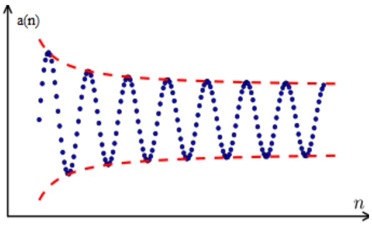
Lecture 3

Infinite Sequences

Definition 1. A numerical sequence (shortly, a sequence) is a function $a : \mathbb{N} \to \mathbb{R}$. We will denote it by the symbol $\{a_n\} \subset \mathbb{R}$, where $a_n = a(n)$, $n \in \mathbb{N}$ is called element of the sequence, n is called index.

An visual representation of a sequence is just given by the plot of the graph of the function $n \mapsto a_n$.



Example. Some sequences can be defined by giving a formula for the nth term. In the following examples we give two descriptions of the sequence: one by using the preceding notation, another by using the defining formula. Notice that n does not have to start at 1.

a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$, **b)** $a_n = \dfrac{(-1)^n(n+1)}{3^n}$,

c) $a_n = \sqrt{n-3}$, $n \geqslant 3$, **d)** $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$.

Example. Here are some sequences that do not have a simple defining equation.

a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.

b) If we let a_n be the digit in the nth decimal place of the number e, then $\{a_n\}$ is a well defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, ...\}$$

c) The Fibonacci sequence is defined recursively by the conditions

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad n \geqslant 3.$$

Each term is the sum of the two preceding terms. The first few terms are

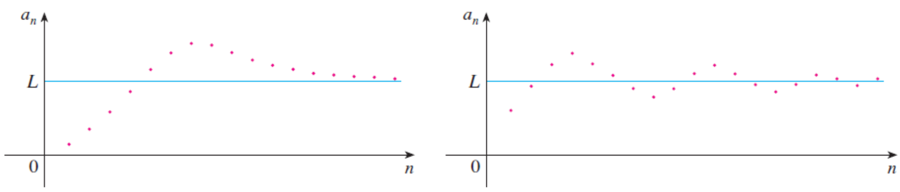
$$\{1, 1, 2, 3, 5, 8, 13, 21, ...\}.$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits.

Definition 2. A sequence $\{a_n\}$ has the limit and we write

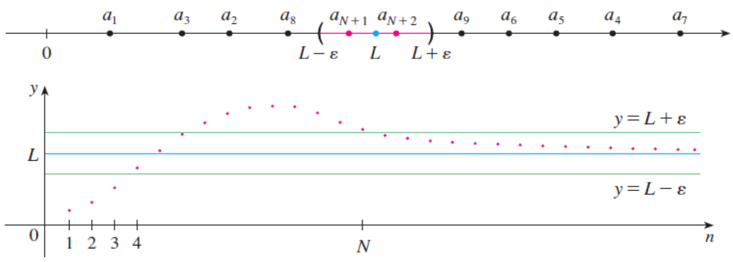
$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to \infty \quad \text{as} \quad n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).



Definition 3. Let $\{a_n\} \subset \mathbb{R}$. We say that $a_n \to L \in \mathbb{R}$ (we read as: $\{a_n\}$ tends to L as n tends to ∞) if

$$\forall_{\varepsilon>0}\exists_{N\in\mathbb{N}} \ \ (n > N \Rightarrow |a_n - L| < \varepsilon).$$



Example. Show that $\frac{1}{n} \to 0$. It is clear that as n gets bigger, $\frac{1}{n}$ gets smaller close to 0. We have to fix $\varepsilon > 0$ and find N such that

$$|\tfrac{1}{n} - 0| < \varepsilon.$$

Now,

$$|\tfrac{1}{n} - 0| < \varepsilon \Leftrightarrow \tfrac{1}{n} < \varepsilon \Leftrightarrow n > \tfrac{1}{\varepsilon} = N.$$

Of course $N = \frac{1}{\varepsilon} \notin \mathbb{N}$, but we may take $N = [\frac{1}{\varepsilon}]+1$, where $[\cdot]$ denotes integer part.

