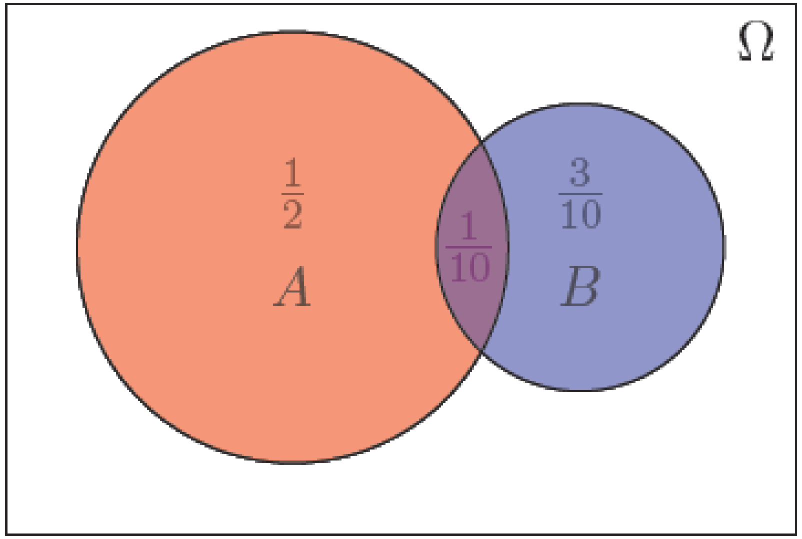
Week 12
Bayesian Inferences

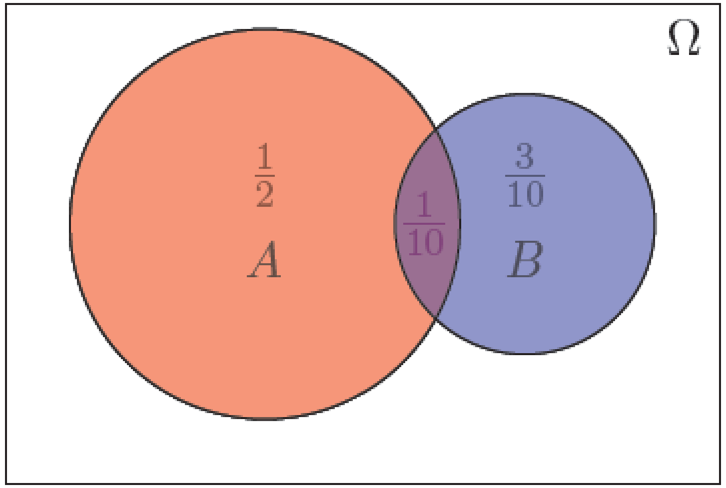
Revision - Probability



For two events A and B, calculate the following:

- $P(A) = \frac{3}{5}$
- $P(B) = \frac{2}{5}$
- $P(A|B) = \frac{1}{4}$
- $P(B|A) = \frac{1}{6}$

Revision – Bayes' Rule



For 2 events A and B,

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)}$$

$$= \frac{1}{6} * \frac{\frac{3}{5}}{\frac{2}{5}} = \frac{1}{4}$$

$$P(A) = \frac{1}{2} + \frac{1}{10} = \frac{3}{5}$$

$$P(B) = \frac{1}{10} + \frac{3}{10} = \frac{2}{5}$$

$$P(A \cap B) = \frac{1}{10}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{10}}{\frac{3}{5}} = \frac{1}{6}$$

Example – Medical Test

The probability of a certain medical test being positive (+) is 90% if a patient has a disease, D. 1% of the population have the disease, and the test records a false positive 5% of the time. If a patient has a positive test result, what is the probability the patient has the disease?

$P(+|D) = 0.9$

$P(D) = 0.01$

$P(+|D') = 0.05$

$P(D \cap +) = P(+|D)P(D) = 0.9(0.01) = 0.009$

$P(+) = P(D \cap +) + P(D' \cap +) = P(+|D)P(D) + P(+|D')P(D') = 0.009 + 0.05(1 – 0.01) = 0.0585$

$$P(D|+) = \frac{P(D \cap +)}{P(+)} = \frac{0.009}{0.0585} = 0.1538$$

Example – Further Medical Test

Suppose that the patient takes the same test again and the result is positive. What is the revised probability that the patient has the disease?

