MIE 1603 / 1653 - Integer Programming
Winter 2019
Assignment #2

Due Date: February 11th, Monday, no later than 11:59PM.

Solve the following problems. You may work in groups of *two* students (from the same course code) or individually. Turn in one solution set with both group members listed on it. Groups must work independently of each other. You *must* cite any references (texts, papers or websites) you have used to help you solve these problems.

Note that some questions will be only for MIE1603 or MIE1653. If so, this is indicated at the beginning of the question. Otherwise, the question is common for both groups. Although you are not responsible (and will not be getting points) from the questions targeting the other group, you are welcome to submit answers for such questions as well if you would like to get feedback. (Solutions, full or partial, will be posted for all questions.)

You can submit your solutions using Quercus (pdf file) or give a hard-copy to the professor or TA in their office hours (or lectures).

1. [25pts] A Canadian national park has decided to build new watchtowers to prevent fires in camping areas. The park has a set of L possible locations to put the watchtower. You need to decide where to construct the watchtowers and their height. In each location $i \in L$ you can construct at most one watchtower of at least m_i meters and at most M_i meters. Constructing a watchtower at location $i \in L$ has a fixed cost f_i and a cost r_i for each extra meter above m_i (e.g., an $m_i + 0.5$ meter watchtower at location i has a total cost of $f_i + 0.5r_i$). The objective is to minimize the total cost of building watchtowers.

   The park has a set of camping areas A that the watchtowers have to monitor. Consider d_{ij} as the distance between location $i \in L$ and camping area $j \in A$. A watchtower $i \in L$ can monitor a camping area $j \in A$ if their distance is less than or equal to three times the height of the watchtower. Each watchtower can monitor at most four camping areas and each camping area needs to be monitored by at least one watchtower. In addition, there is a set of critical camping areas $A_C \subseteq A$ that need at least two watchtowers monitoring them.

   The park is divided into three disjoint sectors. Consider L_s as the watchtower locations of sector $s \in \{1, 2, 3\}$. The park requires each sector to have at least $1/10$ of the total allocated watchtowers and no sector can have more than half of the total allocated watchtowers. Also, if sector 1 has less than k_1 watchtowers, then sectors 2 and 3 together should have at least k_2 watchtowers.

   Write a mixed-integer *linear* programming model for the Canadian National Park problem.

   Solution:

Variables:

$$x_i = \begin{cases} 1 & \text{build watchtower in location } i \\ 0 & o.w. \end{cases},$$

$$z_{ij} = \begin{cases} 1 & \text{watchtower in location } i \text{ monitors camping area } j \\ 0 & o.w. \end{cases},$$

$$y_i = \text{Additional meters of watchtower in location } i,$$

$$w = \begin{cases} 1 & \sum_{i \in L_1} x_i \leq 1 \\ 0 & o.w. \end{cases}.$$

Model:

$$\min \sum_{i \in L} (f_i x_i + r_i y_i) \tag{1}$$

$$s.t. \sum_{j \in A} z_{ij} \leq 4x_i \qquad \forall i \in L \tag{2}$$

$$\sum_{i \in L} z_{ij} \geq 1 \qquad \forall j \in A \tag{3}$$

$$\sum_{i \in L} z_{ij} \geq 2 \qquad \forall j \in A_C \tag{4}$$

$$d_{ij} z_{ij} \leq 3(m_i + y_i) \qquad \forall i \in L,\ j \in A \tag{5}$$

$$\sum_{i \in L_s} x_i \geq \frac{1}{10} \sum_{i \in L} x_i \qquad \forall s \in \{1, 2, 3\} \tag{6}$$

$$\sum_{i \in L_s} x_i \leq \frac{1}{2} \sum_{i \in L} x_i \qquad \forall s \in \{1, 2, 3\} \tag{7}$$

