Lecture 2

Functions

Definition of a Function

Definition 1. A function is a rule that produces a corresponding between one set of elements, called the domain, and a second set of elements, called the range, such that to each element in the domain there corresponds one and only one element in the range.

Definition 2. A symbol that represents an arbitrary number in the domain of a function is called an independent variable. A symbol that represents a number in the range of is called a dependent variable.

There are four ways to represent a function:

- verbally (by description in words),
- numerically (by a table of values),
- visually (by a graph)
- algebraically (by an explicit formula).

Functions specified by equations

Consider the equation:

$$y = x^2 - x, \qquad x \in \mathbb{R}.$$

For each input x we obtain one output y. For example: If $x = 3$, then $y = 6$. If $x = 1$, then $y = 0$. The variable x is called an *independent variable* and y is called a *dependent variable*.

Remark 1. In an equation in two variables, if there corresponds exactly one value of the dependent variable (output) to each value of the independent variable (input), then the equation specifies a function.

Agreement on domains and ranges

Remark 2. If a function is specified by an equation and the domain is not indicated, then we shall assume that the domain is the set of all real number replacement of the independent variable (inputs) that produce real values for the dependent variable (outputs). The range is the set of all outputs corresponding to input values.

Remark 3. For any element x in the domain of the function f, the symbol $f(x)$ represents the element in the range of f corresponding to x in the domain of f. If x is an input value, then $f(x)$ is the corresponding output value; or $f : x \mapsto f(x)$.

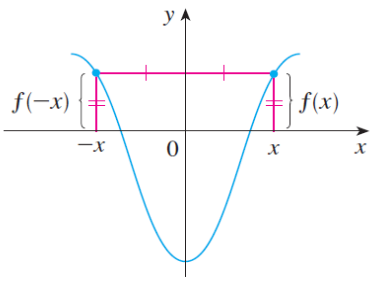
Symmetry

Definition 3. If a function f satisfies $f(-x) = f(x)$ for every number in its domain, then is called an even function.

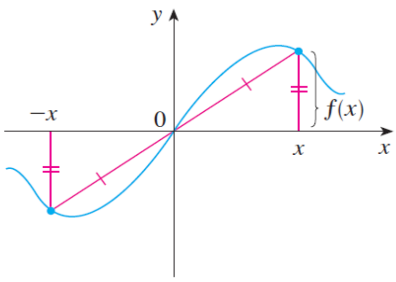
Remark 4. The geometric significance of an even function is that its graph is symmetric with respect to the y-axis.

Definition 4. If f satisfies $f(-x) = -f(x)$ for every number in its domain, then is called an odd function.

Remark 5. The graph of an odd function is symmetric about the origin.



An even function



An odd function

Example. We determine each of the following functions is even, odd, or neither even nor odd.

a) $f(x) = x^5 + x$

$$f(-x) = (-x)^5 + (-x) = -(x^5 + x) = -f(x) \quad \text{odd function}$$

b) $f(x) = 1 - x^4$

$$f(-x) = 1 - (-x)^4 = 1 - x^4 = f(x) \quad \text{even function}$$

c) $f(x) = 2x - x^2$

$$f(-x) = 2(-x) + (-x)^2 = -2x + x^2 \quad \text{neither even nor odd function}$$

Monotonicity

Definition 5. Let X and Y be sets and let $f : X \to Y$ be a function. We say that f is increasing ($\uparrow$) if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \qquad \text{for all} \quad x_1, x_2 \in X.$$

Definition 6. Let X and Y be sets and let $f : X \to Y$ be a function. We say that f is decreasing ($\downarrow$) if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \qquad \text{for all} \quad x_1, x_2 \in X.$$

Definition 7. Let X and Y be sets and let $f : X \to Y$ be a function. The function f is called monotonic if it is either increasing or decreasing.

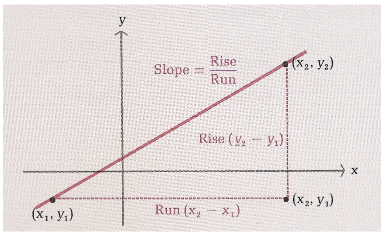
Linear function

Definition 8. The linear function is specified by equations of the form

$$f(x) = ax + b, \qquad \text{where} \quad a, b \in \mathbb{R}.$$

a is the slope of the line and b is the y-intercept.

Graph of the linear function:



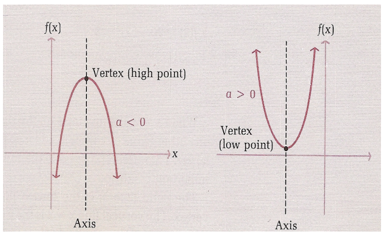
Quadratic function

Definition 9. Any function defined by an equation of the form

$$f(x) = ax^2 + bx + c, \qquad \text{where} \quad a \neq 0 \quad \text{and} \quad b, c \in \mathbb{R}$$

is called a quadratic function.

