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Output regulation for a first-order hyperbolic PIDE with state and sensor delays☆,☆☆

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ABSTRACT

Considering disturbances within domain and at the boundary, a backstepping-based output boundary regulator design is developed for a class of first-order linear hyperbolic partial integro-differential equation (PIDE) in the presence of state and sensor delays. The delays are represented by two transport PDEs, which results in an extended spatial domain where the hyperbolic PIDE, transport PDEs and ordinary differential equation (ODE) are in cascade. The ODE is a finite-dimensional signal model describing exogenous signals. First, a state feedback regulator is realized to achieve a finite time stability by applying an affine Volterra integral transformation. Then, an output regulator is developed on the basis of the nominal plant transfer behavior, which results in an exponential stability. Numerical examples illustrate the performance of the proposed regulators.

1. Introduction

The output regulation design for boundary controlled first-order hyperbolic PIDE systems with state and measurement delays is developed. The first-order hyperbolic PIDE system due to the dominant role of transport by convection frequently appears in engineering process applications, such as plug-flow tubular reactor [23,25], traffic flows [5], heat exchangers [13] and oil drilling applications [20,27]. Therefore, a stabilization control problem for hyperbolic system has been considered by many researchers [4,11,28]. Recently, the infinite-dimensional backstepping technique [19] emerged and was further developed to provide a systematic design methodology for the control of PDE systems. An early paper [18] which addresses the first-order hyperbolic PDEs develops a backstepping-based controller for stabilization of an open-loop unstable hyperbolic PIDE. Along the same line, the contributions in [26] and [7] considered a boundary output-feedback controller design for a linear 2×2 first-order PDE and for a quasilinear 2×2 first-order hyperbolic PDE, respectively. As well as in [12], a backstepping-based output-feedback design approach is introduced for a $n+1$ coupled first-order hyperbolic PDEs. Further, the control problem of general $m+n$ heterodirectional hyperbolic systems are addressed in [15,16]. A new integral transform is introduced in [6] for a first-order hyperbolic PIDE with Fredholm integrals. An adaptive controller and an observer are developed based on swapping for coupled first-order hyperbolic system with unknown parameters in [1,2]. Recently, the research in [3] discusses the robustness of the output feedback for 2×2 hyperbolic PDE with respect to small delays in the actuation and in the measurements. Besides the efforts to address stabilization of these systems, an important control design problem is to consider the asymptotic tracking of reference input signals in the presence of disturbances.

A promising extension of the backsteping method to the regulation problem design, that achieves asymptotic tracking of reference trajectories in the presence of disturbance, is developed in [8,10,22]. Based on the stabilization result of a 2×2 hyperbolic PDE system proposed in [26], a finite-time output feedback regulator is designed [9] and achieves the asymptotic tracking of reference signals in the presence of disturbances. In addition, [29] extends the result for a first-order hyperbolic PIDE with

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Fredholm integrals developed in [6] to an output regulation problem. The key idea of these works is transforming the output regulation problem into a class of PDE-ODE problem, which is considered in an abstract framework in [21].

However, the aforementioned works for hyperbolic PDE systems have not considered the impact of delays in system setting. The existence of delays brings the meaningful modelling framework since sensor delays and/or state delays are frequently present and in addition induce a new challenge to the output regulator design. Along this theme, there are only few references considering the output regulation which account for the impact of delays. In particular, only in [14] the output regulation of an unstable reaction-diffusion PDE with input delay is considered.

Motivated by the above considerations, we extend our previous works [29] and [24] to develop backstepping-based regulator for a first-order hyperbolic PIDE with delays to ensure the output tracking of a reference in the presence of disturbances. An Ordinary Differential Equation (ODE) exo-system is introduced to describe the reference and disturbance signals. We employ the PDE backstepping boundary control to stabilize the system and the exterior state feedforward for output tracking and disturbance rejection. Slightly different from [29], in this paper the rejection of a disturbance at the controlled output is explored. In addition, both state feedback regulation and output feedback regulation are considered in the paper. Due to the state delay and measurement delay, the regulator equation is more complex than one given in [29], which brings greater difficulties for the regulator equation analysis of both state feedback and output feedback. We prove the existence of a unique solution of the regulator equations under some mild assumption on the exo-system properties. The state feedback output regulator achieves a finite time tracking and furthermore, a novelty of the output feedback regulator is that it combines the state observer, which achieves the output tracking of reference input in finite-time. Last but not least, based on the observer design which assumes that the observer used output and the tracking output do not collocate, we prove and demonstrate the output exponentially convergence to the reference signal.

This paper is organized as follows. Section 2 presents the system. In Section 3, the state feedback regulator is designed and the finite-time tracking is proved. Section 4 designs the observers for reference, disturbance and the state, further proves the exponential stability in L^2 norm of the observers. The output feedback regulator based on observer is presented in Section 5 and the supportive simulation results are provided in Section 6. The paper ends with concluding remarks and a discussion of the future work in Section 7.

Notation: Throughout this paper, the partial derivative is denoted as:

$$\partial_\zeta g(\zeta,t)=\frac{\partial g(\zeta,t)}{\partial\zeta},\quad \partial_t g(\zeta,t)=\frac{\partial g(\zeta,t)}{\partial t}. \tag{1}$$

The L^2-norm for $g(\zeta,t)\in L^2[0,1]$ is defined as:

$$\|g(\cdot,t)\|^2=\int_0^1 |g(\zeta,t)|^2\,\mathrm{d}\zeta. \tag{2}$$

The euclidean norm for $X\in\mathbb{R}^n$ is defined as:

$$|X|^2=X^{\mathrm{T}}X. \tag{3}$$

Domains are defined as:

$$\mathcal{T}_1=\left\{(\zeta,\eta)\in\mathbb{R}^2:0\le\zeta\le\eta\le1\right\}, \tag{4}$$

$$\mathcal{T}_2=\{(\zeta,\eta)\in\mathbb{R}^2:0\le\zeta,\ \ \eta\le1\}, \tag{5}$$

$$\mathcal{T}_3=\left\{(\zeta,\eta)\in\mathbb{R}^2:0\le\eta\le\zeta\le1\right\}. \tag{6}$$

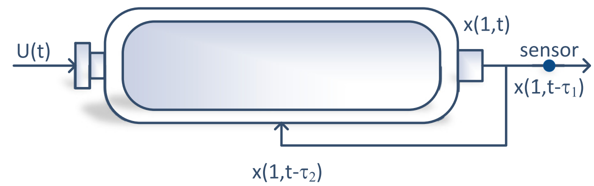

Fig. 1. The sketch of plug-flow tubular reactor.

2. First-order PIDE system

Consider the following linear first-order hyperbolic PIDE with delays, which usually is used to describe a homogeneous tubular reactor with a fraction of the flow funnelled back to the reactor as shown in Fig. 1. The system without disturbance is presented in [24], where the recycle delay to the inlet is ignored in the following discussion. In practical tubular reactor applications with heat recovery, there exists a transportation delay τ_2 which occurs due to transport lag,

$$\begin{aligned}\partial_t x(\zeta,t)&=-a\partial_\zeta x(\zeta,t)+c(\zeta)x(1,t-\tau_2)\\&\quad+\int_\zeta^1 f(\zeta,\eta)x(\eta,t)\,\mathrm{d}\eta+\bar{D}_1(\zeta)d(t),\end{aligned} \tag{7a}$$

$$x(0,t)=U(t)+\bar{D}_2 d(t), \tag{7b}$$

$$\psi(t)=\mathcal{F}x(t)+\bar{D}_3 d(t), \tag{7c}$$

$$y(t)=x(1,t-\tau_1), \tag{7d}$$

for $(\zeta,t)\in[0,1]\times\mathbb{R}^+$, with the initial condition

$$x(\zeta,0)=x_0(\zeta)\in L^2[0,1],\quad \zeta\in[0,1], \tag{8a}$$

$$x(\zeta,h)=0,\quad (\zeta,h)\in[0,1]\times[-\tau_2,0). \tag{8b}$$

The disturbances are denoted by $\bar{D}_1(\zeta)d(t)$, $\bar{D}_2 d(t)$ and $\bar{D}_3 d(t)$, respectively, where $d(t)$ is the disturbance function, $\bar{D}_1\in\mathbb{C}[0,1]$, $\bar{D}_2,\bar{D}_3\in\mathbb{R}$. Here, τ_1 and τ_2 denote the sensor delay and state delay in the system, respectively. Usually, sensors are placed near or at the boundary $\zeta=1$, so it is reasonable that the state delay resulting from the state transport is longer than the delay caused by the sensor, namely, $\tau_1<\tau_2$. $U(t)$ is the control to be designed.

Note that for the more general form, i.e., $\partial_t\breve{x}(\zeta,t)=-\breve{a}(\zeta)\partial_\zeta\breve{x}(\zeta,t)+\breve{b}(\zeta)\breve{x}(\zeta,t)+\breve{c}(\zeta)\breve{x}(1,t-\tau_2)+\int_\zeta^1\breve{f}(\zeta,\eta)$ $\breve{x}(\eta,t)\,\mathrm{d}\eta+\breve{D}_1(\zeta)\breve{d}(t)$, the reaction term $\breve{b}(\zeta)\breve{x}(\zeta,t)$ and coefficient $\breve{a}(\zeta)$'s dependence on ζ can be eliminated by two steps of the change of variables (see more details in [24]).

In order to proceed, the following assumption is made:

Assumption 1. Assume that $f(\zeta,\eta)$, $c(\zeta)$ are real-valued continuous functions in their respective domains, i.e., $f(\zeta,\eta)\in\mathcal{C}(\mathcal{T}_1)$, $c(\zeta)\in\mathcal{C}([0,1])$ and $a>0$ is a constant and $c(1)=0$.

As measurement output $y(t)$ is utilized and $\psi(t)$ represents the output to be controlled. The output $\psi(t)$ can be defined at the boundary, point wise or distributed in-domain as:

$$\mathcal{F}x(t)=\sum_{i=1}^{l}C_i x(\zeta_i,t)+\int_0^1 C(\zeta)x(\zeta,t)\,\mathrm{d}\zeta, \tag{9}$$

for $C_i\in\mathbb{R}$, $\zeta_i\in[0,1]$, $i=1,2,\ldots,l$ denoting the sampled point within domain, and $C(\cdot)$ is a piecewise continuous function. We assume that the resulting output $\psi(t)$ is different from the boundary conditions (7b) and (7d).