$$\sum_{i \in L_1} x_i \geq (k_1 + 1)(1 - w) \tag{8}$$

$$\sum_{i \in L_2} x_i \leq k_2 + |L_2|(1 - w) \tag{9}$$

$$0 \leq y_i \leq x_i(M_i - m_i) \qquad \forall i \in L \tag{10}$$

$$x_i \in \{0, 1\},\ y_i \in \mathbb{R} \qquad \forall i \in L \tag{11}$$

$$z_{ij} \in \{0, 1\} \qquad \forall i \in L,\ j \in A \tag{12}$$

$$w \in \{0, 1\} \tag{13}$$

The objective (1) minimizes the total cost of constructing watchtowers. Constraint (2) enforces that each watchtower should monitor at most 4 camping areas and makes sure that we only use the selected watchtower locations. Constraint (3) forces to monitor each camping area with at least one watchtower and constraint (4) forces to monitor critical areas with at least two watchtowers. Constraint (6) requires that each sector has at least 10% of the total constructed towers and constraint (7) restricts it to be no more than half. Constraints (8) and (9) model the condition that when the watchtowers in L_1 are less than k_1, then the total watchtowers in L_2 have to be more than k_2. Constrain (10) enforce the maximum height of a tower a tower if the location is chosen.

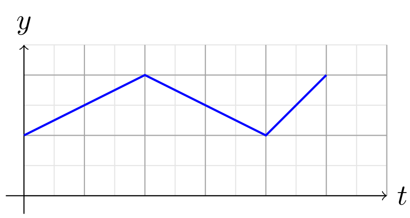
2. [25pts] Consider the Canadian National Park problem presented in Question 1. Assume that the additional height cost of building a wachtower is given by the following piecewise linear functions. There are three piecewise functions, one for the watchtowers in each sector, L_1, L_2 and L_3, receptively. Assume that $a_1, a_2, a_3 > 0$ are constant values and t represents the number of additional meters.

$$g_1(t) = \begin{cases} a_1 t + 1, & 0 \le t \le \frac{1}{a_1} \\ 3 - a_1 t, & \frac{1}{a_1} \le t \le \frac{2}{a_1} \\ -3 + 2a_1 t, & t \ge \frac{2}{a_1} \end{cases} \qquad g_2(t) = \begin{cases} t - 0.5a_2 + 2, & 0 \le t \le \frac{15a_2 - 20}{12} \\ a_2 - 0.2t, & \frac{15a_2-20}{12} \le t \le \frac{5}{2}a_2 \\ -t + 3a_2, & t \ge \frac{5}{2}a_2 \end{cases}$$

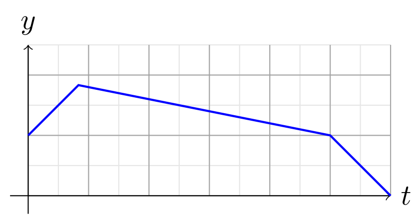
$$g_3(t) = \begin{cases} -a_3 t + 2, & 0 \le t \le \frac{4-a_3}{1+2a_3} \\ \frac{a_3}{2} + 0.5t, & t \ge \frac{4-a_3}{1+2a_3} \end{cases}$$

(a) Draw the piecewise linear functions presented above considering $a_1 = 0.5$, $a_2 = 2$ and $a_3 = 1$. Identify whether or not they are convex functions.

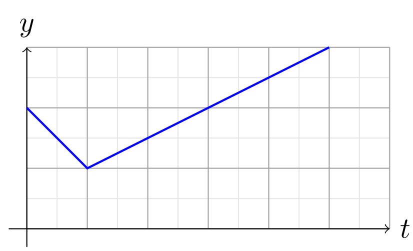
Solution:



(a) $g_1(t)$



(b) $g_2(t)$



(c) $g_3(t)$

In this case we have:

$$g_1(t) = \begin{cases} 0.5t + 1, & 0 \le t \le 2 \\ 3 - 0.5t, & 2 \le t \le 4 \\ -3 + t, & t \ge 4 \end{cases} \qquad g_2(t) = \begin{cases} t + 1, & 0 \le t \le 0.833 \\ 2 - 0.2t, & 0.833 \le t \le 5 \\ -t + 6, & t \ge 5 \end{cases}$$

$$g_3(t) = \begin{cases} -t+2, & 0 \leq t \leq 1 \\ 0.5+0.5t, & t \geq 1 \end{cases}$$

$g_1(t)$ is a general function (neither convex nor concave), $g_2(t)$ is concave and $g_3(t)$ is convex.