The graph of a quadratic function f is a parabola that has its axis parallel to the vertical axis. It opens upward if $a > 0$ and downward if $a < 0$. The intersection point of the axis and parabola is called the vertex.



Graph

If we can find the vertex of the graph, then the rest of the graph can be sketched with relatively few points. We start by transforming the equation

$$f(x) = ax^2 + bx + c$$

into the form

$$f(x) = a(x - h)^2 + k, \qquad \text{where} \quad a, h, k \in \mathbb{R}.$$

Example. Consider the quadratic function given by

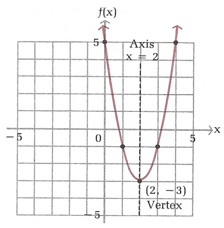
$$f(x) = 2x^2 - 8x + 5.$$

x	$f(x)$
2	-3
1	-1
3	-1
0	5
4	5

We transform the equation by completing the square:

$$\begin{aligned} f(x) &= 2x^2 - 8x + 5 = 2(x^2 - 4x) + 5 = 2(x^2 - 4x + ?) + 5 \\ &= 2(x^2 - 4x + 4) + 5 - 8 = \underbrace{2(x-2)^2}_{never\ negative} - 3. \end{aligned}$$

Therefore, $f(2) = -3$ is the minimum value of $f(x)$ for all x.
Example. The point $(2, -3)$ is the lowest point on the parabola and is also the vertex. We plot the vertex and the axis and a couple of points on either side of the axis to complete the graph.



General result

Remark 6. Quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$

- Axis (of symmetry) of the parabola: $x = -\frac{b}{2a}$.
- Maximum or minimum value of $f(x)$:

$$f\Big(-\frac{b}{2a}\Big) \begin{cases} \text{Minimum if} & a > 0 \\ \text{Maximum if} & a > 0 \end{cases}$$

Quadratic equation

A quadratic equation in one variable is any equation that can be written in the form

$$ax^2 + bx + c = 0, \quad a \neq 0,$$

where x is a variable and a, b, and c are constants. Let us try the method of completing the square on the general quadratic equation. We start by multiplying both sides by $\frac{1}{a}$ to obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Add $-\frac{c}{a}$ to both members

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Now complete the square on left hand-side

$$\left(x^2 + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Quadratic formula

Now we solve by the square root method

$$\begin{aligned} x^2 + \frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}}, \\ x_{1,2} &= -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \qquad \text{if discriminant} \quad \Delta = b^2 - 4ac \geqslant 0. \end{aligned}$$

Remark 7. The quadratic equation $ax^2 + bx + c = 0$ and $a \neq 0$ has

- two real solutions if discriminant $\Delta = b^2 - 4ac$ is positive,
- one real solution if discriminant $\Delta = b^2 - 4ac$ is zero,
- no real solutions if discriminant $\Delta = b^2 - 4ac$ is negative.

Vieta's formulas

The sum of two solutions of the quadratic equation is equal

$$\begin{aligned} x_1 + x_2 &= -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} - \frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a} \\ &= -\frac{b}{a} \end{aligned}$$

and the product of these solutions is

$$\begin{aligned} x_1 \cdot x_2 &= \left(-\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\right) \cdot \left(-\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\right) \\ &= \frac{4ac}{4a^2} = \frac{c}{a}. \end{aligned}$$

Remark 8. Vieta's formula for the quadratic equation $ax^2 + bx + c = 0$

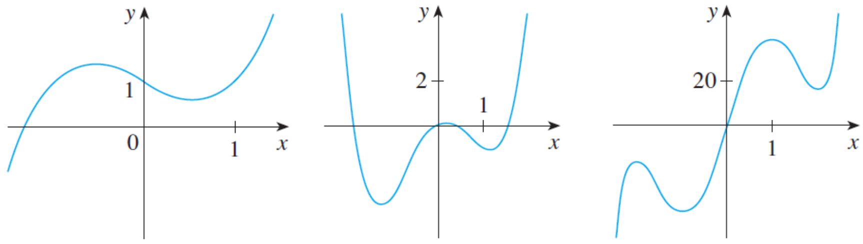
$$x_1 + x_2 = -\frac{b}{a}, \qquad x_1 \cdot x_2 = \frac{c}{a}.$$

Polynomials

Definition 10. A function P is called a polynomial if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where n is a nonnegative integer and the numbers $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants called the coefficients of the polynomial. The domain of any polynomial is $\mathbb{R}$. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.