The exo-system which governs the reference input and the disturbances is

$$\dot{v}(t) = Sv(t), \quad t > 0, \quad v(0) = v_0 \tag{10a}$$

$$r(t) = P_r v(t) = \bar{P}_r v_r(t), \tag{10b}$$

$$d(t) = P_d v(t) = \bar{P}_d v_d(t), \tag{10c}$$

with $v(t) \in \mathbb{R}^n$. Choose a diagonalizable matrix $S = \text{bdiag}(S_r, S_d)$ with all eigenvalues on the imaginary axis, which ensures the bounded and persistently acting exogenous signals, such as constant or trigonometric functions.

Rewriting v as $v = col(v_r, v_d)$, one obtains the reference model $\dot{v}_r(t) = S_r v_r(t), v_r(0) = v_{r0} \in \mathbb{R}^{n_r}$, and the disturbance model $\dot{v}_d(t) = S_d v_d(t), v_d(0) = v_{d0} \in \mathbb{R}^{n_d}$, with $n_r + n_d = n$.

In the paper, we will solve the output regulation problem by designing a stabilizing control such that

$$\lim_{t\to\infty} e_\psi(t) = \lim_{t\to\infty} (\psi(t) - r(t)) = 0, \tag{11}$$

holds independent of the plant initial conditions, the exo-signal model initial conditions or controller initial conditions. To express the delays in state and measurement, we first introduce two transport PDEs, combined with the exogenous system, so the system (7) is rewritten as follows:

$$\begin{aligned}\partial_t x(\zeta, t) = &-a\partial_\zeta x(\zeta, t) + c(\zeta)u_2(0, t)\\ &+ \int_\zeta^1 f(\zeta, \eta)x(\eta, t)\,\mathrm{d}\eta + D_1(\zeta)v(t),\end{aligned} \tag{12a}$$

$$x(0, t) = U(t) + D_2 v(t), \tag{12b}$$

$$\tau_1 \partial_t u_1(\zeta, t) = \partial_\zeta u_1(\zeta, t), \tag{12c}$$

$$u_1(1, t) = x(1, t), \tag{12d}$$

$$(\tau_2 - \tau_1)\partial_t u_2(\zeta, t) = \partial_\zeta u_2(\zeta, t), \tag{12e}$$

$$u_2(1, t) = u_1(0, t), \tag{12f}$$

$$e_\psi(t) = \mathcal{F}x(t) + (D_3 - P_r)v(t), \tag{12g}$$

$$y(t) = u_1(0, t), \tag{12h}$$

where $D_i = \bar{D}_i P_d$, $i = 1, 2, 3$, and e_ψ is the tracking error defined in (11). $u_1(\zeta, t)$ and $u_2(\zeta, t)$ are artificial variables that describe the delayed transmission. Without loss of generality, the initial conditions for $u_1(\zeta, t)$ and $u_2(\zeta, t)$ are $u_1(\zeta, 0) = 0$, $u_2(\zeta, 0) = 0$, and compatibility conditions are $u_1(1, 0) = x(1, 0)$, $u_2(1, 0) = u_1(0, 0)$.

3. State feedback regulation

In this section, we utilize the backstepping method (Section 3.1) in the state feedback regulator design (Section 3.2). The important aspects of finite-time stability of state feedback regulator is proved in Section 3.3.

Follow the technical route of [29], applying the backstepping transformation, the cascade system is mapped into a transitional system coupled with the exo-system. The coupled system is uncoupled via a regulation coordinate change, which results in regulator equations whose solutions are needed in the control gain.

3.1. Backstepping design

First, we introduce an affine Volterra integral transformation as follows:

$$\begin{aligned}z(\zeta, t) &= \mathcal{K}[x, u_1, u_2]\\ &:= x(\zeta, t) - \int_\zeta^1 K_1(\zeta, \eta)x(\eta, t)\,\mathrm{d}\eta\\ &\quad - \int_0^1 K_2(\zeta, \eta)u_1(\eta, t)\,\mathrm{d}\eta - \int_0^1 K_3(\zeta, \eta)u_2(\eta, t)\,\mathrm{d}\eta,\end{aligned} \tag{13}$$

where kernel K_1 is defined on $\mathcal{T}_1$ and K_2, K_3 on $\mathcal{T}_2$. The transformation is invertible [1] (Theorem 1.4) and its inverse transformation is defined as $\mathcal{K}^{-1}[\cdot,\cdot,\cdot]$. According to transformation (13) and boundary condition (12b), the state feedback regulator is given as follows:

$$\begin{aligned}U(t) = &\int_0^1 K_1(0, \eta)x(\eta, t)\,\mathrm{d}\eta + \int_0^1 K_2(0, \eta)u_1(\eta, t)\,\mathrm{d}\eta\\ &+ \int_0^1 K_3(0, \eta)u_2(\eta, t)\,\mathrm{d}\eta + K_v v(t),\end{aligned} \tag{14}$$

with the gain K_v and kernels $K_i(0, \eta)$ for $i = 1, 2, 3$. Therefore, the design includes two steps. In first step, by the backstepping method, one can transform system (12a)-(12g) by the mapping (13) to the following system

$$\dot{v}(t) = Sv(t), \tag{15a}$$

$$\partial_t z(\zeta, t) = -a\partial_\zeta z(\zeta, t) + M(\zeta)v(t), \tag{15b}$$

$$z(0, t) = K_v v(t) + D_2 v(t), \tag{15c}$$

$$\tau_1 \partial_t u_1(\zeta, t) = \partial_\zeta u_1(\zeta, t), \tag{15d}$$

$$u_1(1, t) = z(1, t), \tag{15e}$$

$$\partial_t u_2(\zeta, t) = \frac{1}{\tau_2 - \tau_1}\partial_\zeta u_2(\zeta, t), \tag{15f}$$

$$u_2(1, t) = u_1(0, t), \tag{15g}$$

$$e_\psi(t) = \mathcal{F}\mathcal{K}^{-1}[z, u_1, u_2] + (D_3 - P_r)v(t), \tag{15h}$$

where $M(\zeta) = D_1(\zeta) - \int_\zeta^1 K_1(\zeta, \eta)D_1(\eta)\,\mathrm{d}\eta$. By completing a sequence of tedious calculations, the following kernel equations are obtained:

$$\begin{aligned}\partial_\zeta K_1(\zeta, \eta) = &-\partial_\eta K_1(\zeta, \eta) + \frac{1}{a}f(\zeta, \eta)\\ &- \frac{1}{a}\int_\zeta^\eta K_1(\zeta, s)f(s, \eta)\,\mathrm{d}s,\end{aligned} \tag{16a}$$

$$a\tau_1 \partial_\zeta K_2 = \partial_\eta K_2, \tag{16b}$$

$$a(\tau_2 - \tau_1)\partial_\zeta K_3 = \partial_\eta K_3, \tag{16c}$$

$$a\tau_1 K_1(\zeta, 1) = K_2(\zeta, 1), \tag{16d}$$

$$K_2(1, \eta) = 0, \tag{16e}$$

$$K_3(1, \eta) = 0, \tag{16f}$$

$$\tau_1 K_3(\zeta, 1) = (\tau_2 - \tau_1)K_2(\zeta, 0), \tag{16g}$$

$$\frac{1}{\tau_2 - \tau_1}K_3(\zeta, 0) = \int_\zeta^1 K_1(\zeta, \eta)c(\eta)\,\mathrm{d}\eta - c(\zeta). \tag{16h}$$

Remark 1. Under the Assumption 1 ($c(1) = 0$), it is easy to find that two boundary conditions of K_3 (16f) and (16h) are compatible at (1,0), namely, $K_3(1, 0) = 0$.

Since the transformation (13) is invertible, the corresponding inverse transformation is introduced as:

$$\begin{aligned} x(\zeta,t) = \mathcal{K}^{-1}[z,u_1,u_2] \\ := z(\zeta,t) \ + \int_\zeta^1 L_1(\zeta,\eta)z(\eta,t)\,\mathrm{d}\eta \\ + \int_0^1 L_2(\zeta,\eta)u_1(\eta,t)\,\mathrm{d}\eta \\ + \int_0^1 L_3(\zeta,\eta)u_2(\eta,t)\,\mathrm{d}\eta, \end{aligned} \tag{17}$$

where kernel L_1 is defined on $\mathcal{T}_1$ and L_2, L_3 on $\mathcal{T}_2$, which transform (15b)-(15g) to the original system (12a)-(12f). Similarly, we obtained the following set of equations of inverse kernel $L_i(x, y)$ for $i = 1, 2, 3$,

$$\partial_\zeta L_1(\zeta,\eta)+\partial_\eta L_1(\zeta,\eta) = \frac{1}{a}\int_\zeta^\eta f(\zeta,s)L_1(s,\eta)\,\mathrm{d}s+\frac{1}{a}f(\zeta,\eta), \tag{18a}$$

$$a\tau_1\partial_\zeta L_2(\zeta,\eta) = \partial_\eta L_2(\zeta,\eta)+\tau_1\int_\zeta^1 f(\zeta,s)L_2(s,\eta)\,\mathrm{d}s, \tag{18b}$$

$$a(\tau_2-\tau_1\partial_\zeta)L_3(\zeta,\eta) = \partial_\eta L_3(\zeta,\eta), \tag{18c}$$

$$a\tau_1 L_1(\zeta,1) = L_2(\zeta,1), \tag{18d}$$

$$L_2(1,\eta) = 0, \tag{18e}$$

$$L_3(1,\eta) = 0, \tag{18f}$$

$$\frac{\tau_1}{\tau_2-\tau_1}L_3(\zeta,1) = L_2(\zeta,0), \tag{18g}$$

$$L_3(\zeta,0) = -(\tau_2-\tau_1)c(\zeta). \tag{18h}$$

Remark 2. Under the Assumption 1 ($c(1) = 0$), the two boundary conditions (18f) and (18h) are compatible at (1,0), namely, $L_3(1, 0) = 0$.

Remark 3. The solution existence of the kernel Eqs. (16) and (18) can be proved by the method in Section 3 of [24].

