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1 Combinatorics

Classical Problems

HanoiTower(HT) min steps	$T_n = 2^n - 1$	Regions by n lines	$L_n = n(n+1)/2 + 1$
Regions by n Zig lines	$Z_n = 2n^2 - n + 1$	Joseph Problem (every m-th)	$F_1 = 0,\ F_i = (F_{i-1} + m)\%i$
Joseph Problem (every 2nd)	rotate n 1-bit to left	HanoiTower (no direct A to C)	$T_n = 3^n - 1$
Bounded regions by n lines	$(n^2 - 3n + 2)/2$	Joseph given pos j,find m.(↓con.)	$m \equiv 1 \pmod{\frac{L}{p}}$,
HT min steps A to C clockw.	$Q_n = 2R_{n-1} + 1$	$L(n) = lcm(1, ..., n)$, p prime $\in [\frac{n}{2}, n]$	$m \equiv j + 1 - n \pmod p$
HT min steps C to A clockw.	$R_n = 2R_{n-1} + Q_{n-1} + 2$	$\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$	$\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$
Egyptian Fraction	$\frac{m}{n} = \frac{1}{\lceil n/m \rceil} + (\frac{m}{n} - \frac{1}{\lceil n/m \rceil})$	Farey Seq given m/n, m'/n'	$m'' = \lfloor (n+N)/n' \rfloor m' - m$
Farey Seq given m/n, m''/n''	$m'/n' = \frac{m+m''}{n+n''}$	$m/n = 0/1$, $m'/n' = 1/N$	$n'' = \lfloor (n+N)/n' \rfloor n' - n$
#labeled rooted trees	n^{n-1}	#labeled unrooted trees	n^{n-2}
#SpanningTree of G (no SL)	$C(G) = D(G) - A(G)(\downarrow)$	Stirling's Formula	$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + \frac{1}{12n})$
D : DegMat; A : AdjMat	$Ans = \lvert \det(C - 1r - 1c) \rvert$	Farey Seq	$mn' - m'n = -1$
#heaps of a tree (keys: $1..n$)	$\frac{(n-1)!}{\prod_{i \neq root} \text{size}(i)}$	#ways $0 \to m$ in n steps (never < 0)	$\frac{m+1}{\frac{n+m}{2}+1}\binom{n}{\frac{n+m}{2}}$
$\#seq\langle a_0, ..., a_{mn}\rangle$ of 1's and $(1-m)$'s with sum $+1 = \binom{mn+1}{n}\frac{1}{mn+1} = \binom{mn}{n}\frac{1}{(m-1)n+1}$			$\mathrm{D}_n = n\mathrm{D}_{n-1} + (-1)^n$

Binomial Coefficients

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$, int $n \geq k \geq 0$	$\binom{n}{k} = \binom{n}{n-k}$, int $n \geq 0$, int k	$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$, int $k \neq 0$
$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$, int k	$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$, int m, k	$(x+y)^r = \sum_k \binom{r}{k} x^k y^{r-k}$, int $r \geq 0$ or $\lvert x/y \rvert < 1$
$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$, int k	$\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}$, int n	$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$, int $m, n \geq 0$
$\binom{r+s}{n} = \sum_k \binom{r}{k}\binom{s}{n-k}$, int n	$\sum_{k \leq m} \binom{r}{k}(\frac{r}{2} - k) = \frac{m+1}{2}\binom{r}{m+1}$, int m	$\sum_{k \leq m} \binom{r}{k}(-1)^k = (-1)^m \binom{r-1}{m}$, int m
$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$, int m, n	$\binom{\binom{k}{2}}{2} = 3\binom{k+1}{4}$ | $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$	$\sum_k \binom{l}{m+k}\binom{s}{n+k} = \binom{l+s}{l-m+n}$ int $l \geq 0$, int m, n
$\sum_k \binom{n}{2k} = 2^{n-even(n)}$	$lcm_{i=0}^{n}\binom{n}{i} = \frac{L(n+1)}{n+1}$	$S(n,1) = S(n,n) = n \Rightarrow S(n,k) = \binom{n+1}{k} - \binom{n-1}{k-1}$
$\sum_{i=1}^{n} \binom{n}{i} F_i = F_{2n}$, $F_n = n$-th Fib	$\sum_i \binom{n-i}{i} = F_{n+1}$	