(b) Model this variation of the problem using linear constraints assuming $a_1 = 0.5$, $a_2 = 2$ and $a_3 = 1$. You need to present the modified objective function and the additional constraints and variables used to represent the piecewise costs.

Solution:

I will use the ideal formulation for all the cost functions, but you can use the alternative formulation or the special case for the concave function (since this is a minimization problem). Consider $a_{j,i}$ as the interval values for each location $i \in L$. We will assume that $M_i > 4$ for all $i \in L_1$, $M_i > 5$ for all $i \in L_2$ and $M_i > 1$ for all $i \in L_3$. Then, $a_{0,i} = 0$, $a_{1,i} = 2$, $a_{2,i} = 4$ and $a_{3,i} = M_i$ if $i \in L_1$, $a_{0,i} = 0$, $a_{1,i} = 0.833$, $a_{2,i} = 5$ and $a_{3,i} = M_i$ if $i \in L_2$, and $a_{0,i} = 0$, $a_{1,i} = 1$, $a_{2,i} = M_i$ if $i \in L_3$.

Variables:

$$s_{ij} = \begin{cases} 1 & \text{if } t \text{ is in interval } [a_{j-1,i}, a_{j,i}] \text{ for location } i \\ 0 & o.w. \end{cases}$$

$$t_{ij} = \begin{cases} t & \text{if } t \text{ is in interval } [a_{j-1,i}, a_{j,i}] \text{ for location } i \\ 0 & o.w. \end{cases}$$

$$Z_i = \text{additional height cost for watchtower } i$$

Model:

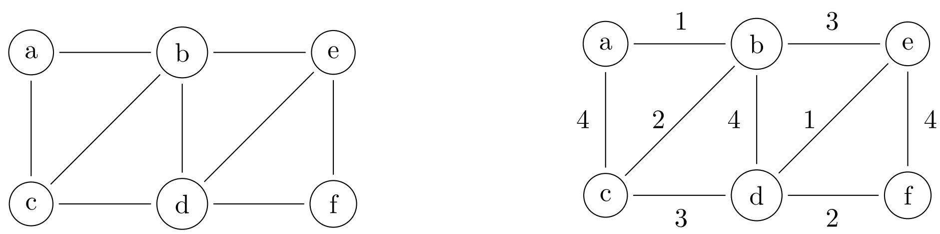
$$
\begin{aligned}
\min\ & \sum_{i \in L} (f_i x_i + Z_i) \\
s.t.\ & (2) - (13) \\
& (0.5t_{i1} + s_{i1}) + (3s_{i2} - 0.5t_{i2}) + (-3s_{i3} + t_{i3}) \leq Z_i && \forall i \in L_1 && (14) \\
& t_{i1} + t_{i2} + t_{i3} = y_i && \forall i \in L_1 && (15) \\
& s_{i1} + s_{i2} + s_{i3} = x_i && \forall i \in L_1 && (16) \\
& a_{j-1,i} \leq t_{ij} \leq a_{j-1,i} s_{ij} && \forall i \in L_1,\ j \in \{1,2,3\} && (17) \\
& (t_{i1} + s_{i1}) + (2s_{i2} - 0.2t_{i2}) + (-t_{i3} + 6s_{i3}) \leq Z_i && \forall i \in L_2 && (18) \\
& t_{i1} + t_{i2} + t_{i3} = y_i && \forall i \in L_2 && (19) \\
& s_{i1} + s_{i2} + s_{i3} = x_i && \forall i \in L_2 && (20) \\
& a_{j-1,i} \leq t_{ij} \leq a_{j-1,i} s_{ij} && \forall i \in L_2,\ j \in \{1,2,3\} && (21) \\
& (-t_{i1} + 2s_{i1}) + (0.5s_{i2} + 0.5t_{i2}) \leq Z_i && \forall i \in L_3 && (22) \\
& t_{i1} + t_{i2} = y_i && \forall i \in L_3 && (23) \\
& s_{i1} + s_{i2} = x_i && \forall i \in L_3 && (24) \\
& a_{j-1,i} \leq t_{ij} \leq a_{j-1,i} s_{ij} && \forall i \in L_2,\ j \in \{1,2\} && (25) \\
& s_{ij} \in \{0,1\} && \forall i \in L,\ j \in \{1,2,3\} \\
& t_{ij} \geq 0 && \forall i \in L,\ j \in \{1,2,3\}
\end{aligned}
$$

