Lecture 1

Preliminaries

Sets

Set properties and set notation

Set is any collection of objects specified in such a way that we can tell whether any given object is or is not in the collection.

- $a \in A$ means *a is an element of set A*
- $\emptyset$ represents *the empty or null set*
- $A \subseteq B$ means *A is a subset of B*
- $A = B$ means *A and B have exactly the same elements*

Examples:

- $\mathbb{N}$ - the set of natural numbers,
- $\mathbb{Z}$ - the set of integers,
- $\mathbb{Q}$ - the set of rational numbers,
- $\mathbb{R}$ - the set of real numbers.

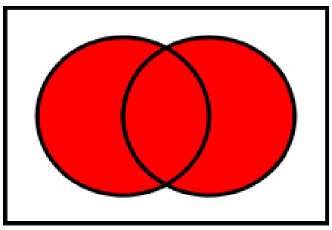
Set operations

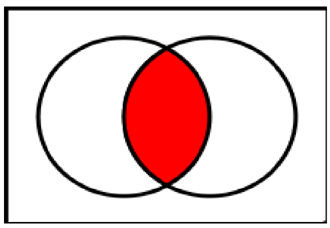
The union of sets A and B is the set of all elements formed by combining all the elements of A and all elements of B into one set. Symbolically,

$$A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}.$$

The intersection of sets A and B is the set of elements in the set A that are also in set B. Symbolically,

$$A \cap B = \{x : x \in A \quad \text{and} \quad x \in B\}.$$

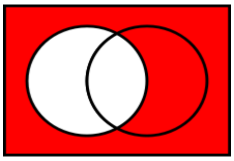




Set operations

The complement of A (relative to U) is the set of all elements in U that are not in A. Symbolically,

$$A^c = \{x \in U : x \notin A\}.$$



Example: If $A = \{3, 6, 9\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{4, 5, 7\}$, and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then

$$\begin{aligned}
A \cup B &= \{3, 4, 5, 6, 7, 9\}, \\
A \cap B &= \{3, 6\}, \\
A \cap C &= \emptyset \qquad A \quad \text{and} \quad C \quad \text{are disjoint}, \\
B^c &= \{1, 2, 8, 9\}.
\end{aligned}$$

Some basic logic notation

Symbol	Read	Example	Also read
$\neg$	not	$\neg p$	p is false
$\wedge$	and	$p \wedge q$	p and q
$\vee$	or	$p \vee q$	p or q
$\Rightarrow$	implies	$p \Rightarrow q$	if p then q
$\Leftrightarrow$	if and only if	$p \Leftrightarrow q$	p iff q

Remark 1. The symbol $\neg$ is called a unary logic operation because it operates on one (albeit possibly compound) statement, say p. The symbols $\wedge, \vee, \Rightarrow, \Leftrightarrow$ are called connectives or binary logic operations, connecting two statements, such as p, q.

Logic Operations

Definition 1. The logical negation $\neg$, which is a unary operation, i.e., acting on one (possibly compound) statement. For example consider the statement $\neg p$, usually read "not p." This is the negation of the statement p. Of course $\neg p$ is not independent of p, but its truth value is based upon that of p; stating that $\neg p$ is true is the same as stating that p is false, and stating that $\neg p$ is false is the same as stating that p is true.

Definition 2. The binary operation $\wedge$ is called the logical conjunction, or just simply and: the statement $p \wedge q$ is usually read "p and q." This compound statement $p \wedge q$ is true exactly when both p and q are true, and false if a component statement is false.

p	q	$\sim p$	$p \vee q$	$p \wedge q$	$p \Rightarrow q$	$p \Leftrightarrow q$
1	1	0	1	1	1	1
1	0	0	1	0	0	0
0	1	1	1	0	1	0
0	0	1	0	0	1	1

Definition 3. The binary operation $\vee$ is called the logical disjunction, or simply or. The statement $p \vee q$ is usually read "p or q." For $p \vee q$ to be true we only need one of the underlying component statements to be true; for $p \vee q$ to be false we need both p and q to be false.

