# Homework 3

*46-921, Fall 2020*

*Due Thursday, September 24 at 9:30 AM EDT*

Please submit a pdf file with your responses to the questions.

**Question 1:**

Suppose that $X_1, X_2, \ldots, X_n$ are iid with the Gamma$(\alpha, \beta)$ distribution. Determine the method of moments estimators for $\alpha$ and $\beta$.

**Comment:** This is an important case because maximum likelihood does not admit a closed form for the estimators for $\alpha$ and $\beta$.

**Question 2:**

Assume that $X_1, X_2, \ldots, X_n$ are iid with mean $\mu$ and variance $\sigma^2$. Both $\mu$ and $\sigma^2$ are unknown. Assume that the sample mean is $\overline{x} = 14.3$ and the sample standard deviation is $s = 4.2$.

a. Without any further information, can you construct a valid 95% confidence interval for $\mu$ if $n = 40$? If so, do it.
b. Without any further information, can you construct a valid 95% confidence interval for $\mu$ if $n = 10$? If so, do it.
c. If I told you that the $X_i$ are normal, can you construct a valid 95% confidence interval for $\mu$ when $n = 10$? If so, do it.

**Question 3:**

Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. from the Poisson$(\lambda)$ distribution.

a. Find the maximum likelihood estimator for $\lambda$.
b. Is the estimator found in part (a) an unbiased estimator of $\lambda$? Explain.
c. What is the standard error of the estimator found in part (a)?

**Question 4:**

During lecture we compared three different estimators for $\lambda$ when working with an iid sample from the Exponential$(\lambda)$ distribution. We concluded that, based on MSE, the "adjusted" method of moments estimator is the best choice.

Now, conduct a simulation experiment to address the following question: In the case where $n = 20$ and $\lambda = 10$, what proportion of the time does the adjusted method of moments estimator come closer to the true value of $\lambda$ than does the "worst" of the three estimators (the method of moments estimator based on the second moment)? Be sure to submit your code and results.

**Question 5:**

Let $X_1, X_2, \ldots, X_n$ be a iid from the following distribution:

$$f_X(x;\theta) = \begin{cases} \dfrac{3x^2}{\theta^3} & 0 \leq x \leq \theta \\ \\ 0 & \text{otherwise} \end{cases}$$

Here are some facts regarding estimation of $\theta$:

- Method of moments estimator is $\hat{\theta}_1 = \dfrac{4\bar{X}}{3}$
- The maximum likelihood estimator is $\hat{\theta}_2 = X_{(n)}$, where $X_{(n)}$ is the maximum of the $X_i$.

Here are some useful results about this distribution that will help you do this problem.

- $E(X) = \dfrac{3}{4}\theta$
- $V(X) = \dfrac{3}{80}\theta^2$
- $f_{X_{(n)}}(x) = \dfrac{3nx^{3n-1}}{\theta^{3n}}, \quad 0 \leq x \leq \theta$
- $V(X_{(n)}) = \dfrac{3n}{(3n+2)(3n+1)^2}\ \theta^2$

**Now, do the following:**

a. Show that $\hat{\theta}_1$ is an unbiased estimator for $\theta$.

b. Show that $\hat{\theta}_2$ is *not* an unbiased estimator for $\theta$, and find $\text{bias}(\hat{\theta}_2) = [E(\hat{\theta}_2) - \theta]$.

c. Show that for $n > 2$, even though $\hat{\theta}_1$ is unbiased and $\hat{\theta}_2$ is not, $\text{MSE}(\hat{\theta}_2) < \text{MSE}(\hat{\theta}_1)$