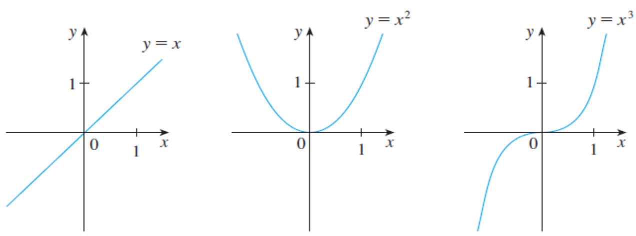
(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Power Functions

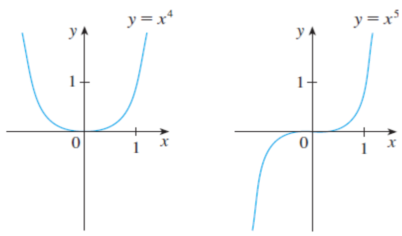
Definition 11. A function of the form $f(x) = x^a$, where a is a constant, is called a power function.

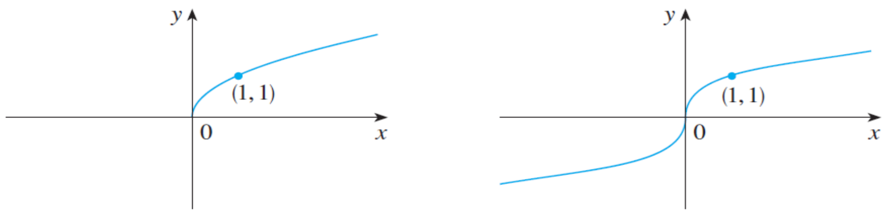
We consider several cases:

- $a = n$, where n is a positive integer, The general shape of the graph of $f(x) = x^n$ depends on whether is even or odd. If n is even, then is an even function and its graph is similar to the parabola . If n is odd, then is an odd function and its graph is similar to that of $y = x^3$. However, that as n increases, the graph of becomes flatter near 0 and steeper when $|x| \geqslant 1$.
- $a = \frac{1}{n}$ The function is a root function. For $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. For other even values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$. For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root). The graph of $y = \sqrt[n]{x}$ for n odd is similar to that of $y = \sqrt[3]{x}$.
- $a = -1$ The function $f(x) = x^{-1}$ is called the reciprocal function. Its graph is a hyperbola with the coordinate axes as its asymptotes.



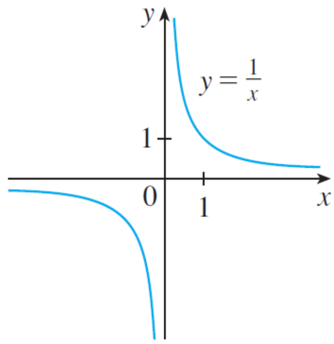
Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$





(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$

Graphs of root functions



The reciprocal function

Rational Functions

Definition 12. A rational function is a ratio of two polynomials:

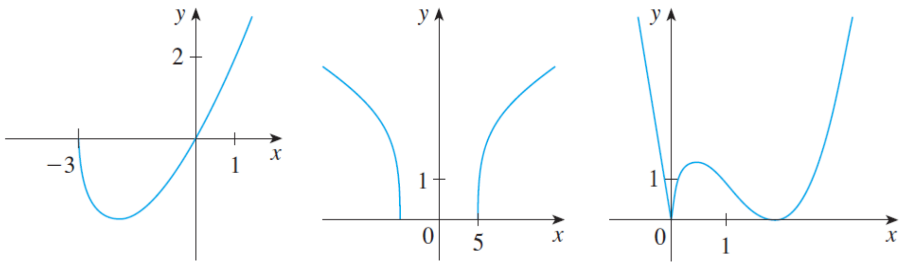
$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of such that $Q(x) \neq 0$.

Example. The function $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$ is a rational function with domain $\mathbb{R} \setminus \{-2, 2\}$.

Algebraic Functions

Definition 13. A function f is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials.



(a) $f(x) = x\sqrt{x+3}$ (b) $g(x) = \sqrt[4]{x^2 - 25}$ (c) $h(x) = x^{2/3}(x-2)^2$

Trigonometric Identities

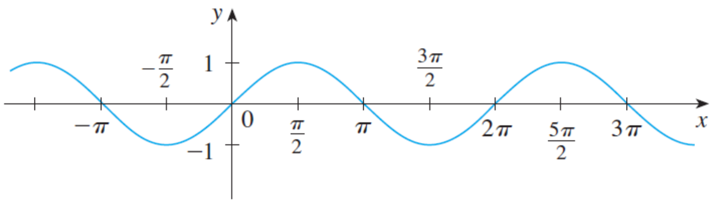
Theorem 14.

