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Output regulation boundary control of first-order coupled linear MIMO hyperbolic PIDE systems

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ABSTRACT
The work addresses the output regulation problem for coupled linear multiple input multiple output (MIMO) hyperbolic partial integro-differential equation systems with disturbances affecting the systems through the space and boundary input. The exosystems are extended to generate ramp signals and general family of polynomial signals. The system decomposition is applied through the state transformation and yields a decoupled equivalent system. Based on the decoupled form, the backstepping transformation is applied and then in the new coordinate, the full state and output-feedback regulators are designed, respectively. For the state feedback regulator, the corresponding regulator equation is obtained and its solvability conditions are provided to facilitate the regulator design and feasibility. The design of observer-based regulator is based on the decoupling of the observer error system into a PDE subsystem and an ODE subsystem so that the backstepping approach achieves stabilisation by eigenvalue assignment leading to design of observer stabilizing gains.

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1. Introduction

Control of first-order hyperbolic PDEs is an active research topic since plentiful processes are modelled by this class of PDE systems. Some relevant stabilisation problems are studied in Krstic and Smyshlyaev (2008), Coron, Vazquez, Krstic, and Bastin (2013), Di Meglio, Vazquez, and Krstic (2013) and Bastin and Coron (2015). Tracking control and disturbance rejection of hyperbolic PDEs are discussed in several important works such as Aamo (2013), Natarajan, Gilliam, and Weiss (2014), Anfinsen and Aamo (2015) and Deutscher (2016). For Riesz-spectral infinite-dimensional systems, a finite-dimensional output-feedback regulator has been developed in Deutscher (2011). Later, backstepping approaches were employed and in the new coordinates the output-feedback regulators have been designed to boundary controlled parabolic PDE systems and 2×2 hyperbolic PDE systems in Deutscher (2015) and Deutscher (2017), respectively. In addition, Fuzzy logic theory was employed to account for the stabilisation and tracking in first-order hyperbolic PDE systems, see Wang, Wu, and Li (2013), Wang and Wu (2014) and Wu, Wang, and Li (2012). For the research of the robust output regulation problems, many important results have been developed by Seppo Pohjolainen and Lassi Paunonen Pohjolainen (1982), Hämäläinen and Pohjolainen (2010), Paunonen and Pohjolainen (2010), Paunonen and Pohjolainen (2014) and Humaloja and Paunonen (2017). Recently, motivated by the work of Byrnes, Laukó, Gilliam, and Shubov (2000), the authors introduced the geometric approach to address the output regulation problem for the regular hyperbolic PDE systems (Xu & Dubljevic, 2016a,1). To complement the work in Deutscher (2011) for Riesz-spectral infinite-dimensional systems, the authors designed a finite-dimensional regulator for the first-order hyperbolic PDE systems (well-known non Riesz-spectral systems) (Xu, Pohjolainen, & Dubljevic, 2017). Moreover, to solve the output regulation problems for general infinite-dimensional systems (including first-order and high-order hyperbolic systems and parabolic systems), new output-feedback and error-feedback regulators are proposed by Xu and Dubljevic (2017a). To address the output regulation problem for boundary controlled hyperbolic PDE systems with Fredholm integral terms and inspired by Deutscher (2015), Xu and Dubljevic introduced an output-feedback regulator and in particular the exosystem is extended to generate polynomial-type reference signals (2017b), where only a scalar hyperbolic PIDE is taken into account.

In this manuscript, the result of Xu and Dubljevic (2017b) is extended and improved to account for the output regulation problem of the first-order coupled multiple input multiple output (MIMO) hyperbolic PIDE systems. It is not straightforward to address the problem for the considered system. To ensure the successful design of proposed regulator, the plant decoupling is first applied and based on the resulting decoupled system a MIMO regulator is proposed and developed. The disturbances rejection and reference signals tracking are achieved simultaneously. In particular, the signal process (exosystem) are extended to generate ramp-like and even polynomial-type exogenous signals more than step-like and sinusoid signals, by taking nilpotent matrix into account. State feedback regulator

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and observer-based regulator are proposed in this work. Regulation equations are obtained for the state feedback regulator and corresponding solvability conditions are provided. In observer-based regulator design, backstepping approach and change of variables are combined such that the observer error system is decoupled into PDE subsystem and ODE subsystem. Then, observer gains are configured such PDE subsystem and ODE subsystem are stabilised, respectively. Moreover, solvability conditions for corresponding *Triangular decoupling BVP* and conditions for observability are studied.

This work is organised as follows. In Section 2, the plant, the exosystem and the output regulation problem are introduced. In Section 3, the state transformation is applied such that an equivalent and completely decoupled plant is obtained and in such a way backstepping approach is applicable. Then, the state feedback regulator problem is discussed in Section 4 and the control law is obtained. Consequently, in Section 5, the observer-based regulator is designed and in particular the observability conditions are discussed. In Section 6, two applications to the parallel-flow heat exchange equations and the reactor equations of plug-flow type are given to demonstrate the performance of proposed regulators, where the reference ramp-like tracking and step-like tracking are realised.

Notation. Suppose that Z and Y are real Hilbert spaces and $A\colon Z \mapsto Y$ is a linear operator, then $D(A)$ stands for the domain of A. $\mathcal{L}(Z, Y)$ denotes the space of all linear and bounded operators from Z to Y . If $Z = Y$, then we write $\mathcal{L}(Z)$. If $A : Z \to Z$, then $\sigma(A)$ is the spectrum of $\mathcal{A}$ (the set of eigenvalues, if $\mathcal{A} \in \mathbb{C}^{nz\times nz}$), $\rho(\mathcal{A}) = \mathbb{C}\backslash\sigma(\mathcal{A})$ is the resolvent set and $R(\lambda; \mathcal{A}) = (\lambda I - \mathcal{A})^{-1} \in \mathcal{L}(X)$ denotes the resolvent operator for $\lambda \in \rho(\mathcal{A})$. The inner product is denoted by $\langle\cdot, \cdot\rangle$. $L_2(0, 1)^m$ with a non-negative integer m is a Hilbert space of an m-dimensional vector of the real functions that are a square integrable over $[0, 1]$.

2. Problem formulation

We consider the following first-order coupled MIMO hyperbolic PIDE systems on the domain $(x, t) \in [0, 1] \times \mathbb{R}^+$:

$$\partial_t z(x,t) = \partial_x z(x,t) + \alpha z(x,t) + g(x)z(0,t) + \int_0^x f(x,y)z(y,t)\,\mathrm{d}y + D(x,t) \tag{1}$$

with the constant matrix $\alpha = \left[\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\right]$ and the disturbances $D = \left[\begin{smallmatrix} m_1(x)d_1(t)\\ m_2(x)d_2(t)\end{smallmatrix}\right]$. The boundary and initial conditions are:

$$z(1,t) = \begin{bmatrix} z_1(1,t)\\ z_2(1,t)\end{bmatrix} = \begin{bmatrix}\chi_1(t) + a_1 d_1(t)\\ \chi_2(t) + a_2 d_2(t)\end{bmatrix}, \quad z(x,0) = \begin{bmatrix} z_{10}(x)\\ z_{20}(x)\end{bmatrix}, \tag{2}$$

where $g \in C^0([0, 1])$ and $f \in C^0(\mathcal{D})$ with $\mathcal{D} = \{(x, y) : 0 \le y \le x \le 1\}$ are known, $z(\cdot, t) = [z_1(\cdot, t)\ \ z_2(\cdot, t)]^{\mathrm{T}} \in Z := L_2(0, 1)^2$ denotes the vector of state variables, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are assumed to be fixed constants such as $(\alpha_1 - \alpha_4)^2 + 4\alpha_2\alpha_3 > 0$ implying that the above system is casual. Without loss of generality, we assume $\alpha_3 \neq 0$ which indicates the coupling of the system. The state space $Z := L_2(0, 1)^2$ is a Hilbert space equipped with the inner product $\langle q, p\rangle = \int_0^1 (q_1(x)p_1(x) + q_2(x)p_2(x))\,\mathrm{d}x$ for $q, p \in Z$. Then, the norm is defined by $\|z\|_Z = \|z\|_2 = \sqrt{\langle z, z\rangle}$. The boundary control inputs are denoted by $\chi_1(t) \in \mathbb{R}$ and $\chi_2(t) \in \mathbb{R}$ and the disturbances are $d_1(t) \in \mathbb{R}$ and $d_2(t) \in \mathbb{R}$. The disturbance input locations specified by $m_1(x), m_2(x) \in C^0([0, 1])$ and $a_1, a_2 \in \mathbb{R}$ are assumed to be known. The output signals to be controlled and the measurements are given by, respectively:

$$\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix} = \mathcal{C}_e z(x,t) = \begin{bmatrix} z_1(x_0,t)\\ z_2(x_0,t)\end{bmatrix}, \quad x_0 \in [0,1], \tag{3}$$

$$\begin{bmatrix} y_{m1}(t)\\ y_{m2}(t)\end{bmatrix} = z(0,t) = \begin{bmatrix} z_1(0,t)\\ z_2(0,t)\end{bmatrix}, \tag{4}$$

where the output operator $\mathcal{C}_e = \mathrm{diag}(\mathcal{C}, \mathcal{C})$ with $\mathcal{C}$ as an evaluation operator at a spatial point $x_0 \in [0, 1]$. The measurements are given by $[y_{m1}(t), y_{m2}(t)]^{\mathrm{T}} \in \mathbb{R}^2$ and the controlled outputs $[y_1(t), y_2(t)]^{\mathrm{T}} \in \mathbb{R}^2$ are not required to be measured.

The reference trajectories for $y_1(t)$ and $y_2(t)$ and disturbances $d_1(t)$ and $d_2(t)$ in (1)–(3) can be generated by the known finite-dimensional exosystem:

$$\dot{v}(t) = Sv(t), \quad v(0) \in \mathbb{C}^{n_v}, \tag{5}$$

$$\begin{aligned} d_1(t) &= p_1^{\mathrm{T}} v(t) = p_{d_1}^{\mathrm{T}} v_d(t),\\ d_2(t) &= p_2^{\mathrm{T}} v(t) = p_{d_2}^{\mathrm{T}} v_d(t), \end{aligned} \tag{6}$$

$$\begin{bmatrix} y_{r1}(t)\\ y_{r2}(t)\end{bmatrix} = \begin{bmatrix} q_1^{\mathrm{T}}\\ q_2^{\mathrm{T}}\end{bmatrix} v(t) = \begin{bmatrix} q_{r1}^{\mathrm{T}}\\ q_{r2}^{\mathrm{T}}\end{bmatrix} v_r(t). \tag{7}$$

For the above exosystem, we make an assumption as following:

Assumption 2.1: *S is a block diagonal matrix having all its eigenvalues on imaginary axis: $S = \mathrm{bdiag}(S_d, S_r)$. In particular, the matrix S_r may contain a nilpotent block S_n with n_n dimension, i.e. $\sigma(S_n) = \{0\}$ and then, the remaining block S_m with n_m dimension in S is diagonalisable and has distinct eigenvalues. Obviously, $n_n + n_m = n_v$ and $S = \mathrm{bdiag}(S_m, S_n)$. Moreover, It is assumed that $\left(\left[\begin{smallmatrix} q_{r1}^{\mathrm{T}}\\ q_{r2}^{\mathrm{T}}\end{smallmatrix}\right], S_r\right)$ is observable and the eigenvalues of S_d are distinct. In particular, the nilpotent block S_n in this manuscript is given by:*

$$S_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & (n_n-1) & 0 \end{bmatrix}. \tag{8}$$

Therefore, the state $v = \mathrm{col}(v_d, v_r)$ can be generated by two signal models $\dot{v}_d(t) = S_d v_d(t), v_d(0) = v_{d0} \in \mathbb{C}^{n_d}$ and $\dot{v}_r(t) = S_r v_r(t), v_r(0) = v_{r0} \in \mathbb{C}^{n_r}$ with $n_d + n_r = n_v$. The exosystem is established to allow the modelling of step-like, sinusoid exogenous ramp-like and even polynomial-type signals (see examples in Section 6). Whereas, the reference trajectories $y_{r1}(t)$ and $y_{r2}(t)$ in (7) are available while disturbances cannot be measured in designed regulator.

Output regulation problem – the control problem defined in this manuscript is as follows: Design an output-feedback regulator such that the following conditions are achieved:

(c1) The closed-loop system operator generates an exponentially stable C_0-semigroup (strongly continuous).
(c2) Let $e_1(t) = y_1(t) - y_{r1}(t)$ and $e_2(t) = y_2(t) - y_{r2}(t)$ denote the tracking errors, then for some $\theta_1 < 0$ and $\theta_2 < 0$,

$$e_1(t) \in L^2_{\theta_1}[0,\infty), e_2(t) \in L^2_{\theta_2}[0,\infty). \tag{9}$$

$L^2_\theta[0,\infty)$ is defined in Rebarber and Weiss (2003):

$$L^2_\theta[0,\infty) = \left\{ h \in L^2_\theta[0,\infty) \Big| \int_0^\infty e^{-2\theta t}|h(t)|^2\, dt < \infty \right\}.$$

3. Decomposition of the system

The equations in (1)–(2) can be written as:

$$\partial_t z(x,t) = Az(x,t) + g(x)z(0,t) + \int_0^x f(x,y)z(y,t)dy + \begin{bmatrix} m_1(x)d_1(t) \\ m_2(x)d_2(t) \end{bmatrix}. \tag{10}$$

The operator A is defined by $Az = \begin{bmatrix} d/dx+\alpha_1 I & \alpha_2 I \\ \alpha_3 I & d/dx+\alpha_4 I \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ on the domain: $D(A) = \{z \in L_2(0,1)^2 : z is a.c. dz/dx \in L_2(0,1)^2$ and $z(0) = 0\}$, where a.c. means absolutely continuous.

The following lemma shows that the plants (1)–(4) is equivalent to a completely decomposed system.

Lemma 3.1: *Under the assumptions:* $(\alpha_1 - \alpha_4)^2 + 4\alpha_2\alpha_3 > 0$, $\alpha_3 \neq 0$, *then there exist* γ *and* δ: $\gamma = (\alpha_1 - \alpha_4) \pm \sqrt{\beta}/2\alpha_3$ *with* $\beta := (\alpha_1 - \alpha_4)^2 + 4\alpha_2\alpha_3 > 0$ *and* $\delta = \alpha_3/(\Upsilon_2 - \Upsilon_1)$ *with* $\Upsilon_1 = (\alpha_1 - \gamma\alpha_3)$ *and* $\Upsilon_2 = (\alpha_4 + \alpha_3\gamma)$, *such that the plant* (1)–(4) *is equivalent to the following decomposed system in the space* $Z := L_2(0,1)^2$:

$$\partial_t \varpi(x,t) = \partial_x \varpi(x,t) + \mathrm{diag}\left(g_1(x), g_2(x)\right)\varpi(0,t) + \int_0^x \mathrm{diag}\left(f_1(x,y), f_2(x,y)\right)\varpi(y,t)\, dy + \begin{bmatrix} \mu_1^T(x) \\ \mu_2^T(x) \end{bmatrix} v(t), \tag{11}$$

$$\varpi(1,t) = \begin{bmatrix} u_1(t) + b_1^T v(t) \\ u_2(t) + b_2^T v(t) \end{bmatrix}, \quad \varpi(x,0) = \begin{bmatrix} \varpi_{10}(x) \\ \varpi_{20}(x) \end{bmatrix}, \tag{12}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \mathcal{C}_e M_\varpi^{-1}\varpi(t) = M_\varpi^{-1}(x_0)\varpi(x_0,t), \tag{13}$$

$$\begin{bmatrix} y_{m1}(t) \\ y_{m2}(t) \end{bmatrix} = M_\varpi^{-1}(0)\varpi(0,t) \tag{14}$$

with the new state $\varpi(x,t) = \begin{bmatrix} \varpi_1(x,t) \\ \varpi_2(x,t) \end{bmatrix} = M_\varpi(x)z(x,t)$ *and* $\begin{bmatrix} \varpi_{10}(x) \\ \varpi_{20}(x) \end{bmatrix} = M_\varpi(x)z(x,0)$, *where* $M_\varpi(x) = \begin{bmatrix} e^{\Upsilon_1 x} & -\gamma e^{\Upsilon_1 x} \\ \delta e^{\Upsilon_2 x} & (1-\delta\gamma)e^{\Upsilon_2 x} \end{bmatrix}$ *is invertible.*

For a proof of Lemma 3.1, consult Appendix 1.