Definition 4. The bi-implication is denoted $p \Leftrightarrow q$, and often read "p if and only if q." This is sometimes also abbreviated "p iff q". It states that p implies q and q implies p simultaneously. Thus truth of p gives truth of q, while truth of q would give truth of p. Furthermore, if p is false, then so must be q, because q being true would have forced p to be true as well. Similarly q false would imply p false (since if p were instead true, so would be q).

Logic Laws

- $\sim (\sim p) \Leftrightarrow p$,
- $\sim (p \Rightarrow q) \Leftrightarrow (p \wedge (\sim q))$,
- $p \vee (\sim p)$,
- $[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$,
- $[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$,
- $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$,
- $[\sim (p \vee q)] \Leftrightarrow [(\sim p) \wedge (\sim q)]$ De Morgan's Law,
- $[\sim (p \wedge q)] \Leftrightarrow [(\sim p) \vee (\sim q)]$ De Morgan's Law.

Quantifiers

Definition 5. The three quantifiers used by nearly every professional mathematician are as follow:

- universal quantifier $\forall$ read "for all," or "for every;"
- existential quantifier $\exists$ read, "there exists;"
- uniqueness quantifier ! read, "unique."

Remark 2. The first two are of equal importance, and far more important than the third which is usually only found after the second.

Quantified statements are usually found in forms such as:

$$(\forall x \in S)P(x), \qquad \text{i.e., for all } x \in S,\ P(x) \text{ is true;}$$
$$(\exists x \in S)P(x), \qquad \text{i.e., there exists an } x \in S \text{ such that } P(x) \text{ is true;}$$
$$(\exists! x \in S)P(x), \qquad \text{i.e., there exists a unique (exactly one) } x \in S$$

such that $P(x)$ is true.

Here S is a set and $P(x)$ is some statement about x. The meanings of these quickly become straightforward. For instance, consider

$$(\forall x \in \mathbb{R})(x + x = 2x): \qquad \text{for all } x \in \mathbb{R}, x + x = 2x;$$
$$(\exists x \in \mathbb{R})(x + 2 = 2): \qquad \text{there exists (an) } x \in \mathbb{R} \text{ such that } x + 2 = 2;$$
$$(\exists! x \in \mathbb{R})(x + 2 = 2): \qquad \text{there exists a unique } x \in \mathbb{R} \text{ such that } x + 2 = 2.$$

Statements with Multiple Quantifiers

Many of the interesting statements in mathematics contain more than one quantifier. To illustrate the mechanics of multiply quantified statements, we will first turn to a more worldly setting. Consider the following sets:

$$M = \{\text{men}\},$$
$$W = \{\text{women}\}.$$

In other words, M is the set of all men, and W the set of all women. Consider the statement

$$(\forall m \in M)(\exists w \in W)[w \text{ loves } m].$$

Set to English, the statement could be written, "for every man there exists a woman who loves him". So if the statement is true, we can in principle arbitrarily choose a man m, and then know that there is a woman w who loves him. It is important that the man m was quantified first. A common syntax that would be used by a logician or mathematician would be to say here that, once our choice of a man is *fixed*, we can in principle find a woman who loves him. Note that the statement allows that different men may need different women to love them, and also that a given man may be loved by more than (but not less than) one woman. Alternatively, consider the statement

$$(\exists w \in W)(\forall m \in M)[w \text{ loves } m].$$

A reasonable English interpretation would be, "there exists a woman who loves every man." We can also consider the statement

$$(\forall m \in M)(\forall w \in W)[w \text{ loves } m].$$

This can be read, "for every man and every woman, the woman loves the man."In other words, every man is loved by every woman. In this case we can reverse the order of quantification:

$$(\forall w \in W)(\forall m \in M)[w \text{ loves } m].$$

Remark 3. If the two quantifiers are the same type - both universal or both existential - then the order does not matter.