- $\sin^2 x + \cos^2 x = 1$,
- $\tan x = \frac{\sin x}{\cos x}$,
- $\cot x = \frac{\cos x}{\sin x}$,
- $\sin(-x) = -\sin x$,
- $\cos(-x) = \cos x$,
- $\sin(x + 2\pi) = \sin x$,
- $\cos(x + 2\pi) = \cos x$,
- $\tan(x + \pi) = \tan x$,
- $\cot(x + \pi) = \cot x$,
- $\sin 2x = 2 \sin x \cos x$,
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

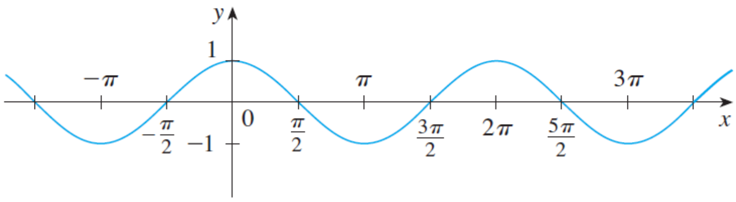
Theorem 15.

- $\sin(x - y) = \sin x \cos y - \cos x \sin y$,
- $\cos(x - y) = \cos x \cos y + \sin x \sin y$,
- $\sin(x + y) = \sin x \cos y + \cos x \cos y$,
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$,

Graphs of Trigonometric Functions

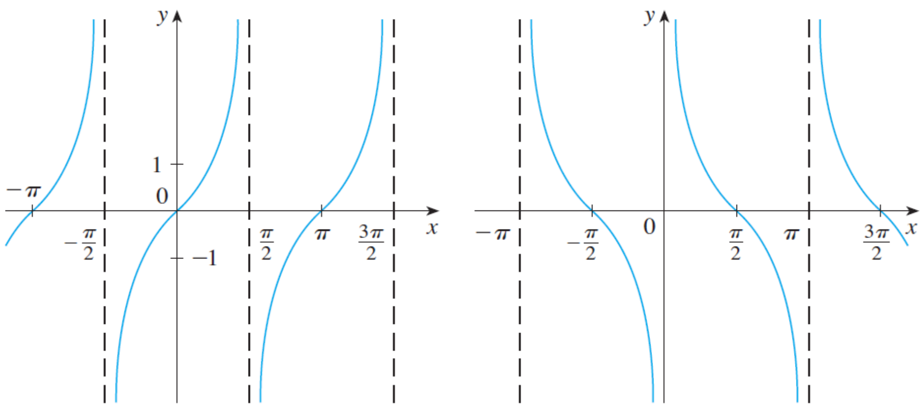


$f(x) = \sin x$



$g(x) = \cos x$

Graphs of Trigonometric Functions



$y = \tan x$ $y = \cot x$

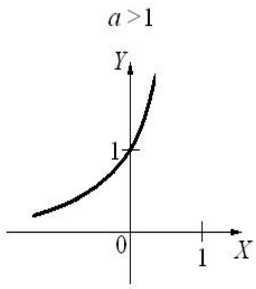
Definition

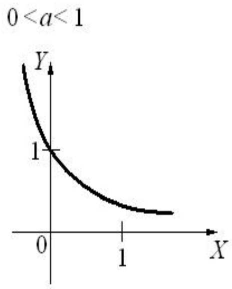
Definition 16. An exponential function is a function defined by an equation of the form $f(x) = a^x$ for $a > 0$ and $a \neq 1$.

- a is a constant called the base,
- the exponent x is a variable,
- the domain of f is the set of real numbers,
- the range of f is the set of positive real numbers.

Exponential functions are referred to as growth functions. They are used to describe the growth of money at compound interest, population growth of people, animals, and bacteria, radioactive decay, the growth of learning skill such as typing or swimming relative to practice.

Graphs





- If $a > 1$, then the exponential function is increasing.
- If $0 < a < 1$, then the exponential function is decreasing.

Basic Exponential Properties

Lemma 17. *Assume that $a, b > 0$ and $a, b \neq 1$, and $m, n \in \mathbb{R}$*

- $a^m = a^n$ *if and only if* $m = n$,
- $(a \cdot b)^m = a^m \cdot b^m$,
- $(a^m)^n = a^{m \cdot n}$,
- $a^{-m} = (\frac{1}{a})^m$,
- $a^m \cdot a^n = a^{m+n}$,
- $\frac{a^m}{a^n} = a^{m-n}$.