Let $\begin{bmatrix} e_{\varpi 1}(t) \\ e_{\varpi 2}(t) \end{bmatrix} = \begin{bmatrix} \varpi_1(x_0,t) \\ \varpi_2(x_0,t) \end{bmatrix} - M_\varpi(x_0)\begin{bmatrix} y_{r1}(t) \\ y_{r2}(t) \end{bmatrix}$ denote the transformed tracking error. Then, the target (9) is equivalent to the following due to the invertibility of $M_\varpi(x)$: for some $\beta_{\varpi 1} < 0$, $\beta_{\varpi 2} < 0$

$$e_{\varpi 1}(t) \in L^2_{\beta_{\varpi 1}}[0,\infty), e_{\varpi 2}(t) \in L^2_{\beta_{\varpi 2}}[0,\infty). \tag{15}$$

4. The state feedback regulator design

In this section, this manuscript addresses the output regulation problem of the systems (11)–(13) by state feedback control law:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \int_0^1 \mathrm{diag}\left(k_1(y), k_2(y)\right)\varpi(y,t)\, dy + \begin{bmatrix} \pi_{v1}^T \\ \pi_{v2}^T \end{bmatrix} v(t) \tag{16}$$

endowed with a feedforward signal of the exosystem states. For the computation of the state feedback gains $k_1(y)$ and $k_2(y)$, the backstepping approach is applied. Controller coordinates $\tilde{\varpi}(x,t) = [\tilde{\varpi}_1(x,t), \tilde{\varpi}_2(x,t)]^T$ are introduced by the *Volterra-type integral transformation*

$$\tilde{\varpi}(x,t) = \mathcal{T}_c(\varpi(x,t))(x) = \begin{bmatrix} \mathcal{T}_{c1}(\varpi_1(t))(x) \\ \mathcal{T}_{c2}(\varpi_2(t))(x) \end{bmatrix} \tag{17}$$

with

$$\mathcal{T}_{c1}(\varpi_1(t))(x) = \varpi_1(x,t) - \int_0^x \kappa_1(x,y)\varpi_1(y,t)\, dy,$$

$$\mathcal{T}_{c2}(\varpi_2(t))(x) = \varpi_2(x,t) - \int_0^x \kappa_2(x,y)\varpi_2(y,t)\, dy.$$

The integral kernel $[\kappa_1(x,y), \kappa_2(x,y)]^T$ is assumed to be the solution of the *control-kernel boundary value problem(BVP)*:

$$\partial_x \begin{bmatrix} \kappa_1(x,y) \\ \kappa_2(x,y) \end{bmatrix} + \partial_y \begin{bmatrix} \kappa_1(x,y) \\ \kappa_2(x,y) \end{bmatrix} = \int_0^x \begin{bmatrix} \kappa_1(x,\zeta)f_1(\zeta,y) \\ \kappa_2(x,\zeta)f_2(\zeta,y) \end{bmatrix} \times d\zeta - \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}, \tag{18}$$

$$\begin{bmatrix} \kappa_1(x,0) \\ \kappa_2(x,0) \end{bmatrix} = \int_0^x \begin{bmatrix} \kappa_1(x,y)g_1(y) \\ \kappa_2(x,y)g_2(y) \end{bmatrix} dy - \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} \tag{19}$$

on the triangular spatial domain $0 \le y \le x \le 1$. Then, a straightforward calculation yields, with $\tilde{\mu}_1^T(x) = \mathcal{T}_{c1}(\mu_1^T)(x)$, $\tilde{\mu}_2^T(x) = \mathcal{T}_{c2}(\mu_2^T)(x)$, the result

$$\partial_t \tilde{\varpi}(x,t) = \partial_x \tilde{\varpi}(x,t) + \begin{bmatrix} \tilde{\mu}_1^T(x) \\ \tilde{\mu}_2^T(x) \end{bmatrix} v(t), \tag{20}$$

$$\tilde{\varpi}(1,t) = \begin{bmatrix} \pi_{v1}^T + b_1^T \\ \pi_{v2}^T + b_2^T \end{bmatrix} v(t) \tag{21}$$

by implementation of the state feedback regulator (16) with $k_1(x) = \kappa_1(1,x)$ and $k_2(x) = \kappa_2(1,x)$. Results in Krstic and Smyshlyaev (2008) imply the existence of unique bounded C^1−solutions $[\kappa_1(x,y), \kappa_2(x,y)]^T$. As a result, the feedback gains $k_1(z)$ and $k_2(z)$ in (16) exist in such a way that the closed-loop systems (11)–(13) for $v(t) \equiv 0$ is exponentially stabilised.

In order to determine feedforward gains π_{v1}^{T} and π_{v2}^{T}, we introduce for (20)–(21) the state error:

$$\tilde{e}(x,t) = \begin{bmatrix} \tilde{e}_1(x,t) \\ \tilde{e}_2(x,t) \end{bmatrix} = \tilde{\varpi}(x,t) - \begin{bmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{bmatrix} v(t), \tag{22}$$

where $\pi_1^{\mathrm{T}}(x)$ and $\pi_2^{\mathrm{T}}(x)$ have to be found. Here, $\left[\begin{smallmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{smallmatrix}\right] v(t)$ describe the behaviour of $\tilde{\varpi}(x,t)$ when achieving the output regulation (9). By combining (5), (20)–(21) and (22), one has: $\partial_t \tilde{e}(x,t) = \partial_x \tilde{e}(x,t)$, $(x,t) \in (0,1) \times \mathbb{R}^+$, if $\left[\begin{smallmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{smallmatrix}\right]$ satisfy the condition: $\partial_x \left[\begin{smallmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{smallmatrix}\right] S - \left[\begin{smallmatrix} \tilde{\mu}_1^{\mathrm{T}}(x) \\ \tilde{\mu}_2^{\mathrm{T}}(x) \end{smallmatrix}\right]$. Then, the boundary condition at $z=1$ for $\tilde{e}(x,t)$ gives: $\tilde{e}(1,t) = 0, t > 0$, if the condition $\left[\begin{smallmatrix} \pi_{v1}^{\mathrm{T}} + b_1^{\mathrm{T}} \\ \pi_{v2}^{\mathrm{T}} + b_2^{\mathrm{T}} \end{smallmatrix}\right] - \left[\begin{smallmatrix} \pi_1^{\mathrm{T}}(1) \\ \pi_2^{\mathrm{T}}(1) \end{smallmatrix}\right] = 0$ holds, see (21). In view of (7), (13), (15), (17) and (22), the corresponding output tracking error e_{ϖ} in the transformed coordinates reduces to: $e_{\varpi}(t) = \mathcal{C}_e \mathcal{T}_c^{-1}(\tilde{e}(x,t))$, if the condition $\mathcal{C}_e \mathcal{T}_c^{-1}\left(\left[\begin{smallmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{smallmatrix}\right]\right) - M_{\varpi}(x_0) \left[\begin{smallmatrix} q_1^{\mathrm{T}} \\ q_2^{\mathrm{T}} \end{smallmatrix}\right] = 0$ holds, where $\mathcal{C}_e$ is defined in Lemma 3.1. Theorem 2 in Krstic and Smyshlyaev (2008) can be utilised to show that the inverse *Volterra-type integral transformation* $\mathcal{T}_c^{-1}$ in $e_{\varpi}(t)$ exits and it is given by:

$$\varpi(x,t) = \mathcal{T}_c^{-1}(\tilde{\varpi}(t))(x) = \begin{bmatrix} \mathcal{T}_{c1}^{-1}(\tilde{\varpi}_1(t))(x) \\ \mathcal{T}_{c2}^{-1}(\tilde{\varpi}_2(t))(x) \end{bmatrix} \tag{23}$$

with

$$\mathcal{T}_{c1}^{-1}(\tilde{\varpi}_1(t))(x) = \tilde{\varpi}_1(x,t) + \int_0^x l_1(x,y)\tilde{\varpi}_1(y,t)\,\mathrm{d}y,$$
$$\mathcal{T}_{c2}^{-1}(\tilde{\varpi}_2(t))(x) = \tilde{\varpi}_2(x,t) + \int_0^x l_2(x,y)\tilde{\varpi}_2(y,t)\,\mathrm{d}y.$$

The inverse kernel $[l_1(x,y), l_2(x,y)]^{\mathrm{T}}$ is bounded C^1 – function and the corresponding inverse *kernel BVP* has the form:

$$\partial_x \begin{bmatrix} l_1(x,y) \\ l_2(x,y) \end{bmatrix} + \partial_y \begin{bmatrix} l_1(x,y) \\ l_2(x,y) \end{bmatrix} = \int_0^x \begin{bmatrix} f_1(z,\zeta) l_1(\zeta,y) \\ f_2(z,\zeta) l_2(\zeta,y) \end{bmatrix} \mathrm{d}\zeta - \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}, \tag{24}$$

$$\begin{bmatrix} l_1(x,0) \\ l_2(x,0) \end{bmatrix} = - \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}. \tag{25}$$

To summarise the above, if the function $\left[\begin{smallmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{smallmatrix}\right]$ satisfies:

$$d_x \begin{bmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{bmatrix} = \begin{bmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{bmatrix} S - \begin{bmatrix} \tilde{\mu}_1^{\mathrm{T}}(x) \\ \tilde{\mu}_2^{\mathrm{T}}(x) \end{bmatrix}, \quad x \in [0,1], \tag{26}$$

$$\mathcal{C}_e \mathcal{T}_c^{-1}\left(\begin{bmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{bmatrix}\right) - M_{\varpi}(x_0) \begin{bmatrix} q_1^{\mathrm{T}} \\ q_2^{\mathrm{T}} \end{bmatrix} = 0. \tag{27}$$

Then, the feedforward gain is chosen as $\left[\begin{smallmatrix} \pi_{v1}^{\mathrm{T}} \\ \pi_{v2}^{\mathrm{T}} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \pi_1^{\mathrm{T}}(1) - b_1^{\mathrm{T}} \\ \pi_2^{\mathrm{T}}(1) - b_2^{\mathrm{T}} \end{smallmatrix}\right]$, then the tracking error system reduces to:

$$\partial_t \tilde{e}(x,t) = \partial_x \tilde{e}(x,t), (x,t) \in (0,1) \times \mathbb{R}^+, \tag{28}$$

$$\tilde{e}(1,t) = 0, t > 0 \tag{29}$$

and the output tracking error in (15) can be expressed as:

$$e_{\varpi}(t) = \mathcal{C}_e \mathcal{T}_c^{-1}\left(\tilde{e}(x,t)\right). \tag{30}$$

The tracking error systems (28)–(29) is exponentially stable, so that the output regulation with closed-loop stability is achieved (see Theorem 4.1). In this manuscript, we call the Equations (26)–(27) as the regulator equations, since they are equivalent to the regulator equations (Sylvester equations) developed for the general class of distributed parameter systems with distributed control in Byrnes et al. (2000). The expression (26) is a very simple ODE and it admits a general solution in closed-form. As a result, one can easily clarify the solvability of the regulator Equations (26)–(27). It is shown that the transfer behaviour of the systems (1)–(4) determines this solvability of regulator Equations (26)–(27). The transfer function can be obtained through the representation of the plant in backstepping coordinates. In this manuscript, the approach in Curtain and Morris (2009) is applied to obtain the transfer function. The next lemma demonstrates the solvability condition for the regulator Equations (26)–(27).

Lemma 4.1 (Regulator equations): *With Υ_1, Υ_2, δ and γ are given in Lemma 3.1, the transfer function of the transfer function $G(s)$ of the plants (1)–(4) from $\left[\begin{smallmatrix} \chi_1(t) \\ \chi_2(t) \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} y_1(t) \\ y_2(t) \end{smallmatrix}\right]$ is $G(s) = \mathcal{C}_e M_{\varpi}^{-1} \left[\begin{smallmatrix} \mathcal{T}_{c1}^{-1}(e^{s(x-1)}) & 0 \\ 0 & \mathcal{T}_{c2}^{-1}(e^{s(x-1)}) \end{smallmatrix}\right] M_{\varpi}(1)$. The regulator Equations (26)–(27) has a unique classical solution if and only if the transfer function $G(\lambda), \forall \lambda \in \sigma(S)$ is invertible.*

Proof: Let us write $\left[\begin{smallmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \tilde{\mu}_1^{\mathrm{T}}(x) \\ \tilde{\mu}_2^{\mathrm{T}}(x) \end{smallmatrix}\right]$ as: $\left[\begin{smallmatrix} \pi_1^{\mathrm{T}}(x) \\ \pi_2^{\mathrm{T}}(x) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \pi_{m1}^{\mathrm{T}}(x) & \pi_{n1}^{\mathrm{T}}(x) \\ \pi_{m2}^{\mathrm{T}}(x) & \pi_{n2}^{\mathrm{T}}(x) \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} \tilde{\mu}_1^{\mathrm{T}}(x) \\ \tilde{\mu}_2^{\mathrm{T}}(x) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \tilde{\mu}_{m1}^{\mathrm{T}}(x) & \tilde{\mu}_{n1}^{\mathrm{T}}(x) \\ \tilde{\mu}_{m2}^{\mathrm{T}}(x) & \tilde{\mu}_{n2}^{\mathrm{T}}(x) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} q_1^{\mathrm{T}}(x) \\ q_2^{\mathrm{T}}(x) \end{smallmatrix}\right] = \left[\begin{smallmatrix} q_{m1}^{\mathrm{T}}(x) & q_{n1}^{\mathrm{T}}(x) \\ q_{m2}^{\mathrm{T}}(x) & q_{n2}^{\mathrm{T}}(x) \end{smallmatrix}\right]$. As a result, the regulator Equations (26)–(27) can be decomposed into the following form:

$$d_x \begin{bmatrix} \pi_{m1}^{\mathrm{T}}(x) \\ \pi_{m2}^{\mathrm{T}}(x) \end{bmatrix} = \begin{bmatrix} \pi_{m1}^{\mathrm{T}}(x) \\ \pi_{m2}^{\mathrm{T}}(x) \end{bmatrix} S_m - \begin{bmatrix} \tilde{\mu}_{m1}^{\mathrm{T}}(x) \\ \tilde{\mu}_{m2}^{\mathrm{T}}(x) \end{bmatrix}, \tag{31}$$

$$\mathcal{C}_e \mathcal{T}_c^{-1}\left(\begin{bmatrix} \pi_{m1}^{\mathrm{T}}(x) \\ \pi_{m2}^{\mathrm{T}}(x) \end{bmatrix}\right) - M_{\varpi}(x_0) \begin{bmatrix} q_{m1}^{\mathrm{T}} \\ q_{m2}^{\mathrm{T}} \end{bmatrix} = 0, \tag{32}$$

$$d_x \begin{bmatrix} \pi_{n1}^{\mathrm{T}}(x) \\ \pi_{n2}^{\mathrm{T}}(x) \end{bmatrix} = \begin{bmatrix} \pi_{n1}^{\mathrm{T}}(x) \\ \pi_{n2}^{\mathrm{T}}(x) \end{bmatrix} S_n - \begin{bmatrix} \tilde{\mu}_{n1}^{\mathrm{T}}(x) \\ \tilde{\mu}_{n2}^{\mathrm{T}}(x) \end{bmatrix}, \tag{33}$$

$$\mathcal{C}_e \mathcal{T}_c^{-1}\left(\begin{bmatrix} \pi_{n1}^{\mathrm{T}}(x) \\ \pi_{n2}^{\mathrm{T}}(x) \end{bmatrix}\right) - M_{\varpi}(x_0) \begin{bmatrix} q_{n1}^{\mathrm{T}} \\ q_{n2}^{\mathrm{T}} \end{bmatrix} = 0. \tag{34}$$