3.2. Regulator design

In order to eliminate the dependency in (15b)-(15h) on $v(t)$, we introduce the bounded invertible change of coordinates

$$\epsilon_1(\zeta,t) = z(\zeta,t)-\Pi_1(\zeta)v(t), \tag{19a}$$

$$\epsilon_2(\zeta,t) = u_1(\zeta,t)-\Pi_2(\zeta)v(t), \tag{19b}$$

$$\epsilon_3(\zeta,t) = u_2(\zeta,t)-\Pi_3(\zeta)v(t), \tag{19c}$$

where $\epsilon(\zeta,t) = \left[\epsilon_1(\zeta,t),\ \epsilon_2(\zeta,t),\ \epsilon_3(\zeta,t)\right]^{\mathrm{T}}$ denotes tracking error, and $\Pi(\zeta) = \left[\Pi_1(\zeta),\ \Pi_2(\zeta),\ \Pi_3(\zeta)\right]^{\mathrm{T}} \in \mathbb{R}^{3n}$ is a function dependent on ζ. Furthermore, by differentiating (19) with respect to time, ones can substitute the (15) into the result, and specify the following:

$$K_v = -D_2+\Pi_1(0). \tag{20}$$

Then, (15b)–(15h) are decoupled from (15a), which yields:

$$\partial_t\epsilon_1(\zeta,t) = -a\partial_\zeta\epsilon_1(\zeta,t), \tag{21a}$$

$$\epsilon_1(0,t) = 0, \tag{21b}$$

$$\tau_1\partial_t\epsilon_{2t}(\zeta,t) = \partial_\zeta\epsilon_{2\zeta}(\zeta,t), \tag{21c}$$

$$\epsilon_2(1,t) = \epsilon_1(1,t), \tag{21d}$$

$$(\tau_2-\tau_1)\partial_t\epsilon_3(\zeta,t) = \partial_\zeta\epsilon_3(\zeta,t), \tag{21e}$$

$$\epsilon_3(1,t) = \epsilon_2(0,t), \tag{21f}$$

$$e_\psi(t) = \mathcal{FK}^{-1}[\epsilon_1(t),\epsilon_2(t),\epsilon_3(t)], \tag{21g}$$

and one obtains the following regulator equations in $\Pi(\zeta)$ which satisfy:

$$a\Pi_1^{'}(\zeta)+\Pi_1(\zeta)S = M(\zeta), \tag{22a}$$

$$\mathcal{FK}^{-1}[\Pi_1,\Pi_2,\Pi_3] = P_r-D_3, \tag{22b}$$

$$\Pi_2'(\zeta) = \tau_1\Pi_2(\zeta)S, \tag{22c}$$

$$\Pi_2(1) = \Pi_1(1), \tag{22d}$$

$$\Pi_3'(\zeta) = (\tau_2-\tau_1)\Pi_3(\zeta)S, \tag{22e}$$

$$\Pi_3(1) = \Pi_2(0). \tag{22f}$$

Lemma 1 (Regulator Equations)**.** *The numerator of transfer matrix $F(s) = N(s)D^{-1}(s)$ of (12a)–(12g) from $U(t)$ to $\Psi(t)$ is*

$$N(s) = \mathcal{FK}^{-1}\left[\mathrm{e}^{-\frac{\zeta}{a}s},\ \mathrm{e}^{-\left(\frac{1}{a}+\tau_1(1-\zeta)\right)s},\ \mathrm{e}^{-\left(\frac{1}{a}+\tau_2-(\tau_2-\tau_1)\zeta\right)s}\right]. \tag{23}$$

The regulator Eqs. (22) have a unique classical solution $\Pi(\zeta) = \left[\Pi_1,\Pi_2,\Pi_3\right]^{\mathrm{T}} \in \mathbb{R}^{3n}$, if and only if $N(\lambda) \neq 0$, where $\forall\lambda \in \sigma(S)$.

Proof. Assume that $\{\phi_k\}_{k=1}^n$ are eigenvectors of S with eigenvalues as$\{\lambda_k\}_{k=1}^n$. By multiplying (22) with ϕ_k, the regulator equations related can be written as

$$a\Pi_{1,k}^{*'}(\zeta)+\Pi_{1,k}^*(\zeta)\lambda_k = M_k^*(\zeta), \tag{24a}$$

$$\mathcal{FK}^{-1}[\Pi_{1,k}^*,\Pi_{2,k}^*,\Pi_{3,k}^*] = P_{r,k}^*-D_{3,k}^*, \tag{24b}$$

$$\Pi_{2,k}^{*'}(\zeta) = \tau_1\lambda_k\Pi_{2,k}^*(\zeta), \tag{24c}$$

$$\Pi_{2,k}^*(1) = \Pi_{1,k}^*(1), \tag{24d}$$

$$\Pi_{3,k}^{*'}(\zeta) = (\tau_2-\tau_1)\lambda_k\Pi_{3,k}^*(\zeta), \tag{24e}$$

$$\Pi_{3,k}^*(1) = \Pi_{2,k}^*(0), \tag{24f}$$

with $\Pi_{i,k}^* = \Pi_i\phi_k$, $P_{r,k}^* = P_r\phi_k$, $D_{3,k}^* = D_3\phi_k$ and $M_k^* = M\phi_k$ for $k = 1, 2, \ldots, n$. Then, the corresponding general solutions are:

$$\Pi_{1,k}^*(\zeta) = \Pi_{1,k}^*(0)\mathrm{e}^{-\frac{\lambda_k}{a}\zeta}+\frac{1}{a}\int_0^\zeta \mathrm{e}^{-\frac{\lambda_k}{a}(\zeta-\eta)}M_k^*(\eta)\,\mathrm{d}\eta, \tag{25a}$$

$$\Pi_{2,k}^*(\zeta) = \Pi_{2,k}^*(0)\mathrm{e}^{\tau_1\lambda_k\zeta}, \tag{25b}$$

$$\Pi_{3,k}^*(\zeta) = \Pi_{3,k}^*(0)\mathrm{e}^{(\tau_2-\tau_1)\lambda_k\zeta}. \tag{25c}$$

Then, substitute the above equations to the boundary conditions (24b), (24d) and (24f), we can get conditions on $\zeta = 0$,

$$\Pi^*_{1,k}(0) = \frac{1}{N(\lambda_k)}\Big(P^*_{r,k} - D^*_{3,k} - \frac{1}{a}\mathcal{F}\mathcal{K}^{-1}\Big[\int_0^\zeta \mathrm{e}^{-\frac{\lambda_k}{a}(\zeta-s)}\cdot M^*_k(s)\,\mathrm{d}s,\ \int_0^1 \mathrm{e}^{-\lambda_k(\tau_1-\tau_1\zeta+\frac{1}{a}-\frac{s}{a})}M^*_k(s)\,\mathrm{d}s,\ \int_0^1 \mathrm{e}^{-\lambda_k(\tau_2-(\tau_2-\tau_1)\zeta+\frac{1}{a}-\frac{s}{a})}M^*_k(s)\,\mathrm{d}s\Big]\Big), \tag{26a}$$

$$\Pi^*_{2,k}(0) = \Pi^*_{1,k}(0)\mathrm{e}^{-\lambda_k(\frac{1}{a}+\tau_1)} + \frac{\mathrm{e}^{-\tau_1\lambda_k}}{a}\int_0^1 \mathrm{e}^{-\frac{\lambda_k}{a}(1-\eta)}M^*_k(\eta)\,\mathrm{d}\eta, \tag{26b}$$

$$\Pi^*_{3,k}(0) = \Pi^*_{1,k}(0)\mathrm{e}^{-\lambda_k(\frac{1}{a}+\tau_2)} + \frac{\mathrm{e}^{-\tau_2\lambda_k}}{a}\int_0^1 \mathrm{e}^{-\frac{\lambda_k}{a}(1-\eta)}M^*_k(\eta)\,\mathrm{d}\eta. \tag{26c}$$

If and only if the solvability condition $N(\lambda) \neq 0$, $\forall\lambda \in \sigma(S)$ holds, ones obtain the solution of $\Pi^*_{1,k}(0)$ from the above equation. Therefore, the solution of the regulator Eq. (22) is solved as $\Pi(\zeta) = [\Pi^*_1(\zeta), \ldots, \Pi^*_n(\zeta)]\ldots[\phi_1, \cdots, \phi_n]^{-1}$. This proof is completed. □

Remark 4. The solvability condition means that no eigenvalue of the exosystem (10) is a "transmission zero" of system (12a)–(12g).

Lemma 1 shows that as long as K_i and L_i for $i = 1, 2, 3$ are calculated from the kernel Eq. (16) and its inverse kernel Eq. (18), the regulator solution $\Pi(\zeta)$ can be easily given. In the following Theorem 1, it states that for the system (12a)–(12g) the regulator (14) achieves output regulation in finite-time.

3.3. Stability analysis

Theorem 1 (State Feedback Regulator). *Let $K_1(\zeta,\eta)$, $K_2(\zeta,\eta)$, $K_3(\zeta,\eta)$ and $\Pi(\zeta)$ be the solutions of the control-kernel Eq. (16) and the solution of the regulator Eq. (22), respectively. Then, the state feedback regulator (14) achieves output regulation (11) for the system (12a)–(12f) and (10) in finite-time, $t > t_c = \tau_2 + 1/a$. Namely, the tracking error $[e_x, e_{u_1}, e_{u_2}]^{\mathrm{T}} = [x, u_1, u_2]^{\mathrm{T}} - \Pi_c[\zeta]v(t)$ with $\Pi_c(\zeta) = \mathcal{K}^{-1}[\Pi_1, \Pi_2, \Pi_3](\zeta)$ satisfies*

$$\Theta_t(t) = 0, \quad t > t_c \tag{27}$$

where $\Theta_t(t) = \|e_x(\cdot,t)\|^2 + \|e_{u_1}(\cdot,t)\|^2 + \|e_{u_2}(\cdot,t)\|^2$.