Example.

a)
$$(2x^3)(3x^5) = 6 \cdot x^{3+5} = 6x^8$$

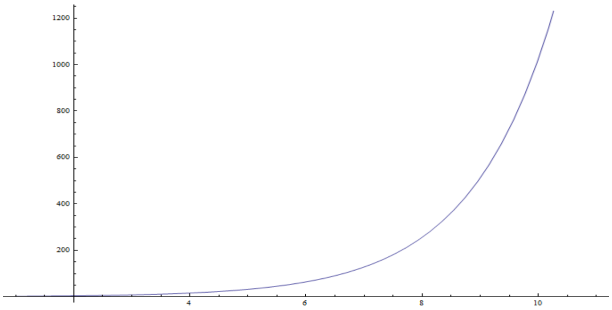
b)
$$\left(\frac{y^{-5}}{y^{-2}}\right)^{-2} = \frac{(y^{-5})^{-2}}{(y^{-2})^{-2}} = \frac{y^{10}}{y^4} = y^6$$

c)
$$\frac{4m^{-3}n^{-5}}{6m^{-4}n^3} = \frac{2m^{-3-(-4)}}{3n^{3-(-5)}} = \frac{2m}{3n^8}$$

d)
$$\left(\frac{2x^{-3}x^3}{n^{-2}}\right)^{-3} = \left(\frac{2}{n^{-2}}\right)^{-3} = \frac{2^{-3}}{n^6} = \frac{1}{8n^6}$$

Applications

Exponential growth: If we start with \$2 and double the amount each day, we would have \$ 2^n after n days. Graph $f(n) = 2^n$ for $1 \leqslant n \leqslant 10$.

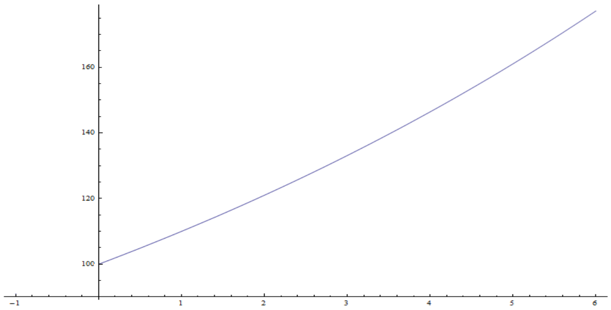


Applications

Compound interest: If the certain amount of money P is invested at r % interest compounded annually, the amount of money A after t years is given by

$$A = P(1+r)^t.$$

Graph the equation for $P =$ \$ 100, $r = 10\%$, and $0 \leqslant t \leqslant 6$. How much money would a person have after 10 years if no interest were withdrawn?

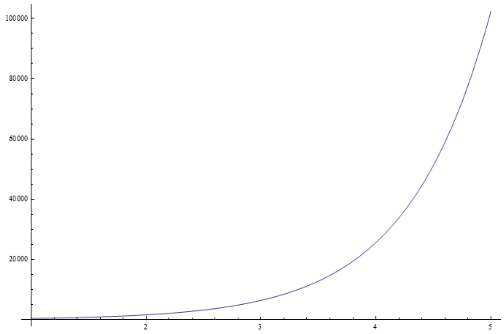


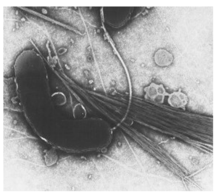
Applications

Bacteria growth: A single cholera bacterium divides every $\frac{1}{2}$ hour to produce two complete cholera bacteria. If we start with 100 bacteria, in t hours we will have

$$A = 100 \cdot 2^{2t}$$

bacteria. Graph this equation for $0 \leqslant t \leqslant 5$.





Base e

For large n

$$\left(1+\frac{1}{n}\right)^n \approx e.$$

We can rise each side to the xth power to obtain

$$\left(1+\frac{1}{n}\right)^{nx} \approx e^x.$$

Example.: If P is invested at r % compounded continuously, then the amount A in the account at the end of t years is given by

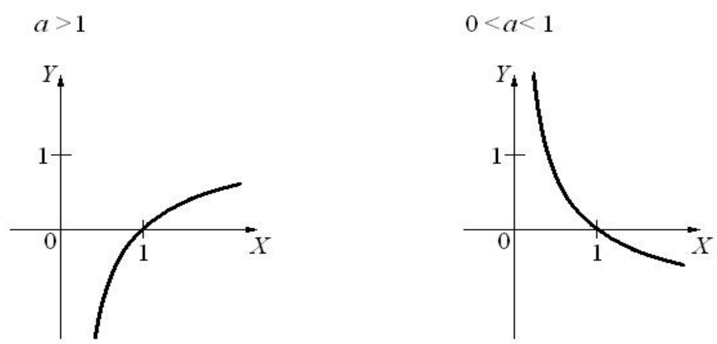
$$A = Pe^{rt}.$$

Logarithmic Function

Definition 18. $y = \log_a x$ if and only if $x = a^y$ for $a > 0$ and $a \neq 1$.

- the domain of a logarithmic function is the set of all positive real numbers,
- the range of a logarithmic function is the set of all real numbers,
- the logarithm of 0 or a negative number is **not defined**.

