QBUS6840 Lecture 9

Seasonal ARIMA Models and Forecast Combination

Discipline of Business Analytics

The University of Sydney Business School

Review of $ARMA(p,q)$ and $ARIMA(p,d,q)$ Processes

- $ARMA(p,q)$ formulation with backshift operators

$$\left(1-\sum_{i=1}^{p}\phi_i B^i\right) Y_t = c + \left(1+\sum_{i=1}^{q}\theta_i B^i\right)\varepsilon_t,$$

- $ARIMA(p,d,q)$ formulation with backshift operators

$$\left(1-\sum_{i=1}^{p}\phi_i B^i\right)(1-B)^d Y_t = c + \left(1+\sum_{i=1}^{q}\theta_i B^i\right)\varepsilon_t$$

- Let $Z_t = (1-B)^d Y_t$, then Z_t is the d-order differencing of Y_t. Hence $ARMA(p,q)$ of Z_t is the $ARIMA(p,d,q)$ of Y_t
- These are nonseasonal models

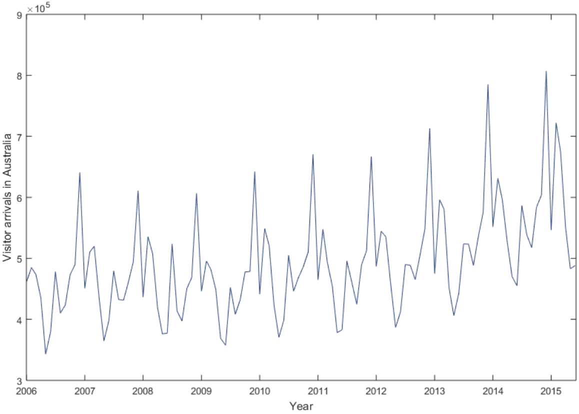
Outline

Seasonal ARIMA Models

- Seasonal ARMA$(P, Q)_m$
- Seasonal ARIMA$(p, d, q)(P, D, Q)_m$

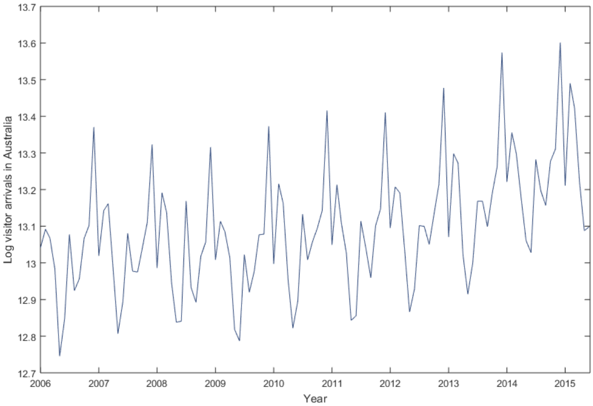
Forecast combinations

Example: Visitor arrivals



Example: Visitor arrivals

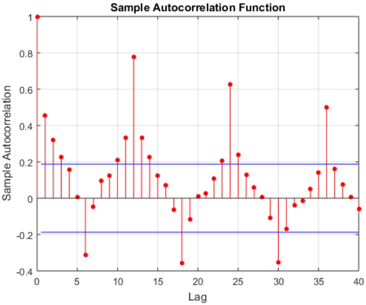
Variance stabilising transform

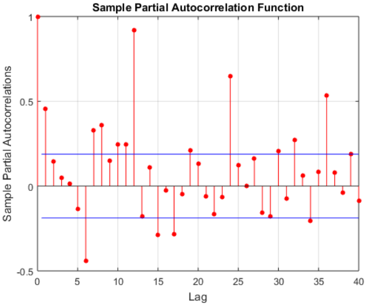


This is the Log transformed data.

