MIE 1603 / 1653 - Integer Programming
Winter 2019
Assignment #3

Due Date: February 23th, Saturday, no later than 11:59PM.

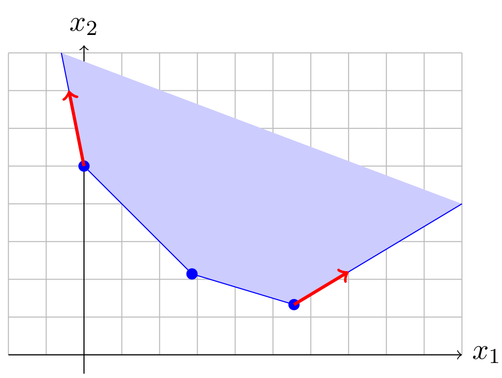
Solve the following problems. You may work in groups of *two* students (from the same course code) or individually. Turn in one solution set with both group members listed on it. Groups must work independently of each other. You *must* cite any references (texts, papers or websites) you have used to help you solve these problems.

You can submit your solutions using Quercus (pdf file) or give a hard-copy to the professor or TA in their office hours (or lectures).

1. [25pts] Consider the following polyhedron

$$P = \left\{ \begin{array}{r} 5x_1 + x_2 \geq 5 \\ x_1 + x_2 \geq 5 \\ 3x_1 + 10x_2 \geq 30 \\ 3x_1 - 5x_2 \leq 10 \\ x_1, x_2 \in \mathbb{R} \end{array} \right\}$$

(a) Draw P. Show in the figure its extreme points and extreme rays.
Solution:



(b) Write down the extreme points and normalized extreme rays. Use the 1-norm for normalizing the extreme rays, i.e., a given vector $v = (v_1, v_2)$ becomes

$$\left(v_1 / (|v_1| + |v_2|), v_2 / (|v_1| + |v_2|) \right)$$

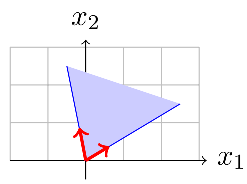
when normalized.
Solution:
Extreme points: (0,5), $(^{15}/_7, ^{20}/_7) = (2.857, 2.143)$ and $(^{50}/_9, ^4/_3) = (5.556, 1.33)$
Extreme rays (norm): $(-^1/_6, ^5/_6) = (-0.166, 0.866)$, $(^5/_8, ^3/_8) = (0.625, 0.375)$

(c) Write down the linear inequalities that define the recession cone and draw the recession cone.

Solution:

The recesion cone is given by:

$$\left\{ \begin{array}{r} 5x_1 + x_2 \geq 5 \\ 3x_1 - 5x_2 \leq 10 \\ x_1, x_2 \in \mathbb{R} \end{array} \right\}$$



(d) Explain why Minkowski's Theorem can be applied to P and use the theorem to express P in terms of its extreme points and extreme rays.

Solution:

We can apply Minkowski's Theorem because P is a polyhedron with $P \neq \emptyset$ and $\operatorname{rank}(A) = 2 = n$, where $A = [-5, -1 \ ; \ -1, -1 \ ; \ -3, -10 \ ; \ 3-5]$. Then, we can write down P in terms of its extreme points and extreme rays as follows:

$$P = \{x \in \mathbb{R}^2 : x = \sum_{i=1}^{3} \lambda_i x^i + \sum_{j=1}^{2} \mu_j r^j, \quad \sum_{i=1}^{3} \lambda_i = 1, \ \lambda_1, \lambda_2, \lambda_3 \geq 0, \ \mu_1, \mu_2 \geq 0\}$$

$$= \left\{ \begin{array}{c} x \in \mathbb{R}^2 : x = \lambda_1 \begin{bmatrix} 0 \\ 5 \end{bmatrix} + \lambda_2 \begin{bmatrix} 15/7 \\ 20/7 \end{bmatrix} + \lambda_3 \begin{bmatrix} 50/9 \\ 4/3 \end{bmatrix} + \mu_1 \begin{bmatrix} -1/6 \\ 5/6 \end{bmatrix} + \mu_2 \begin{bmatrix} 5/8 \\ 3/8 \end{bmatrix}, \\ \sum_{i=1}^{3} \lambda_i = 1, \ \lambda_1, \lambda_2, \lambda_3 \geq 0, \ \mu_1, \mu_2 \geq 0 \end{array} \right\}$$