Graph



- If $a > 1$, then the logarithmic function is increasing.
- If $0 < a < 1$, then the logarithmic function is decreasing.

$$\begin{aligned} \log_a x_1 < \log_a x_2 \Leftrightarrow x_1 < x_2, &\quad \text{if} \quad a > 1, \\ \log_a x_1 < \log_a x_2 \Leftrightarrow x_1 > x_2, &\quad \text{if} \quad a \in (0,1). \end{aligned}$$

Basic Logarithmic Properties

Lemma 19. *Assume that* $a > 0$ and $a \neq 1,\ b, c > 0, d \in \mathbb{R}$

- $a^{\log_a b} = b$
- $\log_a a = 1$
- $\log_a b = \frac{1}{\log_b a}$
- $\log_a b^d = d\ \log_a b$
- $\log_a b \cdot c = \log_a b + \log_a c$
- $\log_a \frac{b}{c} = \log_a b - \log_a c$
- $(\log_a b) = \frac{\log_c b}{\log_c a}$ *for* $c \neq 1$

Example.

a)

$$\begin{aligned} \log_b \frac{wx}{yz} &= \log_b wx - \log_b yz \\ &= \log_w + \log_b x - \log_b y - \log_b z \end{aligned}$$

b) find x

$$\begin{aligned} \frac{3}{2}\log_b 4 - \frac{2}{3}\log_b 8 + \log_b 2 &= x \\ \log_b 4^{\frac{3}{2}} - \log_b 8^{\frac{2}{3}} + \log_b 2 &= \log_b x \\ \log_b \frac{8 \cdot 2}{4} &= \log_b x \\ \log_b 4 &= \log_b x \\ x &= 4 \end{aligned}$$

Common and Natural Logarithms

Logarithmic notation:

- $\log x = \log_{10} x$,
- $\ln x = \log_e x$.

Scientific notation is a way of writing numbers that are too big or too small to be conveniently written in decimal form. For example

$$520 = 5.2 \times 10^2$$

means that this the number of hundreds and its logarithm is between 2 and 3, because

$$10^2 < 520 < 10^3$$

since common logarithm is an increasing function.

Scientific Notation

In general, if

$$x = a \times 10^b, \quad \text{where} \quad a \in (0, 10) \quad \text{and} \quad b \in \mathbb{Z},$$

then

$$\log x = \log(a \times 10^b) = b + \log a, \quad \text{where} \quad 0 < \log a < 1.$$

To compute the logarithm of any number it is enough to know logarithms of numbers from the interval $(0, 10)$. For example

$$3294 = 3.294 \times 10^3,$$

hence

$$\log 3294 = 3 + \log 3.294 \approx 3.5.$$

Applications

Doubling time: How long (to the next whole year) will it take money to double if it is invested at 20% interest compounded annually? We use formula $A = P(1 + r)^t$ for $A = 2P$ and $r = 0.2$. We get

$$2P = P(1 + 0.2)^t.$$

Taking the natural logarithm of both sides and using properties of logarithms, we have

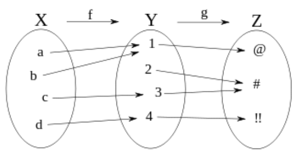
$$t \ln 1.2 = \ln 2$$

and finally

$$t = \frac{\ln 2}{\ln 1.2} \approx 3.8 \approx 4 \quad \text{years.}$$

Note that after 3 years, the money will not be doubled!

Function composition



Example. If $f(x) = \sin x$ and $g(x) = x^2$, then

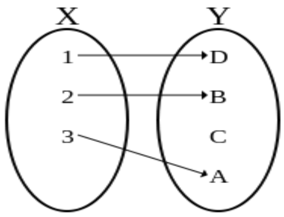
$$f \circ f(x) = \sin(\sin x), \quad f \circ g(x) = \sin x^2, \quad g \circ f(x) = (\sin x)^2, \quad g \circ g(x) = x^4.$$

Example: If an airplane's elevation at time t is given by the function $h(t)$, and the oxygen concentration at elevation x is given by the function $c(x)$, then $(c \circ h)(t)$ describes the oxygen concentration around the plane at time t.

Remark 9. Composition of functions is not commutative, i. e. $f \circ g(x) \neq g \circ f(x)$.

Injective function

Definition 20. Let X and Y be two sets and let $f : X \to Y$ be a function. Then f is injective or one-to-one if and only if whenever $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$ then $x_1 = x_2$.

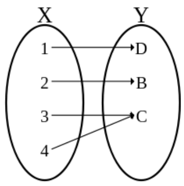


Example.