Constraints (15)-(17) model the cost $g_1(t)$, (18)-(21) the cost $g_2(t)$, and (22)-(25) the cost $g_3(t)$. Notice that constraints (16), (20) and (25) are equal to x_i instead of 1 to make sure that we only pay the additional height cost when a watchtower location is selected.

3. [25pts] In HW1 we defined the *edge numbering problem* and we now consider its variant called the *minimum skipping edge numbering problem*. Consider an undirected graph $G = (N, E)$ and a set of numbers $L = \{1, ..., K\}$. For each node $i \in N$, let $A(i)$ represent all the edges that are incident to the node i. The problem asks you to find a valid edge numbering, i.e., to assign a number from L to each edge such that no adjacent edges (i.e., edges that have a common endpoint) have the same number. For each node $i \in N$, let m_i and M_i denote the minimum and the maximum number in the set of numbers assigned to the edges in $A(i)$, respectively. Then, the cost of node i is defined as the number of labels between m_i and M_i that are skipped, i.e., the number of labels in the set $\{m_a, m_a + 1, \ldots, M_a\}$ that are not assigned to any edge in $A(i)$. The aim is to minimize the total cost (i.e., the sum of the node costs over all nodes).

   As an example, consider the valid edge numbering in Figure 2b. Node a has cost 2 since $m_a = 1$ and $M_a = 4$ and there is no edge in $A(a)$ with label 2 nor label 3. In other words, for node a, the solution skips two numbers (labels) in the set $\{m_a, \ldots, M_a\} = \{1, 2, 3, 4\}$.

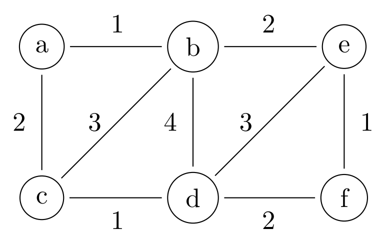
   (a) Write down the cost of each node and the total cost of the solution shown in Figure 2b. Find the optimal edge number solution for the graph in Figure 2a with $K = 4$ and argue why it is optimal.



(a) Example graph. (b) Valid edge numbering.

Figure 2: Example for Question 3 with $K = 4$.

Solution:
Let C_i be the cost of a node $i \in N$. Then, $C_a = 3$, $C_b = 0$, $C_c = 0$, $C_d = 0$, $C_e = 1$ and $C_f = 1$. The total cost is 4. The optimal solution is:



This edge numbering is valid and has cost = 0. Since 0 is a lower bound for the problem, we can gurantee that this is in fact the optimal solution.

(b) Write down a Mixed Integer Linear Programming model for the minimum skipping edge numbering problem, i.e., given any graph $G = (N, E)$, you need to decide how to label the edges such that no adjacent edges have the same number and the total node cost is minimized. You need to use only the following decision variables:

$$x_{ek} = \begin{cases} 1, & \text{if edge } e \text{ is given label } k \\ 0, & \text{otherwise} \end{cases} \quad , \ e \in E, k \in L$$

$$m_i = \text{The minimum number assigned to an edge in } A(i),\ i \in N$$

$$M_i = \text{The maximum number assigned to an edge in } A(i),\ i \in N$$

You are not allowed to define any additional variables for this question.