Thus

$$(\forall m \in M)(\exists w \in W)[w \text{ loves } m] \Leftrightarrow (\forall w \in W)(\forall m \in M)[w \text{ loves } m],$$
$$(\exists m \in M)(\forall w \in W)[w \text{ loves } m] \Leftrightarrow (\exists w \in W)(\forall m \in M)[w \text{ loves } m].$$

In both representations in the existential statements, we are stating that there is at least one man and one woman such that she loves him. Note that in cases where the sets are the same, we can combine two similar quantifications into one, as in

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y = y + x] \Leftrightarrow (\forall x, y \in \mathbb{R})[x + y = y + x].$$

Similarly with existence. The order does matter if the types of quantification are different. Consider

$$(\forall x \in \mathbb{R})(\exists K \in \mathbb{R})(x = 2K) \quad \text{True.}$$

This is read, "for every $x \in \mathbb{R}$, there exists $K \in \mathbb{R}$ such that $x = 2K$." That $K = x/2$ exists (and is actually unique) makes this true, while it would be false if we were to reverse the order of quantification:

$$(\exists K \in \mathbb{R})(\forall x \in \mathbb{R})(x = 2K) \quad \text{False.}$$

The statement claims (erroneously) that there exists $K \in \mathbb{R}$ so that, for every $x \in \mathbb{R}$, $x = 2K$. That is impossible, because no value of K is half of every real number x. For example the value of K which works for $x = 4$ is not the same as the value of K which works for $x = 100$.

Negating Universally and Existentially Quantified Statements

Definition 6. For statements with a single universal or existential quantifier, we have the following negations.

$$\neg[(\forall x \in S)P(x)] \Leftrightarrow (\exists x \in S)[\neg P(x)],$$
$$\neg[(\exists x \in S)P(x)] \Leftrightarrow (\forall x \in S)[\neg P(x)].$$

Remark 4. Thus when we negate such a statement as $(\forall x)P(x)$ or $(\exists x)P(x)$, we change $\forall$ to $\exists$ or vice versa, and negate the statement after the quantifiers.

Example. Negate $(\forall x \in S)[P(x) \Rightarrow Q(x)$.

We will need $\neg(p \Rightarrow q) \Leftrightarrow p \wedge (\neg q)$. We get

$$\begin{aligned}\neg[(\forall x \in S)[P(x) \Rightarrow Q(x)] &\Leftrightarrow (\exists x \in S)[\neg(P(x) \Rightarrow Q(x))] \\ &\Leftrightarrow (\exists x \in S)[P(x) \wedge (\neg Q(x))].\end{aligned}$$

Example. Negate $(\exists x \in S)[P(x) \wedge Q(x)]$.

Here we use $\neg(P \wedge Q) \Leftrightarrow (\neg P) \vee (\neg Q)$, so we can write

$$\neg[(\exists x \in S)(P(x) \wedge Q(x))] \Leftrightarrow (\forall x \in S)[(\neg P(x)) \vee (\neg(Q(x)))].$$

For a typical case of a statement first quantified by $\forall$, and then be $\exists$, we note that we can group these as follows:

$$(\forall x \in R)(\exists y \in S)P(x,y) \Leftrightarrow (\forall x \in R)[(\exists y \in S)P(x,y)].$$

Thus

$$\begin{aligned}\neg[(\forall x \in R)(\exists y \in S)P(x,y)] &\Leftrightarrow \neg[(\forall x \in R)[(\exists y \in S)P(x,y)]] \\ &\Leftrightarrow (\exists x \in R)[\neg[(\exists y \in S)P(x,y)]] \\ &\Leftrightarrow (\exists x \in R)(\forall y \in S)[\neg P(x,y)].\end{aligned}$$

Ultimately we have, in turn, the $\forall$'s become $\exists$'s, the $\exists$'s become $forall$'s, the variables are quantified in the same order as before, and finally the statement P is replaced by its negation $\neg P$. The pattern would continue no matter how many universal and existential quantifiers arise. To summarize for the case of two quantifiers,

$$\begin{aligned}\neg[(\forall x \in R)(\exists y \in S)P(x,y)] &\Leftrightarrow (\exists x \in R)(\forall y \in S)[\neg P(x,y)], \\ \neg[(\exists x \in R)(\forall y \in S)P(x,y)] &\Leftrightarrow (\forall x \in R)(\exists y \in S)[\neg P(x,y)].\end{aligned}$$

Example. Consider the following statement, which is false:

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})[xy = 1].$$

One could say that the statement says every real number x has a real number reciprocal y. This is false, but before that is explained, we compute the negation which must be true:

$$\neg[(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)] \Leftrightarrow (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(xy \neq 1).$$

Indeed, there exists such an x, namely $x = 0$, such that $xy \neq 1$ for all y.