Example: Visitor arrivals

ACF and PACF for the log visitors series

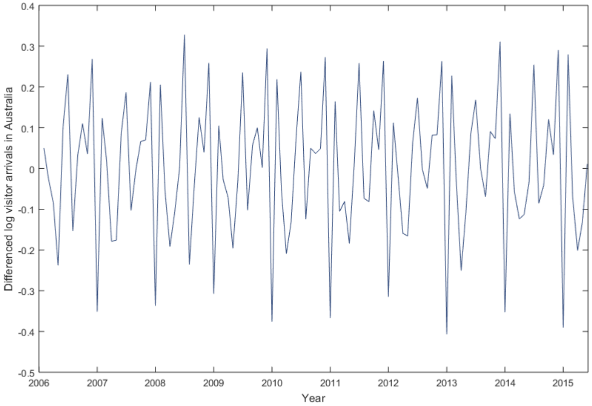




Example: Visitor arrivals

First differenced log visitors series

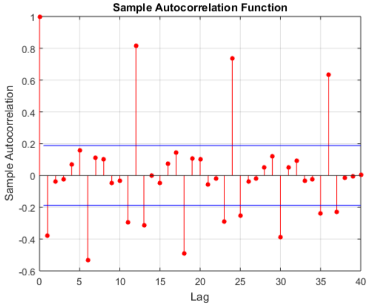
Take the first difference

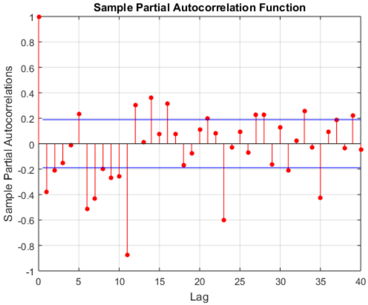


Example: Visitor arrivals

ACF and PACF for the first differenced log visitors series

Is the data stationary yet?





Example: Visitor arrivals

Seasonal differencing

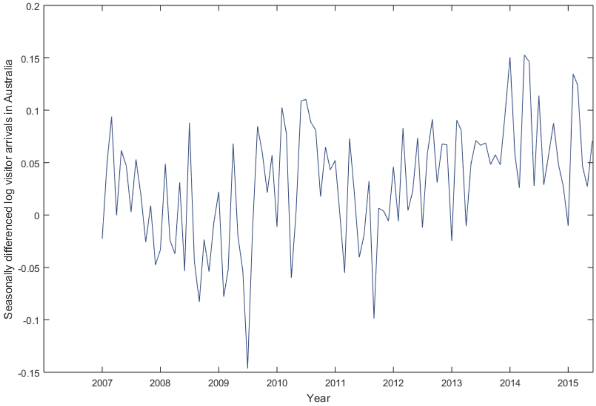
We can use seasonal differencing to remove the nonstationarity caused by the seasonality:

$$y_t - y_{t-12}$$

Example: Visitor arrivals

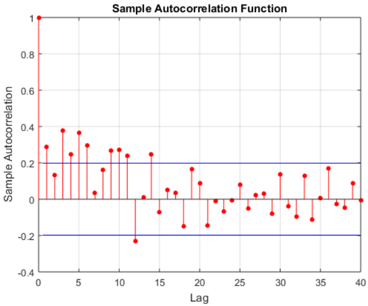
Seasonally differenced log visitors series

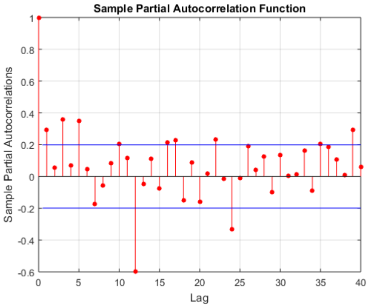
Take the first seasonal difference



Example: Visitor arrivals

ACF and PACF for the seasonally differenced log visitors series





ACF suggests that the transformed series is stationary, but there is clearly a seasonal effect on both ACF and PACF. We can use seasonal ARMA to model this transformed time series

Outline

Seasonal AR(1)$_m$

$$Y_t = c + \Phi_1 Y_{t-m} + \varepsilon_t$$

with m the seasonal frequency.
In the form of B operator

$$(1 - \Phi_1 B^m) Y_t = c + \varepsilon_t$$

Compared to nonseasonal AR(1)

$$Y_t = c + \phi_1 Y_{t-1} + \varepsilon_t$$

Note the use of the notations Φ_1 and ϕ_1.
ACF of AR(1)$_m$

$$\rho_k = \begin{cases} \Phi_1^i, & \text{if } k = i \times m, \ i = 0, 1, \dots \\ 0, & \text{else} \end{cases}$$