Proof. It is easy to obtain that the tracking error system (21a)-(21f) converges to the origin in finite-time $t > t_c = \tau_2 + 1/a$. Namely, $\|\varepsilon_1(\cdot,t)\|^2 + \|\varepsilon_2(\cdot,t)\|^2 + \|\varepsilon_3(\cdot,t)\|^2 = 0$, when $t > t_c$. From (21g), $e_\psi(t) = \mathcal{F}\mathcal{K}^{-1}[\epsilon_1, \epsilon_2, \epsilon_3] = 0$, as $t > t_c$. It means that the output regulation is achieved in finite-time by state feedback control (14). Hence, this concludes the proof. □

The state feedback regulator (14) can be rewritten as follows by replacing $u_1(\cdot,t)$ and $u_2(\cdot,t)$ by $x(1,\cdot)$

$$U(t) = K_v v(t) + \int_0^1 K_1(0,\eta)x(\eta,t)\,\mathrm{d}\eta + \frac{1}{\tau_1}\int_{t-\tau_1}^t K_2\Big(0, 1 - \frac{t-\sigma}{\tau_1}\Big)x(1,\sigma)\,\mathrm{d}\sigma + \frac{1}{\tau_2-\tau_1}\int_{t-\tau_2}^{t-\tau_1} K_3\Big(0, \frac{\sigma+\tau_2-t}{\tau_2-\tau_1}\Big)x(1,\sigma)\,\mathrm{d}\sigma. \tag{28}$$

4. Reference, disturbance and state observer

In the state feedback regulator (28), the state $x(1,\sigma)$, $\sigma \in (t-\tau_1, t)$ cannot be measured due to sensor delay τ_1. Hence, an observer is necessary to estimate the unavailable state for the feedback. In this section, two observers are designed to estimate state $v_r(t)$ from actual reference $r(t)$, and disturbance related state $v_d(t)$ and state $x(\zeta,t)$ estimation from delayed measurement (7d), respectively.

4.1. Reference observer

Assuming that $(\bar{P}_r, S_r)$ is observable, we introduce the following observer for the reference state $v_r(t)$,

$$\dot{\hat{v}}_r(t) = S_r\hat{v}_r(t) + L_r(r(t) - \bar{P}_r\hat{v}_r(t)), \quad t > 0 \tag{29}$$

with the initial condition $\hat{v}_r(0) = \hat{v}_{r0} \in \mathbb{R}^{n_r}$. The corresponding error $\tilde{v}_r(t) = v_r - \hat{v}_r(t)$ satisfies $\dot{\tilde{v}}_r(t) = F_r\tilde{v}_r(t)$, with determining L_r such that $F_r = S_r - L_r\bar{P}_r$ is Hurwitz matrix.

4.2. Disturbance and state observer

Dual with the design of the state output regulator, by introducing a backstepping transformation, the observation error system is mapped to an intermediate system, and further decoupled by the coordinate change, as finally the observation gain is obtained.

Now, we introduce the observer for the disturbance state $v_d(t)$ and the plant state $x(\zeta,t)$ from the delayed measurement as follows:

$$\dot{\hat{v}}_d(t) = S_d\hat{v}_d(t) + L_d(y(t) - \hat{u}_1(0,t)), \tag{30a}$$

$$\partial_t\hat{x}_t(\zeta,t) = -a\partial_\zeta\hat{x}(\zeta,t) + c(\zeta)\hat{u}_2(0,t) + \bar{D}_1(\zeta)\bar{P}_d\hat{v}_d(t) + Q_1(\zeta)(y(t) - \hat{u}_1(0,t)) + \int_\zeta^1 f(\zeta,\eta)\hat{x}(\eta,t)\,\mathrm{d}\eta, \tag{30b}$$

$$\hat{x}(0,t) = U(t) + \bar{D}_2\bar{P}_d\hat{v}_d(t), \tag{30c}$$

$$\tau_1\partial_t\hat{u}_1(\zeta,t) = \partial_\zeta\hat{u}_1(\zeta,t) + Q_2(\zeta)(y(t) - \hat{u}_1(0,t)), \tag{30d}$$

$$\hat{u}_1(1,t) = \hat{x}(1,t), \tag{30e}$$

$$(\tau_2-\tau_1)\partial_t\hat{u}_2(\zeta,t) = \partial_\zeta\hat{u}_2(\zeta,t), \tag{30f}$$

$$\hat{u}_2(1,t) = y(t), \tag{30g}$$

with initial conditions $v_d(0) = \hat{v}_{d0} \in \mathbb{R}^{n_d}$, $\hat{x}(\zeta,0) = \hat{x}_0(\zeta)$, $\hat{u}_1(\zeta,0) = \hat{u}_{10}(\zeta)$ and $\hat{u}_2(\zeta,0) = \hat{u}_{20}(\zeta) \in \mathcal{L}^2(0,1)$. Observer gains are denoted by $L_d \in \mathbb{R}^{n_d\times 1}$ and $Q_i(\zeta) \in \mathcal{L}^2(0,1)$ for $i = 1, 2$. The estimated errors are defined as $\tilde{v}_d = v_d - \hat{v}_d$, $\tilde{x} = x - \hat{x}$, $\tilde{u}_1 = u_1 - \hat{u}_1$, $\tilde{u}_2 = u_2 - \hat{u}_2$, so the error system reads

$$\dot{\tilde{v}}_d(t) = S_d\tilde{v}_d(t) - L_d\tilde{u}_1(0,t), \tag{31a}$$

$$\partial_t\tilde{x}(\zeta,t) = -a\partial_\zeta\tilde{x}(\zeta,t) + c(\zeta)\tilde{u}_2(0,t) + \bar{D}_1(\zeta)\bar{P}_d\tilde{v}_d(t) - Q_1(\zeta)\tilde{u}_1(0,t) + \int_\zeta^1 f(\zeta,\eta)\tilde{x}(\eta,t)\,\mathrm{d}\eta, \tag{31b}$$

$$\tilde{x}(0,t) = \bar{D}_2\bar{P}_d\tilde{v}_d(t), \tag{31c}$$

$$\tau_1\partial_t\tilde{u}_1(\zeta,t) = \partial_\zeta\tilde{u}_1(\zeta,t) - Q_2(\zeta)\tilde{u}_1(0,t), \tag{31d}$$

$$\tilde{u}_1(1,t) = \tilde{x}(1,t), \tag{31e}$$

$$(\tau_2-\tau_1)\partial_t\tilde{u}_2(\zeta,t) = \partial_\zeta\tilde{u}_2(\zeta,t), \tag{31f}$$

$$\tilde{u}_2(1,t) = 0. \tag{31g}$$

First, we introduce two Volterra integral transformations

$$\tilde{x}(\zeta,t)=\mathcal{K}_{o1}[\tilde{z},\tilde{w}] := \tilde{z}(\zeta,t)-\int_\zeta^1 F_1(\zeta,\eta)\tilde{z}(\eta,t)\,\mathrm{d}\eta - \int_0^\zeta F_2(\zeta,\eta)\tilde{w}(\eta,t)\,\mathrm{d}\eta-\int_\zeta^1 F_3(\zeta,\eta)\tilde{w}(\eta,t)\,\mathrm{d}\eta, \tag{32a}$$

$$\tilde{u}_1(\zeta,t)=\mathcal{K}_{o2}[\tilde{w}] := \tilde{w}(\zeta,t)-\int_0^\zeta R(\zeta,\eta)\tilde{w}(\eta,t)\,\mathrm{d}\eta, \tag{32b}$$

where kernel F_1, F_3 are defined on $\mathcal{T}_1$ and F_2, R on $\mathcal{T}_3$, which transform system (31) into a target system

$$\dot{\tilde{v}}_d(t)=S_d\tilde{v}_d(t)-L_d\tilde{w}(0,t), \tag{33a}$$

$$\partial_t\tilde{z}(\zeta,t)=-a\partial_\zeta\tilde{z}(\zeta,t)+\tilde{c}(\zeta)\tilde{u}_2(0,t)-\tilde{Q}_1(\zeta)\tilde{w}(0,t)+\tilde{D}_1(\zeta)\tilde{v}_d(t), \tag{33b}$$

$$\tilde{z}(0,t)=\bar{D}_2\bar{P}_d\tilde{v}_d(t), \tag{33c}$$

$$\tau_1\partial_t\tilde{w}(\zeta,t)=\partial_\zeta\tilde{w}(\zeta,t)-\tilde{Q}_2(\zeta)\tilde{w}(0,t), \tag{33d}$$

$$\tilde{w}(1,t)=\tilde{z}(1,t), \tag{33e}$$

$$(\tau_2-\tau_1)\partial_t\tilde{u}_2(\zeta,t)=\partial_\zeta\tilde{u}_2(\zeta,t), \tag{33f}$$

$$\tilde{u}_2(1,t)=0, \tag{33g}$$

where $\tilde{c}(\zeta)=c(\zeta)-\int_\zeta^1 P_1(\zeta,\eta)c(\eta)\,\mathrm{d}\eta$, $\tilde{D}_1=\bar{D}_1(\zeta)\bar{P}_d-\int_\zeta^1 P_1(\zeta,\eta)\bar{D}_1(\eta)\,\mathrm{d}\eta\bar{P}_d$ and $P_1(\zeta,\eta)$ will be defined later in the inverse transformation. After lengthy calculations, one obtains a final group of observer kernel equations:

$$\partial_\zeta F_1(\zeta,\eta)=-\partial_\eta F_1(\zeta,\eta)-\frac{1}{a}f(\zeta,\eta)+\frac{1}{a}\int_\zeta^\eta f(\zeta,s)F_1(s,\eta)\,\mathrm{d}s, \tag{34a}$$

$$F_1(0,\eta)=0, \tag{34b}$$

$$a\tau_1\partial_\zeta F_2(\zeta,\eta)=\partial_\eta F_2(\zeta,\eta)+\tau_1\int_\zeta^1 f(\zeta,s)F_2(s,\eta)\,\mathrm{d}s, \tag{34c}$$

$$F_2(\zeta,\zeta)=F_3(\zeta,\zeta), \tag{34d}$$

$$a\tau_1\partial_\zeta F_3(\zeta,\eta)=\partial_\eta F_3(\zeta,\eta)+\tau_1\int_\eta^1 f(\zeta,s)F_2(s,\eta)ds+\tau_1\int_\zeta^\eta f(\zeta,s)F_3(s,\eta)ds, \tag{34e}$$

$$F_3(0,\eta)=0, \tag{34f}$$

$$F_3(\zeta,1)=a\tau_1F_1(\zeta,1), \tag{34g}$$

and

$$R_{1\zeta}(\zeta,\eta)+R_{1\eta}(\zeta,\eta)=0, \tag{35a}$$

$$R_1(1,\eta)=F_2(1,\eta), \tag{35b}$$

so that the observer gains are determined by

$$Q_1(\zeta)=\mathcal{K}_{o1}\left[\tilde{Q}_1,\frac{\tilde{Q}_2}{\tau_1}\right](\zeta)-\frac{1}{\tau_1}F_2(\zeta,0), \tag{36a}$$