Solution:

$$
\begin{aligned}
\min\ & \sum_{i \in N} (M_i - m_i - |A(i)| - 1) && && \\
s.t.\ & \sum_{e \in A(i)} x_{ek} \le 1 && \forall i \in N,\ k \in L && (26)\\
& \sum_{k \in L} x_{ek} = 1 && \forall e \in E && (27)\\
& m_i \le \sum_{k \in L} (k \cdot x_{ek}) && \forall i \in N,\ e \in A(i) && (28)\\
& M_i \ge \sum_{k \in L} (k \cdot x_{ek}) && \forall i \in N,\ e \in A(i) && (29)\\
& M_i - m_i \ge |A(i)| - 1 && \forall i \in N && (30)\\
& x_{ek} \in \{0, 1\} && e \in E, k \in L && \\
& 0 \le m_i, M_i \le K && i \in N &&
\end{aligned}
$$

The objective function computes the number of skipped edge labels using M_i, m_i and the number of incident edges to a node i, $|A(i)|$. Constraint (26) makes sure that all incident edges have a different label. Constraint (27) enforce to select one label for each edge. Constraints (28) and (29) compute the minimum and maximum edge numbering for node. Constraint (30) is a valid inequality to make sure that the difference between the maximum and minimum labels is greater than the number of edges incident to a node.

(c) We define $\mathcal{I}(i)$ as the set of intervals defined by all possible feasible m_i and M_i pairs of values that a node can have, i.e.,

$$\mathcal{I}(i) = \{[m, M] :\ m, M \in \mathbb{Z},\ 1 \le m < M \le K,\ M - m \ge |A(i)| - 1\}$$

For example, for the graph in Figure 2a we have $\mathcal{I}(e) = \{[1, 3], [1, 4], [2, 4]\}$. Write down $\mathcal{I}(i)$ for all nodes $i \in N$ of the graph in Figure 2a.

Solution:

$$
\begin{aligned}
\mathcal{I}(a) &= \{[1, 2], [1, 3], [1, 4], [2, 3], [2, 4], [3, 4]\}\\
\mathcal{I}(b) &= \{[1, 4]\}\\
\mathcal{I}(c) &= \{[1, 3], [1, 4], [2, 4]\}\\
\mathcal{I}(d) &= \{[1, 4]\}\\
\mathcal{I}(e) &= \{[1, 3], [1, 4], [2, 4]\}\\
\mathcal{I}(f) &= \{[1, 2], [1, 3], [1, 4], [2, 3], [2, 4], [3, 4]\}
\end{aligned}
$$

(d) Consider the following binary variables:

$$z_{i,[m,M]} = \begin{cases} 1, & \text{if node } i \text{ has min edge value } m \text{ and max edge value M,} \\ 0, & \text{otherwise,} \end{cases}$$

for each node $i \in N$ and $[m, M] \in \mathcal{I}(i)$. Write down an Integer Linear Programming model for the minimum skipping edge numbering problem for the graph in Figure 2a using **only** variables x_{ek} and $z_{i,[m,M]}$.

Solution:

$$\begin{aligned}
\min \ & \sum_{i \in N} \sum_{[m,M] \in \mathcal{I}(i)} (M - m - |A(i)| + 1) z_{i,[m,M]} & & \\
s.t. \ & \sum_{k \in L} x_{ek} = 1 & \forall e \in E & \\
& \sum_{[m,M] \in \mathcal{I}(i)} z_{i,[m,M]} = 1 & \forall i \in N & \quad (31) \\
& \sum_{e \in A(i)} x_{ek} \leq \sum_{\{[m,M] \in \mathcal{I}(i) :\, m \leq k \leq M\}} z_{i,[m,M]} & \forall i \in N,\ k \in L & \quad (32) \\
& x_{ek} \in \{0, 1\} & e \in E, k \in L & \\
& z_{i,[m,M]} \in \{0, 1\} & i \in N,\ [m, M] \in \mathcal{I} &
\end{aligned}$$

This formulation has a similar objective function but now using variables $z_{i,[m,M]}$. Constraint (31) enforces that we only choose one $[m, M]$ interval per node. Constraint (32) makes sure that the choosen edge numberings are inside the choosen node interval.

(e) **(BONUS - 5pts)** Assume that $K = |E|$. Prove that the optimal value of the LP relaxation of your model in part (b) is 0 for any undirected connected graph $G = (N, E)$.

