensions $(0,0,0,0,0,0,1,-1)$ represents a detail at the end of the signal.

The three steps are the transform and compression and inverse transform.

input v $\rightarrow$ coefficients c $\rightarrow$ compressed $\widehat{c}$ $\rightarrow$ compressed $\widehat{v}$
[*lossless*] [*lossy*] [*reconstruct*]

In linear algebra, where everything is perfect, we omit the compression step. The output $\widehat{v}$ is exactly the same as the input v. The transform gives $c = W^{-1}v$ and the reconstruction brings back $v = Wc$. In true signal processing, where nothing is perfect but everything is fast, the transform (lossless) and the compression (which only loses unnecessary information) are absolutely the keys to success. The output is $\widehat{v} = W\widehat{c}$.

I will show those steps for a typical vector like $v = (6, 4, 5, 1)$. Its wavelet coefficients are $c = (4, 1, 1, 2)$. The reconstruction $4\boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3 + 2\boldsymbol{w}_4$ is $v = Wc$:

$$\begin{bmatrix}\mathbf{6}\\\mathbf{4}\\\mathbf{5}\\\mathbf{1}\end{bmatrix} = 4\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} + \begin{bmatrix}1\\-1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\\-1\end{bmatrix} = \begin{bmatrix}1&1&1&0\\1&1&-1&0\\1&-1&0&1\\1&-1&0&-1\end{bmatrix}\begin{bmatrix}\mathbf{4}\\\mathbf{1}\\\mathbf{1}\\\mathbf{2}\end{bmatrix}. \tag{10}$$

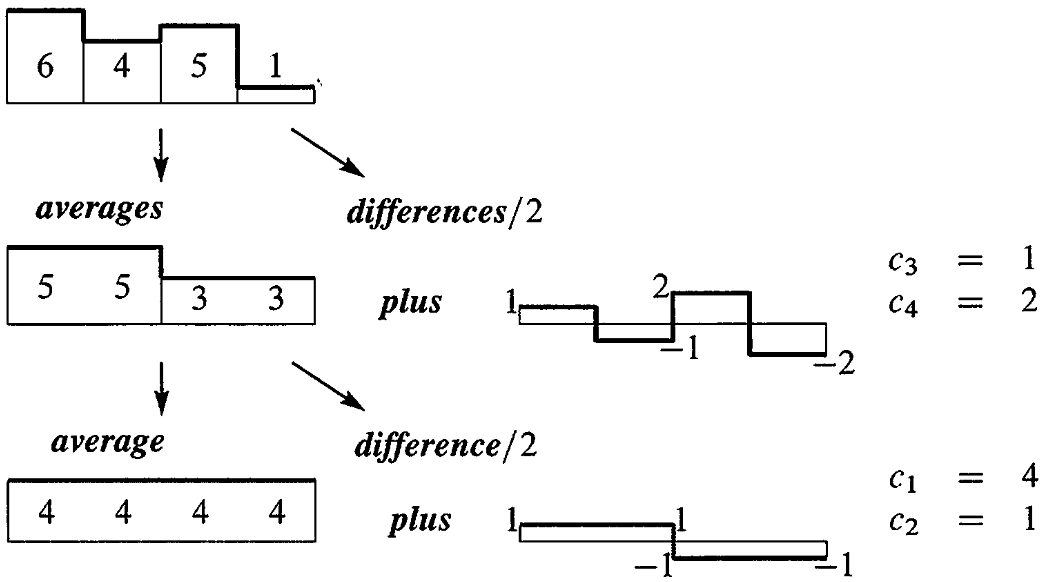
Those coefficients c are $W^{-1}v$. Inverting this basis matrix W is easy because the $\boldsymbol{w}$'s in its columns are orthogonal. But they are not unit vectors, so rescale:

$$W^{-1} = \begin{bmatrix}\frac{1}{4}&&&\\&\frac{1}{4}&&\\&&\frac{1}{2}&\\&&&\frac{1}{2}\end{bmatrix}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&0&0\\0&0&1&-1\end{bmatrix}.$$

The $\frac{1}{4}$'s in the first row of $c = W^{-1}v$ mean that $c_1 = 4$ is the average of $6, 4, 5, 1$.

Example 11 **(Same wavelet basis by recursion)** I can't resist showing you a faster way to find the c's. The special point of the wavelet basis is that you can pick off the details in c_3 and c_4, before the coarse details in c_2 and the overall average in c_1. A picture will explain this "multiscale" method, which is in Chapter 1 of my textbook with Nguyen on *Wavelets and Filter Banks* (Wellesley-Cambridge Press).

Split $v = (6, 4, 5, 1)$ ***into averages and waves at small scale and then large scale:***



Fourier Transform (DFT) = Change to Fourier Basis

The first thing an electrical engineer does with a signal is to take its Fourier transform. For finite vectors we are speaking about the ***Discrete Fourier Transform.*** The DFT involves complex numbers (powers of $e^{2\pi i/n}$). But if we choose $n = 4$, the matrices are small and the only complex numbers are i and $i^3 = -i$. A true electrical engineer would write j instead of i for $\sqrt{-1}$.

Fourier basis w_1 to w_n in the columns of F

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

The first column is the useful flat basis vector $(1, 1, 1, 1)$. It represents the average signal or the direct current (the DC term). It is a wave at zero frequency. The third column is $(1, -1, 1, -1)$, which alternates at the highest frequency. *The Fourier transform decomposes the signal into waves at equally spaced frequencies.*

The Fourier matrix F is absolutely the most important complex matrix in mathematics and science and engineering. Section 10.3 of this book explains the ***Fast Fourier Transform***: it can be seen as a factorization of F into matrices with many zeros. The FFT has revolutionized entire industries, by speeding up the Fourier transform. The beautiful thing is that F^{-1} looks like F, with i changed to $-i$:

Fourier transform v to c
$v = c_1 w_1 + \cdots + c_n w_n = Fc$
Fourier coefficients $c = F^{-1}v$

$$F^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & (-i) & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix} = \frac{1}{4}\overline{F}.$$

The MATLAB command c = fft(v) produces the Fourier coefficients $c_1, \ldots, c_n$ of the vector v. It multiplies v by F^{-1} (fast).

■ REVIEW OF THE KEY IDEAS ■

1. If we know $T(v_1), \ldots, T(v_n)$ for a basis, linearity will determine all other $T(v)$.

2. $\left\{\begin{array}{l} \text{Linear transformation } T \\ \text{Input basis } v_1, \ldots, v_n \\ \text{Output basis } w_1, \ldots, w_m \end{array}\right\} \to \begin{array}{c} \text{Matrix } A \ (m \text{ by } n) \\ \text{represents } T \\ \text{in these bases} \end{array}$

3. The derivative and integral matrices are one-sided inverses: $d(\text{constant})/dx = 0$:

   (Derivative) (Integral) $= I$ is the Fundamental Theorem of Calculus.

4. If A and B represent T and S, and the output basis for S is the input basis for T, then the matrix AB represents the transformation $T(S(u))$.

5. The change of basis matrix M represents $T(v) = v$. Its columns are the coefficients of the output basis expressed in the input basis: $w_j = m_{1j}v_1 + \cdots + m_{nj}v_n$.

■ WORKED EXAMPLES ■

7.2 A Using the standard basis, find the 4 by 4 matrix P that represents a *cyclic permutation* T from $x = (x_1, x_2, x_3, x_4)$ to $T(x) = (x_4, x_1, x_2, x_3)$. Find the matrix for T^2. What is the triple shift $T^3(x)$ and why is $T^3 = T^{-1}$?

Find two real independent eigenvectors of P. What are all the eigenvalues of P?

Solution The first vector $(1, 0, 0, 0)$ in the standard basis transforms to $(0, 1, 0, 0)$ which is the second basis vector. So the first column of P is $(0, 1, 0, 0)$. The other three columns come from transforming the other three standard basis vectors:

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{Then } P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ copies } T.$$

Since we used the standard basis, T *is ordinary multiplication by* P. The matrix for T^2 is a "double cyclic shift" P^2 and it produces (x_3, x_4, x_1, x_2).

The triple shift T^3 will transform $x = (x_1, x_2, x_3, x_4)$ to $T^3(x) = (x_2, x_3, x_4, x_1)$. If we apply T once more we are back to the original x. So $T^4 =$ identity transformation and $P^4 =$ identity matrix.

Two real eigenvectors of P are $(1, 1, 1, 1)$ with eigenvalue $\lambda = 1$ and $(1, -1, 1, -1)$ with eigenvalue $\lambda = -1$. The shift leaves $(1, 1, 1, 1)$ unchanged and it reverses signs in $(1, -1, 1, -1)$. The other eigenvalues are i and $-i$. The determinant is $\lambda_1\lambda_2\lambda_3\lambda_4 = -1$.

Notice that the eigenvalues $1, i, -1, -i$ add to zero (the trace of P). They are the 4th roots of 1, since $\det(P - \lambda I) = \lambda^4 - 1$. They are at angles $0°, 90°, 180°, 270°$ in the complex plane. **The Fourier matrix F is the eigenvector matrix for P.**

7.2 B The space of 2 by 2 matrices has these four "vectors" as a basis:

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

T is the linear transformation that *transposes* every 2 by 2 matrix. What is the matrix A that represents T in this basis (output basis = input basis)? What is the inverse matrix A^{-1}? What is the transformation T^{-1} that inverts the transpose operation?

Solution Transposing those four "basis matrices" just reverses v_2 and v_3:

$$\begin{aligned} T(v_1) &= v_1 \\ T(v_2) &= v_3 \\ T(v_3) &= v_2 \\ T(v_4) &= v_4 \end{aligned} \qquad \text{gives the four columns of} \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The inverse matrix A^{-1} is the same as A. The inverse transformation T^{-1} is the same as T. If we transpose and transpose again, the final output equals the original input.

Problem Set 7.2

Questions 1–4 extend the first derivative example to higher derivatives.

1 The transformation S takes the ***second derivative***. Keep $1, x, x^2, x^3$ as the basis v_1, v_2, v_3, v_4 and also as w_1, w_2, w_3, w_4. Write Sv_1, Sv_2, Sv_3, Sv_4 in terms of the w's. Find the 4 by 4 matrix B for S.

2 What functions have $v'' = \mathbf{0}$? They are in the kernel of the second derivative S. What vectors are in the nullspace of its matrix B in Problem 1?

3 B is not the square of a rectangular first derivative matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ does not allow } A^2.$$

Add a zero row to A, so that output space = input space. Compare A^2 with B. Conclusion: For $B = A^2$ we want output basis = _____ basis. Then $m = n$.

4 (a) The product TS of first and second derivatives produces the *third* derivative. Add zeros to make 4 by 4 matrices, then compute AB.

(b) The matrix B^2 corresponds to S^2 = *fourth* derivative. Why is this zero?

Questions 5–9 are about a particular T and its matrix A.

5 With bases v_1, v_2, v_3 and w_1, w_2, w_3, suppose $T(v_1) = w_2$ and $T(v_2) = T(v_3) = w_1 + w_3$. T is a linear transformation. Find the matrix A and multiply by the vector $(1, 1, 1)$. What is the output from T when the input is $v_1 + v_2 + v_3$?

6 Since $T(v_2) = T(v_3)$, the solutions to $T(v) = \mathbf{0}$ are $v =$ _____. What vectors are in the nullspace of A? Find all solutions to $T(v) = w_2$.

7 Find a vector that is not in the column space of A. Find a combination of w's that is not in the range of T.

8 You don't have enough information to determine T^2. Why is its matrix not necessarily A^2? What more information do you need?

9 Find the rank of A. This is not the dimension of the output space $\mathbf{W}$. It is the dimension of the _____ of T.

Questions 10–13 are about invertible linear transformations.

10 Suppose $T(\boldsymbol{v}_1) = \boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3$ and $T(\boldsymbol{v}_2) = \boldsymbol{w}_2 + \boldsymbol{w}_3$ and $T(\boldsymbol{v}_3) = \boldsymbol{w}_3$. Find the matrix A for T using these basis vectors. What input vector $\boldsymbol{v}$ gives $T(\boldsymbol{v}) = \boldsymbol{w}_1$?

11 Invert the matrix A in Problem 10. Also invert the transformation T—what are $T^{-1}(\boldsymbol{w}_1)$ and $T^{-1}(\boldsymbol{w}_2)$ and $T^{-1}(\boldsymbol{w}_3)$?

12 Which of these are true and why is the other one ridiculous?

(a) $T^{-1}T = I$ (b) $T^{-1}(T(\boldsymbol{v}_1)) = \boldsymbol{v}_1$ (c) $T^{-1}(T(\boldsymbol{w}_1)) = \boldsymbol{w}_1$.

13 Suppose the spaces $\mathbf{V}$ and $\mathbf{W}$ have the same basis $\boldsymbol{v}_1, \boldsymbol{v}_2$.

(a) Describe a transformation T (not I) that is its own inverse.

(b) Describe a transformation T (not I) that equals T^2.

(c) Why can't the same T be used for both (a) and (b)?

Questions 14–19 are about changing the basis.

14 (a) What matrix transforms $(1,0)$ into $(2,5)$ and transforms $(0,1)$ to $(1,3)$?

(b) What matrix transforms $(2,5)$ to $(1,0)$ and $(1,3)$ to $(0,1)$?

(c) Why does no matrix transform $(2,6)$ to $(1,0)$ and $(1,3)$ to $(0,1)$?

15 (a) What matrix M transforms $(1,0)$ and $(0,1)$ to (r,t) and (s,u)?

(b) What matrix N transforms (a,c) and (b,d) to $(1,0)$ and $(0,1)$?

(c) What condition on a, b, c, d will make part (b) impossible?

16 (a) How do M and N in Problem 15 yield the matrix that transforms (a,c) to (r,t) and (b,d) to (s,u)?

(b) What matrix transforms $(2,5)$ to $(1,1)$ and $(1,3)$ to $(0,2)$?

17 If you keep the same basis vectors but put them in a different order, the change of basis matrix M is a _____ matrix. If you keep the basis vectors in order but change their lengths, M is a _____ matrix.

18 The matrix that rotates the axis vectors $(1,0)$ and $(0,1)$ through an angle θ is Q. What are the coordinates (a,b) of the original $(1,0)$ using the new (rotated) axes? This *inverse* can be tricky. Draw a figure or solve for a and b:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + b \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

19 The matrix that transforms $(1,0)$ and $(0,1)$ to $(1,4)$ and $(1,5)$ is $M =$ ____. The combination $a(1,4) + b(1,5)$ that equals $(1,0)$ has $(a,b) = (\ ,\ )$. How are those new coordinates of $(1,0)$ related to M or M^{-1}?

Questions 20–23 are about the space of quadratic polynomials $A + Bx + Cx^2$.

20 The parabola $\boldsymbol{w}_1 = \frac{1}{2}(x^2 + x)$ equals one at $x = 1$, and zero at $x = 0$ and $x = -1$. Find the parabolas $\boldsymbol{w}_2, \boldsymbol{w}_3$, and then find $y(x)$ by linearity.

(a) $\boldsymbol{w}_2$ equals one at $x = 0$ and zero at $x = 1$ and $x = -1$.

(b) $\boldsymbol{w}_3$ equals one at $x = -1$ and zero at $x = 0$ and $x = 1$.

(c) $y(x)$ equals 4 at $x = 1$ and 5 at $x = 0$ and 6 at $x = -1$. Use $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$.

21 One basis for second-degree polynomials is $\boldsymbol{v}_1 = 1$ and $\boldsymbol{v}_2 = x$ and $\boldsymbol{v}_3 = x^2$. Another basis is $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ from Problem 20. Find two change of basis matrices, from the $\boldsymbol{w}$'s to the $\boldsymbol{v}$'s and from the $\boldsymbol{v}$'s to the $\boldsymbol{w}$'s.

22 What are the three equations for A, B, C if the parabola $Y = A + Bx + Cx^2$ equals 4 at $x = a$ and 5 at $x = b$ and 6 at $x = c$? Find the determinant of the 3 by 3 matrix. That matrix transforms values like 4, 5, 6 to parabolas—or is it the other way?

23 Under what condition on the numbers $m_1, m_2, \ldots, m_9$ do these three parabolas give a basis for the space of all parabolas?

$$\boldsymbol{v}_1 = m_1 + m_2x + m_3x^2, \quad \boldsymbol{v}_2 = m_4 + m_5x + m_6x^2, \quad \boldsymbol{v}_3 = m_7 + m_8x + m_9x^2.$$

24 The Gram-Schmidt process changes a basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ to an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$. These are columns in $A = QR$. Show that R is the change of basis matrix from the $\boldsymbol{a}$'s to the $\boldsymbol{q}$'s ($\boldsymbol{a}_2$ is what combination of $\boldsymbol{q}$'s when $A = QR$?).

25 Elimination changes the rows of A to the rows of U with $A = LU$. Row 2 of A is what combination of the rows of U? Writing $A^{\mathrm{T}} = U^{\mathrm{T}}L^{\mathrm{T}}$ to work with columns, the change of basis matrix is $M = L^{\mathrm{T}}$. (We have bases provided the matrices are ____.)

26 Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are **eigenvectors** for T. This means $T(\boldsymbol{v}_i) = \lambda_i \boldsymbol{v}_i$ for $i = 1, 2, 3$. What is the matrix for T when the input and output bases are the $\boldsymbol{v}$'s?

27 Every invertible linear transformation can have I as its matrix! Choose any input basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. For output basis choose $\boldsymbol{w}_i = T(\boldsymbol{v}_i)$. Why must T be invertible?

28 Using $\boldsymbol{v}_1 = \boldsymbol{w}_1$ and $\boldsymbol{v}_2 = \boldsymbol{w}_2$ find the standard matrix for these T's:

(a) $T(\boldsymbol{v}_1) = \boldsymbol{0}$ and $T(\boldsymbol{v}_2) = 3\boldsymbol{v}_1$ (b) $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$ and $T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \boldsymbol{v}_1$.

29 Suppose T is reflection across the x axis and S is reflection across the y axis. The domain $\mathbf{V}$ is the xy plane. If $\boldsymbol{v} = (x, y)$ what is $S(T(\boldsymbol{v}))$? Find a simpler description of the product ST.

30 Suppose T is reflection across the 45° line, and S is reflection across the y axis. If $v = (2,1)$ then $T(v) = (1,2)$. Find $S(T(v))$ and $T(S(v))$. This shows that generally $ST \neq TS$.

31 Show that the product ST of two reflections is a rotation. Multiply these reflection matrices to find the rotation angle:

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}.$$

32 True or false: If we know $T(v)$ for n different nonzero vectors in $\mathbf{R}^n$, then we know $T(v)$ for every vector in $\mathbf{R}^n$.

33 Express $e = (1,0,0,0)$ and $v = (1,-1,1,-1)$ in the wavelet basis, as in equations (8-10). The coefficients c_1, c_2, c_3, c_4 solve $Wc = e$ and $Wc = v$.

34 To represent $v = (7,5,3,1)$ in the wavelet basis, start with $(6,6,2,2)+(1,-1,1,-1)$. Then write $6,6,2,2$ as an overall average plus a difference, using $1,1,1,1$ and $1,1,-1,-1$.

35 What are the eight vectors in the wavelet basis for $\mathbf{R}^8$? They include the long wavelet $(1,1,1,1,-1,-1,-1,-1)$ and the short wavelet $(1,-1,0,0,0,0,0,0)$.

36 Suppose we have two bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ for $\mathbf{R}^n$. If a vector has coefficients b_i in one basis and c_i in the other basis, what is the change of basis matrix in $b = Mc$? Start from

$$b_1 v_1 + \cdots + b_n v_n = Vb = c_1 w_1 + \cdots + c_n w_n = Wc.$$

Your answer represents $T(v) = v$ with input basis of v's and output basis of w's. Because of different bases, the matrix is not I.

Challenge Problems

37 The space $\mathbf{M}$ of 2 by 2 matrices has the basis v_1, v_2, v_3, v_4 in Worked Example **7.2 B**. Suppose T multiplies each matrix by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. What 4 by 4 matrix A represents this transformation T on matrix space?

38 Suppose A is a 3 by 4 matrix of rank $r = 2$, and $T(v) = Av$. Choose input basis vectors v_1, v_2 from the row space of A and v_3, v_4 from the nullspace. Choose output basis $w_1 = Av_1$, $w_2 = Av_2$ in the column space and w_3 from the nullspace of A^{T}. What specially simple matrix represents this T in these special bases?

7.3 Diagonalization and the Pseudoinverse

This section produces better matrices by choosing better bases. When the goal is a diagonal matrix, one way is a basis of *eigenvectors*. The other way is two bases (the input and output bases are different). Those left and right *singular vectors* are orthonormal basis vectors for the four fundamental subspaces of A. They come from the SVD.

By reversing those input and output bases, we will find the "pseudoinverse" of A. This matrix A^+ sends $\mathbf{R}^m$ back to $\mathbf{R}^n$, and it sends column space back to row space.

The truth is that all our great factorizations of A can be regarded as a change of basis. But this is a short section, so we concentrate on the two outstanding examples. In both cases the good matrix is *diagonal*. It is Λ with one basis or Σ with two bases.

1. $S^{-1}AS = \Lambda$ ***when the input and output bases are eigenvectors of*** A.

2. $U^{-1}AV = \Sigma$ ***when those bases are eigenvectors of*** $A^{\mathrm{T}}A$ ***and*** AA^{T}.

You see immediately the difference between Λ and Σ. In Λ the bases are the same. Then $m = n$ and the matrix A must be square. And some square matrices cannot be diagonalized by any S, because they don't have n independent eigenvectors.

In Σ the input and output bases are different. The matrix A can be rectangular. The bases are *orthonormal* because $A^{\mathrm{T}}A$ and AA^{T} are symmetric. Then $U^{-1} = U^{\mathrm{T}}$ and $V^{-1} = V^{\mathrm{T}}$. Every matrix A is allowed, and A has the diagonal form Σ. This is the Singular Value Decomposition (**SVD**) of Section 6.7.

The eigenvector basis is orthonormal only when $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ (a "normal" matrix). That includes symmetric and antisymmetric and orthogonal matrices (*special* might be a better word than normal). In this case the singular values in Σ are the absolute values $\sigma_i = |\lambda_i|$, so that $\Sigma = \mathbf{abs}(\Lambda)$. The two diagonalizations are the same when $A^{\mathrm{T}}A = AA^{\mathrm{T}}$, except for possible factors -1 (real) and $e^{i\theta}$ (complex).

I will just note that the Gram-Schmidt factorization $A = QR$ chooses only *one* new basis. That is the orthogonal output basis given by Q. The input uses the standard basis given by I. We don't reach a diagonal Σ, but we do reach a triangular R. The output basis matrix appears on the left and the input basis appears on the right, in $A = QRI$.

We start with input basis equal to output basis. That will produce S and S^{-1}.

Similar Matrices: A and $S^{-1}AS$ and $W^{-1}AW$

Begin with a square matrix and one basis. The input space $\mathbf{V}$ is $\mathbf{R}^n$ and the output space $\mathbf{W}$ is also $\mathbf{R}^n$. The standard basis vectors are the columns of I. The matrix is n by n, and we call it A. The linear transformation T is "multiplication by A".

Most of this book has been about one fundamental problem—*to make the matrix simple*. We made it triangular in Chapter 2 (by elimination) and Chapter 4 (by Gram-Schmidt). We made it diagonal in Chapter 6 (*by eigenvectors*). Now that change from A to Λ comes from a ***change of basis***: ***Eigenvalue matrix from eigenvector basis.***

Here are the main facts in advance. When you change the basis for **V**, the matrix changes from A to AM. Because **V** is the input space, the matrix M goes on the right (to come first). When you change the basis for **W**, the new matrix is $M^{-1}A$. We are working with the output space so M^{-1} is on the left (to come last).

If you change both bases in the same way, the new matrix is $M^{-1}AM$. The good basis vectors are the eigenvectors of A, when the matrix becomes $S^{-1}AS = \Lambda$.

When the basis contains the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, the matrix for T becomes Λ.

Reason To find column 1 of the matrix, input the first basis vector $\boldsymbol{x}_1$. The transformation multiplies by A. The output is $A\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$. This is λ_1 times the first basis vector plus zero times the other basis vectors. Therefore the first column of the matrix is $(\lambda_1, 0, \ldots, 0)$. ***In the eigenvector basis, the matrix is diagonal.***

Example 1 Project onto the line $y = -x$ that goes from northwest to southeast. The vector $(1, 0)$ projects to $(.5, -.5)$ on that line. The projection of $(0, 1)$ is $(-.5, .5)$:

1. **Standard matrix: Project standard basis** $\quad A = \begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix}$.

2. **Find the diagonal matrix Λ in the eigenvector basis.**

Solution The eigenvectors for this projection are $\boldsymbol{x}_1 = (1, -1)$ and $\boldsymbol{x}_2 = (1, 1)$. The first eigenvector lies on the 135° line and the second is perpendicular (on the 45° line). Their projections are $\boldsymbol{x}_1$ and $\boldsymbol{0}$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$.

2. **Diagonalized matrix: Project eigenvectors** $\quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

3. **Find a third matrix B using another basis $\boldsymbol{v}_1 = \boldsymbol{w}_1 = (2, 0)$ and $\boldsymbol{v}_2 = \boldsymbol{w}_2 = (1, 1)$.**

Solution $\boldsymbol{w}_1$ is not an eigenvector, so the matrix B in this basis will not be diagonal. The first way to compute B follows the rule of Section 7.2:

Find column j of the matrix by writing the projection $T(\boldsymbol{v}_j)$ as a combination of $\boldsymbol{w}$'s.

Apply the projection T to $(2, 0)$. The result is $(1, -1)$ which is $\boldsymbol{w}_1 - \boldsymbol{w}_2$. So the first column of B contains 1 and -1. The second vector $\boldsymbol{w}_2 = (1, 1)$ projects to zero, so the second column of B contains 0 and 0. The eigenvalues must stay at 1 and 0:

3. **Third similar matrix: Project $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$** $\quad B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. $\quad (1)$

The second way to find the same B is more insightful. Use W^{-1} and W to change between the standard basis and the basis of $\boldsymbol{w}$'s. Those change of basis matrices are representing the identity transformation! The product of transformations is just ITI. *The product of matrices is* $B = W^{-1}AW$. This approach shows that B *is* ***similar*** to A.

For any basis $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ find the matrix B in three steps. Change the input basis to the standard basis with W. The matrix in the standard basis is A. Change the output basis back to the $\boldsymbol{w}$'s with W^{-1}. ***Then*** $\boldsymbol{B = W^{-1}AW}$ ***represents*** $\boldsymbol{ITI}$:

$$B_{\boldsymbol{w}\text{'s to }\boldsymbol{w}\text{'s}} = W^{-1}_{\text{standard to }\boldsymbol{w}\text{'s}} \quad A_{\text{standard}} \quad W_{\boldsymbol{w}\text{'s to standard}} \tag{2}$$

A change of basis produces a similarity transformation to $W^{-1}AW$ ***in the matrix.***

Example 2 (continuing with the projection) Apply this $W^{-1}AW$ rule to find B, when the basis $(2, 0)$ and $(1, 1)$ is in the columns of W:

$$W^{-1}AW = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

The $W^{-1}AW$ rule has produced the same B as in equation (1). The matrices A and B are similar. They have the same eigenvalues (1 and 0). And Λ is similar too.

Notice that the projection matrix keeps the property $A^2 = A$ and $B^2 = B$ and $\Lambda^2 = \Lambda$. The second projection doesn't move the first projection.

The Singular Value Decomposition (SVD)

Now the input basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ can be different from the output basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$. In fact the input space $\mathbf{R}^n$ can be different from the output space $\mathbf{R}^m$. Again the best matrix is diagonal (now m by n). To achieve this diagonal matrix Σ, each input vector $\boldsymbol{v}_j$ must transform into a multiple of the output vector $\boldsymbol{u}_j$. That multiple is the *singular value* σ_j on the main diagonal of Σ:

$$\textbf{SVD} \qquad A\boldsymbol{v}_j = \begin{cases} \sigma_j \boldsymbol{u}_j & \text{for } j \le r \\ \mathbf{0} & \text{for } j > r \end{cases} \qquad \text{with orthonormal bases.} \tag{3}$$

The singular values are in the order $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r$. The rank r enters because (by definition) singular values are not zero. The second part of the equation says that $\boldsymbol{v}_j$ is in the nullspace for $j = r + 1, \ldots, n$. This gives the correct number $n - r$ of basis vectors for the nullspace.

Let me connect the matrices with the linear transformations they represent. A and Σ represent the *same transformation*. $A = U\Sigma V^{\mathrm{T}}$ uses the standard bases for $\mathbf{R}^n$ and $\mathbf{R}^m$. The diagonal Σ uses the input basis of $\boldsymbol{v}$'s and the output basis of $\boldsymbol{u}$'s. The orthogonal matrices V and U give the basis changes; they represent the identity transformations (in $\mathbf{R}^n$ and $\mathbf{R}^m$). The product of transformations is ITI, and it is represented in the $\boldsymbol{v}$ and $\boldsymbol{u}$ bases by $U^{-1}AV$ which is Σ.

The matrix Σ in the $\boldsymbol{u}$ and $\boldsymbol{v}$ bases comes from A in the standard bases by $U^{-1}AV$:

$$\Sigma_{\boldsymbol{v}\text{'s to }\boldsymbol{u}\text{'s}} = U^{-1}_{\textit{standard}\text{ to }\boldsymbol{u}\text{'s}} \quad A_{\textit{standard}} \quad V_{\boldsymbol{v}\text{'s to }\textit{standard}}. \tag{4}$$

The SVD chooses orthonormal bases ($U^{-1} = U^{\mathrm{T}}$ and $V^{-1} = V^{\mathrm{T}}$) that diagonalize A.

The two orthonormal bases in the SVD are the eigenvector bases for $A^{\mathrm{T}}A$ (the $\boldsymbol{v}$'s) and AA^{T} (the $\boldsymbol{u}$'s). Since those are symmetric matrices, their unit eigenvectors are orthonormal. Their eigenvalues are the numbers σ_j^2. Equations (10) and (11) in Section 6.7 proved that those bases diagonalize the standard matrix A to produce Σ.

Polar Decomposition

Every complex number has the polar form $re^{i\theta}$. A nonnegative number r multiplies a number on the unit circle. (Remember that $|e^{i\theta}| = |\cos\theta + i\sin\theta| = 1$.) Thinking of these numbers as 1 by 1 matrices, $r \geq 0$ corresponds to a *positive semidefinite matrix* (call it H) and $e^{i\theta}$ corresponds to an *orthogonal matrix* Q. The ***polar decomposition*** extends this factorization to matrices: orthogonal times semidefinite, $A = QH$.

Every real square matrix can be factored into $A = \boldsymbol{QH}$, where Q is ***orthogonal*** and H is ***symmetric positive semidefinite***. If A is invertible, H is positive definite.

For the proof we just insert $V^{\mathrm{T}}V = I$ into the middle of the SVD:

Polar decomposition $$A = U\Sigma V^{\mathrm{T}} = (UV^{\mathrm{T}})(V\Sigma V^{\mathrm{T}}) = (Q)(H). \tag{5}$$

The first factor UV^{T} is Q. The product of orthogonal matrices is orthogonal. The second factor $V\Sigma V^{\mathrm{T}}$ is H. It is positive semidefinite because its eigenvalues are in Σ. If A is invertible then Σ and H are also invertible. ***H is the symmetric positive definite square root of $A^{\mathrm{T}}A$***. Equation (5) says that $H^2 = V\Sigma^2V^{\mathrm{T}} = A^{\mathrm{T}}A$.

There is also a polar decomposition $A = KQ$ in the reverse order. Q is the same but now $K = U\Sigma U^{\mathrm{T}}$. This is the symmetric positive definite square root of AA^{T}.

Example 3 Find the polar decomposition $A = QH$ from its SVD in Section 6.7:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \\ & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U\Sigma V^{\mathrm{T}}.$$

Solution The orthogonal part is $Q = UV^{\mathrm{T}}$. The positive definite part is $H = V\Sigma V^{\mathrm{T}}$. This is also $H = Q^{-1}A$ which is $Q^{\mathrm{T}}A$ because Q is orthogonal:

Orthogonal $$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Positive definite $$H = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 3/\sqrt{2} \end{bmatrix}.$$

In mechanics, the polar decomposition separates the ***rotation*** (in Q) from the ***stretching*** (in H). The eigenvalues of H are the singular values of A. They give the stretching factors. The eigenvectors of H are the eigenvectors of $A^{\mathrm{T}}A$. They give the stretching directions (the principal axes). Then Q rotates those axes.

The polar decomposition just splits the key equation $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ into two steps. The "H" part multiplies $\boldsymbol{v}_i$ by σ_i. The "Q" part swings $\boldsymbol{v}_i$ around into $\boldsymbol{u}_i$.

The Pseudoinverse

By choosing good bases, A multiplies $\boldsymbol{v}_i$ in the row space to give $\sigma_i \boldsymbol{u}_i$ in the column space. A^{-1} must do the opposite! If $A\boldsymbol{v} = \sigma \boldsymbol{u}$ then $A^{-1}\boldsymbol{u} = \boldsymbol{v}/\sigma$. The singular values of A^{-1} are $1/\sigma$, just as the eigenvalues of A^{-1} are $1/\lambda$. The bases are reversed. The $\boldsymbol{u}$'s are in the row space of A^{-1}, the $\boldsymbol{v}$'s are in the column space.

Until this moment we would have added "*if* A^{-1} *exists*." Now we don't. A matrix that multiplies $\boldsymbol{u}_i$ to produce $\boldsymbol{v}_i/\sigma_i$ *does* exist. It is the pseudoinverse A^+:

Pseudoinverse

$$A^+ = V\Sigma^+U^{\mathrm{T}} = \underset{n \text{ by } n}{\begin{bmatrix} \boldsymbol{v}_1 \cdots \boldsymbol{v}_r \cdots \boldsymbol{v}_n \end{bmatrix}} \underset{n \text{ by } m}{\begin{bmatrix} \sigma_1^{-1} & & & \\ & \ddots & & \\ & & \sigma_r^{-1} & \\ & & & \end{bmatrix}} \underset{m \text{ by } m}{\begin{bmatrix} \boldsymbol{u}_1 \cdots \boldsymbol{u}_r \cdots \boldsymbol{u}_m \end{bmatrix}^{\mathrm{T}}}$$

The ***pseudoinverse*** A^+ is an n by m matrix. If A^{-1} exists (we said it again), then A^+ is the same as A^{-1}. In that case $m = n = r$ and we are inverting $U\Sigma V^{\mathrm{T}}$ to get $V\Sigma^{-1}U^{\mathrm{T}}$. The new symbol A^+ is needed when $r < m$ or $r < n$. Then A has no two-sided inverse, but it has a *pseudo*inverse A^+ with that same rank r:

$$A^+\boldsymbol{u}_i = \frac{1}{\sigma_i}\boldsymbol{v}_i \quad \text{for } i \le r \qquad \text{and} \qquad A^+\boldsymbol{u}_i = \mathbf{0} \quad \text{for } i > r.$$

The vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ in the column space of A go back to $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ in the row space. The other vectors $\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m$ are in the left nullspace, and A^+ sends them to zero. When we know what happens to each basis vector $\boldsymbol{u}_i$, we know A^+.

Notice the pseudoinverse Σ^+ of the diagonal matrix Σ. Each σ is replaced by σ^{-1}. The product $\Sigma^+\Sigma$ is as near to the identity as we can get (it is a projection matrix, $\Sigma^+\Sigma$ is partly I and partly 0). We get r 1's. We can't do anything about the zero rows and columns. This example has $\sigma_1 = 2$ and $\sigma_2 = 3$:

$$\Sigma^+\Sigma = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \mathbf{0} \end{bmatrix}.$$

The pseudoinverse A^+ is the n by m matrix that makes AA^+ and A^+A into projections:

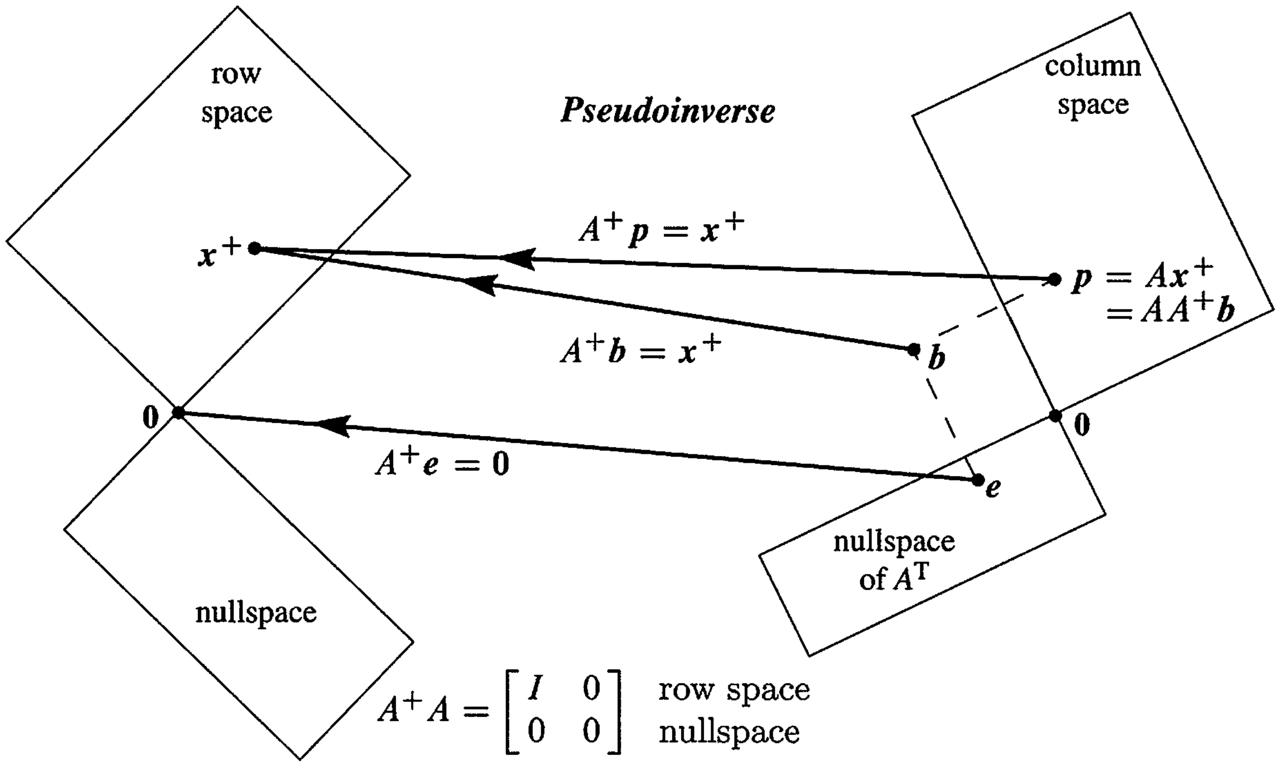


Figure 7.4: Ax^+ in the column space goes back to $A^+Ax^+ = x^+$ in the row space.

Trying for $AA^{-1} = A^{-1}A = I$	$AA^+ =$ projection matrix onto the column space of A $A^+A =$ projection matrix onto the row space of A

Example 4 Find the pseudoinverse of $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. This matrix is not invertible. The rank is 1. The only singular value is $\sqrt{10}$. That is inverted to $1/\sqrt{10}$ in Σ^+:

$$A^+ = V\Sigma^+U^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}.$$

A^+ also has rank 1. Its column space is the row space of A. When A takes $(1,1)$ in the row space to $(4,2)$ in the column space, A^+ does the reverse. $A^+(4,2) = (1,1)$.

Every rank one matrix is a column times a row. With unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, that is $A = \sigma\boldsymbol{u}\boldsymbol{v}^T$. Then the best inverse of a rank one matrix is $A^+ = \boldsymbol{v}\boldsymbol{u}^T/\sigma$. The product AA^+ is $\boldsymbol{u}\boldsymbol{u}^T$, the projection onto the line through $\boldsymbol{u}$. The product A^+A is $\boldsymbol{v}\boldsymbol{v}^T$.

Application to least squares Chapter 4 found the best solution $\widehat{x}$ to an unsolvable system $Ax = b$. The key equation is $A^TA\widehat{x} = A^Tb$, with the assumption that A^TA is invertible. The zero vector was alone in the nullspace.

Now A may have dependent columns (rank $< n$). There can be many solutions to $A^TA\widehat{x} = A^Tb$. ***One solution is*** $x^+ = A^+b$ ***from the pseudoinverse.*** We can check that

A^TAA^+b, is A^Tb, because Figure 7.4 shows that $e = b - AA^+b$ is the part of b in the nullspace of A^T. Any vector in the nullspace of A could be added to x^+, to give another solution $\widehat{x}$ to $A^TA\widehat{x} = A^Tb$. But x^+ will be shorter than any other $\widehat{x}$ (Problem 16):

The shortest least squares solution to $Ax = b$ is $x^+ = A^+b$.

The pseudoinverse A^+ and this best solution x^+ are essential in statistics, because experiments often have a matrix A with **dependent columns**.

■ REVIEW OF THE KEY IDEAS ■

1. Diagonalization $S^{-1}AS = \Lambda$ is the same as a change to the eigenvector basis.
2. The SVD chooses an input basis of v's and an output basis of u's. Those orthonormal bases diagonalize A. This is $Av_i = \sigma_i u_i$, and in matrix form $A = U\Sigma V^T$.
3. Polar decomposition factors A into QH, rotation UV^T times stretching $V\Sigma V^T$.
4. The pseudoinverse $A^+ = V\Sigma^+U^T$ transforms the column space of A back to its row space. A^+A is the identity on the row space (and zero on the nullspace).

■ WORKED EXAMPLES ■

7.3 A If A has rank n (full column rank) then it has a *left inverse* $C = (A^TA)^{-1}A^T$. This matrix C gives $CA = I$. Explain why the pseudoinverse is $A^+ = C$ in this case. If A has rank m (full row rank) then it has a *right inverse* B with $B = A^T(AA^T)^{-1}$. Then $AB = I$. Explain why $A^+ = B$ in this case.

Find B for A_1 and find C for A_2. Find A^+ for all three matrices A_1, A_2, A_3:

$$A_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Solution If A has rank n (independent columns) then A^TA is invertible—this is a key point of Section 4.2. Certainly $C = (A^TA)^{-1}A^T$ multiplies A to give $CA = I$. In the opposite order, $AC = A(A^TA)^{-1}A^T$ is the projection matrix (Section 4.2 again) onto the column space. So C meets the requirements to be A^+: *CA and AC are projections.*

If A has rank m (full row rank) then AA^T is invertible. Certainly A multiplies $B = A^T(AA^T)^{-1}$ to give $AB = I$. In the opposite order, $BA = A^T(AA^T)^{-1}A$ is the projection matrix onto the row space. So B is the pseudoinverse A^+ with rank m.

The example A_1 has full column rank (for C) and A_2 has full row rank (for B):

$$A_1^+ = (A_1^{\mathrm{T}}A_1)^{-1}A_1^{\mathrm{T}} = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 & 2 \end{bmatrix} \qquad A_2^+ = A_2^{\mathrm{T}}(A_2A_2^{\mathrm{T}})^{-1} = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Notice $A_1^+A_1 = [1]$ and $A_2A_2^+ = [1]$. But A_3 (rank 1) has no left or right inverse. **Its rank is not full.** Its pseudoinverse is $A_3^+ = \sigma_1^{-1}\boldsymbol{v}_1\boldsymbol{u}_1^{\mathrm{T}} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]/4$.

Problem Set 7.3

Problems 1–4 compute and use the SVD of a particular matrix (not invertible).

1 (a) Compute $A^{\mathrm{T}}A$ and its eigenvalues and unit eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Find σ_1.

$$\textbf{Rank one matrix} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

(b) Compute AA^{T} and its eigenvalues and unit eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

(c) Verify that $A\boldsymbol{v}_1 = \sigma_1\boldsymbol{u}_1$. Put numbers into the SVD:

$$\boldsymbol{A = U\Sigma V^{\mathrm{T}}} \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}}.$$

2 (a) From the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s in Problem 1 write down orthonormal bases for the four fundamental subspaces of this matrix A.

(b) Describe all matrices that have those same four subspaces. Multiples of A?

3 From U, V, and Σ in Problem 1 find the orthogonal matrix $Q = UV^{\mathrm{T}}$ and the symmetric matrix $H = V\Sigma V^{\mathrm{T}}$. Verify the polar decomposition $A = QH$. This H is only semidefinite because ______. Test $H^2 = A$.

4 Compute the pseudoinverse $A^+ = V\Sigma^+U^{\mathrm{T}}$. The diagonal matrix Σ^+ contains $1/\sigma_1$. Rename the four subspaces (for A) in Figure 7.4 as four subspaces for A^+. Compute the projections $P_{\text{row}} = A^+A$ and $P_{\text{column}} = AA^+$.

Problems 5–9 are about the SVD of an invertible matrix.

5 Compute $A^{\mathrm{T}}A$ and its eigenvalues and unit eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. What are the singular values σ_1 and σ_2 for this matrix A?

$$A = \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}.$$

6 AA^{T} has the same eigenvalues σ_1^2 and σ_2^2 as $A^{\mathrm{T}}A$. Find unit eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Put numbers into the SVD:

$$A = \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^{\mathrm{T}}.$$

7 In Problem 6, multiply columns times rows to show that $A = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}}$. Prove from $A = U\Sigma V^{\mathrm{T}}$ that every matrix of rank r is the sum of r matrices of rank one.

8 From U, V, and Σ find the orthogonal matrix $Q = UV^{\mathrm{T}}$ and the symmetric matrix $K = U\Sigma U^{\mathrm{T}}$. Verify the polar decomposition in reverse order $A = KQ$.

9 The pseudoinverse of this A is the same as _____ because _____.

Problems 10–11 compute and use the SVD of a 1 by 3 rectangular matrix.

10 Compute $A^{\mathrm{T}}A$ and AA^{T} and their eigenvalues and unit eigenvectors when the matrix is $A = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}$. What are the singular values of A?

11 Put numbers into the singular value decomposition of A:

$$A = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} = [\boldsymbol{u}_1] \begin{bmatrix} \sigma_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}^{\mathrm{T}}.$$

Put numbers into the pseudoinverse $V\Sigma^+U^{\mathrm{T}}$ of A. *Compute AA^+ and A^+A*:

$$\textbf{Pseudoinverse} \quad A^+ = \begin{bmatrix} & \\ & \\ & \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} \begin{bmatrix} 1/\sigma_1 \\ 0 \\ 0 \end{bmatrix} [\boldsymbol{u}_1]^{\mathrm{T}}.$$

12 What is the only 2 by 3 matrix that has no pivots and no singular values? What is Σ for that matrix? A^+ is the zero matrix, but what shape?

13 If $\det A = 0$ why is $\det A^+ = 0$? If A has rank r, why does A^+ have rank r?

14 When are the factors in $U\Sigma V^{\mathrm{T}}$ the same as in $Q\Lambda Q^{\mathrm{T}}$? The eigenvalues λ_i must be positive, to equal the σ_i. Then A must be _____ and positive _____.

Problems 15–18 bring out the main properties of A^+ and $x^+ = A^+b$.

15 All matrices in this problem have rank one. The vector $\boldsymbol{b}$ is (b_1, b_2).

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad A^{\mathrm{T}} = \begin{bmatrix} .2 & .1 \\ .2 & .1 \end{bmatrix} \quad AA^{\mathrm{T}} = \begin{bmatrix} .8 & .4 \\ .4 & .2 \end{bmatrix} \quad A^{\mathrm{T}}A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$$

(a) The equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ has many solutions because $A^{\mathrm{T}}A$ is _____.

(b) Verify that $\boldsymbol{x}^+ = A^+\boldsymbol{b} = (.2b_1 + .1b_2, .2b_1 + .1b_2)$ solves $A^{\mathrm{T}}A\boldsymbol{x}^+ = A^{\mathrm{T}}\boldsymbol{b}$.

(c) Add $(1,-1)$ to that x^+ to get another solution to $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. Show that $\|\widehat{x}\|^2 = \|x^+\|^2 + 2$, and x^+ is shorter.

16 ***The vector $x^+ = A^+b$ is the shortest possible solution to $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.*** Reason: The difference $\widehat{x} - x^+$ is in the nullspace of $A^{\mathrm{T}}A$. This is also the nullspace of A, orthogonal to x^+. Explain how it follows that $\|\widehat{x}\|^2 = \|x^+\|^2 + \|\widehat{x} - x^+\|^2$.

17 Every b in $\mathbf{R}^m$ is $p + e$. This is the column space part plus the left nullspace part. Every x in $\mathbf{R}^n$ is $x_r + x_n =$ (row space part) + (nullspace part). Then

$$AA^+p = \_\_\_\_ \qquad AA^+e = \_\_\_\_ \qquad A^+Ax_r = \_\_\_\_ \qquad A^+Ax_n = \_\_\_\_$$

18 Find A^+ and A^+A and AA^+ and x^+ for this 2 by 1 matrix and these b:

$$A = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} [1] \qquad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ and } b = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

19 A general 2 by 2 matrix A is determined by four numbers. If triangular, it is determined by three. If diagonal, by two. If a rotation, by one. An eigenvector, by one. Check that the total count is four for each factorization of A:

$$\textbf{Four numbers in} \quad LU \quad LDU \quad QR \quad U\Sigma V^{\mathrm{T}} \quad S\Lambda S^{-1}.$$

20 Following Problem 19, check that LDL^{T} and $Q\Lambda Q^{\mathrm{T}}$ are determined by *three* numbers. This is correct because the matrix A is now _____.

21 From $A = U\Sigma V^{\mathrm{T}}$ and $A^+ = V\Sigma^+U^{\mathrm{T}}$ explain these splittings into rank 1:

$$A = \sum_1^r \sigma_i u_i v_i^{\mathrm{T}} \qquad A^+ = \sum_1^r \frac{v_i u_i^{\mathrm{T}}}{\sigma_i} \qquad A^+A = \sum_1^r v_i v_i^{\mathrm{T}} \qquad AA^+ = \sum_1^r u_i u_i^{\mathrm{T}}$$

Challenge Problems

22 This problem looks for all matrices A with a given column space in $\mathbf{R}^m$ and a given row space in $\mathbf{R}^n$. Suppose $c_1, \ldots, c_r$ and $b_1, \ldots, b_r$ are bases for those two spaces. Make them columns of C and B. The goal is to show that ***$A = CMB^{\mathrm{T}}$ for an r by r invertible matrix M.*** *Hint: Start from $A = U\Sigma V^{\mathrm{T}}$.* A must have this form:

The first r columns of U and V must be connected to C and B by invertible matrices, because they contain bases for the same column space and row space.

23 A pair of singular vectors v and u will satisfy $Av = \sigma u$ and $A^{\mathrm{T}}u = \sigma v$. This means that the double vector $x = \begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector of what symmetric block matrix? What is the eigenvalue? The SVD of A is equivalent to the diagonalization of that symmetric block matrix.

Chapter 8

Applications

8.1 Matrices in Engineering

This section will show how engineering problems produce symmetric matrices K (often K is positive definite). The "linear algebra reason" for symmetry and positive definiteness is their form $K = A^{\mathrm{T}}A$ and $K = A^{\mathrm{T}}CA$. The "physical reason" is that the expression $\frac{1}{2}\boldsymbol{u}^{\mathrm{T}}K\boldsymbol{u}$ represents *energy*—and energy is never negative. The matrix C, often diagonal, contains positive physical constants like conductance or stiffness or diffusivity.

Our first examples come from mechanical and civil and aeronautical engineering. K is the ***stiffness matrix***, and $K^{-1}\boldsymbol{f}$ is the structure's response to forces $\boldsymbol{f}$ from outside. Section 8.2 turns to electrical engineering—the matrices come from networks and circuits. The exercises involve chemical engineering and I could go on! Economics and management and engineering design come later in this chapter (there the key is optimization).

Engineering leads to linear algebra in two ways, directly and indirectly:

Direct way The physical problem has only a finite number of pieces. The laws connecting their position or velocity are *linear* (movement is not too big or too fast). The laws are expressed by *matrix equations.*

Indirect way The physical system is "continuous". Instead of individual masses, the mass density and the forces and the velocities are functions of x or x, y or x, y, z. The laws are expressed by *differential equations.* ***To find accurate solutions we approximate by finite difference equations or finite element equations.***

Both ways produce matrix equations and linear algebra. I really believe that you cannot do modern engineering without matrices.

Here we present equilibrium equations $K\boldsymbol{u} = \boldsymbol{f}$. With motion, $M d^2\boldsymbol{u}/dt^2 + K\boldsymbol{u} = \boldsymbol{f}$ becomes dynamic. Then we use eigenvalues from $K\boldsymbol{x} = \lambda M\boldsymbol{x}$, or finite differences.

Before explaining the physical examples, may I write down the matrices? The tridiagonal K_0 appears many times in this textbook. Now we will see its applications. These matrices are all symmetric, and the first four are positive definite:

$$K_0 = A_0^{\mathrm{T}} A_0 = \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{2} \end{bmatrix} \qquad A_0^{\mathrm{T}} C_0 A_0 = \begin{bmatrix} c_1 + c_2 & -c_2 & \\ -c_2 & c_2 + c_3 & -c_3 \\ & -c_3 & c_3 + c_4 \end{bmatrix}$$

Fixed-fixed **Spring constants included**

$$K_1 = A_1^{\mathrm{T}} A_1 = \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{1} \end{bmatrix} \qquad A_1^{\mathrm{T}} C_1 A_1 = \begin{bmatrix} c_1 + c_2 & -c_2 & \\ -c_2 & c_2 + c_3 & -c_3 \\ & -c_3 & c_3 \end{bmatrix}$$

Fixed-free **Spring constants included**

$$K_{\textbf{singular}} = \begin{bmatrix} \mathbf{1} & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{1} \end{bmatrix} \qquad K_{\textbf{circular}} = \begin{bmatrix} 2 & -1 & \mathbf{-1} \\ -1 & 2 & -1 \\ \mathbf{-1} & -1 & 2 \end{bmatrix}$$

Free-free

The matrices $K_0, K_1, K_{\textbf{singular}}$, and $K_{\textbf{circular}}$ have $C = I$ for simplicity. This means that all the "spring constants" are $c_i = 1$. We included $A_0^{\mathrm{T}} C_0 A_0$ and $A_1^{\mathrm{T}} C_1 A_1$ to show how the spring constants enter the matrix (without changing its positive definiteness). Our first goal is to show where these stiffness matrices come from.

A Line of Springs

Figure 8.1 shows three masses m_1, m_2, m_3 connected by a line of springs. One case has four springs, with top and bottom fixed. The fixed-free case has only three springs; the lowest mass hangs freely. The **fixed-fixed** problem will lead to K_0 and $A_0^{\mathrm{T}} C_0 A_0$. The **fixed-free** problem will lead to K_1 and $A_1^{\mathrm{T}} C_1 A_1$. A **free-free** problem, with no support at either end, produces the matrix $K_{\textbf{singular}}$.

We want equations for the mass movements $\boldsymbol{u}$ and the tensions (or compressions) $\boldsymbol{y}$:

$$\begin{aligned} \boldsymbol{u} &= (u_1, u_2, u_3) = \textbf{movements of the masses} \text{ (down or up)} \\ \boldsymbol{y} &= (y_1, y_2, y_3, y_4) \text{ or } (y_1, y_2, y_3) = \textbf{tensions in the springs} \end{aligned}$$

When a mass moves downward, its displacement is positive ($u_i > 0$). For the springs, tension is positive and compression is negative ($y_i < 0$). In tension, the spring is stretched so it pulls the masses inward. Each spring is controlled by its own Hooke's Law $y = c\,e$: (*stretching force*) = (*spring constant*) times (*stretching distance*).

Our job is to link these one-spring equations $y = ce$ into a vector equation $K\boldsymbol{u} = f$ for the whole system. The force vector f comes from gravity. The gravitational constant g will multiply each mass to produce forces $f = (m_1 g, m_2 g, m_3 g)$.

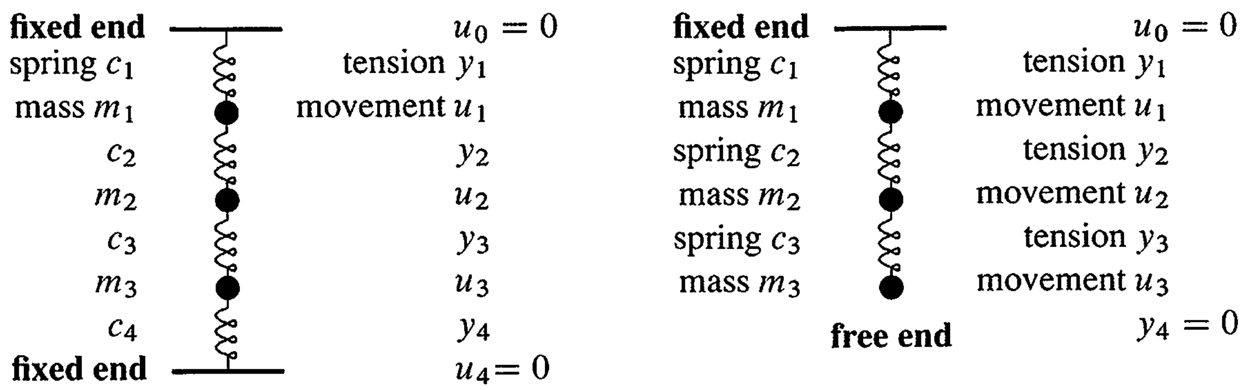


Figure 8.1: Lines of springs and masses: **fixed-fixed** and **fixed-free** ends.

The real problem is to find the stiffness matrix (**fixed-fixed** and **fixed-free**). The best way to create K is in three steps, not one. Instead of connecting the movements $\boldsymbol{u}_i$ directly to the forces, it is much better to connect each vector to the next in this list:

$$
\begin{array}{lcll}
u & = & \textit{Movements} \text{ of } n \text{ masses} & = (u_1, \ldots, u_n) \\
e & = & \textit{Elongations} \text{ of } m \text{ springs} & = (e_1, \ldots, e_m) \\
y & = & \textit{Internal forces} \text{ in } m \text{ springs} & = (y_1, \ldots, y_m) \\
f & = & \textit{External forces} \text{ on } n \text{ masses} & = (f_1, \ldots, f_n)
\end{array}
$$

The framework that connects $\boldsymbol{u}$ to $\boldsymbol{e}$ to $\boldsymbol{y}$ to f looks like this:

$$
\begin{array}{ccc}
\boxed{u} & & \boxed{f} \\
A\downarrow & & \uparrow A^{\mathrm{T}} \\
\boxed{e} & \xrightarrow{C} & \boxed{y}
\end{array}
\qquad
\begin{array}{ll}
e = Au & A \text{ is } m \text{ by } n \\
y = Ce & C \text{ is } m \text{ by } m \\
f = A^{\mathrm{T}}y & A^{\mathrm{T}} \text{ is } n \text{ by } m
\end{array}
$$

We will write down the matrices A and C and A^{T} for the two examples, first with fixed ends and then with the lower end free. Forgive the simplicity of these matrices, it is their form that is so important. Especially the appearance of A together with A^{T}.

The *elongation* $\boldsymbol{e}$ is the stretching distance—how far the springs are extended. Originally there is no stretching—the system is lying on a table. When it becomes vertical and upright, gravity acts. The masses move down by distances u_1, u_2, u_3. Each spring is stretched or compressed by $e_i = u_i - u_{i-1}$, *the difference in displacements of its ends*:

	First spring:	$e_1 = u_1$	(the top is fixed so $u_0 = 0$)
Stretching of	Second spring:	$e_2 = u_2 - u_1$	
each spring	Third spring:	$e_3 = u_3 - u_2$	
	Fourth spring:	$e_4 = \quad - u_3$	(the bottom is fixed so $u_4 = 0$)

If both ends move the same distance, that spring is not stretched: $u_i = u_{i-1}$ and $e_i = 0$. The matrix in those four equations is a 4 by 3 *difference matrix* A, and $\boldsymbol{e} = A\boldsymbol{u}$:

Stretching distances (elongations) $\quad \boldsymbol{e} = A\boldsymbol{u}$ is

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \qquad (1)$$

The next equation $\boldsymbol{y} = C\boldsymbol{e}$ connects spring elongation $\boldsymbol{e}$ with spring tension $\boldsymbol{y}$. *This is Hooke's Law $y_i = c_i e_i$ for each separate spring.* It is the "constitutive law" that depends on the material in the spring. A soft spring has small c, so a moderate force y can produce a large stretching e. Hooke's linear law is nearly exact for real springs, before they are overstretched and the material becomes plastic.

Since each spring has its own law, the matrix in $\boldsymbol{y} = C\boldsymbol{e}$ is a diagonal matrix C:

Hooke's Law $\boldsymbol{y} = C\boldsymbol{e}$

$$\begin{aligned} y_1 &= c_1 e_1 \\ y_2 &= c_2 e_2 \\ y_3 &= c_3 e_3 \\ y_4 &= c_4 e_4 \end{aligned} \quad \text{is} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \qquad (2)$$

Combining $\boldsymbol{e} = A\boldsymbol{u}$ with $\boldsymbol{y} = C\boldsymbol{e}$, the spring forces are $\boldsymbol{y} = CA\boldsymbol{u}$.

Finally comes the *balance equation*, the most fundamental law of applied mathematics. The internal forces from the springs balance the external forces on the masses. Each mass is pulled or pushed by the spring force y_j above it. From below it feels the spring force y_{j+1} plus f_j from gravity. Thus $y_j = y_{j+1} + f_j$ or $f_j = y_j - y_{j+1}$:

Force balance $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$

$$\begin{aligned} f_1 &= y_1 - y_2 \\ f_2 &= y_2 - y_3 \\ f_3 &= y_3 - y_4 \end{aligned} \quad \text{is} \quad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad (3)$$

That matrix is** A^{T}. The equation for balance of forces is $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$.* Nature transposes the rows and columns of the $\boldsymbol{e} - \boldsymbol{u}$ matrix to produce the $\boldsymbol{f} - \boldsymbol{y}$ matrix. This is the beauty of the framework, that A^{T} appears along with A. The three equations combine into $K\boldsymbol{u} = \boldsymbol{f}$, where the ***stiffness matrix is $K = A^{\mathrm{T}}CA$:

$$\left\{ \begin{aligned} \boldsymbol{e} &= A\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{e} \\ \boldsymbol{f} &= A^{\mathrm{T}}\boldsymbol{y} \end{aligned} \right\} \quad \text{combine into} \quad A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f} \quad \text{or} \quad K\boldsymbol{u} = \boldsymbol{f}.$$

In the language of elasticity, $\boldsymbol{e} = A\boldsymbol{u}$ is the **kinematic** equation (for displacement). The force balance $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$ is the **static** equation (for equilibrium). The **constitutive law** is $\boldsymbol{y} = C\boldsymbol{e}$ (from the material). Then $A^{\mathrm{T}}CA$ is n by n = (n by m)(m by m)(m by n).

Finite element programs spend major effort on assembling $K = A^{\mathrm{T}}CA$ from thousands of smaller pieces. We find K for four springs (**fixed-fixed**) by multiplying A^{T} times CA:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ -c_2 & c_2 & 0 \\ 0 & -c_3 & c_3 \\ 0 & 0 & -c_4 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$$

If all springs are identical, with $c_1 = c_2 = c_3 = c_4 = 1$, then $C = I$. The stiffness matrix reduces to $A^{\mathrm{T}}A$. It becomes the special $-1, 2, -1$ matrix:

$$\textbf{With } C = I \qquad K_0 = A_0^{\mathrm{T}}A_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}. \tag{4}$$

Note the difference between $A^{\mathrm{T}}A$ from engineering and LL^{T} from linear algebra. The matrix A from four springs is 4 by 3. The triangular matrix L from elimination is square. The stiffness matrix K is assembled from $A^{\mathrm{T}}A$, and then broken up into LL^{T}. One step is applied mathematics, the other is computational mathematics. Each K is built from rectangular matrices and factored into square matrices.

May I list some properties of $K = A^{\mathrm{T}}CA$? You know almost all of them:

1. K is **tridiagonal**, because mass 3 is not connected to mass 1.
2. K is **symmetric**, because C is symmetric and A^{T} comes with A.
3. K is **positive definite**, because $c_i > 0$ and A has **independent columns**.
4. K^{-1} is a full matrix in equation (5) with **all positive entries**.

That last property leads to an important fact about $\boldsymbol{u} = K^{-1}\boldsymbol{f}$: *If all forces act downwards* $(f_j > 0)$ *then all movements are downwards* $(u_j > 0)$. Notice that "positiveness" is different from "positive definiteness". Here K^{-1} is positive (K is not). Both K and K^{-1} are positive definite.

Example 1 Suppose all $c_i = c$ and $m_j = m$. Find the movements $\boldsymbol{u}$ and tensions $\boldsymbol{y}$.

All springs are the same and all masses are the same. But all movements and elongations and tensions will *not* be the same. K^{-1} includes $\frac{1}{c}$ because $A^{\mathrm{T}}CA$ includes c:

$$\boldsymbol{u} = K^{-1}\boldsymbol{f} = \frac{1}{4c}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} \mathbf{3/2} \\ \mathbf{2} \\ \mathbf{3/2} \end{bmatrix} \tag{5}$$

The displacement u_2, for the mass in the middle, is greater than u_1 and u_3. The units are correct: the force mg divided by force per unit length c gives a length u. Then

$$\boldsymbol{e} = A\boldsymbol{u} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \frac{mg}{c}\begin{bmatrix} \frac{3}{2} \\ 2 \\ \frac{3}{2} \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} \mathbf{3/2} \\ \mathbf{1/2} \\ \mathbf{-1/2} \\ \mathbf{-3/2} \end{bmatrix}.$$

Those elongations add to zero because the ends of the line are fixed. (The sum $u_1 + (u_2 - u_1) + (u_3 - u_2) + (-u_3)$ is certainly zero.) For each spring force y_i we just multiply e_i by c. So y_1, y_2, y_3, y_4 are $\frac{3}{2}mg$, $\frac{1}{2}mg$, $-\frac{1}{2}mg$, $-\frac{3}{2}mg$. The upper two springs are stretched, the lower two springs are compressed.

Notice how $\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{y}$ are computed in that order. We assembled $K = A^{\mathrm{T}}CA$ from rectangular matrices. To find $\boldsymbol{u} = K^{-1}\boldsymbol{f}$, we work with the whole matrix and not its three pieces! The rectangular matrices A and A^{T} do not have (two-sided) inverses.

Warning: *Normally you cannot write* $K^{-1} = A^{-1}C^{-1}(A^{\mathrm{T}})^{-1}$.

The three matrices are mixed together by $A^{\mathrm{T}}CA$, and they cannot easily be untangled. In general, $A^{\mathrm{T}}y = f$ has many solutions. And four equations $Au = e$ would usually have no solution with three unknowns. But $A^{\mathrm{T}}CA$ gives the correct solution to all three equations in the framework. Only when $m = n$ and the matrices are square can we go from $y = (A^{\mathrm{T}})^{-1}f$ to $e = C^{-1}y$ to $u = A^{-1}e$. We will see that now.

Fixed End and Free End

Remove the fourth spring. All matrices become 3 by 3. The pattern does not change! The matrix A loses its fourth row and (of course) A^{T} loses its fourth column. The new stiffness matrix K_1 becomes a product of square matrices:

$$A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

The missing column of A^{T} and row of A multiplied the missing c_4. So the quickest way to find the new $A^{\mathrm{T}}CA$ is to set $c_4 = 0$ in the old one:

FIXED
FREE
$$K_1 = A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}. \tag{6}$$

If $c_1 = c_2 = c_3 = 1$ and $C = I$, this is the $-1, 2, -1$ tridiagonal matrix, except the last entry is 1 instead of 2. The spring at the bottom is free.