$$Q_2(\zeta)=\mathcal{K}_{o2}\left[\tilde{Q}_2\right](\zeta)-R_1(\zeta,0), \tag{36b}$$

where $\tilde{Q}_i$ will be derived later. Transformation (32) is invertible [2] (Theorem 1.4) and its inverse transformation $(\tilde{x},\tilde{u}_1)\mapsto(\tilde{z},\tilde{w})$ is written as:

$$\tilde{z}(\zeta,t)=\mathcal{K}_{o1}^{-1}[\tilde{x},\tilde{u}_1] := \tilde{x}(\zeta,t)-\int_\zeta^1 P_1(\zeta,\eta)\tilde{x}(\eta,t)\,\mathrm{d}\eta-\int_0^\zeta P_2(\zeta,\eta)\tilde{u}_1(\eta,t)\,\mathrm{d}\eta-\int_\zeta^1 P_3(\zeta,\eta)\tilde{u}_1(\eta,t)\,\mathrm{d}\eta, \tag{37a}$$

$$\tilde{w}(\zeta,t)=\mathcal{K}_{o2}^{-1}[\tilde{u}_1] := \tilde{u}_1(\zeta,t)-\int_0^\zeta G(\zeta,\eta)\tilde{u}_1(\eta,t)\,\mathrm{d}\eta, \tag{37b}$$

where kernel P_1, P_3 are defined on $\mathcal{T}_1$ and P_2, G on $\mathcal{T}_3$, satisfy the following equation:

$$\partial_\zeta P_1(\zeta,\eta)+\partial_\eta P_1(\zeta,\eta)=\frac{1}{a}f(\zeta,\eta)-\frac{1}{a}\int_\eta^\zeta P_1(\zeta,s)f(s,\eta)ds, \tag{38a}$$

$$P_1(0,\eta)=0, \tag{38b}$$

$$a\tau_1\partial_\zeta P_2(\zeta,\eta)=\partial_\eta P_2(\zeta,\eta), \tag{38c}$$

$$P_2(\zeta,\zeta)=P_3(\zeta,\zeta), \tag{38d}$$

$$a\tau_1\partial_\zeta P_3(\zeta,\eta)=\partial_\eta P_3(\zeta,\eta), \tag{38e}$$

$$P_3(0,\eta)=0, \tag{38f}$$

$$P_3(\zeta,1)=a\tau_1P_1(\zeta,1), \tag{38g}$$

and

$$G_\zeta+G_\eta=0, \tag{39a}$$

$$G(1,\eta)=P_2(1,\eta). \tag{39b}$$

Remark 5. There exist unique solutions for both the kernel Eqs. (34)–(35) and the inverse kernel Eqs. (38)–(39), the proof is similar to [24].

In the second step, the PDE subsystem (33b)-(33g) is decoupled from the ODE system (33a) by introducing coordinate changes

$$\tilde{\varepsilon}_1(\zeta,t)=\tilde{z}(\zeta,t)-\tilde{\Pi}_1(\zeta)\tilde{v}_d(t), \tag{40a}$$

$$\tilde{\varepsilon}_2(\zeta,t)=\tilde{w}(\zeta,t)-\tilde{\Pi}_2(\zeta)\tilde{v}_d(t), \tag{40b}$$

with $\tilde{\Pi}(\zeta)=\left[\tilde{\Pi}_1(\zeta),\ \tilde{\Pi}_2(\zeta)\right]^{\mathrm{T}}\in\mathbb{R}^{2n_d}$. In the same vein as in previous section by differentiating with respect to time and space, and then substituting (33), which decouples the PDE from the ODE as follows:

$$\dot{\tilde{v}}_d=\left(S_d-L_d\tilde{\Pi}_2(0)\right)\tilde{v}_d(t)-L_d\tilde{\varepsilon}_2(0,t), \tag{41a}$$

$$\partial_t\tilde{\varepsilon}_1=-a\partial_\zeta\tilde{\varepsilon}_1+\tilde{c}(\zeta)\tilde{u}_2(0,t), \tag{41b}$$

$$\tilde{\varepsilon}_1(0,t)=0, \tag{41c}$$

$$\tau_1\partial_t\tilde{\varepsilon}_2=\partial_\zeta\tilde{\varepsilon}_2, \tag{41d}$$

$$\tilde{\varepsilon}_2(1,t)=\tilde{\varepsilon}_1(1,t), \tag{41e}$$

where $\tilde{u}_2$ is determined by (33f)-(33g) and in turn gives the output feedback gains $\tilde{Q}_i$,

$$\tilde{Q}_1(\zeta)=\tilde{\Pi}_1(\zeta)L_d, \tag{42a}$$

$$\tilde{Q}_2(\zeta)=\tau_1\tilde{\Pi}_2(\zeta)L_d, \tag{42b}$$

if $\tilde{\Pi}(\zeta)$ is the solution of the following equations,

$$a\tilde{\Pi}_1'(\zeta) + \tilde{\Pi}_1(\zeta)S_d = \tilde{D}_1(\zeta), \tag{43a}$$

$$\tilde{\Pi}_1(0) = \bar{D}_2\bar{P}_d, \tag{43b}$$

$$\tilde{\Pi}_2'(\zeta) = \tau_1\tilde{\Pi}_2(\zeta)S_d, \tag{43c}$$

$$\tilde{\Pi}_1(1) = \tilde{\Pi}_2(1). \tag{43d}$$

The disturbance observer gain L_d can be determined by the eigenvalue assignment for the matrix $S_d - L_d\tilde{\Pi}_2(0)$. Recalling (36), we obtain the observer gain $Q_i(\zeta)$.

The following lemma states that the decoupling Eq. (43) is a well-posed boundary value problem.

Lemma 2. *There always exists a unique classical solution of decoupling Eq. (43).*

Proof. The decoupling Eq. (43) has the same form as (22a)–(22d), only that (22d) is simplified to a Dirichlet boundary. In consequence, it is easy to obtain from Lemma 1 that the solvable condition of (43) is $e^{\lambda_{d,k}} \neq 0$, $k = 1, 2, \ldots, n_d$, where $\lambda_{d,k} \in \sigma(S_d)$. Obviously, this condition is satisfied regardless of the value of $\lambda_{d,k}$. The proof is completed. □

Lemma 3 (Observability)**.** *The numerator of the transfer function $F_d(s) = N_d(s)D_d^{-1}(s)$ of system (12a)–(12f) and (12h) from $d(t)$ to $y(t)$ is*

$$N_d(s) = \bar{D}_2 + \frac{1}{a}\int_0^1 e^{\frac{1-\eta}{a}}\mathcal{K}_{o1}^{-1}[\bar{D}_1, 0](\eta)\,d\eta. \tag{44}$$

Then, the pair $(\tilde{\Pi}_2(0), S_d)$ is observable if and only if $N_d(\lambda_{d,k})\bar{P}_d\phi_{d,k} \neq 0$, $k = 1, 2, \cdots, n_d$, where $\phi_{d,k}$ and $\lambda_{d,k}$ denote the eigenvectors and corresponding eigenvalues of S_d, respectively.

Proof. According to Theorem 6.2–5 (in [17]), the pair $(\tilde{\Pi}_2(0), S_d)$ is observable if and only if $N_d(\lambda_{d,k})\bar{P}_d\phi_{d_k} = \tilde{\Pi}_2(0)v_{d_k} \neq 0$, $k = 1, 2, \cdots n_d$. By applying the transformations $z(\zeta, t) = \mathcal{K}_{o1}^{-1}[x, u_1]$ and $w(\zeta, t) = \mathcal{K}_{o2}^{-1}[u_1]$ (see (37a) and (37b)) and setting $U(t) \equiv 0$, the system (12a)–(12f) and (12h) has the representation,

$$\begin{aligned}\partial_t z(\zeta, t) = &-a\partial_\zeta z(\zeta, t) + \frac{1}{\tau_1}P_2(\zeta, 0)w(0, t)\\ &+ \mathcal{K}_{o1}^{-1}[c, 0](\zeta)u_2(0, t) + \mathcal{K}_{o1}^{-1}[\bar{D}_1, 0](\zeta)d(t),\end{aligned} \tag{45a}$$

$$z(0, t) = \bar{D}_2 d(t), \tag{45b}$$

$$\tau_1\partial_t w(\zeta, t) = \partial_\zeta w(\zeta, t) + G(\zeta, 0)w(0, t), \tag{45c}$$

$$w(1, t) = z(1, t), \tag{45d}$$

$$(\tau_2 - \tau_1)\partial_t u_2(\zeta, t) = \partial_\zeta u_2(\zeta, t), \tag{45e}$$

$$u_2(1, t) = u_1(0, t). \tag{45f}$$

For this system, the transfer function $F_d(s)$ can be derived in a closed-form, where $D_s(s)$ is an irrational denominator. This proof is completed.

□

4.3. Stability analysis

Theorem 2. *Let $Q_1(\zeta)$ and $Q_2(\zeta)$ given by (36). Then, the observer error system (31) is exponentially stable in the L_2-norm, i.e. there exists $C_e, \alpha_e > 0$ such that observer error $[\tilde{v}_d, \tilde{x}, \tilde{u}_1, \tilde{u}_2]^T$ satisfies:*

$$\Theta_e(t) \le C_e e^{-\alpha_e t}\Theta_e(0), \tag{46}$$

where

$$\Theta_e(t) = |\tilde{v}_d(t)|^2 + \|\tilde{x}(\cdot, t)\|^2 + \|\tilde{u}_1(\cdot, t)\|^2 + \|\tilde{u}_2(\cdot, t)\|^2.$$

Proof. Introduce a Lyapunov function for (41):

$$\begin{aligned}V_1(t) = &\tilde{v}_d^T P\tilde{v}_d + c_1\int_0^1(2-\zeta)\tilde{\varepsilon}_1^2(\zeta, t)d\zeta\\ &+ \frac{a\tau_1 c_1}{4}\int_0^1(1+\zeta)\tilde{\varepsilon}_2^2(\zeta, t)d\zeta\\ &+ (\tau_2 - \tau_1)c_2\int_0^1(1+\zeta)\tilde{u}_2^2(\zeta, t)d\zeta,\end{aligned} \tag{47}$$

where $P = P^T > 0$ is the solution of the equation

$$\big(S_d - L_d\tilde{\Pi}_2(0)\big)^T P + P\big(S_d - L_d\tilde{\Pi}_2(0)\big) = -Q, \tag{48}$$

for some $Q = Q^T > 0$ and $c_i > 0$ for $i = 1, 2$ is to be determined later.