2. [50pts] Two jobs, A and B, must be scheduled in one of T timeslots. B must be scheduled after A, i.e., B cannot be scheduled in either the same timeslot as A or earlier. Note that this means B cannot be scheduled in timeslot 1 and A cannot be scheduled in timeslot T. We use binary variables $x_t = 1$ to indicate that A is scheduled in timeslot t, for $t = 1, ..., T-1$ and binary variable $y_t = 1$ to indicate that B is scheduled in timeslot t, for $t = 1, ..., T$. Clearly, since each job must be scheduled at some point, we require:

$$\sum_{t=1}^{T-1} x_t = 1, \tag{1}$$

and,

$$\sum_{t=2}^{T} y_t = 1. \tag{2}$$

We now have to model the requirement that B must be later than A. One approach is to note that if A is scheduled at some period $t \in \{2, ..., T-1\}$ (i.e., $x_t = 1$), then B cannot be scheduled in period t or earlier,i.e., $y_r = 0$ for each $r = 2, ..., t$. This can be modelled as:

$$x_t \leq 1 - y_r, \qquad \forall t \in \{2, ..., T-1\}, r \in \{2, ..., t\}. \tag{3}$$

Alternatively, one can note that if A is scheduled at some period $t \in \{2, ..., T-1\}$ (i.e., $x_t = 1$), then B should be scheduled in some timeslot $t+1$ or later, i.e., at least one $y_r = 1$ for $r = t+1, ..., T$. This can be modelled as:

$$x_t \leq \sum_{r=t+1}^{T} y_r, \qquad \forall t \in \{2, ..., T-1\}. \tag{4}$$

A third way of thinking it is to note that $\sum_{t=1}^{T-1} t x_t$ gives the timeslot at which A is scheduled, and $\sum_{t=2}^{T} t y_t$ gives the timeslot at which B is scheduled. Thus, we might ask that

$$1 + \sum_{t=1}^{T-1} t x_t \leq \sum_{t=2}^{T} t y_t. \tag{5}$$

Let P_1, P_2, P_3 denote the three formulations obtained from these three different ways of modelling that job A precedes job B, i.e., let

$$\begin{aligned}
P_1 &= \left\{(x, y) \in [0,1]^{(T-1)\times(T-1)} : \text{ (1), (2) and (3) hold}\right\}, \\
P_2 &= \left\{(x, y) \in [0,1]^{(T-1)\times(T-1)} : \text{ (1), (2) and (4) hold}\right\}, \\
P_3 &= \left\{(x, y) \in [0,1]^{(T-1)\times(T-1)} : \text{ (1), (2) and (5) hold}\right\}.
\end{aligned}$$

(a) Prove that P_2 is *better* than P_1.

Solution:

Let $(x, y) \in P_2$. We are going to show that (x, y) satisfies (3). For each $t = 2, \ldots, T-1$, we have that

$$x_t \leq \sum_{i=t+1}^{T} y_i = 1 - \sum_{i=2}^{t} y_i,$$

by (2), and so

$$x_t + \sum_{i=2}^{t} y_i \leq 1.$$

Now for $r \in \{2, \ldots, t\}$, $y_r \leq \sum_{i=2}^{t} y_i$ since $y \geq 0$, so $x_t + y_r \leq 1$. Thus $P_2 \subseteq P_1$.

Taking $x = (0, \frac{1}{4}, \frac{3}{4}, 0, 0, \ldots)$ and $y = (-, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, 0, \ldots, 0)$ gives a point in $P_1 \setminus P_2$, as required. Therefore , $P_2 \subsetneq P_1$, which proves that P_2 is a better formulation than P_1

(b) Prove that for the case $T = 3$, P_2 and P_3 are *equivalent*, i.e., $P_2 = P_3$.

Solution:

To prove the statement we need to show that $P_2 \subseteq P_3$ and $P_3 \subseteq P_2$.

First proof: $P_2 \subseteq P_3$

Let $(x, y) \in P_2$. Then $x_2 \le y_3$, by (4) for $t = 2$, so since $x_1 + x_2 = 1$ by (1)

$$1 + x_1 + 2x_2 = 1 + (1 - x_2) + 2x_2 = 2 + x_2 \le 2 + y_3 = 2(y_2 + y_3) + y_3 = 2y_2 + 3y_3$$

since $y_2 + y_3 = 1$ by (2). This is just (5), so it must be that $(x, y) \in P_3$. Thus $P_2 \subseteq P_3$.

Second proof: $P_3 \subseteq P_2$

Now suppose $(x, y) \in P_3$. Then $1+x_1+2x_2 \le 2y_2+3y_3$. But $x_1+x_2 = 1$ and $y_2+y_3 = 1$, so this implies $1 + x_1 + 2(1 - x_1) \le 2y_2 + 3(1 - y_2)$, i.e. $y_2 \le x_1$. Thus $1 - x_1 \le 1 - y_2$, so $x_2 \le y_3$. Lastly, (4) holds and it must be that $(x, y) \in P_2$. Thus $P_3 \subseteq P_2$.