Assume that $\{\phi_k\}$ with $k = 1, 2, \ldots, n_m$ are eigenvectors of S_m with eigenvalues $\{\lambda_k\}_{k=1,\ldots,n_m}$. Due to Assumption 2.1, S_m

is diagonalisable and there exists a similarity transformation $V^{-1}S_mV = \mathrm{diag}(\lambda_1,\dots,\lambda_{n_m})$ with $V = [\phi_1,\dots,\phi_{n_m}]$. Post-multiplying (26) by V leads to the decoupled set of ODEs:

$$d_x\begin{bmatrix}\pi^*_{m1k}(x)\\ \pi^*_{m2k}(x)\end{bmatrix} = \lambda_{v,i}\begin{bmatrix}\pi^*_{m1k}(x)\\ \pi^*_{m2k}(x)\end{bmatrix} - \begin{bmatrix}\tilde{\mu}^*_{m1k}(x)\\ \tilde{\mu}^*_{m2k}(x)\end{bmatrix},\quad k = 1,2,\dots n_m \tag{35}$$

with $\begin{bmatrix}\pi^*_{m1k}(x)\\ \pi^*_{m2k}(x)\end{bmatrix} = \begin{bmatrix}\pi^T_{m1}(x)\\ \pi^T_{m2}(x)\end{bmatrix}\phi_k$ and $\begin{bmatrix}\tilde{\mu}^*_1(x)\\ \tilde{\mu}^*_2(x)\end{bmatrix} = \begin{bmatrix}\tilde{\mu}^T_{m1}(x)\\ \tilde{\mu}^T_{m2}(x)\end{bmatrix}\phi_k$. Then, for $k = 1,\dots,n_m$, the solutions are given by

$$\begin{bmatrix}\pi^*_{m1k}(x)\\ \pi^*_{m2k}(x)\end{bmatrix} = e^{\lambda_k x}\begin{bmatrix}\pi^*_{m1k}(0)\\ \pi^*_{m2k}(0)\end{bmatrix} - \int_0^x e^{\lambda_k(x-\xi)}\begin{bmatrix}\tilde{\mu}^*_{m1k}(\xi)\\ \tilde{\mu}^*_{m2k}(\xi)\end{bmatrix}d\xi.$$

The condition (32) can be written as: $\mathcal{C}_e\mathcal{T}_c^{-1}\left(\begin{bmatrix}\pi^*_{m1k}\\ \pi^*_{m2k}\end{bmatrix}\right) = M_\varpi(x_0)\begin{bmatrix}q^*_{m1k}\\ q^*_{m2k}\end{bmatrix}$ with $\begin{bmatrix}q^*_{m1k}\\ q^*_{m2k}\end{bmatrix} = \begin{bmatrix}q^T_{m1}\\ q^T_{m2}\end{bmatrix}\phi_k$. Consequently,

$$\mathcal{C}_e\begin{bmatrix}\mathcal{T}_{c1}^{-1}\left(e^{\lambda_k x}\right) & 0\\ 0 & \mathcal{T}_{c2}^{-1}\left(e^{\lambda_k x}\right)\end{bmatrix}\begin{bmatrix}\pi^*_{m1k}(0)\\ \pi^*_{m2k}(0)\end{bmatrix} - \mathcal{C}_e\mathcal{T}_c^{-1}\left(\int_0^x\begin{bmatrix}e^{\lambda_k(x-\xi)}\tilde{\mu}^*_{m1k}(\xi)\\ e^{\lambda_k(x-\xi)}\tilde{\mu}^*_{m2k}(\xi)\end{bmatrix}d\xi\right) = M_\varpi(x_0)\begin{bmatrix}q^*_{m1k}\\ q^*_{m2k}\end{bmatrix}.$$

The invertibility of the matrix $\mathcal{C}_e\begin{bmatrix}\mathcal{T}_{c1}^{-1}(e^{\lambda_k x}) & 0\\ 0 & \mathcal{T}_{c2}^{-1}(e^{\lambda_k x})\end{bmatrix}$ ensures the existence and uniqueness of $\begin{bmatrix}\pi^*_{m1k}(0)\\ \pi^*_{m2k}(0)\end{bmatrix}$ and thus $\begin{bmatrix}\pi^*_{m1k}(x)\\ \pi^*_{m2k}(x)\end{bmatrix}$. Once the solution $\begin{bmatrix}\pi^*_{m1k}(x)\\ \pi^*_{m2k}(x)\end{bmatrix}$ is obtained, one can calculate $\begin{bmatrix}\pi^T_{m1}(x)\\ \pi^T_{m2}(x)\end{bmatrix} = \begin{bmatrix}\pi^*_{m11}(x),\dots,\pi^*_{m1n_m}(x)\\ \pi^*_{m21}(x),\dots,\pi^*_{m2n_m}(x)\end{bmatrix}V^{-1}$.

Now, we turn to the solving of (33)–(34). First, write $\begin{bmatrix}\pi^T_{n1}(x)\\ \pi^T_{n2}(x)\end{bmatrix}$ and $\begin{bmatrix}\tilde{\mu}^T_{n1}(x)\\ \tilde{\mu}^T_{n2}(x)\end{bmatrix}$ as $\begin{bmatrix}\pi_{n1,1}(x),\dots,\pi_{n1,n_n}(x)\\ \pi_{n2,1}(x),\dots,\pi_{n2,n_n}(x)\end{bmatrix}$ and $\begin{bmatrix}\tilde{\mu}_{n1,1}(x),\dots,\tilde{\mu}_{n1,n_n}(x)\\ \tilde{\mu}_{n2,1}(x),\dots,\tilde{\mu}_{n2,n_n}(x)\end{bmatrix}$, respectively. Then (33) can be transformed into a set of cascade ODEs:

$$\begin{aligned} d_x\begin{bmatrix}\pi_{n1,k}(x)\\ \pi_{n2,k}(x)\end{bmatrix} &= \begin{bmatrix}k\pi_{n1,k+1}(x)\\ k\pi_{n2,k+1}(x)\end{bmatrix} - \begin{bmatrix}\tilde{\mu}_{n1,k}(x)\\ \tilde{\mu}_{n2,k}(x)\end{bmatrix},\\ d_x\begin{bmatrix}\pi_{n1,n_n}(x)\\ \pi_{n2,n_n}(x)\end{bmatrix} &= -\begin{bmatrix}\tilde{\mu}_{n1,n_n}(x)\\ \tilde{\mu}_{n2,n_n}(x)\end{bmatrix}\end{aligned} \tag{36}$$

with $k = 1,\dots,(n_n-1)$. Subsequently, the solution is given by

$$\begin{aligned}\begin{bmatrix}\pi_{n1,k}(x)\\ \pi_{n2,k}(x)\end{bmatrix} &= \begin{bmatrix}\pi_{n1,k}(0)\\ \pi_{n2,k}(0)\end{bmatrix} + \int_0^x\left(\begin{bmatrix}k\pi_{n1,k+1}(\xi),\\ k\pi_{n2,k+1}(\xi)\end{bmatrix} - \begin{bmatrix}\tilde{\mu}_{n1,k}(\xi)\\ \tilde{\mu}_{n2,k}(\xi)\end{bmatrix}\right)d\xi, \begin{bmatrix}\pi_{n1,n_n}(x)\\ \pi_{n2,n_n}(x)\end{bmatrix}\\ &= \begin{bmatrix}\pi_{n1,n_n}(0)\\ \pi_{n2,n_n}(0)\end{bmatrix} - \int_0^x\begin{bmatrix}\tilde{\mu}_{n1,k}(\xi)\\ \tilde{\mu}_{n2,k}(\xi)\end{bmatrix}d\xi.\end{aligned}$$

Writing $\begin{bmatrix}q^T_{n1}\\ q^T_{n2}\end{bmatrix} = \begin{bmatrix}q_{n1,1},\dots,q_{n1,n_n}\\ q_{n2,1},\dots,q_{n1,n_n}\end{bmatrix}$ and applying (34) gives:

$$\begin{aligned}&\mathcal{C}_e\begin{bmatrix}\mathcal{T}_{c1}^{-1}\left(e^{0x}\right) & 0\\ 0 & \mathcal{T}_{c2}^{-1}\left(e^{0x}\right)\end{bmatrix}\begin{bmatrix}\pi_{n1,n_n}(0)\\ \pi_{n2,n_n}(0)\end{bmatrix} = M_\varpi(x_0)\begin{bmatrix}q_{n1,n_n}\\ q_{n2,n_n}\end{bmatrix} + \mathcal{C}_e\mathcal{T}_c^{-1}\left(\int_0^x\begin{bmatrix}\tilde{\mu}_{n1,n_n}(\xi)\\ \tilde{\mu}_{n2,n_n}(\xi)\end{bmatrix}d\xi\right),\\ &\mathcal{C}_e\begin{bmatrix}\mathcal{T}_{c1}^{-1}\left(e^{0x}\right) & 0\\ 0 & \mathcal{T}_{c2}^{-1}\left(e^{0x}\right)\end{bmatrix}\begin{bmatrix}\pi_{n1,k}(0)\\ \pi_{n2,k}(0)\end{bmatrix} = M_\varpi(x_0)\begin{bmatrix}q_{n1,k}\\ q_{n2,k}\end{bmatrix} - \mathcal{C}_e\mathcal{T}_c^{-1}\left(\int_0^x\left(\begin{bmatrix}k\pi_{n1,k+1}(\xi)\\ k\pi_{n2,k+1}(\xi)\end{bmatrix} - \begin{bmatrix}\tilde{\mu}_{n1,n_n}(\xi)\\ \tilde{\mu}_{n2,n_n}(\xi)\end{bmatrix}\right)d\xi\right).\end{aligned}$$

Therefore, the invertibility of the matrix $\mathcal{C}_e\begin{bmatrix}\mathcal{T}_{c1}^{-1}(e^{0x}) & 0\\ 0 & \mathcal{T}_{c2}^{-1}(e^{0x})\end{bmatrix}$ ensures the existence and uniqueness of $\begin{bmatrix}\pi_{n1,n_n}(0)\\ \pi_{n2,n_n}(0)\end{bmatrix}$ and $\begin{bmatrix}\pi_{n1,k}(0)\\ \pi_{n2,k}(0)\end{bmatrix}$ given $k = 1,\dots,(n_n-1)$ and thus $\begin{bmatrix}\pi^T_{n1}(x)\\ \pi^T_{n2}(x)\end{bmatrix}$.

The systems (1)–(4) has the representation: $\partial_t\tilde{\varpi}(x,t) = \partial_x\tilde{\varpi}(x,t)$ with the boundary conditions $\tilde{\varpi}(1,t) = \begin{bmatrix}u_1(t) - \int_0^1\kappa_1(1,x)\mathcal{T}_{c1}^{-1}(\tilde{\varpi}_1(x,t))\,dx\\ u_2(t) - \int_0^1\kappa_2(1,x)\mathcal{T}_{c2}^{-1}(\tilde{\varpi}_2(x,t))\,dx\end{bmatrix}$ and the output $\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \mathcal{C}_eM_\varpi^{-1}\mathcal{T}_c^{-1}\tilde{\varpi}(t)$. By applying the method in Curtain and Morris (2009), the transfer function for (11)–(13) from $\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$ to $\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}$ can be calculated: $G_\varpi(s) = \mathcal{C}_eM_\varpi^{-1}\begin{bmatrix}\mathcal{T}_{c1}^{-1}(e^{s(x-1)}) & 0\\ 0 & \mathcal{T}_{c2}^{-1}(e^{s(x-1)})\end{bmatrix}$. Obviously, the invertibility of $G_\varpi(\lambda)$, $\lambda\in\sigma(S)$ implies that $\mathcal{C}_e\begin{bmatrix}\mathcal{T}_{c1}^{-1}(e^{\lambda_k x}) & 0\\ 0 & \mathcal{T}_{c2}^{-1}(e^{\lambda_k x})\end{bmatrix}$ is invertible. From the previous context, the new input can be expressed as $\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} = M_\varpi(1)\begin{bmatrix}\chi_1(t)\\ \chi_2(t)\end{bmatrix}$. The transfer function $G(s)$ in lemma for the original plant from $\begin{bmatrix}\chi_1(t)\\ \chi_2(t)\end{bmatrix}$ to $\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}$ is obtained and this indicates that the solvability condition in the lemma. ■

The condition in Lemma 4.1 implies that the eigenmodes of exosystems (5)–(7) can be transferred to the controlled output $y(t)$ in such a way that the disturbance can be compensated and the reference value can be attained at desired steady state. Consideration of the proof of Lemma 4.1 indicates that the solution $\begin{bmatrix}\pi^T_1(x)\\ \pi^T_2(x)\end{bmatrix}$ of the regulator equations (26)–(27) can be found in closed-form given that the backstepping transformations $\mathcal{T}_c$ and $\mathcal{T}_c^{-1}$ have been determined. Therein, the later transformation is not necessary if the output $y(t)$ is defined at the boundary point $z=0$. The next theorem shows that output regulation in the backstepping coordinates also implies output regulation in the original coordinates. Therein, the dynamics of the tracking error $\begin{bmatrix}\bar{e}_1(x,t)\\ \bar{e}_2(x,t)\end{bmatrix} = \varpi(x,t) - \begin{bmatrix}\bar{\pi}^T_1(x)\\ \bar{\pi}^T_2(x)\end{bmatrix}v(t)$ with $\begin{bmatrix}\bar{\pi}^T_1(x)\\ \bar{\pi}^T_2(x)\end{bmatrix} = \mathcal{T}_c^{-1}\left(\begin{bmatrix}\pi^T_1\\ \pi^T_2\end{bmatrix}\right)(x)$ is exponentially stable.

Theorem 4.1 (State feedback regulator): *Having* $[\kappa_1(x,y)\ \kappa_2(x,y)]^T$ *and* $\begin{bmatrix}\pi^T_1\\ \pi^T_2\end{bmatrix}$ *as solutions of the control-kernel BVP* (19)–(20) *and the regulator Equations* (26)–(27)*, the state feedback regulator* (17) *with* $k_1(y) = \kappa_1(1,y)$, $k_2(y) = \kappa_2(1,y)$ *and*

$\pi_{v1}^{\mathrm{T}} = \pi_1^{\mathrm{T}}(1) - b_1^{\mathrm{T}}$, $\pi_{v2}^{\mathrm{T}} = \pi_2^{\mathrm{T}}(1) - b_2^{\mathrm{T}}$ *achieves output regulation* (16) *for* (1)–(3) *and* (5)–(8) *with an exponentially stable tracking error dynamics in the* L_2*-norm, i.e. the tracking error* $\bar{e}(t) = \{\bar{e}(x,t) = \begin{bmatrix} \bar{e}_1(x,t) \\ \bar{e}_2(x,t) \end{bmatrix}, x \in [0,1]\}$ *satisfies*

$$\|\bar{e}(t)\|_2 \leq M_c(\kappa_1, \kappa_2, l_1, l_2)\,\mathrm{e}^{-\alpha_0 t}\|\bar{e}(0)\|_2, \quad t \in (0, T] \tag{37}$$

for $\bar{e}(0) \in Z$ *with positive constants* $M_c > 0$ *(depending* $\kappa_1, \kappa_2, l_1, l_2$*) and* $\alpha_0 > 0$*. In particular, the finite-time output regulation by the state feedback control* (17) *is achieved.*

The proof of theorem is given in Appendix 2.