Example 2 All $c_i = c$ and all $m_j = m$ in the fixed-free hanging line of springs. Then

$$K_1 = c\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad K_1^{-1} = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

The forces mg from gravity are the same. But the movements change from the previous example because the stiffness matrix has changed:

$$u = K_1^{-1}f = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}.$$

Those movements are greater in this fixed-free case. The number 3 appears in u_1 because all three masses are pulling the first spring down. The next mass moves by that 3 plus an additional 2 from the masses below it. The third mass drops even more $(3 + 2 + 1 = 6)$. The elongations $e = Au$ in the springs display those numbers $3, 2, 1$:

$$e = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Multiplying by c, the forces y in the three springs are $3mg$ and $2mg$ and mg. And the special point of square matrices is that y can be found directly from f! The balance equation $A^{\mathrm{T}}y = f$ determines y immediately, because $m = n$ and A^{T} is square. We are allowed to write $(A^{\mathrm{T}}CA)^{-1} = A^{-1}C^{-1}(A^{\mathrm{T}})^{-1}$:

$$y = (A^{\mathrm{T}})^{-1}f \quad \text{is} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \begin{bmatrix} 3mg \\ 2mg \\ 1mg \end{bmatrix}.$$

Two Free Ends: K is Singular

The first line of springs in Figure 8.2 is free at *both ends*. This means trouble (the whole line can move). The matrix A is 2 by 3, short and wide. Here is $\boldsymbol{e} = A\boldsymbol{u}$:

$$\textbf{FREE-FREE} \qquad \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_2 - u_1 \\ u_3 - u_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \qquad (7)$$

Now there is a nonzero solution to $A\boldsymbol{u} = \mathbf{0}$. ***The masses can move with no stretching of the springs.*** The whole line can shift by $\boldsymbol{u} = (1, 1, 1)$ and this leaves $\boldsymbol{e} = (0, 0)$. A has *dependent columns* and the vector $(1, 1, 1)$ is in its nullspace:

$$A\boldsymbol{u} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \textbf{no stretching}. \qquad (8)$$

$A\boldsymbol{u} = \mathbf{0}$ certainly leads to $A^{\mathrm{T}}CA\boldsymbol{u} = \mathbf{0}$. So $A^{\mathrm{T}}CA$ is only *positive semidefinite*, without c_1 and c_4. The pivots will be c_2 and c_3 and *no third pivot*. The rank is only 2:

$$\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & \\ & c_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \qquad (9)$$

Two eigenvalues will be positive but $\boldsymbol{x} = (1, 1, 1)$ is an eigenvector for $\lambda = 0$. We can solve $A^{\mathrm{T}}CA\boldsymbol{u} = f$ only for special vectors f. The forces have to add to $f_1 + f_2 + f_3 = 0$, or the whole line of springs (with both ends free) will take off like a rocket.

Circle of Springs

A third spring will complete the circle from mass 3 back to mass 1. This doesn't make K invertible—the new matrix is still singular. That stiffness matrix $K_{circular}$ is not tridiagonal, but it is symmetric (always) and *semidefinite*:

$$A^{\mathrm{T}}_{\textbf{circular}}A_{\textbf{circular}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \qquad (10)$$

The only pivots are 2 and $\frac{3}{2}$. The eigenvalues are 3 and 3 and 0. The determinant is zero. The nullspace still contains $\boldsymbol{x} = (1, 1, 1)$, when all the masses move together.

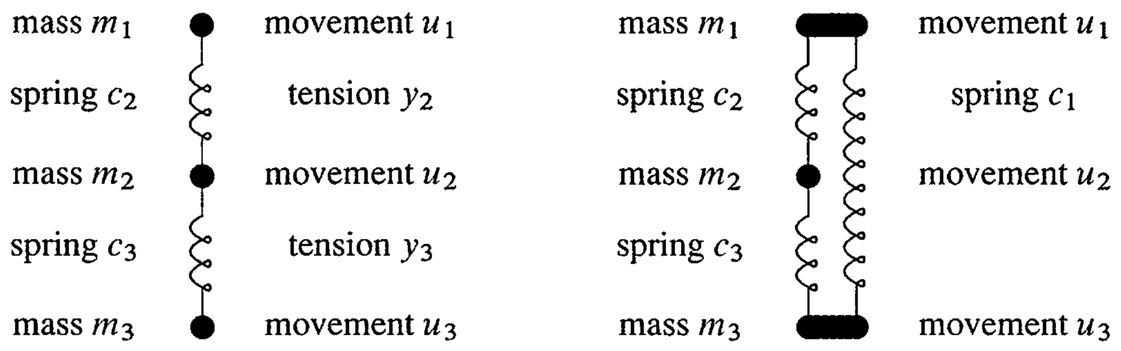


Figure 8.2: **Free-free ends**: A line of springs and a "circle" of springs: *Singular* K's. The masses can move without stretching the springs so $Au = \mathbf{0}$ has nonzero solutions.

This movement vector $(1, 1, 1)$ is in the nullspace of $A_{\textbf{circular}}$ and $K_{\textbf{circular}}$, even after the diagonal matrix C of spring constants is included: the springs are not stretched.

$$(A^{\mathrm{T}}CA)_{\textbf{circular}} = \begin{bmatrix} c_1 + c_2 & -c_2 & -c_1 \\ -c_2 & c_2 + c_3 & -c_3 \\ -c_1 & -c_3 & c_3 + c_1 \end{bmatrix}. \tag{11}$$

Continuous Instead of Discrete

Matrix equations are discrete. Differential equations are continuous. We will see the differential equation that corresponds to the tridiagonal $-1, 2, -1$ matrix $A^{\mathrm{T}}A$. And it is a pleasure to see the boundary conditions that go with K_0 and K_1.

The matrices A and A^{T} ***correspond to the derivatives*** d/dx ***and*** $-d/dx$! Remember that $\boldsymbol{e} = A\boldsymbol{u}$ took differences $u_i - u_{i-1}$, and $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$ took differences $y_i - y_{i+1}$. Now the springs are infinitesimally short, and those differences become derivatives:

$$\frac{u_i - u_{i-1}}{\Delta x} \text{ is like } \frac{du}{dx} \qquad \frac{y_i - y_{i+1}}{\Delta x} \text{ is like } -\frac{dy}{dx}$$

The factor Δx didn't appear earlier—we imagined the distance between masses was 1. To approximate a continuous solid bar, we take many more masses (smaller and closer). Let me jump to the three steps A, C, A^{T} in the continuous model, when there is stretching and Hooke's Law and force balance at every point x:

$$e(x) = Au = \frac{du}{dx} \qquad y(x) = c(x)e(x) \qquad A^{\mathrm{T}}y = -\frac{dy}{dx} = f(x)$$

Combining those equations into $A^{\mathrm{T}}CAu(x) = f(x)$, we have a differential equation not a matrix equation. The line of springs becomes an elastic bar:

$$\textbf{Solid Elastic Bar} \qquad A^{\mathrm{T}}CAu(x) = f(x) \quad \text{is} \quad -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \tag{12}$$

$A^{\mathrm{T}}A$ corresponds to a second derivative. A is a "difference matrix" and $A^{\mathrm{T}}A$ is a "second difference matrix". ***The matrix has*** $-1, 2, -1$ ***and the equation has*** $-d^2u/dx^2$:

$$-u_{i+1} + 2u_i - u_{i-1} \text{ is a } \textbf{second difference} \qquad -\frac{d^2u}{dx^2} \text{ is a } \textbf{second derivative}.$$

Now we see why this symmetric matrix is a favorite. When we meet a first derivative du/dx, we have three choices (*forward, backward, and centered differences*):

$$\frac{du}{dx} \simeq \frac{u(x+\Delta x) - u(x)}{\Delta x} \text{ or } \frac{u(x) - u(x-\Delta x)}{\Delta x} \text{ or } \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}.$$

When we meet d^2u/dx^2, the natural choice is $u(x+\Delta x) - 2u(x) + u(x-\Delta x)$, divided by $(\Delta x)^2$. *Why reverse these signs to* $-1, 2, -1$? Because the positive definite matrix has $+2$ on the diagonal. First derivatives are *anti*symmetric; the transpose has a minus sign. So second differences are negative definite, and we change to $-d^2u/dx^2$.

We have moved from vectors to functions. Scientific computing moves the other way. It starts with a differential equation like (12). Sometimes there is a formula for the solution $u(x)$, more often not. In reality we *create* the discrete matrix K by approximating the continuous problem. Watch how the boundary conditions on u come in! By missing u_0 we treat it (correctly) as zero:

$$\begin{array}{l}\textbf{FIXED}\\ \textbf{FIXED}\end{array} \qquad Au = \frac{1}{\Delta x}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \approx \frac{du}{dx} \quad \text{with} \quad \begin{array}{l} \boldsymbol{u_0 = 0} \\ \boldsymbol{u_4 = 0} \end{array} \qquad (13)$$

Fixing the top end gives the boundary condition $u_0 = 0$. What about the free end, when the bar hangs in the air? Row 4 of A is gone and so is u_4. The boundary condition must come from A^{T}. It is the missing y_4 that we are treating (correctly) as zero:

$$\begin{array}{l}\textbf{FIXED}\\ \textbf{FREE}\end{array} \qquad A^{\mathrm{T}}y = \frac{1}{\Delta x}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \approx -\frac{dy}{dx} \quad \text{with} \quad \begin{array}{l} \boldsymbol{u_0 = 0} \\ \boldsymbol{y_4 = 0} \end{array} \qquad (14)$$

The boundary condition $y_4 = 0$ *at the free end becomes* $du/dx = 0$, since $y = Au$ corresponds to du/dx. The force balance $A^{\mathrm{T}}y = f$ at that end (in the air) is $0 = 0$. The last row of $K_1 u = f$ has entries $-1, 1$ to reflect this condition $du/dx = 0$.

May I summarize this section? I hope this example will help you turn calculus into linear algebra, replacing differential equations by difference equations. If your step Δx is small enough, you will have a totally satisfactory solution.

$$\textbf{The equation is} \quad -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \ \textbf{ with } \ u(0) = 0 \ \textbf{ and } \ \left[u(1) \text{ or } \frac{du}{dx}(1)\right] = 0$$

Divide the bar into N pieces of length Δx. Replace du/dx by Au and $-dy/dx$ by $A^{\mathrm{T}}y$. Now A and A^{T} include $1/\Delta x$. The end conditions are $u_0 = 0$ and $[u_N = 0$ or $y_N = 0]$.

The three steps $-d/dx$ and $c(x)$ and d/dx correspond to A^{T} and C and A:

$$f = A^{\mathrm{T}}y \text{ and } y = Ce \text{ and } e = Au \text{ give } A^{\mathrm{T}}CAu = f.$$

This is a fundamental example in computational science and engineering. Our book concentrates on Step 3 in that process (linear algebra). Now we have taken Step 2.

1. Model the problem by a differential equation
2. Discretize the differential equation to a difference equation
3. Understand and solve the difference equation (and boundary conditions!)
4. Interpret the solution; visualize it; redesign if needed.

Numerical simulation has become a third branch of science, together with experiment and deduction. Designing the Boeing 777 was much less expensive on a computer than in a wind tunnel. Our discussion still has to move from ordinary to partial differential equations, and from linear to nonlinear.

The texts *Introduction to Applied Mathematics* and *Computational Science and Engineering* (Wellesley-Cambridge Press) develop this whole subject further—see the course page **math.mit.edu/18085** with video lectures (also on **ocw.mit.edu**). The principles remain the same, and I hope this book helps you to see the framework behind the computations.

Problem Set 8.1

1 Show that $\det A_0^{\mathrm{T}}C_0A_0 = c_1c_2c_3+c_1c_3c_4+c_1c_2c_4+c_2c_3c_4$. Find also $\det A_1^{\mathrm{T}}C_1A_1$ in the fixed-free example.

2 Invert $A_1^{\mathrm{T}}C_1A_1$ in the fixed-free example by multiplying $A_1^{-1}C_1^{-1}(A_1^{\mathrm{T}})^{-1}$.

3 In the free-free case when $A^{\mathrm{T}}CA$ in equation (9) is singular, add the three equations $A^{\mathrm{T}}CAu = f$ to show that we need $f_1 + f_2 + f_3 = 0$. Find a solution to $A^{\mathrm{T}}CAu = f$ when the forces $f = (-1, 0, 1)$ balance themselves. Find all solutions!

4 Both end conditions for the free-free differential equation are $du/dx = 0$:

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \quad \text{with} \quad \frac{du}{dx} = 0 \text{ at both ends.}$$

Integrate both sides to show that the force $f(x)$ must balance itself, $\int f(x)\,dx = 0$, or there is no solution. The complete solution is one particular solution $u(x)$ plus any constant. The constant corresponds to $u = (1, 1, 1)$ in the nullspace of $A^{\mathrm{T}}CA$.

5 In the fixed-free problem, the matrix A is square and invertible. We can solve $A^{\mathrm{T}}y = f$ separately from $Au = e$. Do the same for the differential equation:

$$\text{Solve } -\frac{dy}{dx} = f(x) \text{ with } y(1) = 0. \quad \text{Graph } y(x) \text{ if } f(x) = 1.$$

6 The 3 by 3 matrix $K_1 = A_1^{\mathrm{T}} C_1 A_1$ in equation (6) splits into three "element matrices" $c_1 E_1 + c_2 E_2 + c_3 E_3$. Write down those pieces, one for each c. Show how they come from *column times row* multiplication of $A_1^{\mathrm{T}} C_1 A_1$. This is how finite element stiffness matrices are actually assembled.

7 For five springs and four masses with both ends fixed, what are the matrices A and C and K? With $C = I$ solve $K\boldsymbol{u} = \text{ones}(4)$.

8 Compare the solution $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ in Problem 7 to the solution of the continuous problem $-u'' = 1$ with $u(0) = 0$ and $u(1) = 0$. The parabola $u(x)$ should correspond at $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ to $\boldsymbol{u}$—is there a $(\Delta x)^2$ factor to account for?

9 Solve the fixed-free problem $-u'' = mg$ with $u(0) = 0$ and $u'(1) = 0$. Compare $u(x)$ at $x = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ with the vector $\boldsymbol{u} = (3mg, 5mg, 6mg)$ in Example 2.

10 Suppose $c_1 = c_2 = c_3 = c_4 = 1$, $m_1 = 2$ and $m_2 = m_3 = 1$. Solve $A^{\mathrm{T}}CA\,\boldsymbol{u} = (2, 1, 1)$ for this fixed-fixed line of springs. Which mass moves the most (largest u)?

11 (MATLAB) Find the displacements $u(1), \ldots, u(100)$ of 100 masses connected by springs all with $c = 1$. Each force is $f(i) = .01$. Print graphs of $\boldsymbol{u}$ with **fixed-fixed** and **fixed-free** ends. Note that diag(ones(n, 1), d) is a matrix with n ones along diagonal d. This print command will graph a vector u:

```
plot(u, '+');   xlabel('mass number');   ylabel('movement');   print
```

12 (MATLAB) Chemical engineering has a first derivative du/dx from fluid velocity as well as d^2u/dx^2 from diffusion. Replace du/dx by a *forward* difference, then a *centered* difference, then a *backward* difference, with $\Delta x = \frac{1}{8}$. Graph your three numerical solutions of

$$-\frac{d^2u}{dx^2} + 10\,\frac{du}{dx} = 1 \quad \text{with} \quad u(0) = u(1) = 0.$$

This ***convection-diffusion equation*** appears everywhere. It transforms to the Black-Scholes equation for option prices in mathematical finance.

Problem 12 is developed into the first MATLAB homework in my 18.085 course on Computational Science and Engineering at MIT. Videos on *ocw.mit.edu*.

8.2 Graphs and Networks

Over the years I have seen one model so often, and I found it so basic and useful, that I always put it first. The model consists of ***nodes connected by edges***. This is called a ***graph***.

Graphs of the usual kind display functions $f(x)$. Graphs of this node-edge kind lead to matrices. This section is about the ***incidence matrix*** of a graph—which tells how the n nodes are connected by the m edges. Normally $m > n$, there are more edges than nodes.

For any m by n matrix there are two fundamental subspaces in $\mathbf{R}^n$ and two in $\mathbf{R}^m$. They are the row spaces and nullspaces of A and A^{T}. Their *dimensions* are related by the most important theorem in linear algebra. The second part of that theorem is the *orthogonality* of the subspaces. Our goal is to show how examples from graphs illuminate the Fundamental Theorem of Linear Algebra.

We review the four subspaces (for any matrix). Then we construct a ***directed graph*** and its ***incidence matrix***. The dimensions will be easy to discover. But we want the subspaces themselves—this is where orthogonality helps. It is essential to connect the subspaces to the graph they come from. By specializing to incidence matrices, the laws of linear algebra become Kirchhoff's laws. Please don't be put off by the words "current" and "voltage" and "Kirchhoff." These rectangular matrices are the best.

Every entry of an incidence matrix is 0 or 1 or -1. This continues to hold during elimination. All pivots and multipliers are ± 1. Therefore both factors in $A = LU$ also contain $0, 1, -1$. So do the nullspace matrices! All four subspaces have basis vectors with these exceptionally simple components. The matrices are not concocted for a textbook, they come from a model that is absolutely essential in pure and applied mathematics.

Here is a first incidence matrix. Notice -1 and 1 in each row. This matrix takes ***differences in voltage***, across six edges of a graph. The voltages are x_1, x_2, x_3, x_4 at the four nodes in Figure 8.4—where we will construct this matrix A. Its echelon form is U:

$$\begin{array}{l} \textbf{Incidence} \\ \textbf{matrix} \\ \textbf{6 edges} \\ \textbf{4 nodes} \end{array} \quad A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{reduces to} \quad U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nullspace of A and U is the line through $\boldsymbol{x} = (1, 1, 1, 1)$. The column spaces of A and U have dimension $r = 3$. The pivot rows are a basis for the row space.

Figure 8.3 shows more—the subspaces are orthogonal. ***Every vector in the nullspace is perpendicular to every vector in the row space***. This comes directly from the m equations $A\boldsymbol{x} = \mathbf{0}$. For A and U above, $\boldsymbol{x} = (1, 1, 1, 1)$ is perpendicular to all rows and thus to the whole row space. Equal voltages produce no current!

I would like to review the Four Fundamental Subspaces before using them. The whole point will be to see their meaning on the network.

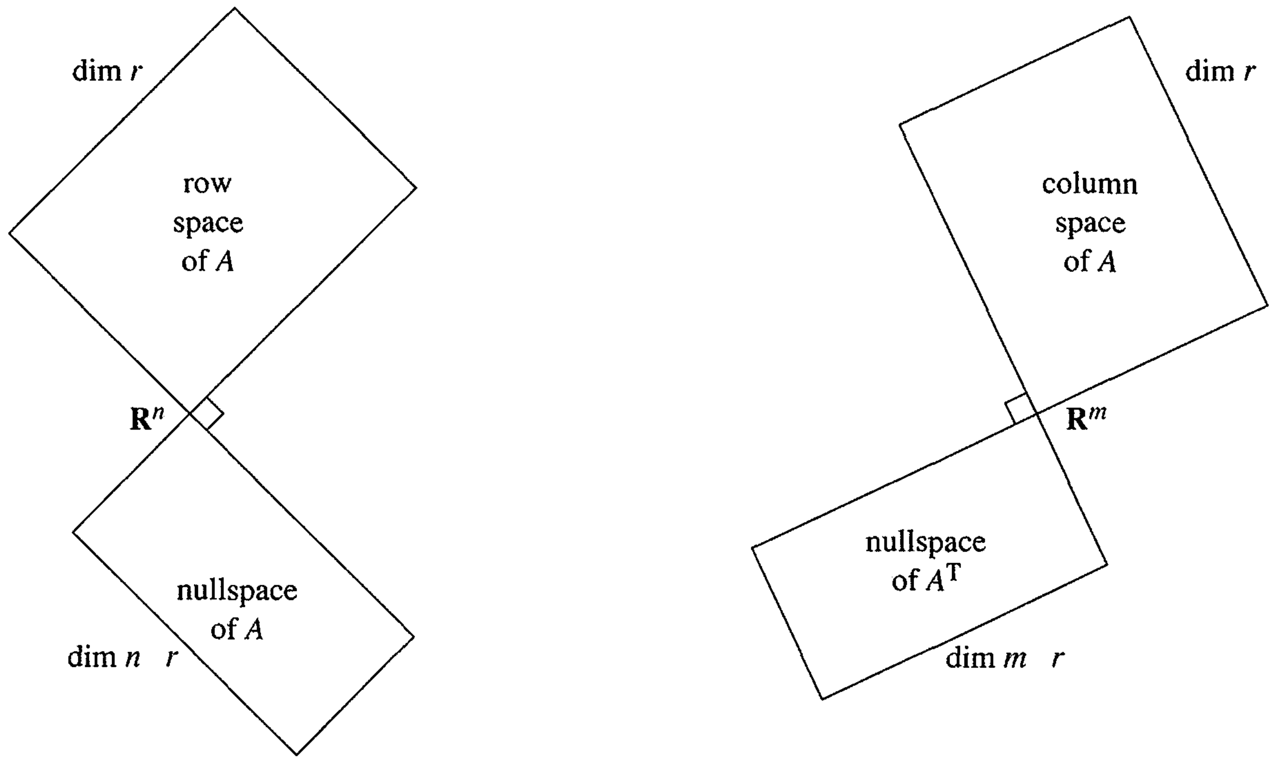


Figure 8.3: **Big picture:** The four subspaces with their dimensions and orthogonality.

Start with an m by n matrix. Its columns are vectors in $\mathbf{R}^m$. Their linear combinations produce the ***column space*** $C(A)$, a subspace of $\mathbf{R}^m$. Those combinations are exactly the matrix-vector products $A\boldsymbol{x}$.

The rows of A are vectors in $\mathbf{R}^n$ (or they would be, if they were column vectors). Their linear combinations produce the ***row space***. To avoid any inconvenience with rows, we transpose the matrix. The row space becomes $C(A^{\mathrm{T}})$, the column space of A^{T}.

The central questions of linear algebra come from these two ways of looking at the same numbers, by columns and by rows.

The ***nullspace*** $N(A)$ contains every $\boldsymbol{x}$ that satisfies $A\boldsymbol{x} = \mathbf{0}$—this is a subspace of $\mathbf{R}^n$. The ***"left" nullspace*** contains all solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Now $\boldsymbol{y}$ has m components, and $N(A^{\mathrm{T}})$ is a subspace of $\mathbf{R}^m$. Written as $\boldsymbol{y}^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$, we are combining rows of A to produce the zero row. The four subspaces are illustrated by Figure 8.3, which shows $\mathbf{R}^n$ on one side and $\mathbf{R}^m$ on the other. The link between them is A.

The information in that figure is crucial. First come the dimensions, which obey the two central laws of linear algebra:

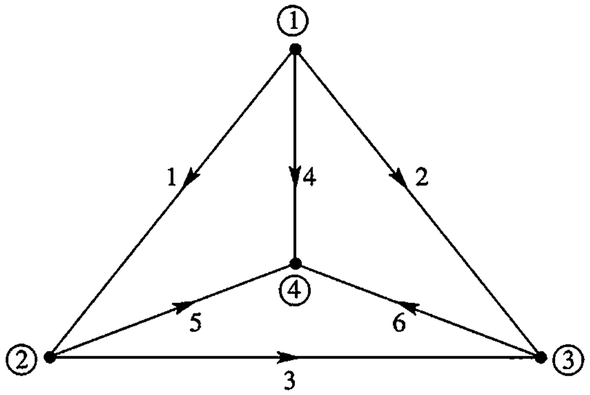
$$\dim \boldsymbol{C}(A) = \dim \boldsymbol{C}(A^{\mathrm{T}}) \quad \text{and} \quad \dim \boldsymbol{C}(A) + \dim \boldsymbol{N}(A) = n.$$

When the row space has dimension r, the nullspace has dimension $n - r$. Elimination leaves these two spaces unchanged, and the echelon form U gives the dimension count. There are r rows and columns with pivots. There are $n - r$ free columns without pivots, and those lead to vectors in the nullspace.

This review of the subspaces applies to any matrix A—only the example was special. Now we concentrate on that example. It is the incidence matrix for a particular graph, and we look to the graph for the meaning of every subspace.

Directed Graphs and Incidence Matrices

Figure 8.4 displays a *graph* with $m = 6$ edges and $n = 4$ nodes, so the matrix A is 6 by 4. It tells which nodes are connected by which edges. The entries -1 and $+1$ also tell the direction of each arrow (this is a *directed* graph). The first row $-1, 1, 0, 0$ of A gives a record of the first edge from node 1 to node 2:



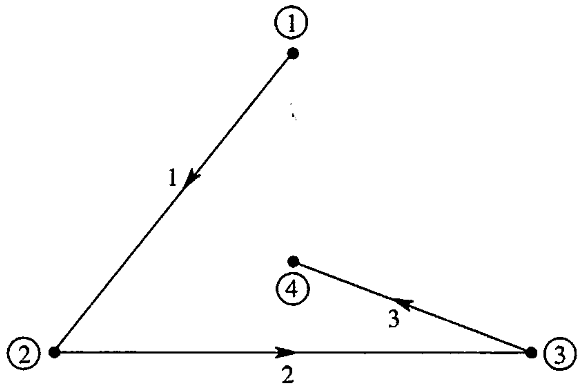
$$\begin{array}{cc} & \textbf{node} \\ & \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ A = & \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \quad \textbf{edge}$$

Figure 8.4a: Complete graph with $m = 6$ edges and $n = 4$ nodes.

Row numbers are edge numbers, column numbers are node numbers.

You can write down A immediately by looking at the graph.

The second graph has the same four nodes but only three edges. Its incidence matrix is 3 by 4:



$$\begin{array}{cc} & \textbf{node} \\ & \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ B = & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \quad \textbf{edge}$$

Figure 8.4b: Tree with 3 edges and 4 nodes and no loops.

The first graph is *complete*—every pair of nodes is connected by an edge. The second graph is a ***tree***—the graph has ***no closed loops***. Those graphs are the two extremes, the maximum number of edges is $\frac{1}{2}n(n-1)$ and the minimum (a tree) is $m = n - 1$.

The rows of B match the nonzero rows of U—the echelon form found earlier. ***Elimination reduces every graph to a tree.*** The loops produce zero rows in U. Look at the loop from edges 1, 2, 3 in the first graph, which leads to a zero row:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Those steps are typical. When two edges share a node, elimination produces the "shortcut edge" without that node. If the graph already has this shortcut edge, elimination gives a row of zeros. When the dust clears we have a tree.

An idea suggests itself: ***Rows are dependent when edges form a loop.*** Independent rows come from trees. This is the key to the row space. We are assuming that the graph is connected, and it makes no fundamental difference which way the arrows go. On each edge, flow with the arrow is "positive." Flow in the opposite direction counts as negative. The flow might be a current or a signal or a force—or even oil or gas or water.

For the column space we look at Ax, which is a vector of differences:

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix}. \tag{1}$$

The unknowns x_1, x_2, x_3, x_4 represent ***potentials*** or ***voltages*** at the nodes. Then Ax gives the ***potential differences*** or ***voltage differences*** across the edges. It is these differences that cause flows. We now examine the meaning of each subspace.

1 The ***nullspace*** contains the solutions to $Ax = \mathbf{0}$. All six potential differences are zero. This means: *All four potentials are equal.* Every x in the nullspace is a constant vector (c, c, c, c). The nullspace of A is a line in $\mathbf{R}^n$—its dimension is $n - r = 1$.

The second incidence matrix B has the same nullspace. It contains $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$:

$$Bx = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can raise or lower all potentials by the same amount c, without changing the differences. There is an "arbitrary constant" in the potentials. Compare this with the same statement for functions. We can raise or lower $f(x)$ by the same amount C, without changing its derivative. There is an arbitrary constant C in the integral.

Calculus adds "$+C$" to indefinite integrals. Graph theory adds (c, c, c, c) to the vector x of potentials. Linear algebra adds any vector x_n in the nullspace to one particular solution of $Ax = b$.

The "$+C$" disappears in calculus when the integral starts at a known point $x = a$. Similarly the nullspace disappears when we set $x_4 = 0$. The unknown x_4 is removed and so are the fourth columns of A and B. Electrical engineers would say that node 4 has been "grounded."

2 The ***row space*** contains all combinations of the six rows. Its dimension is certainly not six. The equation $r + (n - r) = n$ must be $3 + 1 = 4$. The rank is $r = 3$, as we also saw from elimination. After 3 edges, we start forming loops! The new rows are not independent.

How can we tell if $\boldsymbol{v} = (v_1, v_2, v_3, v_4)$ is in the row space? The slow way is to combine rows. The quick way is by orthogonality:

$\boldsymbol{v}$ is in the row space if and only if it is perpendicular to $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ in the nullspace.

The vector $\boldsymbol{v} = (0, 1, 2, 3)$ fails this test—its components add to 6. The vector $(-6, 1, 2, 3)$ passes the test. It lies in the row space because its components add to zero. It equals $6(\text{row } 1) + 5(\text{row } 3) + 3(\text{row } 6)$.

Each row of A adds to zero. This must be true for every vector in the row space.

3 The ***column space*** contains all combinations of the four columns. We expect three independent columns, since there were three independent rows. The first three columns are independent (so are any three). But the four columns add to the zero vector, which says again that $(1, 1, 1, 1)$ is in the nullspace. ***How can we tell if a particular vector $\boldsymbol{b}$ is in the column space of an incidence matrix?***

First answer Try to solve $A\boldsymbol{x} = \boldsymbol{b}$. That misses all the insight. As before, orthogonality gives a better answer. We are now coming to Kirchhoff's two famous laws of circuit theory—the voltage law and current law. Those are natural expressions of "laws" of linear algebra. It is especially pleasant to see the key role of the left nullspace.

Second answer $A\boldsymbol{x}$ is the vector of differences in equation (1). If we add differences around a closed loop in the graph, the cancellation leaves zero. Around the big triangle formed by edges $1, 3, -2$ (the arrow goes backward on edge 2) the differences cancel:

$$\textbf{Voltage Law} \qquad (x_2 - x_1) + (x_3 - x_2) - (x_3 - x_1) = 0.$$

The components of $A\boldsymbol{x}$ add to zero around every loop. When $\boldsymbol{b}$ is in the column space of A, it must obey the same law:

$$\textit{\textbf{Kirchhoff's Law:}} \qquad b_1 + b_3 - b_2 = 0.$$

By testing each loop, we decide whether $\boldsymbol{b}$ is in the column space. $A\boldsymbol{x} = \boldsymbol{b}$ can be solved exactly when the components of $\boldsymbol{b}$ satisfy all the same dependencies as the rows of A. Then elimination leads to $0 = 0$, and $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

4 The ***left nullspace*** contains the solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Its dimension is $m - r = 6 - 3$:

Current Law (KCL)
$$A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (2)$$

The true number of equations is $r = 3$ and not $n = 4$. Reason: The four equations add to $0 = 0$. The fourth equation follows automatically from the first three.

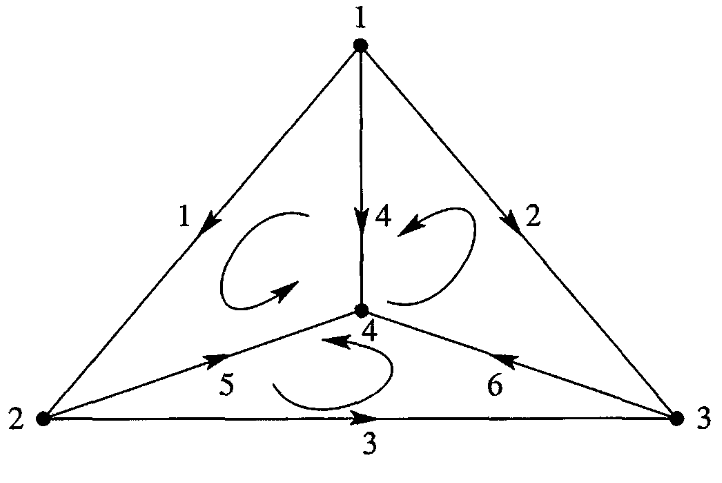
What do the equations mean? The first equation says that $-y_1 - y_2 - y_4 = 0$. ***The net flow into node* 1 *is zero***. The fourth equation says that $y_4 + y_5 + y_6 = 0$. ***Flow into the node minus flow out is zero***. The equations $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ are famous and fundamental:

Kirchhoff's Current Law: ***Flow in equals flow out at each node.***

This law deserves first place among the equations of applied mathematics. It expresses "*conservation*" and "*continuity*" and "*balance*." Nothing is lost, nothing is gained. When currents or forces are in equilibrium, the equation to solve is $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Notice the beautiful fact that the matrix in this balance equation is the transpose of the incidence matrix A.

What are the actual solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$? The currents must balance themselves. The easiest way is to **flow around a loop**. If a unit of current goes around the big triangle (forward on edge 1, forward on 3, backward on 2), the vector is $\boldsymbol{y} = (1, -1, 1, 0, 0, 0)$. This satisfies $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. *Every loop current is a solution to the Current Law.* Around the loop, flow in equals flow out at every node. A smaller loop goes forward on edge 1, forward on 5, back on 4. Then $\boldsymbol{y} = (1, 0, 0, -1, 1, 0)$ is also in the left nullspace.

We expect three independent $\boldsymbol{y}$'s, since $6 - 3 = 3$. The three small loops in the graph are independent. The big triangle seems to give a fourth $\boldsymbol{y}$, but it is the sum of flows around the small loops. The small loops give a basis for the left nullspace.



$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

small loops big loop

Summary The incidence matrix A comes from a connected graph with n nodes and m edges. The row space and column space have dimensions $n-1$. The nullspaces of A and A^{T} have dimension 1 and $m-n+1$:

1 The constant vectors $(c, c, \ldots, c)$ make up the nullspace of A.

2 There are $r = n-1$ independent rows, using edges from any tree.

3 ***Voltage law***: The components of Ax add to zero around every loop.

4 ***Current law***: $A^{\mathrm{T}}y = \mathbf{0}$ is solved by loop currents. $N(A^{\mathrm{T}})$ has dimension $m-r$. ***There are*** $\boldsymbol{m-r=m-n+1}$ ***independent loops in the graph.***

For every graph in a plane, linear algebra yields ***Euler's formula***:

$$(\textbf{\textit{number of nodes}}) - (\textbf{\textit{number of edges}}) + (\textbf{\textit{number of small loops}}) = \mathbf{1}.$$

This is $\boldsymbol{n-m+(m-n+1) = 1}$. The graph in our example has $4-6+3=1$.

A single triangle has (3 nodes) $-$ (3 edges) $+$ (1 loop). On a 10-node tree with 9 edges and no loops, Euler's count is $10-9+0$. All planar graphs lead to the answer 1.

Networks and $A^{\mathrm{T}}CA$

In a real network, the current y along an edge is the product of two numbers. One number is the difference between the potentials x at the ends of the edge. This difference is Ax and it drives the flow. The other number is the "***conductance***" c—which measures how easily flow gets through.

In physics and engineering, c is decided by the material. For electrical currents, c is high for metal and low for plastics. For a superconductor, c is nearly infinite. If we consider elastic stretching, c might be low for metal and higher for plastics. In economics, c measures the capacity of an edge or its cost.

To summarize, the graph is known from its "connectivity matrix" A. This tells the connections between nodes and edges. A ***network*** goes further, and assigns a conductance c to each edge. These numbers $c_1, \ldots, c_m$ go into the "conductance matrix" C—which is diagonal.

For a network of resistors, the conductance is $c = 1/(\text{resistance})$. In addition to Kirchhoff's Laws for the whole system of currents, we have Ohm's Law for each particular current. Ohm's Law connects the current y_1 on edge 1 to the potential difference $x_2 - x_1$ between the nodes:

Ohm's Law: Current along edge = conductance times potential difference.

Ohm's Law for all m currents is $y = -CAx$. The vector Ax gives the potential differences, and C multiplies by the conductances. Combining Ohm's Law with Kirchhoff's Current

Law $A^{\mathrm{T}}y = \mathbf{0}$, we get $A^{\mathrm{T}}CAx = \mathbf{0}$. This is *almost* the central equation for network flows. The only thing wrong is the zero on the right side! The network needs power from outside—a voltage source or a current source—to make something happen.

Note about signs In circuit theory we change from Ax to $-Ax$. The flow is from higher potential to lower potential. There is (positive) current from node 1 to node 2 when $x_1 - x_2$ is positive—whereas Ax was constructed to yield $x_2 - x_1$. The minus sign in physics and electrical engineering is a plus sign in mechanical engineering and economics. Ax versus $-Ax$ is a general headache but unavoidable.

Note about applied mathematics Every new application has its own form of Ohm's law. For elastic structures $y = CAx$ is Hooke's law. The stress y is (elasticity C) times (stretching Ax). For heat conduction, Ax is a temperature gradient. For oil flows it is a pressure gradient. There is a similar law in Section 8.6 for least squares regression in statistics.

My textbooks *Introduction to Applied Mathematics* and *Computational Science and Engineering* (Wellesley-Cambridge Press) are practically built on $A^{\mathrm{T}}CA$. This is the key to equilibrium in matrix equations and also in differential equations. Applied mathematics is more organized than it looks. *I have learned to watch for $A^{\mathrm{T}}CA$.*

We now give an example with a current source. Kirchhoff's Law changes from $A^{\mathrm{T}}y = \mathbf{0}$ to $A^{\mathrm{T}}y = f$, to balance the source f from outside. *Flow into each node still equals flow out.* Figure 8.5 shows the network with its conductances $c_1, \ldots, c_6$, and it shows the current source going into node 1. The source comes out at node 4 to keep the balance (in = out). The problem is: ***Find the currents $y_1, \ldots, y_6$ on the six edges.***

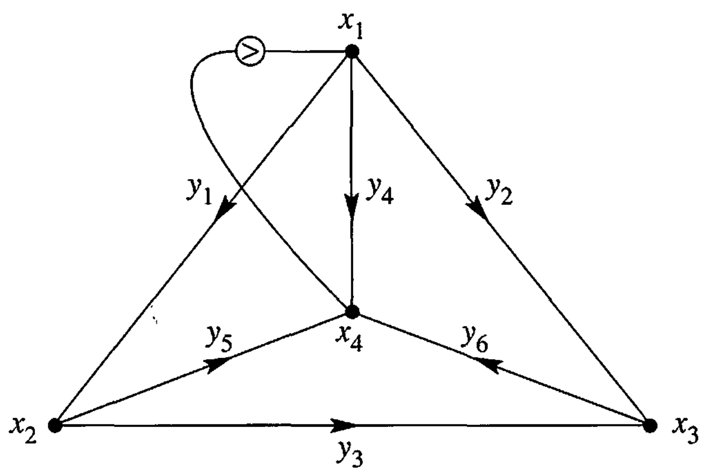


Figure 8.5: The currents in a network with a source S into node 1.

Example 1 All conductances are $c = 1$, so that $C = I$. A current y_4 travels directly from node 1 to node 4. Other current goes the long way from node 1 to node 2 to node 4 (this is $y_1 = y_5$). Current also goes from node 1 to node 3 to node 4 (this is $y_2 = y_6$). We can find the six currents by using special rules for symmetry, or we can do it right by using

$A^{\mathrm{T}}CA$. Since $C = I$, this matrix is $A^{\mathrm{T}}A$, the ***graph Laplacian matrix***:

$$\begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

That last matrix is not invertible! We cannot solve for all four potentials because $(1, 1, 1, 1)$ is in the nullspace. One node has to be grounded. Setting $x_4 = 0$ removes the fourth row and column, and this leaves a 3 by 3 invertible matrix. Now we solve $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$ for the unknown potentials x_1, x_2, x_3, with source S into node 1:

$$\textbf{Voltages} \qquad \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S \\ 0 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S/2 \\ S/4 \\ S/4 \end{bmatrix}.$$

Ohm's Law $\boldsymbol{y} = -CA\boldsymbol{x}$ yields the six currents. Remember $C = I$ and $x_4 = 0$:

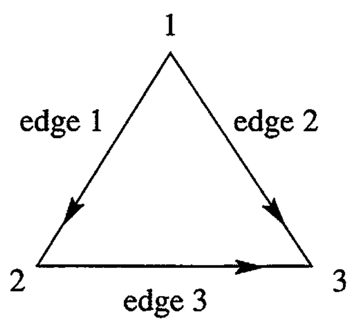
$$\textbf{Currents} \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = - \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} S/2 \\ S/4 \\ S/4 \\ 0 \end{bmatrix} = \begin{bmatrix} S/4 \\ S/4 \\ 0 \\ S/2 \\ S/4 \\ S/4 \end{bmatrix}.$$

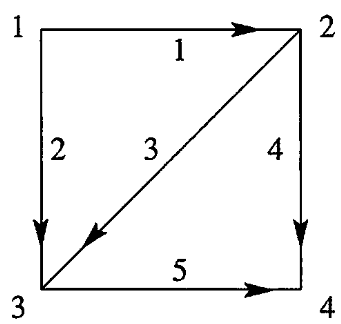
Half the current goes directly on edge 4. That is $y_4 = S/2$. No current crosses from node 2 to node 3. Symmetry indicated $y_3 = 0$ and now the solution proves it.

The same matrix $A^{\mathrm{T}}A$ appears in least squares. Nature distributes the currents to minimize the heat loss. Statistics chooses $\widehat{x}$ to minimize the least squares error.

Problem Set 8.2

Problems 1–7 and 8–14 are about the incidence matrices for these graphs.





1 Write down the 3 by 3 incidence matrix A for the triangle graph. The first row has -1 in column 1 and $+1$ in column 2. What vectors (x_1, x_2, x_3) are in its nullspace? How do you know that $(1, 0, 0)$ is not in its row space?

2 Write down A^{T} for the triangle graph. Find a vector $\boldsymbol{y}$ in its nullspace. The components of $\boldsymbol{y}$ are currents on the edges—how much current is going around the triangle?

3 Eliminate x_1 and x_2 from the third equation to find the echelon matrix U. What tree corresponds to the two nonzero rows of U?

$$\begin{aligned} -x_1 + x_2 &= b_1 \\ -x_1 + x_3 &= b_2 \\ -x_2 + x_3 &= b_3. \end{aligned}$$

4 Choose a vector (b_1, b_2, b_3) for which $A\boldsymbol{x} = \boldsymbol{b}$ can be solved, and another vector $\boldsymbol{b}$ that allows no solution. How are those $\boldsymbol{b}$'s related to $\boldsymbol{y} = (1, -1, 1)$?

5 Choose a vector (f_1, f_2, f_3) for which $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ can be solved, and a vector $\boldsymbol{f}$ that allows no solution. How are those $\boldsymbol{f}$'s related to $\boldsymbol{x} = (1, 1, 1)$? The equation $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is Kirchhoff's _____ law.

6 Multiply matrices to find $A^{\mathrm{T}}A$. Choose a vector $\boldsymbol{f}$ for which $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{f}$ can be solved, and solve for $\boldsymbol{x}$. Put those potentials $\boldsymbol{x}$ and the currents $\boldsymbol{y} = -A\boldsymbol{x}$ and current sources $\boldsymbol{f}$ onto the triangle graph. Conductances are 1 because $C = I$.

7 With conductances $c_1 = 1$ and $c_2 = c_3 = 2$, multiply matrices to find $A^{\mathrm{T}}CA$. For $\boldsymbol{f} = (1, 0, -1)$ find a solution to $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$. Write the potentials $\boldsymbol{x}$ and currents $\boldsymbol{y} = -CA\boldsymbol{x}$ on the triangle graph, when the current source $\boldsymbol{f}$ goes into node 1 and out from node 3.

8 Write down the 5 by 4 incidence matrix A for the square graph with two loops. Find one solution to $A\boldsymbol{x} = \boldsymbol{0}$ and two solutions to $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$.

9 Find two requirements on the b's for the five differences $x_2 - x_1, x_3 - x_1, x_3 - x_2, x_4 - x_2, x_4 - x_3$ to equal b_1, b_2, b_3, b_4, b_5. You have found Kirchhoff's _____ law around the two _____ in the graph.

10 Reduce A to its echelon form U. The three nonzero rows give the incidence matrix for what graph? You found one tree in the square graph—find the other seven trees.

11 Multiply matrices to find $A^{\mathrm{T}}A$ and guess how its entries come from the graph:

(a) The diagonal of $A^{\mathrm{T}}A$ tells how many _____ into each node.

(b) The off-diagonals -1 or 0 tell which pairs of nodes are _____.

12 Why is each statement true about $A^{\mathrm{T}}A$? *Answer for $A^{\mathrm{T}}A$ not A.*

(a) Its nullspace contains $(1, 1, 1, 1)$. Its rank is $n - 1$.

(b) It is positive semidefinite but not positive definite.

(c) Its four eigenvalues are real and their signs are _____.

13 With conductances $c_1 = c_2 = 2$ and $c_3 = c_4 = c_5 = 3$, multiply the matrices $A^{\mathrm{T}}CA$. Find a solution to $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f} = (1, 0, 0, -1)$. Write these potentials $\boldsymbol{x}$ and currents $\boldsymbol{y} = -CA\boldsymbol{x}$ on the nodes and edges of the square graph.

14 The matrix $A^{\mathrm{T}}CA$ is not invertible. What vectors $\boldsymbol{x}$ are in its nullspace? Why does $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$ have a solution if and only if $f_1 + f_2 + f_3 + f_4 = 0$?

15 A connected graph with 7 nodes and 7 edges has how many loops?

16 For the graph with 4 nodes, 6 edges, and 3 loops, add a new node. If you connect it to one old node, Euler's formula becomes $(\quad) - (\quad) + (\quad) = 1$. If you connect it to two old nodes, Euler's formula becomes $(\quad) - (\quad) + (\quad) = 1$.

17 Suppose A is a 12 by 9 incidence matrix from a connected (but unknown) graph.

(a) How many columns of A are independent?

(b) What condition on $\boldsymbol{f}$ makes it possible to solve $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$?

(c) The diagonal entries of $A^{\mathrm{T}}A$ give the number of edges into each node. What is the sum of those diagonal entries?

18 Why does a complete graph with $n = 6$ nodes have $m = 15$ edges? A tree connecting 6 nodes has _____ edges.

Note The ***stoichiometric matrix*** in chemistry is an important "generalized" incidence matrix. Its entries show how much of each chemical species (each column) goes into each reaction (each row).

8.3 Markov Matrices, Population, and Economics

This section is about ***positive matrices***: every $a_{ij} > 0$. The key fact is quick to state: ***The largest eigenvalue is real and positive and so is its eigenvector.*** In economics and ecology and population dynamics and random walks, that fact leads a long way:

$$\textbf{Markov} \quad \lambda_{\max} = 1 \qquad \textbf{Population} \quad \lambda_{\max} > 1 \qquad \textbf{Consumption} \quad \lambda_{\max} < 1$$

$\lambda_{\max}$ controls the powers of A. We will see this first for $\lambda_{\max} = 1$.

Markov Matrices

Suppose we multiply a positive vector $\boldsymbol{u}_0 = (a, 1-a)$ again and again by this A:

Markov matrix $$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \qquad \boldsymbol{u}_1 = A\boldsymbol{u}_0 \qquad \boldsymbol{u}_2 = A\boldsymbol{u}_1 = A^2\boldsymbol{u}_0$$

After k steps we have $A^k\boldsymbol{u}_0$. The vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \ldots$ will approach a "*steady state*" $\boldsymbol{u}_\infty = (.6, .4)$. This final outcome does not depend on the starting vector: ***For every $\boldsymbol{u}_0$ we converge to the same $\boldsymbol{u}_\infty$***. The question is why.

The steady state equation $A\boldsymbol{u}_\infty = \boldsymbol{u}_\infty$ makes $\boldsymbol{u}_\infty$ ***an eigenvector with eigenvalue* 1**:

Steady state $$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}.$$

Multiplying by A does not change $\boldsymbol{u}_\infty$. But this does not explain why all vectors $\boldsymbol{u}_0$ lead to $\boldsymbol{u}_\infty$. Other examples might have a steady state, but it is not necessarily attractive:

Not Markov $$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{has the unattractive steady state} \quad B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In this case, the starting vector $\boldsymbol{u}_0 = (0, 1)$ will give $\boldsymbol{u}_1 = (0, 2)$ and $\boldsymbol{u}_2 = (0, 4)$. The second components are doubled. In the language of eigenvalues, B has $\lambda = 1$ but also $\lambda = 2$— this produces instability. The component of $\boldsymbol{u}$ along that unstable eigenvector is multiplied by λ, and $|\lambda| > 1$ means blowup.

This section is about two special properties of A that guarantee a stable steady state. These properties define a ***Markov matrix***, and A above is one particular example:

Markov matrix
1. ***Every entry of A is nonnegative.***
2. ***Every column of A adds to* 1.**

B did not have Property **2**. When A is a Markov matrix, two facts are immediate:

1. Multiplying a nonnegative $\boldsymbol{u}_0$ by A produces a nonnegative $\boldsymbol{u}_1 = A\boldsymbol{u}_0$.
2. If the components of $\boldsymbol{u}_0$ add to 1, so do the components of $\boldsymbol{u}_1 = A\boldsymbol{u}_0$.

Reason: The components of $\boldsymbol{u}_0$ add to 1 when $\begin{bmatrix}1 & \cdots & 1\end{bmatrix}\boldsymbol{u}_0 = 1$. This is true for each column of A by Property 2. Then by matrix multiplication $[1 \ \ldots \ 1]A = [1 \ \ldots \ 1]$:

$$\textbf{Components of } A\boldsymbol{u}_0 \textbf{ add to 1} \qquad [1 \ \cdots \ 1]A\boldsymbol{u}_0 = [1 \ \cdots \ 1]\boldsymbol{u}_0 = 1.$$

The same facts apply to $\boldsymbol{u}_2 = A\boldsymbol{u}_1$ and $\boldsymbol{u}_3 = A\boldsymbol{u}_2$. ***Every vector $A^k\boldsymbol{u}_0$ is nonnegative with components adding to*** 1. These are "***probability vectors.***" The limit $\boldsymbol{u}_\infty$ is also a probability vector—but we have to prove that there is a limit. We will show that $\lambda_{\text{max}} = 1$ for a positive Markov matrix.

Example 1 The fraction of rental cars in Denver starts at $\frac{1}{50} = .02$. The fraction outside Denver is .98. Every month, 80% of the Denver cars stay in Denver (and 20% leave). Also 5% of the outside cars come in (95% stay outside). This means that the fractions $\boldsymbol{u}_0 = (.02, .98)$ are multiplied by A:

$$\textbf{First month} \qquad A = \begin{bmatrix} .80 & .05 \\ .20 & .95 \end{bmatrix} \quad \text{leads to} \quad \boldsymbol{u}_1 = A\boldsymbol{u}_0 = A\begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .065 \\ .935 \end{bmatrix}.$$

Notice that $.065 + .935 = 1$. All cars are accounted for. Each step multiplies by A:

$$\textbf{Next month} \qquad \boldsymbol{u}_2 = A\boldsymbol{u}_1 = (.09875,\ .90125). \text{ This is } A^2\boldsymbol{u}_0.$$

All these vectors are positive because A is positive. Each vector $\boldsymbol{u}_k$ will have its components adding to 1. The first component has grown from .02 and cars are moving toward Denver. What happens in the long run?

This section involves powers of matrices. The understanding of A^k was our first and best application of diagonalization. Where A^k can be complicated, the diagonal matrix Λ^k is simple. The eigenvector matrix S connects them: A^k equals $S\Lambda^k S^{-1}$. The new application to Markov matrices uses the eigenvalues (in Λ) and the eigenvectors (in S). We will show that $\boldsymbol{u}_\infty$ is an eigenvector corresponding to $\lambda = 1$.

Since every column of A adds to 1, nothing is lost or gained. We are moving rental cars or populations, and no cars or people suddenly appear (or disappear). The fractions add to 1 and the matrix A keeps them that way. The question is how they are distributed after k time periods—which leads us to A^k.

Solution $A^k\boldsymbol{u}_0$ gives the fractions in and out of Denver after k steps. We diagonalize A to understand A^k. The eigenvalues are $\lambda = 1$ and .75 (the trace is 1.75).

$$Ax = \lambda x \qquad A\begin{bmatrix} .2 \\ .8 \end{bmatrix} = 1\begin{bmatrix} .2 \\ .8 \end{bmatrix} \quad \textbf{and} \quad A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = .75\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The starting vector $\boldsymbol{u}_0$ combines $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, in this case with coefficients 1 and .18:

$$\textbf{Combination of eigenvectors} \qquad \boldsymbol{u}_0 = \begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + .18\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Now multiply by A to find $\boldsymbol{u}_1$. The eigenvectors are multiplied by $\lambda_1 = 1$ and $\lambda_2 = .75$:

$$\textbf{Each } \boldsymbol{x} \textbf{ is multiplied by } \boldsymbol{\lambda} \qquad \boldsymbol{u}_1 = 1\begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)(.18)\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Every month, another .75 multiplies the vector x_2. The eigenvector x_1 is unchanged:

After k steps $$u_k = A^k u_0 = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)^k(.18)\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This equation reveals what happens. ***The eigenvector x_1 with $\lambda = 1$ is the steady state.*** The other eigenvector x_2 disappears because $|\lambda| < 1$. The more steps we take, the closer we come to $u_\infty = (.2, .8)$. In the limit, $\frac{2}{10}$ of the cars are in Denver and $\frac{8}{10}$ are outside. This is the pattern for Markov chains, even starting from $u_0 = (0, 1)$:

If A is a *positive* Markov matrix (entries $a_{ij} > 0$, each column adds to **1**), then $\lambda_1 = 1$ is larger than any other eigenvalue. The eigenvector x_1 is the ***steady state***:

$$u_k = x_1 + c_2(\lambda_2)^k x_2 + \cdots + c_n(\lambda_n)^k x_n \quad \textit{always approaches} \quad u_\infty = x_1.$$

The first point is to see that $\lambda = 1$ is an eigenvalue of A. *Reason*: Every column of $A - I$ adds to $1 - 1 = 0$. The rows of $A - I$ add up to the zero row. Those rows are linearly dependent, so $A - I$ is singular. Its determinant is zero and $\lambda = 1$ is an eigenvalue.

The second point is that no eigenvalue can have $|\lambda| > 1$. With such an eigenvalue, the powers A^k would grow. But A^k is also a Markov matrix! A^k has nonnegative entries still adding to 1—and that leaves no room to get large.

A lot of attention is paid to the possibility that another eigenvalue has $|\lambda| = 1$.

Example 2 $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ has no steady state because $\lambda_2 = -1$.

This matrix sends all cars from inside Denver to outside, and vice versa. The powers A^k alternate between A and I. The second eigenvector $x_2 = (-1, 1)$ will be multiplied by $\lambda_2 = -1$ at every step—and does not become smaller: No steady state.

Suppose the entries of A or any power of A are all *positive*—zero is not allowed. In this "regular" or "primitive" case, $\lambda = 1$ is strictly larger than any other eigenvalue. The powers A^k approach the rank one matrix that has the steady state in every column.

Example 3 (**"Everybody moves"**) Start with three groups. At each time step, half of group 1 goes to group 2 and the other half goes to group 3. The other groups also *split in half and move*. Take one step from the starting populations p_1, p_2, p_3:

New populations $$u_1 = Au_0 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}p_2 + \frac{1}{2}p_3 \\ \frac{1}{2}p_1 + \frac{1}{2}p_3 \\ \frac{1}{2}p_1 + \frac{1}{2}p_2 \end{bmatrix}.$$

A is a Markov matrix. Nobody is born or lost. A contains zeros, which gave trouble in Example 2. But after two steps in this new example, the zeros disappear from A^2:

Two-step matrix $$u_2 = A^2 u_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 1$ (because A is Markov) and $\lambda_2 = \lambda_3 = -\frac{1}{2}$. For $\lambda = 1$, ***the eigenvector*** $\boldsymbol{x}_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ ***will be the steady state.*** When three equal populations split in half and move, the populations are again equal. Starting from $\boldsymbol{u}_0 = (8, 16, 32)$, the Markov chain approaches its steady state:

$$\boldsymbol{u}_0 = \begin{bmatrix} 8 \\ 16 \\ 32 \end{bmatrix} \quad \boldsymbol{u}_1 = \begin{bmatrix} 24 \\ 20 \\ 12 \end{bmatrix} \quad \boldsymbol{u}_2 = \begin{bmatrix} 16 \\ 18 \\ 22 \end{bmatrix} \quad \boldsymbol{u}_3 = \begin{bmatrix} 20 \\ 19 \\ 17 \end{bmatrix}.$$

The step to $\boldsymbol{u}_4$ will split some people in half. This cannot be helped. The total population is $8 + 16 + 32 = 56$ at every step. The steady state is 56 times $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. You can see the three populations approaching, but never reaching, their final limits $56/3$.

Challenge Problem 6.7.16 created a Markov matrix A from the number of links between websites. The steady state $\boldsymbol{u}$ will give the Google rankings. ***Google finds $\boldsymbol{u}_\infty$ by a random walk that follows links*** (*random surfing*). That eigenvector comes from counting the fraction of visits to each website—a quick way to compute the steady state.

The size $|\lambda_2|$ of the next largest eigenvalue controls the speed of convergence to steady state.

Perron-Frobenius Theorem

One matrix theorem dominates this subject. The Perron-Frobenius Theorem applies when all $a_{ij} \geq 0$. There is no requirement that columns add to 1. We prove the neatest form, when all $a_{ij} > 0$.

Perron-Frobenius for $A > 0$ ***All numbers in $A\boldsymbol{x} = \lambda_{\max}\boldsymbol{x}$ are strictly positive.***

Proof The key idea is to look at all numbers t such that $A\boldsymbol{x} \geq t\boldsymbol{x}$ for some nonnegative vector $\boldsymbol{x}$ (other than $\boldsymbol{x} = \boldsymbol{0}$). We are allowing inequality in $A\boldsymbol{x} \geq t\boldsymbol{x}$ in order to have many positive candidates t. For the largest value $t_{\max}$ (which is attained), we will show that ***equality holds***: $A\boldsymbol{x} = t_{\max}\boldsymbol{x}$.

Otherwise, if $A\boldsymbol{x} \geq t_{\max}\boldsymbol{x}$ is not an equality, multiply by A. Because A is positive that produces a strict inequality $A^2\boldsymbol{x} > t_{\max}A\boldsymbol{x}$. Therefore the positive vector $\boldsymbol{y} = A\boldsymbol{x}$ satisfies $A\boldsymbol{y} > t_{\max}\boldsymbol{y}$, and $t_{\max}$ could be increased. This contradiction forces the equality $A\boldsymbol{x} = t_{\max}\boldsymbol{x}$, and we have an eigenvalue. Its eigenvector $\boldsymbol{x}$ is positive because on the left side of that equality, $A\boldsymbol{x}$ is sure to be positive.

To see that no eigenvalue can be larger than $t_{\max}$, suppose $A\boldsymbol{z} = \lambda\boldsymbol{z}$. Since λ and $\boldsymbol{z}$ may involve negative or complex numbers, we take absolute values: $|\lambda||\boldsymbol{z}| = |A\boldsymbol{z}| \leq A|\boldsymbol{z}|$ by the "triangle inequality." This $|\boldsymbol{z}|$ is a nonnegative vector, so $|\lambda|$ is one of the possible candidates t. Therefore $|\lambda|$ cannot exceed $t_{\max}$—which must be $\lambda_{\max}$.

Population Growth

Divide the population into three age groups: age < 20, age 20 to 39, and age 40 to 59. At year T the sizes of those groups are n_1, n_2, n_3. Twenty years later, the sizes have changed for two reasons:

1. **Reproduction** $n_1^{\text{new}} = F_1 n_1 + F_2 n_2 + F_3 n_3$ gives a new generation

2. **Survival** $n_2^{\text{new}} = P_1 n_1$ and $n_3^{\text{new}} = P_2 n_2$ gives the older generations

The fertility rates are F_1, F_2, F_3 (F_2 largest). The *Leslie matrix* A might look like this:

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}^{\textbf{new}} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} .04 & \mathbf{1.1} & .01 \\ .98 & 0 & 0 \\ 0 & \mathbf{.92} & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

This is population projection in its simplest form, the same matrix A at every step. In a realistic model, A will change with time (from the environment or internal factors). Professors may want to include a fourth group, age ≥ 60, but we don't allow it.

The matrix has $A \geq 0$ but not $A > 0$. The Perron-Frobenius theorem still applies because $A^3 > 0$. The largest eigenvalue is $\lambda_{\max} \approx 1.06$. You can watch the generations move, starting from $n_2 = 1$ in the middle generation:

$$\textbf{eig}(A) = \begin{matrix} \mathbf{1.06} \\ -1.01 \\ -0.01 \end{matrix} \qquad A^2 = \begin{bmatrix} 1.08 & \mathbf{0.05} & .00 \\ 0.04 & \mathbf{1.08} & .01 \\ 0.90 & 0 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0.10 & \mathbf{1.19} & .01 \\ 0.06 & \mathbf{0.05} & .00 \\ 0.04 & \mathbf{0.99} & .01 \end{bmatrix}.$$

A fast start would come from $\boldsymbol{u}_0 = (0, 1, 0)$. That middle group will reproduce 1.1 and also survive .92. The newest and oldest generations are in $\boldsymbol{u}_1 = (1.1, 0, .92) =$ column 2 of A. Then $\boldsymbol{u}_2 = A\boldsymbol{u}_1 = A^2\boldsymbol{u}_0$ is the second column of A^2. The early numbers (transients) depend a lot on $\boldsymbol{u}_0$, but ***the asymptotic growth rate*** $\lambda_{\max}$ ***is the same from every start.*** Its eigenvector $\boldsymbol{x} = (.63, .58, .51)$ shows all three groups growing steadily together.

Caswell's book on *Matrix Population Models* emphasizes sensitivity analysis. The model is never exactly right. If the F's or P's in the matrix change by 10%, does $\lambda_{\max}$ go below 1 (which means extinction)? Problem 19 will show that a matrix change ΔA produces an eigenvalue change $\Delta\lambda = \boldsymbol{y}^{\mathrm{T}}(\Delta A)\boldsymbol{x}$. Here $\boldsymbol{x}$ and $\boldsymbol{y}^{\mathrm{T}}$ are the right and left eigenvectors of A. So $\boldsymbol{x}$ is a column of S and $\boldsymbol{y}^{\mathrm{T}}$ is a row of S^{-1}.

Linear Algebra in Economics: The Consumption Matrix

A long essay about linear algebra in economics would be out of place here. A short note about one matrix seems reasonable. The ***consumption matrix*** tells how much of each input goes into a unit of output. This describes the manufacturing side of the economy.

Consumption matrix We have n industries like chemicals, food, and oil. To produce a unit of chemicals may require .2 units of chemicals, .3 units of food, and .4 units of oil. Those numbers go into row 1 of the consumption matrix A:

$$\begin{bmatrix} \text{chemical output} \\ \text{food output} \\ \text{oil output} \end{bmatrix} = \begin{bmatrix} .2 & .3 & .4 \\ .4 & .4 & .1 \\ .5 & .1 & .3 \end{bmatrix} \begin{bmatrix} \text{chemical input} \\ \text{food input} \\ \text{oil input} \end{bmatrix}.$$

Row 2 shows the inputs to produce food—a heavy use of chemicals and food, not so much oil. Row 3 of A shows the inputs consumed to refine a unit of oil. The real consumption matrix for the United States in 1958 contained 83 industries. The models in the 1990's are much larger and more precise. We chose a consumption matrix that has a convenient eigenvector.

Now comes the question: Can this economy meet demands y_1, y_2, y_3 for chemicals, food, and oil? To do that, the inputs p_1, p_2, p_3 will have to be higher—because part of $\boldsymbol{p}$ is consumed in producing $\boldsymbol{y}$. The input is $\boldsymbol{p}$ and the consumption is $A\boldsymbol{p}$, which leaves the output $\boldsymbol{p} - A\boldsymbol{p}$. This net production is what meets the demand $\boldsymbol{y}$:

Problem Find a vector $\boldsymbol{p}$ such that $\quad \boldsymbol{p} - A\boldsymbol{p} = \boldsymbol{y} \quad$ or $\quad \boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}$.

Apparently the linear algebra question is whether $I - A$ is invertible. But there is more to the problem. The demand vector $\boldsymbol{y}$ is nonnegative, and so is A. *The production levels in* $\boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}$ *must also be nonnegative*. The real question is:

When is $(I - A)^{-1}$ a nonnegative matrix?

This is the test on $(I - A)^{-1}$ for a productive economy, which can meet any positive demand. If A is small compared to I, then $A\boldsymbol{p}$ is small compared to $\boldsymbol{p}$. There is plenty of output. If A is too large, then production consumes more than it yields. In this case the external demand $\boldsymbol{y}$ cannot be met.

"Small" or "large" is decided by the largest eigenvalue λ_1 of A (which is positive):

If $\lambda_1 > 1$ then $(I - A)^{-1}$ has negative entries
If $\lambda_1 = 1$ then $(I - A)^{-1}$ fails to exist
If $\lambda_1 < 1$ then $(I - A)^{-1}$ is nonnegative as desired.

The main point is that last one. The reasoning uses a nice formula for $(I - A)^{-1}$, which we give now. The most important infinite series in mathematics is the ***geometric series*** $1 + x + x^2 + \cdots$. This series adds up to $1/(1 - x)$ provided x lies between -1 and 1. When $x = 1$ the series is $1 + 1 + 1 + \cdots = \infty$. When $|x| \ge 1$ the terms x^n don't go to zero and the series has no chance to converge.

The nice formula for $(I - A)^{-1}$ is the ***geometric series of matrices***:

Geometric series $\quad (I - A)^{-1} = I + A + A^2 + A^3 + \cdots$.

If you multiply the series $S = I + A + A^2 + \cdots$ by A, you get the same series except for I. Therefore $S - AS = I$, which is $(I - A)S = I$. The series adds to $S = (I - A)^{-1}$ if it converges. ***And it converges if all eigenvalues of A have*** $|\lambda| < 1$.

In our case $A \geq 0$. All terms of the series are nonnegative. Its sum is $(I - A)^{-1} \geq 0$.

Example 4 $A = \begin{bmatrix} .2 & .3 & .4 \\ .4 & .4 & .1 \\ .5 & .1 & .3 \end{bmatrix}$ has $\lambda_{\max} = .9$ and $(I - A)^{-1} = \frac{1}{93}\begin{bmatrix} 41 & 25 & 27 \\ 33 & 36 & 24 \\ 34 & 23 & 36 \end{bmatrix}$.

This economy is productive. A is small compared to I, because $\lambda_{\max}$ is .9. To meet the demand y, start from $p = (I - A)^{-1}y$. Then Ap is consumed in production, leaving $p - Ap$. This is $(I - A)p = y$, and the demand is met.

Example 5 $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ has $\lambda_{\max} = 2$ and $(I - A)^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

This consumption matrix A is too large. Demands can't be met, because production consumes more than it yields. The series $I + A + A^2 + \ldots$ does not converge to $(I - A)^{-1}$ because $\lambda_{\max} > 1$. The series is growing while $(I - A)^{-1}$ is actually negative.

In the same way $1 + 2 + 4 + \cdots$ is not really $1/(1 - 2) = -1$. But not entirely false!

Problem Set 8.3

Questions 1–12 are about Markov matrices and their eigenvalues and powers.