Seasonal MA(1)$_m$

$$Y_t = c + \Theta_1 \varepsilon_{t-m} + \varepsilon_t$$

Seasonal MA(1)$_m$

$$Y_t = c + \Theta_1 \varepsilon_{t-m} + \varepsilon_t$$

In the form of B operator

$$Y_t = c + (1 + \Theta_1 B^m)\varepsilon_t$$

ACF

$$\rho_k = \begin{cases} \frac{\Theta_1}{1+\Theta_1^2}, & \text{if } k = m \\ 0, & k \geq 1 \text{ and } k \neq m. \end{cases}$$

Seasonal ARMA$(P, Q)_m$

$$\left(1 - \sum_{i=1}^{P} \Phi_i B^{im}\right) Y_t = c + \left(1 + \sum_{i=1}^{Q} \Theta_i B^{im}\right) \varepsilon_t,$$

where error terms ε_t are white noise with mean zero and variance σ^2.

- Conditions for stationarity and invertibility are the same as for ARMA. E.g., AR$(1)_m$ is stationary if $|\Phi_1| < 1$

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- For AR$(P)_m$ (which is ARMA$(P, 0)_m$): ACF dies down at lags im, $i = 0, 1, 2, \ldots$ and PACF cuts off after lag Pm.

Seasonal ARMA$(P, Q)_m$

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- For AR$(P)_m$ (which is ARMA$(P, 0)_m$): ACF dies down at lags im, $i = 0, 1, 2, ...$ and PACF cuts off after lag Pm.
- For MA$(Q)_m$ (which is ARMA$(0, Q)_m$): ACF cuts off after lag Qm and PACF dies down at lags im, $i = 0, 1, 2, ...$

Seasonal ARMA$(P, Q)_m$

$$\left(1 - \sum_{i=1}^{P} \Phi_i B^{im}\right) Y_t = c + \left(1 + \sum_{i=1}^{Q} \Theta_i B^{im}\right) \varepsilon_t,$$

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- For AR$(P)_m$ (which is ARMA$(P, 0)_m$): ACF dies down at lags im, $i = 0, 1, 2, \ldots$ and PACF cuts off after lag Pm.
- For MA$(Q)_m$ (which is ARMA$(0, Q)_m$): ACF cuts off after lag Qm and PACF dies down at lags im, $i = 0, 1, 2, \ldots$
- For ARMA$(P, Q)_m$: both ACF and PACF die down at lags im, $i = 0, 1, 2, \ldots$.

Mixed Seasonal ARMA$(p, q)(P, Q)_m$

In practice, many time series can be modeled well by a mixed seasonal ARMA model

$$\left(1 - \sum_{i=1}^{P} \Phi_i B^{im}\right)\left(1 - \sum_{i=1}^{p} \phi_i B^i\right) Y_t = c + \left(1 + \sum_{i=1}^{Q} \Theta_i B^{im}\right)\left(1 + \sum_{i=1}^{q} \theta_i B^i\right) \varepsilon_t$$

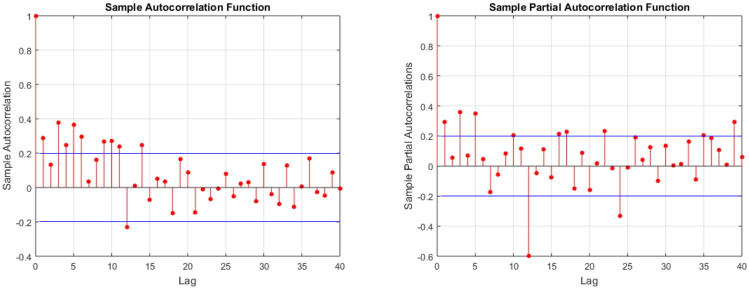
- For example, an ARMA$(0, 1)(1, 0)_{12}$ model is a combination of a seasonal AR$(1)_{12}$ and a non-seasonal MA(1)

$$Y_t - \Phi_1 Y_{t-12} = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

- The behavior of ACF and PACF is a combination of behavior of the seasonal and nonseasonal parts of the model.