Remark 4.1: Since the separated systems (11)–(13) is equivalent to the original plants (1)–(4), then through $\begin{bmatrix} \chi_1(t) \\ \chi_2(t) \end{bmatrix} = M_\varpi^{-1}(1)\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ the equivalent control law for (1)–(4) realising the output regulation (9) can be obtained from (16) as

$$\begin{bmatrix} \chi_1(t) \\ \chi_2(t) \end{bmatrix} = M_\varpi^{-1}(1)\begin{bmatrix} \pi_{v1}^{\mathrm{T}} \\ \pi_{v2}^{\mathrm{T}} \end{bmatrix} v(t) + M_\varpi^{-1}(1)\int_0^1 \begin{bmatrix} k_1(y) & 0 \\ 0 & k_2(y) \end{bmatrix} M_\varpi(y)\begin{bmatrix} z_1(y,t) \\ z_2(y,t) \end{bmatrix}\mathrm{d}y. \tag{38}$$

Correspondingly, Theorem 4.1 means that the regulator solves the finite-time output regulation problem (9).

5. The output-feedback regulator design

This section addresses the output regulator problem such that the output regulation (10) of the plants (1)–(4) is achieved. The common assumption is made that the reference signals $y_{r1}(t)$ and $y_{r2}(t)$ and the measurements $y_{m1}(t)$ and y_{m2} are measurable for the regulator design. With known $y_{r1}(t)$ and $y_{r2}(t)$, the signal process state $v_r(t)$ can be observed through a simple finite-dimensional observer:

$$\dot{\hat{v}}_r(t) = S_r\hat{v}_r(t) + L_r\left(\begin{bmatrix} y_{r1}(t) \\ y_{r2}(t) \end{bmatrix} - \begin{bmatrix} q_{r1}^{\mathrm{T}} \\ q_{r2}^{\mathrm{T}} \end{bmatrix}\hat{v}_r(t)\right), \quad t > 0 \tag{39}$$

with an arbitrary initial condition $\hat{v}_r(0) = \hat{v}_{r0} \in \mathbb{C}^{n_r}$. Under the assumption that $\left(\begin{bmatrix} q_{r1}^{\mathrm{T}} \\ q_{r2}^{\mathrm{T}} \end{bmatrix}, S_r\right)$ is observable, there exists an output injection gain L_r such that the observer error dynamics $e_r = v_r - \hat{v}_r$ attenuates asymptotically.

For the estimation of $v_d(t)$, $z_1(x,t)$ and $z_2(x,t)$, a PDE-ODE coupled observer is designed:

$$\dot{\hat{v}}_d(t) = S_d\hat{v}_d(t) + \begin{bmatrix} l_{d1} & l_{d2} \end{bmatrix}\left(\begin{bmatrix} y_{m1}(t) \\ y_{m2}(t) \end{bmatrix} - \begin{bmatrix} \hat{z}_1(0,t) \\ \hat{z}_2(0,t) \end{bmatrix}\right), \tag{40}$$

$$\partial_t\hat{z}(x,t) = \partial_x\hat{z}(x,t) + \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}\hat{z}(x,t) + g(x)\hat{z}(0,t) + \int_0^x f(x,y)\hat{z}(y,t)\,\mathrm{d}y + \begin{bmatrix} m_1(x)p_{d1}^{\mathrm{T}}(x) \\ m_2(x)p_{d2}^{\mathrm{T}}(x) \end{bmatrix}\hat{v}_d(t) + L(x)\left(\begin{bmatrix} y_{m1}(t) \\ y_{m2}(t) \end{bmatrix} - \hat{z}(0,t)\right), \tag{41}$$

$$\hat{z}(1,t) = \begin{bmatrix} \chi_1(t) \\ \chi_2(t) \end{bmatrix} + \begin{bmatrix} a_1p_{d_1}^{\mathrm{T}} \\ a_2p_{d_2}^{\mathrm{T}} \end{bmatrix}\hat{v}_d(t) \tag{42}$$

with $\hat{z}(x,t) = [\,\hat{z}_1(x,t)\ \hat{z}_2(x,t)\,]^{\mathrm{T}}$ on the domain $(x,t) \in [0,1] \times \mathbb{R}^+$. In order to guarantee the convergence of the observer, the corresponding *observer error system* has to be exponentially stabilised:

$$\dot{e}_d(t) = S_de_d(t) - \begin{bmatrix} l_{d1} & l_{d2} \end{bmatrix}\begin{bmatrix} e_{z1}(0,t) \\ e_{z2}(0,t) \end{bmatrix}, \tag{43}$$

$$\partial_t\begin{bmatrix} e_{z1}(x,t) \\ e_{z2}(x,t) \end{bmatrix} = \partial_x\begin{bmatrix} e_{z1}(x,t) \\ e_{z2}(x,t) \end{bmatrix} + \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}\begin{bmatrix} e_{z1}(x,t) \\ e_{z2}(x,t) \end{bmatrix} + g(x)\begin{bmatrix} e_{z1}(0,t) \\ e_{z2}(0,t) \end{bmatrix} + \int_0^x f(x,y)\begin{bmatrix} e_{z1}(y,t) \\ e_{z2}(y,t) \end{bmatrix}\mathrm{d}y + \begin{bmatrix} m_1(x)p_{d1}^{\mathrm{T}} \\ m_2(x)p_{d2}^{\mathrm{T}} \end{bmatrix}e_d(t) - L(x)\begin{bmatrix} e_{z1}(0,t) \\ e_{z2}(0,t) \end{bmatrix}, \tag{44}$$

$$\begin{bmatrix} e_{z1}(1,t) \\ e_{z2}(1,t) \end{bmatrix} = \begin{bmatrix} a_1p_{d1}^{\mathrm{T}} \\ a_2p_{d2}^{\mathrm{T}} \end{bmatrix}e_d(t). \tag{45}$$

Observe that the observer error systems (43)–(44) has similar structure with (1)–(2), therefore the same state transformation in Section 3 can be applied, i.e. $e_\varpi(x,t) = \begin{bmatrix} e_{\varpi 1}(x,t) \\ e_{\varpi 2}(x,t) \end{bmatrix} = M_\varpi(x)\begin{bmatrix} e_{z1}(x,t) \\ e_{z2}(x,t) \end{bmatrix}$. As a result, the corresponding transformed error system can be obtained:

$$\dot{e}_d(t) = S_de_d(t) - [\widehat{l}_{d1} \quad \widehat{l}_{d2}]e_\varpi(0,t), \tag{46}$$

$$\partial_te_\varpi(x,t) = \partial_xe_\varpi(x,t) + \begin{bmatrix} g_1(x) & 0 \\ 0 & g_2(x) \end{bmatrix}e_\varpi(0,t) + \int_0^x \begin{bmatrix} f_1(x,y) & 0 \\ 0 & f_2(x,y) \end{bmatrix}e_\varpi(y,t)\mathrm{d}y + \begin{bmatrix} \mu_{d1}^{\mathrm{T}}(x) \\ \mu_{d2}^{\mathrm{T}}(x) \end{bmatrix}e_d(t) - L_m(x)e_\varpi(0,t), \tag{47}$$

$$e_\varpi(1,t) = \begin{bmatrix} b_{d1}^{\mathrm{T}} \\ b_{d2}^{\mathrm{T}} \end{bmatrix}e_d(t) \tag{48}$$

with

$$\mu_{d1}^{\mathrm{T}}(x) = \mathrm{e}^{\Upsilon_1 x}\left(m_1(x)p_{d1}^{\mathrm{T}} - \gamma m_2(x)p_{d2}^{\mathrm{T}}\right),$$
$$\mu_{d2}^{\mathrm{T}}(x) = \mathrm{e}^{\Upsilon_2 x}\left(\delta m_1(x)p_{d1}^{\mathrm{T}} + (1-\delta\gamma)\,m_2(x)p_{d2}^{\mathrm{T}}\right),$$
$$b_{d1}^{\mathrm{T}} = \mathrm{e}^{\Upsilon_1}\left(a_1p_{d1}^{\mathrm{T}} - \gamma a_2p_{d2}^{\mathrm{T}}\right),$$
$$b_{d2}^{\mathrm{T}} = \mathrm{e}^{\Upsilon_2}\left(\delta a_1p_{d1}^{\mathrm{T}} + (1-\delta\gamma)\,a_2p_{d2}^{\mathrm{T}}\right).$$

The new output injection gains are expressed in terms of the original injection gains in (44)–(45):

$$[\widehat{l}_{d1} \quad \widehat{l}_{d2}] = [l_{d1} \quad l_{d2}]M_\varpi^{-1}(0), \tag{49}$$

$$L_m(x) = \begin{bmatrix} l_{m1}(x) & l_{m2}(x) \\ l_{m3}(x) & l_{m4}(x) \end{bmatrix} = M_\varpi(x) L(x) M_\varpi^{-1}(0). \quad (50)$$

To solve the stabilisation problem of the *observer error system*, the backstepping approach is applied. The new coordinate $\tilde{e}_\varpi(x,t) = \begin{bmatrix} \tilde{e}_{\varpi 1}(x,t) \\ \tilde{e}_{\varpi 2}(x,t) \end{bmatrix}$ is introduced for the infinite-dimensional subsystems (55)–(56) by inverse *Volterra-type integral transformation:*

$$\begin{aligned} e_\varpi(x,t) &= \tilde{e}_\varpi(x,t) - \int_0^x \begin{bmatrix} p_1(x,y) & 0 \\ 0 & p_2(x,y) \end{bmatrix} \tilde{e}_\varpi(y,t)\,\mathrm{d}y \\ &= \begin{bmatrix} \mathcal{T}_{o1}^{-1}\left(\tilde{e}_{\varpi 1}(t)\right)(x) \\ \mathcal{T}_{o2}^{-1}\left(\tilde{e}_{\varpi 2}(t)\right)(x) \end{bmatrix} = \mathcal{T}_o^{-1}\left(\tilde{e}_\varpi(t)\right)(x) \end{aligned} \quad (51)$$

in such a way that the design of injection gains can be simplified. Assume that the observer kernels $p_1(x,y)$ and $p_2(x,y)$ are the solutions of the inverse *observer kernel BVP:*

$$\begin{aligned} \partial_x \begin{bmatrix} p_1(x,y) \\ p_2(x,y) \end{bmatrix} + \partial_y \begin{bmatrix} p_1(x,y) \\ p_2(x,y) \end{bmatrix} &= \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} \\ &- \int_y^x \begin{bmatrix} f_1(x,\zeta) & 0 \\ 0 & f_2(x,\zeta) \end{bmatrix} \begin{bmatrix} p_1(\zeta,y) \\ p_2(\zeta,y) \end{bmatrix} \mathrm{d}\zeta, \end{aligned} \quad (52)$$

$$\begin{bmatrix} p_1(1,y) \\ p_2(1,y) \end{bmatrix} = 0 \quad (53)$$

on the triangular spatial domain $0 \le y \le x \le 1$. The straightforward derivation yields the following presentation:

$$\dot{e}_d(t) = S_d e_d(t) - [\widehat{l}_{d1} \quad \widehat{l}_{d2}] \tilde{e}_\varpi(0,t), \quad (54)$$

$$\begin{aligned} \partial_t \tilde{e}_\varpi(x,t) = \partial_x \tilde{e}_\varpi(x,t) &- \begin{bmatrix} \tilde{l}_{m1}(x) & \tilde{l}_{m2}(x) \\ \tilde{l}_{m3}(x) & \tilde{l}_{m4}(x) \end{bmatrix} \tilde{e}_\varpi(0,t) \\ &+ \begin{bmatrix} \tilde{\mu}_{d1}^{\mathrm{T}}(x) \\ \tilde{\mu}_{d2}^{\mathrm{T}}(x) \end{bmatrix} e_d(t), \end{aligned} \quad (55)$$

$$\tilde{e}_\varpi(1,t) = \begin{bmatrix} b_{d1}^{\mathrm{T}} \\ b_{d2}^{\mathrm{T}} \end{bmatrix} e_d(t) \quad (56)$$

and the observer gains:

$$\begin{aligned} l_{m1}(x) &= g_1(x) + \mathcal{T}_{o1}^{-1}\left(\tilde{l}_{m1}\right) - p_1(x,0), \\ l_{m2}(x) &= \mathcal{T}_{o1}^{-1}\left(\tilde{l}_{m2}\right), \quad l_{m3}(x) = \mathcal{T}_{o2}^{-1}\left(\tilde{l}_{m3}\right), \\ l_{m4}(x) &= g_2(x) + \mathcal{T}_{o2}^{-1}\left(\tilde{l}_{m4}\right) - p_2(x,0), \end{aligned} \quad (57)$$

where the coefficients are $\tilde{\mu}_{d1}^{\mathrm{T}}(x) = \mathcal{T}_{o1}(\mu_{d1}^{\mathrm{T}})$ and $\tilde{\mu}_{d2}^{\mathrm{T}}(x) = \mathcal{T}_{o2}(\mu_{d2}^{\mathrm{T}})$. The new output injection gains $\tilde{l}_{m1}(x), \tilde{l}_{m2}(x), \tilde{l}_{m3}(x)$ and $\tilde{l}_{m4}(x)$ in (55) and (57) are needed as an additional degree of freedom for the further design. The transformations $\mathcal{T}_{o1}$ and $\mathcal{T}_{o2}$ are given in $\tilde{\mu}_{d1}^{\mathrm{T}}(x)$ and $\tilde{\mu}_{d2}^{\mathrm{T}}(x)$ by *Volterra-type integral transformation*:

$$\begin{aligned} \mathcal{T}_o\left(\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\right)(x) &= \begin{bmatrix} \mathcal{T}_{o1}(h_1)(x) \\ \mathcal{T}_{o2}(h_2)(x) \end{bmatrix} \\ &= \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} + \int_0^x \begin{bmatrix} q_1(x,y) & 0 \\ 0 & q_2(x,y) \end{bmatrix} \\ &\quad \times \begin{bmatrix} h_1(y) \\ h_2(y) \end{bmatrix} \mathrm{d}y \end{aligned} \quad (58)$$

Therein, the corresponding *observer kernel BVP* for $q_1(x,y)$ and $q_2(x,y)$ is:

$$\begin{aligned} \partial_x \begin{bmatrix} q_1(x,y) \\ q_2(x,y) \end{bmatrix} + \partial_y \begin{bmatrix} q_1(x,y) \\ q_2(x,y) \end{bmatrix} &= \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} \\ &+ \int_y^x \begin{bmatrix} f_1(\zeta,y) & 0 \\ 0 & f_2(\zeta,y) \end{bmatrix} \begin{bmatrix} q_1(x,\zeta) \\ q_2(x,\zeta) \end{bmatrix} \mathrm{d}\zeta, \end{aligned} \quad (59)$$

$$\begin{bmatrix} q_1(1,y) \\ q_2(1,y) \end{bmatrix} = 0. \quad (60)$$

In order to decouple the PDE subsystems (55)–(56) from the ODE subsystem (54), the new error coordinates are introduced:

$$\varepsilon_\varpi(x,t) = \begin{bmatrix} \varepsilon_{\varpi 1}(x,t) \\ \varepsilon_{\varpi 2}(x,t) \end{bmatrix} = \begin{bmatrix} \tilde{e}_{\varpi 1}(x,t) \\ \tilde{e}_{\varpi 2}(x,t) \end{bmatrix} - \begin{bmatrix} \tilde{n}_1^{\mathrm{T}}(x) \\ \tilde{n}_2^{\mathrm{T}}(x) \end{bmatrix} e_d(t). \quad (61)$$