$P(+|D) = 0.9$

$P(D) = 0.01$

$P(+|D') = 0.05$

Suppose that you took the test once.   There's a positive outcome.   Given this positive outcome, the probability of the disease is 0.153 from the first test.   Now, you move to the second test. The prior probability (before the second test, i.e, prior to the second test) is now 0.153.

$P(D|+) = 0.153846 \overset{\text{update}}{\Rightarrow} P(D) = 0.153846$ and $P(D') = 1 - 0.153846 = 0.846154$

$P(D \cap +) = P(+|D)P(D) = 0.9(0.153846) = 0.1384614$

$P(+) = P(D \cap +) + P(D' \cap +) = P(+|D)P(D) + P(+|D')P(D') = 0.1385 + 0.05(0.846) = 0.1807691$

$P(D|+) = \frac{P(D \cap +)}{P(+)} = \frac{0.1384614}{0.1807691} = 0.76596$. This is the updated posterior probability of D.

Prior Distributions

The prior distribution is a description of the knowledge about the parameter in question prior to observation of the data.

There are different types of prior distributions which include:

- uninformed prior – you have no prior knowledge
- subjective or informed prior – incorporates information from an expert's opinion or your level of knowledge
- conjugate prior – the same family as the posterior
- improper prior – does not normalise to unity

Theory – Posterior Distribution

Assume that parameter(s) θ describe (part of) the distribution of the data . Then by Bayes' rule:

$$\boxed{P(\theta|) = \frac{P(|\theta)P(\theta)}{P()}}$$

where

- $P(\theta)$ is called the prior probability;
- $P(|\theta)$ is called the likelihood (or sampling distribution);
- $P(\theta|)$ is called the posterior probability;
- $P()$ is the normalising constant.

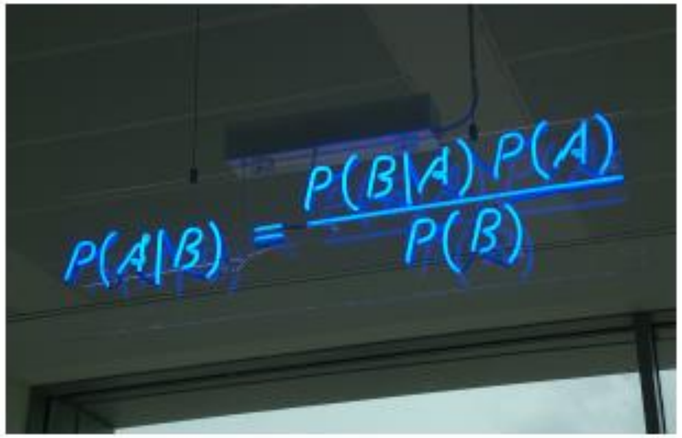
Note that $P()$ is a constant in the equation above so as a consequence, we may only need the prior distribution and the likelihood and use

$$P(\theta|) \propto P(|\theta)P(\theta).$$

The Bayesian Way

There are four steps involved:

1. assume a prior distribution of the parameter θ before analysing the new data set;
2. find an appropriate likelihood function for the observed data – $P(|\theta)$;
3. get the posterior distribution – $P(\theta|)$; and
4. make inference based on the posterior distribution.



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A bit of history

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- So far the majority of what you have been learning in this course is the frequentist approach to statistical inference.
- Frequentist approach was popularised by Fisher, Neyman and Pearson in the early 20th century.
- Bayesian approach uses the rule/theorem by Thomas Bayes at its foundation.
- *Bayesian inference*, unlike Frequentist inference, makes distributional assumptions on the parameters.

Frequentist vs. Bayesian

Frequentist:

- In the frequentist (or classical) approach to statistics, probability is interpreted as long run frequencies (Lecture 2).
- The goal of frequentist inference is to create procedures (for example, methods of estimation) with long run guarantees.
- In frequentist inference, sampling processes are random while parameters are fixed, unknown quantities.

Bayesian:

- In the Bayesian approach, probability is regarded as a measure of subjective degree of belief.
- Bayesian statements are probability statements about the uncertainty of the parameters.
- The data are a given, the uncertainty on parameters can vary.

Maximum likelihood estimation

Suppose that the likelihood function depends on k parameters $\theta_1, \theta_2, \ldots, \theta_k$. Choose as estimates those values of the parameters that maximize the likelihood

$L(x_1, x_2, \ldots, x_n | \theta_1, \theta_2, \ldots, \theta_k)$.