- The identity function $Id : X \to X$ is injective.
- Consider the function $f : \{1; 2; 3\} \to \{3; 5; 6; 9\}$ define by $f(1) = 6, f(2) = 5$, and $f(3) = 9$. Then f is injective since $f(1)$, $f(2)$, and $f(3)$ are all distinct elements.

Surjective function

Definition 21. Let X and Y be sets and let $f : X \to Y$ be a function. We say that f is surjective or onto if the range of f is entire Y (that is, for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$).



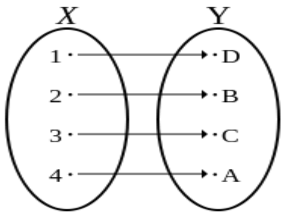
Example.

- Consider the function $f : \{1; 2; 3; 4\} \to \{3; 5; 9\}$ define by $f(1) = 3, f(2) = 5, f(3) = 9$, and $f(4) = 5$. Then f is surjective since the range of f is the entire set $\{3; 5; 9\}$.
- Consider the function $f : \{1; 2; 3; 4\} \to \{3; 5; 9\}$ define by $f(1) = 9, f(2) = 5, f(3) = 9$, and $f(4) = 5$. Then f is not surjective since 3 is not in the range of f.

Bijective and periodic function

Definition 22. Let X and Y be sets and let $f : X \to Y$ be a function. We say that f is bijective if it is injective and surjective.

Definition 23. Let X and Y be sets and let $f : X \to Y$ be a function. We say that f is periodic with period T if $f(x + T) = f(x)$.

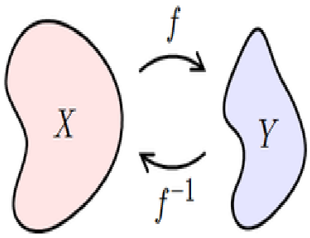


Inverse function

Theorem 24. *Let X and Y be sets and let $f : X \to Y$ be a bijection. Then there exists a unique function $g : Y \to X$ such that*

- *$g(\ f(x)\ ) = x$,*
- *$f(g(y)) = y$.*

The function g is called the inverse of f and it is denoted f^{-1}.



Remark 10. Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \quad \text{does not mean} \quad \frac{1}{f(x)}.$$

The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$.

Remark 11. How to find inverse function of a one-to-one function f

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. To express f^{-1} as a function of x, interchange x and y. This resulting equation is $y = f^{-1}(x)$.

Example. We find the inverse function of $f(x) = x^3 + 2$. We first write

$$y = x^3 + 2.$$

Then we solve this equation for x

$$x^3 = y - 2,$$
$$x = \sqrt[3]{y - 2}.$$

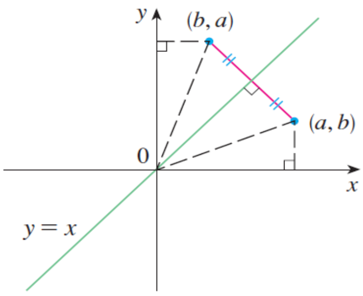
Finally, we interchange x and y

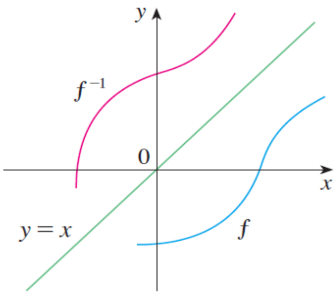
$$y = \sqrt[3]{x - 2}.$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$.

Graph of inverse function

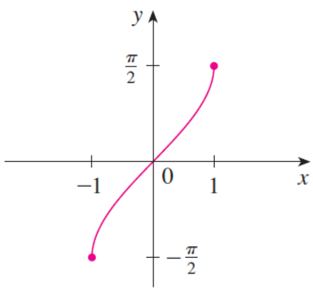
The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f. Since $f^{-1}(b) = a$ if and only if $f(a) = b$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point from by reflecting about the line $y = x$.





Arcsine function

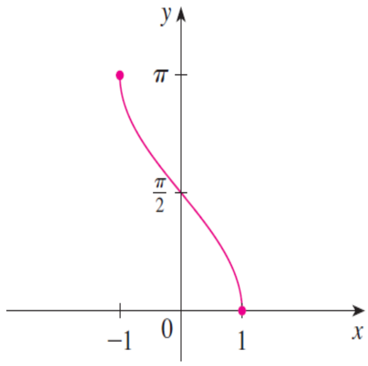
Definition 25. The sine function $y = \sin x$ is not one-to-one. But the function $f(x) = \sin x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, is one-to-one. The inverse function of this restricted sine function f exists and is denoted arcsin. It is called the inverse sine function or the arcsine function. The inverse sine function has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



$y = \sin^{-1} x = \arcsin x$

Arccosine function

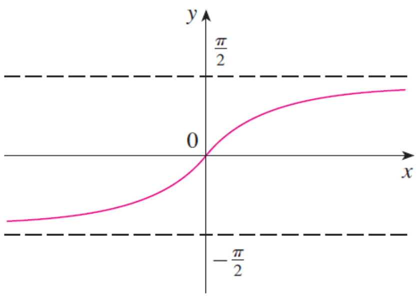
Definition 26. The restricted cosine function $f(x) = \cos x$, $0 \leqslant x \leqslant \pi$ is not one-to-one and so it has an inverse function denoted by arccos. The inverse cosine function has domain $[-1, 1]$ and range $[0, \pi]$.