Applying (61) into (54)–(56) yields a *triangular observer error system* on the domain $(x,t) \in [0,1] \times \mathbb{R}^+$:

$$\begin{aligned} \dot{e}_d(t) &= \left(S_d - [\widehat{l}_{d1} \quad \widehat{l}_{d2}] \begin{bmatrix} \tilde{n}_1^{\mathrm{T}}(0) \\ \tilde{n}_2^{\mathrm{T}}(0) \end{bmatrix}\right) e_d(t) \\ &\quad - [\widehat{l}_{d1} \quad \widehat{l}_{d2}] \varepsilon_\varpi(0,t), \end{aligned} \quad (62)$$

$$\partial_t \varepsilon_\varpi(x,t) = \partial_x \varepsilon_\varpi(x,t), \quad x \in [0,1], \quad (63)$$

$$\varepsilon_\varpi(1,t) = 0, \quad t > 0, \quad (64)$$

if $\begin{bmatrix} \tilde{n}_1^{\mathrm{T}}(x) \\ \tilde{n}_2^{\mathrm{T}}(x) \end{bmatrix}$ is the solution of the following *triangular decoupling BVP* on spatial domain $x \in [0,1]$:

$$d_x \begin{bmatrix} \tilde{n}_1^{\mathrm{T}}(x) \\ \tilde{n}_2^{\mathrm{T}}(x) \end{bmatrix} - \begin{bmatrix} \tilde{n}_1^{\mathrm{T}}(x) \\ \tilde{n}_2^{\mathrm{T}}(x) \end{bmatrix} S_d + \begin{bmatrix} \tilde{\mu}_{d1}^{\mathrm{T}}(x) \\ \tilde{\mu}_{d2}^{\mathrm{T}}(x) \end{bmatrix} = 0, \quad (65)$$

$$\begin{bmatrix} \tilde{n}_1^{\mathrm{T}}(1) \\ \tilde{n}_2^{\mathrm{T}}(1) \end{bmatrix} = \begin{bmatrix} b_{d1}^{\mathrm{T}} \\ b_{d2}^{\mathrm{T}} \end{bmatrix} \quad (66)$$

and $\tilde{l}_{m1}(x), \tilde{l}_{m2}(x), \tilde{l}_{m3}(x)$ and $\tilde{l}_{m4}(x)$ satisfy the condition:

$$\begin{aligned} \tilde{l}_{m1}(x) &= \tilde{n}_1^{\mathrm{T}}(x) \widehat{l}_{d1}, \tilde{l}_{m2}(x) = \tilde{n}_1^{\mathrm{T}}(x) \widehat{l}_{d2}, \\ \tilde{l}_{m3}(x) &= \tilde{n}_2^{\mathrm{T}}(x) \widehat{l}_{d1}, \tilde{l}_{m4}(x) = \tilde{n}_2^{\mathrm{T}}(x) \widehat{l}_{d2}. \end{aligned} \quad (67)$$

The simple form of BVP (65)–(66) makes the following existence result easy to prove.

Lemma 5.1 (Triangular Decoupling BVP): *The triangular decoupling BVP (65)–(66) always has a unique classical solution.*

Proof: The BVP (65)–(66) has the same form with (26)–(27) with replacing S and (27) by S_d and a Dirichlet boundary condition (66). Then, with $\lambda_{d,i}$, $i = 1, \ldots, n_d$ denoting eigenvalues of S_d, the solvability condition $e^{\lambda_{d,i}} \neq 0$ can be obtained from the proof part of Lemma 4.1. The factor that this condition always holds for any value of $\lambda_{d,i}$ yields to the conclusion of lemma. ■

Remark 5.1: From (57) and (67), the observer injection gains for (40)–(42) can be obtained:

$$[l_{d1} \quad l_{d2}] = [\widehat{l}_{d1} \quad \widehat{l}_{d2}] M_{\varpi}(0), \tag{68}$$

$$L(x) = M_{\varpi}^{-1}(x) \begin{bmatrix} l_{m1}(x) & l_{m2}(x) \\ l_{m3}(x) & l_{m4}(x) \end{bmatrix} M_{\varpi}(0) \tag{69}$$

with $l_{m1}(x) = g_1(x) + \mathcal{T}_{o1}^{-1}(\tilde{n}_1^{\mathrm{T}}(x))\widehat{l}_{d1} - p_1(x, 0)$, $l_{m2}(x) = \mathcal{T}_{o1}^{-1}(\tilde{n}_1^{\mathrm{T}}(x))\widehat{l}_{d2}$, $l_{m3}(x) = \mathcal{T}_{o2}^{-1}(\tilde{n}_2^{\mathrm{T}}(x))\widehat{l}_{d1}$ and $l_{m4}(x) = g_2(x) + \mathcal{T}_{o2}^{-1}(\tilde{n}_2^{\mathrm{T}}(x))\widehat{l}_{d2} - p_2(x, 0)$. $\widehat{l}_{d1}$ and $\widehat{l}_{d2}$ can be determined by an eigenvalue assignment for $\left(S_d - [\widehat{l}_{d1} \;\; \widehat{l}_{d2}]\left[\begin{smallmatrix} \tilde{n}_1^{\mathrm{T}}(0) \\ \tilde{n}_2^{\mathrm{T}}(0) \end{smallmatrix}\right]\right)$ given that $\left(\left[\begin{smallmatrix} \tilde{n}_1^{\mathrm{T}}(0) \\ \tilde{n}_2^{\mathrm{T}}(0) \end{smallmatrix}\right], S_d\right)$ is observable.

The observability of $\left(\left[\begin{smallmatrix} \tilde{n}_1^{\mathrm{T}}(0) \\ \tilde{n}_2^{\mathrm{T}}(0) \end{smallmatrix}\right], S_d\right)$ can be assured whenever the conditions in the following lemma are satisfied.

Lemma 5.2 (Observability): *The numerator of the transfer matrix $F_{dm}(s) = N_d(s)/D_d(s)$ of (1)–(2) and (4) from $\left[\begin{smallmatrix} d_1(t) \\ d_2(t) \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} y_{m1}(t) \\ y_{m2}(t) \end{smallmatrix}\right]$ is*

$$N_d(s) = \begin{bmatrix} f_{m1}(s) & f_{m2}(s) \\ f_{m3}(s) & f_{m4}(s) \end{bmatrix} \tag{70}$$

with

$$f_{m1}(s) = a_1 e^{\Upsilon_1 - s} + \int_0^1 e^{-sy}\mathcal{T}_{o1}\left(e^{\Upsilon_1(\cdot)} m_1\right)(y)\,dy,$$

$$f_{m2}(s) = -\gamma a_2 e^{\Upsilon_1 - s} - \gamma \int_0^1 e^{-sy}\mathcal{T}_{o1}\left(e^{\Upsilon_1(\cdot)} m_2\right)(y)\,dy,$$

$$f_{m3}(s) = \delta a_1 e^{\Upsilon_2 - s} + \delta \int_0^1 e^{-sy}\mathcal{T}_{o2}\left(e^{\Upsilon_2(\cdot)} m_1\right)(y)\,dy,$$

$$f_{m4}(s) = (1 - \delta\gamma) a_2 e^{\Upsilon_2 - s} + (1 - \delta\gamma)\int_0^1 e^{-sy}\mathcal{T}_{o2}\left(e^{\Upsilon_2(\cdot)} m_2\right)(y)\,dy.$$

Consequently, with $v_{d,i}$ and $\lambda_{d,i}$, $i = 1, \ldots, n_{d,i}$ denoting eigenfunctions and eigenvalues of S_d, respectively, the pair $(\left[\begin{smallmatrix} \tilde{n}_1^{\mathrm{T}}(0) \\ \tilde{n}_2^{\mathrm{T}}(0) \end{smallmatrix}\right]$, $S_d)$ is observable iff

$$N_d(\lambda_{d,i}) \begin{bmatrix} p_{d1}^{\mathrm{T}} v_{d,i} \\ p_{d2}^{\mathrm{T}} v_{d,i} \end{bmatrix} \neq 0, \quad i = 1, \ldots, n_d. \tag{71}$$

Proof: From the equivalence between (1)–(4) and (11)–(13), then the transfer function from $\left[\begin{smallmatrix} d_1(t) \\ d_2(t) \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} y_{m1}(t) \\ y_{m2}(t) \end{smallmatrix}\right]$ can be computed through (11)–(13). By applying the *Volterra transformation* $\bar{\varpi}(x, t) = \left[\begin{smallmatrix} \bar{\varpi}_1(x,t) \\ \bar{\varpi}_2(x,t) \end{smallmatrix}\right] = \mathcal{T}_o(\varpi(x, t))(x)$, (see (58)), (11)–(12) and (13) becomes

$$\begin{aligned} \partial_t\bar{\varpi}(x, t) &= \partial_x\bar{\varpi}(x, t) + \mathrm{diag}\left(\tilde{g}_1(x), \tilde{g}_2(x)\right)\bar{\varpi}(0, t) \\ &+ \begin{bmatrix} \mathcal{T}_{o1}\left(e^{\Upsilon_1(\cdot)} m_1\right)(x) & -\gamma\mathcal{T}_{o1}\left(e^{\Upsilon_1(\cdot)} m_2\right)(x) \\ \delta\mathcal{T}_{o2}\left(e^{\Upsilon_2(\cdot)} m_1\right)(x) & (1 - \delta\gamma)\mathcal{T}_{o2}\left(e^{\Upsilon_2(\cdot)} m_2\right)(x) \end{bmatrix} \\ &\times \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix}, \end{aligned} \tag{72}$$

$$\bar{\varpi}(1, t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} a_1 e^{\Upsilon_1} & -\gamma a_2 e^{\Upsilon_1} \\ \delta a_1 e^{\Upsilon_2} & (1 - \delta\gamma) a_2 e^{\Upsilon_2} \end{bmatrix} \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix}, \tag{73}$$

$$\begin{bmatrix} y_{m1}(t) \\ y_{m2}(t) \end{bmatrix} = \bar{\varpi}(0, t), \tag{74}$$

where $\tilde{g}_1(x) = \mathcal{T}_{o1}(g_1)(x) - q_1(x, 0)$, $\tilde{g}_2(x) = \mathcal{T}_{o2}(g_2)(x) - q_2(x, 0)$. As a consequence, the representation of transfer function matrix $F_{dm}(s) = N_d(s)/D_d(s)$ can be derived in closed-form, where D_d is an irrational matrix denominator. From Theorem 6.2–5 in Kailath (1980), the above pair is observable iff $\left[\begin{smallmatrix} \tilde{n}_{1,i}^{*}(0) \\ \tilde{n}_{2,i}^{\mathrm{T}}(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \tilde{n}_1^{\mathrm{T}}(0) \\ \tilde{n}_2^{\mathrm{T}}(0) \end{smallmatrix}\right] v_{d,i} \neq 0$, $i = 1, \ldots, n_d$, since the eigenvalues of S_d are distinct. Applying the similar method in the proof of Lemma 4.1, the result $\left[\begin{smallmatrix} \tilde{n}_{1,i}^{*}(0) \\ \tilde{n}_{2,i}^{*}(0) \end{smallmatrix}\right] = N_d(\lambda_{d,i}) \left[\begin{smallmatrix} p_{d1}^{\mathrm{T}} v_{d,i} \\ p_{d2}^{\mathrm{T}} v_{d,i} \end{smallmatrix}\right]$ can be obtained for $i = 1, \ldots, n_d$. This completes the proof. ■

According to Remark 4.1 and results in this section, the output-feedback regulator achieving output regulation can be obtained:

$$\begin{aligned} \begin{bmatrix} \chi_1(t) \\ \chi_2(t) \end{bmatrix} &= M_{\varpi}^{-1}(1) \begin{bmatrix} \pi_{v1}^{\mathrm{T}} \\ \pi_{v2}^{\mathrm{T}} \end{bmatrix} \hat{v}(t) \\ &+ M_{\varpi}^{-1}(1) \int_0^1 \begin{bmatrix} k_1(y) & 0 \\ 0 & k_2(y) \end{bmatrix} M_{\varpi}(y) \begin{bmatrix} \hat{z}_1(y, t) \\ \hat{z}_2(y, t) \end{bmatrix} dy. \end{aligned} \tag{75}$$

consisting of observers (39)–(42) with the feedback (75). Here, $\hat{v}(t) = \mathrm{col}(\hat{v}_r(t), \hat{v}_d(t))$ is state estimation of $v(t)$.

In the next theorem, it is shown that for the separation principle holds for the resulting observer-based regulator achieving output regulation.

Theorem 5.1: *Consider the output-feedback regulator* (39)–(42) *with* (68)–(69) *and* (75). *Then, the output regulation* (9) *is achieved in the norm $\|\cdot\|_{Z_{cl}} = (\|\cdot\|_{\mathbb{C}^{n_r}}^2 + \|\cdot\|_{\mathbb{C}^{n_d}}^2 + \|\cdot\|_Z^2 + \|\cdot\|_Z^2)^{1/2}$, i.e. the closed-loop state $e_{cl} = col(e_r, e_d, e_{\varpi}, \hat{\bar{e}})$ with $\hat{\bar{e}} = \left[\begin{smallmatrix} \hat{\bar{\varpi}}_1 \\ \hat{\bar{\varpi}}_2 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \bar{\pi}_1 \\ \bar{\pi}_2 \end{smallmatrix}\right] \hat{v}(t)$ and $\left[\begin{smallmatrix} \bar{\pi}_1(x) \\ \bar{\pi}_2(x) \end{smallmatrix}\right] = \mathcal{T}_c^{-1}(\left[\begin{smallmatrix} \pi_1 \\ \pi_2 \end{smallmatrix}\right])(x)$ satisfies*

$$\|e_{cl}(t)\|_{Z_{cl}} \leq M_{cl}\, e^{-\alpha_{cl} t}\|e_{cl}(0)\|_{Z_{cl}}, \quad t \geq 0 \tag{76}$$

for all $e_{cl}(0) \in Z_{cl} = \mathbb{C}^{n_r} \oplus \mathbb{C}^{n_d} \oplus Z \oplus Z$ and a constant $M_{cl} > 0$.

For the proof, refer to Appendix 3.

6. Representative physical application

In this section, two examples are given to verify proposed regulators: state and output-feedback regulators. Before moving to the demonstration of detailed examples, we first summarise the algorithm (design steps) for these regulators in Table 1.

6.1 Heat-exchanger equation (state feedback regulator design)

The system of parallel-flow heat exchange equations shown in Figure 2 is considered in this subsection, where $f(x,y)=0$, $g(x)=\bar{b}\,\mathrm{e}^{\eta x}$, $\eta=1, \bar{b}=0.5$ with $\alpha_1=-\alpha_2=-0.4$ and $\alpha_3=-\alpha_4=0.6$. The heat exchanger given in Figure 2 is concurrent with $z_1(x,t)$ and $z_2(x,t)$ denoting temperature distribution along inner and outer pipes at time t. $\chi_1(t)$, $\chi_2(t)$ and $y_1(t)$, $y_2(t)$ are inlet temperatures and outlet temperatures for flows FL_1 and FL_2, respectively.