Formally, $\boxed{\hat{\theta}_{\mathrm{ML}} = \arg\max_{\theta} \mathcal{L}(\theta|) = \arg\max_{\theta} f(|\theta).}$

where θ is the parameter(s) and X is the data.

The maximum likelihood (ML) is a frequentist approach.

Example – maximum likelihood estimator

Given X “successes” in n trials, find the maximum likelihood estimate of the parameter p of the corresponding binomial distribution.

To find the value of p which maximizes L(p) = $\binom{n}{x} p^x(1-p)^{n-x}$, it will be convenient to make use of the fact that the value of p which maximizes L(p) will also maximize

lnL(p) = ln $\binom{n}{x}$ + xln(p) + (n – x)ln(1 – p)

Thus, we get

$\frac{d[\ln L(p)]}{dp} = \frac{x}{p} - \frac{n - x}{1 - p}$

And equating this derivative to 0 and solving for p, we find that the likelihood function has a maximum at p = $\frac{x}{n}$. This is the maximum likelihood estimate of the binomial parameter p, and we refer to $\hat{p} = \frac{x}{n}$ as the corresponding maximum likelihood estimator.

Example – maximum likelihood estimator

If $X_1, X_2, \ldots, X_n$ are the values of a random sample from an exponential population, find the maximum likelihood estimator of its parameter λ.

Since the likelihood function is given by

$L(\lambda) = f(X_1, X_2, \ldots, X_n; \lambda) = \prod_{i=1}^{n} \lambda\, e^{-\lambda X_i} = \lambda^n e^{-\lambda \sum X_i}$,

Differentiation of $\ln L(\lambda) = n\ln(\lambda) - \lambda \sum X_i$ with respect to λ yields

$\frac{d[\ln L(\lambda)]}{d\lambda} = \frac{n}{\lambda} - \sum X_i$

Equating this derivative to 0 and solving for λ, we get the maximum likelihood estimate $\hat{\lambda} = \frac{n}{\sum X_i} = \frac{1}{\overline{X}}$. Hence, the maximum likelihood estimator is $\hat{\lambda} = \frac{1}{\overline{X}}$.

Special probability densities

Uniform distribution

If X ~ Unif(a, b), then $f(x) = \frac{1}{b-a}$; $a < X < b$

$E(X) = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$

Beta distribution

If X ~ Beta(α, β), then $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} X^{\alpha-1}(1-X)^{\beta-1}$; $0 < X < 1$, $\alpha > 0$ and $\beta > 0$.

$E(X) = \frac{\alpha}{\alpha+\beta}$ and $Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Note that Beta(1, 1) = Unif(0, 1)

Maximum A Posteriori Estimate (MAP)

- Maximum a posteriori (MAP) estimate is the mode of the posterior distribution or more formally,

$$\boxed{\hat{\theta}_{\text{MAP}} = \arg\max_{\theta} f(\theta|) = \arg\max_{\theta} f(|\theta) f(\theta).}$$

  where θ is the parameter(s) and X is the data, once again.
- Prior to sampling four fruits from the bag, I have no information so I may assume an uninformative prior, say $p \sim U(0,1)$ and so $f(p) = 1$ for $0 < p < 1$.
- Then the posterior density is given as

$$\begin{aligned} f(p|X) &= \frac{f(X|p)f(p)}{f(X)} = \frac{f(X|p) \cdot 1}{f(X)} \\ &= f(X|p) \end{aligned}$$

- In this case, the likelihood is equal to posterior density and so the MAP estimate is equal to the ML estimate.

Theorem 1

If X is a binomial random variable and the prior distribution of p is a beta distribution with the parameters α and β, then the posterior distribution of $p|X=x$ is a beta distribution with the parameters $\alpha_1 = x + \alpha$ and $\beta_1 = n - x + \beta$.