1 Find the eigenvalues of this Markov matrix (their sum is the trace):

$$A = \begin{bmatrix} .90 & .15 \\ .10 & .85 \end{bmatrix}.$$

What is the steady state eigenvector for the eigenvalue $\lambda_1 = 1$?

2 Diagonalize the Markov matrix in Problem 1 to $A = S\Lambda S^{-1}$ by finding its other eigenvector:

$$A = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 1 & \\ & .75 \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}.$$

What is the limit of $A^k = S\Lambda^k S^{-1}$ when $\Lambda^k = \begin{bmatrix} 1 & 0 \\ 0 & .75^k \end{bmatrix}$ approaches $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$?

3 What are the eigenvalues and steady state eigenvectors for these Markov matrices?

$$A = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix} \quad A = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix} \quad A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

4 For every 4 by 4 Markov matrix, what eigenvector of A^{T} corresponds to the (known) eigenvalue $\lambda = 1$?

5 Every year 2% of young people become old and 3% of old people become dead. (No births.) Find the steady state for

$$\begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k+1} = \begin{bmatrix} .98 & .00 & 0 \\ .02 & .97 & 0 \\ .00 & .03 & 1 \end{bmatrix} \begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_k .$$

6 For a Markov matrix, the sum of the components of $\boldsymbol{x}$ equals the sum of the components of $A\boldsymbol{x}$. If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ with $\lambda \neq 1$, prove that the components of this non-steady eigenvector $\boldsymbol{x}$ add to zero.

7 Find the eigenvalues and eigenvectors of A. Explain why A^k approaches A^∞:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \qquad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Challenge problem: Which Markov matrices produce that steady state $(.6, .4)$?

8 The steady state eigenvector of a permutation matrix is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. This is *not* approached when $\boldsymbol{u}_0 = (0, 0, 0, 1)$. What are $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$? What are the four eigenvalues of P, which solve $\lambda^4 = 1$?

Permutation matrix = Markov matrix $\qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$

9 Prove that the square of a Markov matrix is also a Markov matrix.

10 If $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is a Markov matrix, its eigenvalues are 1 and ______. The steady state eigenvector is $\boldsymbol{x}_1 =$ ______.

11 Complete A to a Markov matrix and find the steady state eigenvector. When A is a symmetric Markov matrix, why is $\boldsymbol{x}_1 = (1, \ldots, 1)$ its steady state?

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ - & - & - \end{bmatrix}.$$

12 A Markov differential equation is not $d\boldsymbol{u}/dt = A\boldsymbol{u}$ but $d\boldsymbol{u}/dt = (A - I)\boldsymbol{u}$. The diagonal is negative, the rest of $A - I$ is positive. The columns add to zero.

Find the eigenvalues of $B = A - I = \begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix}$. Why does $A - I$ have $\lambda = 0$?

When $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ multiply $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, what is the steady state as $t \to \infty$?

Questions 13–15 are about linear algebra in economics.

13 Each row of the consumption matrix in Example 4 adds to .9. Why does that make $\lambda = .9$ an eigenvalue, and what is the eigenvector?

14 Multiply $I + A + A^2 + A^3 + \cdots$ by $I - A$ to show that the series adds to ____ . For $A = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$, find A^2 and A^3 and use the pattern to add up the series.

15 For which of these matrices does $I + A + A^2 + \cdots$ yield a nonnegative matrix $(I - A)^{-1}$? Then the economy can meet any demand:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 4 \\ .2 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}.$$

If the demands are $y = (2, 6)$, what are the vectors $p = (I - A)^{-1} y$?

16 (Markov again) This matrix has zero determinant. What are its eigenvalues?

$$A = \begin{bmatrix} .4 & .2 & .3 \\ .2 & .4 & .3 \\ .4 & .4 & .4 \end{bmatrix}.$$

Find the limits of $A^k u_0$ starting from $u_0 = (1, 0, 0)$ and then $u_0 = (100, 0, 0)$.

17 If A is a Markov matrix, does $I + A + A^2 + \cdots$ add up to $(I - A)^{-1}$?

18 For the Leslie matrix show that $\det(A - \lambda I) = 0$ gives $F_1\lambda^2 + F_2P_1\lambda + F_3P_1P_2 = \lambda^3$. The right side λ^3 is larger as $\lambda \longrightarrow \infty$. The left side is larger at $\lambda = 1$ if $F_1 + F_2P_1 + F_3P_1P_2 > 1$. In that case the two sides are equal at an eigenvalue $\lambda_{\max} > 1$: *growth.*

19 **Sensitivity of eigenvalues**: A matrix change ΔA produces eigenvalue changes $\Delta\Lambda$. *The formula for those changes* $\Delta\lambda_1, \ldots, \Delta\lambda_n$ *is* **diag**$(S^{-1}\,\Delta A\, S)$. **Challenge**: Start from $AS = S\Lambda$. The eigenvectors and eigenvalues change by ΔS and $\Delta\Lambda$:

$$(A{+}\Delta A)(S{+}\Delta S) = (S{+}\Delta S)(\Lambda{+}\Delta\Lambda) \text{ becomes } A(\Delta S){+}(\Delta A)S = S(\Delta\Lambda){+}(\Delta S)\Lambda.$$

Small terms $(\Delta A)(\Delta S)$ and $(\Delta S)(\Delta\Lambda)$ are ignored. *Multiply the last equation by* S^{-1}. From the inner terms, the diagonal part of $S^{-1}(\Delta A)S$ gives $\Delta\Lambda$ as we want. *Why do the outer terms* $S^{-1}\,A\,\Delta S$ *and* $S^{-1}\,\Delta S\,\Lambda$ *cancel on the diagonal?*

Explain $S^{-1}A = \Lambda S^{-1}$ and then $\quad$ **diag**$(\Lambda\, S^{-1}\,\Delta S) =$ **diag**$(S^{-1}\,\Delta S\,\Lambda)$.

20 Suppose $B > A > 0$, meaning that each $b_{ij} > a_{ij} > 0$. How does the Perron-Frobenius discussion show that $\lambda_{\max}(B) > \lambda_{\max}(A)$?

8.4 Linear Programming

Linear programming is linear algebra plus two new ideas: ***inequalities*** and ***minimization***. The starting point is still a matrix equation $Ax = b$. But the only acceptable solutions are *nonnegative*. We require $x \geq 0$ (meaning that no component of x can be negative). The matrix has $n > m$, more unknowns than equations. If there are any solutions $x \geq \mathbf{0}$ to $Ax = b$, there are probably a lot. Linear programming picks the solution $x^* \geq \mathbf{0}$ that minimizes the cost:

The cost is $c_1x_1 + \cdots + c_nx_n$. The winning vector x^* is the nonnegative solution of $Ax = b$ that has smallest cost.

Thus a linear programming problem starts with a matrix A and two vectors b and c:

i) A has $n > m$: for example $A = [\,1 \quad 1 \quad 2\,]$ (one equation, three unknowns)

ii) b has m components for m equations $Ax = b$: for example $b = [\,4\,]$

iii) The ***cost vector c*** has n components: for example $c = [\,5 \quad 3 \quad 8\,]$.

Then the problem is to minimize $c \cdot x$ subject to the requirements $Ax = b$ and $x \geq 0$:

$$\textit{Minimize}\quad 5x_1 + 3x_2 + 8x_3 \quad \textit{subject to}\quad x_1 + x_2 + 2x_3 = 4 \quad \textit{and}\quad x_1, x_2, x_3 \geq 0.$$

We jumped right into the problem, without explaining where it comes from. Linear programming is actually the most important application of mathematics to management. Development of the fastest algorithm and fastest code is highly competitive. You will see that finding x^* is harder than solving $Ax = b$, because of the extra requirements: $x^* \geq 0$ and minimum cost $c^{\mathrm{T}}x^*$. We will explain the background, and the famous *simplex method*, and *interior point methods*, after solving the example.

Look first at the "constraints": $Ax = b$ and $x \geq 0$. The equation $x_1 + x_2 + 2x_3 = 4$ gives a plane in three dimensions. The nonnegativity $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ chops the plane down to a triangle. The solution x^* must lie in the triangle PQR in Figure 8.6.

Inside that triangle, all components of x are positive. On the edges of PQR, one component is zero. At the corners P and Q and R, two components are zero. ***The optimal solution x^* will be one of those corners!*** We will now show why.

The triangle contains all vectors x that satisfy $Ax = b$ and $x \geq 0$. Those x's are called *feasible points*, and the triangle is the ***feasible set***. These points are the allowed candidates in the minimization of $c \cdot x$, which is the final step:

Find x^* in the triangle PQR to minimize the cost $5x_1 + 3x_2 + 8x_3$.

The vectors that have *zero* cost lie on the plane $5x_1 + 3x_2 + 8x_3 = 0$. That plane does not meet the triangle. We cannot achieve zero cost, while meeting the requirements on x. So increase the cost C until the plane $5x_1 + 3x_2 + 8x_3 = C$ does meet the triangle. As C increases, we have *parallel planes moving toward the triangle*.

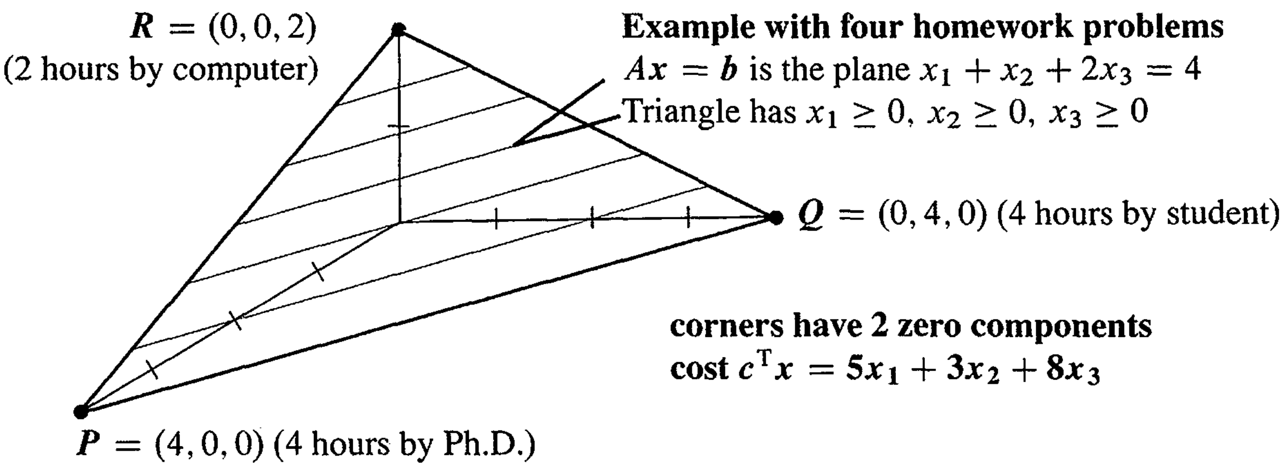


Figure 8.6: The triangle contains all nonnegative solutions: $Ax = b$ and $x \geq 0$. The lowest cost solution x^* is a corner P, Q, or R of this feasible set.

The first plane $5x_1 + 3x_2 + 8x_3 = C$ to touch the triangle has minimum cost C. ***The point where it touches is the solution x^*.*** This touching point must be one of the corners P or Q or R. A moving plane could not reach the inside of the triangle before it touches a corner! So check the cost $5x_1 + 3x_2 + 8x_3$ at each corner:

$$P = (4, 0, 0) \text{ costs } 20 \qquad Q = (0, 4, 0) \text{ costs } 12 \qquad R = (0, 0, 2) \text{ costs } 16.$$

The winner is Q. Then $x^* = (0, 4, 0)$ solves the linear programming problem.

If the cost vector c is changed, the parallel planes are tilted. For small changes, Q is still the winner. For the cost $c \cdot x = 5x_1 + 4x_2 + 7x_3$, the optimum x^* moves to $R = (0, 0, 2)$. The minimum cost is now $7 \cdot 2 = 14$.

Note 1 Some linear programs *maximize profit* instead of minimizing cost. The mathematics is almost the same. The parallel planes start with a large value of C, instead of a small value. They move toward the origin (instead of away), as C gets smaller. *The first touching point is still a corner.*

Note 2 The requirements $Ax = b$ and $x \geq 0$ could be impossible to satisfy. The equation $x_1 + x_2 + x_3 = -1$ cannot be solved with $x \geq 0$. *That feasible set is empty.*

Note 3 It could also happen that the feasible set is *unbounded.* If the requirement is $x_1 + x_2 - 2x_3 = 4$, the large positive vector $(100, 100, 98)$ is now a candidate. So is the larger vector $(1000, 1000, 998)$. The plane $Ax = b$ is no longer chopped off to a triangle. The two corners P and Q are still candidates for x^*, but R moved to infinity.

Note 4 With an unbounded feasible set, the minimum cost could be $-\infty$ (*minus infinity*). Suppose the cost is $-x_1 - x_2 + x_3$. Then the vector $(100, 100, 98)$ costs $C = -102$. The vector $(1000, 1000, 998)$ costs $C = -1002$. We are being paid to include x_1 and x_2, instead of paying a cost. In realistic applications this will not happen. But it is theoretically possible that A, b, and c can produce unexpected triangles and costs.

The Primal and Dual Problems

This first problem will fit A, b, c in that example. The unknowns x_1, x_2, x_3 represent hours of work by a Ph.D. and a student and a machine. The costs per hour are \$5, \$3, and \$8. (*I apologize for such low pay.*) The number of hours cannot be negative: $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$. The Ph.D. and the student get through one homework problem per hour. *The machine solves two problems in one hour.* In principle they can share out the homework, which has four problems to be solved: $x_1 + x_2 + 2x_3 = 4$.

The problem is to finish the four problems at minimum cost $c^{\mathrm{T}}x$.

If all three are working, the job takes one hour: $x_1 = x_2 = x_3 = 1$. The cost is $5 + 3 + 8 = 16$. But certainly the Ph.D. should be put out of work by the student (who is just as fast and costs less—this problem is getting realistic). When the student works two hours and the machine works one, the cost is $6 + 8$ and all four problems get solved. We are on the edge QR of the triangle because the Ph.D. is not working: $x_1 = 0$. But the best point is all work by student (at Q) or all work by machine (at R). In this example the student solves four problems in four hours for \$12—the minimum cost.

With only one equation in $Ax = b$, the corner $(0, 4, 0)$ has only one nonzero component. ***When $Ax = b$ has m equations, corners have m nonzeros.*** We solve $Ax = b$ for those m variables, with $n - m$ free variables set to zero. But unlike Chapter 3, ***we don't know which m variables to choose.***

The number of possible corners is the number of ways to choose m components out of n. This number "n choose m" is heavily involved in gambling and probability. With $n = 20$ unknowns and $m = 8$ equations (still small numbers), the "feasible set" can have $20!/8!12!$ corners. That number is $(20)(19)\cdots(13) = 5{,}079{,}110{,}400$.

Checking three corners for the minimum cost was fine. Checking five billion corners is not the way to go. The simplex method described below is much faster.

The Dual Problem In linear programming, problems come in pairs. There is a minimum problem and a maximum problem—the original and its "dual." The original problem was specified by a matrix A and two vectors b and c. The dual problem transposes A and switches b and c: ***Maximize $b \cdot y$***. Here is the dual to our example:

> **A cheater offers to solve homework problems by selling the answers.** The charge is y dollars per problem, or $4y$ altogether. (Note how $b = 4$ has gone into the cost.) The cheater must be as cheap as the Ph.D. or student or machine: $y \leq 5$ and $y \leq 3$ and $2y \leq 8$. (Note how $c = (5, 3, 8)$ has gone into inequality constraints). The cheater maximizes the income $4y$.

Dual Problem ***Maximize $b \cdot y$ subject to $A^{\mathrm{T}}y \leq c$.***

The maximum occurs when $y = 3$. The income is $4y = 12$. The maximum in the dual problem (\$12) equals the minimum in the original (\$12). *Max = min* is duality.

If either problem has a best vector (x^* or y^*) then so does the other.
Minimum cost $c \cdot x^*$ equals maximum income $b \cdot y^*$

This book started with a row picture and a column picture. The first "duality theorem" was about rank: The number of independent rows equals the number of independent columns. That theorem, like this one, was easy for small matrices. Minimum cost = maximum income is proved in our text *Linear Algebra and Its Applications*. One line will establish the easy half of the theorem: ***The cheater's income $b^{\mathrm{T}}y$ cannot exceed the honest cost***:

$$\textit{If } \; Ax = b, x \geq 0, A^{\mathrm{T}}y \leq c \quad \textit{then} \quad b^{\mathrm{T}}y = (Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y) \leq x^{\mathrm{T}}c. \tag{1}$$

The full duality theorem says that when $b^{\mathrm{T}}y$ reaches its maximum and $x^{\mathrm{T}}c$ reaches its minimum, they are equal: $b \cdot y^* = c \cdot x^*$. Look at the last step in (1), with $\leq$ sign:

The dot product of $x \geq 0$ and $s = c - A^{\mathrm{T}}y \geq 0$ gave $x^{\mathrm{T}}s \geq 0$. This is $x^{\mathrm{T}}A^{\mathrm{T}}y \leq x^{\mathrm{T}}c$.

Equality needs $x^{\mathrm{T}}s = 0$ So the optimal solution has $x_j^* = 0$ or $s_j^* = 0$ for each j.

The Simplex Method

Elimination is the workhorse for linear equations. The simplex method is the workhorse for linear inequalities. We cannot give the simplex method as much space as elimination, but the idea can be clear. *The simplex method goes from one corner to a neighboring corner of lower cost.* Eventually (and quite soon in practice) it reaches the corner of minimum cost.

A ***corner*** is a vector $x \geq 0$ that satisfies the m equations $Ax = b$ with at most m positive components. *The other $n - m$ components are zero.* (Those are the free variables. Back substitution gives the m basic variables. All variables must be nonnegative or x is a false corner.) For a *neighboring corner*, one zero component of x becomes positive and one positive component becomes zero.

The simplex method must decide which component "enters" by becoming positive, and which component "leaves" by becoming zero. That exchange is chosen so as to lower the total cost. This is one step of the simplex method, moving toward x^*.

Here is the overall plan. Look at each zero component at the current corner. If it changes from 0 to 1, the other nonzeros have to adjust to keep $Ax = b$. Find the new x by back substitution and compute the change in the total cost $c \cdot x$. This change is the "reduced cost" r of the new component. The ***entering variable*** is the one that gives the ***most negative*** r. This is the greatest cost reduction for a single unit of a new variable.

Example 1 Suppose the current corner is $P = (4, 0, 0)$, with the Ph.D. doing all the work (the cost is \$20). If the student works one hour, the cost of $x = (3, 1, 0)$ is down to \$18. The reduced cost is $r = -2$. If the machine works one hour, then $x = (2, 0, 1)$ also costs \$18. The reduced cost is also $r = -2$. In this case the simplex method can choose either the student or the machine as the entering variable.

Even in this small example, the first step may not go immediately to the best x^*. The method chooses the entering variable before it knows how much of that variable to include. We computed r when the entering variable changes from 0 to 1, but one unit may be too much or too little. The method now chooses the leaving variable (the Ph.D.). It moves to corner Q or R in the figure.

The more of the entering variable we include, the lower the cost. This has to stop when one of the positive components (which are adjusting to keep $Ax = b$) hits zero. ***The leaving variable** is the first positive x_i to reach zero.* When that happens, a neighboring corner has been found. Then start again (from the new corner) to find the next variables to enter and leave.

When all reduced costs are positive, the current corner is the optimal x^*. No zero component can become positive without increasing $c \cdot x$. No new variable should enter. The problem is solved (and we can show that y^* is found too).

Note Generally x^* is reached in αn steps, where α is not large. But examples have been invented which use an exponential number of simplex steps. Eventually a different approach was developed, which is guaranteed to reach x^* in fewer (but more difficult) steps. The new methods travel through the *interior* of the feasible set.

Example 2 Minimize the cost $c \cdot x = 3x_1 + x_2 + 9x_3 + x_4$. The constraints are $x \geq 0$ and two equations $Ax = b$:

$$\begin{aligned} x_1 + 2x_3 + x_4 &= 4 \qquad & m &= 2 \quad \text{equations} \\ x_2 + x_3 - x_4 &= 2 \qquad & n &= 4 \quad \text{unknowns.} \end{aligned}$$

A starting corner is $x = (4, 2, 0, 0)$ which costs $c \cdot x = 14$. It has $m = 2$ nonzeros and $n - m = 2$ zeros. The zeros are x_3 and x_4. The question is whether x_3 or x_4 should enter (become nonzero). Try one unit of each of them:

$$\begin{array}{ll} \text{If } x_3 = 1 \text{ and } x_4 = 0, & \text{then } x = (2, 1, 1, 0) \text{ costs } 16. \\ \text{If } x_4 = 1 \text{ and } x_3 = 0, & \text{then } x = (3, 3, 0, 1) \text{ costs } 13. \end{array}$$

Compare those costs with 14. The reduced cost of x_3 is $r = 2$, positive and useless. The reduced cost of x_4 is $r = -1$, negative and helpful. *The entering variable is x_4.*

How much of x_4 can enter? One unit of x_4 made x_1 drop from 4 to 3. Four units will make x_1 drop from 4 to zero (while x_2 increases all the way to 6). *The leaving variable is x_1.* The new corner is $x = (0, 6, 0, 4)$, which costs only $c \cdot x = 10$. This is the optimal x^*, but to know that we have to try another simplex step from $(0, 6, 0, 4)$. Suppose x_1 or x_3 tries to enter:

$$\begin{array}{lll} \textbf{Start from the} & \text{If } x_1 = 1 \text{ and } x_3 = 0, & \text{then } x = (1, 5, 0, 3) \text{ costs } 11. \\ \textbf{corner } (\mathbf{0, 6, 0, 4}) & \text{If } x_3 = 1 \text{ and } x_1 = 0, & \text{then } x = (0, 3, 1, 2) \text{ costs } 14. \end{array}$$

Those costs are higher than 10. Both r's are positive—it does not pay to move. The current corner $(0, 6, 0, 4)$ is the solution x^*.

These calculations can be streamlined. Each simplex step solves three linear systems with the same matrix B. (This is the m by m matrix that keeps the m basic columns of A.) When a column enters and an old column leaves, there is a quick way to update B^{-1}. That is how most codes organize the simplex method.

Our text on *Computational Science and Engineering* includes a short code with comments. (The code is also on **math.mit.edu/cse**) The best $\boldsymbol{y}^*$ solves m equations $A^{\mathrm{T}}\boldsymbol{y}^* = \boldsymbol{c}$ in the m components that are nonzero in $\boldsymbol{x}^*$. Then we have optimality $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{s} = 0$ and this is duality: *Either* $x_j^* = 0$ *or the "slack" in* $\boldsymbol{s}^* = \boldsymbol{c} - A^{\mathrm{T}}\boldsymbol{y}^*$ *has* $s_j^* = 0$.

When $\boldsymbol{x}^* = (0, 4, 0)$ was the optimal corner $\boldsymbol{Q}$, the cheater's price was set by $y^* = 3$.

Interior Point Methods

The simplex method moves along the edges of the feasible set, eventually reaching the optimal corner $\boldsymbol{x}^*$. ***Interior point methods move inside the feasible set*** (where $\boldsymbol{x} > \boldsymbol{0}$). These methods hope to go more directly to $\boldsymbol{x}^*$. They work well.

One way to stay inside is to put a barrier at the boundary. Add extra cost as a *logarithm that blows up* when any variable x_j touches zero. The best vector has $\boldsymbol{x} > \boldsymbol{0}$. The number θ is a small parameter that we move toward zero.

Barrier problem $\quad$ **Minimize** $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \theta\,(\log x_1 + \cdots + \log x_n)$ **with** $A\boldsymbol{x} = \boldsymbol{b}$ $\quad$ (2)

This cost is nonlinear (but linear programming is already nonlinear from inequalities). The constraints $x_j \geq 0$ are not needed because $\log x_j$ becomes infinite at $x_j = 0$.

The barrier gives an *approximate problem* for each θ. The m constraints $A\boldsymbol{x} = \boldsymbol{b}$ have Lagrange multipliers $y_1, \ldots, y_m$. This is the good way to deal with constraints.

$\boldsymbol{y}$ from Lagrange
$$L(x, y, \theta) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \theta\,(\textstyle\sum \log x_i) - \boldsymbol{y}^{\mathrm{T}}(A\boldsymbol{x} - \boldsymbol{b}) \tag{3}$$

$\partial L/\partial y = 0$ brings back $A\boldsymbol{x} = \boldsymbol{b}$. The derivatives $\partial L/\partial x_j$ are interesting!

Optimality in barrier pbm
$$\frac{\partial L}{\partial x_j} = c_j - \frac{\theta}{x_j} - (A^{\mathrm{T}}y)_j = 0 \quad \text{which is} \quad \boldsymbol{x_j s_j = \theta}\,. \tag{4}$$

The true problem has $x_j s_j = 0$. The barrier problem has $x_j s_j = \theta$. The solutions $\boldsymbol{x}^*(\theta)$ lie on the ***central path*** to $\boldsymbol{x}^*(0)$. Those n optimality equations $x_j s_j = \theta$ are nonlinear, and we solve them iteratively by Newton's method.

The current $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}$ will satisfy $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ and $A^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}$, *but not* $x_j s_j = \theta$. Newton's method takes a step $\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s}$. By ignoring the second-order term $\Delta\boldsymbol{x}\,\Delta\boldsymbol{s}$ in $(\boldsymbol{x} + \Delta\boldsymbol{x})(\boldsymbol{s} + \Delta\boldsymbol{s}) = \boldsymbol{\theta}$, the corrections in $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}$ come from linear equations:

Newton step
$$\begin{aligned} A\,\Delta\boldsymbol{x} &= 0 \\ A^{\mathrm{T}}\Delta\boldsymbol{y} + \Delta\boldsymbol{s} &= 0 \\ s_j\Delta x_j + x_j\Delta s_j &= \theta - x_j s_j \end{aligned} \tag{5}$$

Newton iteration has quadratic convergence for each θ, and then θ approaches zero. The duality gap $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{s}$ generally goes below 10^{-8} after 20 to 60 steps. The explanation in my *Computational Science and Engineering* textbook takes one Newton step in detail, for the example with four homework problems. I didn't intend that the student should end up doing all the work, but $\boldsymbol{x}^*$ turned out that way.

This interior point method is used almost "as is" in commercial software, for a large class of linear and nonlinear optimization problems.

Problem Set 8.4

1 Draw the region in the xy plane where $x + 2y = 6$ and $x \geq 0$ and $y \geq 0$. Which point in this "feasible set" minimizes the cost $c = x + 3y$? Which point gives maximum cost? Those points are at corners.

2 Draw the region in the xy plane where $x + 2y \leq 6,\ 2x + y \leq 6,\ x \geq 0,\ y \geq 0$. It has four corners. Which corner minimizes the cost $c = 2x - y$?

3 What are the corners of the set $x_1 + 2x_2 - x_3 = 4$ with x_1, x_2, x_3 all ≥ 0? Show that the cost $x_1 + 2x_3$ can be very negative in this feasible set. This is an example of unbounded cost: no minimum.

4 Start at $\boldsymbol{x} = (0, 0, 2)$ where the machine solves all four problems for \$16. Move to $\boldsymbol{x} = (0, 1, \quad)$ to find the reduced cost r (the savings per hour) for work by the student. Find r for the Ph.D. by moving to $\boldsymbol{x} = (1, 0, \quad)$ with 1 hour of Ph.D. work.

5 Start Example 1 from the Ph.D. corner $(4, 0, 0)$ with $\boldsymbol{c}$ changed to $[\,5 \;\; 3 \;\; 7\,]$. Show that r is better for the machine even when the total cost is lower for the student. The simplex method takes two steps, first to the machine and then to the student for $\boldsymbol{x}^*$.

6 Choose a different cost vector $\boldsymbol{c}$ so the Ph.D. gets the job. Rewrite the dual problem (maximum income to the cheater).

7 A six-problem homework on which the Ph.D. is fastest gives a second constraint $2x_1 + x_2 + x_3 = 6$. Then $\boldsymbol{x} = (2, 2, 0)$ shows two hours of work by Ph.D. and student on each homework. Does this $\boldsymbol{x}$ minimize the cost $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ with $\boldsymbol{c} = (5, 3, 8)$?

8 These two problems are also dual. Prove weak duality, that always $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} \leq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$:

Primal problem	Minimize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ with $A\boldsymbol{x} \geq \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$.
Dual problem	Maximize $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}$ with $A^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$ and $\boldsymbol{y} \geq \boldsymbol{0}$.

8.5 Fourier Series: Linear Algebra for Functions

This section goes from finite dimensions to *infinite* dimensions. I want to explain linear algebra in infinite-dimensional space, and to show that it still works. First step: look back. This book began with vectors and dot products and linear combinations. We begin by converting those basic ideas to the infinite case—then the rest will follow.

What does it mean for a vector to have infinitely many components? There are two different answers, both good:

1. The vector becomes $v = (v_1, v_2, v_3, \ldots)$. It could be $(1, \frac{1}{2}, \frac{1}{4}, \ldots)$.

2. The vector becomes a function $f(x)$. It could be $\sin x$.

We will go both ways. Then the idea of Fourier series will connect them.

After vectors come *dot products.* The natural dot product of two infinite vectors $(v_1, v_2, \ldots)$ and $(w_1, w_2, \ldots)$ is an infinite series:

$$\textbf{Dot product} \qquad v \cdot w = v_1 w_1 + v_2 w_2 + \cdots . \tag{1}$$

This brings a new question, which never occurred to us for vectors in $\mathbf{R}^n$. Does this infinite sum add up to a finite number? Does the series converge? Here is the first and biggest difference between finite and infinite.

When $v = w = (1, 1, 1, \ldots)$, the sum certainly does not converge. In that case $v \cdot w = 1 + 1 + 1 + \cdots$ is infinite. Since v equals w, we are really computing $v \cdot v = \|v\|^2 =$ length squared. The vector $(1, 1, 1, \ldots)$ has infinite length. *We don't want that vector.* Since we are making the rules, we don't have to include it. The only vectors to be allowed are those with finite length:

DEFINITION The vector $(v_1, v_2, \ldots)$ is in our infinite-dimensional "***Hilbert space***" if and only if its length $\|v\|$ is finite:

$$\|v\|^2 = v \cdot v = v_1^2 + v_2^2 + v_3^2 + \cdots \text{ must add to a finite number.}$$

Example 1 The vector $v = (1, \frac{1}{2}, \frac{1}{4}, \ldots)$ is included in Hilbert space, because its length is $2/\sqrt{3}$. We have a geometric series that adds to $4/3$. The length of v is the square root:

$$\textbf{Length squared} \qquad v \cdot v = 1 + \tfrac{1}{4} + \tfrac{1}{16} + \cdots = \frac{1}{1 - \frac{1}{4}} = \tfrac{4}{3}.$$

Question If v and w have finite length, how large can their dot product be?

Answer The sum $v \cdot w = v_1 w_1 + v_2 w_2 + \cdots$ also adds to a finite number. We can safely take dot products. The Schwarz inequality is still true:

$$\textbf{Schwarz inequality} \qquad |v \cdot w| \le \|v\| \, \|w\|. \tag{2}$$

The ratio of $v \cdot w$ to $\|v\| \, \|w\|$ is still the cosine of θ (the angle between v and w). Even in infinite-dimensional space, $|\cos \theta|$ is not greater than 1.

Now change over to functions. Those are the "vectors." The space of functions $f(x)$, $g(x), h(x), \ldots$ defined for $0 \leq x \leq 2\pi$ must be somehow bigger than $\mathbf{R}^n$. ***What is the dot product of $f(x)$ and $g(x)$? What is the length of $f(x)$?***

Key point in the continuous case: *Sums are replaced by integrals.* Instead of a sum of v_j times w_j, the dot product is an integral of $f(x)$ times $g(x)$. Change the "dot" to parentheses with a comma, and change the words "dot product" to *inner product*:

DEFINITION The ***inner product*** of $f(x)$ and $g(x)$, and the ***length squared***, are

$$(f, g) = \int_0^{2\pi} f(x)g(x)\,dx \quad \text{and} \quad \|f\|^2 = \int_0^{2\pi} \left(f(x)\right)^2 dx. \tag{3}$$

The interval $[0, 2\pi]$ where the functions are defined could change to a different interval like $[0, 1]$ or $(-\infty, \infty)$. We chose 2π because our first examples are $\sin x$ and $\cos x$.

Example 2 The length of $f(x) = \sin x$ comes from its inner product with itself:

$$(f, f) = \int_0^{2\pi} (\sin\ x)^2\,dx = \pi. \quad \text{The length of } \sin x \text{ is } \sqrt{\pi}.$$

That is a standard integral in calculus—not part of linear algebra. By writing $\sin^2 x$ as $\frac{1}{2} - \frac{1}{2}\cos 2x$, we see it go above and below its average value $\frac{1}{2}$. Multiply that average by the interval length 2π to get the answer π.

More important: $\sin x$ ***and*** $\cos x$ ***are orthogonal in function space***:

Inner product is zero
$$\int_0^{2\pi} \sin x \cos x\,dx = \int_0^{2\pi} \tfrac{1}{2}\sin 2x\,dx = \left[-\tfrac{1}{4}\cos 2x\right]_0^{2\pi} = 0. \tag{4}$$

This zero is no accident. It is highly important to science. The orthogonality goes beyond the two functions $\sin x$ and $\cos x$, to an infinite list of sines and cosines. The list contains $\cos 0x$ (which is 1), $\sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots$.

Every function in that list is orthogonal to every other function in the list.

Fourier Series

The Fourier series of a function $y(x)$ is its expansion into sines and cosines:

$$y(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots. \tag{5}$$

We have an orthogonal basis! The vectors in "function space" are combinations of the sines and cosines. On the interval from $x = 2\pi$ to $x = 4\pi$, all our functions repeat what they did from 0 to 2π. They are "*periodic*." The distance between repetitions is the period 2π.

Remember: The list is infinite. The Fourier series is an infinite series. We avoided the vector $v = (1, 1, 1, \ldots)$ because its length is infinite, now we avoid a function like $\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots$. (*Note*: This is π times the famous **delta function** $\delta(x)$. It is an infinite "spike" above a single point. At $x = 0$ its height $\frac{1}{2} + 1 + 1 + \cdots$ is infinite. At all points inside $0 < x < 2\pi$ the series adds in some average way to zero.) The integral of $\delta(x)$ is 1. But $\int \delta^2(x) = \infty$, so delta functions are excluded from Hilbert space.

Compute the length of a typical sum $f(x)$:

$$\begin{aligned}(f, f) &= \int_0^{2\pi} (a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \cdots)^2 \, dx \\ &= \int_0^{2\pi} (a_0^2 + a_1^2 \cos^2 x + b_1^2 \sin^2 x + a_2^2 \cos^2 2x + \cdots) \, dx \\ \|f\|^2 &= 2\pi a_0^2 + \pi(a_1^2 + b_1^2 + a_2^2 + \cdots).\end{aligned} \tag{6}$$

The step from line 1 to line 2 used orthogonality. All products like $\cos x \cos 2x$ integrate to give zero. Line 2 contains what is left—the integrals of each sine and cosine squared. Line 3 evaluates those integrals. (The integral of 1^2 is 2π, when all other integrals give π.) If we divide by their lengths, our functions become *orthonormal*:

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots \textbf{\textit{ is an orthonormal basis for our function space.}}$$

These are unit vectors. We could combine them with coefficients $A_0, A_1, B_1, A_2, \ldots$ to yield a function $F(x)$. Then the 2π and the π's drop out of the formula for length.

Function length = vector length $\qquad \|F\|^2 = (F, F) = A_0^2 + A_1^2 + B_1^2 + A_2^2 + \cdots. \quad (7)$

Here is the important point, for $f(x)$ as well as $F(x)$. *The function has finite length exactly when the vector of coefficients has finite length.* Fourier series gives us a perfect match between function space and infinite-dimensional Hilbert space. The function is in L^2, its Fourier coefficients are in ℓ^2.

The function space contains $f(x)$ exactly when the Hilbert space contains the vector $v = (a_0, a_1, b_1, \ldots)$ of Fourier coefficients. Both $f(x)$ and v have finite length.

Example 3 Suppose $f(x)$ is a "square wave," equal to 1 for $0 \le x < \pi$. Then $f(x)$ drops to -1 for $\pi \le x < 2\pi$. The $+1$ and -1 repeats forever. This $f(x)$ is an odd function like the sines, and all its cosine coefficients are zero. We will find its Fourier series, containing only sines:

$$\textbf{Square wave} \qquad f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]. \tag{8}$$

The length is $\sqrt{2\pi}$, because at every point $\big(f(x)\big)^2$ is $(-1)^2$ or $(+1)^2$:

$$\|f\|^2 = \int_0^{2\pi} \big(f(x)\big)^2 \, dx = \int_0^{2\pi} 1 \, dx = 2\pi.$$

At $x = 0$ the sines are zero and the Fourier series gives zero. This is half way up the jump from -1 to $+1$. The Fourier series is also interesting when $x = \frac{\pi}{2}$. At this point the square wave equals 1, and the sines in (8) alternate between $+1$ and -1:

$$\textbf{Formula for } \pi \qquad 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right). \tag{9}$$

Multiply by π to find a magical formula $4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$ for that famous number.

The Fourier Coefficients

How do we find the a's and b's which multiply the cosines and sines? For a given function $f(x)$, we are asking for its Fourier coefficients:

$$\textbf{Fourier series} \qquad f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \cdots .$$

Here is the way to find a_1. ***Multiply both sides by*** $\cos x$. ***Then integrate from* 0 *to*** 2π. The key is orthogonality! All integrals on the right side are zero, except for $\cos^2 x$:

$$\textbf{Coefficient } a_1 \qquad \int_0^{2\pi} f(x) \cos x \, dx = \int_0^{2\pi} a_1 \cos^2 x \, dx = \pi a_1. \tag{10}$$

Divide by π and you have a_1. To find any other a_k, multiply the Fourier series by $\cos kx$. Integrate from 0 to 2π. Use orthogonality, so only the integral of $a_k \cos^2 kx$ is left. That integral is πa_k, and divide by π:

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx \quad \text{and similarly} \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx. \tag{11}$$

The exception is a_0. This time we multiply by $\cos 0x = 1$. The integral of 1 is 2π:

$$\textbf{Constant term} \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot 1 \, dx = \textbf{\textit{average value}} \text{ of } f(x). \tag{12}$$

I used those formulas to find the Fourier coefficients for the square wave. The integral of $f(x) \cos kx$ was zero. The integral of $f(x) \sin kx$ was $4/k$ for odd k.

Compare Linear Algebra in $\mathbf{R}^n$

The point to emphasize is how this infinite-dimensional case is so much like the n-dimensional case. Suppose the nonzero vectors $v_1, \ldots, v_n$ are orthogonal. We want to write the vector b (instead of the function $f(x)$) as a combination of those v's:

$$\textbf{Finite orthogonal series} \quad b = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n. \tag{13}$$

Multiply both sides by v_1^{T}. Use orthogonality, so $v_1^{\mathrm{T}} v_2 = 0$. Only the c_1 term is left:

$$\textbf{Coefficient } c_1 \quad v_1^{\mathrm{T}} b = c_1 v_1^{\mathrm{T}} v_1 + 0 + \cdots + 0. \quad \text{Therefore } c_1 = v_1^{\mathrm{T}} b / v_1^{\mathrm{T}} v_1. \tag{14}$$

The denominator $v_1^{\mathrm{T}} v_1$ is the length squared, like π in equation 11. The numerator $v_1^{\mathrm{T}} b$ is the inner product like $\int f(x) \cos kx \, dx$. ***Coefficients are easy to find when the basis***

vectors are orthogonal. We are just doing one-dimensional projections, to find the components along each basis vector.

The formulas are even better when the vectors are orthonormal. Then we have unit vectors. The denominators $v_k^{\mathrm{T}}v_k$ are all 1. You know $c_k = v_k^{\mathrm{T}}b$ in another form:

$$\textbf{Equation for } c\textbf{'s} \quad c_1v_1 + \cdots + c_nv_n = b \quad \text{or} \quad \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = b.$$

The v's are in an orthogonal matrix Q. Its inverse is Q^{T}. That gives the c's:

$$Qc = b \quad \text{yields} \quad c = Q^{\mathrm{T}}b. \quad \text{Row by row this is } c_k = q_k^{\mathrm{T}}b.$$

Fourier series is like having a matrix with infinitely many orthogonal columns. Those columns are the basis functions $1, \cos x, \sin x, \ldots$. After dividing by their lengths we have an "infinite orthogonal matrix." Its inverse is its transpose. Orthogonality is what reduces a series of terms to one single term.

Problem Set 8.5

1 Integrate the trig identity $2\cos jx \cos kx = \cos(j+k)x + \cos(j-k)x$ to show that $\cos jx$ is orthogonal to $\cos kx$, provided $j \neq k$. What is the result when $j = k$?

2 Show that $1, x$, and $x^2 - \frac{1}{3}$ are orthogonal, when the integration is from $x = -1$ to $x = 1$. Write $f(x) = 2x^2$ as a combination of those orthogonal functions.

3 Find a vector $(w_1, w_2, w_3, \ldots)$ that is orthogonal to $v = (1, \frac{1}{2}, \frac{1}{4}, \ldots)$. Compute its length $\|w\|$.

4 The first three *Legendre polynomials* are $1, x$, and $x^2 - \frac{1}{3}$. Choose c so that the fourth polynomial $x^3 - cx$ is orthogonal to the first three. All integrals go from -1 to 1.

5 For the square wave $f(x)$ in Example 3, show that

$$\int_0^{2\pi} f(x)\cos x\, dx = 0 \qquad \int_0^{2\pi} f(x)\sin x\, dx = 4 \qquad \int_0^{2\pi} f(x)\sin 2x\, dx = 0.$$

Which three Fourier coefficients come from those integrals?

6 The square wave has $\|f\|^2 = 2\pi$. Then (6) gives what remarkable sum for π^2?

7 Graph the square wave. Then graph by hand the sum of two sine terms in its series, or graph by machine the sum of 2, 3, and 10 terms. The famous ***Gibbs phenomenon*** is the oscillation that overshoots the jump (this doesn't die down with more terms).

8 Find the lengths of these vectors in Hilbert space:

(a) $v = \left(\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \ldots\right)$

(b) $v = (1, a, a^2, \dots)$

(c) $f(x) = 1 + \sin x$.

9 Compute the Fourier coefficients a_k and b_k for $f(x)$ defined from 0 to 2π:

(a) $f(x) = 1$ for $0 \le x \le \pi$, $f(x) = 0$ for $\pi < x < 2\pi$

(b) $f(x) = x$.

10 When $f(x)$ has period 2π, why is its integral from $-\pi$ to π the same as from 0 to 2π? If $f(x)$ is an *odd* function, $f(-x) = -f(x)$, show that $\int_0^{2\pi} f(x)\,dx$ is zero. Odd functions only have sine terms, even functions have cosines.

11 From trig identities find the only two terms in the Fourier series for $f(x)$:

(a) $f(x) = \cos^2 x$ (b) $f(x) = \cos\left(x + \frac{\pi}{3}\right)$ (c) $f(x) = \sin^3 x$

12 The functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$ are a basis for Hilbert space. Write the derivatives of those first five functions as combinations of the same five functions. What is the 5 by 5 "differentiation matrix" for these functions?

13 Find the Fourier coefficients a_k and b_k of the square pulse $F(x)$ centered at $x = 0$: $F(x) = 1/h$ for $|x| \le h/2$ and $F(x) = 0$ for $h/2 < |x| \le \pi$.

As $h \to 0$, this $F(x)$ approaches a delta function. Find the limits of a_k and b_k.

The Fourier Series section 4.1 of *Computational Science and Engineering* explains the sine series, cosine series, complete series, and complex series $\Sigma c_k e^{ikx}$ on **math.mit.edu/cse**.

8.6 Linear Algebra for Statistics and Probability

Statistics deals with data, often in large quantities. Since data tends to go into rectangular matrices, we expect to see $A^{\mathrm{T}}A$. The least squares problem $A\widehat{x} \approx \boldsymbol{b}$ is ***linear regression.*** The best solution $\widehat{x}$ fits m observations by $n < m$ parameters. This is a fundamental application of linear algebra to statistics.

This section goes beyond $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$. These unweighted equations assume that the measurements $b_1, \ldots, b_m$ are equally reliable. When there is good reason to expect higher accuracy (lower variance) in some b_i, those equations should be weighted more heavily. *With what weights* $w_1, \cdots, w_m$? And if the b_i are not independent, a ***covariance matrix*** Σ gives the statistics of the errors. Here are key topics in this section:

1. Weighted least squares and $A^{\mathrm{T}}CA\widehat{x} = A^{\mathrm{T}}C\boldsymbol{b}$
2. Variances $\sigma_1^2, \ldots, \sigma_m^2$ and the covariance matrix Σ
3. Important probability distributions: binomial, Poisson, and normal
4. Principal Component Analysis (PCA) to find combinations with greatest variance.

Weighted Least Squares

To include weights in the m equations $A\boldsymbol{x} = \boldsymbol{b}$, multiply each equation i by a weight w_i. Put those m weights into a diagonal matrix W. *We are replacing* $A\boldsymbol{x} = \boldsymbol{b}$ *by* $WA\boldsymbol{x} = W\boldsymbol{b}$. The equations are no more and no less solvable—we expect to use least squares.

The least squares equation $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}\boldsymbol{b}$ changes to $(WA)^{\mathrm{T}}WA\widehat{x} = (WA)^{\mathrm{T}}W\boldsymbol{b}$. The matrix $C = W^{\mathrm{T}}W$ is inside $(WA)^{\mathrm{T}}WA$, in the middle of weighted least squares.

Weighted least squares ***$C = W^{\mathrm{T}}W$ is in the n equations for $\widehat{x}$*** $$A^{\mathrm{T}}CA\widehat{x} = A^{\mathrm{T}}C\boldsymbol{b} \tag{1}$$

When $n = 1$ and $A =$ column of 1's, $\widehat{x}$ changes from an average to a weighted average:

Simplest case $$\widehat{x} = \frac{b_1 + \cdots + b_m}{m} \quad \text{changes to} \quad \widehat{x}_W = \frac{w_1^2 b_1 + \cdots + w_m^2 b_m}{w_1^2 + \cdots + w_m^2}. \tag{2}$$

This average $\widehat{x}_W$ gives greatest weight to the observations b_i that have the largest w_i. We always assume that errors have *zero mean.* (Subtract the mean if necessary, so there is no one-sided bias in the measurements.)

How should we choose the weights w_i? This depends on the reliability of b_i. If that observation has variance σ_i^2, then the root mean square error in b_i is σ_i. **When we divide the equations by** $\sigma_1, \ldots \sigma_m$ (left side together with right side), **all variances will equal 1.** So the weight is $w_i = 1/\sigma_i$ and the diagonal of $C = W^{\mathrm{T}}W$ contains the numbers $1/\sigma_i^2$.

The statistically correct matrix is $C = \operatorname{diag}(1/\sigma_1^2, \ldots, 1/\sigma_m^2)$.

This is correct provided the errors e_i and e_j in different equations are statistically independent. If the errors are dependent, off-diagonal entries show up in the covariance matrix Σ. The good choice is still $C = \Sigma^{-1}$ as described in this section.

Mean and Variance

The two crucial numbers for a random variable are its **mean** m and its **variance** σ^2. The "expected value" $\mathrm{E}[e]$ is found from the probabilities $p_1, p_2, \ldots$ of the possible errors $e_1, e_2, \ldots$ (and the variance σ^2 is always measured around the mean).

For a discrete random variable, the error e_j has probability p_j (the p_j add to 1):

$$\textbf{Mean } m = \mathrm{E}[e] = \sum e_j p_j \qquad \textbf{Variance } \sigma^2 = \mathrm{E}[(e-m)^2] = \sum (e_j - m)^2 p_j \qquad (3)$$

Example 1 Flip a fair coin. The result is 1 (for heads) or 0 (for tails). Those events have equal probabilities $p_0 = p_1 = 1/2$. The mean is $m = 1/2$ and the variance is $\sigma^2 = 1/4$:

$$\text{Mean} = (0)\,\frac{1}{2} + (1)\,\frac{1}{2} \qquad \text{Variance} = \left(0 - \frac{1}{2}\right)^2 \frac{1}{2} + \left(1 - \frac{1}{2}\right)^2 \frac{1}{2} = \frac{1}{4}.$$

Example 2 (**Binomial**) Flip the fair coin N times and count heads. With 3 flips, we see $M = 0, 1, 2,$ or 3 heads. The chances are $1/8, 3/8, 3/8, 1/8$. There are three ways to see $M = 2$ heads: HHT, HTH, and THH, and only HHH for $M = 3$ heads.

For all N, the number of ways to see M heads is the binomial coefficient "N *choose* M". Divide by the total number 2^N of all possible outcomes to get the probability for each M:

$$\begin{array}{l}\textbf{\textit{M} heads in}\\ \textbf{\textit{N} coin flips}\end{array} \quad p_M = \frac{1}{2^N}\binom{N}{M} = \frac{1}{2^N}\,\frac{N!}{M!(N-M)!} \qquad \text{Check } \frac{1}{2^3}\,\frac{3!}{2!\,1!} = \frac{3}{8} \qquad (4)$$

Gamblers know this instinctively. The probabilities p_M add to $\left(\frac{1}{2} + \frac{1}{2}\right)^N = 1$. The mean value of the number of heads is $m = N/2$. The variance around m turns out to be $\sigma^2 = N/4$. The standard deviation $\sigma = \sqrt{N}/2$ measures the expected spread around the mean.

Example 3 (**Poisson**) A very unfair coin (small $p << \frac{1}{2}$) is flipped very often (large N). **The product $\lambda = pN$ is kept fixed.** The high probability of tails is $1 - p$ each time. So the chance p_0 of no heads in N flips (tails every time) is $(1-p)^N = (1 - \lambda/N)^N$. For large N this approaches $e^{-\lambda}$. The probability p_j of j heads in N very unfair flips comes out neatly in terms of the crucial number $\lambda = pN$:

$$\textbf{Poisson probabilities} \quad p_j = \frac{\lambda^j}{j!}e^{-\lambda} \qquad \textbf{Mean } m = \lambda \qquad \textbf{Variance } \sigma^2 = \lambda \qquad (5)$$

Poisson applies to counting infrequent events (low p) over a long time T. Then $\lambda = pT$.

A *continuous* random variable will have a probability *density* function $p(x)$ instead of $p_1, p_2, \ldots$. "An outcome between x and $x + dx$ has probability $p(x)\,dx$." The total probability is $\int p(x)\,dx = 1$, since some outcome must happen. Sums become integrals:

$$\textbf{Mean } m = \textbf{Expected value} = \int x p(x)\,dx \qquad \textbf{Variance } \sigma^2 = \int (x-m)^2 p(x)\,dx. \qquad (6)$$

The outstanding example of a probability density function $p(x)$ (called the pdf) is the ***normal distribution*** $\mathrm{N}(0, \sigma)$. This has mean zero by symmetry. Its variance is σ^2:

$$\textbf{Normal (Gaussian)} \qquad p(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/2\sigma^2} \quad \text{with} \quad \int_{-\infty}^{\infty} p(x)\, dx = 1. \qquad (7)$$

The graph of $p(x)$ is the famous bell-shaped curve. The integral of $p(x)$ from $-\sigma$ to σ is the probability that a random sample is less than one standard deviation σ from the mean. This is near $2/3$. MATLAB's **randn** uses the normal distribution with $\sigma = 1$.

This normal $p(x)$ appears everywhere because of the ***Central Limit Theorem***: The average over many independent trials of another distribution (like binomial) will approach a normal distribution as $N \to \infty$. A shift produces $m = 0$ and rescaling produces $\sigma = 1$.

$$\textbf{Normalized headcount} \qquad x = \frac{M - \text{mean}}{\sigma} = \frac{M - N/2}{\sqrt{N}/2} \longrightarrow \text{Normal } \mathrm{N}(0, 1).$$

The Covariance Matrix

Now run m different experiments at once. They might be independent, or there might be some correlation between them. Each measurement $\boldsymbol{b}$ is now a *vector* with m components. Those components are the outputs b_i from the m experiments.

If we measure distances from the means m_i, each error $e_i = b_i - m_i$ has *mean zero*. If two errors e_i and e_j are *independent* (no relation between them), their product $e_i e_j$ also has mean zero. But if the measurements are by the same observer at nearly the same time, the errors e_i and e_j could tend to have the same sign or the same size. ***The errors in the m experiments could be correlated.*** The products $e_i e_j$ are weighted by p_{ij} (their probability): *covariance* $\sigma_{ij} = \sum\sum p_{ij} e_i e_j$. The sum of $e_i^2 p_{ii}$ is the variance σ_i^2:

$$\textbf{Covariance} \qquad \sigma_{ij} = \sigma_{ji} = \mathrm{E}[e_i e_j] = \textbf{expected value of } (e_i \textbf{ times } e_j). \qquad (8)$$

This is the (i, j) and (j, i) entry of the ***covariance matrix*** $\boldsymbol{\Sigma}$. The (i, i) entry is $\sigma_{ii} = \sigma_i^2$.

Example 4 **(Multivariate normal)** For m random variables, the probability density function moves from $p(x)$ to $p(\boldsymbol{b}) = p(b_1, \ldots, b_m)$. The normal distribution with mean zero was controlled by one positive number σ^2. Now $p(\boldsymbol{b})$ is controlled by an m by m positive definite matrix $\boldsymbol{\Sigma}$. This is the covariance matrix and its determinant is $|\boldsymbol{\Sigma}|$:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-x^2/2\sigma^2} \quad \text{becomes} \quad p(\boldsymbol{b}) = \frac{1}{(2\pi)^{m/2}|\boldsymbol{\Sigma}|^{1/2}}\, e^{-\boldsymbol{b}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{b}/2}$$

The integral of $p(\boldsymbol{b})$ over m-dimensional space is 1. The integral of $\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} p(\boldsymbol{b})$ is $\boldsymbol{\Sigma}$.

The good way to handle that exponent $-\boldsymbol{b}^{\mathrm{T}}\Sigma^{-1}\boldsymbol{b}/2$ is to use the eigenvalues and orthonormal eigenvectors of Σ (*linear algebra enters here*). When $\Sigma = Q\Lambda Q^{\mathrm{T}} = Q\Lambda Q^{-1}$, replacing $\boldsymbol{b}$ by $Q\boldsymbol{c}$ will split $p(\boldsymbol{b})$ into m one-dimensional normal distributions:

$$\exp\left(-\boldsymbol{b}^{\mathrm{T}}\Sigma^{-1}\boldsymbol{b}/2\right) = \exp\left(-\boldsymbol{c}^{\mathrm{T}}\Lambda^{-1}\boldsymbol{c}/2\right) = \left(e^{-c_1^2/2\lambda_1}\right)\cdots\left(e^{-c_m^2/2\lambda_m}\right).$$

The determinant has $|\Sigma|^{1/2} = |\Lambda|^{1/2} = (\lambda_1\cdots\lambda_m)^{1/2}$. Each integral over $-\infty < c_i < \infty$ is back to one dimension, where $\lambda = \sigma^2$. Notice the wonderful fact that after any linear transformation (here $\boldsymbol{c} = Q^{-1}\boldsymbol{b}$), we still have a multivariate normal distribution.

We could even reach variances $= 1$ by including $\sqrt{\Lambda}$ in the change from $\boldsymbol{b}$ to z:

Standard normal $\qquad \boldsymbol{b} = \sqrt{\Lambda}Qz$ changes $p(\boldsymbol{b})d\boldsymbol{b}$ to $p(z)dz = \dfrac{e^{-z^{\mathrm{T}}z/2}}{(2\pi)^{m/2}}dz$

This tells us the right weight matrix W to bring $A\boldsymbol{x} = \boldsymbol{b}$ back to ordinary least squares for $WA\boldsymbol{x} = W\boldsymbol{b}$. We want $W\boldsymbol{b}$ to become the standard normal z. So W will be the inverse of $\sqrt{\Lambda}Q$. Better than that, ***$C = W^{\mathrm{T}}W$ is the inverse of $Q\Lambda Q^{\mathrm{T}}$ which is Σ.***

Summary For independent errors, Σ is the diagonal matrix $\mathbf{diag}(\sigma_1^2,\ldots,\sigma_m^2)$. This is the usual choice. The right weights w_i for the equations $A\boldsymbol{x} = \boldsymbol{b}$ are $1/\sigma_1,\ldots,1/\sigma_m$ (this will equalize all variances to 1). The right matrix $C = W^{\mathrm{T}}W$ in the middle of the weighted least squares equations is exactly Σ^{-1}:

Weighted least squares $\qquad A^{\mathrm{T}}\Sigma^{-1}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\Sigma^{-1}\boldsymbol{b} \qquad (9)$

This choice of weighting returns $A\boldsymbol{x} = \boldsymbol{b}$ to a least squares problem $WA\boldsymbol{x} = W\boldsymbol{b}$ with equally reliable and independent errors. The usual equation $(WA)^{\mathrm{T}}WA\widehat{\boldsymbol{x}} = (WA)^{\mathrm{T}}W\boldsymbol{b}$ is the same as (9).

It was Gauss who found this *best linear unbiased estimate* $\widehat{\boldsymbol{x}}$. Unbiased because the mean of $\boldsymbol{x} - \widehat{\boldsymbol{x}}$ is zero, linear because of equation (9), best because the covariance of $\boldsymbol{x} - \widehat{\boldsymbol{x}}$ is as small as possible. That covariance (for error in $\widehat{\boldsymbol{x}}$, not error in $\boldsymbol{b}$!) is important:

Covariance of the best $\widehat{\boldsymbol{x}}$ $\qquad P = \mathrm{E}\left[(x-\widehat{x})(x-\widehat{x})^{\mathrm{T}}\right] = \left(A^{\mathrm{T}}\Sigma^{-1}A\right)^{-1}. \qquad (10)$

Example 5 Your pulse rate is measured ten times by independent doctors, all equally reliable. The mean error of each b_i is zero, and each variance is σ^2. Then $\Sigma = \sigma^2 I$. The ten equations $x = b_i$ produce the 10 by 1 matrix A of all ones. The best estimate $\widehat{x}$ is the average of the ten b_i. *The variance of that average value $\widehat{x}$ is the number P*:

$$P = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1} = \sigma^2/10 \quad \text{so averaging reduces the variance.}$$

This matrix $P = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}$ tells how reliable is the result $\widehat{x}$ of the experiment (Problem 6). P does not depend on the b's in the actual experiment! Those b's have probability distributions. Each experiment produces a sample value of $\widehat{x}$ from a sample $\boldsymbol{b}$.

When a small Σ gives good reliability of the inputs $\boldsymbol{b}$, a small P gives good reliability of the outputs $\widehat{\boldsymbol{x}}$. The key formula $P = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}$ connects those covariances.

Principal Component Analysis

These paragraphs are about finding useful information in a data matrix A. Start by measuring m properties (m features) of n samples. These could be grades in m courses for n students (a row for each course, a column for each student). From each row, subtract its average so the sample means are zero. We look for a ***combination of courses*** and/or ***combination of students*** for which the data provides the most information.

Information is "distance from randomness" and it is measured by **variance**. A large variance in course grades means greater information than a small variance.

The key matrix idea is the Singular Value Decomposition $A = U\Sigma V^{\mathrm{T}}$. We are back again to $A^{\mathrm{T}}A$ and AA^{T}, because their unit eigenvectors are the singular vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ in V and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ in U. The singular values in the diagonal matrix Σ (not the covariance) are in decreasing order and σ_1 is the most important. Weighting the m courses by the components of $\boldsymbol{u}_1$ gives a "***master course***" or "***eigencourse***" with the most significant grades.

Example 6 Suppose the grades A, B, C, F are worth $\mathbf{4, 2, 0, -6}$ points. If each course and each student has one of each grade, then all means are zero. Here is the grade matrix A with $(1, 1, 1, 1)$ in its nullspace (rank 3). To keep integers, the SVD of A will be written as $2U$ times $\Sigma/4$ times $(2V)^{\mathrm{T}}$. So the σ's are $12, 8, 4$:

$$\begin{bmatrix} -6 & 2 & 0 & 4 \\ 0 & 4 & -6 & 2 \\ 4 & 0 & 2 & -6 \\ 2 & -6 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & & \\ & 2 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Weighting the rows (the courses) by $\boldsymbol{u}_1 = \frac{1}{2}(-1, -1, 1, 1)$ will give the *eigencourse*. Weighting the columns (the students) by $\boldsymbol{v}_1 = \frac{1}{2}(1, -1, 1, -1)$ gives the *eigenstudent*. The fraction of the grade matrix that is "explained" by that one course and student is $\sigma_1^2/(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = 9/14$. The σ's in the SVD are the variances σ^2.

I guess this master course is what a Director of Admissions is looking for. If all grades in gym are the same, that row of A will be all zero—and gym is not part of the master course. Probably calculus is a part, but what about students who don't take calculus? The problem of ***missing data*** (holes in the matrix A) is extremely difficult for social sciences and the census and so much of the statistics of experiments.

Gene expression data Determining the functions of genes, and combinations of genes, is a central problem of genetics. Which genes combine to give which properties? Which genes malfunction to give which diseases?

We now have an incredibly fast way to find gene expression data in the lab. A gene microarray is often packed onto an Affymetrix chip, measuring tens of thousands of genes from one sample (one person). The understanding of genetic data (*bioinformatics*) has become a tremendous application of linear algebra.

Problem Set 8.6

1 Which line $Ct + D$ is the best fit to the three independent measurements $1, 2, 4$ at times $t = 0, 1, 2$ if the variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ are $1, 1, 2$? Use weights $w_i = 1/\sigma_i$.

2 In Problem 1, suppose that the third measurement is **totally unreliable**. The variance σ_3^2 becomes infinite. Then the best line will not use _____. Find the line that goes through the first two points and solves the first two equations in $Ax = b$ exactly.

3 In Problem 1, suppose that the third measurement is **totally reliable**. The variance σ_3^2 approaches zero. Now the best line will go through the third point exactly. Choose that line to minimize the sum of squares of the first two errors.

4 A single flip of a fair coin (0 or 1) has mean $m = 1/2$ and variance $\sigma^2 = 1/4$. This was Example 1. For the sum of two flips, the mean is $m = 1$. Compute the variance σ^2 around this mean, using the outcomes $0, 1, 2$ with their probabilities.

5 Instead of adding the flip results, make them two independent experiments. The outcome is $(0, 0), (1, 0), (0, 1)$ or $(1, 1)$. What is the covariance matrix Σ?

6 Change Example 1 so that the coin flip can be unfair. The probability is p for heads and $1 - p$ for tails. Find the mean m and the variance σ^2 of this distribution.

7 For two independent measurements $x = b_1$ and $x = b_2$, the best $\widehat{x}$ should be some weighted average $\widehat{x} = ab_1 + (1 - a)b_2$. When b_1 and b_2 have mean zero and variances σ_1^2 and σ_2^2, the variance of $\widehat{x}$ will be $P = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$. ***Choose the number a that minimizes P***: $dP/da = 0$.

Show that this a gives the $\widehat{x}$ in equation (2) which the text claimed is best, using weights $w_1 = 1/\sigma_1$ and $w_2 = 1/\sigma_2$.

8 The least squares estimate correctly weighted by Σ^{-1} is $\widehat{x} = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}A^{\mathrm{T}}\Sigma^{-1}\boldsymbol{b}$. ***Call that*** $\widehat{x} = L\boldsymbol{b}$. If $\boldsymbol{b}$ contains an error vector $\boldsymbol{e}$, then $\widehat{x}$ contains the error $L\boldsymbol{e}$.

The covariance matrix of those output errors $L\boldsymbol{e}$ is their expected value (average value) $P = \mathrm{E}\left[(L\boldsymbol{e})(L\boldsymbol{e})^{\mathrm{T}}\right] = L\mathrm{E}\left[\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}\right]L^{\mathrm{T}} = L\Sigma L^{\mathrm{T}}$. **Problem**: Do the multiplication $L\Sigma L^{\mathrm{T}}$ to show that P equals $(A^{\mathrm{T}}\Sigma^{-1}A)^{-1}$ as predicted in equation (10).

9 Change the grades to $3, 1, -1, -3$ for A, B, C, F. Show that the SVD of this grade matrix has the same $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2$ (same eigencourses) as in Example 5, but now A has rank 2.

$$\textbf{Grade matrix} \qquad A = \begin{bmatrix} 3 & -1 & 1 & -3 \\ -1 & 3 & -3 & 1 \\ -3 & 1 & -1 & 3 \\ 1 & -3 & 3 & -1 \end{bmatrix}$$

Notes One way to deal with missing entries in A is to complete the matrix to have minimum rank. And statistics makes major use of the pseudoinverse A^+ (which is exactly the left inverse $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ from the normal equation when $A^{\mathrm{T}}A$ is invertible).

8.7 Computer Graphics

Computer graphics deals with images. The images are moved around. Their scale is changed. Three dimensions are projected onto two dimensions. All the main operations are done by matrices—but the shape of these matrices is surprising.

The transformations of three-dimensional space are done with 4 *by* 4 *matrices.* You would expect 3 by 3. The reason for the change is that one of the four key operations cannot be done with a 3 by 3 matrix multiplication. Here are the four operations:

Translation	**(shift the origin to another point $P_0 = (x_0, y_0, z_0)$)**
Rescaling	**(by c in all directions or by different factors c_1, c_2, c_3)**
Rotation	**(around an axis through the origin or an axis through P_0)**
Projection	**(onto a plane through the origin or a plane through P_0).**

Translation is the easiest—just add (x_0, y_0, z_0) to every point. But this is not linear! No 3 by 3 matrix can move the origin. So we change the coordinates of the origin to $(0, 0, 0, 1)$. This is why the matrices are 4 by 4. The "*homogeneous coordinates*" of the point (x, y, z) are $(x, y, z, 1)$ and we now show how they work.

1. Translation Shift the whole three-dimensional space along the vector $\boldsymbol{v}_0$. The origin moves to (x_0, y_0, z_0). This vector $\boldsymbol{v}_0$ is added to every point $\boldsymbol{v}$ in $\mathbf{R}^3$. Using homogeneous coordinates, the 4 by 4 matrix T shifts the whole space by $\boldsymbol{v}_0$:

$$\textbf{\textit{Translation matrix}} \qquad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_0 & y_0 & z_0 & 1 \end{bmatrix}.$$

Important: *Computer graphics works with row vectors.* We have row times matrix instead of matrix times column. You can quickly check that $[0\ \ 0\ \ 0\ \ 1]\,T = [x_0\ \ y_0\ \ z_0\ \ 1]$.

To move the points $(0, 0, 0)$ and (x, y, z) by $\boldsymbol{v}_0$, change to homogeneous coordinates $(0, 0, 0, 1)$ and $(x, y, z, 1)$. Then multiply by T. A row vector times T gives a row vector. ***Every $\boldsymbol{v}$ moves to $\boldsymbol{v} + \boldsymbol{v}_0$***: $[x\ \ y\ \ z\ \ 1]\,T = [x + x_0\ \ y + y_0\ \ z + z_0\ \ 1]$.

The output tells where any $\boldsymbol{v}$ will move. (It goes to $\boldsymbol{v} + \boldsymbol{v}_0$.) Translation is now achieved by a matrix, which was impossible in $\mathbf{R}^3$.

