QBUS6840 Lecture 03

Time Series Decomposition

Discipline of Business Analytics

The University of Sydney Business School

Recap and an example

Last week:

- *Plotting*: time series and scatter plots
- *Components*: trend, cycle, seasonal, and irregular patterns
- *Simple forecasting methods*: use last observation, use last season, drift, mean...
- *Accuracy measures*: MAD, MSE and RMSE, MAPE
- *Model validation*

This lecture: how to **extract the trend and seasonal components**

An example: Keeling curve
https://en.wikipedia.org/wiki/Keeling_Curve,
http://mlg.eng.cam.ac.uk/carl/words/keeling.pdf

Table of contents

Readings

Readings: Online textbook Ch. 6; BOK section 6.3 and Ch. 7

Time Series Decomposition

- Decompose a time series into its four components

  $$y_t = f(T_t, C_t, S_t, R_t)$$

  where T_t, C_t and S_t denote the trend, cycle and seasonal components. R_t is the remainder, often modeled by mathematical models.
- Pioneered by the French government in 1911, then US Bureau of Census and Bureau of Labor Statistics. Developed by many academics around the world.
- The goal of time series decomposition is usually to understand and interpret the time series rather than for forecasting.
  - However, it also helps improve forecast accuracy

Time Series Decomposition

- Note: There might be a cycle component in the time series but this component is often combined into the trend component, as it is more mathematically convenient to model these two components together. Most decomposition methods are developed to estimate this trend-cycle component together.
- From now on, we aim to decompose

$$y_t = f(T_t, S_t, R_t)$$

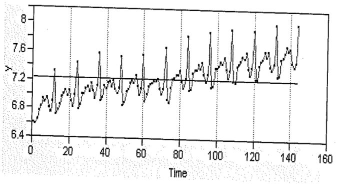
where T_t is the trend-cycle component, and often simply referred to as trend.

Decomposition Models

- Additive decomposition:

$$y_t = T_t + S_t + R_t$$

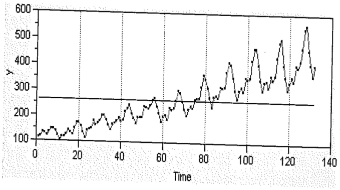
Appropriate when the variation (around the trend, or the seasonal pattern) doesn't change over time



- Multiplicative decomposition

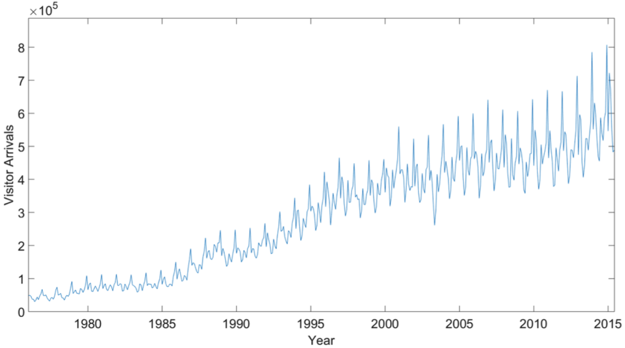
$$y_t = T_t \times S_t \times R_t$$

variation (around the trend, or the seasonal pattern) changes over time



Additive or Multiplicative?

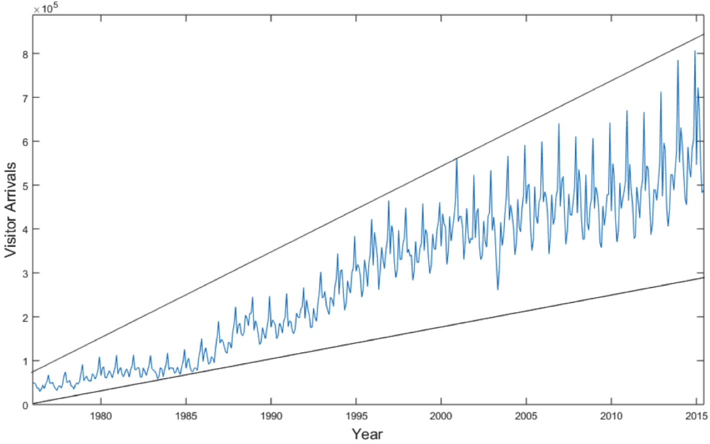
- Is the seasonal variation proportional to the trend?