$X \sim \text{Bin}(n, p)$

$f(x|p) = \binom{n}{x} p^x (1-p)^{n-x}$; $x = 0, 1, 2, \ldots, n$

$h(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$; $0 < p < 1$

$g(p|x) = \frac{f(x|p)h(p)}{f(x)} = \frac{\Gamma(\alpha+n+\beta)}{\Gamma(x+\alpha)\Gamma(n-x+\beta)} p^{x+\alpha-1}(1-p)^{n-x+\beta-1}$; $0 < p < 1$

Theorem 2

If $\overline{X}$ is the mean of a random sample of size n from a normal population with the known variance σ^2 and the prior distribution of μ is a normal distribution with the mean μ_0 and the variance σ_0^2, then the posterior distribution $\mu|\overline{X}=\bar{x}$ is a normal distribution with mean μ_1 and the variance σ_1^2, where

$$\mu_1 = \frac{n\bar{x}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2} \text{ and } \sigma_1^2 = \frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2}$$

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$f(\bar{x}|\mu) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2\right]; \ -\infty < \bar{x} < \infty$$

$$h(\mu) = \frac{1}{\sigma_0\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\mu-\mu_0}{\sigma_0}\right)^2\right]; \ -\infty < \mu < \infty$$

$$g(\mu|\bar{x}) = \frac{f(\bar{x}|\mu)h(\mu)}{f(\bar{x})} = \frac{1}{\sigma_1\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\mu-\mu_1}{\sigma_1}\right)^2\right]; \ -\infty < \mu < \infty$$

Binomial likelihood with Beta prior

- Suppose the likelihood is modelled by $Bin(n, p)$, your prior distribution is $p \sim Beta(\alpha, \beta)$ and your observation is x successes out of n.
- Then the posterior distribution is a Beta distribution with parameters $\alpha + x$ and $\beta + n - x$.
- We call Beta distribution a conjugate prior for the Binomial likelihood function.
- **Formal definition**: if the posterior distribution is in the same family as the prior distribution then the prior and posterior are conjugate distributions and the prior is called a conjugate prior for the likelihood function.
- Here $\hat{p}_{\mathsf{ML}} = \dfrac{x}{n}$ and $\hat{p}_{\mathsf{MAP}} = \dfrac{x + \alpha - 1}{n + \alpha + \beta - 2}$.

Normal likelihood with normal prior

Normal likelihood with Normal prior

- Suppose that the parameter of your interest is μ which is estimated by the mean of the data $\bar{X}$ sampled n times from $N(\mu, \sigma^2)$ with known σ^2. Note: $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.
- You assume a prior: $\mu \sim N(\mu_0, \sigma_0^2)$.
- Then the posterior distribution is given as

$$
\begin{aligned}
f(\mu|\bar{X} = \bar{x}) &\propto f(\bar{X}|\mu) f(\mu) \\
&\propto \exp\left(-\frac{1}{2\sigma^2/n}(\bar{x} - \mu)^2 - \frac{1}{2\sigma_0^2}(\mu - \mu_0)^2\right) \\
&\propto \exp\left(-\frac{1}{2 \cdot \kappa^2}(\mu - \tau)^2\right)
\end{aligned}
$$

where $\kappa^2 = \dfrac{\sigma^2 \sigma_0^2}{\sigma^2 + n\sigma_0^2}$ and $\tau = \dfrac{n\sigma_0^2 \bar{x} + \sigma^2 \mu_0}{\sigma^2 + n\sigma_0^2}$.

Summary

Parameter	Likelihood	Prior	ML Est.	MAP Estimate	Credible Interval
μ	$N\left(\mu, \frac{\sigma^2}{n}\right)$	$N(\mu_0, \sigma_0^2)$	$\bar{x}$	μ_1	$\mu_1 \pm Z_{1-\alpha/2}\sigma_1$
p	Bin(n, p)	Beta(α, β)	$\frac{x}{n}$	$\frac{x + \alpha - 1}{n + \alpha + \beta - 2}$	Out of scope

$$\mu_1 = \frac{n\bar{x}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2} \text{ and } \sigma_1^2 = \frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2}$$

$X \sim \text{Beta}(\alpha, \beta)$, then $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} X^{\alpha-1}(1 - X)^{\beta-1}$; $0 < X < 1$, $\alpha > 0$ and $\beta > 0$.

$E(X) = \frac{\alpha}{\alpha+\beta}$ and $Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$p|X=x$ is a beta distribution with the parameters $\alpha_1 = x + \alpha$ and $\beta_1 = n - x + \beta$.

Example

Kevin, a biology student, poses a statistical model for his scores on standard IQ tests. He thinks that, in general, his scores are normally distributed with unknown mean μ and variance of 80.

