For any probability distribution $p(\theta)$ and any unbiased estimator $\hat{\theta}$ of θ, the Kullback-Leibler (KL) divergence $D_{KL}(p(\theta) \parallel q(\hat{\theta}))$ between $p(\theta)$ and an approximating distribution $q(\hat{\theta})$ can be approximated as:

$$D_{KL}(p(\theta) \parallel q(\hat{\theta})) \approx \int p(\theta) \log \left(\frac{p(\theta)}{\mathcal{N}(\theta, I^{-1}(\theta))} \right) d\theta$$

where $I(\theta)$ is the Fisher information matrix of the parameter θ, and $I^{-1}(\theta)$ is its inverse, which serves as a lower bound for the covariance matrix of the estimator $\hat{\theta}$.

Let $p(\theta)$ be the true distribution of a parameter θ, and let $\hat{\theta}$ be an unbiased estimator of θ. By the Cramér-Rao Lower Bound (CRLB), the covariance matrix $V(\hat{\theta})$ of $\hat{\theta}$ is bounded below by the inverse of the Fisher information matrix, i.e., $V(\hat{\theta}) \succeq I^{-1}(\theta)$.

The KL divergence between $p(\theta)$ and an approximating distribution $q(\hat{\theta})$ is given by:

$$D_{KL}(p(\theta) \parallel q(\hat{\theta})) = \int p(\theta) \log \left(\frac{p(\theta)}{q(\hat{\theta}}) \right) d\theta$$

Applying Jensen's inequality to the logarithm term:

$$\log(x) \leq x - 1$$

This inequality holds for all $x > 0$. Using this inequality, we can rewrite the KL divergence as:

$$D_{KL}(p(\theta) \parallel q(\hat{\theta})) \geq \int p(\theta) \left(\frac{p(\theta)}{q(\hat{\theta}}) - 1 \right) d\theta$$

Recognize that $E_p \left[\frac{p(\theta)}{q(\hat{\theta}}) \right] = \theta$, where E_p denotes the expectation under the distribution $p(\theta)$.

Substitute the expectation back into the inequality:

$$D_{KL}(p(\theta) \parallel q(\hat{\theta})) \geq \theta - 1$$

The inequality $D_{KL}(p(\theta) \parallel q(\hat{\theta})) \geq \theta - 1$ holds for any probability distribution $p(\theta)$ and any approximating distribution $q(\hat{\theta})$. This establishes the approximation of the KL divergence using the inverse Fisher information matrix $I^{-1}(\theta)$.