First, we discuss the equivalent stability property between V_1 and Θ_e. Call now kernel functions $F_i(\zeta, \eta)$, $P_i(\zeta, \eta)$, $R(\zeta, \eta)$, $G(\zeta, \eta)$ and $\tilde{\Pi}_i(\zeta)$ are bounded by $\bar{F}_i$, $\bar{P}_i$, $\bar{R}$, $\bar{G}$ and $\bar{\tilde{\Pi}}_i$ in their respective domains. From (32), (37) and (40) it is easy to find, by using Cauchy-Schwarz inequality, that

$$\|\tilde{x}\|^2 \le 4(1 + \bar{F}_1^2)\|\tilde{z}\|^2 + 4(\bar{F}_2^2 + \bar{F}_3^2)\|\tilde{w}\|^2, \tag{49a}$$

$$\|\tilde{u}_1\|^2 \le 2(1 + \bar{R}^2)\|\tilde{w}\|^2, \tag{49b}$$

$$\|\tilde{z}\|^2 \le 2\|\tilde{\varepsilon}_1\|^2 + 2\bar{\tilde{\Pi}}_1^2|\tilde{v}_d|^2, \tag{49c}$$

$$\|\tilde{w}\|^2 \le 2\|\tilde{\varepsilon}_2\|^2 + 2\bar{\tilde{\Pi}}_2^2|\tilde{v}_d|^2. \tag{49d}$$

$$\|\tilde{z}\|^2 \le 4(1 + \bar{P}_1^2)\|\tilde{x}\|^2 + 4(\bar{P}_2^2 + \bar{P}_3^2)\|\tilde{u}_1\|^2, \tag{49e}$$

$$\|\tilde{w}\|^2 \le 2(1 + \bar{G}^2)\|\tilde{u}_1\|^2, \tag{49f}$$

$$\|\tilde{\varepsilon}_1\|^2 \le 2\|\tilde{z}\|^2 + 2\bar{\tilde{\Pi}}_1^2|\tilde{v}_d|^2, \tag{49g}$$

$$\|\tilde{\varepsilon}_2\|^2 \le 2\|\tilde{w}\|^2 + 2\bar{\tilde{\Pi}}_2^2|\tilde{v}_d|^2. \tag{49h}$$

From (49e)-(49h), we get that

$$V_1 \le \bar{\delta}_e\big(|\tilde{v}_d|^2 + \|\tilde{x}\|^2 + \|\tilde{u}_1\|^2 + \|\tilde{u}_2\|^2\big), \tag{50}$$

where $\bar{\delta_e} = \max\{\lambda_{\max}(P) + 4\bar{\tilde{\Pi}}_1^2 + a\tau_1 c_1\bar{\tilde{\Pi}}_2^2, 2a\tau_1 c_1(1 + \bar{G}^2) + 16c_1(\bar{P}_2^2 + \bar{P}_3^2), 16c_1(1 + \bar{P}_1^2), 2c_2\}$. Similarly, from (49c)-(49d), we have

$$V_1 \ge \frac{\lambda_{\min}(P)}{3}|\tilde{v}_d|^2 + \beta_1\|\tilde{z}\|^2 + \beta_2\|\tilde{w}\|^2 + c_2\|\tilde{u}_2\|^2, \tag{51}$$

where

$$\beta_1 = \frac{\lambda_{\min}(P)}{3}|\tilde{v}_d|^2 + \frac{\min\{\frac{\lambda_{\min}(P)}{3}, \frac{c_1}{2}\}}{\max\{2, 2\bar{\tilde{\Pi}}_1^2\}}, \tag{52a}$$

$$\beta_2 = \frac{\min\{\frac{\lambda_{\min}(P)}{3}, \frac{a\tau c_1}{8}\}}{\max\{2, 2\bar{\tilde{\Pi}}_2^2\}}. \tag{52b}$$

Then, from (49a)-(49b), we obtain

$$\begin{aligned}V_1 \ge &\frac{\lambda_{\min}(P)}{3}|\tilde{v}_d|^2 + \frac{\min\{\beta_1, \beta_2/2\}}{\max\{4(1+\bar{F}_1^2), 4(\bar{F}_2^2 + \bar{F}_3^2)\}}\|\tilde{x}\|^2\\ &+ \frac{\beta_2}{4(1+\bar{R}^2)}\|\tilde{u}_1\|^2 + c_2\|\tilde{u}_2\|^2\\ \ge &\ \underline{\delta}_e\big(|\tilde{v}_d|^2 + \|\tilde{x}\|^2 + \|\tilde{u}_1\|^2 + \|\tilde{u}_2\|^2\big),\end{aligned} \tag{53}$$

where

$$\underline{\delta_e} = \min\left\{\frac{\lambda_{\min}(P)}{3}, \frac{\min\{\beta_1, \beta_2/2\}}{\max\{4(1+\bar{F}_1^2), 4(\bar{F}_2^2+\bar{F}_3^2)\}}, \frac{\beta_2}{4(1+\bar{R}^2)}, c_2\right\}.$$

So we have that

$$\underline{\delta_e}\Theta_e(t) \le V_1 \le \overline{\delta_e}\Theta_e(t). \tag{54}$$

Then, taking the time derivative of (47) and using Young's and Cauchy-Schwarz Inequalities, we obtain

$$\begin{aligned}\dot{V}_1(t) \le &-\frac{\lambda_{\min}(Q)}{2}|\tilde{v}_d|^2 - \left(\frac{ac_1}{4} - \frac{2|PL_d|^2}{\lambda_{min}(Q)}\right)\tilde{\varepsilon}_2^2(0,t)\\ &-\frac{ac_1}{2}\tilde{\varepsilon}_1^2(1,t) - \frac{ac_1}{2}\|\tilde{\varepsilon}_1\|^2 - \frac{ac_1}{4}\|\tilde{\varepsilon}_2\|^2\\ &-\left(c_2 - \frac{4c_1\|\tilde{c}\|^2}{a}\right)\tilde{u}_2^2(0,t) - c_2\|\tilde{u}_2^2\|^2.\end{aligned} \tag{55}$$

Making $c_1 \ge \frac{8|PL_d|^2}{a\lambda_{\min}(Q)}$ and $c_2 \ge \frac{4c_1\|\tilde{c}\|^2}{a}$, we have

$$\begin{aligned}\dot{V}_1 &\le -\frac{\lambda_{\min}(Q)}{2}|\tilde{v}_d|^2 - \frac{ac_1}{2}\|\tilde{\varepsilon}_1\|^2 - \frac{ac_1}{4}\|\tilde{\varepsilon}_2\|^2 - c_2\|\tilde{u}_2^2\|^2\\ &\le -\alpha_e V_1,\end{aligned} \tag{56}$$

where $\alpha_e = \min\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{a}{4}, \frac{1}{2\tau_1}, \frac{1}{2(\tau_2-\tau_1)}\}$. Thus, combining the condition (56), we have,

$$\Theta_e(t) \le C_e e^{-\alpha_e t}\Theta_e(0), \tag{57}$$

where $C_e = \overline{\delta_e}/\underline{\delta_e}$, thus proving the result. □

5. Output feedback regulation

The output feedback regulator is the result of combining the estimated $\hat{v}_r$, $\hat{v}_d$, $\hat{x}$, $\hat{u}_1$ and $\hat{u}_2$ by the observer (29)-(30) with the state feedback regulator designed (14) in Section 3:

$$\begin{aligned}U(t) = &\int_0^1 K_1(0,\eta)\hat{x}(\eta,t)\,d\eta + \int_0^1 K_2(0,\eta)\hat{u}_1(\eta,t)\,d\eta\\ &+\int_0^1 K_3(0,\eta)\hat{u}_2(\eta,t)\,d\eta + K_v\hat{v}(t),\end{aligned} \tag{58}$$

where $\hat{v}(t) = col(\hat{v}_r(t), \hat{v}_d(t))$.

Before discussing the stability of the output feedback closed-loop system, we show the target system of the observer, by applying the transformation $\mathcal{K}[\hat{x}, \hat{u}_1, \hat{u}_2](\zeta)$ on the observer system (30),

$$\dot{\hat{v}}_d(t) = S_d\hat{v}_d(t) + L_d\tilde{w}(0,t), \tag{59a}$$

$$\partial_t\hat{z} = -a\partial_\zeta\hat{z} + M(\zeta)\hat{v}(t) + L_c(\zeta)\tilde{w}(0,t), \tag{59b}$$

$$\hat{z}(0,t) = \Pi_1(0)\hat{v}(t), \tag{59c}$$

$$\tau_1\partial_t\hat{u}_1 = \partial_\zeta\hat{u}_1 - Q_2(\zeta)\tilde{w}(0,t), \tag{59d}$$

$$\hat{u}_1(1,t) = \hat{z}(1,t), \tag{59e}$$

$$(\tau_2-\tau_1)\partial_t\hat{u}_2 = \partial_\zeta\hat{u}_2, \tag{59f}$$

$$\hat{u}_2(1,t) = u_1(0,t), \tag{59g}$$

where

$$L_c = \mathcal{K}\left[Q_1, \frac{Q_2}{\tau_1}, \frac{Q_3}{\tau_2-\tau_1}\right](\zeta) + \frac{K_2(\zeta,0)}{\tau_1} + \frac{K_3(\zeta,1)}{\tau_2-\tau_1},$$

and $M(\zeta)$ is defined in (15b). Then, using the regulation coordinates changes

$$\hat{\varepsilon}_1(\zeta,t) = \hat{z}(\zeta,t) - \Pi_1(\zeta)\hat{v}(t), \tag{60a}$$

$$\hat{\varepsilon}_2(\zeta,t) = \hat{u}_1(\zeta,t) - \Pi_2(\zeta)\hat{v}(t), \tag{60b}$$