2. Scaling To make a picture fit a page, we change its width and height. A Xerox copier will rescale a figure by 90%. In linear algebra, we multiply by .9 times the identity matrix. That matrix is normally 2 by 2 for a plane and 3 by 3 for a solid. In computer graphics, with homogeneous coordinates, the matrix is *one size larger:*

$$\textbf{\textit{Rescale the plane:}}\ \ S = \begin{bmatrix} .9 & & \\ & .9 & \\ & & 1 \end{bmatrix} \qquad \textbf{\textit{Rescale a solid:}}\ \ S = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Important: S is not cI. We keep the "1" in the lower corner. Then $[x, y, 1]$ times S is the correct answer in homogeneous coordinates. The origin stays in its normal position because $[0\ 0\ 1]S = [0\ 0\ 1]$.

If we change that 1 to c, the result is strange. ***The point*** (cx, cy, cz, c) ***is the same as*** $(x, y, z, 1)$. The special property of homogeneous coordinates is that *multiplying by cI does not move the point.* The origin in $\mathbf{R}^3$ has homogeneous coordinates $(0, 0, 0, 1)$ and $(0, 0, 0, c)$ for every nonzero c. This is the idea behind the word "homogeneous."

Scaling can be different in different directions. To fit a full-page picture onto a half-page, scale the y direction by $\frac{1}{2}$. To create a margin, scale the x direction by $\frac{3}{4}$. The graphics matrix is diagonal but not 2 by 2. It is 3 by 3 to rescale a plane and 4 by 4 to rescale a space:

$$\textbf{\textit{Scaling matrices}} \quad S = \begin{bmatrix} \frac{3}{4} & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & 1 \end{bmatrix}.$$

That last matrix S rescales the x, y, z directions by positive numbers c_1, c_2, c_3. The extra column in all these matrices leaves the extra 1 at the end of every vector.

Summary The scaling matrix S is the same size as the translation matrix T. They can be multiplied. To translate and then rescale, multiply $\boldsymbol{v}TS$. To rescale and then translate, multiply $\boldsymbol{v}ST$. Are those different? *Yes.*

The point (x, y, z) in $\mathbf{R}^3$ has homogeneous coordinates $(x, y, z, 1)$ in $\mathbf{P}^3$. This "projective space" is not the same as $\mathbf{R}^4$. It is still three-dimensional. To achieve such a thing, (cx, cy, cz, c) is the same point as $(x, y, z, 1)$. Those points of projective space $\mathbf{P}^3$ are really lines through the origin in $\mathbf{R}^4$.

Computer graphics uses ***affine*** transformations, *linear plus shift.* An affine transformation T is executed on $\mathbf{P}^3$ by a 4 by 4 matrix with a special fourth column:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix} = \begin{bmatrix} T(1,0,0) & 0 \\ T(0,1,0) & 0 \\ T(0,0,1) & 0 \\ T(0,0,0) & 1 \end{bmatrix}.$$

The usual 3 by 3 matrix tells us three outputs, this tells four. The usual outputs come from the inputs $(1, 0, 0)$ and $(0, 1, 0)$ and $(0, 0, 1)$. When the transformation is linear, three outputs reveal everything. When the transformation is affine, the matrix also contains the output from $(0, 0, 0)$. Then we know the shift.

3. Rotation A rotation in $\mathbf{R}^2$ or $\mathbf{R}^3$ is achieved by an orthogonal matrix Q. The determinant is $+1$. (With determinant -1 we get an extra reflection through a mirror.) Include the extra column when you use homogeneous coordinates!

$$\textbf{\textit{Plane rotation}} \quad Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix rotates the plane around the origin. ***How would we rotate around a different point*** $(4,5)$? The answer brings out the beauty of homogeneous coordinates. ***Translate*** $(4,5)$ ***to*** $(0,0)$, ***then rotate by*** θ, ***then translate*** $(0,0)$ ***back to*** $(4,5)$:

$$vT_-RT_+ = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -5 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}.$$

I won't multiply. The point is to apply the matrices one at a time: v translates to vT_-, then rotates to vT_-R, and translates back to vT_-RT_+. Because each point $\begin{bmatrix} x & y & 1 \end{bmatrix}$ is a row vector, T_- acts first. The center of rotation $(4,5)$—otherwise known as $(4,5,1)$—moves first to $(0,0,1)$. Rotation doesn't change it. Then T_+ moves it back to $(4,5,1)$. All as it should be. The point $(4,6,1)$ moves to $(0,1,1)$, then turns by θ and moves back.

In three dimensions, every rotation Q turns around an axis. The axis doesn't move—it is a line of eigenvectors with $\lambda = 1$. Suppose the axis is in the z direction. The 1 in Q is to leave the z axis alone, the extra 1 in R is to leave the origin alone:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} & & & 0 \\ & Q & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now suppose the rotation is around the unit vector $\boldsymbol{a} = (a_1, a_2, a_3)$. With this axis $\boldsymbol{a}$, the rotation matrix Q which fits into R has three parts:

$$Q = (\cos\theta)I + (1-\cos\theta) \begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{bmatrix} - \sin\theta \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}. \quad (1)$$

The axis doesn't move because $\boldsymbol{a}Q = \boldsymbol{a}$. When $\boldsymbol{a} = (0,0,1)$ is in the z direction, this Q becomes the previous Q—for rotation around the z axis.

The linear transformation Q always goes in the upper left block of R. Below it we see zeros, because rotation leaves the origin in place. When those are not zeros, the transformation is affine and the origin moves.

4. Projection In a linear algebra course, most planes go through the origin. In real life, most don't. A plane through the origin is a vector space. The other planes are affine spaces, sometimes called "flats." An affine space is what comes from translating a vector space.

We want to project three-dimensional vectors onto planes. Start with a plane through the origin, whose unit normal vector is $\boldsymbol{n}$. (We will keep $\boldsymbol{n}$ as a column vector.) The vectors in the plane satisfy $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{v} = 0$. ***The usual projection onto the plane is the matrix*** $\boldsymbol{I - nn^{\mathrm{T}}}$. To project a vector, multiply by this matrix. The vector $\boldsymbol{n}$ is projected to zero, and the in-plane vectors $\boldsymbol{v}$ are projected onto themselves:

$$(I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\boldsymbol{n} = \boldsymbol{n} - \boldsymbol{n}(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{n}) = \boldsymbol{0} \quad \text{and} \quad (I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{n}(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{v}) = \boldsymbol{v}.$$

In homogeneous coordinates the projection matrix becomes 4 by 4 (but the origin doesn't move):

$$\textbf{\textit{Projection onto the plane}} \quad \boldsymbol{n}^{\mathrm{T}}\boldsymbol{v} = 0 \qquad P = \begin{bmatrix} & & & 0 \\ & I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now project onto a plane $\boldsymbol{n}^{\mathrm{T}}(\boldsymbol{v}-\boldsymbol{v}_0) = 0$ that does *not* go through the origin. One point on the plane is $\boldsymbol{v}_0$. This is an affine space (or a ***flat***). It is like the solutions to $A\boldsymbol{v} = \boldsymbol{b}$ when the right side is not zero. One particular solution $\boldsymbol{v}_0$ is added to the nullspace—to produce a flat.

The projection onto the flat has three steps. Translate $\boldsymbol{v}_0$ to the origin by T_-. Project along the $\boldsymbol{n}$ direction, and translate back along the row vector $\boldsymbol{v}_0$:

$$\textbf{\textit{Projection onto a flat}} \qquad T_-PT_+ = \begin{bmatrix} I & 0 \\ -\boldsymbol{v}_0 & 1 \end{bmatrix} \begin{bmatrix} I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ \boldsymbol{v}_0 & 1 \end{bmatrix}.$$

I can't help noticing that T_- and T_+ are inverse matrices: translate and translate back. They are like the elementary matrices of Chapter 2.

The exercises will include reflection matrices, also known as *mirror matrices*. These are the fifth type needed in computer graphics. A reflection moves each point twice as far as a projection—***the reflection goes through the plane and out the other side.*** So change the projection $I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$ to $I - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$ for a mirror matrix.

The matrix P gave a "*parallel*" projection. All points move parallel to $\boldsymbol{n}$, until they reach the plane. The other choice in computer graphics is a "*perspective*" projection. This is more popular because it includes foreshortening. With perspective, an object looks larger as it moves closer. Instead of staying parallel to $\boldsymbol{n}$ (and parallel to each other), the lines of projection come *toward the eye*—the center of projection. This is how we perceive depth in a two-dimensional photograph.

The basic problem of computer graphics starts with a scene and a viewing position. Ideally, the image on the screen is what the viewer would see. The simplest image assigns just one bit to every small picture element—called a ***pixel.*** It is light or dark. This gives a black and white picture with no shading. You would not approve. In practice, we assign shading levels between 0 and 2^8 for three colors like red, green, and blue. That means $8 \times 3 = 24$ bits for each pixel. Multiply by the number of pixels, and a lot of memory is needed!

Physically, a *raster frame buffer* directs the electron beam. It scans like a television set. The quality is controlled by the number of pixels and the number of bits per pixel. In this area, one standard text is *Computer Graphics: Principles and Practices* by Foley, Van Dam, Feiner, and Hughes (Addison-Wesley, 1995). The newer books still use homogeneous coordinates to handle translations. My best references were notes by Ronald Goldman and by Tony DeRose.

■ REVIEW OF THE KEY IDEAS ■

1. Computer graphics needs shift operations $T(\boldsymbol{v}) = \boldsymbol{v} + \boldsymbol{v}_0$ as well as linear operations $T(\boldsymbol{v}) = A\boldsymbol{v}$.
2. A shift in $\mathbf{R}^n$ can be executed by a matrix of order $n + 1$, using homogeneous coordinates.
3. The extra component 1 in $[x\ y\ z\ 1]$ is preserved when all matrices have the numbers $0, 0, 0, 1$ as last column.

Problem Set 8.7

1 A typical point in $\mathbf{R}^3$ is $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. The coordinate vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. The coordinates of the point are (x, y, z).

This point in computer graphics is $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ + **origin**. Its homogeneous coordinates are (, , ,). Other coordinates for the same point are (, , ,).

2 A linear transformation T is determined when we know $T(\boldsymbol{i}), T(\boldsymbol{j}), T(\boldsymbol{k})$. For an affine transformation we also need $T($_____$)$. The input point $(x, y, z, 1)$ is transformed to $xT(\boldsymbol{i}) + yT(\boldsymbol{j}) + zT(\boldsymbol{k}) +$ _____.

3 Multiply the 4 by 4 matrix T for translation along $(1, 4, 3)$ and the matrix T_1 for translation along $(0, 2, 5)$. The product TT_1 is translation along _____.

4 Write down the 4 by 4 matrix S that scales by a constant c. Multiply ST and also TS, where T is translation by $(1, 4, 3)$. To blow up the picture around the center point $(1, 4, 3)$, would you use $\boldsymbol{v}ST$ or $\boldsymbol{v}TS$?

5 What scaling matrix S (in homogeneous coordinates, so 3 by 3) would produce a 1 by 1 square page from a standard 8.5 by 11 page?

6 What 4 by 4 matrix would move a corner of a cube to the origin and then multiply all lengths by 2? The corner of the cube is originally at $(1, 1, 2)$.

7 When the three matrices in equation 1 multiply the unit vector $\boldsymbol{a}$, show that they give $(\cos\theta)\boldsymbol{a}$ and $(1 - \cos\theta)\boldsymbol{a}$ and $\mathbf{0}$. Addition gives $\boldsymbol{a}Q = \boldsymbol{a}$ and the rotation axis is not moved.

8 If $\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$, multiply by the three matrices in 1 to get $(\cos\theta)\boldsymbol{b}$ and $\mathbf{0}$ and a vector perpendicular to $\boldsymbol{b}$. So $Q\boldsymbol{b}$ makes an angle θ with $\boldsymbol{b}$. ***This is rotation.***

9 What is the 3 by 3 projection matrix $I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$ onto the plane $\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z = 0$? In homogeneous coordinates add $0, 0, 0, 1$ as an extra row and column in P.

10 With the same 4 by 4 matrix P, multiply $T_- P T_+$ to find the projection matrix onto the plane $\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z = 1$. The translation T_- moves a point on that plane (choose one) to $(0, 0, 0, 1)$. The inverse matrix T_+ moves it back.

11 Project $(3,3,3)$ onto those planes. Use P in Problem 9 and T_-PT_+ in Problem 10.

12 If you project a square onto a plane, what shape do you get?

13 If you project a cube onto a plane, what is the outline of the projection? Make the projection plane perpendicular to a diagonal of the cube.

14 The 3 by 3 mirror matrix that reflects through the plane $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{v} = 0$ is $M = I - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}$. Find the reflection of the point $(3,3,3)$ in the plane $\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z = 0$.

15 Find the reflection of $(3,3,3)$ in the plane $\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z = 1$. Take three steps T_-MT_+ using 4 by 4 matrices: translate by T_- so the plane goes through the origin, reflect the translated point $(3,3,3,1)T_-$ in that plane, then translate back by T_+.

16 The vector between the origin $(0,0,0,1)$ and the point $(x,y,z,1)$ is the difference $\boldsymbol{v} =$ _____. In homogeneous coordinates, vectors end in _____. So we add a _____ to a point, not a point to a point.

17 If you multiply only the *last* coordinate of each point to get (x,y,z,c), you rescale the whole space by the number _____. This is because the point (x,y,z,c) is the same as $(\ ,\ ,\ ,1)$.

Chapter 9

Numerical Linear Algebra

9.1 Gaussian Elimination in Practice

Numerical linear algebra is a struggle for *quick* solutions and also *accurate* solutions. We need efficiency but we have to avoid instability. In Gaussian elimination, the main freedom (always available) is to ***exchange equations***. This section explains when to exchange rows for the sake of speed, and when to do it for the sake of accuracy.

The key to accuracy is to avoid unnecessarily large numbers. Often that requires us to avoid small numbers! A small pivot generally means large multipliers (since we divide by the pivot). A good plan is "***partial pivoting***", to choose the *largest candidate* in each new column as the pivot. We will see why this strategy is built into computer programs.

Other row exchanges are done to save elimination steps. In practice, most large matrices are ***sparse***—almost all entries are zeros. Elimination is fastest when the equations are ordered to put those zeros (as far as possible) *outside the band of nonzeros.* Zeros inside the band "fill in" during elimination—the zeros are destroyed and don't help.

Section 9.2 is about instability that can't be avoided. It is built into the problem, and this sensitivity is measured by the "***condition number***". Then Section 9.3 describes how to solve $Ax = b$ by ***iterations***. Instead of direct elimination, the computer solves an easier equation many times. Each answer x_k leads to the next guess x_{k+1}. For good iterations, like **conjugate gradients**, the x_k converge quickly to $x = A^{-1}b$.

The Fastest Supercomputer

A new supercomputing record was announced by IBM and Los Alamos on May 20, 2008. The Roadrunner was the first to achieve a quadrillion (10^{15}) floating-point operations per second: *a petaflop machine*. The benchmark for this world record was a large dense linear system $Ax = b$: linear algebra.

The LINPACK software does elimination with partial pivoting. The biggest difference from this book is to organize the steps to use large submatrices and never single numbers. Roadrunner is a multicore Linux cluster with very remarkable processors, based on the

Cell Broadband Engine from Sony's PlayStation 3. The market for video games dwarfs scientific computing and led to astonishing acceleration in the chips.

This path to petascale is not the approach taken by IBM's BlueGene. A key issue was to count the standard quad-core processors that a petaflop machine would need: 32,000. The new architecture uses much less power, but its hybrid design has a price: a code needs three separate compilers and explicit instructions to move all the data. Please see the excellent article in *SIAM News* (**siam.org**, July 2008) and the details on **www.lanl.gov/roadrunner**.

The TOP500 project ranks the most powerful computer systems in the world. Roadrunner and BlueGene are #1 and #2 as this page is written in 2009.

Our thinking about matrix calculations is reflected in the highly optimized **BLAS** (*Basic Linear Algebra Subroutines*). They come at levels 1, 2, and 3:

1 Linear combinations of vectors $a\boldsymbol{u} + \boldsymbol{v}$: $O(n)$ work

2 Matrix-vector multiplications $A\boldsymbol{u} + \boldsymbol{v}$: $O(n^2)$ work

3 Matrix-matrix multiplications $AB + C$: $O(n^3)$ work

Level 1 is a single elimination step (multiply row j by ℓ_{ij} and subtract from row i). Level 2 can eliminate a whole column at once. A high performance solver is rich in Level 3 BLAS (AB has $2n^3$ flops and $2n^2$ data, a good ratio of work to talk).

It is *data passing* and *storage retrieval* that limit the speed of parallel processing. The high-velocity cache between main memory and floating-point computation has to be fully used! Top speed demands a ***block matrix approach*** to elimination.

The big change, coming now, is parallel processing at the chip level.

Roundoff Error and Partial Pivoting

Up to now, any pivot (nonzero of course) was accepted. In practice a small pivot is dangerous. A catastrophe can occur when numbers of different sizes are added. Computers keep a fixed number of significant digits (say three decimals, for a very weak machine). The sum 10,000 + 1 is rounded off to 10,000. The "1" is completely lost. Watch how that changes the solution to this problem:

$$\begin{aligned} .0001u + v &= 1 \\ -u + v &= 0 \end{aligned} \qquad \text{starts with coefficient matrix} \qquad A = \begin{bmatrix} .0001 & 1 \\ -1 & 1 \end{bmatrix}.$$

If we accept .0001 as the pivot, elimination adds 10,000 times row 1 to row 2. Roundoff leaves

$$10{,}000v = 10{,}000 \qquad \text{instead of} \qquad 10{,}001v = 10{,}000.$$

The computed answer $v = 1$ is near the true $v = .9999$. But then back substitution puts the wrong v into the equation for u:

$$.0001\,u + 1 = 1 \qquad \text{instead of} \qquad .0001\,u + .9999 = 1.$$

The first equation gives $u = 0$. The correct answer (look at the second equation) is $u = 1.000$. By losing the "1" in the matrix, we have lost the solution. ***The change from* 10,001 *to* 10,000 *has changed the answer from u* = 1 *to u* = 0** (100% error!).

If we exchange rows, even this weak computer finds an answer that is correct to three places:

$$\begin{aligned} -u+v&=0 \\ .0001u+v&=1 \end{aligned} \quad \longrightarrow \quad \begin{aligned} -u+v&=0 \\ v&=1 \end{aligned} \quad \longrightarrow \quad \begin{aligned} u&=1 \\ v&=1. \end{aligned}$$

The original pivots were .0001 and 10,000—badly scaled. After a row exchange the exact pivots are -1 and 1.0001—well scaled. The computed pivots -1 and 1 come close to the exact values. Small pivots bring numerical instability, and the remedy is ***partial pivoting***. The kth pivot is decided when we reach and search column k:

Choose the largest number in row k or below. Exchange its row with row k.

The strategy of *complete pivoting* looks also in later columns for the largest pivot. It exchanges columns as well as rows. This expense is seldom justified, and all major codes use partial pivoting. Multiplying a row or column by a scaling constant can also be very worthwhile. *If the first equation above is* $u + 10{,}000v = 10{,}000$ *and we don't rescale, then* 1 *looks like a good pivot and we would miss the essential row exchange.*

For positive definite matrices, row exchanges are *not* required. It is safe to accept the pivots as they appear. Small pivots can occur, but the matrix is not improved by row exchanges. When its condition number is high, the problem is in the matrix and not in the code. In this case the output is unavoidably sensitive to the input.

The reader now understands how a computer actually solves $Ax = b$***—by elimination with partial pivoting.*** Compared with the theoretical description—***find*** A^{-1} ***and multiply*** $A^{-1}b$—the details took time. But in computer time, elimination is much faster. I believe this algorithm is also the best approach to the algebra of row spaces and nullspaces.

Operation Counts: Full Matrices and Band Matrices

Here is a practical question about cost. *How many separate operations are needed to solve* $Ax = b$ *by elimination*? This decides how large a problem we can afford.

Look first at A, which changes gradually into U. When a multiple of row 1 is subtracted from row 2, we do n operations. The first is a division by the pivot, to find the multiplier ℓ. For the other $n-1$ entries along the row, the operation is a "multiply-subtract". For convenience, we count this as a single operation. If you regard multiplying by ℓ and subtracting from the existing entry as two separate operations, *multiply all our counts by* 2.

The matrix A is n by n. The operation count applies to all $n-1$ rows below the first. Thus it requires n times $n-1$ operations, or $n^2 - n$, to produce zeros below the first pivot. *Check: All* n^2 *entries are changed, except the* n *entries in the first row.*

When elimination is down to k equations, the rows are shorter. We need only $k^2 - k$ operations (instead of $n^2 - n$) to clear out the column below the pivot. This is true for $1 \le k \le n$. The last step requires no operations ($1^2 - 1 = 0$), since the pivot is set and forward elimination is complete. The total count to reach U is the sum of $k^2 - k$ over all values of k from 1 to n:

$$(1^2 + \cdots + n^2) - (1 + \cdots + n) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n^3 - n}{3}.$$

Those are known formulas for the sum of the first n numbers and the sum of the first n squares. Substituting $n = 1$ into $n^3 - n$ gives zero. Substituting $n = 100$ gives a million minus a hundred—then divide by 3. (That translates into one second on a workstation.) We will ignore the last term n in comparison with the larger term n^3, to reach our main conclusion:

The multiply-subtract count for forward elimination (A to U, producing L) is $\frac{1}{3}n^3$.

That means $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions. Doubling n increases this cost by eight (because n is cubed). 100 equations are easy, 1000 are more expensive, 10000 dense equations are close to impossible. We need a faster computer or a lot of zeros or a new idea.

On the right side of the equations, the steps go much faster. We operate on single numbers, not whole rows. ***Each right side needs exactly n^2 operations.*** Down and back up we are solving two triangular systems, $Lc = b$ forward and $Ux = c$ backward. In back substitution, the last unknown needs only division by the last pivot. The equation above it needs two operations—substituting x_n and dividing by *its* pivot. The kth step needs k multiply-subtract operations, and the total for back substitution is

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \tfrac{1}{2}n^2 \quad \text{operations.}$$

The forward part is similar. *The n^2 total exactly equals the count for multiplying $A^{-1}b$!* This leaves Gaussian elimination with two big advantages over $A^{-1}b$:

1 Elimination requires $\frac{1}{3}n^3$ compared to n^3 for A^{-1}.

2 If A is *banded* so are L and U. But A^{-1} is full of nonzeros.

Band Matrices

These counts are improved when A has "*good zeros*". A good zero is an entry that remains zero in L and U. ***The best zeros are at the beginning of a row.*** They require no elimination steps (the multipliers are zero). So we also find those same good zeros in L. That is especially clear for this *tridiagonal matrix* A:

Tridiagonal
Bidiagonal
times
bidiagonal

$$\begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}.$$

Rows 3 and 4 of A begin with zeros. No multiplier is needed, so L has the same zeros. Also columns 3 and 4 *start* with zeros. When a multiple of row 1 is subtracted from row 2, no calculation is required beyond the second column. The rows are short. They stay short! Figure 9.1 shows how a band matrix A has band factors L and U.

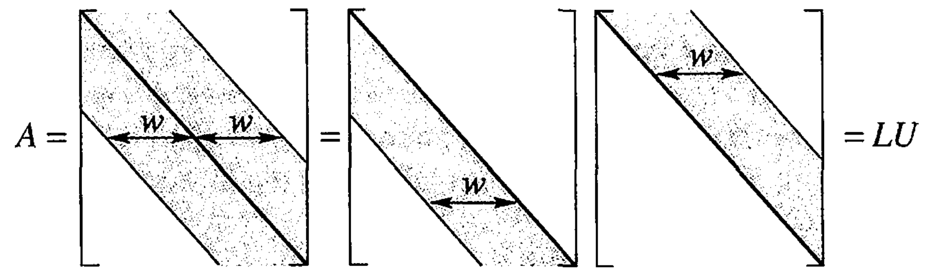


Figure 9.1: $A = LU$ for a band matrix. Good zeros in A *stay zero* in L and U.

These zeros lead to a complete change in the operation count, for "half-bandwidth" w:

A band matrix has $a_{ij} = 0$ when $|i - j| > w$.

Thus $w = 1$ for a diagonal matrix, $w = 2$ for tridiagonal, $w = n$ for dense. The length of the pivot row is at most w. There are no more than $w - 1$ nonzeros below any pivot. Each stage of elimination is complete after $w(w-1)$ operations, and *the band structure survives.* There are n columns to clear out. Therefore:

Elimination on a band matrix (A to L and U) needs less than $w^2 n$ operations.

For a band matrix, the count is proportional to n instead of n^3. It is also proportional to w^2. A full matrix has $w = n$ and we are back to n^3. For an exact count, remember that the bandwidth drops below w in the lower right corner (not enough space):

$$\textbf{Band}\quad \frac{w(w-1)(3n-2w+1)}{3} \qquad \textbf{Dense}\quad \frac{n(n-1)(n+1)}{3} = \frac{n^3 - n}{3}$$

On the right side, to find $\boldsymbol{x}$ from $\boldsymbol{b}$, the cost is about $2wn$ (compared to the usual n^2). *Main point: For a band matrix the operation counts are **proportional to** n.* This is extremely fast. A tridiagonal matrix of order 10,000 is very cheap, provided *we don't compute* A^{-1}. That inverse matrix has no zeros at all:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{has} \quad A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We are actually worse off knowing A^{-1} than knowing L and U. Multiplication by A^{-1} needs the full n^2 steps. Solving $Lc = \boldsymbol{b}$ and $Ux = c$ needs only $2wn$. A band structure is very common in practice, when the matrix reflects connections between near neighbors: $a_{13} = 0$ and $a_{14} = 0$ because 1 is not a neighbor of 3 and 4.

We close with counts for Gauss-Jordan and Gram-Schmidt-Householder:

A^{-1} costs n^3 multiply-subtract steps. ***QR costs $\frac{2}{3}n^3$ steps.***

Start with $AA^{-1} = I$. The jth column of A^{-1} solves $Ax_j = j$th column of I. The left side costs $\frac{1}{3}n^3$ as usual. (This is a one-time cost! L and U are not repeated.) The special saving for the jth column of I comes from its first $j-1$ zeros. No work is required on the right side until elimination reaches row j. The forward cost is $\frac{1}{2}(n-j)^2$ instead of $\frac{1}{2}n^2$. Summing over j, the total for forward elimination on the n right sides is $\frac{1}{6}n^3$. The final multiply-subtract count for A^{-1} is n^3 if we actually want the inverse:

$$\textbf{For}\,A^{-1} \qquad \frac{n^3}{3}\ (L \text{ and } U) + \frac{n^3}{6}\ (\text{forward}) \ + n\left(\frac{n^2}{2}\right) (\text{back substitutions}) \ = \boldsymbol{n^3}. \quad (1)$$

Orthogonalization *(A to Q)*: The key difference from elimination is that *each multiplier is decided by a dot product.* That takes n operations, where elimination just divides by the pivot. Then there are n "multiply-subtract" operations to remove from column k its projection along column $j<k$ (see Section 4.4). The combined cost is $2n$ where for elimination it is n. This factor 2 is the price of orthogonality. We are changing a dot product to zero where elimination changes an entry to zero.

Caution To judge a numerical algorithm, it is **not enough** to count the operations. Beyond "flop counting" is a study of stability (Householder wins) and the flow of data.

Reordering Sparse Matrices

In discussing band matrices, we assumed a constant width w. The rows were in an optimal order. But for most sparse matrices in real computations, the width of the band is *not constant* and there are many zeros inside the band. Those zeros can fill in as elimination proceeds—they are lost. We need to ***renumber the equations to reduce fill-in***, and thereby speed up elimination.

Generally speaking, we want to move zeros to early rows and columns. Later rows and columns are shorter anyway. The "approximate minimum degree" algorithm in sparse MATLAB is *greedy*—it chooses the row to eliminate without counting all the consequences. We may reach a nearly full matrix near the end, but the total operation count to reach LU is still much smaller. To renumber for an absolute minimum of nonzeros in L and U is an NP-hard problem, much too expensive, and **amd** is a good compromise.

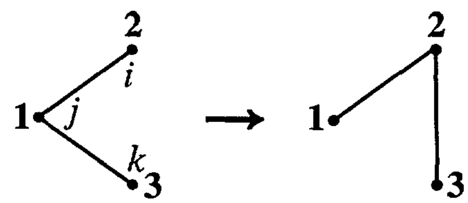
We only need the *positions* of the nonzeros, not their exact values. Think of the n rows as n nodes in a graph. *Node i is connected to node j if $a_{ij} \neq 0$.* Watch to see how elimination can create a new edge from i to k. This means that a zero is filled in, which we are trying to avoid:

When a_{kj} is eliminated, a multiple of the pivot row $j=1$ is subtracted from row $k=3$.

If a_{ji} was nonzero in row j, then a_{ki} becomes nonzero in the new row k. A new edge.

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & \mathbf{0} \\ -2 & \mathbf{0} & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$a_{32} = \mathbf{0} \qquad\qquad a_{32} = \mathbf{2}$$



$a_{32} = \mathbf{0}$ **before** $\qquad a_{32} \neq \mathbf{0}$ **after**

In this example, the 1's change the 0's into 2's. Those entries fill in.

The graph shows each step—look at the eliminationmovie on **math.mit.edu/18086.** The command **nnz**(L) counts the nonzero multipliers in the lower triangular L, **find** (L) will list them, and **spy**(L) shows them all.

The matrix in the movie is the 2D version of our $-1, 2, -1$ matrix. Instead of second differences along a line, the matrix has x and y differences on a plane grid. Each point is connected to its four nearest neighbors. But it is impossible to number all the points so that neighbors stay together. If we number by rows of the grid, there is a long wait to come around to the gridpoint above.

The goal of **colamd** and **symamd** is a better ordering (permutation P) that reduces fill-in for PA and PAP^{T}—*by choosing the pivot with the fewest nonzeros below it.*

Fast Orthogonalization

There are three ways to reach the important factorization $A = QR$. Gram-Schmidt works to find the orthonormal vectors in Q. Then R is upper triangular because of the order of Gram-Schmidt steps. Now we look at better methods (Householder and Givens), which use a product of specially simple Q's that we *know* are orthogonal.

Elimination gives $A = LU$, orthogonalization gives $A = QR$. We don't want a triangular L, we want an orthogonal Q. L is a product of E's, with 1's on the diagonal and the multiplier ℓ_{ij} below. *Q will be a product of orthogonal matrices.*

There are two simple orthogonal matrices to take the place of the E's. The ***reflection matrices*** $I - 2uu^{\mathrm{T}}$ are named after Householder. The ***plane rotation matrices*** are named after Givens. The simple matrix that rotates the xy plane by θ is Q_{21}:

Givens rotation
$$Q_{21} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Use Q_{21} the way you used E_{21}, to produce a zero in the $(2, 1)$ position. That determines the angle θ. Bill Hager gives this example in *Applied Numerical Linear Algebra*:

$$Q_{21}A = \begin{bmatrix} .6 & .8 & 0 \\ -.8 & .6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 90 & -153 & 114 \\ 120 & -79 & -223 \\ 200 & -40 & 395 \end{bmatrix} = \begin{bmatrix} 150 & -155 & -110 \\ 0 & 75 & -225 \\ 200 & -40 & 395 \end{bmatrix}.$$

The zero came from $-.8(90) + .6(120)$. No need to find θ, what we needed was $\cos\theta$:

$$\cos\theta = \frac{90}{\sqrt{90^2 + 120^2}} \qquad \text{and} \qquad \sin\theta = \frac{-120}{\sqrt{90^2 + 120^2}}. \tag{2}$$

Now we attack the $(3, 1)$ entry. The rotation will be in rows and columns 3 and 1. The numbers $\cos\theta$ and $\sin\theta$ are determined from 150 and 200, instead of 90 and 120.

$$Q_{31}Q_{21}A = \begin{bmatrix} .6 & 0 & .8 \\ 0 & 1 & 0 \\ -.8 & 0 & .6 \end{bmatrix} \begin{bmatrix} 150 & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 200 & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 250 & -125 & 250 \\ 0 & 75 & -225 \\ 0 & 100 & 325 \end{bmatrix}.$$

One more step to R. The $(3,2)$ entry has to go. The numbers $\cos\theta$ and $\sin\theta$ now come from 75 and 100. The rotation is now in rows and columns 2 and 3:

$$Q_{32}Q_{31}Q_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .6 & .8 \\ 0 & -.8 & .6 \end{bmatrix} \begin{bmatrix} 250 & -125 & \cdot \\ 0 & 75 & \cdot \\ 0 & 100 & \cdot \end{bmatrix} = \begin{bmatrix} 250 & -125 & 250 \\ 0 & 125 & 125 \\ 0 & 0 & 375 \end{bmatrix}.$$

We have reached the upper triangular R. What is Q? Move the plane rotations Q_{ij} to the other side to find $A = QR$—just as you moved the elimination matrices E_{ij} to the other side to find $A = LU$:

$$Q_{32}Q_{31}Q_{21}A = R \qquad \text{means} \qquad A = (Q_{21}^{-1}Q_{31}^{-1}Q_{32}^{-1})R = QR. \tag{3}$$

The inverse of each Q_{ij} is Q_{ij}^{T} (rotation through $-\theta$). The inverse of E_{ij} was not an orthogonal matrix! LU and QR are similar but not the same.

Householder reflections are faster because each one clears out a whole column below the diagonal. Watch how the first column $\boldsymbol{a}_1$ of A becomes column $\boldsymbol{r}_1$ of R:

Reflection by H_1
$H_1 = I - 2\boldsymbol{u}_1\boldsymbol{u}_1^{\mathrm{T}}$

$$H_1\,\boldsymbol{a}_1 = \begin{bmatrix} \|\boldsymbol{a}_1\| \\ 0 \\ \cdot \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -\|\boldsymbol{a}_1\| \\ 0 \\ \cdot \\ 0 \end{bmatrix} = \boldsymbol{r}_1. \tag{4}$$

The length was not changed, and $\boldsymbol{u}_1$ is in the direction of $\boldsymbol{a}_1 - \boldsymbol{r}_1$. We have $n-1$ entries in the unit vector $\boldsymbol{u}_1$ to get $n-1$ zeros in $\boldsymbol{r}_1$. (Rotations had one angle θ to get one zero.) When we reach column k, $n-k$ available choices in the unit vector $\boldsymbol{u}_k$ lead to $n-k$ zeros in $\boldsymbol{r}_k$. *We just store the $\boldsymbol{u}$'s and $\boldsymbol{r}$'s to know Q and R:*

$$\textbf{Inverse of } H_i \textbf{ is } H_i \quad (H_{n-1}\ldots H_1)A = R \;\text{ means }\; A = (H_1\ldots H_{n-1})R = QR. \tag{5}$$

This is how LAPACK improves on Gram-Schmidt. Q is *exactly* orthogonal.

Section 9.3 explains how $A = QR$ is used in the other big computation of linear algebra—***the eigenvalue problem***. The factors QR are reversed to give $A_1 = RQ$ which is $Q^{-1}AQ$. Since A_1 is similar to A, the eigenvalues are unchanged. Then A_1 is factored into Q_1R_1, and reversing the factors gives A_2. Amazingly, the entries below the diagonal get smaller in $A_1, A_2, A_3, \ldots$ and we can identify the eigenvalues. This is the "QR method" for $A\boldsymbol{x} = \lambda\boldsymbol{x}$, a big success of numerical linear algebra.

Problem Set 9.1

1 Find the two pivots with and without row exchange to maximize the pivot:

$$A = \begin{bmatrix} .001 & 0 \\ 1 & 1000 \end{bmatrix}.$$

With row exchanges to maximize pivots, why are no entries of L larger than 1? Find a 3 by 3 matrix A with all $|a_{ij}| \le 1$ and $|\ell_{ij}| \le 1$ but third pivot $= 4$.

2 Compute the exact inverse of the Hilbert matrix A by elimination. Then compute A^{-1} again by rounding all numbers to three figures:

$$\textbf{Ill-conditioned matrix} \qquad A = \mathsf{hilb(3)} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

3 For the same A compute $\boldsymbol{b} = A\boldsymbol{x}$ for $\boldsymbol{x} = (1, 1, 1)$ and $\boldsymbol{x} = (0, 6, -3.6)$. A small change $\Delta\boldsymbol{b}$ produces a large change $\Delta\boldsymbol{x}$.

4 Find the eigenvalues (by computer) of the 8 by 8 Hilbert matrix $a_{ij} = 1/(i + j - 1)$. In the equation $A\boldsymbol{x} = \boldsymbol{b}$ with $\|\boldsymbol{b}\| = 1$, how large can $\|\boldsymbol{x}\|$ be? If $\boldsymbol{b}$ has roundoff error less than 10^{-16}, how large an error can this cause in $\boldsymbol{x}$? See Section 9.2.

5 For back substitution with a band matrix (width w), show that the number of multiplications to solve $U\boldsymbol{x} = \boldsymbol{c}$ is approximately wn.

6 If you know L and U and Q and R, is it faster to solve $LU\boldsymbol{x} = \boldsymbol{b}$ or $QR\boldsymbol{x} = \boldsymbol{b}$?

7 Show that the number of multiplications to invert an upper triangular n by n matrix is about $\frac{1}{6}n^3$. Use back substitution on the columns of I, upward from 1's.

8 Choosing the largest available pivot in each column (partial pivoting), factor each A into $PA = LU$:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

9 Put 1's on the three central diagonals of a 4 by 4 tridiagonal matrix. Find the cofactors of the six zero entries. Those entries are nonzero in A^{-1}.

10 (Suggested by C. Van Loan.) Find the LU factorization and solve by elimination when $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$:

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon \\ 2 \end{bmatrix}.$$

The true $\boldsymbol{x}$ is $(1, 1)$. Make a table to show the error for each ε. Exchange the two equations and solve again—the errors should almost disappear.

11 (a) Choose $\sin\theta$ and $\cos\theta$ to triangularize A, and find R:

$$\textbf{Givens rotation} \qquad Q_{21}A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} = R.$$

(b) Choose $\sin\theta$ and $\cos\theta$ to make QAQ^{-1} triangular. What are the eigenvalues?

12 When A is multiplied by a plane rotation Q_{ij}, which n^2 entries of A are changed? When $Q_{ij}A$ is multiplied on the right by Q_{ij}^{-1}, which entries are changed now?

13 How many multiplications and how many additions are used to compute $Q_{ij}A$? Careful organization of the whole sequence of rotations gives $\frac{2}{3}n^3$ multiplications and $\frac{2}{3}n^3$ additions—the same as for QR by reflectors and twice as many as for LU.

Challenge Problems

14 (**Turning a robot hand**) The robot produces any 3 by 3 rotation A from plane rotations around the x, y, z axes. Then $Q_{32}Q_{31}Q_{21}A = R$, where A is orthogonal so R is I! The three robot turns are in $A = Q_{21}^{-1}Q_{31}^{-1}Q_{32}^{-1}$. The three angles are "Euler angles" and $\det Q = 1$ to avoid reflection. Start by choosing $\cos\theta$ and $\sin\theta$ so that

$$Q_{21}A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{is zero in the } (2,1) \text{ position.}$$

15 Create the 10 by 10 second difference matrix $K = \textbf{toeplitz}([2\ -1\ \textbf{zeros}(1,8)])$. Permute rows and columns randomly by $KK = K(\textbf{randperm}(10), \textbf{randperm}(10))$. Factor by $[L, U] = \textbf{lu}(K)$ and $[LL, UU] = \textbf{lu}(KK)$, and count nonzeros by $\textbf{nnz}(L)$ and $\textbf{nnz}(LL)$. In this case L is in perfect tridiagonal order, but not LL.

16 Another ordering for this matrix K colors the meshpoints alternately red and black. This permutation P changes the normal $1, \ldots, 10$ to $1, 3, 5, 7, 9, 2, 4, 6, 8, 10$:

$$\textbf{Red-black ordering} \qquad PKP^{\mathrm{T}} = \begin{bmatrix} 2I & D \\ D^{\mathrm{T}} & 2I \end{bmatrix}. \qquad \textit{Find the matrix } D.$$

So many interesting experiments are possible. If you send good ideas they can go on the linear algebra website math.mit.edu/linearalgebra. I also recommend learning the command $B = \textbf{sparse}(A)$, after which $\textbf{find}(B)$ will list the nonzero entries and $\boldsymbol{\ell}\textbf{u}(B)$ will factor B using that sparse format for L and U. Only the nonzeros are computed, where ordinary (dense) MATLAB computes all the zeros too.

17 Jeff Stuart has created a student activity that brilliantly demonstrates ill-conditioning:

$$\begin{bmatrix} 1 & 1.0001 \\ 1 & 1.0000 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.0001 + e \\ 3.0000 + E \end{bmatrix} \quad \begin{matrix} \textbf{With errors} \\ \boldsymbol{e} \textbf{ and } \boldsymbol{E} \end{matrix} \quad \begin{matrix} x = 2 - 10000(e - E) \\ y = 1 + 10000(e - E) \end{matrix}$$

The algebra shows how errors $\boldsymbol{e}$ and $\boldsymbol{E}$ are amplified by 10000 unless $\boldsymbol{e} = \boldsymbol{E}$.

As always, the solution of a 2 by 2 system is the meeting point of two lines. The neat idea is to replace mathematical lines by ***long sticks held by students.*** The sticks for these two equations are almost parallel, and A is almost singular. Perpendicular sticks come from well-conditioned equations.

In Stuart's *Shake a Stick* activity, the students plot where the sticks cross (after multiple shakes). See ***www.plu.edu/~stuartjl*** for the wild movements of that crossing point (x, y), when the sticks are nearly parallel.

9.2 Norms and Condition Numbers

How do we measure the size of a matrix? For a vector, the length is $\|x\|$. For a matrix, ***the norm is*** $\|A\|$. This word "*norm*" is sometimes used for vectors, instead of length. It is always used for matrices, and there are many ways to measure $\|A\|$. We look at the requirements on all "matrix norms" and then choose one.

Frobenius squared all the $|a_{ij}|^2$ and added; his norm $\|A\|_F$ is the square root. This treats A like a long vector with n^2 components: sometimes useful, but not the choice here.

I prefer to start with a vector norm. The triangle inequality says that $\|x+y\|$ is not greater than $\|x\| + \|y\|$. The length of $2x$ or $-2x$ is doubled to $2\|x\|$. The same rules will apply to matrix norms:

$$\|A+B\| \le \|A\| + \|B\| \quad \text{and} \quad \|cA\| = |c|\,\|A\|. \tag{1}$$

The second requirements for a matrix norm are new, because matrices multiply. The norm $\|A\|$ controls the growth from x to Ax, and from B to AB:

$$\textbf{Growth factor } \|A\| \qquad \|Ax\| \le \|A\|\,\|x\| \quad \text{and} \quad \|AB\| \le \|A\|\,\|B\|. \tag{2}$$

This leads to a natural way to define $\|A\|$, the norm of a matrix:

$$\textit{The norm of } A \textit{ is the largest ratio } \|Ax\|/\|x\|: \quad \|A\| = \max_{x \ne 0} \frac{\|Ax\|}{\|x\|}. \tag{3}$$

$\|Ax\|/\|x\|$ is never larger than $\|A\|$ (its maximum). This says that $\|Ax\| \le \|A\|\,\|x\|$.

Example 1 If A is the identity matrix I, the ratios are $\|x\|/\|x\|$. Therefore $\|I\| = 1$. If A is an orthogonal matrix Q, lengths are again preserved: $\|Qx\| = \|x\|$. The ratios still give $\|Q\| = 1$. An orthogonal Q is good to compute with: errors don't grow.

Example 2 The norm of a diagonal matrix is its largest entry (using absolute values):

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{has norm} \quad \|A\| = 3. \quad \text{The eigenvector} \quad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{has} \quad Ax = 3x.$$

The eigenvalue is 3. For this A (but not all A), the largest eigenvalue equals the norm.

For a positive definite symmetric matrix the norm is $\|A\| = \lambda_{\max}(A)$.

Choose x to be the eigenvector with maximum eigenvalue. Then $\|Ax\|/\|x\|$ equals $\lambda_{\max}$. The point is that no other x can make the ratio larger. The matrix is $A = Q\Lambda Q^T$, and the orthogonal matrices Q and Q^T leave lengths unchanged. So the ratio to maximize is really $\|\Lambda x\|/\|x\|$. The norm is the largest eigenvalue in the diagonal Λ.

Symmetric matrices Suppose A is symmetric but not positive definite. $A = Q\Lambda Q^T$ is still true. Then the norm is the largest of $|\lambda_1|$, $|\lambda_2|$, ..., $|\lambda_n|$. We take absolute values, because the norm is only concerned with length. For an eigenvector $\|Ax\| = \|\lambda x\| = |\lambda|$ times $\|x\|$. The x that gives the maximum ratio is the eigenvector for the maximum $|\lambda|$.

Unsymmetric matrices If A is not symmetric, its eigenvalues may not measure its true size. *The norm can be larger than any eigenvalue.* A very unsymmetric example has $\lambda_1 = \lambda_2 = 0$ but its norm is not zero:

$$\|A\| > \lambda_{\max} \quad A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{has norm} \quad \|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = 2.$$

The vector $x = (0, 1)$ gives $Ax = (2, 0)$. The ratio of lengths is $2/1$. This is the maximum ratio $\|A\|$, even though x is not an eigenvector.

It is the ***symmetric matrix*** $A^{\mathrm{T}}A$, not the unsymmetric A, that has eigenvector $x = (0, 1)$. The norm is really decided by ***the largest eigenvalue of*** $A^{\mathrm{T}}A$:

The norm of A (symmetric or not) ***is the square root of*** $\lambda_{\max}(A^{\mathrm{T}}A)$***:***

$$\|A\|^2 = \max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \max_{x \neq 0} \frac{x^{\mathrm{T}}A^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} = \lambda_{\max}(A^{\mathrm{T}}A). \tag{4}$$

The unsymmetric example with $\lambda_{\max}(A) = 0$ has $\lambda_{\max}(A^{\mathrm{T}}A) = 4$:

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \text{ leads to } A^{\mathrm{T}}A = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \text{ with } \lambda_{\max} = 4. \text{ So the norm is } \|A\| = \sqrt{4}.$$

For any A Choose x to be the eigenvector of $A^{\mathrm{T}}A$ with largest eigenvalue $\lambda_{\max}$. The ratio in equation (4) is $x^{\mathrm{T}}A^{\mathrm{T}}Ax = x^{\mathrm{T}}(\lambda_{\max})x$ divided by $x^{\mathrm{T}}x$. This is $\lambda_{\max}$.

No x can give a larger ratio. The symmetric matrix $A^{\mathrm{T}}A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and orthonormal eigenvectors $q_1, q_2, \ldots, q_n$. Every x is a combination of those vectors. Try this combination in the ratio and remember that $q_i^{\mathrm{T}}q_j = 0$:

$$\frac{x^{\mathrm{T}}A^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} = \frac{(c_1q_1 + \cdots + c_nq_n)^{\mathrm{T}}(c_1\lambda_1q_1 + \cdots + c_n\lambda_nq_n)}{(c_1q_1 + \cdots + c_nq_n)^{\mathrm{T}}(c_1q_1 + \cdots + c_nq_n)} = \frac{c_1^2\lambda_1 + \cdots + c_n^2\lambda_n}{c_1^2 + \cdots + c_n^2}.$$

The maximum ratio $\lambda_{\max}$ is when all c's are zero, except the one that multiplies $\lambda_{\max}$.

Note 1 The ratio in equation (4) is the ***Rayleigh quotient*** for the symmetric matrix $A^{\mathrm{T}}A$. Its maximum is the largest eigenvalue $\lambda_{\max}(A^{\mathrm{T}}A)$. The minimum ratio is $\lambda_{\min}(A^{\mathrm{T}}A)$. If you substitute any vector x into the Rayleigh quotient $x^{\mathrm{T}}A^{\mathrm{T}}Ax/x^{\mathrm{T}}x$, you are guaranteed to get a number between $\lambda_{\min}(A^{\mathrm{T}}A)$ and $\lambda_{\max}(A^{\mathrm{T}}A)$.

Note 2 The norm $\|A\|$ equals the ***largest singular value*** $\sigma_{\max}$ of A. The singular values $\sigma_1, \ldots, \sigma_r$ are the square roots of the positive eigenvalues of $A^{\mathrm{T}}A$. So certainly $\sigma_{\max} = (\lambda_{\max})^{1/2}$. Since U and V are orthogonal in $A = U\Sigma V^{\mathrm{T}}$, the norm is $\|A\| = \sigma_{\max}$.

The Condition Number of A

Section 9.1 showed that roundoff error can be serious. Some systems are sensitive, others are not so sensitive. The sensitivity to error is measured by the ***condition number***. This is the first chapter in the book which intentionally introduces errors. We want to estimate how much they change $\boldsymbol{x}$.

The original equation is $A\boldsymbol{x} = \boldsymbol{b}$. Suppose the right side is changed to $\boldsymbol{b} + \Delta\boldsymbol{b}$ because of roundoff or measurement error. The solution is then changed to $\boldsymbol{x} + \Delta\boldsymbol{x}$. Our goal is to estimate the change $\Delta\boldsymbol{x}$ in the solution from the change $\Delta\boldsymbol{b}$ in the equation. Subtraction gives the *error equation* $A(\Delta\boldsymbol{x}) = \Delta\boldsymbol{b}$:

$$\text{Subtract } A\boldsymbol{x} = \boldsymbol{b} \text{ from } A(\boldsymbol{x} + \Delta\boldsymbol{x}) = \boldsymbol{b} + \Delta\boldsymbol{b} \quad \text{to find} \quad A(\Delta\boldsymbol{x}) = \Delta\boldsymbol{b}. \tag{5}$$

The error is $\Delta\boldsymbol{x} = A^{-1}\Delta\boldsymbol{b}$. It is large when A^{-1} is large (then A is nearly singular). The error $\Delta\boldsymbol{x}$ is especially large when $\Delta\boldsymbol{b}$ points in the worst direction—which is amplified most by A^{-1}. ***The worst error has*** $\|\Delta\boldsymbol{x}\| = \|A^{-1}\|\,\|\Delta\boldsymbol{b}\|$.

This error bound $\|A^{-1}\|$ has one serious drawback. If we multiply A by 1000, then A^{-1} is divided by 1000. The matrix looks a thousand times better. But a simple rescaling cannot change the reality of the problem. It is true that $\Delta\boldsymbol{x}$ will be divided by 1000, but so will the exact solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. The ***relative error*** $\|\Delta\boldsymbol{x}\|/\|\boldsymbol{x}\|$ will stay the same. It is this relative change in $\boldsymbol{x}$ that should be compared to the relative change in $\boldsymbol{b}$.

Comparing relative errors will now lead to the "condition number" $c = \|A\|\,\|A^{-1}\|$. Multiplying A by 1000 does not change this number, because A^{-1} is divided by 1000 and the condition number c stays the same. It measures the sensitivity of $A\boldsymbol{x} = \boldsymbol{b}$.

The solution error is less than $c = \|A\|\,\|A^{-1}\|$ ***times the problem error:***

$$\textbf{Condition number } c \qquad \frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \le c\,\frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|}. \tag{6}$$

If the problem error is ΔA (error in A instead of $\boldsymbol{b}$), ***still*** c ***controls*** $\Delta\boldsymbol{x}$***:***

$$\textbf{Error } \Delta A \textbf{ in } A \qquad \frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x} + \Delta\boldsymbol{x}\|} \le c\,\frac{\|\Delta A\|}{\|A\|}. \tag{7}$$

Proof The original equation is $\boldsymbol{b} = A\boldsymbol{x}$. The error equation (5) is $\Delta\boldsymbol{x} = A^{-1}\Delta\boldsymbol{b}$. Apply the key property $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$ of matrix norms:

$$\|\boldsymbol{b}\| \le \|A\|\,\|\boldsymbol{x}\| \qquad \text{and} \qquad \|\Delta\boldsymbol{x}\| \le \|A^{-1}\|\,\|\Delta\boldsymbol{b}\|.$$

Multiply the left sides to get $\|\boldsymbol{b}\|\,\|\Delta\boldsymbol{x}\|$, and multiply the right sides to get $c\|\boldsymbol{x}\|\,\|\Delta\boldsymbol{b}\|$. Divide both sides by $\|\boldsymbol{b}\|\,\|\boldsymbol{x}\|$. The left side is now the relative error $\|\Delta\boldsymbol{x}\|/\|\boldsymbol{x}\|$. The right side is now the upper bound in equation (6).

The same condition number $c = \|A\|\,\|A^{-1}\|$ appears when the error is in the matrix. We have ΔA instead of $\Delta \boldsymbol{b}$ in the error equation:

$$\text{Subtract } A\boldsymbol{x} = \boldsymbol{b} \text{ from } (A + \Delta A)(\boldsymbol{x} + \Delta \boldsymbol{x}) = \boldsymbol{b} \text{ to find } A(\Delta \boldsymbol{x}) = -(\Delta A)(\boldsymbol{x} + \Delta \boldsymbol{x}).$$

Multiply the last equation by A^{-1} and take norms to reach equation (7):

$$\|\Delta \boldsymbol{x}\| \le \|A^{-1}\|\,\|\Delta A\|\,\|\boldsymbol{x} + \Delta \boldsymbol{x}\| \quad \text{or} \quad \frac{\|\Delta \boldsymbol{x}\|}{\|\boldsymbol{x} + \Delta \boldsymbol{x}\|} \le \|A\|\,\|A^{-1}\|\frac{\|\Delta A\|}{\|A\|}.$$

Conclusion Errors enter in two ways. They begin with an error ΔA or $\Delta \boldsymbol{b}$—a wrong matrix or a wrong $\boldsymbol{b}$. This problem error is amplified (a lot or a little) into the solution error $\Delta \boldsymbol{x}$. That error is bounded, relative to $\boldsymbol{x}$ itself, by the condition number c.

The error $\Delta \boldsymbol{b}$ depends on computer roundoff and on the original measurements of $\boldsymbol{b}$. The error ΔA also depends on the elimination steps. Small pivots tend to produce large errors in L and U. Then $L + \Delta L$ times $U + \Delta U$ equals $A + \Delta A$. When ΔA or the condition number is very large, the error $\Delta \boldsymbol{x}$ can be unacceptable.

Example 3 When A is symmetric, $c = \|A\|\,\|A^{-1}\|$ comes from the eigenvalues:

$$A = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \text{ has norm } 6. \qquad A^{-1} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \text{ has norm } \tfrac{1}{2}.$$

This A is symmetric positive definite. Its norm is $\lambda_{\max} = 6$. The norm of A^{-1} is $1/\lambda_{\min} = \frac{1}{2}$. Multiplying norms gives the *condition number* $\|A\|\,\|A^{-1}\| = \lambda_{\max}/\lambda_{\min}$:

$$\textbf{Condition number for positive definite } A \qquad c = \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{6}{2} = 3.$$

Example 4 Keep the same A, with eigenvalues 6 and 2. To make $\boldsymbol{x}$ small, choose $\boldsymbol{b}$ along the first eigenvector $(1, 0)$. To make $\Delta \boldsymbol{x}$ large, choose $\Delta \boldsymbol{b}$ along the second eigenvector $(0, 1)$. Then $\boldsymbol{x} = \frac{1}{6}\boldsymbol{b}$ and $\Delta \boldsymbol{x} = \frac{1}{2}\Delta \boldsymbol{b}$. The ratio $\|\Delta \boldsymbol{x}\|/\|\boldsymbol{x}\|$ is exactly $c = 3$ times the ratio $\|\Delta \boldsymbol{b}\|/\|\boldsymbol{b}\|$.

This shows that the worst error allowed by the condition number $\|A\|\,\|A^{-1}\|$ can actually happen. Here is a useful rule of thumb, experimentally verified for Gaussian elimination: *The computer can lose* $\log c$ *decimal places to roundoff error.*

Problem Set 9.2

1 Find the norms $\|A\| = \lambda_{\max}$ and condition numbers $c = \lambda_{\max}/\lambda_{\min}$ of these positive definite matrices:

$$\begin{bmatrix} .5 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

2 Find the norms and condition numbers from the square roots of $\lambda_{\max}(A^{\mathrm{T}}A)$ and $\lambda_{\min}(A^{\mathrm{T}}A)$. Without positive definiteness in A, we go to $A^{\mathrm{T}}A$!

$$\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

3 Explain these two inequalities from the definitions (3) of $\|A\|$ and $\|B\|$:

$$\|ABx\| \le \|A\|\,\|Bx\| \le \|A\|\,\|B\|\,\|x\|.$$

From the ratio of $\|ABx\|$ to $\|x\|$, deduce that $\|AB\| \le \|A\|\,\|B\|$. This is the key to using matrix norms. The norm of A^n is never larger than $\|A\|^n$.

4 Use $\|AA^{-1}\| \le \|A\|\,\|A^{-1}\|$ to prove that the condition number is at least 1.

5 Why is I the only symmetric positive definite matrix that has $\lambda_{\max} = \lambda_{\min} = 1$? Then the only other matrices with $\|A\| = 1$ and $\|A^{-1}\| = 1$ must have $A^{\mathrm{T}}A = I$. Those are _____ matrices: perfectly conditioned.

6 Orthogonal matrices have norm $\|Q\| = 1$. If $A = QR$ show that $\|A\| \le \|R\|$ and also $\|R\| \le \|A\|$. Then $\|A\| = \|Q\|\,\|R\|$. Find an example of $A = LU$ with $\|A\| < \|L\|\|U\|$.

7 (a) Which famous inequality gives $\|(A+B)x\| \le \|Ax\| + \|Bx\|$ for every x?

(b) Why does the definition (3) of matrix norms lead to $\|A+B\| \le \|A\| + \|B\|$?

8 Show that if λ is any eigenvalue of A, then $|\lambda| \le \|A\|$. Start from $Ax = \lambda x$.

9 The "*spectral radius*" $\rho(A) = |\lambda_{\max}|$ is the largest absolute value of the eigenvalues. Show with 2 by 2 examples that $\rho(A+B) \le \rho(A) + \rho(B)$ and $\rho(AB) \le \rho(A)\rho(B)$ can both be *false*. The spectral radius is not acceptable as a norm.

10 (a) Explain why A and A^{-1} have the same condition number.

(b) Explain why A and A^{T} have the same norm, based on $\lambda(A^{\mathrm{T}}A)$ and $\lambda(AA^{\mathrm{T}})$.

11 Estimate the condition number of the ill-conditioned matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$.

12 Why is the determinant of A no good as a norm? Why is it no good as a condition number?

13 (Suggested by C. Moler and C. Van Loan.) Compute $b - Ay$ and $b - Az$ when

$$b = \begin{bmatrix} .217 \\ .254 \end{bmatrix} \quad A = \begin{bmatrix} .780 & .563 \\ .913 & .659 \end{bmatrix} \quad y = \begin{bmatrix} .341 \\ -.087 \end{bmatrix} \quad z = \begin{bmatrix} .999 \\ -1.0 \end{bmatrix}.$$

Is y closer than z to solving $Ax = b$? Answer in two ways: Compare the *residual* $b - Ay$ to $b - Az$. Then compare y and z to the true $x = (1, -1)$. Both answers can be right. Sometimes we want a small residual, sometimes a small Δx.

14 (a) Compute the determinant of A in Problem 13. Compute A^{-1}.

(b) If possible compute $\|A\|$ and $\|A^{-1}\|$ and show that $c > 10^6$.

Problems 15–19 are about vector norms other than the usual $\|x\| = \sqrt{x \cdot x}$.

15 The "ℓ^1 norm" and the "ℓ^∞ norm" of $x = (x_1, \ldots, x_n)$ are

$$\|x\|_1 = |x_1| + \cdots + |x_n| \quad \text{and} \quad \|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

Compute the norms $\|x\|$ and $\|x\|_1$ and $\|x\|_\infty$ of these two vectors in $\mathbf{R}^5$:

$$x = (1, 1, 1, 1, 1) \qquad x = (.1, .7, .3, .4, .5).$$

16 Prove that $\|x\|_\infty \le \|x\| \le \|x\|_1$. Show from the Schwarz inequality that the ratios $\|x\|/\|x\|_\infty$ and $\|x\|_1/\|x\|$ are never larger than $\sqrt{n}$. Which vector $(x_1, \ldots, x_n)$ gives ratios equal to $\sqrt{n}$?

17 All vector norms must satisfy the *triangle inequality*. Prove that

$$\|x + y\|_\infty \le \|x\|_\infty + \|y\|_\infty \qquad \text{and} \qquad \|x + y\|_1 \le \|x\|_1 + \|y\|_1.$$

18 Vector norms must also satisfy $\|cx\| = |c| \, \|x\|$. The norm must be positive except when $x = \mathbf{0}$. Which of these are norms for vectors (x_1, x_2) in $\mathbf{R}^2$?

$$\|x\|_A = |x_1| + 2|x_2| \qquad \|x\|_B = \min\,(|x_1|, |x_2|)$$

$$\|x\|_C = \|x\| + \|x\|_\infty \qquad \|x\|_D = \|Ax\| \quad \text{(this answer depends on } A\text{)}.$$

Challenge Problems

19 Show that $x^{\mathrm{T}}y \le \|x\|_1 \, \|y\|_\infty$ by choosing components $y_i = \pm 1$ to make $x^{\mathrm{T}}y$ as large as possible.

20 The eigenvalues of the $-1, 2, -1$ difference matrix K are $\lambda = 2 - 2\cos\,(j\pi/n{+}1)$. Estimate $\lambda_{\min}$ and $\lambda_{\max}$ and $c = \mathbf{cond}(K) = \lambda_{\max}/\lambda_{\min}$ as n increases: $c \approx Cn^2$ with what constant C?

Test this estimate with $\mathbf{eig}(K)$ and $\mathbf{cond}(K)$ for $n = 10, 100, 1000$.

21 For unsymmetric matrices, the spectral radius $\rho = \max|\lambda_i|$ is not a norm (Problem 9). But still $\|A^n\|$ grows or decays like ρ^n for large n. Compare those numbers for $A = [1 \quad 1; \quad 0 \quad 1.1]$ using the command **norm**.

In particular $A^n \to 0$ when $\rho < 1$. This is the key to Section 9.3 with $A = S^{-1}T$.

9.3 Iterative Methods and Preconditioners

Up to now, our approach to $Ax = b$ has been direct. We accepted A as it came. We attacked it by elimination with row exchanges. This section is about ***iterative methods***, which replace A by a simpler matrix S. The difference $T = S - A$ is moved over to the right side of the equation. The problem becomes easier to solve, with S instead of A. But there is a price—*the simpler system has to be solved over and over.*

An iterative method is easy to invent. Just split A (carefully) into $S - T$.

$$\textbf{Rewrite } Ax = b \qquad Sx = Tx + b. \tag{1}$$

The novelty is to solve (1) iteratively. Each guess x_k leads to the next x_{k+1}:

$$\textbf{Pure iteration} \qquad Sx_{k+1} = Tx_k + b. \tag{2}$$

Start with any x_0. Then solve $Sx_1 = Tx_0 + b$. Continue to the second iteration $Sx_2 = Tx_1 + b$. A hundred iterations are very common—often more. Stop when (and if!) the new vector x_{k+1} is sufficiently close to x_k—or when the **residual** $r_k = b - Ax_k$ is near zero. We choose the stopping test. Our hope is to get near the true solution, more quickly than by elimination. When the sequence x_k converges, its limit $x = x_\infty$ does solve equation (1). The proof is to let $k \to \infty$ in equation (2).

The two goals of the splitting $A = S - T$ are ***speed per step*** and ***fast convergence***. The speed of each step depends on S and the speed of convergence depends on $S^{-1}T$:

1 Equation (2) should be easy to solve for x_{k+1}. The "***preconditioner***" S could be the diagonal or triangular part of A. A fast way uses $S = L_0U_0$, where those factors have many zeros compared to the exact $A = LU$. This is "*incomplete LU*".

2 The difference $x - x_k$ (this is the error e_k) should go quickly to zero. Subtracting equation (2) from (1) cancels b, and it leaves the ***equation for the error e_k***:

$$\textbf{Error equation} \qquad Se_{k+1} = Te_k \ \text{ which means } \ e_{k+1} = S^{-1}Te_k. \tag{3}$$

At every step the error is multiplied by $S^{-1}T$. If $S^{-1}T$ is small, its powers go quickly to zero. But what is "small"?

The extreme splitting is $S = A$ and $T = 0$. Then the first step of the iteration is the original $Ax = b$. Convergence is perfect and $S^{-1}T$ is zero. But the cost of that step is what we wanted to avoid. The choice of S is a battle between speed per step (a simple S) and fast convergence (S close to A). Here are some popular choices:

J $S =$ diagonal part of A (the iteration is called *Jacobi's method*)

GS $S =$ lower triangular part of A including the diagonal (*Gauss-Seidel method*)

SOR $S =$ combination of Jacobi and Gauss-Seidel (*successive overrelaxation*)

ILU $S =$ approximate L times approximate U (*incomplete LU method*).

Our first question is pure linear algebra: ***When do the x_k's converge to x?*** The answer uncovers the number $|\lambda|_{\max}$ that controls convergence. In examples of **J** and **GS** and **SOR**, we will compute this "*spectral radius*" $|\lambda|_{\max}$. It is the largest eigenvalue of the ***iteration matrix*** $B = S^{-1}T$.

The Spectral Radius $\rho(B)$ Controls Convergence

Equation (3) is $e_{k+1} = S^{-1}Te_k$. Every iteration step multiplies the error by the same matrix $B = S^{-1}T$. The error after k steps is $e_k = B^k e_0$. ***The error approaches zero if the powers of $B = S^{-1}T$ approach zero.*** It is beautiful to see how the eigenvalues of B—the largest eigenvalue in particular—control the matrix powers B^k.

The powers B^k approach zero if and only if every eigenvalue of B has $|\lambda| < 1$. ***The rate of convergence is controlled by the spectral radius of B:*** $\rho = \max|\lambda(B)|$.

The test for convergence is $|\lambda|_{\max} < 1$. Real eigenvalues must lie between -1 and 1. Complex eigenvalues $\lambda = a + ib$ must have $|\lambda|^2 = a^2 + b^2 < 1$. (Chapter 10 will discuss complex numbers.) The spectral radius "*rho*" is the largest distance from 0 to the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $B = S^{-1}T$. This is $\rho = |\lambda|_{\max}$.

To see why $|\lambda|_{\max} < 1$ is necessary, suppose the starting error e_0 happens to be an eigenvector of B. After one step the error is $Be_0 = \lambda e_0$. After k steps the error is $B^k e_0 = \lambda^k e_0$. If we start with an eigenvector, we continue with that eigenvector—and it grows or decays with the powers λ^k. *This factor λ^k goes to zero when* $|\lambda| < 1$. Since this condition is required of every eigenvalue, we need $\rho = |\lambda|_{\max} < 1$.

To see why $|\lambda|_{\max} < 1$ is sufficient for the error to approach zero, suppose e_0 is a combination of eigenvectors:

$$e_0 = c_1 x_1 + \cdots + c_n x_n \quad \text{leads to} \quad e_k = c_1(\lambda_1)^k x_1 + \cdots + c_n(\lambda_n)^k x_n. \tag{4}$$

This is the point of eigenvectors! They grow independently, each one controlled by its eigenvalue. When we multiply by B, the eigenvector x_i is multiplied by λ_i. If all $|\lambda_i| < 1$ then equation (4) ensures that e_k goes to zero.

Example 1 $\quad B = \begin{bmatrix} .6 & .5 \\ .6 & .5 \end{bmatrix}$ has $\lambda_{\max} = 1.1 \qquad B' = \begin{bmatrix} .6 & 1.1 \\ 0 & .5 \end{bmatrix}$ has $\lambda_{\max} = .6$

B^2 is 1.1 times B. Then B^3 is $(1.1)^2$ times B. The powers of B will blow up. Contrast with the powers of B'. The matrix $(B')^k$ has $(.6)^k$ and $(.5)^k$ on its diagonal. The off-diagonal entries also involve $\rho^k = (.6)^k$, which sets the speed of convergence.