Mixed Seasonal ARMA$(p, q)(P, Q)_m$: Example

Transformed series of visitor arrival data



We can conclude that the seasonal part is MA$(1)_{12}$. The orders p and q of the non-seasonal part ARMA(p, q) are not clear – need to do model selection

Outline

Seasonal ARIMA Models

Seasonal ARMA$(P, Q)_m$

Seasonal ARIMA$(p, d, q)(P, D, Q)_m$

Forecast combinations

Seasonal ARIMA$(p, d, q)(P, D, Q)_m$ models

- Seasonal ARMA and mixed seasonal ARMA require stationarity
- For non-stationary time series Y_t, by taking d-order difference

$$\nabla^d(Y_t) = (1 - B)^d(Y_t)$$

  and D-order seasonal difference

$$\nabla_m^D(Y_t) = (1 - B^m)^D(Y_t)$$

  we can arrive at a transformed time series Z_t which is stationary
- By combining difference and seasonal difference with mixed seasonal ARMA, we have a very general seasonal autoregressive integrated moving average (Seasonal ARIMA) model, also called Seasonal Box-Jenkins models.

Seasonal ARIMA$(p, d, q)(P, D, Q)_m$ models

$$\text{ARIMA} \quad \underbrace{(p, d, q)}_{\substack{\uparrow \\ \left(\begin{array}{l}\text{Non-seasonal part}\\ \text{of the model}\end{array}\right)}} \quad \underbrace{(P, D, Q)_m}_{\substack{\uparrow \\ \left(\begin{array}{l}\text{Seasonal part}\\ \text{of the model}\end{array}\right)}}$$

where $m =$ seasonal period (e.g. $m = 12$).

$$\begin{aligned}
&(1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p)(1-\Phi_1 B^{m}-\Phi_2 B^{2m}-\cdots-\Phi_P B^{Pm})(1-B)^d(1-B^m)^D Y_t \\
=&c+(1+\theta_1 B+\theta_2 B^2+\cdots+\theta_q B^q)(1+\Theta_1 B^{m}+\Theta_2 B^{2m}+\cdots+\Theta_Q B^{Qm})\epsilon_t
\end{aligned}$$

Seasonal Box-Jenkins models: Example

$$\underbrace{(1-\phi_1 B)}_{\text{Non-seasonal AR(1)}}\ \underbrace{(1-\Phi_1 B^4)}_{\text{Seasonal AR(1)}}\ \underbrace{(1-B)}_{\text{Non-seasonal difference}}\ \underbrace{(1-B^4)}_{\text{Seasonal difference}}y_t \;=\; \underbrace{(1+\theta_1 B)}_{\text{Non-seasonal MA(1)}}\ \underbrace{(1+\Theta_1 B^4)}_{\text{Seasonal MA(1)}}e_t.$$

The above is an $ARIMA(1,1,1)(1,1,1)_4$ model (with $c = 0$)

Seasonal Box-Jenkins models: example

$ARIMA(1,0,0)(0,1,1)_{12}$ for monthly data:

$$(1-\phi_1 B)(1-B^{12})Y_t = c + (1+\Theta_1 B^{12})\varepsilon_t$$

This is equivalent to

$$Y_t - Y_{t-12} = c + \phi_1(Y_{t-1} - Y_{t-13}) + \varepsilon_t + \Theta_1\varepsilon_{t-12}$$

Seasonal Box-Jenkins models: example

$ARIMA(1,0,0)(1,0,0)_{12}$ for monthly data:

$$(1-\phi_1 B)(1-\Phi_1 B^{12})Y_t = c + \varepsilon_t$$

Or write it out

$$Y_t = c + \phi_1 Y_{t-1} + \Phi_1 Y_{t-12} - \phi_1 \Phi_1 Y_{t-13} + \varepsilon_t$$