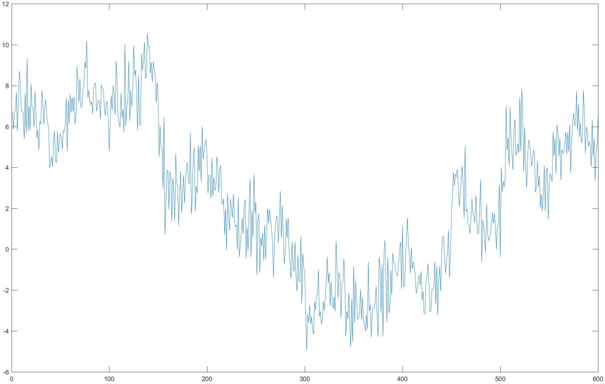
Short-term visitor arrivals to Australia (monthly data), available at the Australian Bureau of Statistics
http://www.abs.gov.au/ausstats/abs@.nsf/mf/3401.0

Additive or Multiplicative?

- A multiplicative model seems best for the visitor arrival series

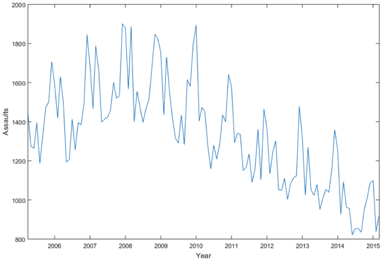


Additive or Multiplicative?



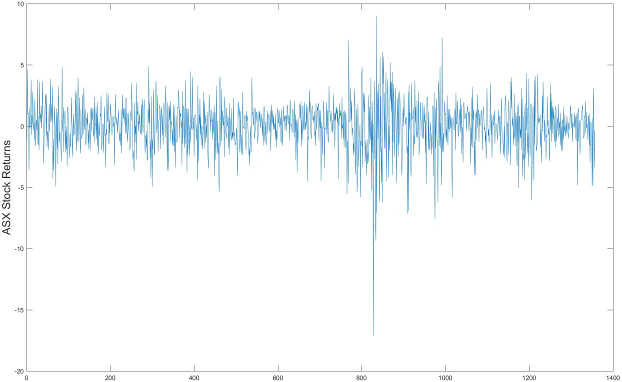
Additive or multiplicative?

- Alcohol related assaults in NSW
- Data from the NSW Bureau of Crime Statistics and Research: `www.bocsar.nsw.gov.au/Pages/bocsar_pages/Alcohol_Related_Violence.aspx`



Additive or multiplicative?

Australian stock returns ASX200, 1992-2018.

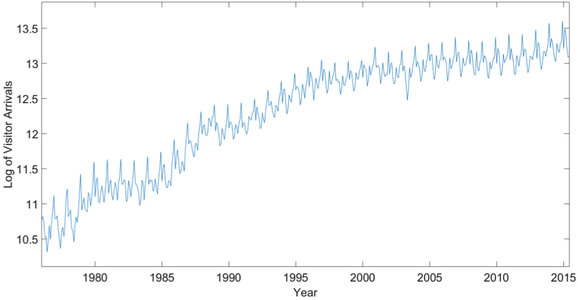


Downloaded from Yahoo Finance

`https://au.finance.yahoo.com/quote/%5EAXJO/`

Multiplicative Model to Additive Model

- We can convert a multiplicative model into an additive model by noting that
$$y_t = T_t \times S_t \times R_t$$
implies
$$\log y_t = \log T_t + \log S_t + \log R_t$$
- Another way to visualize whether a multiplicative model is adequate is to plot the log series. [compare this to Slide 9]



Seasonal adjustment

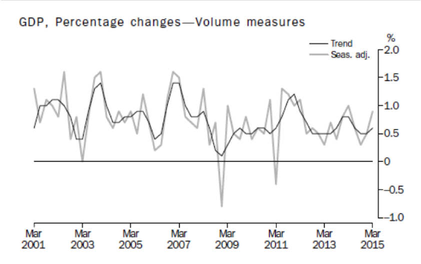
- Seasonal adjustment is a statistical technique that estimates and removes the influences of predictable seasonal patterns, in order to reveal how time series values change from time to time.
- We obtain the seasonally adjusted data if the seasonal component is removed from the original time series

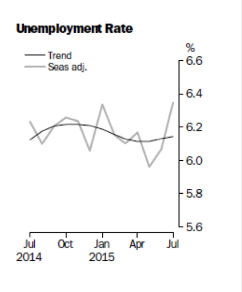
$$\tilde{y}_t = y_t - S_t, \quad \text{or} \quad \tilde{y}_t = y_t / S_t$$

- Working with seasonally adjusted data also leads to more accurate forecasts.