Note There is a technical difficulty when B does not have n independent eigenvectors. (To produce this effect in B', change .5 to .6.) The starting error e_0 may not be a combination of eigenvectors—there are too few for a basis. Then diagonalization is impossible and equation (4) is not correct. We turn to the *Jordan form* when eigenvectors are missing:

$$\textbf{Jordan form } J \qquad B = MJM^{-1} \quad \text{and} \quad B^k = MJ^kM^{-1}. \tag{5}$$

Section 6.6 shows how J and J^k are made of "blocks" with one repeated eigenvalue:

$$\text{The powers of a 2 by 2 block in } J \text{ are} \quad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}.$$

If $|\lambda| < 1$ then these powers approach zero. The extra factor k from a double eigenvalue is overwhelmed by the decreasing factor λ^{k-1}. This applies to all Jordan blocks. A block of size $S + 1$ has $k^S\, \lambda^{k-S}$ in J^k, which also approaches zero when $|\lambda| < 1$.

Diagonalizable or not: Convergence $B^k \to 0$ and its speed depend on $\rho = |\lambda|_{\max} < 1$.

Jacobi versus Gauss-Seidel

We now solve a specific 2 by 2 problem. Watch for that number $|\lambda|_{\max}$.

$$Ax = b \qquad \begin{aligned} 2u - v &= 4 \\ -u + 2v &= -2 \end{aligned} \qquad \text{has the solution} \quad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{6}$$

The first splitting is ***Jacobi's method.*** Keep the *diagonal* of A on the left side (this is S). Move the off-diagonal part of A to the right side (this is T). Then iterate:

$$\textbf{Jacobi iteration} \qquad Sx_{k+1} = Tx_k + b \qquad \begin{aligned} 2u_{k+1} &= v_k + 4 \\ 2v_{k+1} &= u_k - 2. \end{aligned}$$

Start from $u_0 = v_0 = 0$. The first step finds $u_1 = 2$ and $v_1 = -1$. Keep going:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 3/2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1/4 \end{bmatrix} \quad \begin{bmatrix} 15/8 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1/16 \end{bmatrix} \quad \text{approaches} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

This shows convergence. At steps 1, 3, 5 the second component is $-1, -1/4, -1/16$. The error is multiplied by $\frac{1}{4}$ every two steps. The components $0, 3/2, 15/8$ have errors $2, \frac{1}{2}, \frac{1}{8}$. Those also drop by 4 in each two steps. *The error equation is $Se_{k+1} = Te_k$:*

$$\textbf{Error equation} \qquad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} e_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_k \quad \text{or} \quad e_{k+1} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} e_k. \tag{7}$$

That last matrix $S^{-1}T$ has eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$. So its spectral radius is $\rho(B) = \frac{1}{2}$:

$$B = S^{-1}T = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad \text{has } |\lambda|_{\max} = \tfrac{1}{2} \quad \text{and} \quad \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

Two steps multiply the error by $\frac{1}{4}$ exactly, in this special example. The important message is this: Jacobi's method works well when the main diagonal of A is large compared to the off-diagonal part. The diagonal part is S, the rest is $-T$. We want the diagonal to dominate and $S^{-1}T$ to be small.

The eigenvalue $\lambda = \frac{1}{2}$ is unusually small. Ten iterations reduce the error by $2^{10} = 1024$. More typical and more expensive is $|\lambda|_{\max} = .99$ or $.999$.

The ***Gauss-Seidel method*** keeps the whole lower triangular part of A as S:

$$\textbf{Gauss-Seidel} \qquad \begin{aligned} 2u_{k+1} \quad\quad\quad &= v_k + 4 \\ -u_{k+1} + 2v_{k+1} &= \quad\; -2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} u_{k+1} &= \tfrac{1}{2}v_k \quad + 2 \\ v_{k+1} &= \tfrac{1}{2}u_{k+1} - 1. \end{aligned} \tag{8}$$

Notice the change. The new u_{k+1} from the first equation is used *immediately* in the second equation. With Jacobi, we saved the old u_k until the whole step was complete. With Gauss-Seidel, the new values enter right away and the old u_k is destroyed. This cuts the storage in half. It also speeds up the iteration (usually). And it costs no more than the Jacobi method.

Starting from $(0, 0)$, the exact answer $(2, 0)$ is reached in one step. That is an accident I did not expect. Test the iteration from another start $u_0 = 0$ and $v_0 = -1$:

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 3/2 \\ -1/4 \end{bmatrix} \quad \begin{bmatrix} 15/8 \\ -1/16 \end{bmatrix} \quad \begin{bmatrix} 63/32 \\ -1/64 \end{bmatrix} \quad \text{approaches} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The errors in the first component are 2, 1/2, 1/8, 1/32. The errors in the second component are $-1, -1/4, -1/16, -1/32$. We divide by 4 in *one step* not two steps. ***Gauss-Seidel is twice as fast as Jacobi.*** We have $\rho_{\text{GS}} = (\rho_{\text{J}})^2$.

This double speed is true for every positive definite tridiagonal matrix. Anything is possible when A is strongly nonsymmetric—Jacobi is sometimes better, and both methods might fail. Our example is small and A is positive definite tridiagonal:

$$S = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad S^{-1}T = \begin{bmatrix} \mathbf{0} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}.$$

The Gauss-Seidel eigenvalues are 0 and $\frac{1}{4}$. Compare with $\frac{1}{2}$ and $-\frac{1}{2}$ for Jacobi.

With a small push we can explain the ***successive overrelaxation method*** (**SOR**). The new idea is to introduce a parameter ω (omega) into the iteration. Then choose this number ω to make the spectral radius of $S^{-1}T$ as small as possible.

Rewrite $Ax = b$ as $\omega Ax = \omega b$. The matrix S in **SOR** has the diagonal of the original A, but below the diagonal we use ωA. On the right side T is $S - \omega A$:

$$\textbf{SOR} \qquad \begin{aligned} 2u_{k+1} \quad\quad\quad &= (2 - 2\omega)u_k + \quad \omega v_k + 4\omega \\ -\omega u_{k+1} + 2v_{k+1} &= \quad\quad\quad\quad (2 - 2\omega)v_k - 2\omega. \end{aligned} \tag{9}$$

This looks more complicated to us, but the computer goes as fast as ever. Each new u_{k+1} from the first equation is used immediately to find v_{k+1} in the second equation. This is like Gauss-Seidel, with an adjustable number ω. The key matrix is $S^{-1}T$:

$$\textbf{SOR iteration matrix} \qquad S^{-1}T = \begin{bmatrix} 1 - \omega & \frac{1}{2}\omega \\ \frac{1}{2}\omega(1 - \omega) & 1 - \omega + \frac{1}{4}\omega^2 \end{bmatrix}. \tag{10}$$

The determinant is $(1 - \omega)^2$. At the best ω, both eigenvalues turn out to equal $7 - 4\sqrt{3}$, which is close to $(\frac{1}{4})^2$. Therefore **SOR** is twice as fast as Gauss-Seidel in this example. In other examples **SOR** can converge ten or a hundred times as fast.

I will put on record the most valuable test matrix of order n. It is our favorite -1, 2, -1 tridiagonal matrix K. The diagonal is $2I$. Below and above are -1's. Our example had $n = 2$, which leads to $\cos\frac{\pi}{3} = \frac{1}{2}$ as the Jacobi eigenvalue found above. Notice especially that this eigenvalue is squared for Gauss-Seidel:

The splittings of the $-1, 2, -1$ matrix K of order n yield these eigenvalues of B:

Jacobi ($S = 0, 2, 0$ matrix): $S^{-1}T$ has $|\lambda|_{\max} = \cos\dfrac{\pi}{n+1}$

Gauss-Seidel ($S = -1, 2, 0$ matrix): $S^{-1}T$ has $|\lambda|_{\max} = \left(\cos\dfrac{\pi}{n+1}\right)^2$

SOR (with the best ω): $S^{-1}T$ has $|\lambda|_{\max} = \left(\cos\dfrac{\pi}{n+1}\right)^2 \Big/ \left(1 + \sin\dfrac{\pi}{n+1}\right)^2$.

Let me be clear: For the -1, 2, -1 matrix you should not use any of these iterations! Elimination is very fast (exact LU). Iterations are intended for large sparse matrices—when a high percentage of the entries are zero. The not good zeros are inside the band, which is wide. They become nonzero in the exact L and U, which is why elimination becomes expensive.

We mention one more splitting. The idea of "***incomplete*** LU" is to set the small nonzeros in L and U *back to zero*. This leaves triangular matrices L_0 and U_0 which are again sparse. The splitting has $S = L_0U_0$ on the left side. Each step is quick:

Incomplete LU $\quad L_0U_0\boldsymbol{x}_{k+1} = (L_0U_0 - A)\boldsymbol{x}_k + \boldsymbol{b}.$

On the right side we do sparse matrix-vector multiplications. Don't multiply L_0 times U_0, those are matrices. Multiply $\boldsymbol{x}_k$ by U_0 and then multiply that vector by L_0. On the left side we do forward and back substitutions. If L_0U_0 is close to A, then $|\lambda|_{\max}$ is small. A few iterations will give a close answer.

Multigrid and Conjugate Gradients

I cannot leave the impression that Jacobi and Gauss-Seidel are great methods. Generally the "low-frequency" part of the error decays very slowly, and many iterations are needed. Here are two ideas that bring tremendous improvement. Multigrid can solve problems of size n in $O(n)$ steps. With a good preconditioner, conjugate gradients becomes one of the most popular and powerful algorithms in numerical linear algebra.

Multigrid Solve smaller problems (often coming from coarser grids and doubled step-sizes Δx and Δy). Each iteration will be cheaper and convergence will be faster. Then interpolate between the values computed on the coarse grid to get a quick and close head-start on the full-size problem. Multigrid might go 4 levels down and back.

Conjugate gradients An ordinary iteration like $x_{k+1} = x_k - Ax_k + b$ involves multiplication by A at each step. If A is sparse, this is not too expensive: Ax_k is what we are willing to do. It adds one more basis vector to the growing "Krylov spaces" that contain our approximations. But x_{k+1} is **not the best combination** of $x_0, Ax_0, \ldots, A^k x_0$. The ordinary iterations are simple but far from optimal.

The conjugate gradient method chooses **the best combination** x_k at every step. The extra cost (beyond one multiplication by A) is not great. We will give the CG iteration, emphasizing that this method was created for a *symmetric positive definite matrix.* When A is not symmetric, one good choice is GMRES. When $A = A^{\mathrm{T}}$ is not positive definite, there is MINRES. A world of high-powered iterative methods has been created around the idea of making optimal choices of each successive x_k.

My textbook *Computational Science and Engineering* describes multigrid and CG in much more detail. Among books on numerical linear algebra, Trefethen-Bau is deservedly popular (others are terrific too). Golub-Van Loan is a level up.

The Problem Set reproduces the five steps in each conjugate gradient cycle from x_{k-1} to x_k. We compute that new approximation x_k, the new residual $r_k = b - Ax_k$, and the new search direction d_k to look for the next x_{k+1}.

I wrote those steps for the original matrix A. But a **preconditioner** S can make convergence much faster. Our original equation is $Ax = b$. The preconditioned equation is $S^{-1}Ax = S^{-1}b$. Small changes in the code give the *preconditioned conjugate gradient method*—the leading iterative method to solve positive definite systems.

The biggest competition is direct elimination, with the equations reordered to take maximum advantage of many zeros in A. It is not easy to outperform Gauss.

Iterative Methods for Eigenvalues

We move from $Ax = b$ to $Ax = \lambda x$. Iterations are an option for linear equations. They are a necessity for eigenvalue problems. The eigenvalues of an n by n matrix are the roots of an nth degree polynomial. The determinant of $A - \lambda I$ starts with $(-\lambda)^n$. This book must not leave the impression that eigenvalues should be computed that way! Working from $\det(A - \lambda I) = 0$ is a very poor approach—except when n is small.

For $n > 4$ there is no formula to solve $\det(A - \lambda I) = 0$. Worse than that, the λ's can be very unstable and sensitive. It is much better to work with A itself, gradually making it diagonal or triangular. (Then the eigenvalues appear on the diagonal.) Good computer codes are available in the LAPACK library—individual routines are free on **www.netlib.org/lapack**. This library combines the earlier LINPACK and EISPACK, with many improvements (to use matrix-matrix operations in the Level 3 BLAS). It is a collection of Fortran 77 programs for linear algebra on high-performance computers. For your computer and mine, a high quality matrix package is all we need. For supercomputers with parallel processing, move to ScaLAPACK and block elimination.

We will briefly discuss the power method and the QR method (chosen by LAPACK) for computing eigenvalues. It makes no sense to give full details of the codes.

1 Power methods and inverse power methods. Start with any vector $\boldsymbol{u}_0$. Multiply by A to find $\boldsymbol{u}_1$. Multiply by A again to find $\boldsymbol{u}_2$. If $\boldsymbol{u}_0$ is a combination of the eigenvectors, then A multiplies each eigenvector $\boldsymbol{x}_i$ by λ_i. After k steps we have $(\lambda_i)^k$:

$$\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n. \tag{11}$$

As the power method continues, ***the largest eigenvalue begins to dominate.*** The vectors $\boldsymbol{u}_k$ point toward that dominant eigenvector. We saw this for Markov matrices in Chapter 8:

$$A = \begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix} \quad \text{has} \quad \lambda_{\max} = 1 \quad \text{with eigenvector} \quad \begin{bmatrix} .75 \\ .25 \end{bmatrix}.$$

Start with $\boldsymbol{u}_0$ and multiply at every step by A:

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \boldsymbol{u}_1 = \begin{bmatrix} .9 \\ .1 \end{bmatrix}, \ \boldsymbol{u}_2 = \begin{bmatrix} .84 \\ .16 \end{bmatrix} \quad \text{is approaching} \quad \boldsymbol{u}_\infty = \begin{bmatrix} .75 \\ .25 \end{bmatrix}.$$

The speed of convergence depends on the *ratio* of the second largest eigenvalue λ_2 to the largest λ_1. We don't want λ_1 to be small, we want λ_2/λ_1 to be small. Here $\lambda_2 = .6$ and $\lambda_1 = 1$, giving good speed. For large matrices it often happens that $|\lambda_2/\lambda_1|$ is very close to 1. Then the power method is too slow.

Is there a way to find the *smallest* eigenvalue—which is often the most important in applications? Yes, by the *inverse* power method: Multiply $\boldsymbol{u}_0$ by A^{-1} instead of A. Since we never want to compute A^{-1}, we actually solve $A\boldsymbol{u}_1 = \boldsymbol{u}_0$. By saving the LU factors, the next step $A\boldsymbol{u}_2 = \boldsymbol{u}_1$ is fast. Step k has $A\boldsymbol{u}_k = \boldsymbol{u}_{k-1}$:

$$\textbf{Inverse power method} \qquad \boldsymbol{u}_k = A^{-k}\boldsymbol{u}_0 = \frac{c_1\boldsymbol{x}_1}{(\lambda_1)^k} + \cdots + \frac{c_n\boldsymbol{x}_n}{(\lambda_n)^k}. \tag{12}$$

Now the *smallest* eigenvalue $\lambda_{\min}$ is in control. When it is very small, the factor $1/\lambda_{\min}^k$ is large. For high speed, we make $\boldsymbol{\lambda}_{\min}$ even smaller by shifting the matrix to $A - \lambda^* I$.

That shift doesn't change the eigenvectors. (λ^* might come from the diagonal of A, even better is a Rayleigh quotient $x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$). If λ^* is close to $\lambda_{\min}$ then $(A - \lambda^* I)^{-1}$ has the very large eigenvalue $(\lambda_{\min} - \lambda^*)^{-1}$. Each ***shifted inverse power step*** multiplies the eigenvector by this big number, and that eigenvector quickly dominates.

2 The QR Method This is a major achievement in numerical linear algebra. Fifty years ago, eigenvalue computations were slow and inaccurate. We didn't even realize that solving $\det(A - \lambda I) = 0$ was a terrible method. Jacobi had suggested earlier that A should gradually be made triangular—then the eigenvalues appear automatically on the diagonal. He used 2 by 2 rotations to produce off-diagonal zeros. (Unfortunately the previous zeros can become nonzero again. But Jacobi's method made a partial comeback with parallel computers.) At present the ***QR method*** is the leader in eigenvalue computations and we describe it briefly.

The basic step is to factor A, whose eigenvalues we want, into QR. Remember from Gram-Schmidt (Section 4.4) that Q has orthonormal columns and R is triangular. For eigenvalues the key idea is: ***Reverse Q and R.*** The new matrix (same λ's) is $A_1 = RQ$.

The eigenvalues are not changed in RQ *because* $A = QR$ *is similar to* $A_1 = Q^{-1}AQ$:

$$\mathbf{A_1 = RQ \text{ has the same } \lambda} \qquad QRx = \lambda x \quad \text{gives} \quad RQ(Q^{-1}x) = \lambda(Q^{-1}x). \tag{13}$$

This process continues. Factor the new matrix A_1 into Q_1R_1. Then reverse the factors to R_1Q_1. This is the similar matrix A_2 and again no change in the eigenvalues. Amazingly, those eigenvalues begin to show up on the diagonal. Often the last entry of A_4 holds an accurate eigenvalue. In that case we remove the last row and column and continue with a smaller matrix to find the next eigenvalue.

Two extra ideas make this method a success. One is to shift the matrix by a multiple of I, before factoring into QR. Then RQ is shifted back:

$$\text{Factor } A_k - c_kI \text{ into } Q_kR_k. \text{ The next matrix is } A_{k+1} = R_kQ_k + c_kI.$$

A_{k+1} has the same eigenvalues as A_k, and the same as the original $A_0 = A$. A good shift chooses c near an (unknown) eigenvalue. That eigenvalue appears more accurately on the diagonal of A_{k+1}—which tells us a better c for the next step to A_{k+2}.

The other idea is to obtain off-diagonal zeros before the QR method starts. An elimination step E will do it, or a Eivens rotation, but don't forget E^{-1} (to keep λ):

$$EAE^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ \mathbf{1} & 4 & 5 \\ \mathbf{1} & \mathbf{6} & 7 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ \mathbf{1} & 9 & 5 \\ \mathbf{0} & \mathbf{4} & 2 \end{bmatrix}. \ \textbf{Same } \boldsymbol{\lambda}\textbf{'s.}$$

We must leave those nonzeros 1 and 4 along *one subdiagonal*. More E's could remove them, but E^{-1} would fill them in again. This is a "***Hessenberg matrix***" (one nonzero subdiagonal). The zeros in the lower left corner will stay zero through the QR method. The operation count for each QR factorization drops from $O(n^3)$ to $O(n^2)$.

Golub and Van Loan give this example of one shifted QR step on a Hessenberg matrix. The shift is $7I$, taking 7 from all diagonal entries (then shifting back for A_1):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & .001 & 7 \end{bmatrix} \quad \text{leads to} \quad A_1 = \begin{bmatrix} -.54 & 1.69 & 0.835 \\ .31 & 6.53 & -6.656 \\ 0 & .00002 & 7.012 \end{bmatrix}.$$

Factoring $A-7I$ into QR produced $A_1 = RQ+7I$. Notice the very small number .00002. The diagonal entry 7.012 is almost an exact eigenvalue of A_1, and therefore of A. Another QR step on A_1 with shift by $7.012I$ would give terrific accuracy.

For large sparse matrices I would look to ARPACK. Problems 27–29 describe the Arnoldi iteration that orthogonalizes the basis—each step has only three terms when A is symmetric. The matrix becomes tridiagonal and still orthogonally similar to the original A: a wonderful start for computing eigenvalues.

Problem Set 9.3

Problems 1–12 are about iterative methods for $Ax = b$.

1 Change $Ax = b$ to $x = (I - A)x + b$. What are S and T for this splitting? What matrix $S^{-1}T$ controls the convergence of $x_{k+1} = (I - A)x_k + b$?

2 If λ is an eigenvalue of A, then _____ is an eigenvalue of $B = I - A$. The real eigenvalues of B have absolute value less than 1 if the real eigenvalues of A lie between _____ and _____.

3 Show why the iteration $x_{k+1} = (I - A)x_k + b$ does not converge for $A = \left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]$.

4 Why is the norm of B^k never larger than $\|B\|^k$? Then $\|B\| < 1$ guarantees that the powers B^k approach zero (convergence). No surprise since $|\lambda|_{\max}$ is below $\|B\|$.

5 If A is singular then all splittings $A = S - T$ must fail. From $Ax = \mathbf{0}$ show that $S^{-1}Tx = x$. So this matrix $B = S^{-1}T$ has $\lambda = 1$ and fails.

6 Change the 2's to 3's and find the eigenvalues of $S^{-1}T$ for Jacobi's method:

$$Sx_{k+1} = Tx_k + b \quad \text{is} \quad \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_k + b.$$

7 Find the eigenvalues of $S^{-1}T$ for the Gauss-Seidel method applied to Problem 6:

$$\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + b.$$

Does $|\lambda|_{\max}$ for Gauss-Seidel equal $|\lambda|^2_{\max}$ for Jacobi?

8 For any 2 by 2 matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ show that $|\lambda|_{\max}$ equals $|bc/ad|$ for Gauss-Seidel and $|bc/ad|^{1/2}$ for Jacobi. We need $ad \neq 0$ for the matrix S to be invertible.

9 The best ω produces two equal eigenvalues for $S^{-1}T$ in the **SOR** method. Those eigenvalues are $\omega - 1$ because the determinant is $(\omega - 1)^2$. Set the trace in equation (10) equal to $(\omega - 1) + (\omega - 1)$ and find this optimal ω.

10 Write a computer code (MATLAB or other) for the Gauss-Seidel method. You can define S and T from A, or set up the iteration loop directly from the entries a_{ij}. Test it on the $-1, 2, -1$ matrices A of order 10, 20, 50 with $b = (1, 0, \ldots, 0)$.

11 The Gauss-Seidel iteration at component i uses earlier parts of x^{new}:

$$\textbf{Gauss-Seidel} \qquad x_i^{\text{new}} = x_i^{\text{old}} + \frac{1}{a_{ii}}\Big(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{\text{new}} - \sum_{j=i}^{n} a_{ij}x_j^{\text{old}}\Big).$$

If every $x_i^{\text{new}} = x_i^{\text{old}}$ how does this show that the solution x is correct? How does the formula change for Jacobi's method? For **SOR** insert ω outside the parentheses.

12 The **SOR** splitting matrix S is the same as for Gauss-Seidel except that the diagonal is divided by ω. Write a program for **SOR** on an n by n matrix. Apply it with $\omega = 1$, 1.4, 1.8, 2.2 when A is the $-1, 2, -1$ matrix of order $n = 10$.

13 Divide equation (11) by λ_1^k and explain why $|\lambda_2/\lambda_1|$ controls the convergence of the power method. Construct a matrix A for which this method *does not converge*.

14 The Markov matrix $A = \begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix}$ has $\lambda = 1$ and .6, and the power method $\boldsymbol{u}_k = A^k \boldsymbol{u}_0$ converges to $\begin{bmatrix} .75 \\ .25 \end{bmatrix}$. Find the eigenvectors of A^{-1}. What does the inverse power method $\boldsymbol{u}_{-k} = A^{-k}\boldsymbol{u}_0$ converge to (after you multiply by $.6^k$)?

15 The tridiagonal matrix of size $n - 1$ with diagonals $-1, 2, -1$ has eigenvalues $\lambda_j = 2 - 2\cos(j\pi/n)$. Why are the smallest eigenvalues approximately $(j\pi/n)^2$? The inverse power method converges at the speed $\lambda_1/\lambda_2 \approx 1/4$.

16 For $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ apply the power method $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ three times starting with $\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What eigenvector is the power method converging to?

17 In Problem 11 apply the *inverse* power method $\boldsymbol{u}_{k+1} = A^{-1}\boldsymbol{u}_k$ three times with the same $\boldsymbol{u}_0$. What eigenvector are the $\boldsymbol{u}_k$'s approaching?

18 In the QR method for eigenvalues, show that the 2, 1 entry drops from $\sin\theta$ in $A = QR$ to $-\sin^3\theta$ in RQ. *(Compute R and RQ.)* This "cubic convergence" makes the method a success:

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix} = QR = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & ? \\ 0 & ? \end{bmatrix}.$$

19 If A is an orthogonal matrix, its QR factorization has $Q =$ _____ and $R =$ _____. Therefore $RQ =$ _____. These are among the rare examples when the QR method goes nowhere.

20 The shifted QR method factors $A - cI$ into QR. Show that the next matrix $A_1 = RQ + cI$ equals $Q^{-1}AQ$. Therefore A_1 has the _____ eigenvalues as A (but is closer to triangular).

21 When $A = A^{\mathrm{T}}$, the "*Lanczos method*" finds a's and b's and orthonormal $\boldsymbol{q}$'s so that $A\boldsymbol{q}_j = b_{j-1}\boldsymbol{q}_{j-1} + a_j\boldsymbol{q}_j + b_j\boldsymbol{q}_{j+1}$ (with $\boldsymbol{q}_0 = \boldsymbol{0}$). Multiply by $\boldsymbol{q}_j^{\mathrm{T}}$ to find a formula for a_j. The equation says that $AQ = QT$ where T is a tridiagonal matrix.

22 The equation in Problem 21 develops from this loop with $b_0 = 1$ and $\boldsymbol{r}_0 =$ any $\boldsymbol{q}_1$:

$$\boldsymbol{q}_{j+1} = \boldsymbol{r}_j/b_j;\ j = j+1;\ a_j = \boldsymbol{q}_j^{\mathrm{T}}A\boldsymbol{q}_j;\ \boldsymbol{r}_j = A\boldsymbol{q}_j - b_{j-1}\boldsymbol{q}_{j-1} - a_j\boldsymbol{q}_j;\ b_j = \|\boldsymbol{r}_j\|.$$

Write a code and test it on the $-1, 2, -1$ matrix A. $Q^{\mathrm{T}}Q$ should be I.

23 Suppose A is *tridiagonal and symmetric in the QR method.* From $A_1 = Q^{-1}AQ$ show that A_1 is symmetric. Write $A_1 = RAR^{-1}$ to show that A_1 is also tridiagonal. (If the lower part of A_1 is proved tridiagonal then by symmetry the upper part is too.)

Symmetric tridiagonal matrices are the best way to start in the QR method.

Questions 24–26 are about quick ways to estimate the location of the eigenvalues.

24 If the sum of $|a_{ij}|$ along every row is less than 1, explain this proof that $|\lambda| < 1$. Suppose $Ax = \lambda x$ and $|x_i|$ is larger than the other components of x. Then $|\Sigma a_{ij}x_j|$ is less than $|x_i|$. That means $|\lambda x_i| < |x_i|$ so $|\lambda| < 1$.

(Gershgorin circles) Every eigenvalue of A is in one or more of n circles. Each circle is centered at a diagonal entry a_{ii} with radius $r_i = \Sigma_{j\neq i}|a_{ij}|$.

This follows from $(\lambda - a_{ii})x_i = \Sigma_{j\neq i}a_{ij}x_j$. If $|x_i|$ is larger than the other components of x, this sum is at most $r_i|x_i|$. Dividing by $|x_i|$ leaves $|\lambda - a_{ii}| \leq r_i$.

25 What bound on $|\lambda|_{\max}$ does Problem 24 give for these matrices? What are the three Gershgorin circles that contain all the eigenvalues? Those circles show immediately that K is at least positive semidefinite (*actually definite*) and A has $\lambda_{\max} = 1$.

$$A = \begin{bmatrix} .3 & .5 & .2 \\ .3 & .4 & .3 \\ .4 & .1 & .5 \end{bmatrix} \qquad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

26 These matrices are diagonally dominant because each $a_{ii} > r_i$ = absolute sum along the rest of row i. From the Gershgorin circles containing all λ's, show that diagonally dominant matrices are invertible.

$$A = \begin{bmatrix} 1 & .3 & .4 \\ .3 & 1 & .5 \\ .4 & .5 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 2 & 5 \end{bmatrix}.$$

Problems 27–30 present two fundamental iterations. Each step involves Aq or Ad.

The key point for large matrices is that matrix-vector multiplication is much faster than matrix-matrix multiplication. A crucial construction starts with a vector b. Repeated multiplication will produce $Ab, A^2b, \ldots$ but those vectors are far from orthogonal. The "**Arnoldi iteration**" creates an orthonormal basis $q_1, q_2, \ldots$ for the same space by the Gram-Schmidt idea: *orthogonalize each new Aq_n against the previous $q_1, \ldots, q_{n-1}$*. The "Krylov space" spanned by $b, Ab, \ldots, A^{n-1}b$ then has a much better basis $q_1, \ldots, q_n$.

Here in pseudocode are two of the most important algorithms in numerical linear algebra: Arnoldi gives a good basis and CG gives a good approximation to $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Arnoldi Iteration	**Conjugate Gradient Iteration for Positive Definite A**	
$\boldsymbol{q}_1 = \boldsymbol{b}/\lVert\boldsymbol{b}\rVert$	$\boldsymbol{x}_0 = 0, \boldsymbol{r}_0 = \boldsymbol{b}, \boldsymbol{d}_0 = \boldsymbol{r}_0$	
for $n = 1$ **to** $N-1$	**for** $n = 1$ **to** N	
$\quad \boldsymbol{v} = A\boldsymbol{q}_n$	$\quad \alpha_n = (\boldsymbol{r}_{n-1}^{\mathrm{T}}\boldsymbol{r}_{n-1})/(\boldsymbol{d}_{n-1}^{\mathrm{T}}A\boldsymbol{d}_{n-1})$	step length $\boldsymbol{x}_{n-1}$ to $\boldsymbol{x}_n$
$\quad$ **for** $j = 1$ **to** n	$\quad \boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \alpha_n\boldsymbol{d}_{n-1}$	approximate solution
$\quad\quad h_{jn} = \boldsymbol{q}_j^{\mathrm{T}}\boldsymbol{v}$	$\quad \boldsymbol{r}_n = \boldsymbol{r}_{n-1} - \alpha_n A\boldsymbol{d}_{n-1}$	new residual $\boldsymbol{b} - A\boldsymbol{x}_n$
$\quad\quad \boldsymbol{v} = \boldsymbol{v} - h_{jn}\boldsymbol{q}_j$	$\quad \beta_n = (\boldsymbol{r}_n^{\mathrm{T}}\boldsymbol{r}_n)/(\boldsymbol{r}_{n-1}^{\mathrm{T}}\boldsymbol{r}_{n-1})$	improvement this step
$\quad h_{n+1,n} = \lVert\boldsymbol{v}\rVert$	$\quad \boldsymbol{d}_n = \boldsymbol{r}_n + \beta_n\boldsymbol{d}_{n-1}$	next search direction
$\quad \boldsymbol{q}_{n+1} = \boldsymbol{v}/h_{n+1,n}$	*% Notice: only* 1 *matrix-vector multiplication* $A\boldsymbol{q}$ *and* $A\boldsymbol{d}$	

For conjugate gradients, the residuals $\boldsymbol{r}_n$ are orthogonal and the search directions are A-orthogonal: all $\boldsymbol{d}_j^{\mathrm{T}}A\boldsymbol{d}_k = 0$. The iteration solves $A\boldsymbol{x} = \boldsymbol{b}$ by minimizing the error $\boldsymbol{e}^{\mathrm{T}}A\boldsymbol{e}$ over all vectors in the Krylov subspace. It is a fantastic algorithm.

27 For the diagonal matrix $A = \mathrm{diag}([1 \;\; 2 \;\; 3 \;\; 4])$ and the vector $\boldsymbol{b} = (1, 1, 1, 1)$, go through one Arnoldi step to find the orthonormal vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

28 Arnoldi's method is finding Q so that $AQ = QH$ (column by column):

$$AQ = \begin{bmatrix} A\boldsymbol{q}_1 & \cdots & A\boldsymbol{q}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_N \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & \cdot & h_{1N} \\ h_{21} & h_{22} & \cdot & h_{2N} \\ 0 & h_{32} & \cdot & \cdot \\ 0 & 0 & \cdot & h_{NN} \end{bmatrix} = QH$$

H is a "Hessenberg matrix" with one nonzero subdiagonal. Here is the crucial fact when A is symmetric: ***The matrix*** $H = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ ***is symmetric and therefore tridiagonal.*** Explain that sentence.

29 This tridiagonal H (when A is symmetric) gives the **Lanczos iteration**:

$$\textbf{Three terms only} \qquad \boldsymbol{q}_{j+1} = (A\boldsymbol{q}_j - h_{j,j}\boldsymbol{q}_j - h_{j-1,j}\boldsymbol{q}_{j-1})/h_{j+1,j}$$

From $H = Q^{-1}AQ$, why are the eigenvalues of H the same as the eigenvalues of A? For large matrices, the "Lanczos method" computes the leading eigenvalues by stopping at a smaller tridiagonal matrix H_k. The QR method in the text is applied to compute the eigenvalues of H_k.

30 Apply the conjugate gradient method to solve $A\boldsymbol{x} = \boldsymbol{b} = \mathbf{ones}(100, 1)$, where A is the $-1, 2, -1$ second difference matrix $A = \mathbf{toeplitz}([2 \;\; -1 \;\; \mathbf{zeros}(1, 98)])$. Graph $\boldsymbol{x}_{10}$ and $\boldsymbol{x}_{20}$ from CG, along with the exact solution $\boldsymbol{x}$. (Its 100 components are $x_i = (ih - i^2h^2)/2$ with $h = 1/101$. "**plot**$(i, x(i))$" should produce a parabola.)

Chapter 10

Complex Vectors and Matrices

10.1 Complex Numbers

A complete presentation of linear algebra must include complex numbers. Even when the matrix is real, ***the eigenvalues and eigenvectors are often complex***. Example: A 2 by 2 rotation matrix has no real eigenvectors. Every vector in the plane turns by θ—its direction changes. But the rotation matrix has complex eigenvectors $(1, i)$ and $(1, -i)$.

Notice that those eigenvectors are connected by changing i to $-i$. For a real matrix, the eigenvectors come in "conjugate pairs." The eigenvalues of rotation by θ are also conjugate complex numbers $e^{i\theta}$ and $e^{-i\theta}$. We must move from $\mathbf{R}^n$ to $\mathbf{C}^n$.

The second reason for allowing complex numbers goes beyond λ and x to the matrix A. ***The matrix itself may be complex.*** We will devote a whole section to the most important example—*the Fourier matrix.* Engineering and science and music and economics all use Fourier series. In reality the series is finite, not infinite. Computing the coefficients in $c_1 e^{ix} + c_2 e^{i2x} + \cdots + c_n e^{inx}$ is a linear algebra problem.

This section gives the main facts about complex numbers. It is a review for some students and a reference for everyone. Everything comes from $i^2 = -1$. The Fast Fourier Transform applies the amazing formula $e^{2\pi i} = 1$. Add angles when $e^{i\theta}$ multiplies $e^{i\theta}$:

The square of $e^{2\pi i/4} = i$ ***is*** $e^{4\pi i/4} = -1$. ***The fourth power of*** $e^{2\pi i/4}$ ***is*** $e^{2\pi i} = 1$.

Adding and Multiplying Complex Numbers

Start with the imaginary number i. Everybody knows that $x^2 = -1$ has no real solution. When you square a real number, the answer is never negative. So the world has agreed on a solution called i. (Except that electrical engineers call it j.) Imaginary numbers follow the normal rules of addition and multiplication, with one difference. ***Replace*** i^2 ***by*** -1.

A complex number (say $3 + 2i$) ***is the sum of a real number*** (3) ***and a pure imaginary number*** ($2i$). Addition keeps the real and imaginary parts separate. Multiplication uses $i^2 = -1$:

$$\textbf{Add:} \quad (3+2i)+(3+2i)=6+4i$$
$$\textbf{Multiply:} \quad (3+2i)(1-i)=3+2i-3i-2i^2=5-i.$$

If I add $3+i$ to $1-i$, the answer is 4. The real numbers $3+1$ stay separate from the imaginary numbers $i-i$. We are adding the vectors $(3,1)$ and $(1,-1)$.

The number $(1+i)^2$ is $1+i$ times $1+i$. The rules give the surprising answer $2i$:

$$(1+i)(1+i)=1+i+i+i^2=2i.$$

In the complex plane, $1+i$ is at an angle of 45°. It is like the vector $(1,1)$. When we square $1+i$ to get $2i$, the angle doubles to 90°. If we square again, the answer is $(2i)^2=-4$. The 90° angle doubled to 180°, the direction of a negative real number.

A real number is just a complex number $z=a+bi$, with zero imaginary part: $b=0$. A pure imaginary number has $a=0$:

The ***real part*** is $\quad a = \text{Re}\,(a+bi)$. The ***imaginary part*** is $\quad b=\text{Im}\,(a+bi)$.

The Complex Plane

Complex numbers correspond to points in a plane. Real numbers go along the x axis. Pure imaginary numbers are on the y axis. ***The complex number*** $\boldsymbol{3+2i}$ ***is at the point with coordinates*** $\mathbf{(3,2)}$. The number zero, which is $0+0i$, is at the origin.

Adding and subtracting complex numbers is like adding and subtracting vectors in the plane. The real component stays separate from the imaginary component. The vectors go head-to-tail as usual. The complex plane $\mathbf{C}^1$ is like the ordinary two-dimensional plane $\mathbf{R}^2$, except that we multiply complex numbers and we didn't multiply vectors.

Now comes an important idea. ***The complex conjugate of*** $\boldsymbol{3+2i}$ ***is*** $\boldsymbol{3-2i}$. The complex conjugate of $z=1-i$ is $\bar{z}=1+i$. In general the conjugate of $z=a+bi$ is $\bar{z}=a-bi$. **(Some writers use a "*bar*" on the number and others use a "*star*":** $\bar{z}=z^*$.) The imaginary parts of z and "z bar" have opposite signs. In the complex plane, $\bar{z}$ is the image of z on the other side of the real axis.

Two useful facts. ***When we multiply conjugates*** $\bar{z}_1$ ***and*** $\bar{z}_2$, ***we get the conjugate of*** z_1z_2. When we add $\bar{z}_1$ and $\bar{z}_2$, we get the conjugate of z_1+z_2:

$$\bar{z}_1+\bar{z}_2=(3-2i)+(1+i)=4-i. \text{ This is the conjugate of } z_1+z_2=4+i.$$
$$\bar{z}_1\times\bar{z}_2=(3-2i)\times(1+i)=5+i. \text{ This is the conjugate of } z_1\times z_2=5-i.$$

Adding and multiplying is exactly what linear algebra needs. By taking conjugates of $A\boldsymbol{x}=\lambda\boldsymbol{x}$, when A is real, we have another eigenvalue $\bar{\lambda}$ and its eigenvector $\bar{\boldsymbol{x}}$:

$$\textbf{\textit{If }} A\boldsymbol{x}=\lambda\boldsymbol{x} \textbf{\textit{ and }} A \textbf{\textit{ is real then }} A\bar{\boldsymbol{x}}=\bar{\lambda}\bar{\boldsymbol{x}}. \tag{1}$$

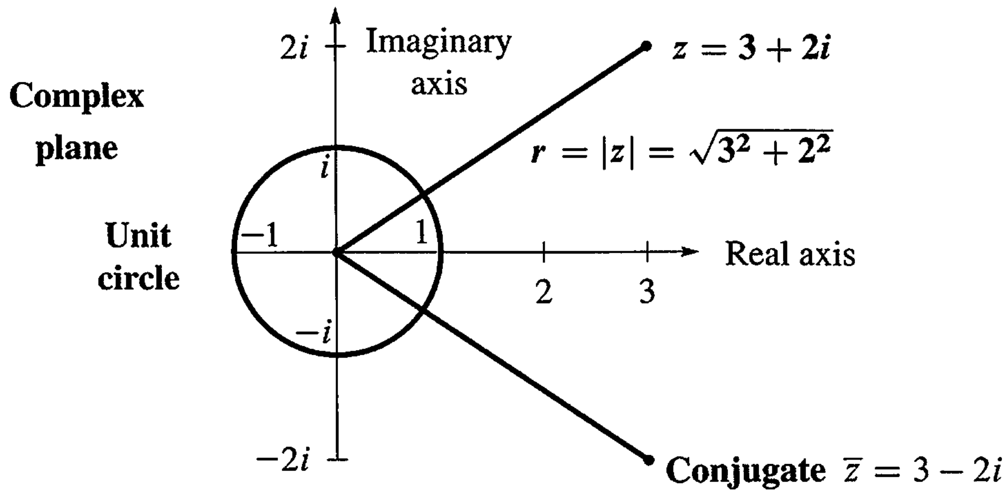


Figure 10.1: The number $z = a + bi$ corresponds to the point (a, b) and the vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$.

Something special happens when $z = 3 + 2i$ combines with its own complex conjugate $\bar{z} = 3 - 2i$. The result from adding $z + \bar{z}$ or multiplying $z\bar{z}$ is always real:

$$\begin{aligned} z + \bar{z} &= \textbf{real} \qquad (3 + 2i) + (3 - 2i) = 6 \quad \text{(real)} \\ z\bar{z} &= \textbf{real} \qquad (3 + 2i) \times (3 - 2i) = 9 + 6i - 6i - 4i^2 = 13 \quad \text{(real).} \end{aligned}$$

The sum of $z = a + bi$ and its conjugate $\bar{z} = a - bi$ is the real number $2a$. The product of z times $\bar{z}$ is the real number $a^2 + b^2$:

$$\textbf{Multiply } z \textbf{ times } \bar{z} \qquad (a + bi)(a - bi) = a^2 + b^2. \tag{2}$$

The next step with complex numbers is $1/z$. How to divide by $a + ib$? The best idea is to multiply by $\bar{z}/\bar{z}$. That produces $z\bar{z}$ in the denominator, which is $a^2 + b^2$:

$$\frac{1}{a + ib} = \frac{1}{a + ib}\frac{a - ib}{a - ib} = \frac{a - ib}{a^2 + b^2} \qquad \frac{1}{3 + 2i} = \frac{1}{3 + 2i}\frac{3 - 2i}{3 - 2i} = \frac{3 - 2i}{13}.$$

In case $a^2 + b^2 = 1$, this says that $(a + ib)^{-1}$ is $a - ib$. ***On the unit circle,* $1/z$ equals $\bar{z}$.** Later we will say: $1/e^{i\theta}$ is $e^{-i\theta}$ (the conjugate). A better way to multiply and divide is to use the polar form with distance r and angle θ.

The Polar Form $re^{i\theta}$

The square root of $a^2 + b^2$ is $|z|$. This is the ***absolute value*** (or ***modulus***) of the number $z = a + ib$. The square root $|z|$ is also written r, because it is the distance from 0 to z. ***The real number r in the polar form gives the size of the complex number*** z:

The absolute value of $\quad z = a + ib \quad$ is $\quad |z| = \sqrt{a^2 + b^2}$. $\quad$ This is called r.

The absolute value of $\quad z = 3 + 2i \quad$ is $\quad |z| = \sqrt{3^2 + 2^2}$. $\quad$ This is $r = \sqrt{13}$.

The other part of the polar form is the angle θ. The angle for $z = 5$ is $\theta = 0$ (because this z is real and positive). The angle for $z = 3i$ is $\pi/2$ radians. The angle for a negative $z = -9$ is π radians. ***The angle doubles when the number is squared.*** The polar form is excellent for multiplying complex numbers (not good for addition).

When the distance is r and the angle is θ, trigonometry gives the other two sides of the triangle. The real part (along the bottom) is $a = r\cos\theta$. The imaginary part (up or down) is $b = r\sin\theta$. Put those together, and the rectangular form becomes the polar form:

$$\textit{\textbf{The number}} \quad \boldsymbol{z = a + ib} \qquad \textit{\textbf{is also}} \quad \boldsymbol{z = r\cos\theta + ir\sin\theta}. \quad \textit{\textbf{This is}}\ \boldsymbol{re^{i\theta}}$$

Note: $\boldsymbol{\cos\theta + i\sin\theta}$ ***has absolute value*** $\boldsymbol{r = 1}$ ***because*** $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$. Thus $\cos\theta + i\sin\theta$ lies on the circle of radius 1—*the unit circle.*

Example 1 Find r and θ for $z = 1 + i$ and also for the conjugate $\overline{z} = 1 - i$.

Solution The absolute value is the same for z and $\overline{z}$. For $z = 1+i$ it is $r = \sqrt{1+1} = \sqrt{2}$:

$$|z|^2 = 1^2 + 1^2 = 2 \qquad \text{and also} \qquad |\overline{z}|^2 = 1^2 + (-1)^2 = 2.$$

The distance from the center is $\sqrt{2}$. What about the angle? The number $1 + i$ is at the point $(1, 1)$ in the complex plane. The angle to that point is $\pi/4$ radians or $45°$. The cosine is $1/\sqrt{2}$ and the sine is $1/\sqrt{2}$. Combining r and θ brings back $z = 1 + i$:

$$r\cos\theta + ir\sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + i\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1 + i.$$

The angle to the conjugate $1-i$ can be positive or negative. We can go to $7\pi/4$ radians which is $315°$. Or we can go *backwards through a negative angle*, to $-\pi/4$ radians or $-45°$. ***If*** $\boldsymbol{z}$ ***is at angle*** $\boldsymbol{\theta}$***, its conjugate*** $\boldsymbol{\overline{z}}$ ***is at*** $\boldsymbol{2\pi - \theta}$ ***and also at*** $\boldsymbol{-\theta}$***.***

We can freely add 2π or 4π or -2π to any angle! Those go full circles so the final point is the same. This explains why there are infinitely many choices of θ. Often we select the angle between zero and 2π radians. But $-\theta$ is very useful for the conjugate $\overline{z}$.

Powers and Products: Polar Form

Computing $(1 + i)^2$ and $(1 + i)^8$ is quickest in polar form. That form has $r = \sqrt{2}$ and $\theta = \pi/4$ (or $45°$). If we square the absolute value to get $r^2 = 2$, and double the angle to get $2\theta = \pi/2$ (or $90°$), we have $(1 + i)^2$. For the eighth power we need r^8 and 8θ:

$$\boldsymbol{(1+i)^8} \qquad r^8 = 2\cdot 2\cdot 2\cdot 2 = 16 \text{ and } 8\theta = 8\cdot\frac{\pi}{4} = 2\pi.$$

This means: $(1 + i)^8$ has absolute value 16 and angle 2π. *The eighth power of* $1 + i$ *is the real number* 16.

Powers are easy in polar form. So is multiplication of complex numbers.

The polar form of z^n has absolute value r^n. The angle is n times θ:

$$\textbf{\textit{The nth power of}} \quad z = r(\cos\theta + i\sin\theta) \quad \textbf{\textit{is}} \quad z^n = r^n(\cos n\theta + i\sin n\theta). \tag{3}$$

In that case z multiplies itself. In all cases, ***multiply r's and add the angles***:

$$r(\cos\theta + i\sin\theta) \text{ times } r'(\cos\theta' + i\sin\theta') = rr'\big(\cos(\theta+\theta') + i\sin(\theta+\theta')\big). \tag{4}$$

One way to understand this is by trigonometry. Concentrate on angles. Why do we get the double angle 2θ for z^2?

$$(\cos\theta + i\sin\theta) \times (\cos\theta + i\sin\theta) = \cos^2\theta + i^2\sin^2\theta + 2i\sin\theta\cos\theta.$$

The real part $\cos^2\theta - \sin^2\theta$ is $\cos 2\theta$. The imaginary part $2\sin\theta\cos\theta$ is $\sin 2\theta$. Those are the "double angle" formulas. They show that θ in z becomes 2θ in z^2.

There is a second way to understand the rule for z^n. It uses the only amazing formula in this section. Remember that $\cos\theta + i\sin\theta$ has absolute value 1. The cosine is made up of even powers, starting with $1 - \frac{1}{2}\theta^2$. The sine is made up of odd powers, starting with $\theta - \frac{1}{6}\theta^3$. The beautiful fact is that $e^{i\theta}$ combines both of those series into $\cos\theta + i\sin\theta$:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \quad \text{becomes} \quad e^{i\theta} = 1 + i\theta + \frac{1}{2}i^2\theta^2 + \frac{1}{6}i^3\theta^3 + \cdots$$

Write -1 for i^2 to see $1 - \frac{1}{2}\theta^2$. ***The complex number*** $e^{i\theta}$ ***is*** $\cos\theta + i\sin\theta$:

$$\textbf{\textit{Euler's Formula}} \quad e^{i\theta} = \cos\theta + i\sin\theta \quad \text{gives} \quad z = r\cos\theta + ir\sin\theta = re^{i\theta} \tag{5}$$

The special choice $\theta = 2\pi$ gives $\cos 2\pi + i\sin 2\pi$ which is 1. Somehow the infinite series $e^{2\pi i} = 1 + 2\pi i + \frac{1}{2}(2\pi i)^2 + \cdots$ adds up to 1.

Now multiply $e^{i\theta}$ times $e^{i\theta'}$. Angles add for the same reason that exponents add:

$$e^2 \text{ times } e^3 \text{ is } e^5 \qquad e^{i\theta} \text{ times } e^{i\theta} \text{ is } e^{2i\theta} \qquad e^{i\theta} \text{ times } e^{i\theta'} \text{ is } e^{i(\theta+\theta')}$$

The powers $(re^{i\theta})^n$ are equal to $r^n e^{in\theta}$. They stay on the unit circle when $r = 1$ and $r^n = 1$. Then we find n different numbers whose nth powers equal 1:

$$\textbf{\textit{Set}} \quad w = e^{2\pi i/n}. \quad \textbf{\textit{The nth powers of}} \ 1, w, w^2, \ldots, w^{n-1} \ \textbf{\textit{all equal 1.}}$$

Those are the "nth roots of 1." They solve the equation $z^n = 1$. They are equally spaced around the unit circle in Figure 10.2b, where the full 2π is divided by n. Multiply their angles by n to take nth powers. That gives $w^n = e^{2\pi i}$ which is 1. Also $(w^2)^n = e^{4\pi i} = 1$. Each of those numbers, to the nth power, comes around the unit circle to 1.

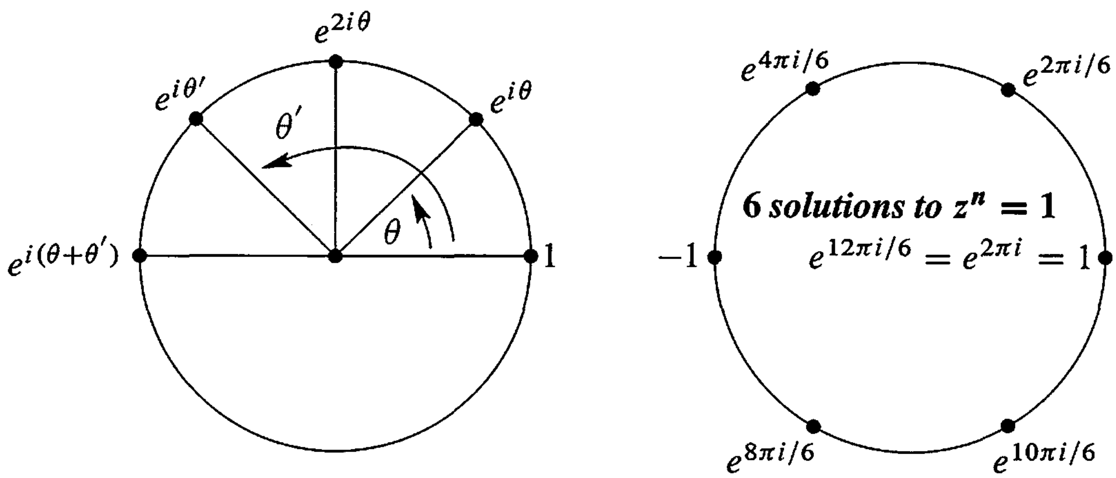


Figure 10.2: (a) Multiplying $e^{i\theta}$ times $e^{i\theta'}$. (b) The nth power of $e^{2\pi i/n}$ is $e^{2\pi i} = 1$.

These n roots of 1 are the key numbers for signal processing. The Discrete Fourier Transform uses w and its powers. Section 10.3 shows how to decompose a vector (a signal) into n frequencies by the Fast Fourier Transform.

■ REVIEW OF THE KEY IDEAS ■

1. Adding $a + ib$ to $c + id$ is like adding $(a, b) + (c, d)$. Use $i^2 = -1$ to multiply.
2. The conjugate of $z = a + bi = re^{i\theta}$ is $\bar{z} = z^* = a - bi = re^{-i\theta}$.
3. z times $\bar{z}$ is $re^{i\theta}$ times $re^{-i\theta}$. This is $r^2 = |z|^2 = a^2 + b^2$ (real).
4. Powers and products are easy in polar form $z = re^{i\theta}$. *Multiply* r*'s and add* θ*'s.*

Problem Set 10.1

Questions 1–8 are about operations on complex numbers.

1 Add and multiply each pair of complex numbers:

(a) $2+i, 2-i$ (b) $-1+i, -1+i$ (c) $\cos\theta + i\sin\theta, \cos\theta - i\sin\theta$

2 Locate these points on the complex plane. Simplify them if necessary:

(a) $2+i$ (b) $(2+i)^2$ (c) $\frac{1}{2+i}$ (d) $|2+i|$

3 Find the absolute value $r = |z|$ of these four numbers. If θ is the angle for $6-8i$, what are the angles for the other three numbers?

(a) $6-8i$ (b) $(6-8i)^2$ (c) $\frac{1}{6-8i}$ (d) $(6+8i)^2$

4 If $|z| = 2$ and $|w| = 3$ then $|z \times w| =$ ____ and $|z + w| \leq$ ____ and $|z/w| =$ ____ and $|z - w| \leq$ ____.

5 Find $a + ib$ for the numbers at angles $30°, 60°, 90°, 120°$ on the unit circle. If w is the number at $30°$, check that w^2 is at $60°$. What power of w equals 1?

6 If $z = r\cos\theta + ir\sin\theta$ then $1/z$ has absolute value ____ and angle ____. Its polar form is ____. Multiply $z \times 1/z$ to get 1.

7 The complex multiplication $M = (a + bi)(c + di)$ is a 2 by 2 real multiplication

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}.$$

The right side contains the real and imaginary parts of M. Test $M = (1+3i)(1-3i)$.

8 $A = A_1 + iA_2$ is a complex n by n matrix and $\boldsymbol{b} = \boldsymbol{b}_1 + i\boldsymbol{b}_2$ is a complex vector. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x}_1 + i\boldsymbol{x}_2$. Write $A\boldsymbol{x} = \boldsymbol{b}$ as a real system of size $2n$:

Complex n by n
Real $2n$ by $2n$
$$\begin{bmatrix} \\ \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}.$$

Questions 9–16 are about the conjugate $\bar{z} = a - ib = re^{-i\theta} = z^*$.

9 Write down the complex conjugate of each number by changing i to $-i$:

(a) $2 - i$ (b) $(2 - i)(1 - i)$ (c) $e^{i\pi/2}$ (which is i)
(d) $e^{i\pi} = -1$ (e) $\frac{1+i}{1-i}$ (which is also i) (f) $i^{103} =$ ____.

10 The sum $z + \bar{z}$ is always ____. The difference $z - \bar{z}$ is always ____. Assume $z \neq 0$. The product $z \times \bar{z}$ is always ____. The ratio $z/\bar{z}$ always has absolute value ____.

11 For a real matrix, the conjugate of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is $A\bar{\boldsymbol{x}} = \bar{\lambda}\bar{\boldsymbol{x}}$. This proves two things: $\bar{\lambda}$ is another eigenvalue and $\bar{\boldsymbol{x}}$ is its eigenvector. Find the eigenvalues $\lambda, \bar{\lambda}$ and eigenvectors $\boldsymbol{x}, \bar{\boldsymbol{x}}$ of $A = [a \;\; b; \;\; -b \;\; a]$.

12 The eigenvalues of a real 2 by 2 matrix come from the quadratic formula:

$$\det\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

gives the two eigenvalues $\lambda = \left[a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}\right]/2$.

(a) If $a = b = d = 1$, the eigenvalues are complex when c is ____.

(b) What are the eigenvalues when $ad = bc$?

(c) The two eigenvalues (plus sign and minus sign) are not always conjugates of each other. Why not?

13 In Problem 12 the eigenvalues are not real when $(\text{trace})^2 = (a+d)^2$ is smaller than _____. Show that the λ's *are* real when $bc > 0$.

14 Find the eigenvalues and eigenvectors of this permutation matrix:

$$P_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{has} \quad \det(P_4 - \lambda I) = \underline{\qquad}.$$

15 Extend P_4 above to P_6 (five 1's below the diagonal and one in the corner). Find $\det(P_6 - \lambda I)$ and the six eigenvalues in the complex plane.

16 A real skew-symmetric matrix ($A^{\mathrm{T}} = -A$) has pure imaginary eigenvalues. First proof: If $Ax = \lambda x$ then block multiplication gives

$$\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ ix \end{bmatrix} = i\lambda \begin{bmatrix} x \\ ix \end{bmatrix}.$$

This block matrix is symmetric. Its eigenvalues must be _____! So λ is _____.

Questions 17–24 are about the form $re^{i\theta}$ of the complex number $r\cos\theta + ir\sin\theta$.

17 Write these numbers in Euler's form $re^{i\theta}$. Then square each number:

(a) $1 + \sqrt{3}i$ (b) $\cos 2\theta + i \sin 2\theta$ (c) $-7i$ (d) $5 - 5i$.

18 Find the absolute value and the angle for $z = \sin\theta + i\cos\theta$ (careful). Locate this z in the complex plane. Multiply z by $\cos\theta + i\sin\theta$ to get _____.

19 Draw all eight solutions of $z^8 = 1$ in the complex plane. What is the rectangular form $a + ib$ of the root $z = \overline{w} = \exp(-2\pi i/8)$?

20 Locate the cube roots of 1 in the complex plane. Locate the cube roots of -1. Together these are the sixth roots of _____.

21 By comparing $e^{3i\theta} = \cos 3\theta + i \sin 3\theta$ with $(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$, find the "triple angle" formulas for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

22 Suppose the conjugate $\overline{z}$ is equal to the reciprocal $1/z$. What are all possible z's?

23 (a) Why do e^i and i^e both have absolute value 1?

(b) In the complex plane put stars near the points e^i and i^e.

(c) The number i^e could be $(e^{i\pi/2})^e$ or $(e^{5i\pi/2})^e$. Are those equal?

24 Draw the paths of these numbers from $t = 0$ to $t = 2\pi$ in the complex plane:

(a) e^{it} (b) $e^{(-1+i)t} = e^{-t}e^{it}$ (c) $(-1)^t = e^{t\pi i}$.

10.2 Hermitian and Unitary Matrices

The main message of this section can be presented in one sentence: ***When you transpose a complex vector z or matrix A, take the complex conjugate too.*** Don't stop at z^{T} or A^{T}. Reverse the signs of all imaginary parts. From a column vector with $z_j = a_j + ib_j$, the good row vector is the *conjugate transpose* with components $a_j - ib_j$:

$$\textbf{Conjugate transpose} \qquad \bar{z}^{\mathrm{T}} = \begin{bmatrix} \bar{z}_1 & \cdots & \bar{z}_n \end{bmatrix} = \begin{bmatrix} a_1 - ib_1 & \cdots & a_n - ib_n \end{bmatrix}. \tag{1}$$

Here is one reason to go to $\bar{z}$. The length squared of a real vector is $x_1^2 + \cdots + x_n^2$. The length squared of a complex vector is *not* $z_1^2 + \cdots + z_n^2$. With that wrong definition, the length of $(1, i)$ would be $1^2 + i^2 = 0$. A nonzero vector would have zero length—not good. Other vectors would have complex lengths. Instead of $(a + bi)^2$ we want $a^2 + b^2$, the *absolute value squared.* This is $(a + bi)$ times $(a - bi)$.

For each component we want z_j times $\bar{z}_j$, which is $|z_j|^2 = a_j^2 + b_j^2$. That comes when the components of z multiply the components of $\bar{z}$:

$$\textbf{Length squared} \quad \begin{bmatrix} \bar{z}_1 & \cdots & \bar{z}_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = |z_1|^2 + \cdots + |z_n|^2. \quad \textbf{This is} \quad \bar{z}^{\mathrm{T}} z = \|z\|^2. \tag{2}$$

Now the squared length of $(1, i)$ is $1^2 + |i|^2 = 2$. The length is $\sqrt{2}$. The squared length of $(1 + i, 1 - i)$ is 4. The only vectors with zero length are zero vectors.

The length $\|z\|$ is the square root of $\bar{z}^{\mathrm{T}} z = z^{\mathrm{H}} z = |z_1|^2 + \cdots + |z_n|^2$

Before going further we replace two symbols by one symbol. Instead of a bar for the conjugate and T for the transpose, we just use a superscript H. Thus $\bar{z}^{\mathrm{T}} = z^{\mathrm{H}}$. This is "$z$ Hermitian," the *conjugate transpose* of z. The new word is pronounced "Hermeeshan." The new symbol applies also to matrices: The conjugate transpose of a matrix A is A^{H}.

Another popular notation is A^*. The MATLAB transpose command $'$ automatically takes complex conjugates (A' is A^{H}).

The vector z^{H} is $\bar{z}^{\mathrm{T}}$. The matrix A^{H} is $\bar{A}^{\mathrm{T}}$, the conjugate transpose of A:

$$A^{\mathrm{H}} = \text{“A Hermitian”} \quad \text{If} \quad A = \begin{bmatrix} 1 & i \\ 0 & 1+i \end{bmatrix} \quad \text{then} \quad A^{\mathrm{H}} = \begin{bmatrix} 1 & 0 \\ -i & 1-i \end{bmatrix}$$

Complex Inner Products

For real vectors, the length squared is $x^{\mathrm{T}}x$—*the inner product of x with itself.* For complex vectors, the length squared is $z^{\mathrm{H}}z$. It will be very desirable if $z^{\mathrm{H}}z$ is the inner product of z with itself. To make that happen, the complex inner product should use the conjugate transpose (not just the transpose). The inner product sees no change when the vectors are real, but there is a definite effect from choosing $\bar{u}^{\mathrm{T}}$, when u is complex:

DEFINITION The inner product of real or complex vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$:

$$\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = \begin{bmatrix} \overline{u}_1 & \cdots & \overline{u}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \overline{u}_1 v_1 + \cdots + \overline{u}_n v_n. \tag{3}$$

With complex vectors, $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$ is different from $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u}$. *The order of the vectors is now important.* In fact $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u} = \overline{v}_1 u_1 + \cdots + \overline{v}_n u_n$ is the complex conjugate of $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$. We have to put up with a few inconveniences for the greater good.

Example 1 The inner product of $\boldsymbol{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ with $\boldsymbol{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is $[\,1 \quad -i\,] \begin{bmatrix} i \\ 1 \end{bmatrix} = 0$.

Example 1 is surprising. Those vectors $(1, i)$ and $(i, 1)$ don't look perpendicular. But they are. ***A zero inner product still means that the*** (complex) ***vectors are orthogonal.*** Similarly the vector $(1, i)$ is orthogonal to the vector $(1, -i)$. Their inner product is $1 - 1 = 0$. We are correctly getting zero for the inner product—where we would be incorrectly getting zero for the length of $(1, i)$ if we forgot to take the conjugate.

Note We have chosen to conjugate the first vector $\boldsymbol{u}$. Some authors choose the second vector $\boldsymbol{v}$. Their complex inner product would be $\boldsymbol{u}^{\mathrm{T}}\overline{\boldsymbol{v}}$. It is a free choice, as long as we stick to it. We wanted to use the single symbol $^{\mathrm{H}}$ in the next formula too:

The inner product of $A\boldsymbol{u}$ with $\boldsymbol{v}$ equals the inner product of $\boldsymbol{u}$ with $A^{\mathrm{H}}\boldsymbol{v}$:

$$A^{\mathrm{H}} = \text{“adjoint” of } A \qquad (A\boldsymbol{u})^{\mathrm{H}}\boldsymbol{v} = \boldsymbol{u}^{\mathrm{H}}(A^{\mathrm{H}}\boldsymbol{v}). \tag{4}$$

The conjugate of $A\boldsymbol{u}$ is $\overline{A\boldsymbol{u}}$. Transposing it gives $\overline{\boldsymbol{u}}^{\mathrm{T}}\overline{A}^{\mathrm{T}}$ as usual. This is $\boldsymbol{u}^{\mathrm{H}}A^{\mathrm{H}}$. Everything that should work, does work. The rule for $^{\mathrm{H}}$ comes from the rule for $^{\mathrm{T}}$. That applies to products of matrices:

The conjugate transpose of AB is $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$.

We constantly use the fact that $(a - ib)(c - id)$ is the conjugate of $(a + ib)(c + id)$.

Hermitian Matrices

Among real matrices, the *symmetric matrices* form the most important special class: $A = A^{\mathrm{T}}$. They have real eigenvalues and a full set of orthogonal eigenvectors. The diagonalizing matrix S is an orthogonal matrix Q. Every symmetric matrix can be written as $A = Q\Lambda Q^{-1}$ and also as $A = Q\Lambda Q^{\mathrm{T}}$ (because $Q^{-1} = Q^{\mathrm{T}}$). All this follows from $a_{ij} = a_{ji}$, when A is real.

Among complex matrices, the special class contains the ***Hermitian matrices***: $A = A^{\mathrm{H}}$. The condition on the entries is $a_{ij} = \overline{a_{ji}}$. In this case we say that "*A is* Hermitian." *Every real symmetric matrix is Hermitian*, because taking its conjugate has no effect. The next matrix is also Hermitian, $A = A^{\mathrm{H}}$:

Example 2 $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ The main diagonal is real since $a_{ii} = \overline{a_{ii}}$. Across it are conjugates $3+3i$ and $3-3i$.

This example will illustrate the three crucial properties of all Hermitian matrices.

If $A = A^{\mathrm{H}}$ and z is any vector, the number $z^{\mathrm{H}}Az$ is real.

Quick proof: $z^{\mathrm{H}}Az$ is certainly 1 by 1. Take its conjugate transpose:

$$(z^{\mathrm{H}}Az)^{\mathrm{H}} = z^{\mathrm{H}}A^{\mathrm{H}}(z^{\mathrm{H}})^{\mathrm{H}} \quad \text{which is } z^{\mathrm{H}}Az \text{ again.}$$

This used $A = A^{\mathrm{H}}$. So the number $z^{\mathrm{H}}Az$ equals its conjugate and must be real. Here is that "energy" $z^{\mathrm{H}}Az$ in our example:

$$\begin{bmatrix} \overline{z}_1 & \overline{z}_2 \end{bmatrix} \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{2\overline{z}_1 z_1 + 5\overline{z}_2 z_2}_{\textit{diagonal}} + \underbrace{(3-3i)\overline{z}_1 z_2 + (3+3i) z_1 \overline{z}_2}_{\textit{off-diagonal}}.$$

The terms $2|z_1|^2$ and $5|z_2|^2$ from the diagonal are both real. The off-diagonal terms are conjugates of each other—so their sum is real. (The imaginary parts cancel when we add.) The whole expression $z^{\mathrm{H}}Az$ is real, and this will make λ real.

Every eigenvalue of a Hermitian matrix is real.

Proof Suppose $Az = \lambda z$. *Multiply both sides by z^{H} to get $z^{\mathrm{H}}Az = \lambda z^{\mathrm{H}}z$.* On the left side, $z^{\mathrm{H}}Az$ is real. On the right side, $z^{\mathrm{H}}z$ is the length squared, real and positive. So the ratio $\lambda = z^{\mathrm{H}}Az/z^{\mathrm{H}}z$ is a real number. Q.E.D.