Expert opinion is that the IQ of biology students, μ, is a normal random variable, with mean 110 and variance 120.

Kevin took the test and scored 98.

$X \sim N(\mu, \sigma^2=80)$ and $\mu \sim N(\mu_0=110, \sigma_0^2=120)$

What is an estimate of Kevin's IQ?

$\hat{\mu} = \bar{X} = 98$ since $n = 1$ (The classical estimate of μ is given by the sample mean, which happens to be the ML estimate).

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Kevin took the test and scored 98.

$X \sim N(\mu, \sigma^2{=}80)$ and $\mu \sim N(\mu_0{=}110, \sigma_0^2{=}120)$

What is an 95% interval estimate of μ?

$\overline{X} \pm Z_{1\text{-}\alpha/2}\frac{\sigma}{\sqrt{n}} = 98 \pm 1.96\frac{\sqrt{80}}{\sqrt{1}} = 98 \pm 17.53 = [80.47, 115.53]$

$W = 115.53 - 80.47 = 35.06$

Example

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Kevin took the test and scored 98.

$X \sim N(\mu, \sigma^2=80)$ and $\mu \sim N(\mu_0=110, \sigma_0^2=120)$

What is the posterior distribution of $\mu | \bar{x}$?

$$\mu_1 = \frac{n\bar{x}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2} = \frac{1(98)(120) + 110(80)}{1(120) + 80} = 102.8 \text{ and } \sigma_1^2 = \frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2} = \frac{80(120)}{1(120) + 80} = 48$$

$\mu | \bar{x} \sim N(\mu_1=102.8, \sigma_1^2=48)$

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Kevin took the test and scored 98.

$X \sim N(\mu, \sigma^2=80)$ and $\mu \sim N(\mu_0=110, \sigma_0^2=120)$

What is the MAP estimate of μ?

$$\hat{\mu}_{MAP} = \mu_1 = \frac{n\bar{x}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2} = \frac{1(98)(120) + 110(80)}{1(120) + 80} = 102.8$$

Example

Kevin, a biology student, poses a statistical model for his scores on standard IQ tests. He thinks that, in general, his scores are normally distributed with unknown mean μ and variance of 80.

Expert opinion is that the IQ of biology students, μ, is a normal random variable, with mean 110 and variance 120.

Kevin took the test and scored 98.

$X \sim N(\mu, \sigma^2=80)$ and $\mu \sim N(\mu_0=110, \sigma_0^2=120)$

What is an 95% credible interval of μ?

$$\mu_1 = \frac{n\bar{x}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2} = \frac{1(98)(120) + 110(80)}{1(120) + 80} = 102.8 \text{ and } \sigma_1^2 = \frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2} = \frac{80(120)}{1(120) + 80} = 48$$

$\mu_1 \pm Z_{1\text{-}\alpha/2}\sigma_1 = 102.8 \pm 1.96\sqrt{48} = 102.8 \pm 13.579 = [89.221, 116.379]$

$W = 116.379 – 89.221 = 27.158$

The credible interval (W = 27.158) is shorter than the Z-confidence interval (W = 35.06) because the posterior variance (48) is smaller than the likelihood variance (80); this is a consequence of the incorporation of information from the prior distribution.

Example

Kevin, a biology student, poses a statistical model for his scores on standard IQ tests. He thinks that, in general, his scores are normally distributed with unknown mean μ and variance of 80.

Expert opinion is that the IQ of biology students, μ, is a normal random variable, with mean 110 and variance 120.

Kevin took the test and scored 98.

$X \sim N(\mu, \sigma^2=80)$ and $\mu \sim N(\mu_0=110, \sigma_0^2=120)$

What happens to the $\hat{\mu}_{MAP}$ estimate when the prior variance increases indefinitely?

$$\hat{\mu}_{MAP} = \mu_1 = \frac{n\bar{x}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2} = \frac{\frac{n\bar{x}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2}}{\frac{n\sigma_0^2 + \sigma^2}{n\sigma_0^2}} = \frac{\bar{x} + \frac{\mu_0\sigma^2}{n\sigma_0^2}}{1+\frac{\sigma^2}{n\sigma_0^2}} \to \bar{x} = \hat{\mu}_{ML} \text{ as } \sigma_0^2 \to \infty$$