Seasonal adjustment

- Used by the Australian Bureau of Statistics (ABS) to adjust series such as: Building approvals, unemployment rate, labour force, change in gross domestic product, average weekly earnings, population growth (by state).





- Seasonal adjustment helps better reveal the change in these time series

Moving Average Methods

- MA is often used to smooth out noise and reveal the trend component.
- Smoothing with a moving average of order 3 (denoted as MA-3): Given the time series $\{Y_1, Y_2, ..., Y_T\}$, its smoothed time series is

$$T_t = \frac{Y_{t-1} + Y_t + Y_{t+1}}{3}, \quad t = 2, ..., T-1.$$

- Smoothing with a moving average of order 5 (denoted as MA-5):

$$T_t = \frac{Y_{t-2} + Y_{t-1} + Y_t + Y_{t+1} + Y_{t+2}}{5}, \quad t = 3, ..., T-2.$$

- How to define MA-k, k odd?

Moving Average Methods

- Example: Smoothing a time series using MA-5

$$\mathcal{T} = \{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}\}$$

- The time point we can start smooth is $t = 3$, and the last time point we can smooth is $t = 8$. The smoothed time series is

$$\widehat{\mathcal{T}} = \{-, -, T_3, T_4, T_5, T_6, T_7, T_8, -, -\}$$

losing two values on both sides, where we calculate them as

$$T_3 = \frac{Y_1 + Y_2 + Y_3 + Y_4 + Y_5}{5}, \cdots, \; T_8 = \frac{Y_6 + Y_7 + Y_8 + Y_9 + Y_{10}}{5}$$

Symmetrically centred MA

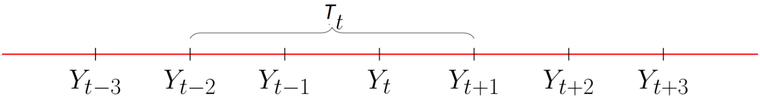
- An MA is said to be symmetrically centred if the same number of observations on either side of Y_t, together with Y_t, are averaged to compute T_t.
- It's easy to see that a MA-k, with k and odd number is symmetrically centred.
- How can we make a MA-k with an even k to be symmetrically centred?

Even Order Moving Average

- MA-5 is symmetrically centred

$$T_t = \frac{Y_{t-2} + Y_{t-1} + Y_t + Y_{t+1} + Y_{t+2}}{5}$$

- Consider possible MA-4. Which way?



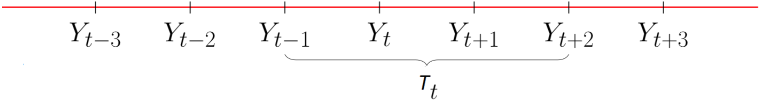
$$T_t = \frac{Y_{t-2} + Y_{t-1} + Y_t + Y_{t+1}}{4}$$

Even Order Moving Average

- MA-5 is symmetrically centred

$$T_t = \frac{Y_{t-2} + Y_{t-1} + Y_t + Y_{t+1} + Y_{t+2}}{5}$$

- Consider possible MA-4. Which way?



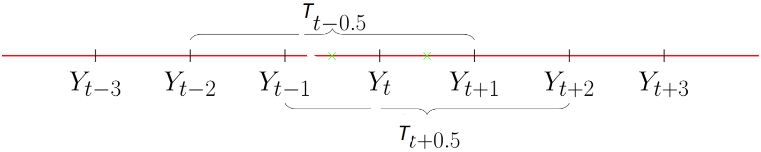
$$T_t = \frac{Y_{t-1} + Y_t + Y_{t+1} + Y_{t+2}}{4}$$