The example above has eigenvalues $\lambda = 8$ and $\lambda = -1$, real because $A = A^{\mathrm{H}}$:

$$\begin{vmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 - |3+3i|^2$$
$$= \lambda^2 - 7\lambda + 10 - 18 = (\lambda - 8)(\lambda + 1).$$

The eigenvectors of a Hermitian matrix are orthogonal (when they correspond to different eigenvalues). If $Az = \lambda z$ and $Ay = \beta y$ then $y^{\mathrm{H}}z = 0$.

Proof Multiply $Az = \lambda z$ on the left by y^{H}. Multiply $y^{\mathrm{H}}A^{\mathrm{H}} = \beta y^{\mathrm{H}}$ on the right by z:

$$y^{\mathrm{H}}Az = \lambda y^{\mathrm{H}}z \quad \text{and} \quad y^{\mathrm{H}}A^{\mathrm{H}}z = \beta y^{\mathrm{H}}z. \tag{5}$$

The left sides are equal because $A = A^{\mathrm{H}}$. Therefore the right sides are equal. Since β is different from λ, the other factor $y^{\mathrm{H}}z$ must be zero. The eigenvectors are orthogonal, as in our example with $\lambda = 8$ and $\beta = -1$:

$$(A - 8I)z = \begin{bmatrix} -6 & 3-3i \\ 3+3i & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$(A + I)y = \begin{bmatrix} 3 & 3-3i \\ 3+3i & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}.$$

Take the inner product of those eigenvectors y and z:

Orthogonal eigenvectors $$y^{\mathrm{H}}z = \begin{bmatrix} 1+i & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = 0.$$

These eigenvectors have squared length $1^2 + 1^2 + 1^2 = 3$. After division by $\sqrt{3}$ they are unit vectors. They were orthogonal, now they are ***orthonormal***. They go into the columns of the *eigenvector matrix* S, which diagonalizes A.

When A is real and symmetric, S is Q—an orthogonal matrix. Now A is complex and Hermitian. Its eigenvectors are complex and orthonormal. ***The eigenvector matrix S is like Q, but complex.*** We now assign a new name "unitary" and a new letter U to a complex orthogonal matrix.

Unitary Matrices

A ***unitary matrix*** U is a (complex) square matrix that has ***orthonormal columns***. U is the complex equivalent of Q. The eigenvectors of A give a perfect example:

Unitary matrix $$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

This U is also a Hermitian matrix. I didn't expect that! The example is almost too perfect. We will see that the eigenvalues of this U must be 1 and -1.

The matrix test for real orthonormal columns was $Q^{\mathrm{T}}Q = I$. When Q^{T} multiplies Q, the zero inner products appear off the diagonal. In the complex case, Q becomes U. The columns show themselves as orthonormal when U^{H} multiplies U. The inner products of the columns are again 1 and 0. They fill up $\boldsymbol{U^{\mathrm{H}}U = I}$:

Every matrix U with orthonormal columns has $\boldsymbol{U^{\mathrm{H}}U = I}$.

If U is square, it is a unitary matrix. Then $\boxed{\boldsymbol{U^{\mathrm{H}} = U^{-1}}}$.

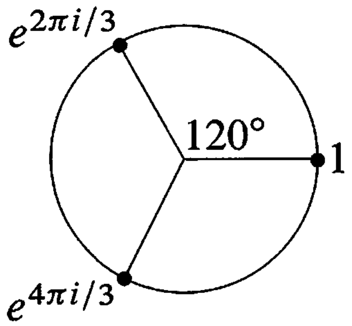
Suppose U (with orthonormal columns) multiplies any z. The vector length stays the same, because $z^{\mathrm{H}}U^{\mathrm{H}}Uz = z^{\mathrm{H}}z$. If z is an eigenvector of U we learn something more: ***The eigenvalues of unitary (and orthogonal) matrices all have absolute value*** $|\lambda| = 1$.

If U is unitary then $\|Uz\| = \|z\|$. Therefore $Uz = \lambda z$ leads to $|\lambda| = 1$.

Our 2 by 2 example is both Hermitian ($U = U^{\mathrm{H}}$) and unitary ($U^{-1} = U^{\mathrm{H}}$). That means real eigenvalues ($\lambda = \overline{\lambda}$), and it means $|\lambda| = 1$. A real number with absolute value 1 has only two possibilities: *The eigenvalues are* 1 *or* -1.

Since the trace is zero for our U, one eigenvalue is $\lambda = 1$ and the other is $\lambda = -1$.

Example 3 The 3 by 3 ***Fourier matrix*** is in Figure 10.3. Is it Hermitian? Is it unitary? F_3 is certainly symmetric. It equals its transpose. But it doesn't equal its conjugate transpose—*it is not Hermitian.* If you change i to $-i$, you get a different matrix.



Fourier matrix $$F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}.$$

Figure 10.3: The cube roots of 1 go into the Fourier matrix $F = F_3$.

Is F unitary? *Yes.* The squared length of every column is $\frac{1}{3}(1 + 1 + 1)$ (unit vector). The first column is orthogonal to the second column because $1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$. This is the sum of the three numbers marked in Figure 10.3.

Notice the symmetry of the figure. If you rotate it by 120°, the three points are in the same position. Therefore their sum S also stays in the same position! The only possible sum in the same position after 120° rotation is $S = 0$.

Is column 2 of F orthogonal to column 3? Their dot product looks like

$$\tfrac{1}{3}(1 + e^{6\pi i/3} + e^{6\pi i/3}) = \tfrac{1}{3}(1 + 1 + 1).$$

This is not zero. The answer is wrong because we forgot to take complex conjugates. The complex inner product uses $^{\mathrm{H}}$ not $^{\mathrm{T}}$:

$$\begin{aligned}(\text{column } 2)^{\mathrm{H}}(\text{column } 3) &= \tfrac{1}{3}(1 \cdot 1 + e^{-2\pi i/3} e^{4\pi i/3} + e^{-4\pi i/3} e^{2\pi i/3}) \\ &= \tfrac{1}{3}(1 + e^{2\pi i/3} + e^{-2\pi i/3}) = 0.\end{aligned}$$

So we do have orthogonality. ***Conclusion: F is a unitary matrix.***

The next section will study the n by n Fourier matrices. Among all complex unitary matrices, these are the most important. When we multiply a vector by F, we are computing its ***Discrete Fourier Transform.*** When we multiply by F^{-1}, we are computing the *inverse transform.* The special property of unitary matrices is that $F^{-1} = F^{\mathrm{H}}$. The inverse transform only differs by changing i to $-i$:

Change i to $-i$ $$F^{-1} = F^{\mathrm{H}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{-4\pi i/3} \\ 1 & e^{-4\pi i/3} & e^{-2\pi i/3} \end{bmatrix}.$$

Everyone who works with F recognizes its value. The last section of the book will bring together Fourier analysis and complex numbers and linear algebra.

This section ends with a table to translate between real and complex—for vectors and for matrices:

Real versus Complex

$$\begin{array}{rcl}
\mathbf{R}^n\text{: vectors with } n \text{ real components} & \leftrightarrow & \mathbf{C}^n\text{: vectors with } n \text{ complex components} \\
\text{length: } \|x\|^2 = x_1^2 + \cdots + x_n^2 & \leftrightarrow & \text{length: } \|z\|^2 = |z_1|^2 + \cdots + |z_n|^2 \\
\text{transpose: } (A^{\mathrm{T}})_{ij} = A_{ji} & \leftrightarrow & \text{conjugate transpose: } (A^{\mathrm{H}})_{ij} = \overline{A_{ji}} \\
\text{product rule: } (AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}} & \leftrightarrow & \text{product rule: } (AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}} \\
\text{dot product: } \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = x_1y_1 + \cdots + x_ny_n & \leftrightarrow & \text{inner product: } \boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = \overline{u}_1v_1 + \cdots + \overline{u}_nv_n \\
\text{reason for } A^{\mathrm{T}}\text{: } (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) & \leftrightarrow & \text{reason for } A^{\mathrm{H}}\text{: } (A\boldsymbol{u})^{\mathrm{H}}\boldsymbol{v} = \boldsymbol{u}^{\mathrm{H}}(A^{\mathrm{H}}\boldsymbol{v}) \\
\text{orthogonality: } \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0 & \leftrightarrow & \text{orthogonality: } \boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = 0 \\
\text{symmetric matrices: } A = A^{\mathrm{T}} & \leftrightarrow & \text{Hermitian matrices: } A = A^{\mathrm{H}} \\
A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}(\text{real } \Lambda) & \leftrightarrow & A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}} \ (\text{real } \Lambda) \\
\text{skew-symmetric matrices: } K^{\mathrm{T}} = -K & \leftrightarrow & \text{skew-Hermitian matrices } K^{\mathrm{H}} = -K \\
\text{orthogonal matrices: } Q^{\mathrm{T}} = Q^{-1} & \leftrightarrow & \text{unitary matrices: } U^{\mathrm{H}} = U^{-1} \\
\text{orthonormal columns: } Q^{\mathrm{T}}Q = I & \leftrightarrow & \text{orthonormal columns: } U^{\mathrm{H}}U = I \\
(Q\boldsymbol{x})^{\mathrm{T}}(Q\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} \text{ and } \|Q\boldsymbol{x}\| = \|\boldsymbol{x}\| & \leftrightarrow & (U\boldsymbol{x})^{\mathrm{H}}(U\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{H}}\boldsymbol{y} \text{ and } \|U\boldsymbol{z}\| = \|\boldsymbol{z}\|
\end{array}$$

The columns and also the eigenvectors of Q and U are orthonormal. Every $|\lambda| = 1$.

Problem Set 10.2

1 Find the lengths of $\boldsymbol{u} = (1+i, 1-i, 1+2i)$ and $\boldsymbol{v} = (i, i, i)$. Also find $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$ and $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u}$.

2 Compute $A^{\mathrm{H}}A$ and AA^{H}. Those are both _____ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

3 Solve $Az = \mathbf{0}$ to find a vector in the nullspace of A in Problem 2. Show that z is orthogonal to the columns of A^{H}. Show that z is *not* orthogonal to the columns of A^{T}. ***The good row space is no longer*** $\boldsymbol{C}(A^{\mathrm{T}})$. Now it is $\boldsymbol{C}(A^{\mathrm{H}})$.

4 Problem 3 indicates that the four fundamental subspaces are $C(A)$ and $N(A)$ and _____ and _____. Their dimensions are still r and $n-r$ and r and $m-r$. They are still orthogonal subspaces. *The symbol* $^{\mathrm{H}}$ *takes the place of* $^{\mathrm{T}}$.

5 (a) Prove that $A^{\mathrm{H}}A$ is always a Hermitian matrix.

(b) If $Az = \mathbf{0}$ then $A^{\mathrm{H}}Az = \mathbf{0}$. If $A^{\mathrm{H}}Az = \mathbf{0}$, multiply by z^{H} to prove that $Az = \mathbf{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are _____. Therefore $A^{\mathrm{H}}A$ is an invertible Hermitian matrix when the nullspace of A contains only $z = 0$.

6 True or false (give a reason if true or a counterexample if false):

(a) If A is a real matrix then $A + iI$ is invertible.

(b) If A is a Hermitian matrix then $A + iI$ is invertible.

(c) If U is a unitary matrix then $A + iI$ is invertible.

7 When you multiply a Hermitian matrix by a real number c, is cA still Hermitian? Show that iA is skew-Hermitian when A is Hermitian. The 3 by 3 Hermitian matrices are a subspace provided the "scalars" are real numbers.

8 Which classes of matrices does P belong to: invertible, Hermitian, unitary?

$$P = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}.$$

Compute P^2, P^3, and P^{100}. What are the eigenvalues of P?

9 Find the unit eigenvectors of P in Problem 8, and put them into the columns of a unitary matrix F. What property of P makes these eigenvectors orthogonal?

10 Write down the 3 by 3 circulant matrix $C = 2I + 5P$. It has the same eigenvectors as P in Problem 8. Find its eigenvalues.

11 If U and V are unitary matrices, show that U^{-1} is unitary and also UV is unitary. Start from $U^{\mathrm{H}}U = I$ and $V^{\mathrm{H}}V = I$.

12 How do you know that the determinant of every Hermitian matrix is real?

13 The matrix $A^{\mathrm{H}}A$ is not only Hermitian but also positive definite, when the columns of A are independent. Proof: $z^{\mathrm{H}}A^{\mathrm{H}}Az$ is positive if z is nonzero because _____.

14 Diagonalize this Hermitian matrix to reach $A = U\Lambda U^{\mathrm{H}}$:

$$A = \begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix}.$$

15 Diagonalize this skew-Hermitian matrix to reach $K = U\Lambda U^{\mathrm{H}}$. All λ's are ____:

$$K = \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}.$$

16 Diagonalize this orthogonal matrix to reach $Q = U\Lambda U^{\mathrm{H}}$. Now all λ's are ____:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

17 Diagonalize this unitary matrix V to reach $V = U\Lambda U^{\mathrm{H}}$. Again all λ's are ____:

$$V = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}.$$

18 If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthonormal basis for $\mathbf{C}^n$, the matrix with those columns is a ____ matrix. Show that any vector $\boldsymbol{z}$ equals $(\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{z})\boldsymbol{v}_1 + \cdots + (\boldsymbol{v}_n^{\mathrm{H}}\boldsymbol{z})\boldsymbol{v}_n$.

19 The functions e^{-ix} and e^{ix} are orthogonal on the interval $0 \le x \le 2\pi$ because their inner product is $\int_0^{2\pi}$ ____ $= 0$.

20 The vectors $\boldsymbol{v} = (1, i, 1)$, $\boldsymbol{w} = (i, 1, 0)$ and $\boldsymbol{z} =$ ____ are an orthogonal basis for ____.

21 If $A = R + iS$ is a Hermitian matrix, are its real and imaginary parts symmetric?

22 The (complex) dimension of $\mathbf{C}^n$ is ____. Find a non-real basis for $\mathbf{C}^n$.

23 Describe all 1 by 1 and 2 by 2 Hermitian matrices and unitary matrices.

24 How are the eigenvalues of A^{H} related to the eigenvalues of the square complex matrix A?

25 If $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{u} = 1$ show that $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ is Hermitian and also unitary. The rank-one matrix $\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ is the projection onto what line in $\mathbf{C}^n$?

26 If $A + iB$ is a unitary matrix (A and B are real) show that $Q = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is an orthogonal matrix.

27 If $A + iB$ is Hermitian (A and B are real) show that $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric.

28 Prove that the inverse of a Hermitian matrix is also Hermitian (transpose $A^{-1}A = I$).

29 Diagonalize this matrix by constructing its eigenvalue matrix Λ and its eigenvector matrix S:

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix} = A^{\mathrm{H}}.$$

30 A matrix with orthonormal eigenvectors has the form $A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$. *Prove that* $AA^{\mathrm{H}} = A^{\mathrm{H}}A$. These are exactly the **normal matrices**. Examples are Hermitian, skew-Hermitian, and unitary matrices. Construct a 2 by 2 normal matrix by choosing complex eigenvalues in Λ.

10.3 The Fast Fourier Transform

Many applications of linear algebra take time to develop. It is not easy to explain them in an hour. The teacher and the author must choose between completing the theory and adding new applications. Often the theory wins, but this section is an exception. It explains the most valuable numerical algorithm in the last century.

We want to multiply quickly by F and F^{-1}, the Fourier matrix and its inverse. This is achieved by the Fast Fourier Transform. An ordinary product Fc uses n^2 multiplications (F has n^2 entries). The FFT needs only n times $\frac{1}{2}\log_2 n$. We will see how.

The FFT has revolutionized signal processing. Whole industries are speeded up by this one idea. Electrical engineers are the first to know the difference—they take your Fourier transform as they meet you (if you are a function). Fourier's idea is to represent f as a sum of harmonics $c_k e^{ikx}$. The function is seen in *frequency space* through the coefficients c_k, instead of *physical space* through its values $f(x)$. The passage backward and forward between c's and f's is by the Fourier transform. Fast passage is by the FFT.

Roots of Unity and the Fourier Matrix

Quadratic equations have two roots (or one repeated root). Equations of degree n have n roots (counting repetitions). This is the Fundamental Theorem of Algebra, and to make it true we must allow complex roots. This section is about the very special equation $z^n = 1$. The solutions z are the "nth roots of unity." They are n evenly spaced points around the unit circle in the complex plane.

Figure 10.4 shows the eight solutions to $z^8 = 1$. Their spacing is $\frac{1}{8}(360°) = 45°$. The first root is at $45°$ or $\theta = 2\pi/8$ radians. ***It is the complex number $w = e^{i\theta} = e^{i2\pi/8}$.*** We call this number w_8 to emphasize that it is an 8th root. You could write it in terms of $\cos\frac{2\pi}{8}$ and $\sin\frac{2\pi}{8}$, but don't do it. The seven other 8th roots are $w^2, w^3, \ldots, w^8$, going around the circle. Powers of w are best in polar form, because we work only with the angles $\frac{2\pi}{8}, \frac{4\pi}{8}, \ldots, \frac{16\pi}{8} = 2\pi$.

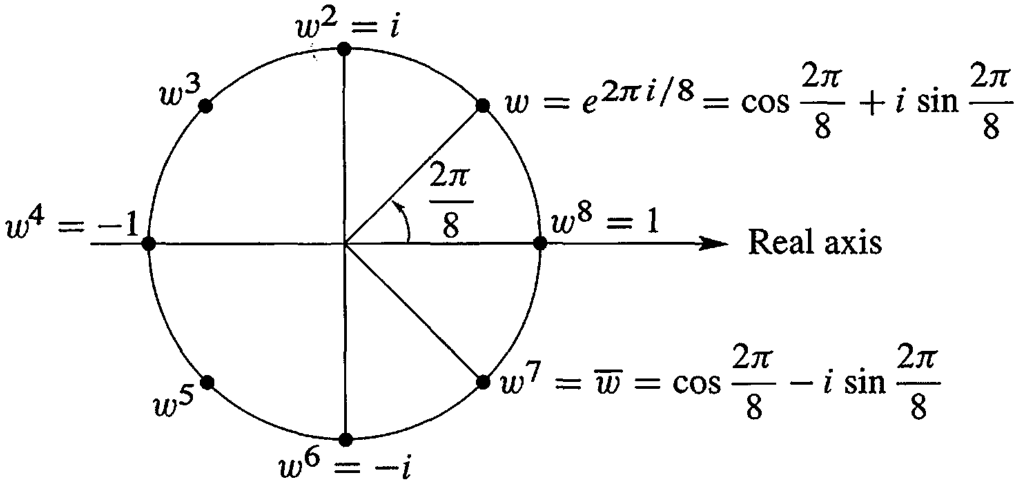


Figure 10.4: The eight solutions to $z^8 = 1$ are $1, w, w^2, \ldots, w^7$ with $w = (1+i)/\sqrt{2}$.

The fourth roots of 1 are also in the figure. They are $i, -1, -i, 1$. The angle is now $2\pi/4$ or $90°$. The first root $w_4 = e^{2\pi i/4}$ is nothing but i. Even the square roots of 1 are seen, with $w_2 = e^{i2\pi/2} = -1$. Do not despise those square roots 1 and -1. The idea behind the FFT is to go from an 8 by 8 Fourier matrix (containing powers of w_8) to the 4 by 4 matrix below (with powers of $w_4 = i$). The same idea goes from 4 to 2. By exploiting the connections of F_8 down to F_4 and up to F_{16} (and beyond), the FFT makes multiplication by F_{1024} very quick.

We describe the *Fourier matrix*, first for $n = 4$. Its rows contain powers of 1 and w and w^2 and w^3. These are the fourth roots of 1, and their powers come in a special order.

Fourier matrix $n = 4$

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}.$$

The matrix is symmetric ($F = F^{\mathrm{T}}$). It is *not* Hermitian. Its main diagonal is not real. But $\frac{1}{2}F$ is a ***unitary matrix***, which means that $(\frac{1}{2}F^{\mathrm{H}})(\frac{1}{2}F) = I$:

The columns of F give $\boldsymbol{F^{\mathrm{H}}F = 4I}$. ***Its inverse is*** $\boldsymbol{\frac{1}{4}F^{\mathrm{H}}}$ ***which is*** $\boldsymbol{F^{-1} = \frac{1}{4}\overline{F}}$.

The inverse changes from $w = i$ to $\overline{w} = -i$. That takes us from F to $\overline{F}$. When the Fast Fourier Transform gives a quick way to multiply by F, it does the same for F^{-1}.

The unitary matrix is $U = F/\sqrt{n}$. We avoid that $\sqrt{n}$ and just put $\frac{1}{n}$ outside F^{-1}. The main point is to multiply F times the Fourier coefficients c_0, c_1, c_2, c_3:

4-point Fourier series

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = Fc = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{1}$$

The input is four complex coefficients c_0, c_1, c_2, c_3. The output is four function values y_0, y_1, y_2, y_3. The first output $y_0 = c_0 + c_1 + c_2 + c_3$ is the value of the Fourier series at $x = 0$. *The second output is the value of that series* $\sum c_k e^{ikx}$ *at* $x = 2\pi/4$:

$$y_1 = c_0 + c_1 e^{i2\pi/4} + c_2 e^{i4\pi/4} + c_3 e^{i6\pi/4} = c_0 + c_1 w + c_2 w^2 + c_3 w^3.$$

The third and fourth outputs y_2 and y_3 are the values of $\sum c_k e^{ikx}$ at $x = 4\pi/4$ and $x = 6\pi/4$. These are *finite* Fourier series! They contain $n = 4$ terms and they are evaluated at $n = 4$ points. Those points $x = 0, 2\pi/4, 4\pi/4, 6\pi/4$ are equally spaced.

The next point would be $x = 8\pi/4$ which is 2π. Then the series is back to y_0, because $e^{2\pi i}$ is the same as $e^0 = 1$. Everything cycles around with period 4. In this world $2 + 2$ is 0 because $(w^2)(w^2) = w^0 = 1$. We will follow the convention that j *and* k *go from* 0 *to* $n - 1$ (instead of 1 to n). The "zeroth row" and "zeroth column" of F contain all ones.

The n by n Fourier matrix contains powers of $w = e^{2\pi i/n}$:

$$F_n c = \begin{bmatrix} 1 & 1 & 1 & \cdot & 1 \\ 1 & w & w^2 & \cdot & w^{n-1} \\ 1 & w^2 & w^4 & \cdot & w^{2(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & w^{n-1} & w^{2(n-1)} & \cdot & w^{(n-1)^2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \cdot \\ y_{n-1} \end{bmatrix} = y. \qquad (2)$$

F_n is symmetric but not Hermitian. ***Its columns are orthogonal***, and $F_n \overline{F}_n = nI$. *Then* F_n^{-1} *is* $\overline{F}_n/n$. The inverse contains powers of $\overline{w}_n = e^{-2\pi i/n}$. Look at the pattern in F:

The entry in row j, column k is w^{jk}. Row zero and column zero contain $w^0 = 1$.

When we multiply c by F_n, we sum the series at n points. *When we multiply y by F_n^{-1}, we find the coefficients c from the function values y.* In MATLAB that command is c = fft(y). The matrix F passes from "frequency space" to "physical space."

Important note. Many authors prefer to work with $\omega = e^{-2\pi i/N}$, which is the *complex conjugate* of our w. (They often use the Greek omega, and I will do that to keep the two options separate.) With this choice, their DFT matrix contains powers of ω not w. It is conj (F) = complex conjugate of our F. This takes us to frequency space.

$\overline{F}$ is a completely reasonable choice! MATLAB uses $\omega = e^{-2\pi i/N}$. The DFT matrix fft(eye(N)) contains powers of this number $\omega = \overline{w}$. **The Fourier matrix with w's reconstructs y from c.** ***The matrix $\overline{F}$ with ω's computes Fourier coefficients as* fft(y).**

Also important. When a function $f(x)$ has period 2π, and we change x to $e^{i\theta}$, the function is defined around the unit circle (where $z = e^{i\theta}$). Then the Discrete Fourier Transform from y to c is matching n values of this $f(z)$ by a polynomial $p(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}$.

Interpolation Find $c_0, \ldots, c_{n-1}$ so that $p(z) = f(z)$ at n points $z = 1, \ldots, w^{n-1}$

The Fourier matrix is the Vandermonde matrix for interpolation at those n points.

One Step of the Fast Fourier Transform

We want to multiply F times c as quickly as possible. Normally a matrix times a vector takes n^2 separate multiplications—the matrix has n^2 entries. You might think it is impossible to do better. (If the matrix has zero entries then multiplications can be skipped. But the Fourier matrix has no zeros!) By using the special pattern w^{jk} for its entries, F can be factored in a way that produces many zeros. This is the **FFT**.

The key idea is to connect F_n with the half-size Fourier matrix $F_{n/2}$. Assume that n is a power of 2 (say $n = 2^{10} = 1024$). We will connect F_{1024} to F_{512}—or rather to ***two***

copies of F_{512}. When $n = 4$, the key is in the relation between these matrices:

$$\boldsymbol{F_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \boldsymbol{F_2} & \\ & \boldsymbol{F_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}.$$

On the left is F_4, with no zeros. On the right is a matrix that is half zero. The work is cut in half. But wait, those matrices are not the same. We need two sparse and simple matrices to complete the FFT factorization:

Factors for FFT
$$F_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}. \tag{3}$$

The last matrix is a permutation. It puts the even c's (c_0 and c_2) ahead of the odd c's (c_1 and c_3). The middle matrix performs half-size transforms F_2 and F_2 on the evens and odds. The matrix at the left combines the two half-size outputs—in a way that produces the correct full-size output $\boldsymbol{y} = F_4\boldsymbol{c}$.

The same idea applies when $n = 1024$ and $m = \frac{1}{2}n = 512$. The number w is $e^{2\pi i/1024}$. It is at the angle $\theta = 2\pi/1024$ on the unit circle. The Fourier matrix F_{1024} is full of powers of w. The first stage of the FFT is the great factorization discovered by Cooley and Tukey (and foreshadowed in 1805 by Gauss):

$$\boldsymbol{F_{1024}} = \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} \boldsymbol{F_{512}} & \\ & \boldsymbol{F_{512}} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}. \tag{4}$$

I_{512} is the identity matrix. D_{512} is the diagonal matrix with entries $(1, w, \ldots, w^{511})$. The two copies of F_{512} are what we expected. Don't forget that they use the 512th root of unity (which is nothing but w^2!!) The permutation matrix separates the incoming vector $\boldsymbol{c}$ into its even and odd parts $\boldsymbol{c}' = (c_0, c_2, \ldots, c_{1022})$ and $\boldsymbol{c}'' = (c_1, c_3, \ldots, c_{1023})$.

Here are the algebra formulas which say the same thing as the factorization of F_{1024}:

(FFT) Set $m = \frac{1}{2}n$. The first m and last m components of $\boldsymbol{y} = F_n\boldsymbol{c}$ combine the half-size transforms $\boldsymbol{y}' = F_m\boldsymbol{c}'$ and $\boldsymbol{y}'' = F_m\boldsymbol{c}''$. Equation (4) shows this step from n to $m = n/2$ as $I\boldsymbol{y}' + D\boldsymbol{y}''$ and $I\boldsymbol{y}' - D\boldsymbol{y}''$:

$$\begin{aligned} y_j &= y'_j + w_n^j y''_j, \quad j = 0, \ldots, m-1 \\ y_{j+m} &= y'_j - w_n^j y''_j, \quad j = 0, \ldots, m-1. \end{aligned} \tag{5}$$

Split $\boldsymbol{c}$ into $\boldsymbol{c}'$ and $\boldsymbol{c}''$, transform them by F_m into $\boldsymbol{y}'$ and $\boldsymbol{y}''$, and reconstruct $\boldsymbol{y}$.

Those formulas come from separating even c_{2k} from odd c_{2k+1}:

$$y_j = \sum_0^{n-1} w^{jk} c_k = \sum_0^{m-1} w^{2jk} c_{2k} + \sum_0^{m-1} w^{j(2k+1)} c_{2k+1} \text{ with } m = \tfrac{1}{2}n. \tag{6}$$

The even c's go into $c' = (c_0, c_2, \ldots)$ and the odd c's go into $c'' = (c_1, c_3, \ldots)$. Then come the transforms $F_m c'$ and $F_m c''$. The key is $\boldsymbol{w_n^2 = w_m}$. This gives $w_n^{2jk} = w_m^{jk}$.

Rewrite $$y_j = \sum w_m^{jk} c'_k + (w_n)^j \sum w_m^{jk} c''_k = y'_j + (w_n)^j y''_j. \tag{7}$$

For $j \geq m$, the minus sign in (5) comes from factoring out $(w_n)^m = -1$.

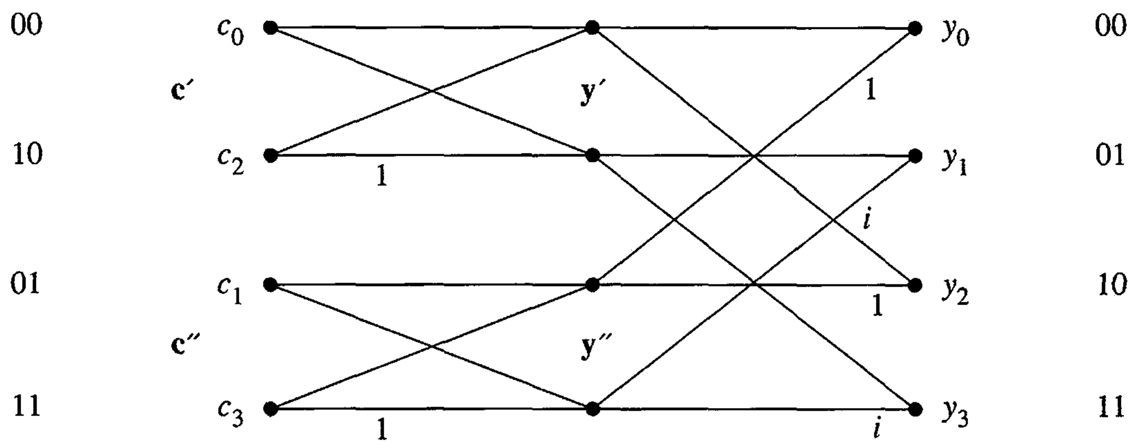
MATLAB easily separates even c's from odd c's and multiplies by w_n^j. We use conj(F) or equivalently MATLAB's inverse transform ifft, because fft is based on $\omega = \overline{w} = e^{-2\pi i/n}$. Problem 17 shows that F and conj(F) are linked by permuting rows.

FFT step from n to $n/2$ in MATLAB

```
y' = ifft (c(0 : 2 : n - 2)) * n/2;
y'' = ifft (c(1 : 2 : n - 1)) * n/2;
d = w.^(0 : n/2 - 1)';
y = [y' + d.* y''; y' - d.* y''];
```

The flow graph shows c' and c'' going through the half-size F_2. Those steps are called "*butterflies*," from their shape. Then the outputs y' and y'' are combined (multiplying y'' by $1, i$ and also by $-1, -i$) to produce $y = F_4 c$.

This reduction from F_n to two F_m's almost cuts the work in half—you see the zeros in the matrix factorization. That reduction is good but not great. The full idea of the **FFT** is much more powerful. It saves much more than half the time.



The Full FFT by Recursion

If you have read this far, you have probably guessed what comes next. We reduced F_n to $F_{n/2}$. ***Keep going to*** $F_{n/4}$. The matrices F_{512} lead to F_{256} (in four copies). Then 256 leads to 128. *That is recursion.* It is a basic principle of many fast algorithms, and here is the second stage with four copies of $F = F_{256}$ and $D = D_{256}$:

$$\begin{bmatrix} F_{512} & \\ & F_{512} \end{bmatrix} = \begin{bmatrix} I & D & & \\ I & -D & & \\ & & I & D \\ & & I & -D \end{bmatrix} \begin{bmatrix} F & & & \\ & F & & \\ & & F & \\ & & & F \end{bmatrix} \begin{bmatrix} \text{pick} & 0, 4, 8, \cdots \\ \text{pick} & 2, 6, 10, \cdots \\ \text{pick} & 1, 5, 9, \cdots \\ \text{pick} & 3, 7, 11, \cdots \end{bmatrix}.$$

We will count the individual multiplications, to see how much is saved. Before the **FFT** was invented, the count was the usual $n^2 = (1024)^2$. This is about a million multiplications. I am not saying that they take a long time. The cost becomes large when we have many, many transforms to do—which is typical. Then the saving by the FFT is also large:

The final count for size $n = 2^\ell$ is reduced from n^2 to $\frac{1}{2}n\ell$.

The number 1024 is 2^{10}, so $\ell = 10$. The original count of $(1024)^2$ is reduced to (5)(1024). The saving is a factor of 200. A million is reduced to five thousand. That is why the FFT has revolutionized signal processing.

Here is the reasoning behind $\frac{1}{2}n\ell$. There are ℓ levels, going from $n = 2^\ell$ down to $n = 1$. Each level has $n/2$ multiplications from the diagonal D's, to reassemble the half-size outputs from the lower level. This yields the final count $\frac{1}{2}n\ell$, which is $\frac{1}{2}n \log_2 n$.

One last note about this remarkable algorithm. There is an amazing rule for the order that the c's enter the FFT, after all the even-odd permutations. Write the numbers 0 to $n-1$ in binary (base 2). *Reverse the order of their digits.* The complete picture shows the bit-reversed order at the start, the $\ell = \log_2 n$ steps of the recursion, and the final output $y_0, \ldots, y_{n-1}$ which is F_n times $\boldsymbol{c}$.

The book ends with that very fundamental idea, a matrix multiplying a vector.

Thank you for studying linear algebra. I hope you enjoyed it, and I very much hope you will use it. It was a pleasure to write about this tremendously useful subject.

Problem Set 10.3

1 Multiply the three matrices in equation (3) and compare with F. In which six entries do you need to know that $i^2 = -1$?

2 Invert the three factors in equation (3) to find a fast factorization of F^{-1}.

3 F is symmetric. So transpose equation (3) to find a new Fast Fourier Transform!

4 All entries in the factorization of F_6 involve powers of w_6 = sixth root of 1:

$$F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & \\ & F_3 \end{bmatrix} \begin{bmatrix} & P & \end{bmatrix}.$$

Write down these matrices with $1, w_6, w_6^2$ in D and $w_3 = w_6^2$ in F_3. Multiply!

5 If $\boldsymbol{v} = (1,0,0,0)$ and $\boldsymbol{w} = (1,1,1,1)$, show that $F\boldsymbol{v} = \boldsymbol{w}$ and $F\boldsymbol{w} = 4\boldsymbol{v}$. Therefore $F^{-1}\boldsymbol{w} = \boldsymbol{v}$ and $F^{-1}\boldsymbol{v} =$ _____.

6 What is F^2 and what is F^4 for the 4 by 4 Fourier matrix?

7 Put the vector $\boldsymbol{c} = (1,0,1,0)$ through the three steps of the FFT to find $\boldsymbol{y} = F\boldsymbol{c}$. Do the same for $\boldsymbol{c} = (0,1,0,1)$.

8 Compute $y = F_8 c$ by the three FFT steps for $c = (1,0,1,0,1,0,1,0)$. Repeat the computation for $c = (0,1,0,1,0,1,0,1)$.

9 If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the _____ and _____ roots of 1.

10 (a) Draw all the sixth roots of 1 on the unit circle. Prove they add to zero.

(b) What are the three cube roots of 1? Do they also add to zero?

11 The columns of the Fourier matrix F are the *eigenvectors* of the cyclic permutation P. Multiply PF to find the eigenvalues λ_1 to λ_4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}.$$

This is $PF = F\Lambda$ or $P = F\Lambda F^{-1}$. The eigenvector matrix (usually S) is F.

12 The equation $\det(P - \lambda I) = 0$ is $\lambda^4 = 1$. This shows again that the eigenvalue matrix Λ is _____. Which permutation P has eigenvalues = cube roots of 1?

13 (a) Two eigenvectors of C are $(1,1,1,1)$ and $(1,i,i^2,i^3)$. Find the eigenvalues.

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad C \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} = e_2 \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}.$$

(b) $P = F\Lambda F^{-1}$ immediately gives $P^2 = F\Lambda^2 F^{-1}$ and $P^3 = F\Lambda^3 F^{-1}$. Then $C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3 = F(c_0 I + c_1 \Lambda + c_2 \Lambda^2 + c_3 \Lambda^3) F^{-1} = FEF^{-1}$. That matrix E in parentheses is diagonal. It contains the _____ of C.

14 Find the eigenvalues of the "periodic" $-1, 2, -1$ matrix from $E = 2I - \Lambda - \Lambda^3$, with the eigenvalues of P in Λ. The -1's in the corners make this matrix periodic:

$$C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \text{has } c_0 = 2, c_1 = -1, c_2 = 0, c_3 = -1.$$

15 ***Fast convolution.*** To multiply C times a vector x, we can multiply $F(E(F^{-1}x))$ instead. The direct way uses n^2 separate multiplications. Knowing E and F, the second way uses only $n \log_2 n + n$ multiplications. How many of those come from E, how many from F, and how many from F^{-1}?

16 Why is row i of $\overline{F}$ the same as row $N - i$ of F (numbered 0 to $N - 1$)?

Solutions to Selected Exercises

Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

4 $3v + w = (7, 5)$ and $cv + dw = (2c + d, c + 2d)$.

6 The components of every $cv + dw$ add to zero. $c = 3$ and $d = 9$ give $(3, 3, -6)$.

9 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$.

11 Four more corners $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Centers of faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$.

12 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

13 Sum = zero vector. Sum = $-2{:}00$ vector = $8{:}00$ vector. 2:00 is 30° from horizontal $= (\cos \frac{\pi}{6}, \sin \frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

16 All combinations with $c + d = 1$ are on the line that passes through v and w. The point $V = -v + 2w$ is on that line but it is beyond w.

17 All vectors $cv + cw$ are on the line passing through $(0, 0)$ and $u = \frac{1}{2}v + \frac{1}{2}w$. That line continues out beyond $v + w$ and back beyond $(0, 0)$. With $c \geq 0$, half of this line is removed, leaving a *ray* that starts at $(0, 0)$.

20 (a) $\frac{1}{3}u + \frac{1}{3}v + \frac{1}{3}w$ is the center of the triangle between u, v and w; $\frac{1}{2}u + \frac{1}{2}w$ lies between u and w (b) To fill the triangle keep $c \geq 0$, $d \geq 0$, $e \geq 0$, and $c + d + e = \mathbf{1}$.

22 The vector $\frac{1}{2}(u + v + w)$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

25 (a) For a line, choose $u = v = w =$ any nonzero vector (b) For a plane, choose u and v in different directions. A combination like $w = u + v$ is in the same plane.

Problem Set 1.2, page 19

3 Unit vectors $v/\|v\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $w/\|w\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{v}{\|v\|} \cdot \frac{w}{\|w\|} = \frac{24}{25}$. The vectors $w, u, -w$ make $0°, 90°, 180°$ angles with w.

4 (a) $v \cdot (-v) = -1$ (b) $(v + w) \cdot (v - w) = v \cdot v + w \cdot v - v \cdot w - w \cdot w = 1 + (\ \ ) - (\ \ ) - 1 = 0$ so $\theta = 90°$ (notice $v \cdot w = w \cdot v$) (c) $(v - 2w) \cdot (v + 2w) = v \cdot v - 4w \cdot w = 1 - 4 = -3$.

6 All vectors $w = (c, 2c)$ are perpendicular to v. All vectors (x, y, z) with $x+y+z = 0$ lie on a *plane*. All vectors perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a *line*.

9 If $v_2w_2/v_1w_1 = -1$ then $v_2w_2 = -v_1w_1$ or $v_1w_1+v_2w_2 = v\cdot w = 0$: perpendicular!

11 $v \cdot w < 0$ means angle $> 90°$; these w's fill half of 3-dimensional space.

12 $(1, 1)$ perpendicular to $(1, 5) - c(1, 1)$ if $6 - 2c = 0$ or $c = 3$; $v \cdot (w - cv) = 0$ if $c = v \cdot w/v \cdot v$. Subtracting cv is the key to perpendicular vectors.

15 $\frac{1}{2}(x + y) = (2 + 8)/2 = 5$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = 8/10$.

17 $\cos\alpha = 1/\sqrt{2}, \cos\beta = 0, \cos\gamma = -1/\sqrt{2}$. For any vector v, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|v\|^2 = 1$.

21 $2v\cdot w \le 2\|v\|\|w\|$ leads to $\|v+w\|^2 = v\cdot v+2v\cdot w+w\cdot w \le \|v\|^2+2\|v\|\|w\|+\|w\|^2$. This is $(\|v\| + \|w\|)^2$. Taking square roots gives $\|v + w\| \le \|v\| + \|w\|$.

22 $v_1^2w_1^2 + 2v_1w_1v_2w_2 + v_2^2w_2^2 \le v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2$ which is $(v_1w_2 - v_2w_1)^2 \ge 0$.

23 $\cos\beta = w_1/\|w\|$ and $\sin\beta = w_2/\|w\|$. Then $\cos(\beta-a) = \cos\beta\cos\alpha+\sin\beta\sin\alpha = v_1w_1/\|v\|\|w\| + v_2w_2/\|v\|\|w\| = v \cdot w/\|v\|\|w\|$. This is $\cos\theta$ because $\beta - \alpha = \theta$.

24 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True: $.96 < 1$.

28 Three vectors in the plane could make angles $> 90°$ with each other: $(1, 0), (-1, 4), (-1, -4)$. Four vectors could not do this (360° total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$?

29 Try $v = (1, 2, -3)$ and $w = (-3, 1, 2)$ with $\cos\theta = \frac{-7}{14}$ and $\theta = 120°$. Write $v \cdot w = xz + yz + xy$ as $\frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$. If $x + y + z = 0$ this is $-\frac{1}{2}(x^2 + y^2 + z^2) = -\frac{1}{2}\|v\|\|w\|$. Then $v \cdot w/\|v\|\|w\| = -\frac{1}{2}$.

Problem Set 1.3, page 29

1 $2s_1 + 3s_2 + 4s_3 = (2, 5, 9)$. The same vector b comes from S times $x = (2, 3, 4)$:

$$\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}(\text{row } 1)\cdot x\\ (\text{row } 2)\cdot x\\ (\text{row } 2)\cdot x\end{bmatrix} = \begin{bmatrix}2\\5\\9\end{bmatrix}.$$

2 The solutions are $y_1 = 1, y_2 = 0, y_3 = 0$ (right side = column 1) and $y_1 = 1, y_2 = 3, y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2.

4 The combination $0w_1 + 0w_2 + 0w_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $w_2 = (w_1 + w_3)/2$ so one combination that gives zero is $\frac{1}{2}w_1 - w_2 + \frac{1}{2}w_3$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $r_2 = \frac{1}{2}(r_1 + r_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual.

7 All three rows are perpendicular to the solution x (the three equations $r_1 \cdot x = 0$ and $r_2\cdot x = 0$ and $r_3\cdot x = 0$ tell us this). Then the whole plane of the rows is perpendicular to x (the plane is also perpendicular to all multiples cx).

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $C\boldsymbol{x} = \boldsymbol{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{any constant vector.}$$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the nth power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative nt^{n-1}. The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{matrix} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 \qquad &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ -x_4 &= b_5 \end{aligned}$$

Add equations $1, 3, 5$
The left side of the sum is zero
The right side is $b_1 + b_3 + b_5$
There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d are equal!

Problem Set 2.1, page 40

1 The columns are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ and $\boldsymbol{b} = (2, 3, 4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The columns are changed; but same combination.

4 If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

6 Equation 1 + equation 2 − equation 3 is now $0 = -4$. Line misses plane; *no solution*.

8 Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3, 3, 3, 2)$ is $\boldsymbol{x} = (0, 0, 1, 2)$ if A has columns $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$. The equations are $x + y + z + t = 3, y + z + t = 3, z + t = 3, t = 2$.

11 $A\boldsymbol{x}$ equals $(14, 22)$ and $(0, 0)$ and $(9, 7)$.

14 $2x + 3y + z + 5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [2 \;\; 3 \;\; 1 \;\; 5]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane*.

16 $90°$ rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $180°$ rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

18 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

22 The dot product $A\boldsymbol{x} = \begin{bmatrix} 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1 \text{ by } 3)(3 \text{ by } 1)$ is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

23 $A = [1 \;\; 2 \;\; ; \;\; 3 \;\; 4]$ and $\boldsymbol{x} = [5 \;\; -2]'$ and $\boldsymbol{b} = [1 \;\; 7]'$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as zero.

25 **ones**$(4, 4) *$ **ones**$(4, 1) = [4 \;\; 4 \;\; 4 \;\; 4]'$; $B * \boldsymbol{w} = [10 \;\; 10 \;\; 10 \;\; 10]'$.

28 The row picture shows four *lines* in the 2D plane. The column picture is in *four-dimensional* space. No solution unless the right side is a combination of *the two columns.*

29 $\boldsymbol{u}_7, \boldsymbol{v}_7, \boldsymbol{w}_7$ are all close to $(.6, .4)$. Their components still add to 1.

30 $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$ = *steady state* $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

31 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1,1,1) = (15, 15, 15)$; $M_4(1,1,1,1) = (34, 34, 34, 34)$ because $1 + 2 + \cdots + 16 = 136$ which is $4(34)$.

32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*

33 $\boldsymbol{w} = (5, 7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$.

34 $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

35 $\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x}$ = sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices. 6 row orders $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 51

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

14 Subtract 2 times row 1 from row 2 to reach $(d - 10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system becomes singular.

15 The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. A solution is $(1,1,-1)$.

17 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 2 = column 1, then column 2 has no pivot.

19 Row 2 becomes $3y-4z=5$, then row 3 becomes $(q+4)z=t-5$. If $q=-4$ the system is singular — no third pivot. Then if $t=5$ the third equation is $0=0$. Choosing $z=1$ the equation $3y-4z=5$ gives $y=3$ and equation 1 gives $x=-9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1+2=$ row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=4$. No parallel planes but still no solution.

25 $a=2$ (equal columns), $a=4$ (equal rows), $a=0$ (zero column).

28 $A(2,:)=A(2,:)-3*A(1,:)$ will subtract 3 times row 1 from row 2.

29 Pivots 2 and 3 can be arbitrarily large. I believe their averages are infinite! *With row exchanges* in MATLAB's lu code, the averages are much more stable (and should be predictable, also for randn with normal instead of uniform probability distribution).

30 If $A(5,5)$ is 7 not 11, then the last pivot will be 0 not 4.

31 Row j of U is a combination of rows $1,\ldots,j$ of A. If $A\boldsymbol{x}=\boldsymbol{0}$ then $U\boldsymbol{x}=\boldsymbol{0}$ (not true if $\boldsymbol{b}$ replaces $\boldsymbol{0}$). U is the diagonal of A when A is *lower triangular.*

Problem Set 2.3, page 63

1 $E_{21}=\begin{bmatrix}1&0&0\\-5&1&0\\0&0&1\end{bmatrix}$, $E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&7&1\end{bmatrix}$, $P=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$.

3 $\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$ $M=E_{32}E_{31}E_{21}=\begin{bmatrix}1&0&0\\-4&1&0\\10&-2&1\end{bmatrix}$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot.*

9 $M=\begin{bmatrix}1&0&0\\0&0&1\\-1&1&0\end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13}=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$; $\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}$; $E_{31}E_{13}=\begin{bmatrix}2&0&1\\0&1&0\\1&0&1\end{bmatrix}$. Test on the identity matrix!

12 The first product is $\begin{bmatrix}9&8&7\\6&5&4\\3&2&1\end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix}1&2&3\\0&1&-2\\0&2&-3\end{bmatrix}$.

14 E_{21} has $-\ell_{21}=\frac{1}{2}$, E_{32} has $-\ell_{32}=\frac{2}{3}$, E_{43} has $-\ell_{43}=\frac{3}{4}$. Otherwise the E's match I.

18 $EF=\begin{bmatrix}1&0&0\\a&1&0\\b&c&1\end{bmatrix}$, $FE=\begin{bmatrix}1&0&0\\a&1&0\\b+ac&c&1\end{bmatrix}$, $E^2=\begin{bmatrix}1&0&0\\2a&1&0\\2b&0&1\end{bmatrix}$, $F^3=\begin{bmatrix}1&0&0\\0&1&0\\0&3c&1\end{bmatrix}$.

22 (a) $\sum a_{3j}x_j$ (b) $a_{21}-a_{11}$ (c) $a_{21}-2a_{11}$ (d) $(E_{21}Ax)_1 = (Ax)_1 = \sum a_{1j}x_j$.

25 The last equation becomes $0 = 3$. If the original 6 is 3, then row 1 + row 2 = row 3.

27 (a) No solution if $d = 0$ and $c \neq 0$ (b) Many solutions if $d = 0 = c$. No effect from a, b.

28 $A = AI = A(BC) = (AB)C = IC = C$. That middle equation is crucial.

30 $EM = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ then $FEM = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ then $EFEM = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ then $EEFEM = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = B$. So after inverting with $E^{-1} = A$ and $F^{-1} = B$ this is $M = ABAAB$.

Problem Set 2.4, page 75

2 (a) A (column 3 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 4 of B) (d) (Row 1 of C)D(column 1 of E).

5 (a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

7 (a) True (b) False (c) True (d) False.

9 $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF) = (EA)F$: Matrix multiplication is *associative*.

11 (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

15 (a) mn (use every entry of A) (b) $mnp = p \times$part (a) (c) n^3 (n^2 dot products).

16 (a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

18 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

19 (a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

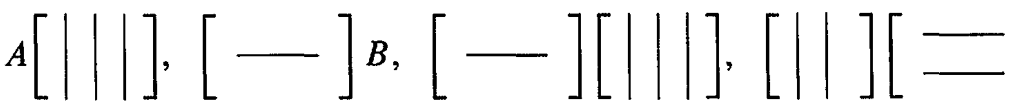
22 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;

$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$. You can find more examples.

24 $(A_1)^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$, $(A_2)^n = 2^{n-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $(A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$.

27 (a) (row 3 of A)$\cdot$(column 1 of B) and (row 3 of A)$\cdot$(column 2 of B) are both zero.

(b) $\begin{bmatrix} x \\ x \\ 0 \end{bmatrix}\begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ x \\ x \end{bmatrix}\begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$: **both upper**.

28 A times B with cuts 

30 In 29, $c = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$, $D - cb/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ in the lower corner of EA.

32 A times $X = [\,x_1 \;\; x_2 \;\; x_3\,]$ will be the identity matrix $I = [\,Ax_1 \;\; Ax_2 \;\; Ax_3\,]$.

33 $b = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ gives $x = 3x_1 + 5x_2 + 8x_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ will have those $x_1 = (1,1,1), x_2 = (0,1,1), x_3 = (0,0,1)$ as columns of its "inverse" A^{-1}.

35 $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$, **aba, ada** **cba, cda** **bab, bcb** **dab, dcb** **abc, adc** **cbc, cdc** **bad, bcd** **dad, dcd** These show 16 2-step paths in the graph

Problem Set 2.5, page 89

1 $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

7 (a) In $Ax = (1,0,0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) Right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 (a) The vector $x = (1,1,-1)$ solves $Ax = \mathbf{0}$ (b) After elimination, columns 1 and 2 end in zeros. Then so does column 3 = column 1 + 2: no third pivot.

12 Multiply $C = AB$ on the left by A^{-1} and on the right by C^{-1}. Then $A^{-1} = BC^{-1}$.

14 $B^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract column 2 of A^{-1} from column 1.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$. The inverse of each matrix is the other divided by $ad - bc$

18 $A^2B = I$ can also be written as $A(AB) = I$. Therefore A^{-1} is AB.

21 Six of the sixteen $0-1$ matrices are invertible, including all four with three 1's.

22 $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$;
$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$.

24 $\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$.

27 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the pattern); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

31 Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)}\begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

33 $x = (1,1,\ldots,1)$ has $Px = Qx$ so $(P-Q)x = \mathbf{0}$.

34 $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

35 A can be invertible with diagonal zeros. B is singular because each row adds to zero.

38 The three Pascal matrices have $P = LU = LL^{\mathrm{T}}$ and then $\mathrm{inv}(P) = \mathrm{inv}(L^{\mathrm{T}})\mathrm{inv}(L)$.

42 $MM^{-1} = (I_n - UV)\,(I_n + U(I_m - VU)^{-1}V)$ (this is testing formula **3**)
$= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V$ (keep simplifying)
$= I_n - UV + U(I_m - VU)(I_m - VU)^{-1}V = I_n$ (formulas **1, 2, 4** are similar)

43 4 by 4 still with $T_{11} = 1$ has pivots 1, 1, 1, 1; reversing to $T^* = UL$ makes $T^*_{44} = 1$.

44 Add the equations $C\boldsymbol{x} = \boldsymbol{b}$ to find $0 = b_1 + b_2 + b_3 + b_4$. Same for $F\boldsymbol{x} = \boldsymbol{b}$.

Problem Set 2.6, page 102

3 $\ell_{31} = 1$ and $\ell_{32} = 2$ (and $\ell_{33} = 1$): reverse steps to get $A\boldsymbol{u} = \boldsymbol{b}$ from $U\boldsymbol{x} = \boldsymbol{c}$: 1 times $(x+y+z = 5)$+2 times $(y+2z = 2)$+1 times $(z = 2)$ gives $x+3y+6z = 11$.

4 $Lc = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$; $U\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{bmatrix}\begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$; $\boldsymbol{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$.

6 $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} = U$. Then $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} U$ is the same as $E_{21}^{-1}E_{32}^{-1}U = LU$. The multipliers $\ell_{21}, \ell_{32} = 2$ fall into place in L.

10 $c = 2$ leads to zero in the second pivot position: exchange rows and not singular. $c = 1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

12 $A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDU$; $\boldsymbol{U}$ **is** $\boldsymbol{L}^{\mathrm{T}}$

$$\begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & -4 & \\ & & 4 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}.$$

14 $\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}$. Need $\begin{matrix} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{matrix}$

15 $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\boldsymbol{c} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ gives $\boldsymbol{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Then $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ gives $\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

$A\boldsymbol{x} = \boldsymbol{b}$ is $LU\boldsymbol{x} = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$. Forward to $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boldsymbol{c}$.

18 (a) Multiply $LDU = L_1D_1U_1$ by inverses to get $L_1^{-1}LD = D_1U_1U^{-1}$. The left side is lower triangular, the right side is upper triangular $\Rightarrow$ both sides are diagonal.
(b) L, U, L_1, U_1 have diagonal 1's so $D = D_1$. Then $L_1^{-1}L$ and U_1U^{-1} are both I.

20 A tridiagonal T has 2 nonzeros in the pivot row and only one nonzero below the pivot (one operation to find ℓ and then one for the new pivot!). $T =$ bidiagonal L times bidiagonal U.

23 The 2 by 2 upper submatrix A_2 has the first two pivots 5, 9. Reason: Elimination on A starts in the upper left corner with elimination on A_2.

24 The upper left blocks all factor at the same time as A: A_k is L_kU_k.

25 The i, j entry of L^{-1} is j/i for $i \geq j$. And $L_{i\,i-1}$ is $(1-i)/i$ below the diagonal

26 $(K^{-1})_{ij} = j(n-i+1)/(n+1)$ for $i \geq j$ (and symmetric): $(n+1)K^{-1}$ looks good.

Problem Set 2.7, page 115

2 $(AB)^{\mathrm{T}}$ is not $A^{\mathrm{T}}B^{\mathrm{T}}$ *except when* $AB = BA$. Transpose that to find: $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$.

4 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. The diagonal of $A^{\mathrm{T}}A$ has dot products of columns of A with themselves. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ A = zero matrix.

6 $M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A$ and $B^{\mathrm{T}} = C$ and $D^{\mathrm{T}} = D$.

8 The 1 in row 1 has n choices; then the 1 in row 2 has $n-1$ choices ... ($n!$ overall).

10 $(3,1,2,4)$ and $(2,3,1,4)$ keep 4 in place; 6 more even P's keep 1 or 2 or 3 in place; $(2,1,4,3)$ and $(3,4,1,2)$ exchange 2 pairs. $(1,2,3,4)$, $(4,3,2,1)$ make 12 even P's.

14 The i,j entry of PAP is the $n-i+1, n-j+1$ entry of A. Diagonal will reverse order.

18 (a) $5+4+3+2+1 = 15$ independent entries if $A = A^{\mathrm{T}}$ (b) L has 10 and D has 5; total 15 in LDL^{T} (c) Zero diagonal if $A^{\mathrm{T}} = -A$, leaving $4+3+2+1 = 10$ choices.

20 $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}\begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = LDL^{\mathrm{T}}.$$

22 $\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}$; $\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$

24 $PA = LU$ is $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}$. If we wait to exchange and a_{12} is the pivot, $A = L_1P_1U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

26 One way to decide even vs. odd is to count all pairs that P has in the wrong order. Then P is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

31 $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$; $A^{\mathrm{T}}y = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix}\begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$ $\begin{matrix} \text{1 truck} \\ \text{1 plane} \end{matrix}$

32 $Ax \cdot y$ is the *cost* of inputs while $x \cdot A^{\mathrm{T}}y$ is the *value* of outputs.

33 $P^3 = I$ so three rotations for 360°; P rotates around $(1,1,1)$ by 120°.

36 These are groups: Lower triangular with diagonal 1's, diagonal invertible D, permutations P, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

37 Certainly B^{T} is northwest. B^2 is a full matrix! B^{-1} is southeast: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. The rows of B are in reverse order from a lower triangular L, so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the *columns* in reverse order from L^{-1}. So B^{-1} is *southeast*. Northwest $B = PL$ times southeast PU is $(PLP)U$ = upper triangular.

38 There are $n!$ permutation matrices of order n. Eventually *two powers of P must be the same*: If $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \le n!$

$$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix} \text{ is 5 by 5 with } P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } P^6 = I.$$

Problem Set 3.1, page 127

1 $\boldsymbol{x} + \boldsymbol{y} \ne \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \ne (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \ne c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$ (b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(2 + \mathbf{1}) = 8$. The zero vector is the number 1.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) Matrices whose main diagonal is all zero.

9 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$ (b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions.

11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

15 (a) Two planes through $(0, 0, 0)$ probably intersect in a line through $(0, 0, 0)$ (b) The plane and line probably intersect in the point $(0, 0, 0)$ (c) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in both $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are in both subspaces.

20 (a) Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Solution only if $b_3 = -b_1$.

23 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.

$$[A \;\; \boldsymbol{b}] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix} \begin{matrix} \text{(larger column space)} \\ \text{(no solution to } A\boldsymbol{x} = \boldsymbol{b}) \end{matrix} \quad \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix} \begin{matrix} (\boldsymbol{b} \text{ is in column space}) \\ (A\boldsymbol{x} = \boldsymbol{b} \text{ has a solution}) \end{matrix}$$

25 The solution to $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\boldsymbol{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

30 (a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\boldsymbol{S} + \boldsymbol{T}$, then $\boldsymbol{u} = \boldsymbol{s}_1 + \boldsymbol{t}_1$ and $\boldsymbol{v} = \boldsymbol{s}_2 + \boldsymbol{t}_2$. So $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$ is also in $\boldsymbol{S} + \boldsymbol{T}$. And so is $c\boldsymbol{u} = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$: *a subspace*.

(b) If $\boldsymbol{S}$ and $\boldsymbol{T}$ are different lines, then $\boldsymbol{S} \cup \boldsymbol{T}$ is just the two lines (*not a subspace*) but $\boldsymbol{S} + \boldsymbol{T}$ is the whole plane that they span.

31 If $\boldsymbol{S} = \boldsymbol{C}(A)$ and $\boldsymbol{T} = \boldsymbol{C}(B)$ then $\boldsymbol{S} + \boldsymbol{T}$ is the column space of $M = [\,A \;\; B\,]$.

32 The columns of AB are combinations of the columns of A. So all columns of $[\,A \;\; AB\,]$ are already in $\boldsymbol{C}(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ when A is *invertible*.

Problem Set 3.2, page 140

2 (a) Free variables x_2, x_4, x_5 and solutions $(-2, 1, 0, 0, 0)$, $(0, 0, -2, 1, 0)$, $(0, 0, -3, 0, 1)$ (b) Free variable x_3: solution $(1, -1, 1)$. Special solution for each free variable.

4 $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, R has the same nullspace as U and A.

6 (a) Special solutions $(3,1,0)$ and $(5,0,1)$ (b) $(3,1,0)$. Total of pivot and free is n.

8 $R = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ with $I = [\,1\,]$; $R = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

10 (a) Impossible row 1 (b) $A =$ invertible (c) $A =$ all ones (d) $A = 2I, R = I$.

14 If column 1 = column 5 then x_5 is a free variable. Its special solution is $(-1,0,0,0,1)$.

16 The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

20 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1,0,1,0,1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (a line in $\mathbf{R}^5$).

24 This construction is impossible: 2 pivot columns and 2 free variables, only 3 columns.

26
30 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$ and also (a)(b)(c) are all false. Notice rref$(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

32 Any zero rows come after these rows: $R = [\,1 \;\; -2 \;\; -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

33 (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) All 8 matrices are R's!

35 The nullspace of $B = [\,A \;\; A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{y} \end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

36 If $C\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection.*

37 *Currents*: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops.

Problem Set 3.3, page 151

1 (a) and (c) are correct; (d) is false because R might have 1's in nonpivot columns.

3 $R_A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_B = \begin{bmatrix} R_A & R_A \end{bmatrix}$ $R_C \longrightarrow \begin{bmatrix} R_A & 0 \\ 0 & R_A \end{bmatrix} \longrightarrow$ Zero rows go to the bottom

5 I think $R_1 = A_1, R_2 = A_2$ is true. But $R_1 - R_2$ may have -1's in some pivots.

7 Special solutions in $N = [-2 \;\; -4 \;\; 1 \;\; 0; -3 \;\; -5 \;\; 0 \;\; 1]$ and $[1 \;\; 0 \;\; 0; 0 \;\; -2 \;\; 1]$.

13 P has rank r (the same as A) because elimination produces the same pivot columns.

14 The rank of R^{T} is also r. The example matrix A has rank 2 with invertible S:

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix} \qquad P^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix} \qquad S^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

16 $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})(\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}) = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w})\boldsymbol{z}^{\mathrm{T}}$ has rank one unless the inner product is $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.

18 If we know that $\text{rank}(B^{\mathrm{T}}A^{\mathrm{T}}) \leq \text{rank}(A^{\mathrm{T}})$, then since rank stays the same for transposes, (apologies that this fact is not yet proved), we have $\text{rank}(AB) \leq \text{rank}(A)$.

20 Certainly A and B have at most rank 2. Then their product AB has at most rank 2. Since BA is 3 by 3, it cannot be I even if $AB = I$.

21 (a) A and B will both have the same nullspace and row space as the R they share.
(b) A equals an *invertible* matrix times B, when they share the same R. A key fact!

22 $A = \text{(pivot columns)(nonzero rows of } R) = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} +$
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$. $B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{matrix} \text{columns} \\ \text{times rows} \end{matrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$

26 The m by n matrix Z has r ones to start its main diagonal. Otherwise Z is all zeros.

27 $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r \text{ by } r & r \text{ by } n-r \\ m-r \text{ by } r & m-r \text{ by } n-r \end{bmatrix}$; $\textbf{rref}(R^{\mathrm{T}}) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; $\textbf{rref}(R^{\mathrm{T}}R) = \text{same } R$

28 The *row-column reduced echelon form* is always $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; I is r by r.

Problem Set 3.4, page 163

2 $\begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - 3\mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - 2\mathbf{b}_1 \end{bmatrix}$ Then $[\,R \;\; \boldsymbol{d}\,] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$
$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\boldsymbol{C}(A) = $ line through $(2, 6, 4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{x}_p = \boldsymbol{d} = (5, 0, 0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

4 $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + x_2(-3, 1, 0, 0) + x_4(0, 0, -2, 1)$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{x}_p$
(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

8 (a) Every $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$: *independent rows*, only the zero combination gives $\mathbf{0}$.
(b) Need $b_3 = 2b_2$, because $(\text{row } 3) - 2(\text{row } 2) = \mathbf{0}$.

12 (a) $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\mathbf{0}$ solve $A\boldsymbol{x} = \mathbf{0}$ (b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \mathbf{0}, A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be $\boldsymbol{x}_p$
(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (length 2)
(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \mathbf{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a *free* variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \mathbf{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists.* The column space is $\mathbf{R}^3$. An example is $A = [\,I \quad F\,]$ for any 3 by 2 matrix F.

18 Rank = 2; rank = 3 unless $q = 2$ (then rank = 2). Transpose has the same rank!

25 (a) $r < m$, always $r \le n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

28 $$\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}; \ \boldsymbol{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}.$$

Free $x_2 = 0$ gives $\boldsymbol{x}_p = (-1, 0, 2)$ because the pivot columns contain I.

30 $$\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -\mathbf{4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}; \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}; \boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

36 If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \mathbf{0}$). If $\boldsymbol{b}$ = column 1 of A, $\boldsymbol{x} = (1, 0, \ldots, 0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $A = C$!

Problem Set 3.5, page 178

2 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors are on the plane $(1, 1, 1, 1) \cdot \boldsymbol{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column 1 $= \mathbf{0}$; if $d = 0$ then b(column 1) $- a$(column 2) $= \mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A.

8 If $c_1(\boldsymbol{w}_2 + \boldsymbol{w}_3) + c_2(\boldsymbol{w}_1 + \boldsymbol{w}_3) + c_3(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\boldsymbol{w}_1 + (c_1 + c_3)\boldsymbol{w}_2 + (c_1 + c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. *False.* The zero vector is always in the row space.

15 The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n $(m \ge n)$.

18 (a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.

20 One basis is $(2, 1, 0)$, $(-3, 0, 1)$. A basis for the intersection with the xy plane is $(2, 1, 0)$. The normal vector $(1, -2, 3)$ is a basis for the line perpendicular to the plane.

22 (a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in $\mathbf{S}$.

25 Rank 2 if $c = 0$ and $d = 2$; rank 2 except when $c = d$ or $c = -d$.

28 $$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

32 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

34 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

37 The subspace of matrices that have $AS = SA$ has dimension *three*.

39 If the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A. If $[\,A \;\; \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent, $\boldsymbol{b}$ *is* a combination of those columns. In this case $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

41 $I = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$. The six P's are dependent.

42 The dimension of $\mathbf{S}$ is (a) zero when $\boldsymbol{x} = \mathbf{0}$ (b) one when $\boldsymbol{x} = (1, 1, 1, 1)$ (c) three when $\boldsymbol{x} = (1, 1, -1, -1)$ because all rearrangements have $x_1 + \cdots + x_4 = 0$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** $\dim \mathbf{S} = 2$.

43 The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\mathbf{0}$. **To be proved**: All coefficients = zero.

Key idea: The part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$, so the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now $\boldsymbol{x} - \boldsymbol{x} = \mathbf{0}$ uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \mathbf{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

44 The inputs to an m by n matrix fill $\mathbf{R}^n$. The outputs (column space!) have dimension r. The nullspace has $n - r$ special solutions. The formula becomes $r + (n - r) = n$.

Problem Set 3.6, page 190

1 (a) Row and column space dimensions $= 5$, nullspace dimension $= 4$, $\dim(N(A^{\mathrm{T}})) = 2$ sum $= 16 = m + n$ (b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$.