Famous Numbers

Catalan	$C_0 = 1$, $C_n = \frac{1}{n+1}\binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1} = \frac{4n-2}{n+1} C_{n-1}$	
Stirling 1st kind	$\left[{0 \atop 0}\right] = 1$, $\left[{n \atop 0}\right] = \left[{0 \atop n}\right] = 0$, $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$	#perms of n objs with exactly k cycles
Stirling 2nd kind	$\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$, $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$	#ways to partition n objs into k nonempty sets
Euler	$\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, $\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$	#perms of n objs with exactly k ascents
Euler 2nd Order	$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-k-1)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$	#perms of $1, 1, 2, 2, ..., n, n$ with exactly k ascents
Bell	$B_1 = 1$, $B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k} = \sum_{k=0}^{n} \left\{{n \atop k}\right\}$	#partitions of $1..n$ (Stirling 2nd, no limit on k)

The Twelvefold Way (Putting n balls into k boxes)

Balls Boxes	same same	distinguishable same	same distinguishable	distinguishable distinguishable	Remarks
-	$\mathrm{p}_k(n)$	$\sum_{i=0}^{k} \left\{{n \atop i}\right\}$	$\binom{n+k-1}{k-1}$	k^n	$\mathrm{p}_k(n)$: #partitions of n into $\leq k$ positive parts
size ≥ 1	$\mathrm{p}(n,k)$	$\left\{{n \atop k}\right\}$	$\binom{n-1}{k-1}$	$k!\left\{{n \atop k}\right\}$	$\mathrm{p}(n,k)$: #partitions of n into k positive parts
size ≤ 1	$[n \leq k]$	$[n \leq k]$	$\binom{k}{n}$	$n!\binom{k}{n}$	$[cond]$: 1 if $cond = true$, else 0

Burnside's Lemma: $L = \frac{1}{\lvert G \rvert}\sum_{k=1}^{n} \lvert Z_k \rvert = \frac{1}{\lvert G \rvert}\sum_{a_i \in G} C_1(a_i)$. Z_k : the set of permutations in G under which k stays stable; $C_1(a_i)$: the number of cycles of order 1 in a_i. **Pólya's Theorem:** The number of colorings of n objects with m colors $L = \frac{1}{\lvert \overline{G} \rvert}\sum_{g_i \in \overline{G}} m^{c(g_i)}$. $\overline{G}$: the group over n objects; $c(g_i)$:the number of cycles in g_i.

Regular Polyhedron Coloring **with at most n colors (up to isomorph)**

Description	*Formula*	Remarks	
vertices of octahedron or **faces of cube**	$(n^6 + 3n^4 + 12n^3 + 8n^2)/24$		(V, F, E)
vertices of cube or **faces of octahedron**	$(n^8 + 17n^4 + 6n^2)/24$	tetrahedron:	$(4, 4, 6)$
edges of cube or **edges of octahedron**	$(n^{12} + 6n^7 + 3n^6 + 8n^4 + 6n^3)/24$	cube:	$(8, 6, 12)$
vertices or **faces of tetrahedron**	$(n^4 + 11n^2)/12$	octahedron:	$(6, 8, 12)$
edges of tetrahedron	$(n^6 + 3n^4 + 8n^2)/12$	dodecahedron:	$(20, 12, 30)$
vertices of icosahedron or **faces of dodecahedron**	$(n^{12} + 15n^6 + 44n^4)/60$	icosahedron	$(12, 20, 30)$
vertices of dodecahedron or **faces of icosahedron**	$(n^{20} + 15n^{10} + 20n^8 + 24n^4)/60$		
edges of dodecahedron or **edges of icosahedron**	$(n^{30} + 15n^{16} + 20n^{10} + 24n^6)/60$	*This line may be wrong.*	