Even Order Moving Average

- MA-5 is symmetrically centred

$$T_t = \frac{Y_{t-2} + Y_{t-1} + Y_t + Y_{t+1} + Y_{t+2}}{5}$$

- Work out the half time smoothing, then average them



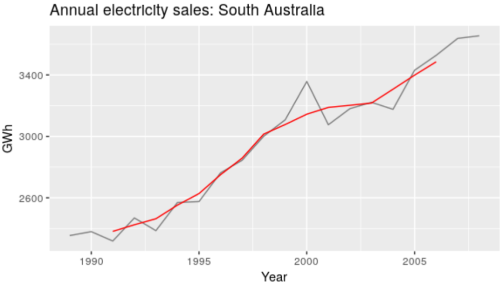
$$T_{t-0.5} = \frac{Y_{t-2} + Y_{t-1} + Y_t + Y_{t+1}}{4};\ T_{t+0.5} = \frac{Y_{t-1} + Y_t + Y_{t+1} + Y_{t+2}}{4}$$

$$T_t = \frac{T_{t-0.5} + T_{t+0.5}}{2} = \frac{1}{8}Y_{t+2} + \frac{1}{4}(Y_{t+1} + Y_t + Y_{t-1}) + \frac{1}{8}Y_{t-2}$$

- That is, we do a two-layer MA: MA-4, followed by MA-2.

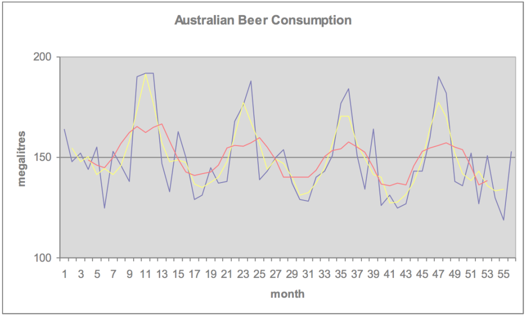
MA for estimating trend

The smoothed time series (red curve) produced by MA-5. This can be viewed as the trend component. The grey line is the data.



Selecting k

- Heavier smoothing VS Responsiveness
- Useful to compare results with different k
- What will happen here if $k = 12$ or higher?
- Demo in `Lecture03_Example01.py`



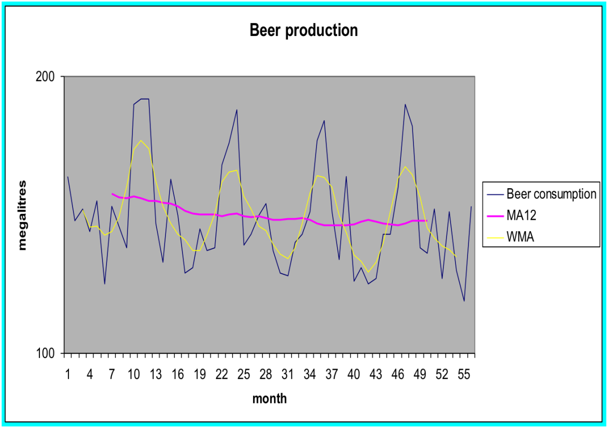
— MA 3 — MA 7

Example: Quarterly product sales

- Notice we lose 2 data points at the beginning of the series and 2 data at the end

t	Y_t		T_t	St
1	897			
2	476			
2.5		564.2		
3	376		573.1	0.656
3.5		582		
4	509		588.625	0.865
4.5		595.25		
5	967		599.125	1.614
5.5		603		
6	529		585.75	0.903
6.5		568.5		
7	407		558.125	0.729
7.5		547.75		
8	371		532.5	0.697
8.5		517.25		
9	884		504	1.75
9.5		490.75		
10	407			
11	301			

When k increase, we lose more data on both sides



See demo in `Lecture03_Example01.py`

Four Minutes Exercise

- Show that the formula for a centered MA-6 is

$$T_t = \frac{1}{12}(Y_{t+3} + Y_{t-3}) + \frac{1}{6}(Y_{t+2} + Y_{t+1} + Y_t + Y_{t-1} + Y_{t-2})$$

- The formula for a centered MA-k (k is an even number) is

$$T_t = \frac{1}{2k}(Y_{t+k/2} + Y_{t-k/2}) + \frac{1}{k}\big(Y_{t+k/2-1} + Y_{t+k/2-2} + \cdots + Y_t + \cdots + Y_{t-k/2+2} + Y_{t-k/2+1}\big)$$