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r + (n - r)$ must be 3 (c) $[1 \;\; 1]$ (d) $\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$

(e) *Impossible* Row space = column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension. Section 4.1 will prove $N(A)$ and $N(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

6 A: dim **2, 2, 2, 1**: Rows $(0, 3, 3, 3)$ and $(0, 1, 0, 1)$; columns $(3, 0, 1)$ and $(3, 0, 0)$; nullspace $(1, 0, 0, 0)$ and $(0, -1, 0, 1)$; $N(A^{\mathrm{T}})$ $(0, 1, 0)$. B: dim **1, 1, 0, 2** Row space (1), column space $(1, 4, 5)$, nullspace: empty basis, $N(A^{\mathrm{T}})$ $(-4, 1, 0)$ and $(-5, 0, 1)$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same (b) Same column space and left nullspace. Same rank (dimension of column space).

11 (a) No solution means that $r < m$. Always $r \le n$. Can't compare m and n (b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \mathbf{0}$ is in both $N(A)$ and $C(A^{\mathrm{T}})$.

16 If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$.

18 Row 3$-$2 row 2$+$ row 1 $=$ zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

20 (a) Special solutions $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ are perpendicular to the rows of R (and then ER). (b) $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 1 independent solution = last row of E^{-1}. ($E^{-1}A = R$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank < 2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

24 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y} = \mathbf{0}$.

26 The rows of $C = AB$ are combinations of the rows of B. So rank $C \leq$ rank B. Also rank $C \leq$ rank A, because the columns of C are combinations of the columns of A.

29 $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1$.

30 The subspaces for $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B = cA$ with $c \neq 0$.

31 (a) $AX = 0$ if each column of X is a multiple of $(1,1,1)$; dim(nullspace) $= 3$. (b) If $AX = B$ then all columns of B add to zero; dimension of the B's $= 6$. (c) $3 + 6 = \dim(M^{3\times 3}) = 9$ entries in a 3 by 3 matrix.

32 The key is equal row spaces. First row of $A =$ combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F = G$.

Problem Set 4.1, page 202

1 Both nullspace vectors are orthogonal to the row space vector in $\mathbf{R}^3$. The column space is perpendicular to the nullspace of A^{T} (two lines in $\mathbf{R}^2$ because rank $= 1$).

3 (a) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$ (b) Impossible, $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ not orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ is impossible: not perpendicular (d) Need $A^2 = 0$; take $A = \left[\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right]$ (e) $(1,1,1)$ in the nullspace (columns add to $\mathbf{0}$) and also row space; no such matrix.

6 Multiply the equations by $y_1, y_2, y_3 = 1, 1, -1$. Equations add to $0 = 1$ so no solution: $\boldsymbol{y} = (1,1,-1)$ is in the left nullspace. $A\boldsymbol{x} = \boldsymbol{b}$ would need $0 = (\boldsymbol{y}^{\mathrm{T}}A)\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 1$.

8 $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r$ is in the row space and $\boldsymbol{x}_n$ is in the nullspace. Then $A\boldsymbol{x}_n = \mathbf{0}$ and $A\boldsymbol{x} = A\boldsymbol{x}_r + A\boldsymbol{x}_n = A\boldsymbol{x}_r$. All $A\boldsymbol{x}$ are in $\boldsymbol{C}(A)$.

9 $A\boldsymbol{x}$ is always in the *column space* of A. If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ is also in the nullspace of A^{T}. So $A\boldsymbol{x}$ is perpendicular to itself. Conclusion: $A\boldsymbol{x} = \mathbf{0}$ if $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$.

10 (a) With $A^{\mathrm{T}} = A$, the column and row spaces are the same (b) $\boldsymbol{x}$ is in the nullspace and $\boldsymbol{z}$ is in the column space = row space: so these "eigenvectors" have $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{z} = 0$.

12 $\boldsymbol{x}$ splits into $\boldsymbol{x}_r + \boldsymbol{x}_n = (1,-1) + (1,1) = (2,0)$. Notice $N(A^{\mathrm{T}})$ is a plane $(1,0) = (1,1)/2 + (1,-1)/2 = \boldsymbol{x}_r + \boldsymbol{x}_n$.

13 $V^{\mathrm{T}}W =$ zero makes each basis vector for $\boldsymbol{V}$ orthogonal to each basis vector for $\boldsymbol{W}$. Then every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$ (combinations of the basis vectors).

14 $Ax = B\widehat{x}$ means that $[\,A \quad B\,]\begin{bmatrix} x \\ -\widehat{x} \end{bmatrix} = \mathbf{0}$. Three homogeneous equations in four unknowns always have a nonzero solution. Here $x = (3,1)$ and $\widehat{x} = (1,0)$ and $Ax = B\widehat{x} = (5,6,5)$ is in both column spaces. Two planes in $\mathbf{R}^3$ must share a line.

16 $A^{\mathrm{T}}y = \mathbf{0}$ leads to $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}A^{\mathrm{T}}y = 0$. Then $y \perp Ax$ and $N(A^{\mathrm{T}}) \perp C(A)$.

18 $S^{\perp}$ is the nullspace of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. Therefore $S^{\perp}$ is a *subspace* even if S is not.

21 For example $(-5,0,1,1)$ and $(0,1,-1,0)$ span $S^{\perp}$=nullspace of $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$.

23 x in $V^{\perp}$ is perpendicular to any vector in V. Since V contains all the vectors in S, x is also perpendicular to any vector in S. So every x in $V^{\perp}$ is also in $S^{\perp}$.

28 (a) $(1,-1,0)$ is in both planes. Normal vectors are perpendicular, but planes still intersect! (b) Need *three* orthogonal vectors to span the whole orthogonal complement. (c) Lines can meet at the zero vector without being orthogonal.

30 When $AB = 0$, the column space of B is contained in the nullspace of A. Therefore the dimension of $C(B) \le$ dimension of $N(A)$. This means $\operatorname{rank}(B) \le 4 - \operatorname{rank}(A)$.

31 null(N') produces a basis for the *row space* of A (perpendicular to $N(A)$).

32 We need $r^{\mathrm{T}}n = 0$ and $c^{\mathrm{T}}\ell = 0$. All possible examples have the form acr^{T} with $a \neq 0$.

33 Both r's orthogonal to both n's, both c's orthogonal to both ℓ's, each pair independent. All A's with these subspaces have the form $[c_1\ c_2]M[r_1\ r_2]^{\mathrm{T}}$ for a 2 by 2 invertible M.

Problem Set 4.2, page 214

1 (a) $a^{\mathrm{T}}b/a^{\mathrm{T}}a = 5/3$; $p = 5a/3$; $e = (-2,1,1)/3$ (b) $a^{\mathrm{T}}b/a^{\mathrm{T}}a = -1$; $p = a$; $e = \mathbf{0}$.

3 $P_1 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P_1 b = \frac{1}{3}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$. $P_2 = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$ and $P_2 b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

6 $p_1 = (\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$ and $p_2 = (\frac{4}{9}, \frac{4}{9}, -\frac{2}{9})$ and $p_3 = (\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$. So $p_1 + p_2 + p_3 = b$.

9 Since A is invertible, $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$: project on all of $\mathbf{R}^2$.

11 (a) $p = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b = (2,3,0)$, $e = (0,0,4)$, $A^{\mathrm{T}}e = \mathbf{0}$ (b) $p = (4,4,6)$, $e = \mathbf{0}$.

15 $2A$ has the same column space as A. $\widehat{x}$ for $2A$ is *half* of $\widehat{x}$ for A.

16 $\frac{1}{2}(1,2,-1) + \frac{3}{2}(1,0,1) = (2,1,1)$. So b is in the plane. Projection shows $Pb = b$.

18 (a) $I - P$ is the projection matrix onto $(1,-1)$ in the perpendicular direction to $(1,1)$ (b) $I - P$ projects onto the plane $x + y + z = 0$ perpendicular to $(1,1,1)$.

20 $e = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $Q = \frac{ee^{\mathrm{T}}}{e^{\mathrm{T}}e} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, $I - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

21 $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2 = A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. So $P^2 = P$. Pb is in the column space (where P projects). Then its projection $P(Pb)$ is Pb.

24 The nullspace of A^{T} is *orthogonal* to the column space $\boldsymbol{C}(A)$. So if $A^{\mathrm{T}}\boldsymbol{b} = \mathbf{0}$, the projection of $\boldsymbol{b}$ onto $\boldsymbol{C}(A)$ should be $\boldsymbol{p} = \mathbf{0}$. Check $P\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}\mathbf{0}$.

28 $P^2 = P = P^{\mathrm{T}}$ give $P^{\mathrm{T}}P = P$. Then the $(2,2)$ entry of P equals the $(2,2)$ entry of $P^{\mathrm{T}}P$ which is the length squared of column 2.

29 $A = B^{\mathrm{T}}$ has independent columns, so $A^{\mathrm{T}}A$ (which is BB^{T}) must be invertible.

30 (a) The column space is the line through $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $P_C = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \dfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$.

(b) The row space is the line through $\boldsymbol{v} = (1,2,2)$ and $P_R = \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}/\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}$. Always $P_C A = A$ (columns of A project to themselves) and $AP_R = A$. Then $P_C A P_R = A$!

31 The error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ must be perpendicular to all the $\boldsymbol{a}$'s.

32 Since $P_1\boldsymbol{b}$ is in $\boldsymbol{C}(A)$, $P_2(P_1\boldsymbol{b})$ equals $P_1\boldsymbol{b}$. So $P_2P_1 = P_1 = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ where $\boldsymbol{a} = (1,2,0)$.

33 If $P_1P_2 = P_2P_1$ then $\boldsymbol{S}$ is contained in $\boldsymbol{T}$ or $\boldsymbol{T}$ is contained in $\boldsymbol{S}$.

34 BB^{T} is invertible as in Problem 29. Then $(A^{\mathrm{T}}A)(BB^{\mathrm{T}})$ = product of r by r invertible matrices, so rank r. AB can't have rank $< r$, since A^{T} and B^{T} cannot increase the rank. *Conclusion*: A (m by r of rank r) times B (r by n of rank r) produces AB of rank r.

Problem Set 4.3, page 226

1 $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ gives $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$, $E = \|\boldsymbol{e}\|^2 = \mathbf{44}$

5 $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$. $A^{\mathrm{T}} = [1 \;\; 1 \;\; 1 \;\; 1]$ and $A^{\mathrm{T}}A = [4]$. $A^{\mathrm{T}}\boldsymbol{b} = [36]$ and $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = 9$ = best height C. Errors $\boldsymbol{e} = (-9,-1,-1,11)$.

7 $A = [0 \;\; 1 \;\; 3 \;\; 4]^{\mathrm{T}}$, $A^{\mathrm{T}}A = [26]$ and $A^{\mathrm{T}}\boldsymbol{b} = [112]$. Best $D = 112/26 = 56/13$.

8 $\widehat{x} = 56/13$, $\boldsymbol{p} = (56/13)(0,1,3,4)$. $(C,D) = (9, 56/13)$ don't match $(C,D) = (1,4)$. Columns of A were not perpendicular so we can't project separately to find C and D.

9 Parabola Project $\boldsymbol{b}$ 4D to 3D $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$.

11 (a) The best line $x = 1 + 4t$ gives the center point $\widehat{b} = 9$ when $\widehat{t} = 2$.

(b) The first equation $Cm + D\sum t_i = \sum b_i$ divided by m gives $C + D\widehat{t} = \widehat{b}$.

13 $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b} - A\boldsymbol{x}) = \widehat{\boldsymbol{x}} - \boldsymbol{x}$. When $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ averages to $\mathbf{0}$, so does $\widehat{\boldsymbol{x}} - \boldsymbol{x}$.

14 The matrix $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^{\mathrm{T}}$ is $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^{\mathrm{T}}A(A^{\mathrm{T}}A)^{-1}$. When the average of $(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^{\mathrm{T}}$ is $\sigma^2 I$, the average of $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^{\mathrm{T}}$ will be the *output covariance matrix* $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\sigma^2 A(A^{\mathrm{T}}A)^{-1}$ which simplifies to $\sigma^2(A^{\mathrm{T}}A)^{-1}$.

16 $\frac{1}{10}b_{10} + \frac{9}{10}\widehat{x}_9 = \frac{1}{10}(b_1 + \cdots + b_{10})$. Knowing $\widehat{x}_9$ avoids adding all b's.

18 $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \boldsymbol{0}$.

21 $\boldsymbol{e}$ is in $\boldsymbol{N}(A^{\mathrm{T}})$; $\boldsymbol{p}$ is in $\boldsymbol{C}(A)$; $\widehat{\boldsymbol{x}}$ is in $\boldsymbol{C}(A^{\mathrm{T}})$; $\boldsymbol{N}(A) = \{\boldsymbol{0}\}$ = zero vector only.

23 The square of the distance between points on two lines is $E = (y-x)^2 + (3y-x)^2 + (1+x)^2$. Derivatives $\frac{1}{2}\partial E/\partial x = 3x - 4y + 1 = 0$ and $\frac{1}{2}\partial E/\partial y = -4x + 10y = 0$. The solution is $x = -5/7, y = -2/7$; $E = 2/7$, and the minimum distance is $\sqrt{2/7}$.

25 3 points on a line: *Equal slopes* $(b_2-b_1)/(t_2-t_1) = (b_3-b_2)/(t_3-t_2)$. Linear algebra: Orthogonal to $(1,1,1)$ and (t_1,t_2,t_3) is $\boldsymbol{y} = (t_2-t_3, t_3-t_1, t_1-t_2)$ in the left nullspace. $\boldsymbol{b}$ is in the column space. Then $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ is the same equal slopes condition written as $(b_2-b_1)(t_3-t_2) = (b_3-b_2)(t_2-t_1)$.

27 The shortest link connecting two lines in space is *perpendicular to those lines.*

28 Only 1 plane contains $\boldsymbol{0}, \boldsymbol{a}_1, \boldsymbol{a}_2$ unless $\boldsymbol{a}_1, \boldsymbol{a}_2$ are *dependent*. Same test for $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

Problem Set 4.4, page 239

3 (a) $A^{\mathrm{T}}A$ will be $16I$ (b) $A^{\mathrm{T}}A$ will be diagonal with entries $1, 4, 9$.

6 Q_1Q_2 is orthogonal because $(Q_1Q_2)^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_1^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_2 = I$.

8 If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are *orthonormal* vectors in $\mathbf{R}^5$ then $(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_1 + (\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_2$ is closest to $\boldsymbol{b}$.

11 (a) Two *orthonormal* vectors are $\boldsymbol{q}_1 = \frac{1}{10}(1,3,4,5,7)$ and $\boldsymbol{q}_2 = \frac{1}{10}(-7,3,4,-5,1)$ (b) Closest in the plane: *project* $QQ^{\mathrm{T}}(1,0,0,0,0) = (0.5, -0.18, -0.24, 0.4, 0)$.

13 The multiple to subtract is $\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}$. Then $\boldsymbol{B} = \boldsymbol{b} - \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = (4,0) - 2\cdot(1,1) = (2,-2)$.

14 $$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} \begin{bmatrix} \|\boldsymbol{a}\| & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b} \\ 0 & \|\boldsymbol{B}\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = QR.$$

15 (a) $\boldsymbol{q}_1 = \frac{1}{3}(1,2,-2)$, $\boldsymbol{q}_2 = \frac{1}{3}(2,1,2)$, $\boldsymbol{q}_3 = \frac{1}{3}(2,-2,-1)$ (b) The nullspace of A^{T} contains $\boldsymbol{q}_3$ (c) $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(1,2,7) = (1,2)$.

16 The projection $\boldsymbol{p} = (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a} = 14\boldsymbol{a}/49 = 2\boldsymbol{a}/7$ is closest to $\boldsymbol{b}$; $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\| = \boldsymbol{a}/7$ is $(4,5,2,2)/7$. $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (-1,4,-4,-4)/7$ has $\|\boldsymbol{B}\| = 1$ so $\boldsymbol{q}_2 = \boldsymbol{B}$.

18 $\boldsymbol{A} = \boldsymbol{a} = (1,-1,0,0)$; $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (\frac{1}{2}, \frac{1}{2}, -1, 0)$; $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{p}_A - \boldsymbol{p}_B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. Notice the pattern in those orthogonal $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. In $\mathbf{R}^5$, $\boldsymbol{D}$ would be $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1)$.

20 (a) *True* (b) *True*. $Q\boldsymbol{x} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2$. $\|Q\boldsymbol{x}\|^2 = x_1^2 + x_2^2$ because $\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 = 0$.

21 The orthonormal vectors are $\boldsymbol{q}_1 = (1,1,1,1)/2$ and $\boldsymbol{q}_2 = (-5,-1,1,5)/\sqrt{52}$. Then $\boldsymbol{b} = (-4,-3,3,0)$ projects to $\boldsymbol{p} = (-7,-3,-1,3)/2$. And $\boldsymbol{b} - \boldsymbol{p} = (-1,-3,7,-3)/2$ is orthogonal to both $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

22 $A = (1,1,2)$, $B = (1,-1,0)$, $C = (-1,-1,1)$. These are not yet unit vectors.

26 $(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 = \frac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}$ because $\boldsymbol{q}_2 = \frac{\boldsymbol{B}}{\|\boldsymbol{B}\|}$ and the extra $\boldsymbol{q}_1$ in $\boldsymbol{C}^*$ is orthogonal to $\boldsymbol{q}_2$.

28 There are mn multiplications in (11) and $\frac{1}{2}m^2n$ multiplications in each part of (12).

30 The wavelet matrix W has orthonormal columns. Notice $W^{-1} = W^{\mathrm{T}}$ in Section 7.3.

32 $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ across plane $y + z = 0$.

33 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal and zeros elsewhere.

Problem Set 5.1, page 251

1 $\det(2A) = 8$; $\det(-A) = (-1)^4 \det A = \frac{1}{2}$; $\det(A^2) = \frac{1}{4}$; $\det(A^{-1}) = 2 = \det(A^{\mathrm{T}})^{-1}$.

5 $|J_5| = 1$, $|J_6| = -1$, $|J_7| = -1$. Determinants $1, 1, -1, -1$ repeat so $|J_{101}| = 1$.

8 $Q^{\mathrm{T}}Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so det can't blow up.

10 If the entries in every row add to zero, then $(1, 1, \ldots, 1)$ is in the nullspace: singular A has det $= 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

11 $CD = -DC \Rightarrow \det CD = (-1)^n \det DC$ and *not* $-\det DC$. If n is even we can have an invertible CD.

14 $\det(A) = 36$ and the 4 by 4 second difference matrix has det $= 5$.

15 The first determinant is 0, the second is $1 - 2t^2 + t^4 = (1 - t^2)^2$.

17 Any 3 by 3 skew-symmetric K has $\det(K^{\mathrm{T}}) = \det(-K) = (-1)^3\det(K)$. This is $-\det(K)$. But always $\det(K^{\mathrm{T}}) = \det(K)$, so we must have $\det(K) = 0$ for 3 by 3.

21 Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

23 $\det(A) = 10$, $A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$, $\det(A^2) = 100$, $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$ has det $\frac{1}{10}$. $\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$ when $\lambda = \mathbf{2}$ or $\lambda = \mathbf{5}$; those are eigenvalues.

27 $\det A = abc$, $\det B = -abcd$, $\det C = a(b - a)(c - b)$ by doing elimination.

32 Typical determinants of $\mathsf{rand}(n)$ are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. $\mathsf{randn}(n)$ with normal distribution gives $10^{31}, 10^{78}, 10^{186}$, Inf which means $\geq 2^{1024}$. MATLAB allows $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives Inf!

Problem Set 5.2, page 263

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = -1$, independent.

4 $a_{11}a_{23}a_{32}a_{44}$ gives -1, because $2 \leftrightarrow 3$, $a_{14}a_{23}a_{32}a_{41}$ gives $+1$, $\det A = 1 - 1 = 0$; $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 64 - 16 = 48$.

6 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms are sure zeros (b) 15 terms must be zero.

8 Some term $a_{1\alpha}a_{2\beta} \cdots a_{n\omega}$ in the big formula is not zero! Move rows $1, 2, \ldots, n$ into rows $\alpha, \beta, \ldots, \omega$. Then these nonzero a's will be on the main diagonal.

9 To get $+1$ for the even permutations the matrix needs an *even* number of -1's. For the odd P's the matrix needs an *odd* number of -1's. So six 1's and det $= 6$ are impossible five 1's and one -1 will give $AC = (ad - bc)I = (\det A)I$ max(det) $= 4$.

11 $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$. $\det B = 1(0) + 2(42) + 3(-35) = -21$. Puzzle: $\det D = 441 = (-21)^2$. *Why?*

12 $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore $A^{-1} = \frac{1}{4}C^{\mathrm{T}} = C^{\mathrm{T}}/\det A$.

13 (a) $C_1 = 0$, $C_2 = -1$, $C_3 = 0$, $C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row 1 then cofactors of column 1. Therefore $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$.

15 The $1,1$ cofactor of the n by n matrix is E_{n-1}. The $1,2$ cofactor has a single 1 in its first column, with cofactor E_{n-2}: sign gives $-E_{n-2}$. So $E_n = E_{n-1} - E_{n-2}$. Then E_1 to E_6 is $1, 0, -1, -1, 0, 1$ and this cycle of six will repeat: $E_{100} = E_4 = -1$.

16 The $1,1$ cofactor of the n by n matrix is F_{n-1}. The $1,2$ cofactor has a 1 in column 1, with cofactor F_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the $1,2$ entry to find $F_n = F_{n-1} + F_{n-2}$ (so these determinants are Fibonacci numbers).

19 Since x, x^2, x^3 are all in the same row, they are never multiplied in $\det V_4$. The determinant is zero at $x = a$ or b or c, so $\det V$ has factors $(x-a)(x-b)(x-c)$. Multiply by the cofactor V_3. The Vandermonde matrix $V_{ij} = (x_i)^{j-1}$ is for fitting a polynomial $p(x) = \boldsymbol{b}$ at the points x_i. It has $\det V =$ product of all $x_k - x_m$ for $k > m$.

20 $G_2 = -1$, $G_3 = 2$, $G_4 = -3$, and $G_n = (-1)^{n-1}(n-1) =$ (product of the λ's).

24 (a) All L's have $\det = 1$; $\det U_k = \det A_k = 2, 6, -6$ (b) Pivots $5, 6/5, 7/6$.

25 Problem 23 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$ which is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

27 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. Derivative with respect to $a_{11} =$ cofactor C_{11}.

29 There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs: $+ (1,1)(2,2)(3,3)(4,4) + (1,2)(2,1)(3,4)(4,3) - (1,2)(2,1)(3,3)(4,4) - (1,1)(2,2)(3,4)(4,3) - (1,1)(2,3)(3,2)(4,4)$. Total -1.

32 The problem is to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}.$$

33 The difference from 20 to 19 multiplies its 3 by 3 cofactor $= 1$: then det drops by 1.

34 (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one of those choices will be zero.

Problem Set 5.3, page 278

2 (a) $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = c/(ad - bc)$ (b) $y = \det B_2/\det A = (fg - id)/D$.

3 (a) $x_1 = 3/0$ and $x_2 = -2/0$: *no solution* (b) $x_1 = x_2 = \mathbf{0}/\mathbf{0}$: *undetermined.*

4 (a) $x_1 = \det([\,\boldsymbol{b} \ \ \boldsymbol{a}_2 \ \ \boldsymbol{a}_3\,])/\det A$, if $\det A \neq 0$ (b) The determinant is linear in its first column so $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3| + x_2|\boldsymbol{a}_2\ \boldsymbol{a}_2\ \boldsymbol{a}_3| + x_3|\boldsymbol{a}_3\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$. The last two determinants are zero because of repeated columns, leaving $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$ which is $x_1 \det A$.

6 (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$ (b) $\frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. An invertible symmetric matrix has a symmetric inverse.

8 $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. This is $(\det A)I$ and $\det A = 3$. The $1,3$ cofactor of A is 0. Multiplying by 4 or 100: no change.

9 If we know the cofactors and $\det A = 1$, then $C^{\mathrm{T}} = A^{-1}$ and also $\det A^{-1} = 1$. Now A is the inverse of C^{T}, so A can be found from the cofactor matrix for C.

11 The cofactors of A are integers. Division by $\det A = \pm 1$ gives integer entries in A^{-1}.

15 For $n = 5$, C contains *25* cofactors and each 4 by 4 cofactor has *24* terms. Each term needs *3* multiplications: total 1800 multiplications vs.125 for Gauss-Jordan.

17 Volume$=\begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 20$. Area of faces length of cross product $= \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \begin{matrix} -2i - 2j + 8k \\ \text{length} = 6\sqrt{2} \end{matrix}$

18 (a) Area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5$ (b) $5 +$ new triangle area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 5 + 7 = 12$.

21 The maximum volume is $L_1L_2L_3L_4$ reached when the edges are orthogonal in $\mathbf{R}^4$. With entries 1 and -1 all lengths are $\sqrt{4} = 2$. The maximum determinant is $2^4 = 16$, achieved in Problem 20. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can't be achieved.

23 $A^{\mathrm{T}}A = \begin{bmatrix} a^{\mathrm{T}} \\ b^{\mathrm{T}} \\ c^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a^{\mathrm{T}}a & 0 & 0 \\ 0 & b^{\mathrm{T}}b & 0 \\ 0 & 0 & c^{\mathrm{T}}c \end{bmatrix}$ has $\begin{matrix} \det A^{\mathrm{T}}A & = & (\|a\|\,\|b\|\,\|c\|)^2 \\ \det A & = & \pm\|a\|\,\|b\|\,\|c\| \end{matrix}$

25 The n-dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. Coefficients from $(2 + x)^n$ in Worked Example **2.4A**. Cube from $2I$ has volume 2^n.

26 The pyramid has volume $\frac{1}{6}$. The 4-dimensional pyramid has volume $\frac{1}{24}$ (and $\frac{1}{n!}$ in $\mathbf{R}^n$)

31 Base area 10, height 2, volume 20.

35 $S = (2, 1, -1)$, area $\|PQ \times PS\| = \|(-2, -2, -1)\| = 3$. The other four corners can be $(0, 0, 0)$, $(0, 0, 2)$, $(1, 2, 2)$, $(1, 1, 0)$. The volume of the tilted box is $|\det| = 1$.

39 $AC^{\mathrm{T}} = (\det A)I$ gives $(\det A)(\det C) = (\det A)^n$. Then $\det A = (\det C)^{1/3}$ with $n = 4$. With $\det A^{-1}$ is $1/\det A$, construct A^{-1} using the cofactors. *Invert to find A.*

Problem Set 6.1, page 293

1 The eigenvalues are 1 and 0.5 for A, 1 and 0.25 for A^2, 1 and 0 for A^∞. Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (the trace is now $0.2 + 0.3$). Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $x_1 = (1, 1)$ and $x_2 = (2, -1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved in section 6.6, Problems 18-19).

8 (a) Multiply Ax to see λx which reveals λ (b) Solve $(A - \lambda I)x = \mathbf{0}$ to find x.

10 A has $\lambda_1 = 1$ and $\lambda_2 = .4$ with $x_1 = (1,2)$ and $x_2 = (1,-1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 Columns of $A - \lambda_1 I$ are in the nullspace of $A - \lambda_2 I$ because $M = (A-\lambda_2 I)(A-\lambda_1 I)$ = zero matrix [this is the *Cayley-Hamilton Theorem* in Problem 6.2.32]. Notice that M has *zero eigenvalues* $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_1) = 0$ and $(\lambda_2 - \lambda_2)(\lambda_2 - \lambda_1) = 0$.

13 (a) $Pu = (uu^{\mathrm{T}})u = u(u^{\mathrm{T}}u) = u$ so $\lambda = 1$ (b) $Pv = (uu^{\mathrm{T}})v = u(u^{\mathrm{T}}v) = \mathbf{0}$ (c) $x_1 = (-1,1,0,0)$, $x_2 = (-3,0,1,0)$, $x_3 = (-5,0,0,1)$ all have $Px = 0x = \mathbf{0}$.

15 The other two eigenvalues are $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$; the three eigenvalues are $1, 1, -1$.

16 Set $\lambda = 0$ in $\det(A - \lambda I) = (\lambda_1 - \lambda)\ldots(\lambda_n - \lambda)$ to find $\det A = (\lambda_1)(\lambda_2)\cdots(\lambda_n)$.

17 $\lambda_1 = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc})$ and $\lambda_2 = \frac{1}{2}(a + d - \sqrt{})$ add to $a + d$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = (\lambda - 3)(\lambda - 4) = \lambda^2 - 7\lambda + 12$.

19 (a) rank = 2 (b) $\det(B^{\mathrm{T}}B) = 0$ (d) eigenvalues of $(B^2 + I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{5}$.

20 Last rows are $-28, 11$ (check trace and det) and $6, -11, 6$ (to match $\det(C - \lambda I)$).

22 $\lambda = 1$ (for Markov), 0 (for singular), $-\frac{1}{2}$ (so sum of eigenvalues = trace = $\frac{1}{2}$).

23 $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Always A^2 is the zero matrix if $\lambda = 0$ and 0, by the Cayley-Hamilton Theorem in Problem 6.2.32.

28 B has $\lambda = -1, -1, -1, 3$ and C has $\lambda = 1, 1, 1, -3$. Both have det $= -3$.

32 (a) u is a basis for the nullspace, v and w give a basis for the column space
(b) $x = (0, \frac{1}{3}, \frac{1}{5})$ is a particular solution. Add any cu from the nullspace
(c) If $Ax = u$ had a solution, u would be in the column space: wrong dimension 3.

34 $\det(P - \lambda I) = 0$ gives the equation $\lambda^4 = 1$. This reflects the fact that $P^4 = I$. The solutions of $\lambda^4 = 1$ are $\lambda = 1, i, -1, -i$. The real eigenvector $x_1 = (1,1,1,1)$ is not changed by the permutation P. Three more eigenvectors are (i, i^2, i^3, i^4) and $(1,-1,1,-1)$ and $(-i, (-i)^2, (-i)^3, (-i)^4)$.

36 $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$ give $\det \lambda_1\lambda_2 = 1$ and trace $\lambda_1 + \lambda_2 = -1$. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \dfrac{2\pi}{3}$ has this trace and det. So does every $M^{-1}AM$!

Problem Set 6.2, page 307

1 $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

3 If $A = S\Lambda S^{-1}$ then the eigenvalue matrix for $A + 2I$ is $\Lambda + 2I$ and the eigenvector matrix is still S. $A + 2I = S(\Lambda + 2I)S^{-1} = S\Lambda S^{-1} + S(2I)S^{-1} = A + 2I$.

4 (a) False: don't know λ's (b) True (c) True (d) False: need eigenvectors of S

6 The columns of S are nonzero multiples of $(2,1)$ and $(0,1)$: either order. Same for A^{-1}.

8 $A = S\Lambda S^{-1} = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}1 & -\lambda_2\\ -1 & \lambda_1\end{bmatrix}$. $S\Lambda^k S^{-1} =$
$\frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{bmatrix}\begin{bmatrix}1 & -\lambda_2\\ -1 & \lambda_1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\textit{2nd component} \text{ is } F_k\\ (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)\end{bmatrix}$.

9 (a) $A = \begin{bmatrix}.5 & .5\\ 1 & 0\end{bmatrix}$ has $\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2}$ with $x_1 = (1,1),\ x_2 = (1,-2)$

(b) $A^n = \begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}1^n & 0\\ 0 & (-.5)^n\end{bmatrix}\begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{bmatrix} \to A^\infty = \begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{1}{3}\end{bmatrix}$

12 (a) False: don't know λ (b) True: an eigenvector is missing (c) True.

13 $A = \begin{bmatrix}8 & 3\\ -3 & 2\end{bmatrix}$ (or other), $A = \begin{bmatrix}9 & 4\\ -4 & 1\end{bmatrix}$, $A = \begin{bmatrix}10 & 5\\ -5 & 0\end{bmatrix}$; only eigenvectors are $x = (c, -c)$.

15 $A^k = S\Lambda^k S^{-1}$ approaches zero **if and only if every $|\lambda| < 1$**; $A_1^k \to A_1^\infty, A_2^k \to 0$.

17 $\Lambda = \begin{bmatrix}.9 & 0\\ 0 & .3\end{bmatrix}$, $S = \begin{bmatrix}3 & -3\\ 1 & 1\end{bmatrix}$; $A_2^{10}\begin{bmatrix}3\\ 1\end{bmatrix} = (.9)^{10}\begin{bmatrix}3\\ 1\end{bmatrix}$, $A_2^{10}\begin{bmatrix}3\\ -1\end{bmatrix} = (.3)^{10}\begin{bmatrix}3\\ -1\end{bmatrix}$,
$A_2^{10}\begin{bmatrix}6\\ 0\end{bmatrix} = (.9)^{10}\begin{bmatrix}3\\ 1\end{bmatrix} + (.3)^{10}\begin{bmatrix}3\\ -1\end{bmatrix}$ because $\begin{bmatrix}6\\ 0\end{bmatrix}$ is the sum of $\begin{bmatrix}3\\ 1\end{bmatrix} + \begin{bmatrix}3\\ -1\end{bmatrix}$.

19 $B^k = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 4\end{bmatrix}^k\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix} = \begin{bmatrix}5^k & 5^k - 4^k\\ 0 & 4^k\end{bmatrix}$.

21 trace $ST = (aq + bs) + (cr + dt)$ is equal to $(qa + rc) + (sb + td) =$ trace TS. Diagonalizable case: the trace of $S\Lambda S^{-1}$ = trace of $(\Lambda S^{-1})S = \Lambda$: *sum of the* λ's.

24 The A's form a subspace since cA and $A_1 + A_2$ all have the same S. When $S = I$ the A's with those eigenvectors give the subspace of diagonal matrices. Dimension 4.

26 Two problems: The nullspace and column space can overlap, so x could be in both. There may not be r independent eigenvectors in the column space.

27 $R = S\sqrt{\Lambda}S^{-1} = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ has $R^2 = A$. $\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, trace is not real.

Note that $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$ can have $\sqrt{-1} = i$ and $-i$, trace 0, real square root $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$.

28 $A^{\mathrm{T}} = A$ gives $x^{\mathrm{T}}ABx = (Ax)^{\mathrm{T}}(Bx) \le \|Ax\|\|Bx\|$ by the Schwarz inequality. $B^{\mathrm{T}} = -B$ gives $-x^{\mathrm{T}}BAx = (Bx)^{\mathrm{T}}(Ax) \le \|Ax\|\|Bx\|$. Add to get Heisenberg's Uncertainty Principle when $AB - BA = I$. Position-momentum, also time-energy.

32 If $A = S\Lambda S^{-1}$ then $(A - \lambda_1 I)\cdots(A - \lambda_n I)$ equals $S(\Lambda - \lambda_1 I)\cdots(\Lambda - \lambda_n I)S^{-1}$. The factor $\Lambda - \lambda_j I$ is zero in row j. *The product is zero in all rows =* ***zero matrix.***

33 $\lambda = 2, -1, 0$ are in Λ and the eigenvectors are in S (below). $A^k = S\Lambda^k S^{-1}$ is

$$\begin{bmatrix}2 & 1 & 0\\ 1 & -1 & 1\\ 1 & -1 & -1\end{bmatrix}\Lambda^k\frac{1}{6}\begin{bmatrix}2 & 1 & 1\\ 2 & -2 & -2\\ 0 & 3 & -3\end{bmatrix} = \frac{2^k}{6}\begin{bmatrix}4 & 2 & 2\\ 2 & 1 & 1\\ 2 & 1 & 1\end{bmatrix} + \frac{(-1)^k}{3}\begin{bmatrix}1 & -1 & -1\\ -1 & 1 & 1\\ -1 & 1 & 1\end{bmatrix}$$

Check $k = 4$. The $(2,2)$ entry of A^4 is $2^4/6 + (-1)^4/3 = 18/6 = 3$. The 4-step paths that begin and end at node 2 are 2 to 1 to 1 to 1 to 2, 2 to 1 to 2 to 1 to 2, and 2 to 1 to 3 to 1 to 2. Much harder to find the eleven 4-step paths that start and end at node 1.

35 B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1 and $B^4 = I$. C has $\lambda = (1 \pm \sqrt{3}i)/2$. This is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ and $C^{1024} = -C$.

37 Columns of S times rows of ΛS^{-1} will give r rank-1 matrices ($r =$ rank of A).

Problem Set 6.3, page 325

1 $\boldsymbol{u}_1 = e^{4t}\begin{bmatrix}1\\0\end{bmatrix}$, $\boldsymbol{u}_2 = e^t\begin{bmatrix}1\\-1\end{bmatrix}$. If $\boldsymbol{u}(0) = (5, -2)$, then $\boldsymbol{u}(t) = 3e^{4t}\begin{bmatrix}1\\0\end{bmatrix} + 2e^t\begin{bmatrix}1\\-1\end{bmatrix}$.

4 $d(v+w)/dt = (w-v)+(v-w) = \mathbf{0}$, so the total $v+w$ is constant. $A = \begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$ has $\begin{matrix}\lambda_1 = 0\\ \lambda_2 = -2\end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$; $\begin{matrix}v(1) = 20 + 10e^{-2} & v(\infty) = 20\\ w(1) = 20 - 10e^{-2} & w(\infty) = 20\end{matrix}$

8 $\begin{bmatrix}6 & -2\\2 & 1\end{bmatrix}$ has $\lambda_1 = 5$, $\boldsymbol{x}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\lambda_2 = 2$, $\boldsymbol{x}_2 = \begin{bmatrix}1\\2\end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches $20/10$; e^{5t} dominates.

12 $A = \begin{bmatrix}0 & 1\\-9 & 6\end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector $(1, 3)$.

14 When A is skew-symmetric, $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\|$ is $\|\boldsymbol{u}(0)\|$. So e^{At} is *orthogonal.*

15 $\boldsymbol{u}_p = 4$ and $\boldsymbol{u}(t) = ce^t + 4$; $\quad \boldsymbol{u}_p = \begin{bmatrix}4\\2\end{bmatrix}$ and $\boldsymbol{u}(t) = c_1e^t\begin{bmatrix}1\\t\end{bmatrix} + c_2e^t\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}4\\2\end{bmatrix}$.

16 Substituting $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ gives $ce^{ct}\boldsymbol{v} = Ae^{ct}\boldsymbol{v} - e^{ct}\boldsymbol{b}$ or $(A - cI)\boldsymbol{v} = \boldsymbol{b}$ or $\boldsymbol{v} = (A - cI)^{-1}\boldsymbol{b} =$ particular solution. If c is an eigenvalue then $A - cI$ is not invertible.

20 The solution at time $t + T$ is also $e^{A(t+T)}\boldsymbol{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

21 $\begin{bmatrix}1 & 4\\0 & 0\end{bmatrix} = \begin{bmatrix}1 & 4\\0 & -1\end{bmatrix}\begin{bmatrix}\mathbf{1} & 0\\0 & \mathbf{0}\end{bmatrix}\begin{bmatrix}1 & 4\\0 & -1\end{bmatrix}; \begin{bmatrix}1 & 4\\0 & -1\end{bmatrix}\begin{bmatrix}e^t & 0\\0 & \mathbf{1}\end{bmatrix}\begin{bmatrix}1 & 4\\0 & -1\end{bmatrix} = \begin{bmatrix}e^t & 4e^t - 4\\0 & 1\end{bmatrix}$.

22 $A^2 = A$ gives $e^{At} = I + At + \frac{1}{2}At^2 + \cdots = I + (e^t - 1)A = \begin{bmatrix}e^t & e^t - 1\\0 & 1\end{bmatrix}$.

24 $A = \begin{bmatrix}1 & 1\\0 & 3\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}\mathbf{1} & 0\\0 & \mathbf{3}\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{2}\\0 & \frac{1}{2}\end{bmatrix}$. Then $e^{At} = \begin{bmatrix}e^t & \frac{1}{2}(e^{3t} - e^t)\\0 & e^{3t}\end{bmatrix}$.

26 (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t} \neq 0$.

27 $(x, y) = (e^{4t}, e^{-4t})$ is a growing solution. The correct matrix for the exchanged $\boldsymbol{u} = (y, x)$ is $\begin{bmatrix}2 & -2\\-4 & 0\end{bmatrix}$. It *does* have the same eigenvalues as the original matrix.

28 Centering produces $\boldsymbol{U}_{n+1} = \begin{bmatrix}1 & \Delta t\\-\Delta t & 1 - (\Delta t)^2\end{bmatrix}\boldsymbol{U}_n$. At $\Delta t = 1$, $\lambda = e^{i\pi/3}$ and $e^{-i\pi/3}$ both have $\lambda^6 = 1$ so $A^6 = I$. $\boldsymbol{U}_6 = A^6\boldsymbol{U}_0$ comes exactly back to $\boldsymbol{U}_0$.

29 First A has $\lambda = \pm i$ and $A^4 = I$. Second A has $\lambda = -1, -1$ and $A^n = (-1)^n\begin{bmatrix}1 - 2n & -2n\\2n & 2n + 1\end{bmatrix}$ Linear growth.

30 With $a = \Delta t/2$ the trapezoidal step is $U_{n+1} = \dfrac{1}{1+a^2}\begin{bmatrix} 1-a^2 & 2a \\ -2a & 1-a^2 \end{bmatrix} U_n$.

Orthonormal columns $\Rightarrow$ orthogonal matrix $\Rightarrow \|U_{n+1}\| = \|U_n\|$

31 (a) $(\cos A)x = (\cos\lambda)x$ (b) $\lambda(A) = 2\pi$ and 0 so $\cos\lambda = 1, 1$ and $\cos A = I$ (c) $u(t) = 3(\cos 2\pi t)(1,1) + 1(\cos 0t)(1,-1)$ [$u' = Au$ has **exp**, $u'' = Au$ has **cos**]

Problem Set 6.4, page 337

3 $\lambda = 0, 4, -2$; unit vectors $\pm(0,1,-1)/\sqrt{2}$ and $\pm(2,1,1)/\sqrt{6}$ and $\pm(1,-1,-1)/\sqrt{3}$.

5 $Q = \dfrac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. The columns of Q are unit eigenvectors of A. Each unit eigenvector could be multiplied by -1

8 If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is symmetric then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ gives $\Lambda = 0$. The only symmetric A is $Q\,0\,Q^{\mathrm{T}}$ = zero matrix.

10 If x is not real then $\lambda = x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ is *not* always real. Can't assume real eigenvectors!

11 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2\begin{bmatrix} \frac12 & -\frac12 \\ -\frac12 & \frac12 \end{bmatrix} + 4\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0\begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25\begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$

14 M is skew-symmetric and orthogonal; λ's must be $i, i, -i, -i$ to have trace zero.

16 (a) If $Az = \lambda y$ and $A^{\mathrm{T}}y = \lambda z$ then $B[y;\ -z] = [-Az;\ A^{\mathrm{T}}y] = -\lambda[y;\ -z]$. So $-\lambda$ is also an eigenvalue of B. (b) $A^{\mathrm{T}}Az = A^{\mathrm{T}}(\lambda y) = \lambda^2 z$. (c) $\lambda = -1, -1, 1, 1$; $x_1 = (1,0,-1,0)$, $x_2 = (0,1,0,-1)$, $x_3 = (1,0,1,0)$, $x_4 = (0,1,0,1)$.

19 A has $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B has $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$. Perpendicular for A. Not perpendicular for B since $B^{\mathrm{T}} \neq B$

21 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True from $A^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}}$ (c) True from $A^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ (d) False!

22 A and A^{T} have the same λ's but the *order* of the x's can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $x_1 = (1, i)$ first for A but $x_1 = (1, -i)$ first for A^{T}.

23 A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. A allows $QR, S\Lambda S^{-1}, Q\Lambda Q^{\mathrm{T}}$; B allows $S\Lambda S^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$.

24 Symmetry gives $Q\Lambda Q^{\mathrm{T}}$ if $b = 1$; repeated λ and no S if $b = -1$; singular if $b = 0$.

25 Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so $\lambda = \pm 1$. Then $A = \pm I$ or
$$A = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

27 The roots of $\lambda^2 + b\lambda + c = 0$ differ by $\sqrt{b^2 - 4c}$. For $\det(A + tB - \lambda I)$ we have $b = -3 - 8t$ and $c = 2 + 16t - t^2$. The minimum of $b^2 - 4c$ is $1/17$ at $t = 2/17$. Then $\lambda_2 - \lambda_1 = 1/\sqrt{17}$.

29 (a) $A = Q\Lambda\overline{Q}^{\mathrm{T}}$ times $\overline{A}^{\mathrm{T}} = Q\overline{\Lambda}^{\mathrm{T}}\overline{Q}^{\mathrm{T}}$ equals $\overline{A}^{\mathrm{T}}$ times A because $\Lambda\overline{\Lambda}^{\mathrm{T}} = \overline{\Lambda}^{\mathrm{T}}\Lambda$ (diagonal!) (b) step 2: The 1, 1 entries of $\overline{T}^{\mathrm{T}}\,T$ and $T\overline{T}^{\mathrm{T}}$ are $|a|^2$ and $|a|^2 + |b|^2$. This makes $b = 0$ and $T = \Lambda$.

30 a_{11} is $\begin{bmatrix} q_{11} \dots q_{1n}\end{bmatrix}\begin{bmatrix}\lambda_1\overline{q}_{11} \dots \lambda_n\overline{q}_{1n}\end{bmatrix}^{\mathrm{T}} \le \lambda_{\max}\left(|q_{11}|^2 + \cdots + |q_{1n}|^2\right) = \lambda_{\max}$.

31 (a) $x^{\mathrm{T}}(Ax) = (Ax)^{\mathrm{T}}x = x^{\mathrm{T}}A^{\mathrm{T}}x = -x^{\mathrm{T}}Ax$. (b) $\overline{z}^{\mathrm{T}}Az$ is pure imaginary, its real part is $x^{\mathrm{T}}Ax + y^{\mathrm{T}}Ay = 0 + 0$ (c) $\det A = \lambda_1 \dots \lambda_n \ge 0$: pairs of λ's $= ib, -ib$.

Problem Set 6.5, page 350

3 Positive definite for $-3 < b < 3$ $\quad \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 9-b^2\end{bmatrix} = \begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9-b^2\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}$

Positive definite for $c > 8$ $\quad \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\ 0 & c-8\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & c-8\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} = LDL^{\mathrm{T}}$.

4 $f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2$; $x^2 + 6xy + 9y^2 = (x + 3y)^2$.

8 $A = \begin{bmatrix}3 & 6\\ 6 & 16\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$. Pivots 3, 4 outside squares, ℓ_{ij} inside. $x^{\mathrm{T}}Ax = 3(x + 2y)^2 + 4y^2$

10 $A = \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $\quad B = \begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$ is singular; $B\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

12 A is positive definite for $c > 1$; determinants $c, c^2 - 1, (c-1)^2(c+2) > 0$. B is *never* positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

14 The eigenvalues of A^{-1} are positive because they are $1/\lambda(A)$. And the entries of A^{-1} pass the determinant tests. And $x^{\mathrm{T}}A^{-1}x = (A^{-1}x)^{\mathrm{T}}A(A^{-1}x) > 0$ for all $x \ne \mathbf{0}$.

17 If a_{jj} were smaller than all λ's, $A - a_{jj}I$ would have all eigenvalues > 0 (positive definite). But $A - a_{jj}I$ has a *zero* in the (j, j) position; impossible by Problem 16.

21 A is positive definite when $s > 8$; B is positive definite when $t > 5$ by determinants.

22 $R = \dfrac{\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}}{\sqrt{2}}\begin{bmatrix}\sqrt{9} & \\ & \sqrt{1}\end{bmatrix}\dfrac{\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}}{\sqrt{2}} = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$; $R = Q\begin{bmatrix}4 & 0\\ 0 & 2\end{bmatrix}Q^{\mathrm{T}} = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$.

24 The ellipse $x^2 + xy + y^2 = 1$ has axes with half-lengths $1/\sqrt{\lambda} = \sqrt{2}$ and $\sqrt{2/3}$.

25 $A = C^{\mathrm{T}}C = \begin{bmatrix}9 & 3\\ 3 & 5\end{bmatrix}$; $\begin{bmatrix}4 & 8\\ 8 & 25\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}4 & 0\\ 0 & 9\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ and $C = \begin{bmatrix}2 & 4\\ 0 & 3\end{bmatrix}$

29 $H_1 = \begin{bmatrix}6x^2 & 2x\\ 2x & 2\end{bmatrix}$ is positive definite if $x \ne 0$; $F_1 = (\frac{1}{2}x^2 + y)^2 = 0$ on the curve $\frac{1}{2}x^2 + y = 0$; $H_2 = \begin{bmatrix}6x & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ is indefinite, $(0, 1)$ is a saddle point of F_2.

31 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a "trough" staying at zero on the line $2x + 3y = 0$.

32 Orthogonal matrices, exponentials e^{At}, matrices with det $= 1$ are groups. Examples of subgroups are orthogonal matrices with det $= 1$, exponentials e^{An} for integer n.

34 The five eigenvalues of K are $2 - 2\cos\frac{k\pi}{6} = 2 - \sqrt{3}, 2 - 1, 2, 2 + 1, 2 + \sqrt{3}$: product of eigenvalues $= 6 = \det K$.

Problem Set 6.6, page 360

1 $B = GCG^{-1} = GF^{-1}AFG^{-1}$ so $M = FG^{-1}$. C similar to A and $B \Rightarrow A$ similar to B.

6 *Eight families* of similar matrices: six matrices have $\lambda = 0, 1$ (one family); three matrices have $\lambda = 1, 1$ and three have $\lambda = 0, 0$ (two families each!); one has $\lambda = 1, -1$; one has $\lambda = 2, 0$; two have $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$ (they are in one family).

7 (a) $(M^{-1}AM)(M^{-1}x) = M^{-1}(Ax) = M^{-1}\mathbf{0} = \mathbf{0}$ (b) The nullspaces of A and of $M^{-1}AM$ have the same *dimension*. Different vectors and different bases.

8 Same Λ Same S But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues $\lambda = 0, 0$.

10 $J^2 = \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix}$ and $J^k = \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix}$; $J^0 = I$ and $J^{-1} = \begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}$.

14 (1) Choose M_i = reverse diagonal matrix to get $M_i^{-1}J_iM_i = M_i^{\mathrm{T}}$ in each block (2) M_0 has those diagonal blocks M_i to get $M_0^{-1}JM_0 = J^{\mathrm{T}}$. (3) $A^{\mathrm{T}} = (M^{-1})^{\mathrm{T}}J^{\mathrm{T}}M^{\mathrm{T}}$ equals $(M^{-1})^{\mathrm{T}}M_0^{-1}JM_0M^{\mathrm{T}} = (MM_0M^{\mathrm{T}})^{-1}A(MM_0M^{\mathrm{T}})$, and A^{T} *is similar to* A.

17 (a) *False*: Diagonalize a nonsymmetric $A = S\Lambda S^{-1}$. Then Λ is symmetric and similar (b) *True*: A singular matrix has $\lambda = 0$. (c) *False*: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar (they have $\lambda = \pm 1$) (d) *True*: Adding I increases all eigenvalues by 1

18 $AB = B^{-1}(BA)B$ so AB is similar to BA. *If* $ABx = \lambda x$ *then* $BA(Bx) = \lambda(Bx)$.

19 Diagonal blocks 6 by 6, 4 by 4; AB has the same eigenvalues as BA plus $6 - 4$ zeros.

22 $A = MJM^{-1}, A^n = MJ^nM^{-1} = 0$ (each J^k has 1's on the kth diagonal). $\det(A - \lambda I) = \lambda^n$ so $J^n = 0$ by the Cayley-Hamilton Theorem.

Problem Set 6.7, page 371

1 $A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix}\begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}} = \dfrac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}}\begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}\dfrac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}}$

4 $A^{\mathrm{T}}A = AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $\sigma_1^2 = \dfrac{3 + \sqrt{5}}{2}, \sigma_2^2 = \dfrac{3 - \sqrt{5}}{2}$. But A is indefinite
$\sigma_1 = (1 + \sqrt{5})/2 = \lambda_1(A)$, $\sigma_2 = (\sqrt{5} - 1)/2 = -\lambda_2(A)$; $u_1 = v_1$ but $u_2 = -v_2$.

5 A proof that ***eigshow*** finds the SVD. When $V_1 = (1, 0), V_2 = (0, 1)$ the demo finds AV_1 and AV_2 at some angle θ. A 90° turn by the mouse to $V_2, -V_1$ finds AV_2 and $-AV_1$ at the angle $\pi - \theta$. Somewhere between, the constantly orthogonal v_1 and v_2 must produce Av_1 and Av_2 *at angle* $\pi/2$. Those orthogonal directions give u_1 and u_2.

9 $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$, which means that $\Sigma = I$.

14 The smallest change in A is to set its smallest singular value σ_2 to zero.

15 The singular values of $A + I$ are *not* $\sigma_j + 1$. Need eigenvalues of $(A + I)^{\mathrm{T}}(A + I)$.

17 $A = U\Sigma V^{\mathrm{T}} =$ [cosines including u_4] **diag(sqrt**$(2 - \sqrt{2}, 2, 2 + \sqrt{2})$**)** [sine matrix]$^{\mathrm{T}}$. $AV = U\Sigma$ says that differences of sines in V are cosines in U times σ's.

Problem Set 7.1, page 380

3 $T(v) = (0,1)$ and $T(v) = v_1v_2$ are not linear.

4 (a) $S(T(v)) = v$ (b) $S(T(v_1) + T(v_2)) = S(T(v_1)) + S(T(v_2))$.

5 Choose $v = (1,1)$ and $w = (-1,0)$. $T(v) + T(w) = (0,1)$ but $T(v+w) = (0,0)$.

7 (a) $T(T(v)) = v$ (b) $T(T(v)) = v + (2,2)$ (c) $T(T(v)) = -v$ (d) $T(T(v)) = T(v)$.

10 Not invertible: (a) $T(1,0) = \mathbf{0}$ (b) $(0,0,1)$ is not in the range (c) $T(0,1) = \mathbf{0}$.

12 Write v as a combination $c(1,1) + d(2,0)$. Then $T(v) = c(2,2) + d(0,0)$. $T(v) = (4,4)$; $(2,2)$; $(2,2)$; if $v = (a,b) = b(1,1) + \frac{a-b}{2}(2,0)$ then $T(v) = b(2,2) + (0,0)$.

16 No matrix A gives $A\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. To professors: Linear transformations on matrix space come from **4** by **4** matrices. Those in Problems 13–15 were special.

17 (a) True (b) True (c) True (d) False.

19 $T(T^{-1}(M)) = M$ so $T^{-1}(M) = A^{-1}MB^{-1}$.

20 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical because $T(1,0) = (a_{11}, 0)$.

27 Also **30** emphasizes that circles are transformed to ellipses (see figure in Section 6.7).

29 (a) $ad - bc = 0$ (b) $ad - bc > 0$ (c) $|ad - bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0,0)$.)

Problem Set 7.2, page 395

3 (Matrix $A)^2 = B$ when (transformation $T)^2 = S$ and output basis = input basis.

5 $T(v_1 + v_2 + v_3) = 2w_1 + w_2 + 2w_3$; A times $(1,1,1)$ gives $(2,1,2)$.

6 $v = c(v_2 - v_3)$ gives $T(v) = \mathbf{0}$; nullspace is $(0, c, -c)$; solutions $(1,0,0) + (0,c,-c)$.

8 For $T^2(v)$ we would need to know $T(w)$. If the w's equal the v's, the matrix is A^2.

12 (c) is wrong because w_1 is not generally in the input space.

14 (a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ = inverse of (a) (c) $A\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ must be $2A\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

16 $MN = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$.

18 $(a,b) = (\cos\theta, -\sin\theta)$. Minus sign from $Q^{-1} = Q^{\mathrm{T}}$.

20 $w_2(x) = 1 - x^2$; $w_3(x) = \frac{1}{2}(x^2 - x)$; $y = 4w_1 + 5w_2 + 6w_3$.

23 The matrix M with these nine entries must be invertible.

27 If T is not invertible, $T(v_1), \ldots, T(v_n)$ is not a basis. We couldn't choose $w_i = T(v_i)$.

30 S takes (x,y) to $(-x,y)$. $S(T(v)) = (\mathbf{-1}, \mathbf{2})$. $S(v) = (-2,1)$ and $T(S(v)) = (\mathbf{1}, \mathbf{-2})$.

34 The last step writes 6, 6, 2, 2 as the overall average 4, 4, 4, 4 plus the difference 2, 2, −2, −2. Therefore $c_1 = 4$ and $c_2 = 2$ and $c_3 = 1$ and $c_4 = 1$.

35 The wavelet basis is $(1,1,1,1,1,1,1,1)$ and the long wavelet and two medium wavelets $(1,1,-1,-1,0,0,0,0)$, $(0,0,0,0,1,1,-1,-1)$ and 4 wavelets with a single pair $1,-1$.

36 If $V\boldsymbol{b} = W\boldsymbol{c}$ then $\boldsymbol{b} = V^{-1}W\boldsymbol{c}$. The change of basis matrix is $V^{-1}W$.

37 Multiplication by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with this basis is represented by 4 by 4 $A = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix}$

38 If $\boldsymbol{w}_1 = A\boldsymbol{v}_1$ and $\boldsymbol{w}_2 = A\boldsymbol{v}_2$ then $a_{11} = a_{22} = 1$. All other entries will be zero.

Problem Set 7.3, page 406

1 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has $\lambda = 50$ and 0, $\boldsymbol{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $\sigma_1 = \sqrt{50}$.

$A\boldsymbol{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 5 \\ 15 \end{bmatrix} = \sigma_1\boldsymbol{u}_1$ and $A\boldsymbol{v}_2 = \boldsymbol{0}$. $\boldsymbol{u}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $AA^{\mathrm{T}}\boldsymbol{u}_1 = 50\,\boldsymbol{u}_1$.

3 $A = QH = \frac{1}{\sqrt{50}}\begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix}\frac{1}{\sqrt{50}}\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$. H is semidefinite because A is singular.

4 $A^+ = V\begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}U^{\mathrm{T}} = \frac{1}{50}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$; $A^+A = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$, $AA^+ = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$.

7 $\begin{bmatrix} \sigma_1\boldsymbol{u}_1 & \sigma_2\boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \end{bmatrix} = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. In general this is $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r\boldsymbol{u}_r\boldsymbol{v}_r^{\mathrm{T}}$.

9 A^+ is A^{-1} because A is invertible. Pseudoinverse equals inverse when A^{-1} exists!

11 $A = [1]\,[5 \;\; 0 \;\; 0]V^{\mathrm{T}}$ and $A^+ = V\begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix}$; $A^+A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $AA^+ = [1]$

13 If $\det A = 0$ then $\operatorname{rank}(A) < n$; thus $\operatorname{rank}(A^+) < n$ and $\det A^+ = 0$.

16 $\boldsymbol{x}^+$ in the row space of A is perpendicular to $\widehat{\boldsymbol{x}} - \boldsymbol{x}^+$ in the nullspace of $A^{\mathrm{T}}A =$ nullspace of A. The right triangle has $c^2 = a^2 + b^2$.

17 $AA^+\boldsymbol{p} = \boldsymbol{p}$, $AA^+\boldsymbol{e} = \boldsymbol{0}$, $A^+A\boldsymbol{x}_r = \boldsymbol{x}_r$, $A^+A\boldsymbol{x}_n = \boldsymbol{0}$.

19 L is determined by ℓ_{21}. Each eigenvector in S is determined by one number. The counts are $1+3$ for LU, $1+2+1$ for LDU, $1+3$ for QR, $1+2+1$ for $U\Sigma V^{\mathrm{T}}$, $2+2+0$ for $S\Lambda S^{-1}$.

22 Keep only the r by r corner Σ_r of Σ (the rest is all zero). Then $A = U\Sigma V^{\mathrm{T}}$ has the required form $A = \widehat{U}M_1\Sigma_r M_2^{\mathrm{T}}\widehat{V}^{\mathrm{T}}$ with an invertible $M = M_1\Sigma_r M_2^{\mathrm{T}}$ in the middle.

23 $\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} A\boldsymbol{v} \\ A^{\mathrm{T}}\boldsymbol{u} \end{bmatrix} = \sigma\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$. The singular values of A are *eigenvalues* of this block matrix.

Problem Set 8.1, page 418

3 The rows of the free-free matrix in equation (9) add to $[0\ \ 0\ \ 0]$ so the right side needs $f_1+f_2+f_3=0$. $\boldsymbol{f}=(-1,0,1)$ gives $c_2u_1-c_2u_2=-1, c_3u_2-c_3u_3=-1, 0=0$. Then $\boldsymbol{u}_{\text{particular}}=(-c_2^{-1}-c_3^{-1},-c_3^{-1},0)$. Add any multiple of $\boldsymbol{u}_{\text{nullspace}}=(1,1,1)$.

4 $\displaystyle\int -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right)dx=-\left[c(x)\frac{du}{dx}\right]_0^1=0$ (bdry cond) so we need $\displaystyle\int f(x)\,dx=0$.

6 Multiply $A_1^{\mathrm{T}}C_1A_1$ as columns of A_1^{T} times c's times rows of A_1. The first 3 by 3 "*element matrix*" $c_1E_1=[1\ \ 0\ \ 0]^{\mathrm{T}}c_1[1\ \ 0\ \ 0]$ has c_1 in the top left corner.

8 The solution to $-u''=1$ with $u(0)=u(1)=0$ is $u(x)=\frac{1}{2}(x-x^2)$. At $x=\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$ this gives $\boldsymbol{u}=2,3,3,2$ (discrete solution in Problem 7) times $(\Delta x)^2=1/25$.

11 Forward/backward/centered for du/dx has a big effect because that term has the large coefficient. MATLAB: E = diag(ones(6, 1), 1); K = 64 $*$ (2$*$ eye(7) $-$ E $-$ E'); D = 80 $*$ ($E-$ eye(7)); (K + D)\ones(7, 1); % forward; (K $-$ D')\ones(7, 1); % backward; (K + $D/2$ $-$ $D'/2$)\ones(7, 1); % centered is usually the best: more accurate

Problem Set 8.2, page 428

1 $A=\begin{bmatrix}-1&1&0\\-1&0&1\\0&-1&1\end{bmatrix}$; nullspace contains $\begin{bmatrix}c\\c\\c\end{bmatrix}$; $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{0}$ for $\boldsymbol{y}=(1,-1,1)$; current along edge 1, edge 3, back on edge 2 (full loop).

5 Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{f}$ is solvable for $\boldsymbol{f}=(1,-1,0)$ and not solvable for $\boldsymbol{f}=(1,0,0)$; $\boldsymbol{f}$ must be orthogonal to $(1,1,1)$ in the nullspace: $f_1+f_2+f_3=0$.

6 $A^{\mathrm{T}}A\boldsymbol{x}=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}3\\-3\\0\end{bmatrix}=\boldsymbol{f}$ produces $\boldsymbol{x}=\begin{bmatrix}1\\-1\\0\end{bmatrix}+\begin{bmatrix}c\\c\\c\end{bmatrix}$; potentials $\boldsymbol{x}=1,-1,0$ and currents $-A\boldsymbol{x}=2,1,-1$; $\boldsymbol{f}$ sends 3 units from node 2 into node 1.

7 $A^{\mathrm{T}}\begin{bmatrix}1&&\\&2&\\&&2\end{bmatrix}A=\begin{bmatrix}3&-1&-2\\-1&3&-2\\-2&-2&4\end{bmatrix}$; $\boldsymbol{f}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ yields $\boldsymbol{x}=\begin{bmatrix}5/4\\1\\7/8\end{bmatrix}+$ any $\begin{bmatrix}c\\c\\c\end{bmatrix}$; potentials $\boldsymbol{x}=\frac{5}{4},1,\frac{7}{8}$ and currents $-CA\boldsymbol{x}=\frac{1}{4},\frac{3}{4},\frac{1}{4}$.

9 Elimination on $A\boldsymbol{x}=\boldsymbol{b}$ always leads to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b}=0$ in the zero rows of U and R: $-b_1+b_2-b_3=0$ and $b_3-b_4+b_5=0$ (those $\boldsymbol{y}$'s are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage* Law around the two *loops*.

11 $A^{\mathrm{T}}A=\begin{bmatrix}2&-1&-1&0\\-1&3&-1&-1\\-1&-1&3&-1\\0&-1&-1&2\end{bmatrix}$ diagonal entry = number of edges into the node
the trace is 2 times the number of nodes
off-diagonal entry $=-1$ if nodes are connected
$A^{\mathrm{T}}A$ is the **graph Laplacian**, $A^{\mathrm{T}}CA$ is **weighted** by C

13 $A^{\mathrm{T}}CA\boldsymbol{x}=\begin{bmatrix}4&-2&-2&0\\-2&8&-3&-3\\-2&-3&8&-3\\0&-3&-3&6\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$ gives four potentials $\boldsymbol{x}=(\frac{5}{12},\frac{1}{6},\frac{1}{6},0)$
I grounded $x_4=0$ and solved for $\boldsymbol{x}$
currents $\boldsymbol{y}=-CA\boldsymbol{x}=(\frac{2}{3},\frac{2}{3},0,\frac{1}{2},\frac{1}{2})$

17 (a) 8 independent columns (b) f must be orthogonal to the nullspace so f's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

Problem Set 8.3, page 437

2 $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; $A^\infty = \begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.

3 $\lambda = 1$ and $.8$, $x = (1, 0)$; 1 and $-.8$, $x = (\frac{5}{9}, \frac{4}{9})$; $1, \frac{1}{4}$, and $\frac{1}{4}$, $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

5 The steady state eigenvector for $\lambda = 1$ is $(0, 0, 1)$ = everyone is dead.

6 Add the components of $Ax = \lambda x$ to find sum $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.

7 $(.5)^k \to 0$ gives $A^k \to A^\infty$; any $A = \begin{bmatrix} .6+.4a & .6-.6a \\ .4-.4a & .4+.6a \end{bmatrix}$ with $\begin{matrix} a \leq 1 \\ .4+.6a \geq 0 \end{matrix}$

9 M^2 is still nonnegative; $[1 \ \cdots \ 1]M = [1 \ \cdots \ 1]$ so multiply on the right by M to find $[1 \ \cdots \ 1]M^2 = [1 \ \cdots \ 1] \Rightarrow$ columns of M^2 add to 1.

10 $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $x_1 = (b, 1-a)$.

12 B has $\lambda = 0$ and $-.5$ with $x_1 = (.3, \ .2)$ and $x_2 = (-1, 1)$; A has $\lambda = 1$ so $A - I$ has $\lambda = 0$. $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t} x_1 = c_1 x_1$.

13 $x = (1, 1, 1)$ is an eigenvector when the row sums are equal; $Ax = (.9, .9, .9)$.

15 The first two A's have $\lambda_{\max} < 1$; $p = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.

16 $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state $(.3, .3, .4)$ and $(30, 30, 40)$.

17 *No*, A has an eigenvalue $\lambda = 1$ and $(I - A)^{-1}$ does not exist.

19 Λ times $S^{-1}\Delta S$ has the same diagonal as $S^{-1}\Delta S$ times Λ because Λ is diagonal.

20 If $B > A > 0$ and $Ax = \lambda_{\max}(A)x > 0$ then $Bx > \lambda_{\max}(A)x$ and $\lambda_{\max}(B) > \lambda_{\max}(A)$.

Problem Set 8.4, page 446

1 Feasible set = line segment $(6, 0)$ to $(0, 3)$; minimum cost at $(6, 0)$, maximum at $(0, 3)$.

2 Feasible set has corners $(0, 0), (6, 0), (2, 2), (0, 6)$. Minimum cost $2x - y$ at $(6, 0)$.

3 Only two corners $(4, 0, 0)$ and $(0, 2, 0)$; let $x_i \to -\infty$, $x_2 = 0$, and $x_3 = x_1 - 4$.

4 From $(0, 0, 2)$ move to $x = (0, 1, 1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $x = (1, 0, 1.5)$ with cost $5(1) + 8(1.5) = \$17$; $r = 1$ means more expensive.