This is because informally

$$(1-\phi_1 B)(1-\Phi_1 B^{12}) = 1 - \phi_1 B - \Phi_1 B^{12} + \phi_1 \Phi_1 B^{13}$$

Hence

$$(1-\phi_1 B)(1-\Phi_1 B^{12})Y_t = Y_t - \phi_1 Y_{t-1} - \Phi_1 Y_{t-12} + \phi_1 \Phi_1 Y_{t-13}$$

Seasonal Box-Jenkins models: Example

$ARIMA(1,1,1)(1,1,0)_{12}$ model:

$$(1-\phi_1 B)(1-\Phi_1 B^{12})(1-B)(1-B^{12})Y_t = c + (1+\theta_1 B)\varepsilon_t$$

Seasonal Box-Jenkins models: Example

$ARIMA(1,1,1)(1,1,0)_{12}$ model:

$$(1-\phi_1 B)(1-\Phi_1 B^{12})(1-B)(1-B^{12})Y_t = c + (1+\theta_1 B)\varepsilon_t$$

The transformed series is $Z_t = (1-B)(1-B^{12})Y_t$, whose model is mixed seasonal ARMA$(1,1)(1,0)_{12}$

$$(1-\phi_1 B)(1-\Phi_1 B^{12})Z_t = c + (1+\theta_1 B)\varepsilon_t$$

Seasonal Box-Jenkins models: Example

$ARIMA(1,1,1)(1,1,0)_{12}$ model:

$$(1-\phi_1 B)(1-\Phi_1 B^{12})(1-B)(1-B^{12})Y_t = c + (1+\theta_1 B)\varepsilon_t$$

The transformed series is $Z_t = (1-B)(1-B^{12})Y_t$, whose model is mixed seasonal ARMA$(1,1)(1,0)_{12}$

$$(1-\phi_1 B)(1-\Phi_1 B^{12})Z_t = c + (1+\theta_1 B)\varepsilon_t$$

or

$$(1-\phi_1 B-\Phi_1 B^{12}+\phi_1\Phi_1 B^{13})Z_t = c + (1+\theta_1 B)\varepsilon_t$$

Seasonal Box-Jenkins models: Example

$ARIMA(1,1,1)(1,1,0)_{12}$ model:

$$(1-\phi_1 B)(1-\Phi_1 B^{12})(1-B)(1-B^{12})Y_t = c + (1+\theta_1 B)\varepsilon_t$$

The transformed series is $Z_t = (1-B)(1-B^{12})Y_t$, whose model is mixed seasonal ARMA$(1,1)(1,0)_{12}$

$$(1-\phi_1 B)(1-\Phi_1 B^{12})Z_t = c + (1+\theta_1 B)\varepsilon_t$$

or

$$(1-\phi_1 B - \Phi_1 B^{12} + \phi_1 \Phi_1 B^{13})Z_t = c + (1+\theta_1 B)\varepsilon_t$$

Hence

$$Z_t = \phi_1 B Z_t + \Phi_1 B^{12} Z_t - \phi_1 \Phi_1 B^{13} Z_t + c + \varepsilon_t + \theta_1 B \varepsilon_t$$

or

$$Z_t = \phi_1 Z_{t-1} + \Phi_1 Z_{t-12} - \phi_1 \Phi_1 Z_{t-13} + c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Seasonal Box-Jenkins models: example

Because

$$Z_t = (1-B)(1-B^{12})Y_t = (Y_t - Y_{t-1}) - (Y_{t-12} - Y_{t-13}),$$

we have

$$\begin{aligned}(Y_t - Y_{t-1}) - (Y_{t-12} - Y_{t-13}) =& c + \phi_1 \left[(Y_{t-1} - Y_{t-2}) - (Y_{t-13} - Y_{t-14})\right] \\ &+ \Phi_1 \left[(Y_{t-12} - Y_{t-13}) - (Y_{t-24} - Y_{t-25})\right] \\ &- \phi_1\Phi_1 \left[(Y_{t-13} - Y_{t-14}) - (Y_{t-25} - Y_{t-26})\right] \\ &+ \varepsilon_t + \theta_1\varepsilon_{t-1}.\end{aligned}$$