Write the formula for $k = 8$

Weighted Moving Averages

- MA is a special case of Weighted Moving Averages (WMA)
- WMA-k, $k = 2m + 1$ odd

  $$T_t = w_{t-m} Y_{t-m} + w_{t-m+1} Y_{t-m+1} + \cdots Y_t + \cdots + w_{t+m} Y_{t+m}$$

  where $\sum w_i = 1$.
- Example of a WMA-5

  $$T_t = 0.15 Y_{t+2} + 0.2 Y_{t+1} + 0.3 Y_t + 0.2 Y_{t-1} + 0.15 Y_{t-2}$$

- A centered MA-4 is a WMA-5

  $$T_t = \frac{1}{8}(Y_{t+2} + Y_{t-2}) + \frac{1}{4}(Y_{t+1} + Y_t + Y_{t-1})$$

Estimating the Seasonal Component (Multiplicative)

Model Assumption:

$$Y_t = T_t \times S_t \times R_t.$$

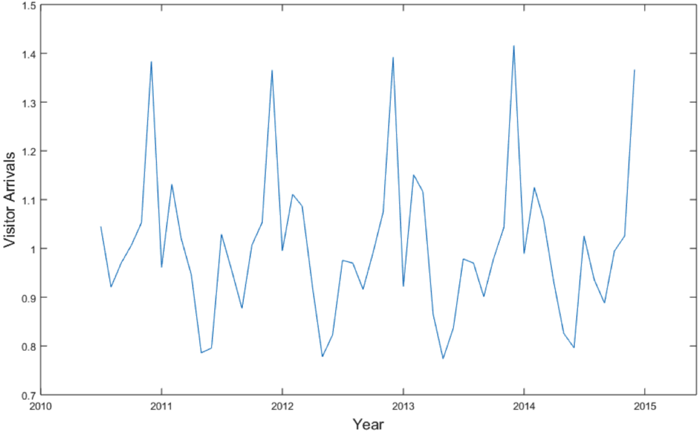
Let M be the seasonal frequency.

- Estimate the trend $\widehat{T_t}$ by smoothing the data (using, e.g., a centred MA-M). Compute the de-trended series $Y_t/\widehat{T_t}$.
- For each season m, $m = 1, ..., M$, compute the average of the de-trended values for that season. Denote these values by $\bar{s}_1, ..., \bar{s}_M$.
- Normalise these M values to make sure that they add up to M:
  - compute the normalising constant $c = M/(\bar{s}_1 + \bar{s}_2 + ... + \bar{s}_M)$.
  - compute the seasonal indexes:

$$\bar{S}_m = c \times \bar{s}_m, \quad m = 1, 2, ..., M.$$

- The seasonal component $\widehat{S}_t$ is obtained by concatenating these seasonal indexes.

Example: Detrended series $\frac{Y_t}{\widehat{T}_t}$ (2010-2015)



Example: Calculation

- Use only two years for demonstration
- First take the average of two same months (of the detrended series)
- Calculate the normalized constant by

$$c = \frac{12}{\text{sum of All month values}}$$
$$= 12/(0.96 + \cdots + 1.39) = 1.004$$

- Month indices
  $\overline{S}_1 = 0.96 * 1.004 = 0.964, \ldots,$
  $\overline{S}_{12} = 1.39 * 1.004 = 1.40$
- The seasonal component is

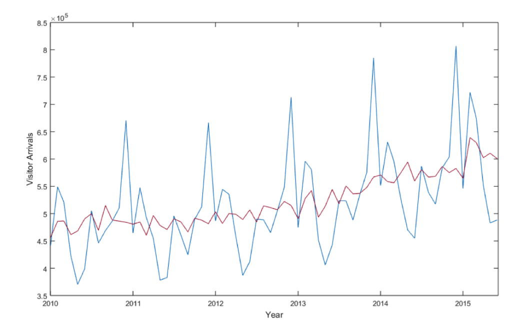
$$\{\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_{12}, \widehat{S}_{13}, \ldots, \widehat{S}_{24}, \ldots\}$$
$$= \{\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_{12}, \overline{S}_1, \cdots, \overline{S}_{12}, \cdots\}$$

	2013	2014	Average	Indices ($\overline{S}_m$)
January	0.92	0.99	0.96	0.964
February	1.15	1.13	$\rightarrow$ 1.14	1.145
March	1.12	1.06	1.09	1.094
April	0.86	0.93	0.90	0.904
May	0.77	0.83	0.80	0.803
June	0.84	0.80	0.82	0.823
July	0.98	1.03	1.00	1.004
August	0.97	0.94	0.95	0.954
September	0.90	0.89	0.89	0.894
October	0.98	0.99	0.99	0.994
November	1.04	1.03	1.03	1.034
December	1.42	1.27	1.39	1.394