The boundary controlled system is given by

$$\partial_t \begin{bmatrix} z_1(x,t) \\ z_2(x,t) \end{bmatrix} = \partial_x \begin{bmatrix} z_1(x,t) \\ z_2(x,t) \end{bmatrix} + \begin{bmatrix} -0.4 & 0.4 \\ 0.6 & -0.6 \end{bmatrix} \begin{bmatrix} z_1(x,t) \\ z_2(x,t) \end{bmatrix}$$

$$+\, g(x) \begin{bmatrix} z_1(0,t) \\ z_2(0,t) \end{bmatrix} + \begin{bmatrix} m_1(x)d_1(t) \\ m_2(x)d_2(t) \end{bmatrix},$$

$$\times\ (x,t) \in [0,1] \times \mathbb{R}^+$$

$$\begin{bmatrix} z_1(1,t) \\ z_2(1,t) \end{bmatrix} = \begin{bmatrix} \chi_1(t)+a_1d_1(t) \\ \chi_2(t)+a_2d_2(t) \end{bmatrix}, \quad \begin{bmatrix} z_1(x,0) \\ z_2(x,0) \end{bmatrix} = \begin{bmatrix} z_{10}(x) \\ z_{20}(x) \end{bmatrix}$$

with initial conditions $z_{10}(x)=(1+0.4\,\mathrm{e}^x)\sin(3\pi x)$ and $z_{20}(x)=(1-0.6\,\mathrm{e}^x)\sin(3\pi x)$. Without loss of generality, the controlled outputs are in-domain and pointwise with $x_0=0.5$, see (3). For simplicity, it is assumed that no disturbance exists, i.e. $d_1(t)=0$ and $d_2(t)=0$. The reference signals for $y_1(t)$ and $y_2(t)$ are step-like and sinusoidalwith $y_{r1}(t)=4$ and $y_{r2}(t)=9\sin(w_0t)$, $w_0=2$. Therefore, it can be modelled by (5) and (7), with $v_{r0}=[4\ \ 0\ \ 3]^{\mathrm{T}}$, $q_{r1}^{\mathrm{T}}=[1\ \ 0\ \ 0]$, $q_{r2}^{\mathrm{T}}=[0\ \ 3\ \ 0]$ and $S_r=\begin{bmatrix} 0 & 0 & \\ 0 & 2 & \\ 0 & -2 & 0 \end{bmatrix}$. In this example, our target is to design a full state feedback regulator (39) such that thetracking target (10) is achieved exponentially. In this case, the new parameters in Lemma 3.1 are obtained: $\beta=1$, $\gamma=1$, $\delta=0.6$, $\Upsilon_1=-1$ and $\Upsilon_2=0$. Thereby, the new coefficients in the equivalent systems (11)–(13) are computed: $g_1(x)=0.5$, $g_2(x)=0.5\,\mathrm{e}^x$ and $f_1(x,y)=f_2(x,y)=0$. As shown in Krstic and Smyshlyaev (2008), the solution of the *control-kernel boundary value problem (BVP)* (18)-(19) is given by $\kappa_1(x,y)=-0.5\,\mathrm{e}^{0.5(x-y)}$ and $\kappa_2(x,y)=-0.5\,\mathrm{e}^{1.5(x-y)}$. Moreover, the solution of the *kernel BVP* (24)–(25) is computed as $l_p(x,y)=-0.5$ and $l_q(x,y)=-0.5\,\mathrm{e}^{(x-y)}$. Then, through Lemma 4.1 and (26)–(27), the feedforward gains are obtained. The simulation is performed in MATLAB using finite difference schemes with time interval $\Delta t=0.001$ and space interval $\Delta x=0.02$.

The state feedback regulator can be applied and the results are shown in Figures 1–5.

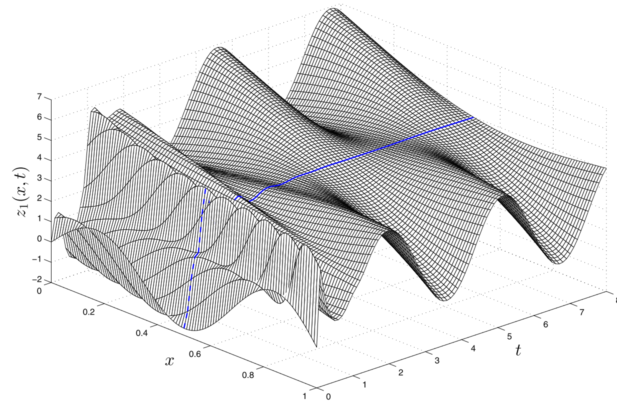

Figure 1. The evolution of state $z_1(x,t)$ under the state feedback control law (38).

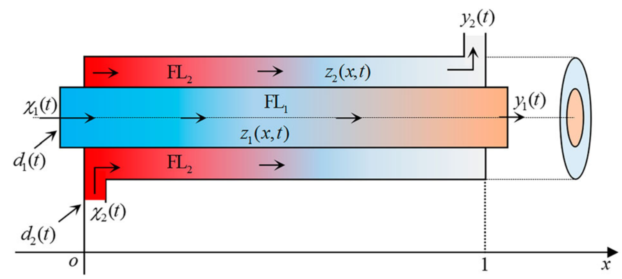

Figure 2. Concurrent heat-exchanger systems geometry.

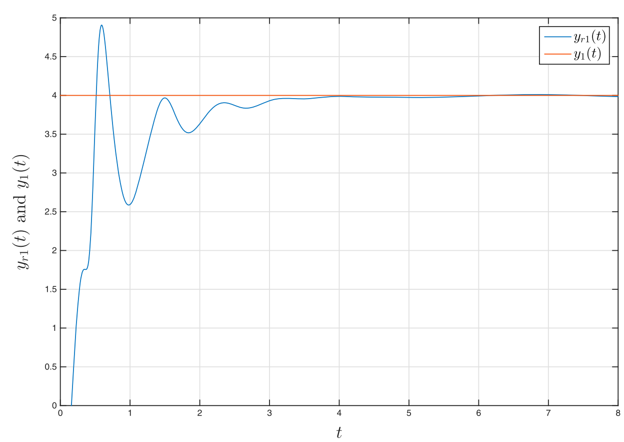

Figure 3. The reference signal $y_{r1}(t)$ and the controlled output $y_1(t)=z_1(0.5,t)$ under the state feedback control law (38).

Table 1. Tuning parameters design for the state and output-feedback regulators.

Algorithm: Design of state and output-feedback regulators
step 1: Based on Lemma 3.1, compute $\Upsilon_1, \Upsilon_2, \delta, \gamma$ and M_{ϖ};   **step 2**: Solve (18)–(19) for κ_1, κ_2 and further k_1 and k_2 in (16);   **step 3**: Solve regulator Equation (26)–(27) for $\pi_1^{\mathrm{T}}, \pi_2^{\mathrm{T}}$ and thus $\pi_{v1}^{\mathrm{T}}, \pi_{v2}^{\mathrm{T}}$ in (16);   **step 4**: Through (38) in **Remark** 4.1, construct the state feedback regulator.   **step 5**: Solve decoupling BVP (65)–(66) for $\tilde{n}_1^{\mathrm{T}}$ and $\tilde{n}_2^{\mathrm{T}}$;   **step 6**: Determine $\widehat{l}_{d2}$ and $\widehat{l}_{d2}$ to stabilise the matrix $(S_d - [\widehat{l}_{d1}\ \ \widehat{l}_{d2}]\begin{bmatrix} \tilde{n}_1^{\mathrm{T}}(0) \\ \tilde{n}_2^{\mathrm{T}}(0) \end{bmatrix})$;   **step 7**: Through (68), compute l_{d1} and l_{d2} for (40);   **step 8**: According to (67), compute $\tilde{l}_{m1}, \tilde{l}_{m2}, \tilde{l}_{m3}$ and $\tilde{l}_{m4}$, and further calculate l_{m1}, l_{m2}, l_{m3} and l_{m4} from (57);   **step 9**: Through (69) or (50), compute $L(x)$ for (41);   **step 10**: Find L_r to stabilise the matrix $S_r - L_r\begin{bmatrix} q_{r1}^{\mathrm{T}} \\ q_{r2}^{\mathrm{T}} \end{bmatrix}$;   **step 11**: Based on the separation principle, construct the output-feedback regulator consisting of (39), (40), (41), (42) and (75).

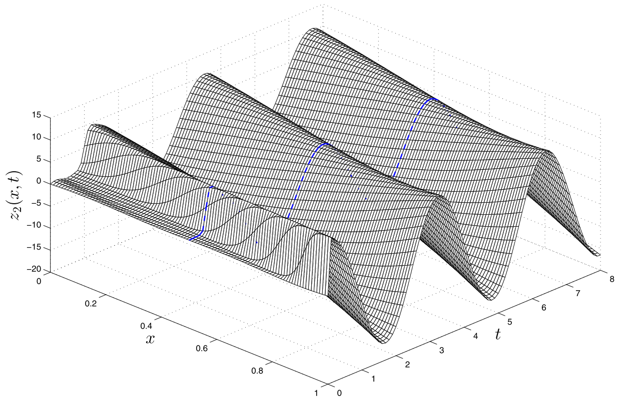

Figure 4. The evolution of state $z_2(x,t)$ under the state feedback control law (38).

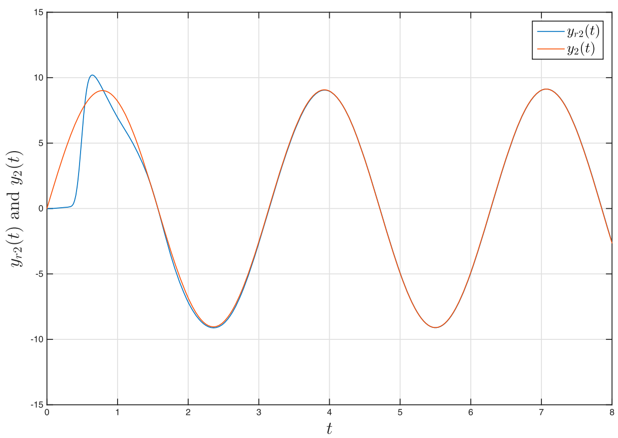

Figure 5. The reference signal $y_{r2}(t)$ and the controlled output $y_2(t) = z_2(0.5,t)$ under the state feedback control law (38).

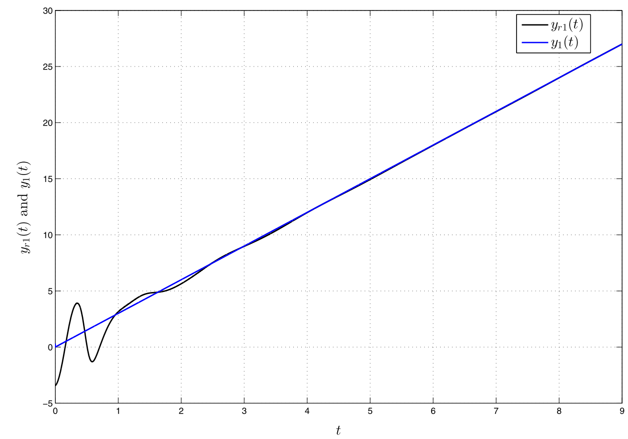

Figure 6. The ramp-like reference signal $y_{r1}(t) = 3t$ and the controlled output $y_1(t) = z_2(0.5,t)$ under the output-feedback control law (75).

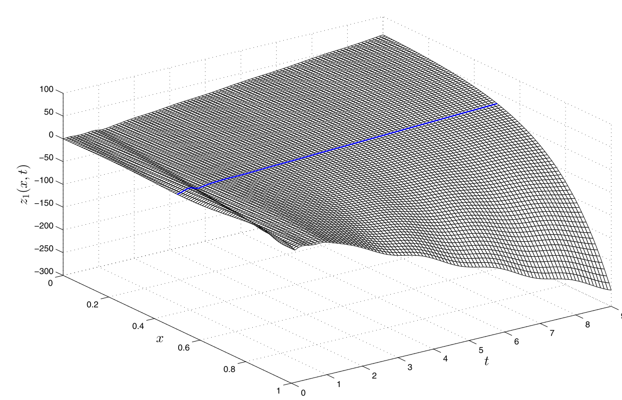

Figure 7. The evolution of state $z_1(x,t)$ under the output-feedback control law (75).

6.2 Reactor equations of plug-flow type (output-feedback regulator design)

In this section we consider the system of reactor equations of plug-flow where $f(x,y) = \nu\, e^{\theta(x-y)}, \nu > 0, \theta \in \mathbb{R}$, $g(x) \equiv 0$ with $\alpha_2 = \alpha_4 = 0, \alpha_1 = -\kappa < 0$ and $\alpha_3 = \sigma\kappa$ for some $\sigma, \kappa > 0$. In this example, the parameters are given as $\nu = 5, \theta = 4, \kappa = 2$ and $\sigma = 4$. Then, the boundary controlled system is given on the domain $(x,t) \in [0,1] \times \mathbb{R}^+$ by

$$\partial_t \begin{bmatrix} z_1(x,t) \\ z_2(x,t) \end{bmatrix} = \partial_x \begin{bmatrix} z_1(x,t) \\ z_2(x,t) \end{bmatrix} + \begin{bmatrix} -\kappa & 0 \\ \sigma\kappa & 0 \end{bmatrix} \begin{bmatrix} cz_1(x,t) \\ z_2(x,t) \end{bmatrix} + \int_0^x \nu\, e^{\theta(x-y)} \begin{bmatrix} z_1(y,t) \\ z_2(y,t) \end{bmatrix} dy + \begin{bmatrix} m_1(x)d_1(t) \\ m_2(x)d_2(t) \end{bmatrix}, \tag{77}$$

$$\begin{bmatrix} z_1(1,t) \\ z_2(1,t) \end{bmatrix} = \begin{bmatrix} \chi_1(t) + a_1 d_1(t) \\ \chi_2(t) + a_2 d_2(t) \end{bmatrix}, \begin{bmatrix} z_1(x,0) \\ z_2(x,0) \end{bmatrix} = \begin{bmatrix} z_{10}(x) \\ z_{20}(x) \end{bmatrix} \tag{78}$$

with initial conditions $z_{10}(x) = x(1.5 - 2x), z_{20}(x) = 2x(1 + 3x)$. The controlled outputs are in-domain pointwise $x_0 = 0.75$. For simplicity, we assume sinusoid disturbances effecting the entire system through the entire space and the input boundary, e.g. $d_1(t) = 2sin(3t)$ and $d_2(t) = 3cos(3t)$. Therefore, the disturbance model is second order with $S_d = \left[\begin{smallmatrix} 0 & 3 \\ -3 & 0 \end{smallmatrix}\right]$, $p_{d_1}^{\mathrm{T}} = [2\ \ 0]$ and $p_{d_2}^{\mathrm{T}} = [0\ \ 3]$. The disturbance locations are given by $m_1(x) = 0.4\, e^x$ and $m_2(x) = 2x$. The reference signals for $y_1(t)$ and $y_2(t)$ are assumed to be a ramp-like and step-like, e.g. $y_{r1}(t) = 3t$ and $y_{r2}(t) = 4$. These reference signals can be generated by the signal process (5)–(7) with $S_r = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, $q_{r1}^{\mathrm{T}} = [0\ \ 3]$ and $q_{r2}^{\mathrm{T}} = [4\ \ 0]$. In this example, the main objective is to design output-feedback regulator such that the tracking errors attenuate to zero exponentially despite of large disturbances $d_1(t)$ and $d_2(t)$. By solving *Triangular Decoupling BVP* (65)–(66), the solution $\left[\begin{smallmatrix} \tilde{n}_1^{\mathrm{T}}(x) \\ \tilde{n}_2^{\mathrm{T}}(x) \end{smallmatrix}\right]$ are obtained. According to Remark 5.1, $[\widehat{l}_{d1}\ \ \widehat{l}_{d2}]$ can be found such that $(S_d - [\widehat{l}_{d1}\ \ \widehat{l}_{d2}] \left[\begin{smallmatrix} \tilde{n}_1^{\mathrm{T}}(0) \\ \tilde{n}_2^{\mathrm{T}}(0) \end{smallmatrix}\right])$ is exponentially stable. From Remark 5.1, the observer gains $L(x)$ and $[l_{d1}\ \ l_{d2}] = [\widehat{l}_{d1}\ \ \widehat{l}_{d2}]$ can be directly computed. In Figures 6–9, the output-feedback regulator is applied and its performance is verified via the tracking results. Moreover, Figure 8 also show the disturbance rejection behaviour clearly since the plant output converges to a constant steady state in presence of time varying (sinusoid) disturbances.