Solution:

Since a $LB = 0$ is a valid lower bound ofr the problem, it is sufficient to show that we can always construct a solution with cost $= 0$.

Conside an assigment $\hat{(x)}_{ek} = 1/K$ for all $e \in E$ and $k \in L$. Also, let

$$\hat{m}_i = \begin{cases} (K-1)/2 & \text{if } K \geq 2|A(i)| - 1 \\ K - |A(i)| & o.w. \end{cases}$$

and $\hat{M}_i = \hat{m}_i + |A(i)| - 1$ for all $i \in N$. It is easy to see that $(\hat{x}, \hat{m}, \hat{M})$ is feasible for the system of equations in part (b). Moreover, this solution has cost $= 0$. □

4. [25pts] Once again, consider the *minimum skipping edge numbering problem* defined above for an undirected graph $G = (N, E)$. For any subset of nodes $S \subseteq N$, we define $E(S)$ as the set of edges incident to only nodes in S, i.e., $E(S) = \{\{i, j\} \in E : i, j \in S\}$. For any number $k \in L$ and $S \subseteq N$ with an odd number of nodes, we can define a set of valid *blossom inequalities* for our problem as follows:

$$\sum_{e \in E(S)} x_{ek} \leq \frac{|S| - 1}{2} \qquad S \subseteq V \text{ s.t. } |S| \text{ odd}, k \in L$$

Any feasible solution to our minimum skipping edge numbering problem satisfy all of the above blossom inequalities.

As an example, consider the graph in Figure 2a, and let $S = \{a, b, d\}$ and $k = 3$. In this case, $E(S) = \{\{a, b\}, \{b, d\}\}$ and the blossom inequality is $x_{\{a,b\},3} + x_{\{b,d\},3} \leq 1$, which is in fact a valid inequality for our problem.

(a) What is the maximum number of blossom inequalities that you can have for the graph in Figure 2a? Explain. Explicitly enumerate all the blossom inequalities for the above graph.

Solution:

We need to consider the subsets $S \subseteq N$ with cardinality 1, 3 and 5. Notice that if $|S| = 1$ the constraint reduce to $0 \leq 0$, so it is not necessary to count those constraints. We have $\binom{6}{3} = 20$ sets of cardinality 3, and $\binom{6}{5} = 6$ sets of cardinality 5. Hence, there are $26 \times 4 = 104$ blossom inequalities for this graph.

The set of valid S with $|S| = 3$ is $\mathcal{S}_3 = \{\{a, b, c\}, \{a, b, d\}, ..., \{d, e, f\}\}$, and the set of valid S with cardinality $|S| = 5$ is $\mathcal{S}_5 = \{\{a, b, c, d, e\}, ..., \{b, c, d, e, f\}\}$. The blossom inequalities are:

$$\sum_{e \in E(S)} x_{ek} \leq 1 \qquad \forall k \in \{1, 2, 3, 4\},\ S \in \mathcal{S}_3$$

$$\sum_{e \in E(S)} x_{ek} \leq 2 \qquad \forall k \in \{1, 2, 3, 4\},\ S \in \mathcal{S}_5$$

(b) Consider a fractional solution for the graph in Figure 2a where $x_{ek} = 0.25$ for all $e \in E$ and $k \in L$ (where $K = 4$). Does this solution violate any blossom inequality? If so, write a blossom inequality that will cut this solution off.

Solution:

There is no blossom inequality that will cut this fractional point. First, let's consider the blossom inequalities with $|S| = 3$ and with the biggest $E(S)$ set, e.g., $S = \{a, b, c\}$. These sets have at most 3 edges, so:

$$\sum_{e \in E(S)} x_{ek} = 0.75 \leq 1$$

i.e., the point satisfies these constraints, and so, any blossom inequality with $|S| = 3$ and $E(S) < 3$. Now, consider the sets S with $|S| = 5$ and largest $E(S)$, e.g., $S = \{a, b, c, d, e\}$. These sets have at most 7 edges, so:

$$\sum_{e \in E(S)} x_{ek} = 1.75 \leq 2$$

i.e., the point satisfies these constraints, and so, any blossom inequality with $|S| = 5$ and $E(S) < 7$.