$$\hat{\varepsilon}_3(\zeta,t) = \hat{u}_2(\zeta,t) - \Pi_3(\zeta)\hat{v}(t), \tag{60c}$$

which gives

$$\dot{\tilde{v}}_d = \left(S_d - L_d\tilde{\Pi}_2(0)\right)\tilde{v}_d(t) - L_d\tilde{\varepsilon}_2(0,t), \tag{61a}$$

$$\begin{aligned}\partial_t\hat{\varepsilon}_1(\zeta,t) = &-a\partial_\zeta\hat{\varepsilon}_1(\zeta,t) + b_1(\zeta)\tilde{\varepsilon}_2(0,t)\\ &+b_2(\zeta)\tilde{v}_d + b_3(\zeta)\tilde{v}_r,\end{aligned} \tag{61b}$$

$$\hat{\varepsilon}_1(0,t) = 0, \tag{61c}$$

$$\begin{aligned}\tau_1\partial_t\hat{\varepsilon}_2(\zeta,t) = &\partial_\zeta\hat{\varepsilon}_2(\zeta,t) + b_4(\zeta)\tilde{\varepsilon}_2(0,t)\\ &+b_5(\zeta)\tilde{v}_d + b_6(\zeta)\tilde{v}_r,\end{aligned} \tag{61d}$$

$$\hat{\varepsilon}_2(1,t) = \hat{\varepsilon}_1(1,t), \tag{61e}$$

$$\begin{aligned}(\tau_2-\tau_1)\partial_t\hat{\varepsilon}_3 = &\partial_\zeta\hat{\varepsilon}_3 + b_7(\zeta)\tilde{\varepsilon}_2(0,t)\\ &+b_8(\zeta)\tilde{v}_d + b_9(\zeta)\tilde{v}_r,\end{aligned} \tag{61f}$$

$$\hat{\varepsilon}_3(1,t) = \hat{\varepsilon}_2(0,t) + \tilde{\varepsilon}_2(0) + \tilde{\Pi}_2(0)\tilde{v}_d, \tag{61g}$$

where

$$\begin{aligned}&b_1 = L_c(\zeta) - \Pi_{1d}(\zeta)L_d, \quad b_2(\zeta) = b_1\tilde{\Pi}_2(0),\\ &b_3 = -\Pi_{1r}(\zeta)L_r\bar{P}_r,\\ &b_4 = -(\tau_1\Pi_{2d}(\zeta)L_d + Q_2(\zeta)), \quad b_5 = b_4\tilde{\Pi}_2(0),\\ &b_6 = -\tau_1\Pi_{2r}(\zeta)L_r\bar{P}_r,\\ &b_7 = -(\tau_2-\tau_1)\Pi_{3d}(\zeta)L_d, \quad b_8 = b_7\tilde{\Pi}_2(0),\\ &b_9 = -(\tau_2-\tau_1)\Pi_{3r}(\zeta)L_r\bar{P}_r.\end{aligned}$$

Theorem 3 (Output Feedback Regulator)**.** *Consider the regulator (58) and the observers (29), (30) with the observer gains (36). The output feedback (58) achieves output regulation (11) for the system (10) and (12) with an exponentially stable tracking and observer error dynamics in the L_2-norm, i.e., let $[\hat{e}_x, \hat{e}_{u_1}, \hat{e}_{u_2}]^T = [\hat{x}, \hat{u}_1, \hat{u}_2]^T - \Pi_c(\zeta)\hat{v}$ with $\Pi_c(\zeta) = \mathcal{K}^{-1}[\Pi](\zeta)$, then there exists C_o, $\alpha_o > 0$ such that the state $e_o = col(\tilde{v}_r, \tilde{v}_d, \tilde{x}, \tilde{u}_1, \tilde{u}_2, \hat{e}_x, \hat{e}_{u_1}, \hat{x}_{u_2})$ satisfies*

$$\Theta_o(t) \le C_o e^{-\alpha_o t}\Theta_o(0). \tag{62}$$

where $\Theta_o(t) = |\tilde{v}_r(t)|^2 + \|\tilde{v}_d(t)\|^2 + \|\tilde{x}(\cdot,t)\|^2 + \|\tilde{u}_1(\cdot,t)\|^2 + \|\tilde{u}_2(\cdot,t)\|^2 + \|\hat{e}_x(\cdot,t)\|^2 + \|\hat{e}_{u_1}(\cdot,t)\|^2 + \|\hat{e}_{u_2}(\cdot,t)\|^2$.

Proof. Introduce a Lyapunov function for system (41b)-(41e) and (61),

$$\begin{aligned}V_2(t) = &\tilde{v}_r^T P_{or}\tilde{v}_r + \tilde{v}_d^T P_{od}\tilde{v}_d + c_3\int_0^1(2-\zeta)\tilde{\varepsilon}_1^2(\zeta,t)d\zeta\\ &+\frac{a\tau_1 c_3}{4}\int_0^1(1+\zeta)\tilde{\varepsilon}_2^2(\zeta,t)d\zeta\\ &+c_4(\tau_2-\tau_1)\int_0^1(1+\zeta)\tilde{u}_2^2(\zeta,t)d\zeta\\ &+c_5\int_0^1(2-\zeta)\hat{\varepsilon}_1^2(\zeta,t)d\zeta\\ &+\frac{a\tau_1 c_5}{4}\int_0^1(1+\zeta)\hat{\varepsilon}_2^2(\zeta,t)d\zeta\\ &+c_6(\tau_2-\tau_1)\int_0^1(1+\zeta)\hat{\varepsilon}_3^2(\zeta,t)d\zeta,\end{aligned} \tag{63}$$

where $P_r = P_r^T > 0$, $P_d = P_d^T > 0$ are the solutions of the equations

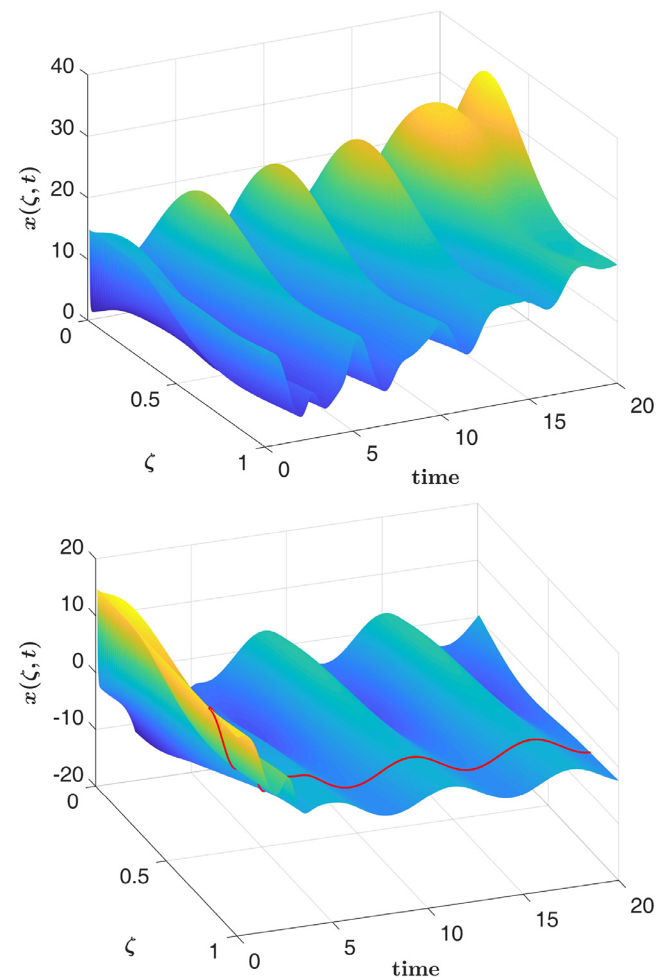

Fig. 2. Top: the open-loop evolution of $x(\zeta,t)$, namely, $U(t)=0$. Bottom: the closed-loop evolution of $x(\zeta,t)$ under the state feedback controller (14).

$$\left(S_r-L_r\bar{P}_r\right)^{\mathrm{T}}P_{or}+P_{or}\left(S_r-L_r\bar{P}_r\right)=-Q_r,$$
$$\left(S_d-L_d\tilde{\Pi}_2(0)\right)^{\mathrm{T}}P_{od}+P_{od}\left(S_d-L_d\tilde{\Pi}_2(0)\right)=-Q_d,$$

for some $Q_r=Q_r^{\mathrm{T}}>0$, $Q_d=Q_d^{\mathrm{T}}>0$ and $c_i>0$ for $i=3,\dots,6$ are to be determined later.

Similar to the proof of (54) in Theorem 2, V_2 and Θ_o are equivalent in the L_2-norm, namely, there exists positive constants $\underline{\delta_o}$ and $\overline{\delta_o}$, such that

$$\underline{\delta_o}\Theta_o(t)\le V_2\le\overline{\delta_o}\Theta_o(t). \tag{64}$$

Taking the time derivative of (63), with Young's and Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\dot{V}_2\le&-\frac{ac_3}{2}\|\tilde{\varepsilon}_1\|^2-\frac{ac_3}{4}\|\tilde{\varepsilon}_2\|^2-c_4\|\tilde{u}_2\|^2\\&-\left(c_4-\frac{4c_3\|\tilde{c}\|^2}{a}\right)\tilde{u}_2^2(0,t)\\&-\frac{ac_5}{2}\|\hat{\varepsilon}_1\|^2-\frac{ac_5}{8}\|\hat{\varepsilon}_2\|^2-c_6\|\hat{\varepsilon}_3\|^2\\&-\left(\frac{ac_5}{4}-6c_6\right)\hat{\varepsilon}_2^2(0,t)-\left(\frac{ac_3}{4}-\frac{2|P_{od}L_d|^2}{\lambda_{\min}(Q_d)}\right.\\&\left.-\frac{24\bar{b}_1^2}{a}c_5-6a\bar{b}_4^2c_5-24\bar{b}_7^2c_6-6c_6\right)\tilde{\varepsilon}_2^2(0,t)\\&-\left(\frac{\lambda_{\min}(Q_d)}{2}-\frac{24\bar{b}_2^2}{a}c_5-6a\bar{b}_5^2c_5-24\bar{b}_8^2c_6\right.\\&\left.-6c_6\tilde{\Pi}_2^2(0)\right)|\tilde{v}_d|^2-\left(\lambda_{\min}(Q_r)-\frac{24\bar{b}_3^2}{a}c_5\right.\\&\left.-6a\bar{b}_6^2c_5-24\bar{b}_9c_6\right)|\tilde{v}_r|^2,\end{aligned} \tag{65}$$