Remark 6.1: If observe results shown in Figures 3, 5, 6 and 8, it can be noticed that tracking performances on sinusoid and ramp-like reference signals are better compared with step-like reference signals. This qualitative different behaviour is due to

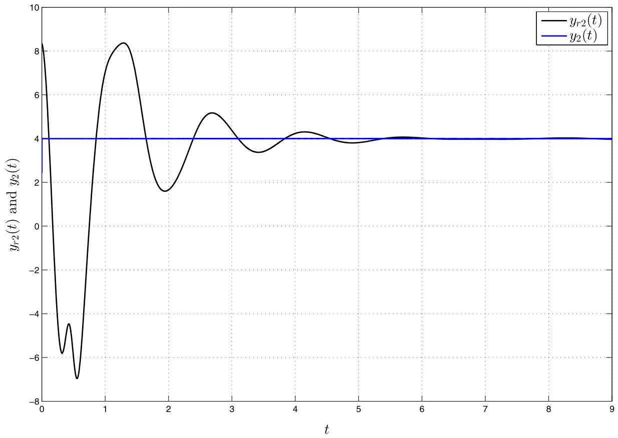

Figure 8. The step-like reference signal $y_{r2}(t) = 4$ and the controlled output $y_2(t) = z_2(0.5, t)$ under the output-feedback control law (75).

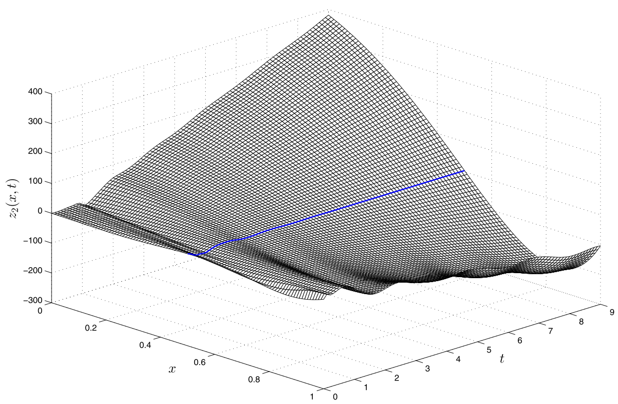

Figure 9. The evolution of state $z_2(x, t)$ under the output-feedback control law (75).

the smooth nature of reference signals applied, such as differentiability. For example, sinusoid reference signals are infinitely differentiable and the corresponding inputs change the plant dynamical performances smoothly. However, in the case of step-like signals, they lack the property of infinite smoothness, and thus their resulting feedforward inputs unevenly effect the dynamical behaviour.

7. Conclusion

This work proposed combination of backstepping approach and internal model theory account for the design problems of output regulators. In order to make the proposed methods more realistic, we extend the exosystem so that it can generates ramp and more general polynomial signals as well as step-like and sinusoidal signals. To ensure the feasibility of the considered regulators, corresponding regulator equations and triangular decoupling BVP are taken into account and solvability conditions are established. In particular, for the observer-based feedforward regulator design, the observability of observer is explored and the corresponding conditions are provided. In simulation part, two examples – heat exchanger and plug-flow type reactor equations are considered to verify the performances of proposed regulators. During the design of the proposed regulator, the disturbance location as well as the frequency has to be known. Moreover, model uncertainties may deteriorate the tracking (and disturbance rejection) behaviour. It is still of interest to consider robust output regulator design for DPS systems effected by disturbance with unknown frequency.

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No potential conflict of interest was reported by the authors.

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Appendix 1. Proof of Lemma 3.1

First apply a similarity transformation in order to get an equivalent description whose system operator is triangular. Consider the state transformation defined by the linear operator $J := \begin{bmatrix} I & \gamma I \\ 0 & I \end{bmatrix}$, where γ is a constant to be determined, then new state $\begin{bmatrix} w_1(x,t) \\ w_2(x,t) \end{bmatrix} = J^{-1}\begin{bmatrix} z_1(x,t) \\ z_2(x,t) \end{bmatrix}$ is obtained. Applying the similarity transformation J to the operator A yields the triangular operator $\tilde{A} := J^{-1}AJ$ given by

$$\tilde{A} = \begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}x} + (\alpha_1 - \alpha_3\gamma) I & 0 \\ \alpha_3 I & \dfrac{\mathrm{d}}{\mathrm{d}x} + (\alpha_4 + \alpha_3\gamma) I \end{bmatrix}, \tag{A1}$$

if γ is the solution of the following equation:

$$\alpha_3\gamma^2 - (\alpha_1 - \alpha_4)\gamma - \alpha_2 = 0, \tag{A2}$$

i.e. $\gamma = (\alpha_1 - \alpha_4) \pm \sqrt{\beta}/2\alpha_3$, with $\beta := (\alpha_1 - \alpha_4)^2 + 4\alpha_2\alpha_3 > 0$. Substituting new state $[w_1(x,t), w_2(x,t)]^{\mathrm{T}}$ into the original systems (1)–(4), the description is obtained:

$$\partial_t \begin{bmatrix} w_1(x,t) \\ w_2(x,t) \end{bmatrix} = \partial_x \begin{bmatrix} w_1(x,t) \\ w_2(x,t) \end{bmatrix} + \begin{bmatrix} \Upsilon_1 & 0 \\ \alpha_3 & \Upsilon_2 \end{bmatrix}\begin{bmatrix} w_1(x,t) \\ w_2(x,t) \end{bmatrix} + g(x)\begin{bmatrix} w_1(0,t) \\ w_2(0,t) \end{bmatrix},$$
$$+ \int_0^x f(x,y)\begin{bmatrix} w_1(y,t) \\ w_2(y,t) \end{bmatrix}\mathrm{d}y + \begin{bmatrix} m_1(x)d_1(t) - \gamma m_2(x)d_2(t) \\ m_2(x)d_2(t) \end{bmatrix}, \tag{A3}$$

$$\begin{bmatrix} w_1(1,t) \\ w_2(1,t) \end{bmatrix} = \begin{bmatrix} U_1(t) + a_1d_1(t) - \gamma a_2d_2(t) \\ U_2(t) + a_2d_2(t) \end{bmatrix},$$
$$\begin{bmatrix} w_1(x,0) \\ w_2(x,0) \end{bmatrix} = J^{-1}\begin{bmatrix} z_{10}(x) \\ z_{20}(x) \end{bmatrix}, \tag{A4}$$

$$\begin{aligned} y_1(t) &= w_1(x_0,t) + \gamma w_2(x_0,t), \\ y_2(t) &= w_2(x_0,t) \end{aligned} \tag{A5}$$

with $\Upsilon_1 = (\alpha_1 - \gamma\alpha_3)$, $\Upsilon_2 = (\alpha_4 + \alpha_3\gamma)$, $U_1(t) = \chi_1(t) - \gamma\chi_2(t)$ and $U_2(t) = \chi_2(t)$. In order to decouple the system completely, the following coordinates are introduced:

$$\begin{bmatrix} \tilde{w}_1(x,t) \\ \tilde{w}_2(x,t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \delta & I \end{bmatrix}\begin{bmatrix} w_1(x,t) \\ w_2(x,t) \end{bmatrix} \tag{A6}$$

in which δ has to be determined. By substituting $[\tilde{w}_1(x,t), \tilde{w}_2(x,t)]^{\mathrm{T}}$ in (A3)–(A5), the resulting decoupled system can be obtained:

$$\partial_t \begin{bmatrix} \tilde{w}_1(x,t) \\ \tilde{w}_2(x,t) \end{bmatrix} = \partial_x \begin{bmatrix} \tilde{w}_1(x,t) \\ \tilde{w}_2(x,t) \end{bmatrix} + \begin{bmatrix} \Upsilon_1 & 0 \\ 0 & \Upsilon_2 \end{bmatrix}\begin{bmatrix} \tilde{w}_1(x,t) \\ \tilde{w}_2(x,t) \end{bmatrix} + g(x)\begin{bmatrix} \tilde{w}_1(0,t) \\ \tilde{w}_2(0,t) \end{bmatrix},$$
$$+ \int_0^x f(x,y)\begin{bmatrix} \tilde{w}_1(y,t) \\ \tilde{w}_2(y,t) \end{bmatrix}\mathrm{d}y + \begin{bmatrix} m_1(x)d_1(t) - \gamma m_2(x)d_2(t), \\ \delta m_1(x)d_1(t) + (1 - \delta\gamma)\, m_2(x)d_2(t) \end{bmatrix}, \tag{A7}$$

$$\begin{bmatrix} \tilde{w}_1(1,t) \\ \tilde{w}_2(1,t) \end{bmatrix} = \begin{bmatrix} U_1(t) + a_1d_1(t) - \gamma a_2d_2(t) \\ U_2(t) + \delta U_1(t) + \delta a_1d_1(t) + (1 - \delta\gamma)\, a_2d_2(t) \end{bmatrix},$$
$$\begin{bmatrix} \tilde{w}_1(x,0) \\ \tilde{w}_2(x,0) \end{bmatrix} = \begin{bmatrix} I & -\gamma I \\ \delta & (1 - \delta\gamma)\, I \end{bmatrix}\begin{bmatrix} z_{10}(x) \\ z_{20}(x) \end{bmatrix}, \tag{A8}$$

$$\begin{aligned} y_1(t) &= (1 - \gamma\delta)\, \tilde{w}_1(x_0,t) + \gamma\tilde{w}_2(x_0,t), \\ y_2(t) &= \tilde{w}_2(\tilde{x}_0,t) - \delta\tilde{w}_1(\tilde{x}_0,t), \end{aligned} \tag{A9}$$

if $\Upsilon_1\delta + \alpha_3 = \Upsilon_2\delta$ is satisfied, i.e. $\delta = \pm\alpha_3/\sqrt{\beta}$. Furthermore, we introduce the new variables for (A7)–(A9):

$$\varpi_1(x,t) = \mathrm{e}^{\Upsilon_1 x}\tilde{w}_1(x,t), \varpi_2(x,t) = \mathrm{e}^{\Upsilon_2 x}\tilde{w}_2(x,t). \tag{A10}$$

As a consequence, the decomposed systems (11)–(13) can be obtained and therein, the disturbance input locations become to

$$b_1^{\mathrm{T}} = \mathrm{e}^{\Upsilon_1}\left(a_1p_1^{\mathrm{T}} - \gamma a_2p_2^{\mathrm{T}}\right), \quad b_2^{\mathrm{T}} = \mathrm{e}^{\Upsilon_2}\left(\delta a_1p_1^{\mathrm{T}} + (1 - \delta\gamma)\, a_2p_2^{\mathrm{T}}\right),$$
$$\mu_1^{\mathrm{T}}(x) = \mathrm{e}^{\Upsilon_1 x}\left(m_1(x)p_1^{\mathrm{T}} - \gamma m_2(x)p_2^{\mathrm{T}}\right),$$
$$\mu_2^{\mathrm{T}}(x) = \mathrm{e}^{\Upsilon_2 x}\left(\delta m_1(x)p_1^{\mathrm{T}} + (m_2(x) - \delta\gamma m_2(x))\, p_2^{\mathrm{T}}\right)$$

the new process coefficients are given by $g_1(x) = g(x)\,\mathrm{e}^{\Upsilon_1 x}$, $g_2(x) = g(x)\,\mathrm{e}^{\Upsilon_2 x}$, $f_1(x,y) = f(x,y)\,\mathrm{e}^{\Upsilon_1(x-y)}$, and $f_2(x,y) = f(x,y)\,\mathrm{e}^{\Upsilon_2(x-y)}$. The inputs in new coordinate reduce to $u_1(t) = \mathrm{e}^{\Upsilon_1}U_1(t)$ and $u_2(t) = \mathrm{e}^{\Upsilon_2}(U_2(t) + \delta U_1(t))$.

Appendix 2. Proof of Theorem 4.1

The tracking error systems (28)–(29) can be solved for

$$\tilde{e}(t) = \begin{cases} \tilde{e}_0(x+t), x+t \le 1 \\ 0, x+t > 1 \end{cases} \tag{A11}$$

for $t \ge 0$, $x \in [0,1]$. Given the defined unbounded output operator $\mathcal{C}_e$ and the boundedness of the kernel $\begin{bmatrix} l_1(x,y) \\ l_2(x,y) \end{bmatrix}$, and based on *Cauchy–Schwarz inequality*, there exists a positive constant $M(l_1, l_2) > 0$ (depending on l_1, l_2) such that the norms

$$\begin{aligned} \left\|\tilde{e}(\cdot,t)\right\|_2 &\le \left\|\tilde{e}(0)\right\|_2 \\ |e(t)| &= \left|CT_c^{-1}\left(\tilde{e}(t)\right)\right| \\ &\le \left\|\tilde{e}(x_0,t)\right\|_2 \\ &\quad + \left\|\int_0^{x_0} \mathrm{diag}\left(l_1(x_0,y), l_2(x_0,y)\right)\tilde{e}(y,t)\,\mathrm{d}y\right\|_2 \\ &\le (1 + M(l_1,l_2))\left\|\tilde{e}(0)\right\|_2 \end{aligned}$$

hold for all $t > 0$. In particular,

$$\begin{aligned} \left\|\tilde{e}(\cdot,t)\right\|_2 &= 0, \quad \forall t > 1, \\ |e(t)| &= 0, \quad \forall t > 1. \end{aligned}$$

Obviously, this proves the output regulation (15) for the considered unbounded operator $\mathcal{C}_e$ within finite-time. To prove (37), one can apply

Cauchy–Schwarz inequality and obtain

$$\begin{aligned}\left\|\tilde{e}(\cdot,t)\right\|_2 &= \|T_c\left(\bar{e}(\cdot,t)\right)\|_2\\ &\le \|\bar{e}(\cdot,t)\|_2 + \left\|\int_0^x \mathrm{diag}\left(\kappa_1(x,y),\kappa_2(x,y)\right)\bar{e}(y,t)\,\mathrm{d}y\right\|_2\\ &\le \|\bar{e}(\cdot,t)\|_2 + \sqrt{\begin{array}{c}\int_0^1\int_0^x\left|\bar{e}^{\mathrm{T}}(y,t)\mathrm{diag}\right.\\ \left.\left(\kappa_1(x,y),\kappa_2(x,y)\right)^2\bar{e}(y,t)\right|\mathrm{d}y\,\mathrm{d}x\end{array}}\\ &\le \|\bar{e}(\cdot,t)\|_2 + \sqrt{\int_0^1\int_0^x\left|\bar{\kappa}(x,y)\right|^2\mathrm{d}y\,\mathrm{d}x}\|\bar{e}(\cdot,t)\|_2,\end{aligned}$$

where $\bar{\kappa}(x,y)=\max(|\kappa_1(x,y)|,|\kappa_2(x,y)|)$. Hence, there exists a positive number $c_0(\kappa_1,\kappa_2)$ such that $\|\tilde{e}(\cdot,t)\|_2 \le (1+c_0(\kappa_1,\kappa_2))\|\bar{e}(\cdot,t)\|_2$ and thus $\|\tilde{e}(\cdot,0)\|_2 \le (1+c_0(\kappa_1,\kappa_2))\|\bar{e}(\cdot,0)\|_2$. In addition, because the kernel $\begin{bmatrix} l_1(x,y)\\ l_2(x,y)\end{bmatrix}$ are also bounded in the direct transformation (24). Therefore, there exists a positive constant $c_1(l_1,l_2)$ such that $\|\bar{e}(\cdot,t)\|_2 \le (1+c_1(l_1,l_2))\|\tilde{e}(\cdot,t)\|_2$. As a result, this results in (38) with $M_c(\kappa_1,\kappa_2,l_1,l_2) = (1+c_1(l_1,l_2))(1+c_o(\kappa_1,\kappa_2))$. Moreover, $\|\bar{e}(\cdot,t)\|_2=0,\forall t>1$, which indicates the finite-time stability of error dynamics.