Definition 7. A compound statement formed by the component statements $p_1, p_2, ..., p_n$ is called a tautology iff its truth table column consists entirely of entries with truth value 1 for each of the 2^n possible truth value combinations (1 and 0) of the component statements.

Definition 8. A compound statement formed by the component statements $p_1, p_2, ..., p_n$ is called a contradiction iff its truth table column consists entirely of entries with truth value 0 for each of the 2^n possible truth value combinations (1 and 0) of the component statements.

Intervals

Notation	Set description	Picture
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leqslant x \leqslant b\}$	
$[a, b)$	$\{x \mid a \leqslant x < b\}$	
$(a, b]$	$\{x \mid a < x \leqslant b\}$	
(a, ∞)	$\{x \mid x > a\}$	
$[a, \infty)$	$\{x \mid x \geqslant a\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leqslant b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

Absolute value

For any real number x the absolute value or modulus of x is denoted by $|x|$ and is defined as

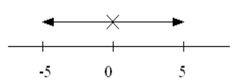
$$|x| = \begin{cases} x, & \text{if} \quad x \geqslant 0, \\ -x, & \text{if} \quad x < 0. \end{cases}$$

The absolute value of x is always either positive or zero, but **NEVER** negative. Fundamental properties:

- $|a| = 0 \Leftrightarrow a = 0$,
- $|a \cdot b| = |a| \cdot |b|$,
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$,
- $\sqrt{x^2} = |x|$,
- $|a + b| \leqslant |a| + |b|$ (triangle inequality),
- $|x^a| = |x|^a$.

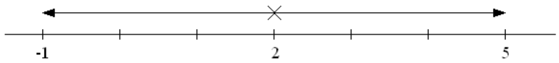
The concept of absolute value is very strongly connected to the concept of distance. The absolute value of a number is that number's distance from 0 on the number line.

Example. Solve the equation $|x| = 5$. Look at the number line:



Both -5 and 5 are the required distance from zero. The solution set for the equation is $\{-5, 5\}$.

Example. Solve the equation $|x - 2| = 3$. Look at the number line:



Both -1 and 5 are the required distance from 2. The solution set for the equation is $\{-1, 5\}$.

Remark 5. In general, whenever you have an equation of the form $|A| = B$, with $B > 0$, then the solutions can be found by solving the following equations:

$$A = -B \qquad \text{and} \qquad A = B.$$

Note that here A and B represent any algebraic expressions.

Example. Solve the equation $|2x + 5| = 3x - 1$.

$$2x + 5 = -3x + 1 \qquad \text{and} \qquad 2x + 5 = 3x - 1.$$

Hence, the solution set for equation is $\{-\frac{4}{5}, 6\}$.
Example. Solve the inequality $|7x-6| < 15$. We write this as a 3-way inequality

$$-15 < 7x - 6 < 15.$$

We get

$$-\frac{9}{7} < x < 3.$$

Example. Solve the inequality $|3x - 10| \geqslant 4$. Note the two inequalities without absolute values

$$3x - 10 \leqslant -4 \qquad \text{or} \qquad 3x - 10 \geqslant 4.$$

Hence, we obtain

$$x \leqslant 2 \qquad \text{or} \qquad x \geqslant \frac{14}{3}.$$

Remark 6. To solve an inequality of the form $|A| \leqslant B$, with $B \geqslant 0$, just solve the equivalent three-way inequality:

$$-B \leqslant A \leqslant B.$$

In this case, B can be zero, as long as A is also zero.

Remark 7. To solve an inequality of the form $|A| \geqslant B$, with $B \geqslant 0$, take the union of the solutions to the two inequalities:

$$A \leqslant -B \qquad \text{or} \qquad A \geqslant B.$$