2 Number Theory

Classical Theorems

exp of p in $n!$ is $\sum_{i\geq 1}[\frac{n}{p^i}]$	$p_n \sim n\log n$; $\forall_{n>1}\exists_{n<p<2n} : p$ is prime	$\pi(x) \sim \frac{x}{\log x}$; $\mathrm{Norm}(\alpha\beta) = \mathrm{Norm}(\alpha)\cdot\mathrm{Norm}(\beta)$
$\mathrm{lcm}(a,b) = \frac{ab}{\gcd(a,b)}$	$a \equiv b \pmod{x, y} \Rightarrow a \equiv b \pmod{\mathrm{lcm}(x,y)}$	All prime factors of $2^{2^n}+1$ have form $2^{n+2}k+1$
$(2^a-1, 2^b-1) = 2^{(a,b)}-1$	$ac \equiv bc \pmod{m} \Rightarrow a \equiv b(\bmod \frac{m}{\gcd(c,m)})$	n-plygn drawable $\Leftrightarrow n = 2^k\prod F_i$, F_i fermatNum
p_i is prime, $\prod_{p_i\leq n} p_i < 4^n$	$W \equiv d + [2.6m - 0.2] - 2C + Y + \left[\frac{Y}{4}\right] + \left[\frac{C}{4}\right] (\%7)$. $m = 11, 12, 1$ for Ja,Fe,Ma. J&F∈lastyear	

Classical Theorems

p prime $\Leftrightarrow (p-1)! \equiv -1(\%p)$	$a \perp m \Rightarrow a^{\phi(m)} = 1(\%m)$	Min general idx $\lambda(n)$: $\forall_a : a^{\lambda(n)} \equiv 1(\%n)$
$\sum_{d\mid n}\phi(d) = \sum_{d\mid n}\phi(n/d) = n$	$\sum_{m\perp n, m<n} m = \frac{n\phi(n)}{2}$	$\sum_{i=1}^{n}\sigma_0(i) = 2\sum_{i=1}^{[\sqrt{n}]}[n/j] - [\sqrt{n}]^2$
$(\sum_{d\mid n}\sigma_0(d))^2 = \sum_{d\mid n}\sigma_0(d)^3$	$\sum_{d\mid n} n\sigma_1(d)/d = \sum_{d\mid n} d\sigma_0(d)$	$[\sqrt{n}]$ Newton: $y = [\frac{x+[n/x]}{2}]$, $x_0 = 2^{[\frac{\log_2(n)+2}{2}]}$
$\sigma_0(p_1^{e_1}\cdots p_s^{e_s}) = \prod_{i=1}^{s}(e_i+1)$	$\sigma_1(p_1^{e_1}\cdots p_s^{e_s}) = \prod_{i=1}^{s}\frac{p_i^{e_i+1}-1}{p_i-1}$	$r_1 = 4$, $r_k \equiv r_{k-1}^2 - 2(\%M_p)$, M_p prime $\Leftrightarrow r_{p-1} \equiv 0(\%M_p)$
$\mu(p_1p_2\cdots p_s) = (-1)^s$, else 0	$\sum_{d\mid n}\mu(d) = 1$ if $n = 1$, else 0	$F(n) = \sum_{d\mid n} f(d) \Leftrightarrow f(n) = \sum_{d\mid n}\mu(d)F(\frac{n}{d})$
$n = \sum_{d\mid n}\mu(\frac{n}{d})\sigma_1(d)$	$n = 2, 4, p^t, 2p^t \Leftrightarrow n$ has p_roots	$a \perp n$, then $a^i \equiv a^j(\%n) \Leftrightarrow i \equiv j(\%\mathrm{ord}_n(a))$
$1 = \sum_{d\mid n}\mu(\frac{n}{d})\sigma_0(d)$	$r = \mathrm{ord}_n(a)$, $\mathrm{ord}_n(a^u) = \frac{r}{\gcd(r,u)}$	r p_root of n, then r^u is p_root of $n \Leftrightarrow u \perp \phi(n)$
$\mathrm{ord}_n(a) = \mathrm{ord}_n(a^{-1})$	r p_root of $n \Leftrightarrow r^{-1}$ p_root of n	n has p_roots $\Leftrightarrow n$ has $\phi(\phi(n))$ p_roots
$a^n \equiv a^{\phi(m)+n\%\phi(m)}(\%m)$,nbig	$\lambda(2^t) = 2^{t-2}$, $\lambda(p^t) = \phi(p^t) = (p-1)p^{t-1}$, $\lambda(2^{t_0}p_1^{t_1}\cdots p_m^{t_m}) = lcm(\lambda(2^{t0}), \phi(p_1^{t_1}), \cdots, \phi(p_m^{t_m}))$	
$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2}(\%p)$	Legendre sym $\left(\frac{a}{p}\right) = 1$ if a is quad residue $\%p$; -1 if a is non-residue; 0 if $a = 0$	
$a \equiv b(\%p) \Rightarrow \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$	$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$; $\left(\frac{a^2}{p}\right) = 1$	$a \perp p$, s from $a, 2a, ..., \frac{p-1}{2}a(\%p)$ are $> \frac{p}{2} \Rightarrow \left(\frac{a}{p}\right) = (-1)^s$
$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$	Gauss Integer $\pi = a + bi$. $\mathrm{Norm}(\pi) = p$ prime $\Rightarrow \pi$ and $\overline{\pi}$ prime, p not prime	