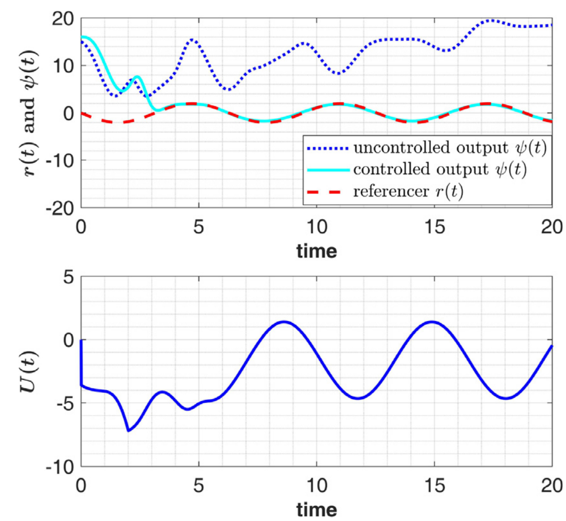

Fig. 3. Top: the reference trajectory $r(t)=2\sin(t+\pi)$ (red dashed line); the output $\psi(t)=x(0.8,t)$ under the state feedback controller (14) (royal blue solid line); the output $\psi(t)=x(0.8,t)$ without controller (blue dotted line). Bottom: control input $U(t)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

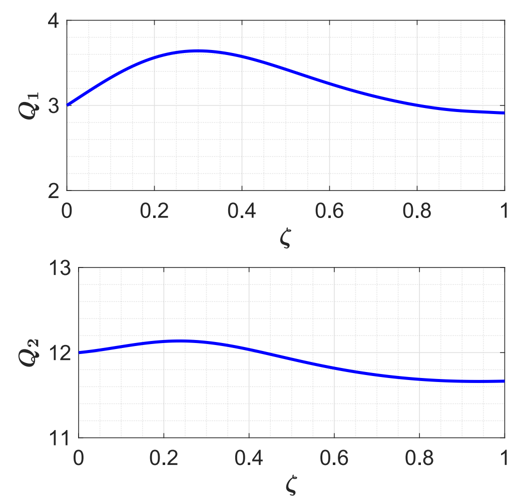

Fig. 4. The numerical solution of oberver gains $Q_1(\zeta)$ in (36a) and $Q_2(\zeta)$ in (36b).

where $\bar{b}_i$ are the bounds of $b_i(\zeta)$ in their domains. Taking

$$c_3=\frac{8}{a}\left(\frac{2|P_{od}L_d|^2}{\lambda_{\min}(Q_d)}-\frac{24\bar{b}_1^2}{a}c_5+6a\bar{b}_4^2c_5+24\bar{b}_7^2c_6+6c_6\right),$$
$$c_4\ge\frac{4c_3\|\tilde{c}\|^2}{a},\quad c_6=\frac{ac_5}{48},$$
$$c_5\le\min\left\{\frac{2a\lambda_{\min}(Q_r)}{48\bar{b}_3^2+12a^2\bar{b}_6^2+a^2\bar{b}_9^2},\ \frac{a\lambda_{\min}(Q_d)}{192\bar{b}_2^2+48a^2\bar{b}_5^2+4a^2\bar{b}_8^2+a^2\tilde{\Pi}_2^2(0)}\right\},$$

we get $\dot{V}_2\le\alpha_oV_2$, where

$$\alpha_o=\min\left\{\frac{\lambda_{\min}(Q_r)}{2\lambda_{\max}(P_{or})},\frac{\lambda_{\min}(Q_d)}{4\lambda_{\max}(P_{od})},\frac{a}{4},\frac{1}{4\tau_1},\frac{1}{2(\tau_2-\tau_1)}\right\}.$$

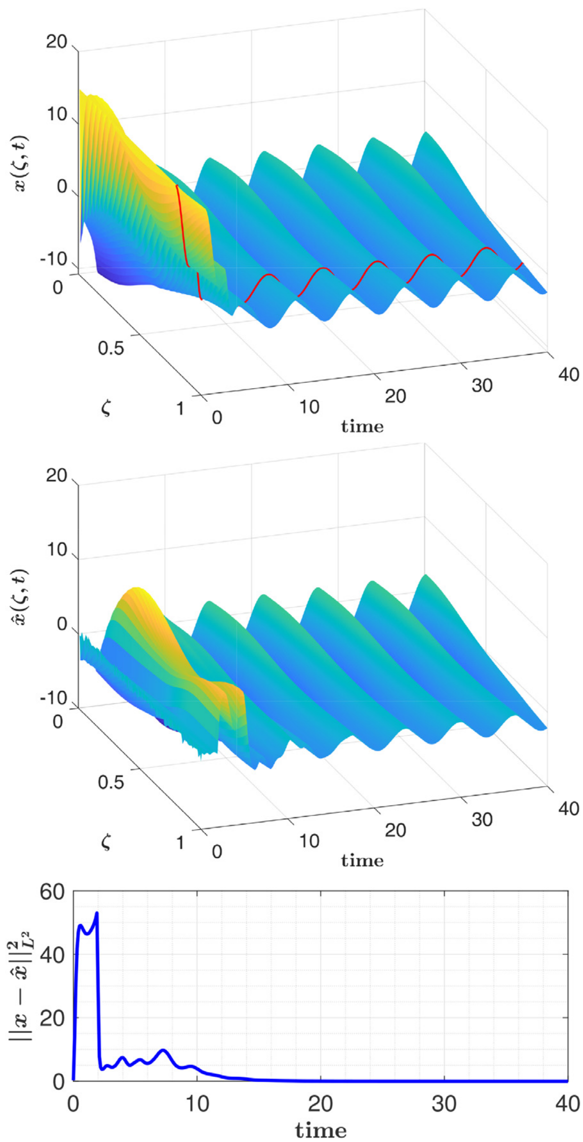

Fig. 5. The close-loop evolution with the output feedback regulator (58). Actual state $x(\zeta,t)$ (top); estimate state $\hat{x}(\zeta,t)$ by the observer (middle); L_2 norm of the error $\tilde{x}$ (bottom).

Thus, combining the condition (64), we have,

$$\Theta_o(t) \le C_o e^{-\alpha_o t}\Theta_o(t), \tag{66}$$

where $C_o = \overline{\delta_o}/\underline{\delta_o}$, thus proving the result. □

6. Numerical simulation

To demonstrate the effectiveness of the proposed output regulator, we provide some simulation examples. In this section, the parameters in PIDE system (7) are set as $a = \frac{1}{3}$, $\tau_1 = 1, \tau_2 = 2$, $c(\zeta) = 1-\zeta$, $f(\zeta,\eta) = \cos(2\pi\zeta) + 2\sin(2\pi\eta)$, $\bar{D}_1(\zeta) = 0.5e^{-\zeta}$, $\bar{D}_2 = \bar{D}_3 = 1$. The output to be controlled $\psi(t) = x(0.8,t) + d(t)$ is given by (7c) with $C_i = 0$, $i = 1,\dots,l$ and $C = \delta(z-0.8)$. The reference signal $r(t)$ is sinusoidal $r(t) = 2\sin(t+\pi)$ and disturbance signal is a constant $d(t) = 1$, which are designed by the exo-system (10), with

$$S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{67a}$$

$$P_r = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad P_d = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{67b}$$

The Crank-Nicolson method is utilized to simulate where the time step $\Delta t = 0.01$ and space step $\Delta\zeta = 0.01$. As shown in Fig. 2 (top), system open-loop response $x(\zeta,t)$ exhibits unstable behavior when the control input $U(t) = 0$.

6.1. Example 1. state feedback regulator

Firstly, an example of the PIDE system (7) combining the external signal system (10) implementing the full state feedback regulator (14) is provided. The numerical solutions of kernel are obtained by using the upwind scheme. After that, as discussed in Lemma 1, one easily yields the regulator gain $K_v = [-0.9181, 0.4205, -2.0249]$. The initial condition of the system is set as $x(\zeta,0) = 15$. The integral required for control adopts trapezoidal integral method. As Figs. 2 and 3 show, the system response not only realizes the convergence, but also the output tracking of the reference signal within finite time $T > 3.5$. The control input $U(t)$ is described in Fig. 3 (bottom). It is worth mentioning that the controller has the same frequency as the reference signal.

6.2. Example 2. output feedback regulator

Secondly, to implement the output feedback regulator (58), which is constructed by the estimated state $\hat{v}_r$, $\hat{v}_d$, $\hat{x}$, $\hat{u}_1$, $\hat{u}_2$, measurement state $y(t)$ and reference signal $r(t)$ are used to the system (12). To estimate v_r and v_d, output injection gains L_r in (29) and L_d in (30a) are designed as $L_r = [7,9]$, $L_d = 3$ such that the spectrum $\sigma(F_r) = \{-5,-2\}$ and $\sigma(F_d) = \{-3\}$. Further, according to (36) combining the observer gains F_i, P_i, it is easily to yield the observer gains Q_1 and Q_2, as is exhibited in Fig. 4. Moreover, the initial values of observer $(\hat{x}(\zeta,0), \hat{u}_1(\zeta,0), \hat{u}_2(\zeta,0))$ are given as random numbers between 0 and 1. The closed-loop evolution of the actual state, the estimated state and the norm of observer error are shown in Fig. 5. It can be seen that the output feedback regulator using the observer is performing as good as the full-state regulator.

7. Conclusion

In this paper, an output regulator which can compensate for the state and sensor delay for a first-order hyperbolic PIDE plant is designed. Two transport PDEs are used first to transform delays and then an exo-system that defined the reference and the disturbance signals is introduced, which results an ODE and two PDEs cascade system. The state feedback regulation, the observers for the reference, disturbance and state, and the output regulator are established by combing the backstepping and the regulation coordinates change. The resulted regulator equations for the state feedback, the observer and the output regulations have a unique solution under some mild conditions. The numerical simulations are presented to support the theoretical statements. Further research aims to address design of finite-time output regulator and the robust output regulation subject to parameters variation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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