Seasonally Adjusted Series

- We now calculate the seasonally adjusted series (taking off seasonal components)
- For monthly data, there are only 12 different $\widehat{S}_t$ values. For example, when $t = 37$, we know this is a January, so $\widehat{S}_t = \overline{S}_{Jan}$ (ie. $m = 1$)

$$Y_t = T_t \times S_t \times R_t \Rightarrow \widehat{T_t \times R_t} = \frac{Y_t}{\widehat{S}_t}$$

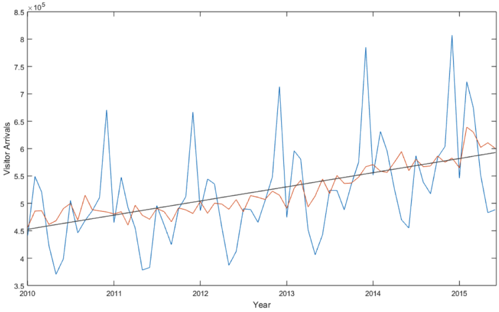


Modelling Trend

- Sometimes, it's desirable to model the trend (of the seasonally adjusted series). For example, forecasting requires a parametric model for trend.
- Common models are

$$T_t = \beta_0 + \beta_1 t$$
$$T_t = \beta_0 + \beta_1 t + \beta_2 t^2$$

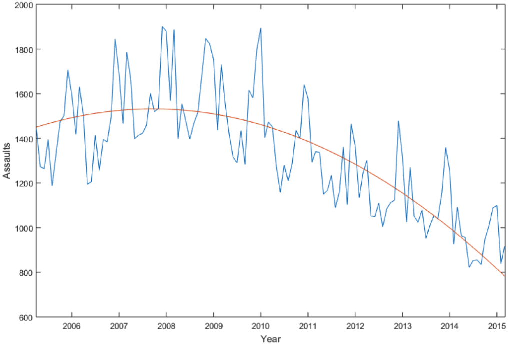
The coefficients are estimated by fitting the seasonally adjusted series on time.



In the picture above, we fit a linear model $T_t = \beta_0 + \beta_1 t$ to the seasonally adjusted data (in red).

Modelling Trend

Example: A quadratic trend model



Here, we fit a quadratic model

$$T_t = \beta_0 + \beta_1 t + \beta_2 t^2$$

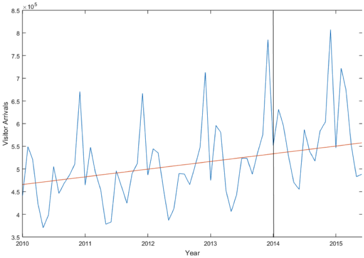
to the seasonally adjusted data of the NSW Alcohol-related Assaults data.

Forecasting Future Values (Multiplicative)

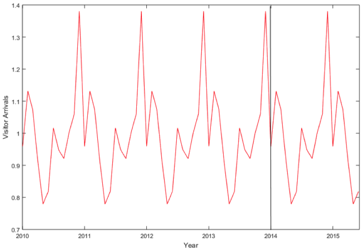
- Forecasts are
$$\widehat{y}_{t+h} = \widehat{T}_{t+h} \times \widehat{S}_{t+h}$$
where $\widehat{T}_{t+h}$ is estimate of T_{t+h} using the trend model, $\widehat{S}_{t+h}$ is seasonal estimate.
- For example, suppose $t = 38$-th month and we wish to forecast for the next two months, which means $h = 1, 2$. Hence $t + h = 39$ and 40 respectively. First use trend formula, e.g., calculate
$$\widehat{T}_{t+h} = \widehat{T}_{38+1} = \widehat{\beta}_0 + \widehat{\beta}_1(38 + 1) = \widehat{\beta}_0 + 39 * \widehat{\beta}_1$$
$$\widehat{T}_{t+h} = \widehat{T}_{38+2} = \widehat{\beta}_0 + 40 * \widehat{\beta}_1$$
- As $t + h = 39$ means March and $t + h = 40$ April, we will have $\widehat{S}_{39} = \overline{S}_{Mar}$ and $\widehat{S}_{40} = \overline{S}_{Apr}$ in the final forecast calculation