3 Probability

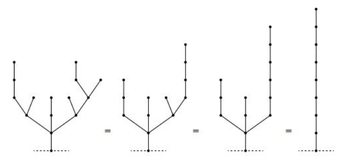
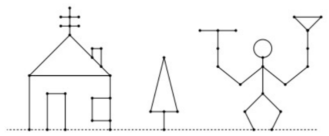
Classical Formulae

Ballot.Always $\#A > k\#B$	$Pr = \frac{a-kb}{a+b}$	Ballot.Always $\#B - \#A \leq k$	$Pr = 1 - \frac{a!b!}{(a+k+1)!(b-k-1)!}$
Ballot.Always $\#A \geq k\#B$	$Pr = \frac{a+1-kb}{a+1}$	Ballot.Always $\#A \geq \#B + k$	$Num = \frac{a-k+1-b}{a-k+1}\binom{a+b-k}{b}$
$E(X+Y) = EX + EY$	$E(\alpha X) = \alpha EX$	X,Y indep. $\Leftrightarrow E(XY) = (EX)(EY)$	

4 Game Theory

Classical Games (❶ last one wins (normal); ❷ last one loses (misère))

Name	Description	Criteria / Opt.strategy	Remarks
NIM	n piles of objs. One can take any number of objs from any pile (i.e. set of possible moves for the i-th pile is $M = [pile_i]$, $[x] := \{1, 2, ..., \lfloor x\rfloor\}$).	$SG = \otimes_{i=1}^{n} pile_i$. Strategy: ❶ make the Nim-Sum 0 by *decreasing* a heap; ❷ the same, except when the normal move would only leave heaps of size 1. In that case, leave an odd number of 1's.	The result of ❷ is the same as ❶, opposite if all piles are 1's. Many games are essentially NIM.
NIM (powers)	$M = \{a^m \mid m \geq 0\}$	If a odd: $SG_n = n\%2$	If a even: $SG_n = 2$, if $n \equiv a\%(a+1)$; $SG_n = n\%(a+1)\%2$, else.
NIM (half)	$M_① = [\frac{pile_i}{2}]$ $M_② = [\lceil\frac{pile_i}{2}\rceil, pile_i]$	①$SG_{2n} = n$, $SG_{2n+1} = SG_n$ ②$SG_0 = 0$, $SG_n = [\log_2 n] + 1$	
NIM (divisors)	$M_①$ = divisors of $pile_i$ $M_②$ = proper divisors of $pile_i$	①$SG_0 = 0$, $SG_n = SG_{②,n} + 1$ ②$SG_1 = 0$, SG_n = number of 0's at the end of n_{binary}	
Subtraction Game	$M_① = [k]$ $M_② = S$ (finite) $M_③ = S \cup \{pile_i\}$	$SG_{①,n} = n \bmod (k+1)$. ❶lose if $SG = 0$; ❷lose if $SG = 1$. $SG_{③,n} = SG_{②,n} + 1$	For any finite M, SG of one pile is eventually periodic.
Moore's NIM_k	One can take any number of objs from at most k piles.	❶Write $pile_i$ in binary, sum up in base $k+1$ without carry. Losing if the result is 0.	❷ If all piles are 1's, losing iff $n \equiv 1\%(k+1)$. Otherwise the result is the same as ❶.