(c) Now consider the following fractional solution for the graph in Figure 2a:

$$\begin{array}{llll}
x_{\{a,b\},1} = 0.75 & x_{\{a,b\},2} = 0.00 & x_{\{a,b\},3} = 0.00 & x_{\{a,b\},4} = 0.25 \\
x_{\{a,c\},1} = 0.25 & x_{\{a,c\},2} = 0.25 & x_{\{a,c\},3} = 0.50 & x_{\{a,c\},4} = 0.00 \\
x_{\{b,c\},1} = 0.25 & x_{\{b,c\},2} = 0.25 & x_{\{b,c\},3} = 0.25 & x_{\{b,c\},4} = 0.25 \\
x_{\{b,d\},1} = 0.00 & x_{\{b,d\},2} = 0.50 & x_{\{b,d\},3} = 0.50 & x_{\{b,d\},4} = 0.00 \\
x_{\{b,e\},1} = 0.00 & x_{\{b,e\},2} = 0.25 & x_{\{b,e\},3} = 0.25 & x_{\{b,e\},4} = 0.50 \\
x_{\{c,d\},1} = 0.00 & x_{\{c,d\},2} = 0.25 & x_{\{c,d\},3} = 0.25 & x_{\{c,d\},4} = 0.50 \\
x_{\{d,e\},1} = 0.25 & x_{\{d,e\},2} = 0.00 & x_{\{d,e\},3} = 0.25 & x_{\{d,e\},4} = 0.50 \\
x_{\{d,f\},1} = 0.75 & x_{\{d,f\},2} = 0.25 & x_{\{d,f\},3} = 0.00 & x_{\{d,f\},4} = 0.00 \\
x_{\{e,f\},1} = 0.25 & x_{\{e,f\},2} = 0.75 & x_{\{e,f\},3} = 0.00 & x_{\{e,f\},4} = 0.00
\end{array}$$

Does this solution violate any blossom inequality? If so, write a blossom inequality that will cut this solution off.

Solution:

The solution violates several blossom inequalities. For example, $S = \{a, b, c\}$ with $k = 1$, $S = \{d, e, , f\}$ with $k = 1$, and $S = \{b, c, d, e, f\}$ with $k = 2$.

(d) **(MIE1603)** Now consider any solution x' (integer or fractional) for the problem above. Write down a mixed-integer linear programming model to find a blossom inequality that is violated at x', if any. By looking at your model's optimal solution and/or optimal value, you should be able to provide a blossom inequality that is violated at x', or should argue that no such violated inequality exists.

Solution:

We want to find a $k \in L$ and a set of nodes $S \subseteq N$ with $|S|$ an odd value such that:

$$2 \sum_{e \in E(S)} x'_{ek} - |S| + 1 \geq 0$$

Considering these variables:

$$w_i = \begin{cases} 1 & \text{if node } i \text{ is in set } S \\ 0 & o.w. \end{cases}, \qquad y_e = \begin{cases} 1 & \text{if edge } e \text{ is in set } E(S) \\ 0 & o.w. \end{cases}$$

$z =$ auxiliary variables to make sure that $|S|$ is an odd number

For each $k \in L$ we need to solve the following problem:

$$\begin{aligned}
v_k^* = \max\ & 2 \sum_{e \in E} y_e x'_{ek} - \sum_{i \in N} w_i + 1 & \\
s.t.\ & y_e \leq w_i + w_j - 1 & \forall e = \{i, j\} \in E \\
& \sum_{i \in N} w_i = 2z + 1 & \\
& 1 \leq z \leq \left\lceil \frac{|N|}{2} \right\rceil + 1 & \\
& w_i, y_e \in \{0, 1\} & \forall i \in N,\ e \in E
\end{aligned}$$

If $v_k^* \geq 0$, then we found a blossom inequality with S given by w^*. Otherwise, there is no blossom inequality that separates the fractional solution for number k.