$y = \cos^{-1} x = \arccos x$

Arctangent function

Definition 27. The tangent function can be made one-to-one by restricting it to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus the inverse tangent function is defined as the inverse of the function $f(x) = \tan x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$. It is denoted by arctan. The inverse tangent function has domain $\mathbb{R}$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



$y = \tan^{-1} x = \arctan x$

Transformations of functions

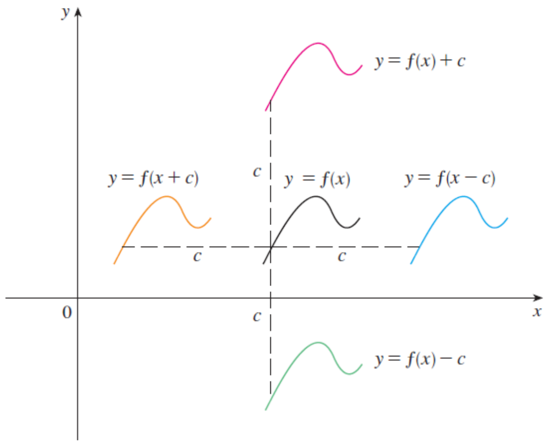
Remark 12. Vertical and Horizontal Shifts: Suppose $c > 0$. To obtain the graph of

- $y = f(x) + c$ shift the graph of $y = f(x)$ a distance c units upward,
- $y = f(x) - c$ shift the graph of $y = f(x)$ a distance c units downward,
- $y = f(x - c)$ shift the graph of $y = f(x)$ a distance c units to the right,
- $y = f(x + c)$ shift the graph of $y = f(x)$ a distance c units to the left.

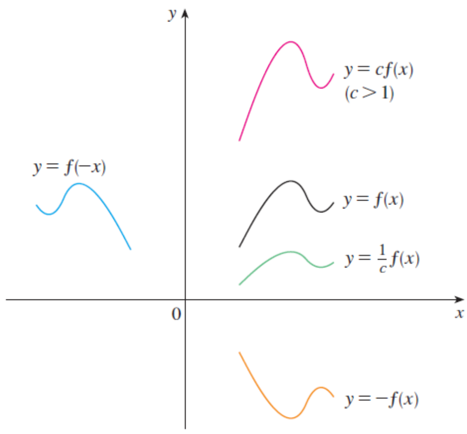
Remark 13. Vertical and Horizontal Stretching and Reflecting: Suppose $c > 1$. To obtain the graph of

- $y = cf(x)$ stretch the graph of $y = f(x)$ vertically by a factor of c,

- $y = \frac{1}{c}f(x)$ shrink the graph of $y = f(x)$ vertically by a factor of c,
- $y = f(cx)$ shrink the graph of $y = f(x)$ horizontally by a factor of c,
- $y = f\left(\frac{x}{c}\right)$ stretch the graph of $y = f(x)$ horizontally by a factor of c,
- $y = -f(x)$ reflect the graph of $y = f(x)$ about the x-axis,
- $y = f(-x)$ reflect the graph of $y = f(x)$ about the y-axis.

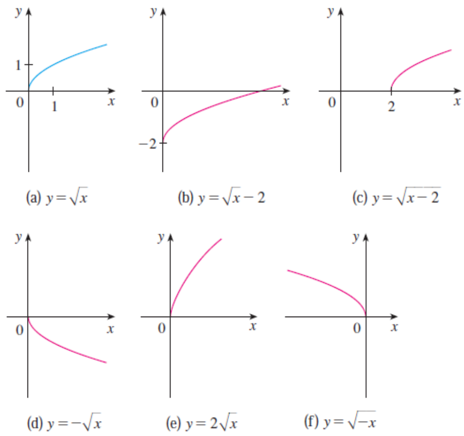


Translating the graph of f



Stretching and reflecting the graph of f

Example. Given the graph of $y = \sqrt{x}$, use transformation to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, $y = \sqrt{-x}$.



(a) $y=\sqrt{x}$ (b) $y=\sqrt{x}-2$ (c) $y=\sqrt{x-2}$

(d) $y=-\sqrt{x}$ (e) $y=2\sqrt{x}$ (f) $y=\sqrt{-x}$