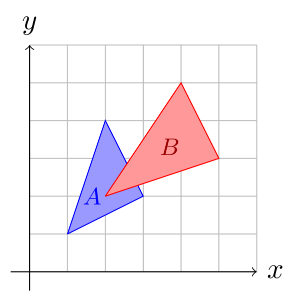
We conclude that $P_2 = P_3$.

(c) Show that P_2 and P_3 are *incomparable*, i.e., neither P_2 nor P_3 is better than the other, in general.
(Hint: to find a point which is in each set but not the other, it is sufficient to consider $T = 4$.)

Solution:

For $T = 4$, taking $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -)$, $y = (-, 0, 1, 0)$ yields a point in $P_3 \setminus P_2$, while taking $x = (0, \frac{1}{2}, \frac{1}{2}, -)$, $y = (-, \frac{1}{2}, 0, \frac{1}{2})$ yields a point in $P_2 \setminus P_3$. Therefore, neither P_2 nor P_3 is better than the other.

3. [25pts] Consider region X that is given as the **union** of the shaded regions (red and blue) in the following figure:



That is, $X = A \cup B$. Notice that the triangle A has extreme points (1,1), (2,4) and (3,2), while triangle B has extreme points (2,2), (4,5) amd (5,3).

(a) Write down an *ideal* integer programming formulation for the problem $\min\{c^\top x : x \in X\}$.

Solution:

First, we need to find the constraints that define A and B.

$$A = \left\{ (x_1, y) \in \mathbb{R}^n : \begin{array}{l} x_2 - 3x_1 \leq -2 \\ 2x_2 - x_1 \geq 1 \\ x_2 + 2x_1 \leq 8 \end{array} \right\}, \quad B = \left\{ (x_1, x_2) \in \mathbb{R}^n : \begin{array}{l} 2x_2 - 3x_1 \leq -2 \\ 3x_2 - x_1 \geq 4 \\ x_2 + 2x_1 \leq 13 \end{array} \right\}.$$

Consider the following variables:

$$y_A = \begin{cases} 1 & \text{point } x \text{ is in set } A \\ 0 & o.w \end{cases}, \qquad y_B = \begin{cases} 1 & \text{point } x \text{ is in set } B \\ 0 & o.w \end{cases},$$
$$z^A = \text{value of } x \text{ in set } A,$$
$$z^B = \text{value of } x \text{ in set } B.$$

Ideal Formulation:

$$\begin{aligned}
\min\ & c^\top x \\
s.t.\ & z_2^A - 3z_1^A \leq -2y_A \\
& 2z_2^A - z_1^A \geq 1y_A \\
& z_2^A + 2z_1^A \leq 8y_A \\
& 2z_2^B - 3z_1^B \leq -2y_B \\
& 3z_2^B - z_1^B \geq 4y_B \\
& z_2^B + 2z_1^B \leq 13y_B \\
& y_A + y_B = 1 \\
& x_1 = z_1^A + z_1^B \\
& x_2 = z_2^A + z_2^B \\
& y_A \leq z_1^A \leq 5y_A \\
& y_B \leq z_1^B \leq 5y_B \\
& y_A \leq z_2^A \leq 5y_A \\
& y_B \leq z_2^B \leq 5y_B \\
& 1 \leq x_1, x_2 \leq 5 \\
& y_A, y_B \in \{0, 1\}
\end{aligned}$$

(b) Write down an alternative integer programming formulation for the problem $\min\{c^\top x : x \in X\}$.
Solution:

Big-M coefficients:

$$M^A = \max\{x_2 - 3x_1 + 2\ ;\ -2x_2 + x_1 + 1\ ;\ x_2 + 2x_1 - 8 :\ 1 \leq x_1, x_2 \leq 5\} = 7.$$
$$M^B = \max\{2x_2 - 3x_1 + 2\ ;\ -3x_2 + x_1 + 4\ ;\ x_2 + 2x_1 - 13 :\ 1 \leq x_1, x_2 \leq 5\} = 9.$$

Alternative Formulation:

$$
\begin{aligned}
\min\ & c^\top x \\
s.t.\ & x_2 - 3x_1 + 2 \leq 7y_A \\
& -2x_2 + x_1 + 1 \leq 7y_A \\
& x_2 + 2x_1 - 8 \leq 7y_A \\
& 2x_2 - 3x_1 + 2 \leq 9y_B \\
& -3x_2 + x_1 + 4 \leq 9y_B \\
& x_2 + 2x_1 - 13 \leq 9y_B \\
& y_A + y_B = 1 \\
& 1 \leq x_1, x_2 \leq 5 \\
& y_A, y_B \in \{0, 1\}
\end{aligned}
$$