Forecasting T and S

Trend $\widehat{T}_{t+h}$

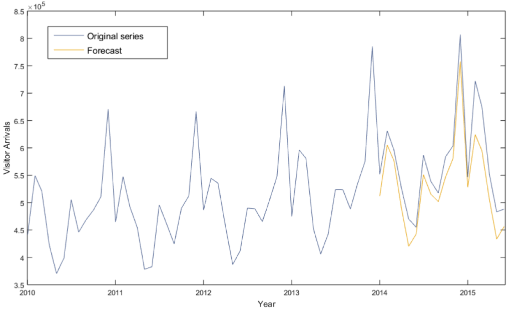


Seasonal $\widehat{S}_{t+h}$



Visitor Arrival Forecast

- We use data from 2010 to 2013 to estimate the multiplicative model, and then forecast arrivals from Jan 2014 to June 2015.
- Does the forecast look fine?



Estimating the seasonal component (additive)

Model Assumption:

$$Y_t = T_t + S_t + R_t.$$

Let M be the seasonal frequency.

- Estimate the trend $\widehat{T_t}$ by smoothing the data (using, e.g., a centred MA-M). Compute the de-trended series $Y_t - \widehat{T}_t$.
- For each season m, $m = 1, ..., M$, compute the average $\bar{s}_m$ of the de-trended values for that season.
- Normalise these M values to make sure that they add up to 0
  - compute the mean $\mu = \frac{1}{M}(\bar{s}_1 + \bar{s}_2 + ... + \bar{s}_M)$.
  - compute the seasonal indexes:

$$\bar{S}_m = \bar{s}_m - \mu, \quad m = 1, 2, ..., M.$$

- The seasonal component $\widehat{S}_t$ is obtained by concatenating these seasonal indexes.

Estimating the seasonal component (additive)

- The seasonally adjusted data is

$$d_t = Y_t - \widehat{S}_t$$

- As before, we can find a parametric trend model by fitting a tren model to d_t. E.g., using the linear trend model

$$d_t = \beta_0 + \beta_1 t + R_t \Rightarrow \widehat{T}_t = \widehat{\beta}_0 + \widehat{\beta}_1 t.$$

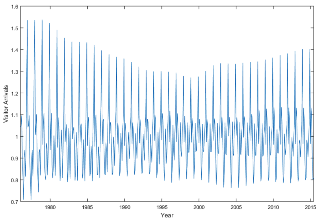
The X11 Method

- The X11 method is a more sophisticated time series decomposition method, developed by the U.S. Bureau of Census in 1965.
- Integrated into software packages such as SAS, R and EViews.
- It seasonally adjusts data and estimates the components of a time series.
- For the X11 method, there must be at least three years of observations in the input data sets.
- Unfortunately I have not found out a Python implementation of the X11 method. However, we can import and call R functions in Python using r2py.

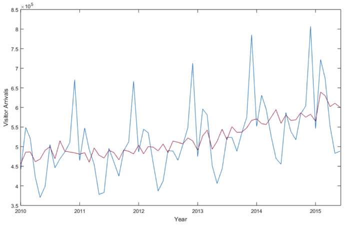
X11 Algorithm Steps

The detailed algorithm can be found in Bleikh and Yong https://books.google.com.au/books?id=WfefCwAAQBAJ&pg=PP66&lpg=PP66&dq=The+X11+method&source=bl&ots=H_q3UTyeSH&sig=mv9clb6sIUqbNzzWk69X6pbBKxI&hl=en&sa=X&ved=0ahUKEwi4xP-gsrDLAhWIHpQKHRO3Chc4ChDoAQgaMAA#v=onepage&q=The%20X11%20method&f=false

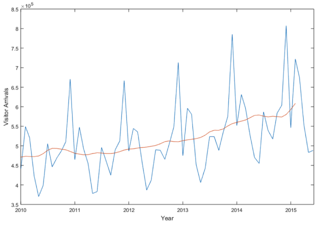
The X11 Method



X11 Seasonal Indexes $\widehat{S}_t$



X11 Seasonally Adjusted Series



X11 Trend Estimate

Summary

In this lecture, you have

- looked at multivariate and additive decomposition models for time series
- learned how to use moving average for smoothing time series, and estimating trend
- learned methods for seasonal adjustment.