Staircase NIM	n piles in a line. One can take any number of objs from $pile_i$, $i > 0$ to $pile_{i-1}$	Losing if the NIM formed by the odd-indexed piles is losing(i.e. $\otimes_{i=0}^{(n-1)/2} pile_{2i+1} = 0$)	
Lasker's NIM	Two possible moves: 1.take any number of objs; 2.split a pile into two (no obj removed)	$SG_n = n,$ if $n \equiv 1, 2(\%4)$ $SG_n = n + 1,$ if $n \equiv 3(\%4)$ $SG_n = n - 1,$ if $n \equiv 0(\%4)$	
Kayles	Two possible moves: 1.take 1 or 2 objs; 2.split a pile into two (after removing objs)	SG_n for small n can be computed recursively. SG_n for $n \in [72, 83]$: 4 1 2 8 1 4 7 2 1 8 2 7	SG_n becomes periodic from the 72-th item with period length 12.
Dawson's Chess	n boxes in a line. One can occupy a box if its neighbours are not occupied.	SG_n for $n \in [1, 18]$: 1 1 2 0 3 1 1 0 3 3 2 2 4 0 5 2 2 3	Period = 34 from the 52-th item.
Wythoff's Game	**Two** piles of objs. One can take any number of objs from either pile, or take the *same* number from *both* piles.	$n_k = \lfloor k\phi \rfloor = \lfloor m_k \phi \rfloor - m_k$ $m_k = \lfloor k\phi^2 \rfloor = \lceil n_k \phi \rceil = n_k + k$ $\phi := \frac{1+\sqrt{5}}{2}$. (n_k, m_k) is the k-th losing position.	n_k and m_k form a pair of complementary Beatty Sequences (since $\frac{1}{\phi} + \frac{1}{\phi^2} = 1$). Every $x > 0$ appears either in n_k or in m_k.
Mock Turtles	n coins in a line. One can turn over 1, 2 or 3 coins, with the rightmost from head to tail.	$SG_n = 2n$, if ones($2n$) odd; $SG_n = 2n + 1$, else. ones(x): the number of 1's in x_{binary}	SG_n for $n \in [0, 10]$ (leftmost position is 0): 1 2 4 7 8 11 13 14 16 19 21
Ruler	n coins in a line. One can turn over any *consecutive* coins, with the rightmost from head to tail.	SG_n = the largest power of 2 dividing n. This is implemented as $n\&{-n}$(lowbit)	SG_n for $n \in [1, 10]$: 1 2 1 4 1 2 1 8 1 2
Hackenbush-tree	Given a forest of rooted trees, one can take an edge and remove the part which becomes unrooted.	At every branch, one can replace the branches by a non-branching stalk of length equal to their nim-sum.	
Hackenbush-graph		Vertices on any circuit can be *fused* without changing SG of the graph. Fusion: two neighbouring vertices into one, and bend the edge into a loop.