5 $c = [3 \ \ 5 \ \ 7]$ has minimum cost 12 by the Ph.D. since $x = (4, 0, 0)$ is minimizing. The dual problem maximizes $4y$ subject to $y \leq 3$, $y \leq 5$, $y \leq 7$. Maximum = 12.

8 $y^{\mathrm{T}} b \leq y^{\mathrm{T}} A x = (A^{\mathrm{T}} y)^{\mathrm{T}} x \leq c^{\mathrm{T}} x$. The first inequality needed $y \geq 0$ and $Ax - b \geq 0$.

Problem Set 8.5, page 451

1 $\int_0^{2\pi} \cos((j+k)x)\,dx = \left[\frac{\sin((j+k)x)}{j+k}\right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos((j-k)x)\,dx = 0$ Notice $j-k \neq 0$ in the denominator. If $j=k$ then $\int_0^{2\pi} \cos^2 jx\,dx = \pi$.

4 $\int_{-1}^{1}(1)(x^3-cx)\,dx = 0$ and $\int_{-1}^{1}(x^2-\frac{1}{3})(x^3-cx)\,dx = 0$ for all c (odd functions). Choose c so that $\int_{-1}^{1} x(x^3-cx)\,dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^{1} = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.

5 The integrals lead to the Fourier coefficients $a_1 = 0,\ b_1 = 4/\pi,\ b_2 = 0$.

6 From eqn. (3) $a_k = 0$ and $b_k = 4/\pi k$ $(odd\ k)$. The square wave has $\|f\|^2 = 2\pi$. Then eqn. (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots)$. That infinite series equals $\pi^2/8$.

8 $\|v\|^2 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 2$ so $\|v\| = \sqrt{2}$; $\|v\|^2 = 1+a^2+a^4+\cdots = 1/(1-a^2)$ so $\|v\| = 1/\sqrt{1-a^2}$; $\int_0^{2\pi}(1+2\sin x+\sin^2 x)\,dx = 2\pi+0+\pi$ so $\|f\| = \sqrt{3\pi}$.

9 (a) $f(x) = (1+\text{square wave})/2$ so the a's are $\frac{1}{2}, 0, 0, \ldots$ and the b's are $2/\pi, 0, -2/3\pi, 0, 2/5\pi, \ldots$ (b) $a_0 = \int_0^{2\pi} x\,dx/2\pi = \pi$, all other $a_k = 0, b_k = -2/k$.

11 $\cos^2 x = \frac{1}{2}+\frac{1}{2}\cos 2x$; $\cos(x+\frac{\pi}{3}) = \cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$.

13 $a_0 = \dfrac{1}{2\pi}\int F(x)\,dx = \dfrac{1}{2\pi}, a_k = \dfrac{\sin(kh/2)}{\pi kh/2} \to \dfrac{1}{\pi}$ for delta function; all $b_k = 0$.

Problem Set 8.6, page 458

3 If $\sigma_3 = 0$ the third equation is exact.

4 $0, 1, 2$ have probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ and $\sigma^2 = (0-1)^2\frac{1}{4} + (1-1)^2\frac{1}{2} + (2-1)^2\frac{1}{4} = \frac{1}{2}$.

5 Mean $(\frac{1}{2}, \frac{1}{2})$. Independent flips lead to $\Sigma = \mathbf{diag}(\frac{1}{4}, \frac{1}{4})$. Trace $= \sigma^2_{\text{total}} = \frac{1}{2}$.

6 Mean $m = p_0$ and variance $\sigma^2 = (1-p_0)^2 p_0 + (0-p_0)^2(1-p_0) = p_0(1-p_0)$.

7 Minimize $P = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$ at $P' = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 0; a = \sigma_2^2/(\sigma_1^2+\sigma_2^2)$ recovers equation (2) for the statistically correct choice with minimum variance.

8 Multiply $L\Sigma L^{\mathrm{T}} = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}A^{\mathrm{T}}\Sigma^{-1}\Sigma\Sigma^{-1}A(A^{\mathrm{T}}\Sigma^{-1}A)^{-1} = P = (A^{\mathrm{T}}\Sigma^{-1}A)^{-1}$.

9 Row 3 $=$ −row 1 and row 4 $=$ −row 2: A has rank 2.

Problem Set 8.7, page 464

1 (x, y, z) has homogeneous coordinates (cx, cy, cz, c) for $c = 1$ and all $c \neq 0$.

4 $S = \text{diag}\,(c, c, c, 1)$; row 4 of ST and TS is $1, 4, 3, 1$ and $c, 4c, 3c, 1$; use vTS!

5 $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$ for a 1 by 1 square, starting from an 8.5 by 11 page.

9 $\boldsymbol{n} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ has $P = I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$. Notice $\|\boldsymbol{n}\| = 1$.

10 We can choose $(0,0,3)$ on the plane and multiply $T_-PT_+ = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 6 & 6 & 3 & 9 \end{bmatrix}$.

11 $(3,3,3)$ projects to $\frac{1}{3}(-1,-1,4)$ and $(3,3,3,1)$ projects to $(\frac{1}{3},\frac{1}{3},\frac{5}{3},1)$. Row vectors!

13 That projection of a cube onto a plane produces a hexagon.

14 $(3,3,3)(I - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} = \left(-\frac{11}{3},-\frac{11}{3},-\frac{1}{3}\right)$.

15 $(3,3,3,1) \to (3,3,0,1) \to (-\frac{7}{3},-\frac{7}{3},-\frac{8}{3},1) \to (-\frac{7}{3},-\frac{7}{3},\frac{1}{3},1)$.

17 Space is rescaled by $1/c$ because (x,y,z,c) is the same point as $(x/c,y/c,z/c,1)$.

Problem Set 9.1, page 472

1 Without exchange, pivots .001 and 1000; with exchange, 1 and -1. When the pivot is larger than the entries below it, all $|\ell_{ij}| = |\text{entry/pivot}| \le 1$. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

4 The largest $\|\boldsymbol{x}\| = \|A^{-1}\boldsymbol{b}\|$ is $\|A^{-1}\| = 1/\lambda_{\min}$ since $A^{\mathrm{T}} = A$; largest error $10^{-16}/\lambda_{\min}$.

5 Each row of U has at most w entries. Then w multiplications to substitute components of $\boldsymbol{x}$ (already known from below) and divide by the pivot. Total for n rows $< wn$.

6 The triangular L^{-1}, U^{-1}, R^{-1} need $\frac{1}{2}n^2$ multiplications. Q needs n^2 to multiply the right side by $Q^{-1} = Q^{\mathrm{T}}$. So $QR\boldsymbol{x} = \boldsymbol{b}$ takes 1.5 times longer than $LU\boldsymbol{x} = \boldsymbol{b}$.

7 $UU^{-1} = I$: Back substitution needs $\frac{1}{2}j^2$ multiplications on column j, using the j by j upper left block. Then $\frac{1}{2}(1^2 + 2^2 + \cdots + n^2) \approx \frac{1}{2}(\frac{1}{3}n^3) =$ total to find U^{-1}.

10 With 16-digit floating point arithmetic the errors $\|\boldsymbol{x} - \boldsymbol{x}_{\text{computed}}\|$ for $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$ are of order $10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}$.

11 (a) $\cos\theta = \frac{1}{\sqrt{10}}$, $\sin\theta = \frac{-3}{\sqrt{10}}$, $R = Q_{21}A = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}$ (b) $\lambda = 4$; use $-\theta$ $\boldsymbol{x} = (1,-3)/\sqrt{10}$

13 $Q_{ij}A$ uses $4n$ multiplications (2 for each entry in rows i and j). By factoring out $\cos\theta$, the entries 1 and $\pm\tan\theta$ need only $2n$ multiplications, which leads to $\frac{2}{3}n^3$ for QR.

Problem Set 9.2, page 478

1 $\|A\| = 2$, $\|A^{-1}\| = 2, c = 4$; $\|A\| = 3$, $\|A^{-1}\| = 1, c = 3$; $\|A\| = 2 + \sqrt{2} = \lambda_{\max}$ for positive definite A, $\|A^{-1}\| = 1/\lambda_{\min}$, $c = (2+\sqrt{2})/(2-\sqrt{2}) = 5.83$.

3 For the first inequality replace $\boldsymbol{x}$ by $B\boldsymbol{x}$ in $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$; the second inequality is just $\|B\boldsymbol{x}\| \le \|B\|\|\boldsymbol{x}\|$. Then $\|AB\| = \max(\|AB\boldsymbol{x}\|/\|\boldsymbol{x}\|) \le \|A\|\|B\|$.

7 The triangle inequality gives $\|A\boldsymbol{x} + B\boldsymbol{x}\| \le \|A\boldsymbol{x}\| + \|B\boldsymbol{x}\|$. Divide by $\|\boldsymbol{x}\|$ and take the maximum over all nonzero vectors to find $\|A + B\| \le \|A\| + \|B\|$.

8 If $Ax = \lambda x$ then $\|Ax\|/\|x\| = |\lambda|$ for that particular vector x. When we maximize the ratio over all vectors we get $\|A\| \geq |\lambda|$.

13 The residual $b - Ay = (10^{-7}, 0)$ is much smaller than $b - Az = (.0013, .0016)$. But z is much closer to the solution than y.

14 $\det A = 10^{-6}$ so $A^{-1} = 10^3 \begin{bmatrix} 659 & -563 \\ -913 & 780 \end{bmatrix}$: $\|A\| > 1$, $\|A^{-1}\| > 10^6$, then $c > 10^6$.

16 $x_1^2 + \cdots + x_n^2$ is not smaller than $\max(x_i^2)$ and not larger than $(|x_1| + \cdots + |x_n|)^2 = \|x\|_1^2$. $x_1^2 + \cdots + x_n^2 \leq n \max(x_i^2)$ so $\|x\| \leq \sqrt{n}\|x\|_\infty$. Choose $y_i = \operatorname{sign} x_i = \pm 1$ to get $\|x\|_1 = x \cdot y \leq \|x\|\|y\| = \sqrt{n}\|x\|$. $x = (1, \ldots, 1)$ has $\|x\|_1 = \sqrt{n}\,\|x\|$.

Problem Set 9.3, page 489

2 If $Ax = \lambda x$ then $(I - A)x = (1 - \lambda)x$. Real eigenvalues of $B = I - A$ have $|1 - \lambda| < 1$ provided λ is between 0 and 2.

6 Jacobi has $S^{-1}T = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{3}$. Small problem, fast convergence.

7 Gauss-Seidel has $S^{-1}T = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{9}$ which is $(|\lambda|_{\max}$ for Jacobi$)^2$.

9 Set the trace $2 - 2\omega + \frac{1}{4}\omega^2$ equal to $(\omega - 1) + (\omega - 1)$ to find $\omega_{\text{opt}} = 4(2 - \sqrt{3}) \approx 1.07$. The eigenvalues $\omega - 1$ are about .07, a big improvement.

15 In the jth component of Ax_1, $\lambda_1 \sin \frac{j\pi}{n+1} = 2 \sin \frac{j\pi}{n+1} - \sin \frac{(j-1)\pi}{n+1} - \sin \frac{(j+1)\pi}{n+1}$. The last two terms combine into $-2 \sin \frac{j\pi}{n+1} \cos \frac{\pi}{n+1}$. Then $\lambda_1 = 2 - 2\cos \frac{\pi}{n+1}$.

17 $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ gives $u_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $u_2 = \frac{1}{9}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $u_3 = \frac{1}{27}\begin{bmatrix} 14 \\ 13 \end{bmatrix} \to u_\infty = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

18 $R = Q^{\mathrm{T}}A = \begin{bmatrix} 1 & \cos\theta \sin\theta \\ 0 & -\sin^2\theta \end{bmatrix}$ and $A_1 = RQ = \begin{bmatrix} \cos\theta(1 + \sin^2\theta) & -\sin^3\theta \\ -\sin^3\theta & -\cos\theta\sin^2\theta \end{bmatrix}$.

20 If $A - cI = QR$ then $A_1 = RQ + cI = Q^{-1}(QR + cI)Q = Q^{-1}AQ$. No change in eigenvalues because A_1 is similar to A.

21 Multiply $Aq_j = b_{j-1}q_{j-1} + a_j q_j + b_j q_{j+1}$ by q_j^{T} to find $q_j^{\mathrm{T}}Aq_j = a_j$ (because the q's are orthonormal). The matrix form (multiplying by columns) is $AQ = QT$ where T is *tridiagonal*. The entries down the diagonals of T are the a's and b's.

23 If A is symmetric then $A_1 = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ is also symmetric. $A_1 = RQ = R(QR)R^{-1} = RAR^{-1}$ has R and R^{-1} upper triangular, so A_1 cannot have nonzeros on a lower diagonal than A. If A is tridiagonal and symmetric then (by using symmetry for the upper part of A_1) the matrix $A_1 = RAR^{-1}$ is also tridiagonal.

26 If each center a_{ii} is larger than the circle radius r_i (this is diagonal dominance), then 0 is outside all circles: not an eigenvalue so A^{-1} exists.

Problem Set 10.1, page 498

2 In polar form these are $\sqrt{5}e^{i\theta}$, $5e^{2i\theta}$, $\frac{1}{\sqrt{5}}e^{-i\theta}$, $\sqrt{5}$.

4 $|z \times w| = 6$, $|z+w| \le 5$, $|z/w| = \frac{2}{3}$, $|z-w| \le 5$.

5 $a+ib = \frac{\sqrt{3}}{2}+\frac{1}{2}i, \frac{1}{2}+\frac{\sqrt{3}}{2}i, i, -\frac{1}{2}+\frac{\sqrt{3}}{2}i$; $w^{12}=1$.

9 $2+i$; $(2+i)(1+i) = 1+3i$; $e^{-i\pi/2} = -i$; $e^{-i\pi} = -1$; $\frac{1-i}{1+i} = -i$; $(-i)^{103} = i$.

10 $z+\bar{z}$ is real; $z-\bar{z}$ is pure imaginary; $z\bar{z}$ is positive; $z/\bar{z}$ has absolute value 1.

12 (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; λ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a+d$ when $ad = bc$ (c) the λ's can be real and different.

13 Complex λ's when $(a+d)^2 < 4(ad-bc)$; write $(a+d)^2-4(ad-bc)$ as $(a-d)^2+4bc$ which is positive when $bc > 0$.

14 $\det(P-\lambda I) = \lambda^4 - 1 = 0$ has $\lambda = 1, -1, i, -i$ with eigenvectors $(1,1,1,1)$ and $(1,-1,1,-1)$ and $(1,i,-1,-i)$ and $(1,-i,-1,i)$ = columns of Fourier matrix.

16 The symmetric block matrix has real eigenvalues; so $i\lambda$ is real and λ is pure imaginary.

18 $r = 1$, angle $\frac{\pi}{2}-\theta$; multiply by $e^{i\theta}$ to get $e^{i\pi/2} = i$.

21 $\cos 3\theta = \text{Re}[(\cos\theta + i\sin\theta)^3] = \cos^3\theta - 3\cos\theta\sin^2\theta$; $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

23 e^i is at angle $\theta = 1$ on the unit circle; $|i^e| = 1^e$; Infinitely many $i^e = e^{i(\pi/2+2\pi n)e}$.

24 (a) Unit circle (b) Spiral in to $e^{-2\pi}$ (c) Circle continuing around to angle $\theta = 2\pi^2$.

Problem Set 10.2, page 506

3 z = multiple of $(1+i, 1+i, -2)$; $Az = \mathbf{0}$ gives $z^{\mathrm{H}}A^{\mathrm{H}} = \mathbf{0}^{\mathrm{H}}$ so z (not $\bar{z}$!) is orthogonal to all columns of A^{H} (using complex inner product z^{H} times columns of A^{H}).

4 The four fundamental subspaces are now $\boldsymbol{C}(A), \boldsymbol{N}(A), \boldsymbol{C}(A^{\mathrm{H}}), \boldsymbol{N}(A^{\mathrm{H}})$. A^{H} ***and not*** A^{T}.

5 (a) $(A^{\mathrm{H}}A)^{\mathrm{H}} = A^{\mathrm{H}}A^{\mathrm{HH}} = A^{\mathrm{H}}A$ again (b) If $A^{\mathrm{H}}Az = \mathbf{0}$ then $(z^{\mathrm{H}}A^{\mathrm{H}})(Az) = 0$. This is $\|Az\|^2 = 0$ so $Az = \mathbf{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are always the ***same***.

6 (a) False (c) False $A = U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) True: $-i$ is not an eigenvalue when $A = A^{\mathrm{H}}$.

10 $(1,1,1)$, $(1, e^{2\pi i/3}, e^{4\pi i/3})$, $(1, e^{4\pi i/3}, e^{2\pi i/3})$ are orthogonal (complex inner product!) because P is an orthogonal matrix—and therefore its eigenvector matrix is unitary.

11 $C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} = 2 + 5P + 4P^2$ has the Fourier eigenvector matrix F.

The eigenvalues are $2+5+4 = 11$, $2+5e^{2\pi i/3}+4e^{4\pi i/3}$, $2+5e^{4\pi i/3}+4e^{8\pi i/3}$.

13 Determinant = product of the eigenvalues *(all real)*. And $A = A^{\mathrm{H}}$ gives $\det A = \overline{\det A}$.

15 $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$.

18 $V = \frac{1}{L}\begin{bmatrix} 1+\sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{L}\begin{bmatrix} 1+\sqrt{3} & 1-i \\ -1-i & 1+\sqrt{3} \end{bmatrix}$ with $L^2 = 6+2\sqrt{3}$.
Unitary means $|\lambda| = 1$. $V = V^{\mathrm{H}}$ gives real λ. Then trace zero gives $\lambda = 1$ and -1.

19 The $\boldsymbol{v}$'s are columns of a unitary matrix U, so U^{H} is U^{-1}. Then $z = UU^{\mathrm{H}}z =$ (multiply by columns) $= \boldsymbol{v}_1(\boldsymbol{v}_1^{\mathrm{H}}z) + \cdots + \boldsymbol{v}_n(\boldsymbol{v}_n^{\mathrm{H}}z)$: a typical orthonormal expansion.

20 Don't multiply $(e^{-ix})(e^{ix})$. Conjugate the first, then $\int_0^{2\pi} e^{2ix}\,dx = [e^{2ix}/2i]_0^{2\pi} = 0$.

21 $R + iS = (R+iS)^{\mathrm{H}} = R^{\mathrm{T}} - iS^{\mathrm{T}}$; R is symmetric but S is skew-symmetric.

24 $[\,1\,]$ and $[\,-1\,]$; any $[\,e^{i\theta}\,]$; $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$; $\begin{bmatrix} w & e^{i\phi}\overline{z} \\ -z & e^{i\phi}\overline{w} \end{bmatrix}$ with $|w|^2 + |z|^2 = 1$ and any angle ϕ

27 Unitary $U^{\mathrm{H}}U = I$ means $(A^{\mathrm{T}} - iB^{\mathrm{T}})(A+iB) = (A^{\mathrm{T}}A + B^{\mathrm{T}}B) + i(A^{\mathrm{T}}B - B^{\mathrm{T}}A) = I$. $A^{\mathrm{T}}A + B^{\mathrm{T}}B = I$ and $A^{\mathrm{T}}B - B^{\mathrm{T}}A = 0$ which makes the block matrix orthogonal.

30 $A = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix} = S\Lambda S^{-1}$. Note real $\lambda = 1$ and 4.

Problem Set 10.3, page 514

8 $c \to (1,1,1,1,0,0,0,0) \to (4,0,0,0,0,0,0,0) \to (4,0,0,0,4,0,0,0) = F_8 c$.
$C \to (0,0,0,0,1,1,1,1) \to (0,0,0,0,4,0,0,0) \to (4,0,0,0,-4,0,0,0) = F_8 C$.

9 If $w^{64} = 1$ then w^2 is a 32nd root of 1 and $\sqrt{w}$ is a 128th root of 1: Key to FFT.

13 $e_1 = c_0 + c_1 + c_2 + c_3$ and $e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3$; E contains the four eigenvalues of $C = FEF^{-1}$ because F contains the eigenvectors.

14 Eigenvalues $e_1 = 2 - 1 - 1 = 0$, $e_2 = 2 - i - i^3 = 2$, $e_3 = 2 - (-1) - (-1) = 4$, $e_4 = 2 - i^3 - i^9 = 2$. Just transform column 0 of C. Check trace $0 + 2 + 4 + 2 = 8$.

15 Diagonal E needs n multiplications, Fourier matrix F and F^{-1} need $\frac{1}{2}n\log_2 n$ multiplications each by the **FFT**. The total is much less than the ordinary n^2 for C times $\boldsymbol{x}$.

Conceptual Questions for Review

Chapter 1

1.1 Which vectors are linear combinations of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$?

1.2 Compare the dot product of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$ to the product of their lengths. Which is larger? Whose inequality?

1.3 What is the cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ in Question 1.2? What is the cosine of the angle between the x-axis and $\boldsymbol{v}$?

Chapter 2

2.1 Multiplying a matrix A times the column vector $\boldsymbol{x} = (2, -1)$ gives what combination of the columns of A? How many rows and columns in A?

2.2 If $A\boldsymbol{x} = \boldsymbol{b}$ then the vector $\boldsymbol{b}$ is a linear combination of what vectors from the matrix A? In vector space language, $\boldsymbol{b}$ lies in the _____ space of A.

2.3 If A is the 2 by 2 matrix $\left[\begin{smallmatrix} 2 & 1 \\ 6 & 6 \end{smallmatrix}\right]$ what are its pivots?

2.4 If A is the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ how does elimination proceed? What permutation matrix P is involved?

2.5 If A is the matrix $\left[\begin{smallmatrix} 2 & 1 \\ 6 & 3 \end{smallmatrix}\right]$ find $\boldsymbol{b}$ and $\boldsymbol{c}$ so that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution and $A\boldsymbol{x} = \boldsymbol{c}$ has a solution.

2.6 What 3 by 3 matrix L adds 5 times row 2 to row 3 and then adds 2 times row 1 to row 2, when it multiplies a matrix with three rows?

2.7 What 3 by 3 matrix E subtracts 2 times row 1 from row 2 and then subtracts 5 times row 2 from row 3? How is E related to L in Question 2.6?

2.8 If A is 4 by 3 and B is 3 by 7, how many *row times column* products go into AB? How many *column times row* products go into AB? How many separate small multiplications are involved (the same for both)?

2.9 Suppose $A = \left[\begin{smallmatrix} I & U \\ 0 & I \end{smallmatrix}\right]$ is a matrix with 2 by 2 blocks. What is the inverse matrix?

2.10 How can you find the inverse of A by working with $[\,A \quad I\,]$? If you solve the n equations Ax = columns of I then the solutions x are columns of _____.

2.11 How does elimination decide whether a square matrix A is invertible?

2.12 Suppose elimination takes A to U (upper triangular) by row operations with the multipliers in L (lower triangular). Why does the last row of A agree with the last row of L times U?

2.13 What is the factorization (from elimination with possible row exchanges) of any square invertible matrix?

2.14 What is the transpose of the inverse of AB?

2.15 How do you know that the inverse of a permutation matrix is a permutation matrix? How is it related to the transpose?

Chapter 3

3.1 What is the column space of an invertible n by n matrix? What is the nullspace of that matrix?

3.2 If every column of A is a multiple of the first column, what is the column space of A?

3.3 What are the two requirements for a set of vectors in $\mathbf{R}^n$ to be a subspace?

3.4 If the row reduced form R of a matrix A begins with a row of ones, how do you know that the other rows of R are zero and what is the nullspace?

3.5 Suppose the nullspace of A contains only the zero vector. What can you say about solutions to $Ax = b$?

3.6 From the row reduced form R, how would you decide the rank of A?

3.7 Suppose column 4 of A is the sum of columns 1, 2, and 3. Find a vector in the nullspace.

3.8 Describe in words the complete solution to a linear system $Ax = b$.

3.9 If $Ax = b$ has exactly one solution for every b, what can you say about A?

3.10 Give an example of vectors that span $\mathbf{R}^2$ but are not a basis for $\mathbf{R}^2$.

3.11 What is the dimension of the space of 4 by 4 symmetric matrices?

3.12 Describe the meaning of *basis* and *dimension* of a vector space.

3.13 Why is every row of A perpendicular to every vector in the nullspace?

3.14 How do you know that a column $\boldsymbol{u}$ times a row $\boldsymbol{v}^{\mathrm{T}}$ (both nonzero) has rank 1?

3.15 What are the dimensions of the four fundamental subspaces, if A is 6 by 3 with rank 2?

3.16 What is the row reduced form R of a 3 by 4 matrix of all 2's?

3.17 Describe a *pivot column* of A.

3.18 True? The vectors in the left nullspace of A have the form $A^{\mathrm{T}}\boldsymbol{y}$.

3.19 Why do the columns of every invertible matrix yield a basis?

Chapter 4

4.1 What does the word *complement* mean about orthogonal subspaces?

4.2 If $\boldsymbol{V}$ is a subspace of the 7-dimensional space $\mathbf{R}^7$, the dimensions of $\boldsymbol{V}$ and its orthogonal complement add to _____.

4.3 The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is the vector _____.

4.4 The projection matrix onto the line through $\boldsymbol{a}$ is $P =$ _____.

4.5 The key equation to project $\boldsymbol{b}$ onto the column space of A is the *normal equation* _____.

4.6 The matrix $A^{\mathrm{T}}A$ is invertible when the columns of A are _____.

4.7 The least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ minimizes what error function?

4.8 What is the connection between the least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ and the idea of projection onto the column space?

4.9 If you graph the best straight line to a set of 10 data points, what shape is the matrix A and where does the projection $\boldsymbol{p}$ appear in the graph?

4.10 If the columns of Q are orthonormal, why is $Q^{\mathrm{T}}Q = I$?

4.11 What is the projection matrix P onto the columns of Q?

4.12 If Gram-Schmidt starts with the vectors $\boldsymbol{a} = (2,0)$ and $\boldsymbol{b} = (1,1)$, which two orthonormal vectors does it produce? If we keep $\boldsymbol{a} = (2,0)$ does Gram-Schmidt always produce the same two orthonormal vectors?

4.13 True? Every permutation matrix is an orthogonal matrix.

4.14 The inverse of the orthogonal matrix Q is _____.

Chapter 5

5.1 What is the determinant of the matrix $-I$?

5.2 Explain how the determinant is a linear function of the first row.

5.3 How do you know that $\det A^{-1} = 1/\det A$?

5.4 If the pivots of A (with no row exchanges) are 2, 6, 6, what submatrices of A have known determinants?

5.5 Suppose the first row of A is $0, 0, 0, 3$. What does the "big formula" for the determinant of A reduce to in this case?

5.6 Is the ordering $(2, 5, 3, 4, 1)$ even or odd? What permutation matrix has what determinant, from your answer?

5.7 What is the cofactor C_{23} in the 3 by 3 elimination matrix E that subtracts 4 times row 1 from row 2? What entry of E^{-1} is revealed?

5.8 Explain the meaning of the cofactor formula for $\det A$ using column 1.

5.9 How does Cramer's Rule give the first component in the solution to $I\boldsymbol{x} = \boldsymbol{b}$?

5.10 If I combine the entries in row 2 with the cofactors from row 1, why is $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ automatically zero?

5.11 What is the connection between determinants and volumes?

5.12 Find the cross product of $\boldsymbol{u} = (0, 0, 1)$ and $\boldsymbol{v} = (0, 1, 0)$ and its direction.

5.13 If A is n by n, why is $\det(A - \lambda I)$ a polynomial in λ of degree n?

Chapter 6

6.1 What equation gives the eigenvalues of A without involving the eigenvectors? How would you then find the eigenvectors?

6.2 If A is singular what does this say about its eigenvalues?

6.3 If A times A equals $4A$, what numbers can be eigenvalues of A?

6.4 Find a real matrix that has no real eigenvalues or eigenvectors.

6.5 How can you find the sum and product of the eigenvalues directly from A?

6.6 What are the eigenvalues of the rank one matrix $[\,1\ \ 2\ \ 1\,]^{\mathrm{T}}[\,1\ \ 1\ \ 1\,]$?

6.7 Explain the diagonalization formula $A = S\Lambda S^{-1}$. Why is it true and when is it true?

6.8 What is the difference between the algebraic and geometric multiplicities of an eigenvalue of A? Which might be larger?

6.9 Explain why the trace of AB equals the trace of BA.

6.10 How do the eigenvectors of A help to solve $du/dt = Au$?

6.11 How do the eigenvectors of A help to solve $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$?

6.12 Define the matrix exponential e^A and its inverse and its square.

6.13 If A is symmetric, what is special about its eigenvectors? Do any other matrices have eigenvectors with this property?

6.14 What is the diagonalization formula when A is symmetric?

6.15 What does it mean to say that A is *positive definite*?

6.16 When is $B = A^{\mathrm{T}}A$ a positive definite matrix (A is real)?

6.17 If A is positive definite describe the surface $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$ in $\mathbf{R}^n$.

6.18 What does it mean for A and B to be *similar*? What is sure to be the same for A and B?

6.19 The 3 by 3 matrix with ones for $i \geq j$ has what Jordan form?

6.20 The SVD expresses A as a product of what three types of matrices?

6.21 How is the SVD for A linked to $A^{\mathrm{T}}A$?

Chapter 7

7.1 Define a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ and give one example.

7.2 If the upper middle house on the cover of the book is the original, find something nonlinear in the transformations of the other eight houses.

7.3 If a linear transformation takes every vector in the input basis into the next basis vector (and the last into zero), what is its matrix?

7.4 Suppose we change from the standard basis (the columns of I) to the basis given by the columns of A (invertible matrix). What is the change of basis matrix M?

7.5 Suppose our new basis is formed from the eigenvectors of a matrix A. What matrix represents A in this new basis?

7.6 If A and B are the matrices representing linear transformations S and T on $\mathbf{R}^n$, what matrix represents the transformation from $\boldsymbol{v}$ to $S(T(\boldsymbol{v}))$?

7.7 Describe five important factorizations of a matrix A and explain when each of them succeeds (what conditions on A?).

GLOSSARY: A DICTIONARY FOR LINEAR ALGEBRA

Adjacency matrix of a graph. Square matrix with $a_{ij} = 1$ when there is an edge from node i to node j; otherwise $a_{ij} = 0$. $A = A^{\mathrm{T}}$ when edges go both ways (undirected).

Affine transformation $Tv = Av + v_0 =$ linear transformation plus shift.

Associative Law $(AB)C = A(BC)$. Parentheses can be removed to leave ABC.

Augmented matrix $[\,A \;\; \boldsymbol{b}\,]$. $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is in the column space of A; then $[\,A \;\; \boldsymbol{b}\,]$ has the same rank as A. Elimination on $[\,A \;\; \boldsymbol{b}\,]$ keeps equations correct.

Back substitution. Upper triangular systems are solved in reverse order x_n to x_1.

Basis for V. Independent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ whose linear combinations give each vector in V as $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \ldots + c_d\boldsymbol{v}_d$. V has many bases, each basis gives unique c's. A vector space has many bases!

Big formula for n by n determinants. Det(A) is a sum of $n!$ terms. For each term: Multiply one entry from each row and column of A: rows in order $1, \ldots, n$ and column order given by a permutation P. Each of the $n!$ P's has a $+$ or $-$ sign.

Block matrix. A matrix can be partitioned into matrix blocks, by cuts between rows and/or between columns. **Block multiplication** of AB is allowed if the block shapes permit.

Cayley-Hamilton Theorem. $p(\lambda) = \det(A - \lambda I)$ has $p(A) =$ *zero matrix.*

Change of basis matrix M. The old basis vectors $\boldsymbol{v}_j$ are combinations $\sum m_{ij}\boldsymbol{w}_i$ of the new basis vectors. The coordinates of $c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = d_1\boldsymbol{w}_1 + \cdots + d_n\boldsymbol{w}_n$ are related by $\boldsymbol{d} = M\boldsymbol{c}$. (For $n = 2$ set $\boldsymbol{v}_1 = m_{11}\boldsymbol{w}_1 + m_{21}\boldsymbol{w}_2$, $\boldsymbol{v}_2 = m_{12}\boldsymbol{w}_1 + m_{22}\boldsymbol{w}_2$.)

Characteristic equation $\det(A - \lambda I) = 0$. The n roots are the eigenvalues of A.

Cholesky factorization $A = C^{\mathrm{T}}C = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}}$ for positive definite A.

Circulant matrix C. Constant diagonals wrap around as in cyclic shift S. Every circulant is $c_0 I + c_1 S + \cdots + c_{n-1}S^{n-1}$. $C\boldsymbol{x} =$ **convolution** $\boldsymbol{c} * \boldsymbol{x}$. Eigenvectors in F.

Cofactor C_{ij}. Remove row i and column j; multiply the determinant by $(-1)^{i+j}$.

Column picture of $A\boldsymbol{x} = \boldsymbol{b}$. The vector $\boldsymbol{b}$ becomes a combination of the columns of A. The system is solvable only when $\boldsymbol{b}$ is in the column space $\boldsymbol{C}(A)$.

Column space $\boldsymbol{C}(A)$ = space of all combinations of the columns of A.

Commuting matrices $AB = BA$. If diagonalizable, they share n eigenvectors.

Companion matrix. Put $c_1, \ldots, c_n$ in row n and put $n-1$ ones just above the main diagonal. Then $\det(A - \lambda I) = \pm(c_1 + c_2\lambda + c_3\lambda^2 + \cdots + c_n\lambda^{n-1} - \lambda^n)$.

Complete solution $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$ to $A\boldsymbol{x} = \boldsymbol{b}$. (Particular $\boldsymbol{x}_p$) + ($\boldsymbol{x}_n$ in nullspace).

Complex conjugate $\overline{z} = a - ib$ for any complex number $z = a + ib$. Then $z\overline{z} = |z|^2$.

Condition number $cond(A) = c(A) = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min}$. In $Ax = b$, the relative change $\|\delta x\|/\|x\|$ is less than $cond(A)$ times the relative change $\|\delta b\|/\|b\|$. Condition numbers measure the *sensitivity* of the output to change in the input.

Conjugate Gradient Method. A sequence of steps (end of Chapter 9) to solve positive definite $Ax = b$ by minimizing $\frac{1}{2}x^{\mathrm{T}}Ax - x^{\mathrm{T}}b$ over growing Krylov subspaces.

Covariance matrix Σ. When random variables x_i have mean = average value = 0, their covariances Σ_{ij} are the averages of $x_i x_j$. With means $\overline{x}_i$, the matrix Σ = mean of $(x - \overline{x})(x - \overline{x})^{\mathrm{T}}$ is positive (semi)definite; Σ is diagonal if the x_i are independent.

Cramer's Rule for $Ax = b$. B_j has b replacing column j of A; $x_j = \det B_j / \det A$

Cross product $u \times v$ in $\mathbf{R}^3$. Vector perpendicular to u and v, length $\|u\|\|v\||\sin\theta| =$ area of parallelogram, $u \times v =$ "determinant" of $[\, i \;\; j \;\; k\,;\, u_1 \;\; u_2 \;\; u_3\,;\, v_1 \;\; v_2 \;\; v_3\,]$.

Cyclic shift S. Permutation with $s_{21} = 1, s_{32} = 1, \ldots$, finally $s_{1n} = 1$. Its eigenvalues are the nth roots $e^{2\pi ik/n}$ of 1; eigenvectors are columns of the Fourier matrix F.

Determinant $|A| = \det(A)$. Defined by $\det I = 1$, sign reversal for row exchange, and linearity in each row. Then $|A| = 0$ when A is singular. Also $|AB| = |A||B|$ and $|A^{-1}| = 1/|A|$ and $|A^{\mathrm{T}}| = |A|$. The big formula for $\det(A)$ has a sum of $n!$ terms, the cofactor formula uses determinants of size $n - 1$, volume of box $= |\det(A)|$.

Diagonal matrix D. $d_{ij} = 0$ if $i \neq j$. **Block-diagonal**: zero outside square blocks D_{ii}.

Diagonalizable matrix A. Must have n independent eigenvectors (in the columns of S; automatic with n different eigenvalues). Then $S^{-1}AS = \Lambda =$ eigenvalue matrix.

Diagonalization $\Lambda = S^{-1}AS$. $\Lambda =$ eigenvalue matrix and $S =$ eigenvector matrix of A. A must have n independent eigenvectors to make S invertible. All $A^k = S\Lambda^k S^{-1}$.

Dimension of vector space $\dim(V) =$ number of vectors in any basis for V.

Distributive Law $A(B + C) = AB + AC$. Add then multiply, or multiply then add.

Dot product = Inner product $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$. Complex dot product is $\overline{x}^{\mathrm{T}}y$. Perpendicular vectors have $\overline{x}^{\mathrm{T}}y = 0$. $(AB)_{ij} =$ (row i of A)$^{\mathrm{T}}$(column j of B).

Echelon matrix U. The first nonzero entry (the pivot) in each row comes in a later column than the pivot in the previous row. All zero rows come last.

Eigenvalue λ **and eigenvector** x. $Ax = \lambda x$ with $x \neq \mathbf{0}$ so $\det(A - \lambda I) = 0$.

Elimination. A sequence of row operations that reduces A to an upper triangular U or to the reduced form $R = \mathsf{rref}(A)$. Then $A = LU$ with multipliers ℓ_{ij} in L, or $PA = LU$ with row exchanges in P, or $EA = R$ with an invertible E.

Elimination matrix = Elementary matrix E_{ij}. The identity matrix with an extra $-\ell_{ij}$ in the i, j entry ($i \neq j$). Then $E_{ij}A$ subtracts ℓ_{ij} times row j of A from row i.

Ellipse (or ellipsoid) $x^{\mathrm{T}}Ax = 1$. A must be positive definite; the axes of the ellipse are eigenvectors of A, with lengths $1/\sqrt{\lambda}$. (For $\|x\| = 1$ the vectors $y = Ax$ lie on the ellipse $\|A^{-1}y\|^2 = y^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}y = 1$ displayed by eigshow; axis lengths σ_i.)

Exponential $e^{At} = I + At + (At)^2/2! + \cdots$ has derivative Ae^{At}; $e^{At}u(0)$ solves $u' = Au$.

Factorization $A = LU$. If elimination takes A to U *without row exchanges*, then the lower triangular L with multipliers ℓ_{ij} (and $\ell_{ii} = 1$) brings U back to A.

Fast Fourier Transform (FFT). A factorization of the Fourier matrix F_n into $\ell = \log_2 n$ matrices S_i times a permutation. Each S_i needs only $n/2$ multiplications, so $F_n\boldsymbol{x}$ and $F_n^{-1}\boldsymbol{c}$ can be computed with $n\ell/2$ multiplications. Revolutionary.

Fibonacci numbers $0, 1, 1, 2, 3, 5, \ldots$ satisfy $F_n = F_{n-1} + F_{n-2} = (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)$. Growth rate $\lambda_1 = (1+\sqrt{5})/2$ is the largest eigenvalue of the Fibonacci matrix $\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]$.

Four Fundamental Subspaces $\boldsymbol{C}(A), \boldsymbol{N}(A), \boldsymbol{C}(A^{\mathrm{T}}), \boldsymbol{N}(A^{\mathrm{T}})$. Use $\overline{A}^{\mathrm{T}}$ for complex A.

Fourier matrix F. Entries $F_{jk} = e^{2\pi ijk/n}$ give orthogonal columns $\overline{F}^{\mathrm{T}} F = nI$. Then $y = Fc$ is the (inverse) Discrete Fourier Transform $y_j = \sum c_k e^{2\pi ijk/n}$.

Free columns of A. Columns without pivots; these are combinations of earlier columns.

Free variable x_i. Column i has no pivot in elimination. We can give the $n - r$ free variables any values, then $A\boldsymbol{x} = \boldsymbol{b}$ determines the r pivot variables (if solvable!).

Full column rank $r = n$. Independent columns, $\boldsymbol{N}(A) = \{\boldsymbol{0}\}$, no free variables.

Full row rank $r = m$. Independent rows, at least one solution to $A\boldsymbol{x} = \boldsymbol{b}$, column space is all of $\mathbf{R}^m$. *Full rank* means full column rank or full row rank.

Fundamental Theorem. The nullspace $\boldsymbol{N}(A)$ and row space $\boldsymbol{C}(A^{\mathrm{T}})$ are orthogonal complements in $\mathbf{R}^n$(perpendicular from $A\boldsymbol{x} = \boldsymbol{0}$ with dimensions r and $n - r$). Applied to A^{T}, the column space $\boldsymbol{C}(A)$ is the orthogonal complement of $\boldsymbol{N}(A^{\mathrm{T}})$ in $\mathbf{R}^m$.

Gauss-Jordan method. Invert A by row operations on $[\,A \quad I\,]$ to reach $[\,I \quad A^{-1}\,]$.

Gram-Schmidt orthogonalization $A = QR$. Independent columns in A, orthonormal columns in Q. Each column $\boldsymbol{q}_j$ of Q is a combination of the first j columns of A (and conversely, so R is upper triangular). Convention: $\mathrm{diag}(R) > \boldsymbol{0}$.

Graph G. Set of n nodes connected pairwise by m edges. A **complete graph** has all $n(n-1)/2$ edges between nodes. A **tree** has only $n - 1$ edges and no closed loops.

Hankel matrix H. Constant along each antidiagonal; h_{ij} depends on $i + j$.

Hermitian matrix $A^{\mathrm{H}} = \overline{A}^{\mathrm{T}} = A$. Complex analog $\overline{a_{ji}} = a_{ij}$ of a symmetric matrix.

Hessenberg matrix H. Triangular matrix with one extra nonzero adjacent diagonal.

Hilbert matrix hilb(n). Entries $H_{ij} = 1/(i+j-1) = \int_0^1 x^{i-1}x^{j-1}dx$. Positive definite but extremely small $\lambda_{\min}$ and large condition number: H is *ill-conditioned*.

Hypercube matrix P_L^2. Row $n + 1$ counts corners, edges, faces, ... of a cube in $\mathbf{R}^n$.

Identity matrix I **(or** I_n**)**. Diagonal entries $= 1$, off-diagonal entries $= 0$.

Incidence matrix of a directed graph. The m by n edge-node incidence matrix has a row for each edge (node i to node j), with entries -1 and 1 in columns i and j.

Indefinite matrix. A symmetric matrix with eigenvalues of both signs ($+$ and $-$).

Independent vectors $v_1, \ldots, v_k$. No combination $c_1v_1 + \cdots + c_kv_k$ = zero vector unless all $c_i = 0$. If the v's are the columns of A, the only solution to $Ax = \mathbf{0}$ is $x = \mathbf{0}$.

Inverse matrix A^{-1}. Square matrix with $A^{-1}A = I$ and $AA^{-1} = I$. No inverse if $\det A = 0$ and $\operatorname{rank}(A) < n$ and $Ax = \mathbf{0}$ for a nonzero vector x. The inverses of AB and A^{T} are $B^{-1}A^{-1}$ and $(A^{-1})^{\mathrm{T}}$. Cofactor formula $(A^{-1})_{ij} = C_{ji}/\det A$.

Iterative method. A sequence of steps intended to approach the desired solution.

Jordan form $J = M^{-1}AM$. If A has s independent eigenvectors, its "generalized" eigenvector matrix M gives $J = \operatorname{diag}(J_1, \ldots, J_s)$. The block J_k is $\lambda_k I_k + N_k$ where N_k has 1's on diagonal 1. Each block has one eigenvalue λ_k and one eigenvector.

Kirchhoff's Laws. *Current Law*: net current (in minus out) is zero at each node. *Voltage Law*: Potential differences (voltage drops) add to zero around any closed loop.

Kronecker product (tensor product) $A \otimes B$. Blocks $a_{ij}B$, eigenvalues $\lambda_p(A)\lambda_q(B)$.

Krylov subspace $K_j(A, b)$. The subspace spanned by $b, Ab, \ldots, A^{j-1}b$. Numerical methods approximate $A^{-1}b$ by x_j with residual $b - Ax_j$ in this subspace. A good basis for K_j requires only multiplication by A at each step.

Least squares solution $\widehat{x}$. The vector $\widehat{x}$ that minimizes the error $\|e\|^2$ solves $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. Then $e = b - A\widehat{x}$ is orthogonal to all columns of A.

Left inverse A^+. If A has full column rank n, then $A^+ = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ has $A^+A = I_n$.

Left nullspace $N(A^{\mathrm{T}})$. Nullspace of A^{T} = "left nullspace" of A because $y^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$.

Length $\|x\|$. Square root of $x^{\mathrm{T}}x$ (Pythagoras in n dimensions).

Linear combination $cv + dw$ or $\sum c_jv_j$. Vector addition and scalar multiplication.

Linear transformation T. Each vector v in the input space transforms to $T(v)$ in the output space, and linearity requires $T(cv + dw) = c\,T(v) + d\,T(w)$. Examples: Matrix multiplication Av, differentiation and integration in function space.

Linearly dependent $v_1, \ldots, v_n$. A combination other than all $c_i = 0$ gives $\sum c_iv_i = \mathbf{0}$.

Lucas numbers $L_n = 2, 1, 3, 4, \ldots$ satisfy $L_n = L_{n-1} + L_{n-2} = \lambda_1^n + \lambda_2^n$, with $\lambda_1, \lambda_2 = (1 \pm \sqrt{5})/2$ from the Fibonacci matrix $\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]$. Compare $L_0 = 2$ with $F_0 = 0$.

Markov matrix M. All $m_{ij} \ge 0$ and each column sum is 1. Largest eigenvalue $\lambda = 1$. If $m_{ij} > 0$, the columns of M^k approach the steady state eigenvector $Ms = s > \mathbf{0}$.

Matrix multiplication AB. The i, j entry of AB is (row i of A)·(column j of B) = $\sum a_{ik}b_{kj}$. By columns: Column j of AB = A times column j of B. By rows: row i of A multiplies B. Columns times rows: AB = sum of (column k)(row k). All these equivalent definitions come from the rule that AB times x equals A times Bx.

Minimal polynomial of A. The lowest degree polynomial with $m(A)$ = zero matrix. This is $p(\lambda) = \det(A - \lambda I)$ if no eigenvalues are repeated; always $m(\lambda)$ divides $p(\lambda)$.

Multiplication $Ax = x_1(\text{column } 1) + \cdots + x_n(\text{column } n)$ = combination of columns.

Multiplicities AM and GM. The algebraic multiplicity AM of λ is the number of times λ appears as a root of $\det(A - \lambda I) = 0$. The geometric multiplicity GM is the number of independent eigenvectors for λ (= dimension of the eigenspace).

Multiplier ℓ_{ij}. The pivot row j is multiplied by ℓ_{ij} and subtracted from row i to eliminate the i, j entry: ℓ_{ij} = (entry to eliminate) / (jth pivot).

Network. A directed graph that has constants $c_1, \ldots, c_m$ associated with the edges.

Nilpotent matrix N. Some power of N is the zero matrix, $N^k = 0$. The only eigenvalue is $\lambda = 0$ (repeated n times). Examples: triangular matrices with zero diagonal.

Norm $\|A\|$. The "ℓ^2 norm" of A is the maximum ratio $\|Ax\|/\|x\| = \sigma_{\max}$. Then $\|Ax\| \leq \|A\|\|x\|$ and $\|AB\| \leq \|A\|\|B\|$ and $\|A + B\| \leq \|A\| + \|B\|$. **Frobenius norm** $\|A\|_F^2 = \sum\sum a_{ij}^2$. The ℓ^1 and ℓ^∞ norms are largest column and row sums of $|a_{ij}|$.

Normal equation $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. Gives the least squares solution to $Ax = b$ if A has full rank n (independent columns). The equation says that (columns of A)$\cdot(b - A\widehat{x}) = 0$.

Normal matrix. If $NN^{\mathrm{T}} = N^{\mathrm{T}}N$, then N has orthonormal (complex) eigenvectors.

Nullspace $N(A)$ = All solutions to $Ax = \mathbf{0}$. Dimension $n - r$ = (# columns) − rank.

Nullspace matrix N. The columns of N are the $n - r$ special solutions to $As = \mathbf{0}$.

Orthogonal matrix Q. Square matrix with orthonormal columns, so $Q^{\mathrm{T}} = Q^{-1}$. Preserves length and angles, $\|Qx\| = \|x\|$ and $(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$. All $|\lambda| = 1$, with orthogonal eigenvectors. Examples: Rotation, reflection, permutation.

Orthogonal subspaces. Every v in V is orthogonal to every w in W.

Orthonormal vectors $q_1, \ldots, q_n$. Dot products are $q_i^{\mathrm{T}}q_j = 0$ if $i \neq j$ and $q_i^{\mathrm{T}}q_i = 1$. The matrix Q with these orthonormal columns has $Q^{\mathrm{T}}Q = I$. If $m = n$ then $Q^{\mathrm{T}} = Q^{-1}$ and $q_1, \ldots, q_n$ is an **orthonormal basis** for $\mathbf{R}^n$: every $v = \sum(v^{\mathrm{T}}q_j)q_j$.

Outer product uv^{T} = column times row = rank one matrix.

Partial pivoting. In each column, choose the largest available pivot to control roundoff; all multipliers have $|\ell_{ij}| \leq 1$. See *condition number*.

Particular solution x_p. Any solution to $Ax = b$; often x_p has free variables = 0.

Pascal matrix P_S = pascal(n) = the symmetric matrix with binomial entries $\binom{i+j-2}{i-1}$. $P_S = P_L P_U$ all contain Pascal's triangle with det = 1 (see Pascal in the index).

Permutation matrix P. There are $n!$ orders of $1, \ldots, n$. The $n!$ P's have the rows of I in those orders. PA puts the rows of A in the same order. P is *even* or *odd* ($\det P = 1$ or -1) based on the number of row exchanges to reach I.

Pivot columns of A. Columns that contain pivots after row reduction. These are *not* combinations of earlier columns. The pivot columns are a basis for the column space.

Pivot. The diagonal entry (*first nonzero*) at the time when a row is used in elimination.

Plane (or hyperplane) in $\mathbf{R}^n$. Vectors x with $a^{\mathrm{T}}x = 0$. Plane is perpendicular to $a \neq \mathbf{0}$.

Polar decomposition $A = QH$. Orthogonal Q times positive (semi)definite H.

Positive definite matrix A. Symmetric matrix with positive eigenvalues and positive pivots. *Definition*: $x^{\mathrm{T}}Ax > 0$ unless $x = \mathbf{0}$. Then $A = LDL^{\mathrm{T}}$ with diag(D)> 0.

Projection $p = a(a^{\mathrm{T}}b/a^{\mathrm{T}}a)$ **onto the line through** a. $P = aa^{\mathrm{T}}/a^{\mathrm{T}}a$ has rank 1.

Projection matrix P **onto subspace** S. Projection $p = Pb$ is the closest point to b in S, error $e = b - Pb$ is perpendicular to S. $P^2 = P = P^{\mathrm{T}}$, eigenvalues are 1 or 0, eigenvectors are in S or $S^{\perp}$. If columns of A = basis for S then $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

Pseudoinverse A^+ **(Moore-Penrose inverse).** The n by m matrix that "inverts" A from column space back to row space, with $N(A^+) = N(A^{\mathrm{T}})$. A^+A and AA^+ are the projection matrices onto the row space and column space. Rank(A^+) = rank(A).

Random matrix rand(n) or randn(n). MATLAB creates a matrix with random entries, uniformly distributed on $[\,0 \quad 1\,]$ for rand and standard normal distribution for randn.

Rank one matrix $A = uv^{\mathrm{T}} \neq 0$. Column and row spaces = lines cu and cv.

Rank $r(A)$ = number of pivots = dimension of column space = dimension of row space.

Rayleigh quotient $q(x) = x^{\mathrm{T}}Ax/x^{\mathrm{T}}x$ for symmetric A: $\lambda_{\min} \leq q(x) \leq \lambda_{\max}$. Those extremes are reached at the eigenvectors x for $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$.

Reduced row echelon form R = rref(A). Pivots = 1; zeros above and below pivots; the r nonzero rows of R give a basis for the row space of A.

Reflection matrix (Householder) $Q = I - 2uu^{\mathrm{T}}$. Unit vector u is reflected to $Qu = -u$. All x in the plane mirror $u^{\mathrm{T}}x = 0$ have $Qx = x$. Notice $Q^{\mathrm{T}} = Q^{-1} = Q$.

Right inverse A^+. If A has full row rank m, then $A^+ = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ has $AA^+ = I_m$.

Rotation matrix $R = \left[\begin{smallmatrix} c & -s \\ s & c \end{smallmatrix}\right]$ rotates the plane by θ and $R^{-1} = R^{\mathrm{T}}$ rotates back by $-\theta$. Eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$, eigenvectors are $(1, \pm i)$. $c, s = \cos\theta,\ \sin\theta$.

Row picture of $Ax = b$. Each equation gives a plane in $\mathbf{R}^n$; the planes intersect at x.

Row space $C(A^{\mathrm{T}})$ = all combinations of rows of A. Column vectors by convention.

Saddle point of $f(x_1, \ldots, x_n)$. A point where the first derivatives of f are zero and the second derivative matrix ($\partial^2 f/\partial x_i \partial x_j$ = **Hessian matrix**) is indefinite.

Schur complement $S = D - CA^{-1}B$. Appears in block elimination on $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

Schwarz inequality $|v \cdot w| \leq \|v\|\,\|w\|$.Then $|v^{\mathrm{T}}Aw|^2 \leq (v^{\mathrm{T}}Av)(w^{\mathrm{T}}Aw)$ for pos def A.

Semidefinite matrix A. (Positive) semidefinite: all $x^{\mathrm{T}}Ax \geq 0$, all $\lambda \geq 0$; A = any $R^{\mathrm{T}}R$.

Similar matrices A **and** B. Every $B = M^{-1}AM$ has the same eigenvalues as A.

Simplex method for linear programming. The minimum cost vector x^* is found by moving from corner to lower cost corner along the edges of the feasible set (where the constraints $Ax = b$ and $x \geq \mathbf{0}$ are satisfied). Minimum cost at a corner!

Singular matrix A. A square matrix that has no inverse: det(A) = 0.

Singular Value Decomposition (SVD) $A = U\Sigma V^{\mathrm{T}}$ = **(orthogonal)(diag)(orthogonal)** First r columns of U and V are orthonormal bases of $C(A)$ and $C(A^{\mathrm{T}})$, $Av_i = \sigma_i u_i$ with singular value $\sigma_i > 0$. Last columns are orthonormal bases of nullspaces.

Skew-symmetric matrix K. The transpose is $-K$, since $K_{ij} = -K_{ji}$. Eigenvalues are pure imaginary, eigenvectors are orthogonal, e^{Kt} is an orthogonal matrix.

Solvable system $Ax = b$. The right side b is in the column space of A.

Spanning set. Combinations of $v_1, \ldots, v_m$ fill the space. The columns of A span $C(A)$!

Special solutions to $As = 0$. One free variable is $s_i = 1$, other free variables $= 0$.

Spectral Theorem $A = Q\Lambda Q^{\mathrm{T}}$. Real symmetric A has real λ's and orthonormal q's.

Spectrum of A = the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. **Spectral radius** = max of $|\lambda_i|$.

Standard basis for $\mathbf{R}^n$. Columns of n by n identity matrix (written i, j, k in $\mathbf{R}^3$).

Stiffness matrix If x gives the movements of the nodes, Kx gives the internal forces. $K = A^{\mathrm{T}}CA$ where C has spring constants from Hooke's Law and Ax = stretching.

Subspace S **of** V. Any vector space inside V, including V and Z = {zero vector only}.

Sum $V + W$ **of subspaces.** Space of all (v in V) + (w in W). **Direct sum**: $V \cap W = \{0\}$.

Symmetric factorizations $A = LDL^{\mathrm{T}}$ **and** $A = Q\Lambda Q^{\mathrm{T}}$. Signs in Λ = signs in D.

Symmetric matrix A. The transpose is $A^{\mathrm{T}} = A$, and $a_{ij} = a_{ji}$. A^{-1} is also symmetric.

Toeplitz matrix. Constant down each diagonal = time-invariant (shift-invariant) filter.

Trace of A = sum of diagonal entries = sum of eigenvalues of A. Tr AB = Tr BA.

Transpose matrix A^{T}. Entries $A^{\mathrm{T}}_{ij} = A_{ji}$. A^{T} is n by m, $A^{\mathrm{T}}A$ is square, symmetric, positive semidefinite. The transposes of AB and A^{-1} are $B^{\mathrm{T}}A^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$.

Triangle inequality $\|u + v\| \le \|u\| + \|v\|$. For matrix norms $\|A + B\| \le \|A\| + \|B\|$.

Tridiagonal matrix T: $t_{ij} = 0$ if $|i - j| > 1$. T^{-1} has rank 1 above and below diagonal.

Unitary matrix $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}} = U^{-1}$. Orthonormal columns (complex analog of Q).

Vandermonde matrix V. $Vc = b$ gives coefficients of $p(x) = c_0 + \cdots + c_{n-1}x^{n-1}$ with $p(x_i) = b_i$. $V_{ij} = (x_i)^{j-1}$ and det V = product of $(x_k - x_i)$ for $k > i$.

Vector v **in** $\mathbf{R}^n$. Sequence of n real numbers $v = (v_1, \ldots, v_n)$ = point in $\mathbf{R}^n$.

Vector addition. $v + w = (v_1 + w_1, \ldots, v_n + w_n)$ = diagonal of parallelogram.

Vector space V. Set of vectors such that all combinations $cv + dw$ remain within V. Eight required rules are given in Section 3.1 for scalars c, d and vectors v, w.

Volume of box. The rows (or the columns) of A generate a box with volume $|\det(A)|$.

Wavelets $w_{jk}(t)$. Stretch and shift the time axis to create $w_{jk}(t) = w_{00}(2^j t - k)$.

MATRIX FACTORIZATIONS

1. $A = LU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{pivots on the diagonal} \end{pmatrix}$

   Requirements: No row exchanges as Gaussian elimination reduces A to U.

2. $A = LDU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{pivot matrix} \\ D \text{ is diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{1's on the diagonal} \end{pmatrix}$

   Requirements: No row exchanges. The pivots in D are divided out to leave 1's on the diagonal of U. If A is symmetric then U is L^{T} and $A = LDL^{\mathrm{T}}$.

3. $PA = LU$ (permutation matrix P to avoid zeros in the pivot positions).

   Requirements: A is invertible. Then P, L, U are invertible. P does all of the row exchanges in advance, to allow normal LU. Alternative: $A = L_1 P_1 U_1$.

4. $EA = R$ (m by m invertible E) (any matrix A) = rref(A).

   Requirements: None! *The reduced row echelon form* R has r pivot rows and pivot columns. The only nonzero in a pivot column is the unit pivot. The last $m - r$ rows of E are a basis for the left nullspace of A; they multiply A to give zero rows in R. The first r columns of E^{-1} are a basis for the column space of A.

5. $A = C^{\mathrm{T}}C$ = (lower triangular) (upper triangular) with $\sqrt{D}$ on both diagonals

   Requirements: A is symmetric and positive definite (all n pivots in D are positive). This *Cholesky factorization* C = chol(A) has $C^{\mathrm{T}} = L\sqrt{D}$, so $C^{\mathrm{T}}C = LDL^{\mathrm{T}}$.

6. $A = QR$ = (orthonormal columns in Q) (upper triangular R).

   Requirements: A has independent columns. Those are *orthogonalized* in Q by the Gram-Schmidt or Householder process. If A is square then $Q^{-1} = Q^{\mathrm{T}}$.

7. $A = S\Lambda S^{-1}$ = (eigenvectors in S) (eigenvalues in Λ) (left eigenvectors in S^{-1}).

   Requirements: A must have n linearly independent eigenvectors.

8. $A = Q\Lambda Q^{\mathrm{T}}$ = (orthogonal matrix Q) (real eigenvalue matrix Λ) (Q^{T} is Q^{-1}).

   Requirements: A is *real and symmetric*. This is the Spectral Theorem.

9. $A = MJM^{-1}$ = (generalized eigenvectors in M) (Jordan blocks in J) (M^{-1}).

Requirements: A is any square matrix. This *Jordan form* J has a block for each independent eigenvector of A. Every block has only one eigenvalue.

10. $A = U\Sigma V^{\mathrm{T}} = \begin{pmatrix} \text{orthogonal} \\ U \text{ is } m \times n \end{pmatrix} \begin{pmatrix} m \times n \text{ singular value matrix} \\ \sigma_1, \ldots, \sigma_r \text{ on its diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ V \text{ is } n \times n \end{pmatrix}.$

Requirements: None. This *singular value decomposition* (SVD) has the eigenvectors of AA^{T} in U and eigenvectors of $A^{\mathrm{T}}A$ in V; $\sigma_i = \sqrt{\lambda_i(A^{\mathrm{T}}A)} = \sqrt{\lambda_i(AA^{\mathrm{T}})}$.

11. $A^+ = V\Sigma^+U^{\mathrm{T}} = \begin{pmatrix} \text{orthogonal} \\ n \times n \end{pmatrix} \begin{pmatrix} n \times m \text{ pseudoinverse of } \Sigma \\ 1/\sigma_1, \ldots, 1/\sigma_r \text{ on diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ m \times m \end{pmatrix}.$

Requirements: None. The *pseudoinverse* A^+ has A^+A = projection onto row space of A and AA^+ = projection onto column space. The shortest least-squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\widehat{\boldsymbol{x}} = A^+\boldsymbol{b}$. This solves $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

12. $A = QH$ = (orthogonal matrix Q) (symmetric positive definite matrix H).

Requirements: A is invertible. This *polar decomposition* has $H^2 = A^{\mathrm{T}}A$. The factor H is semidefinite if A is singular. The reverse polar decomposition $A = KQ$ has $K^2 = AA^{\mathrm{T}}$. Both have $Q = UV^{\mathrm{T}}$ from the SVD.

13. $A = U\Lambda U^{-1}$ = (unitary U) (eigenvalue matrix Λ) (U^{-1} which is $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}}$).

Requirements: A is *normal*: $A^{\mathrm{H}}A = AA^{\mathrm{H}}$. Its orthonormal (and possibly complex) eigenvectors are the columns of U. Complex λ's unless $A = A^{\mathrm{H}}$: Hermitian case.

14. $A = UTU^{-1}$ = (unitary U) (triangular T with λ's on diagonal) ($U^{-1} = U^{\mathrm{H}}$).

Requirements: *Schur triangularization* of any square A. There is a matrix U with orthonormal columns that makes $U^{-1}AU$ triangular: Section 6.4.

15. $F_n = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{n/2} & \\ & F_{n/2} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}$ = one step of the (recursive) **FFT**.

Requirements: F_n = Fourier matrix with entries w^{jk} where $w^n = 1$: $F_n\overline{F}_n = nI$. D has $1, w, \ldots, w^{n/2-1}$ on its diagonal. For $n = 2^\ell$ the *Fast Fourier Transform* will compute $F_n x$ with only $\frac{1}{2}n\ell = \frac{1}{2}n\log_2 n$ multiplications from ℓ stages of D's.

MATLAB TEACHING CODES

These Teaching Codes are directly available from **web.mit.edu/ 18.06**

cofactor	Compute the n by n matrix of cofactors.
cramer	Solve the system $Ax = b$ by Cramer's Rule.
deter	Matrix determinant computed from the pivots in $PA = LU$.
eigen2	Eigenvalues, eigenvectors, and $\det(A - \lambda I)$ for 2 by 2 matrices.
eigshow	Graphical demonstration of eigenvalues and singular values.
eigval	Eigenvalues and their multiplicity as roots of $\det(A - \lambda I) = 0$.
eigvec	Compute as many linearly independent eigenvectors as possible.
elim	Reduction of A to row echelon form R by an invertible E.
findpiv	Find a pivot for Gaussian elimination (used by **plu**).
fourbase	Construct bases for all four fundamental subspaces.
grams	Gram-Schmidt orthogonalization of the columns of A.
house	2 by 12 matrix giving corner coordinates of a house.
inverse	Matrix inverse (if it exists) by Gauss-Jordan elimination.
leftnull	Compute a basis for the left nullspace.
linefit	Plot the least squares fit to m given points by a line.
lsq	Least squares solution to $Ax = b$ from $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.
normal	Eigenvalues and orthonormal eigenvectors when $A^{\mathrm{T}}A = AA^{\mathrm{T}}$.
nulbasis	Matrix of special solutions to $Ax = 0$ (basis for nullspace).
orthcomp	Find a basis for the orthogonal complement of a subspace.
partic	Particular solution of $Ax = b$, with all free variables zero.
plot2d	Two-dimensional plot for the house figures.
plu	Rectangular $PA = LU$ factorization with row exchanges.
poly2str	Express a polynomial as a string.
project	Project a vector b onto the column space of A.
projmat	Construct the projection matrix onto the column space of A.
randperm	Construct a random permutation.
rowbasis	Compute a basis for the row space from the pivot rows of R.
samespan	Test whether two matrices have the same column space.
signperm	Determinant of the permutation matrix with rows ordered by p.
slu	LU factorization of a square matrix using *no row exchanges.*
slv	Apply **slu** to solve the system $Ax = b$ allowing no row exchanges.
splu	Square $PA = LU$ factorization *with row exchanges.*
splv	The solution to a square, invertible system $Ax = b$.
symmeig	Compute the eigenvalues and eigenvectors of a symmetric matrix.
tridiag	Construct a tridiagonal matrix with constant diagonals a, b, c.

Index

With the single heading* "Matrix" *this page indexes the active life of linear algebra.

Index of Symbols

Linear Algebra Websites

math.mit.edu/linearalgebra Dedicated to help readers and teachers working with this book

ocw.mit.edu MIT's OpenCourseWare site including video lectures in 18.06 and 18.085-6

web.mit.edu/18.06 Current and past exams and homeworks with extra materials

wellesleycambridge.com Ordering information for books by Gilbert Strang

LINEAR ALGEBRA IN A NUTSHELL

((*The matrix* A *is* n *by* n))

Nonsingular	**Singular**
A is invertible	A is not invertible
The columns are independent	The columns are dependent
The rows are independent	The rows are dependent
The determinant is not zero	The determinant is zero
$A\boldsymbol{x} = \mathbf{0}$ has one solution $\boldsymbol{x} = \mathbf{0}$	$A\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions
$A\boldsymbol{x} = \boldsymbol{b}$ has one solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$	$A\boldsymbol{x} = \boldsymbol{b}$ has no solution or infinitely many
A has n (nonzero) pivots	A has $r < n$ pivots
A has full rank $r = n$	A has rank $r < n$
The reduced row echelon form is $R = I$	R has at least one zero row
The column space is all of $\mathbf{R}^n$	The column space has dimension $r < n$
The row space is all of $\mathbf{R}^n$	The row space has dimension $r < n$
All eigenvalues are nonzero	Zero is an eigenvalue of A
$A^{\mathrm{T}}A$ is symmetric positive definite	$A^{\mathrm{T}}A$ is only semidefinite
A has n (positive) singular values	A has $r < n$ singular values

This book is designed to help students understan
central problems of linear algebra:

$Ax = b$	n by n	Chapters 1-2	Linear systems
$Ax = b$	m by n	Chapters 3-4	Least squares
$Ax = \lambda x$	n by n	Chapters 5-6	Eigenvalues
$Av = \sigma u$	m by n	Chapters 6-7	Singular values

The diagram on the front cover shows the four fundamental subspaces for the matrix A. Those subspaces lead to the Fundamental Theorem of Linear Algebra:

1. The dimensions of the four subspaces
2. The orthogonality of the two pairs
3. The best bases for all four subspaces

This is the textbook that accompanies the author's video lectures and the review material on MIT's OpenCourseWare.

ocw.mit.edu **and** ***web.mit.edu/18.06***

Many universities and colleges (and now high schools) use this textbook. Chapters 7-10 are for a second course on linear algebra.

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