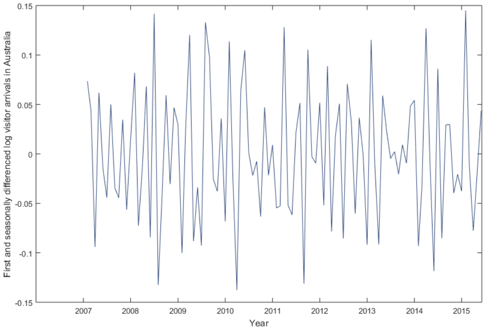
Finally,

$$\begin{aligned}Y_t =& c + Y_{t-1} + (Y_{t-12} - Y_{t-13}) + \phi_1 \left[(Y_{t-1} - Y_{t-2}) - (Y_{t-13} - Y_{t-14})\right] \\ &+ \Phi_1 \left[(Y_{t-12} - Y_{t-13}) - (Y_{t-24} - Y_{t-25})\right] \\ &- \phi_1\Phi_1 \left[(Y_{t-13} - Y_{t-14}) - (Y_{t-25} - Y_{t-26})\right] \\ &+ \varepsilon_t + \theta_1\varepsilon_{t-1}.\end{aligned}$$

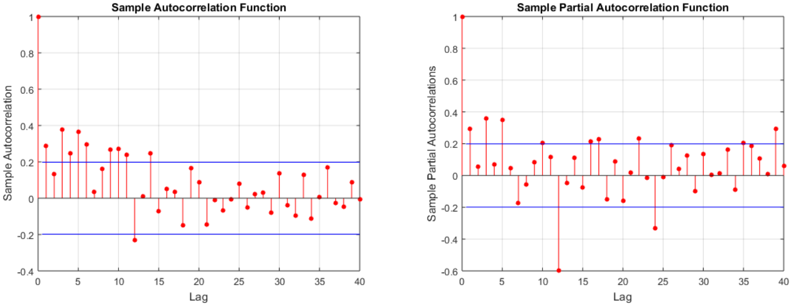
Seasonal ARIMA models: visitor data

We saw that both first difference and seasonal difference were needed to transform the log visitors series Y_t into a stationary series Z_t

$$Z_t = (1 - B^{12})(1 - B)Y_t = (Y_t - Y_{t-1}) - (Y_{t-12} - Y_{t-13})$$



Seasonal ARIMA models: visitor data



The seasonal part is $MA(1)_{12}$, for the non-seasonal part, let's try ARMA(2,2). Then we have an $ARIMA(2,1,2)(0,1,1)_{12}$ model

Seasonal ARIMA models

Estimation

$ARIMA(2,1,2)(0,1,1)_{12}$ model:

$$(1-\phi_1 B-\phi_2 B^2)(1-B)(1-B^{12})Y_t = c+(1+\theta_1 B+\theta_2 B^2)(1+\Theta_1 B^{12})\varepsilon_t$$

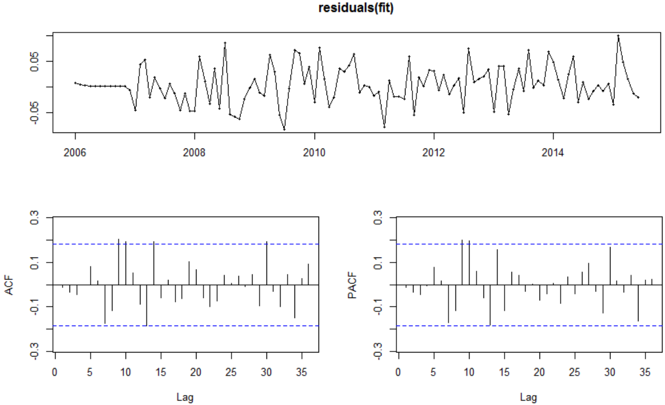
Estimated coefficients (using R):

	ar1	ar2	ma1	ma2	sma1
	-0.7817	-0.3154	-0.0300	-0.4007	-0.7471
s.e.	0.2212	0.1227	0.2213	0.1909	0.1073

log likelihood=178.99, AIC=-345.97, AICc=-345.08, BIC=-330.28.