Appendix 3. Proof of Theorem 5.1

Define the state transformation: $\hat{\tilde{\varpi}}(x,t)=\begin{bmatrix}\hat{\tilde{\varpi}}_1\\ \hat{\tilde{\varpi}}_2\end{bmatrix}=\mathcal{T}_c\left(M_\varpi\begin{bmatrix}\hat{z}_1(t)\\ \hat{z}_2(t)\end{bmatrix}\right)(x)$, then the observer (41)–(42) reduces to

$$\partial_t\hat{\tilde{\varpi}}(x,t)=\partial_x\hat{\tilde{\varpi}}(x,t)+\begin{bmatrix}\tilde{\mu}_1^{\mathrm{T}}(x)\\ \tilde{\mu}_2^{\mathrm{T}}(x)\end{bmatrix}\hat{v}(t)+\tilde{L}_{mc}(x)e_\varpi(0,t), \tag{A12}$$

$$\hat{\tilde{\varpi}}(1,t)=\begin{bmatrix}\pi_1^{\mathrm{T}}(1)\\ \pi_2^{\mathrm{T}}(1)\end{bmatrix}\hat{v}(t), \tag{A13}$$

where $\tilde{L}_{mc}(x)=\begin{bmatrix}\tilde{l}_{mc1}(x) & \tilde{l}_{mc2}(x)\\ \tilde{l}_{mc3}(x) & \tilde{l}_{mc4}(x)\end{bmatrix}$ with $\tilde{l}_{mci}(x)=\mathcal{T}_c(l_{mci})(x),\ i=1,2,3,4.$ Define $\hat{\varepsilon}_\varpi=\hat{\tilde{\varpi}}-\begin{bmatrix}\pi_1^{\mathrm{T}}(x)\\ \pi_2^{\mathrm{T}}(x)\end{bmatrix}\hat{v}(t)$ and $\begin{bmatrix}\pi_1^{\mathrm{T}}\\ \pi_2^{\mathrm{T}}\end{bmatrix}=\begin{bmatrix}\pi_{11}^{\mathrm{T}} & \pi_{12}^{\mathrm{T}}\\ \pi_{21}^{\mathrm{T}} & \pi_{22}^{\mathrm{T}}\end{bmatrix}$ and then $\hat{\varepsilon}_\varpi = T_c(\hat{\bar{e}})(x)$, the the following is derived:

$$\partial_t\hat{\varepsilon}_\varpi(x,t)=\partial_x\hat{\varepsilon}_\varpi(x,t)+\tilde{a}_1(x)\varepsilon_\varpi(0,t)+\tilde{a}_2(x)e_d(t)-\tilde{a}_3(x)e_r(t) \tag{A14}$$

with the boundary condition $\hat{\varepsilon}_\varpi(1,t)=0$, where the parameters are given by $\tilde{a}_1(x)=\tilde{L}_{mc}(x)-\begin{bmatrix}\pi_{11}^{\mathrm{T}}(x)\\ \pi_{21}^{\mathrm{T}}(x)\end{bmatrix}\left[\,l_{d1}\ \ l_{d2}\,\right]M_\varpi^{-1}(0)$, $\tilde{a}_2(x)=\tilde{a}_1(x)\begin{bmatrix}\tilde{n}_1^{\mathrm{T}}(0)\\ \tilde{n}_2^{\mathrm{T}}(0)\end{bmatrix}$ and $\tilde{a}_3(x)=\begin{bmatrix}\pi_{12}^{\mathrm{T}}(x)\\ \pi_{22}^{\mathrm{T}}(x)\end{bmatrix}L_r\begin{bmatrix}q_{r1}^{\mathrm{T}}\\ q_{r2}^{\mathrm{T}}\end{bmatrix}$. In the derivation, $\varepsilon_\varpi(x,t)=\tilde{e}_\varpi(x,t)-\begin{bmatrix}\tilde{n}_1^{\mathrm{T}}(x)\\ \tilde{n}_2^{\mathrm{T}}(x)\end{bmatrix}$ $e_d(t)$ and $\tilde{e}_\varpi(0,t)=e_\varpi(0,t)$ were applied. Denote $\widehat{l}_d=\left[\,\widehat{l}_{d1}\ \ \widehat{l}_{d2}\,\right]$, $\tilde{S}_d = S_d-[\widehat{l}_{d1}\ \ \widehat{l}_{d2}]\begin{bmatrix}\tilde{n}_1^{\mathrm{T}}(0)\\ \tilde{n}_2^{\mathrm{T}}(0)\end{bmatrix}$ and $\tilde{S}_r=S_r-L_r\begin{bmatrix}q_{r1}^{\mathrm{T}}\\ q_{r2}^{\mathrm{T}}\end{bmatrix}$. Then, according to (39) and (62), the dynamics of $e_r(t)$ and $e_d(t)$ are rewritten as:

$$\dot{e}_r(t)=\tilde{S}_re_r(t),$$

$$\dot{e}_d(t)=\tilde{S}_de_d(t)-\widehat{l}_d\varepsilon_\varpi(0,t).$$

Then, defining the state of closed-loop system as $\hat{e}_{cl}:=col(e_r,e_d,\varepsilon_\varpi,\hat{\varepsilon}_\varpi)$ in the Hilbert space $Z_{cl}:=\mathbb{C}^{n_r}\oplus\mathbb{C}^{n_d}\oplus Z\oplus Z$ with the standard inner product. Consider Lyapunov candidate:

$$\begin{aligned}V(t)&=e_r^{\mathrm{T}}(t)P_0e_r(t)+e_d^{\mathrm{T}}(t)P_1e_d(t)\\ &\quad+o_1\int_0^1(1+x)\varepsilon_\varpi^{\mathrm{T}}(x,t)\,\varepsilon_\varpi(x,t)\,\mathrm{d}x\\ &\quad+o_2\int_0^1(1+x)\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\hat{\varepsilon}_\varpi(x,t)\,\mathrm{d}x,\end{aligned} \tag{A15}$$

where $o_1>0$, $o_2>0$, $P_0=P_0^{\mathrm{T}}>0$ and $P_1=P_1^{\mathrm{T}}>0$. In particular, the positive definite matrices P_0 and P_1 are solutions of the following Lyapunov equations:

$$P_0\tilde{S}_r+\tilde{S}_rP_0=-Q_r,$$

$$P_1\tilde{S}_d+\tilde{S}_dP_1=-Q_d$$

for some positive definite matrices $Q_r=Q_r^{\mathrm{T}}>0$ and $Q_d=Q_d^{\mathrm{T}}>0$, which can be chosen freely.

$$\begin{aligned}\dot{V}(t)&=e_r^{\mathrm{T}}(t)\left(\tilde{S}_r^{\mathrm{T}}P_0+P_0\tilde{S}_r^{\mathrm{T}}\right)e_r(t)+e_d^{\mathrm{T}}(t)\left(\tilde{S}_d^{\mathrm{T}}P_1+P_1\tilde{S}_d\right)e_d(t)\\ &\quad-2e_d^{\mathrm{T}}(t)P_1\widehat{l}_d\varepsilon_\varpi(0,t)-o_1\varepsilon_\varpi^{\mathrm{T}}(0,t)\,\varepsilon_\varpi(0,t)\\ &\quad-o_1\int_0^1\varepsilon_\varpi^{\mathrm{T}}(x,t)\,\varepsilon_\varpi(x,t)\,dx-o_2\hat{\varepsilon}_\varpi^{\mathrm{T}}(0,t)\,\hat{\varepsilon}_\varpi(0,t)\\ &\quad-o_2\int_0^1\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\hat{\varepsilon}_\varpi(x,t)\,dx+2o_2\int_0^1(1+x)\hat{\varepsilon}_\varpi^{\mathrm{T}}\\ &\qquad\times(x,t)\,\tilde{a}_1(x)\,\{\mathrm{d}x\varepsilon_\varpi(0,t)\\ &\quad+2o_2\int_0^1(1+x)\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\tilde{a}_2(x)\mathrm{d}xe_d(t)\\ &\quad-2o_2\int_0^1(1+x)\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\tilde{a}_3(x)\mathrm{d}xe_r(t)\\ &\le-\frac{1}{2}e_r^{\mathrm{T}}(t)Q_re_r(t)-\frac{1}{2}e_d^{\mathrm{T}}(t)Q_de_d(t)-o_2\hat{\varepsilon}_\varpi^{\mathrm{T}}(0,t)\,\hat{\varepsilon}_\varpi(0,t)\\ &\quad-o_1\int_0^1\varepsilon_\varpi^{\mathrm{T}}(x,t)\,\varepsilon_\varpi(x,t)\,\mathrm{d}x-\frac{1}{4}o_2\int_0^1\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\hat{\varepsilon}_\varpi(x,t)\,\mathrm{d}x\\ &\quad-\frac{1}{2}o_1\varepsilon_\varpi^{\mathrm{T}}(0,t)\,\varepsilon_\varpi(0,t)-2e_d^{\mathrm{T}}(t)P_1\widehat{l}_d\varepsilon_\varpi(0,t)-\frac{1}{4}e_d^{\mathrm{T}}(t)Q_de_d(t)\\ &\quad-\frac{1}{4}o_2\int_0^1\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\hat{\varepsilon}_\varpi(x,t)\,\mathrm{d}x\\ &\quad+2o_2\int_0^1(1+x)\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\tilde{a}_1(x)\,\mathrm{d}x\varepsilon_\varpi(0,t)-\frac{1}{2}o_1\varepsilon_\varpi^{\mathrm{T}}(0,t)\,\varepsilon_\varpi(0,t)\\ &\quad-\frac{1}{4}o_2\int_0^1\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\hat{\varepsilon}_\varpi(x,t)\,\mathrm{d}x\\ &\quad+2o_2\int_0^1(1+x)\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\tilde{a}_2(x)\,\mathrm{d}xe_d(t)-\frac{1}{4}e_d^{\mathrm{T}}(t)Q_de_d(t)\\ &\quad-\frac{1}{4}o_2\int_0^1\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\hat{\varepsilon}_\varpi(x,t)\,\mathrm{d}x\\ &\quad-2o_2\int_0^1(1+x)\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\tilde{a}_3(x)\mathrm{d}xe_r(t)-\frac{1}{2}e_r^{\mathrm{T}}(t)Q_re_r(t).\end{aligned}$$

Because o_1 and o_2 are chosen arbitrarily, we choose o_1 sufficiently large and o_2 sufficiently small to guarantee that

$$\frac{1}{2}o_1\varepsilon_\varpi^{\mathrm{T}}(0,t)\,\varepsilon_\varpi(0,t)+2e_d^{\mathrm{T}}(t)P_1\widehat{l}_d\varepsilon_\varpi(0,t)+\frac{1}{4}e_d^{\mathrm{T}}(t)Q_de_d(t)\ge 0$$

$$\frac{1}{4}o_2\int_0^1\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\hat{\varepsilon}_\varpi(x,t)\,\mathrm{d}x-2o_2\int_0^1(1+x)\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\tilde{a}_1(x)\,\mathrm{d}x\varepsilon_\varpi(0,t)+\frac{1}{2}o_1\varepsilon_\varpi^{\mathrm{T}}(0,t)\,\varepsilon_\varpi(0,t)\ge 0$$

$$\frac{1}{4}o_2\int_0^1\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\hat{\varepsilon}_\varpi(x,t)\,\mathrm{d}x-2o_2\int_0^1(1+x)\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\tilde{a}_2(x)\,\mathrm{d}xe_d(t)+\frac{1}{4}e_d^{\mathrm{T}}(t)Q_de_d(t)\ge 0$$

$$\frac{1}{4}o_2\int_0^1\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\hat{\varepsilon}_\varpi(x,t)\,\mathrm{d}x+2o_2\int_0^1(1+x)\hat{\varepsilon}_\varpi^{\mathrm{T}}(x,t)\,\tilde{a}_3(x)\,\mathrm{d}xe_r(t)+\frac{1}{2}e_r^{\mathrm{T}}(t)Q_re_r(t)\ge 0.$$

Consequently, it is straightforward to have:

$$\dot{V}(t) \leq -\frac{1}{2}e_r^{\mathrm{T}}(t)Q_r e_r(t) - \frac{1}{2}e_d^{\mathrm{T}}(t)Q_d e_d(t)$$
$$-o_1\int_0^1 \varepsilon_{\varpi}^{\mathrm{T}}(x,t)\,\varepsilon_{\varpi}(x,t)\,\mathrm{d}x - \frac{1}{4}o_2\int_0^1 \hat{\varepsilon}_{\varpi}^{\mathrm{T}}(x,t)\,\hat{\varepsilon}_{\varpi}(x,t)\,\mathrm{d}x. \quad \text{(A16)}$$

From (A15) and (A16), there exits a positive constant $\alpha_v > 0$ such that

$$\dot{V}(t) \leq \alpha_v V(t). \quad \text{(A17)}$$

Hence, the systems for $\hat{e}_{cl} := col(e_r, e_d, \varepsilon_{\varpi}, \hat{\varepsilon}_{\varpi})$ are for $t > 1$ exponentially stable in L_2-norm. Therefore, there exists positive constants $\hat{M}_{cl} > 0$ and $\alpha_{cl} > 0$ such that

$$\left\|\hat{e}_{cl}\right\|_{Z_{cl}} \leq \hat{M}_{cl}\,\mathrm{e}^{-\alpha_{cl}t}\left(\|e_r(0)\|_{C^{n_r}} + \|e_d(0)\|_{C^{n_d}} + \|\varepsilon_{\varpi}(0)\|_2 + \left\|\hat{\varepsilon}_{\varpi}(0)\right\|_2\right).$$

Due to $\varepsilon_{\varpi}(x,t) = \tilde{e}_{\varpi}(x,t) - \begin{bmatrix}\tilde{n}_1^{\mathrm{T}}(x)\\ \tilde{n}_2^{\mathrm{T}}(x)\end{bmatrix} e_d(t)$, there is a positive constant $\tilde{M}_{cl} > 0$ for $\tilde{e}_{cl} := col(e_r, e_d, \tilde{e}_{\varpi}, \hat{\varepsilon}_{\varpi})$ such that

$$\left\|\tilde{e}_{cl}\right\|_{Z_{cl}} \leq \hat{M}_{cl}\,\mathrm{e}^{-\alpha_{cl}t}\left(\|e_r(0)\|_{C^{n_r}} + \|e_d(0)\|_{C^{n_d}} + \left\|\tilde{e}_{\varpi}(0)\right\|_2 + \left\|\hat{\varepsilon}_{\varpi}(0)\right\|_2\right).$$

Since the transformations $\mathcal{T}_c$ and $\mathcal{T}_c^{-1}$ are bounded, there exist positive constants $c_5(\kappa_1, \kappa_2)$ and $c_6(l_1, l_2)$ such that $\|\hat{\varepsilon}_{\varpi}(\cdot,t)\|_2 \leq (1 + c_5(\kappa_1,\kappa_2))\|\hat{\tilde{e}}(\cdot,t)\|_2$ and $\|\hat{\tilde{e}}(\cdot,t)\|_2 \leq (1 + c_6(l_1,l_2))\|\hat{\varepsilon}_{\varpi}(\cdot,t)\|_2$. Similarly, the boundedness of transformations $\mathcal{T}_o$ and $\mathcal{T}_o^{-1}$ indicates that there exist positive constants $c_3(q_1,q_2)$ and $c_4(p_1,p_2)$ such that $\|\tilde{e}_{\varpi}(\cdot,t)\|_2 \leq (1 + c_3(q_1,q_2))\|e_{\varpi}(\cdot,t)\|_2$ and $\|e_{\varpi}(\cdot,t)\|_2 \leq (1 + c_4(p_1,p_2))\|\tilde{e}_{\varpi}(\cdot,t)\|_2$. This results in (76) with $M_{cl} = \hat{M}_{cl}(1 + c_n)(1 + c_m)$ where c_n and c_m are given by $c_m = \max(c_3, c_5)$ and $c_n = \max(c_4, c_6)$ and this completes the proof.