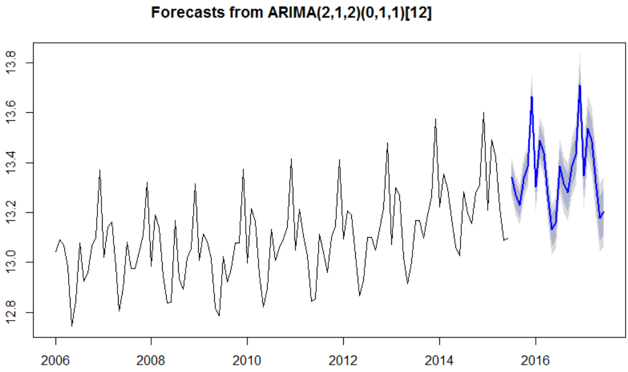
Seasonal ARIMA models

$ARIMA(2,1,2)(0,1,1)_{12}$ model: residuals



Seasonal ARIMA models

$ARIMA(2,1,2)(0,1,1)_{12}$ model: forecasts



Seasonal ARIMA models

See a detailed Python example in Lecture09_Example01.py

Outline

Seasonal ARIMA Models

Seasonal ARMA$(P, Q)_m$

Seasonal ARIMA$(p, d, q)(P, D, Q)_m$

Forecast combinations

BBC: The Code - The Wisdom of the Crowd

https://www.youtube.com/watch?v=i0ucwX7Z1HU

Forecast combinations

Introduction

Classical reference:

Bates, J. M., and C. W. J. Granger (1969). The combination of forecasts, *Operational Research Quarterly*, 20, 451–468.

They provide the following illustration:

TABLE 1. ERRORS IN FORECASTS (ACTUAL LESS ESTIMATED) OF PASSENGER MILES FLOWN, 1953

Month	Brown's exponential smoothing forecast errors	Box–Jenkins adaptive forecasting errors	Combined forecast ($\frac{1}{2}$ Brown + $\frac{1}{2}$ Box–Jenkins) errors
Jan	1	−3	−1
Feb.	6	−10	−2
March	18	24	21
April	18	22	20
May	3	−9	−3
June	−17	−22	−19·5
July	−24	10	−7
Aug.	−16	2	−7
Sept.	−12	−11	−11·5
Oct.	−9	−10	−9·5
Nov.	−12	−12	−12
Dec.	−13	−7	−10
Variance of errors	196	188	150

Forecast combinations
Introduction

- It is possible to combine unbiased forecasts $\hat{y}^{(i)}_{T+1|T}$ from models $i = 1, \ldots, m$.
- The models can be various ARIMA type of models or a set of ARIMA models, exponential smoothing models and regression models for example.
- $\hat{y}^{(i)}_{T+1|T}$, $i = 1, \ldots, m$ could also be m expert forecasts.

Forecasting combinations

Weights

- The forecasts can be combined as follows

$$\hat{y}^{c}_{T+1|T} = \sum_{i=1}^{m} w_i \; \hat{y}^{(i)}_{T+1|T}$$

- The simplest way is to set $w_i = \frac{1}{m}$, then you are using a simple average.
- Simple averages often work surprisingly well.
- The question is how to combine forecasts “optimally”?

Forecasting combinations

There is extensive empirical evidence in favour of combinations as a forecasting strategy.

- Forecasting combinations offer diversification gains that make them very useful compared to relying on a single model, as we have just seen.
- There may be structural breaks in the data, making it plausible that combining models with different levels of adaptability will lead to better results than relying on a single model.

Combining predictions from multiple models are useful beyond time series, e.g. combining climate forecasts[1], ensembles in stats and machine learning.

[1] https://journals.ametsoc.org/jcli/article/23/10/2739/32016/Challenges-in-Combining-Projections-from-Multiple

Recap

In the last 3 lectures:

- Autoregressive processes, AR(p)
- Moving average processes, MA(q)
- ARMA and ARIMA processes
- Seasonal and mixed ARMA/ARIMA models

Next lectures: (Recurrent) neural networks for time series

Thank you!