Definition 4. Let $\{a_n\} \subset \mathbb{R}$. We say that $a_n \to \infty$ if

$$\forall_{M>0}\exists_{N\in\mathbb{N}} \ \ (n > N \Rightarrow a_n \geqslant M).$$

We write also $\lim\limits_{n\to\infty} a_n = \infty$. Similarly is defined $\lim\limits_{n\to\infty} a_n = -\infty$

$$\forall_{M<0}\exists_{N\in\mathbb{N}} \ \ (n > N \Rightarrow a_n \leqslant M).$$

Example. Show that $\frac{n^2+1}{n} \to \infty$. Fix $M > 0$, we have to find N such that

$$\frac{n^2+1}{n} \geqslant M.$$

Studying the inequality, we get

$$\frac{n^2+1}{n} \geqslant M \Leftrightarrow n^2 - Mn + 1 \geqslant 0.$$

This is a second degree inequality in n. Let $\Delta = M^2 - 4$. We have

- if $\Delta < 0$ the inequality is fulfilled for every $n \in \mathbb{N}$, this means we can take $N = 0$,
- if $\Delta \geqslant 0$, the solutions are

$$n \leqslant \frac{M-\sqrt{\Delta}}{2}, \quad n \geqslant \frac{M+\sqrt{\Delta}}{2}.$$

The first has at maximum a finite number of solutions which are natural numbers. But if we consider the second one and take $N = \frac{M+\sqrt{\Delta}}{2}$, we see that any $n \geqslant N$ is a solution. In any case we are able to find an initial index N.

A sequence having a finite limit is shortly called convergent, while if the limit is infinite is called divergent. There's a third possible situation, because not all the sequences are convergent or divergent. **Example.** Consider the sequence

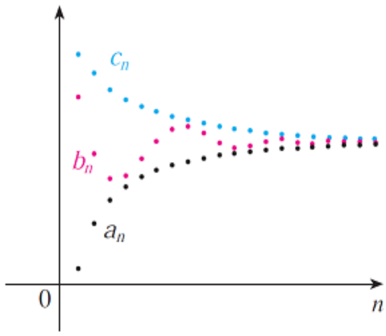
$$a_n = (-1)^n, \quad \text{that is} \quad +1; -1; +1; -1; \dots$$

It seems evident that such a sequence cannot have a limit.

Theorem 5 (Limit Laws). *If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then*

- $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$,
- $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$,
- $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n$,
- $\lim\limits_{n\to\infty} a_n b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$,
- $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ *if* $\lim\limits_{n\to\infty} b_n \neq 0$,
- $\lim\limits_{n\to\infty} a_n^p = [\lim\limits_{n\to\infty} a_n]^p$ *if* $p > 0$ *and* $a_n > 0$.

Theorem 6 (Squeeze Theorem). *If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.*



Theorem 7. *If* $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Theorem 8. *We have*

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad \textit{if} \quad r > 0.$$

Example. Find $\lim_{n\to\infty} \frac{n}{n+1}$. Divide numerator and denominator by the highest power of n that occurs in the denominator and then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1.$$

Example. Is the sequence $a_n = \frac{n}{\sqrt{n+10}}$ convergent or divergent? We divide numerator and denominator by n

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n}+\frac{10}{n^2}}} = \infty$$

because the numerator is constant and the denominator approaches 0. So $\{a_n\}$ is divergent.

Example. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists. We first calculate the limit of the absolute value

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Theorem 9. *If* $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L, *then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Example. Find $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$. Because the sine function is continuous at 0, we write

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin 0 = 0.$$

Example. Discuss the convergence of the sequence $a_n = \frac{n!}{n^n}$, where $n! = 1 \cdot 2 \cdot \ldots \cdot n$. Let us write

$$a_n = \frac{1 \cdot 2 \cdot \ldots \cdot n}{n \cdot n \cdot \ldots \cdot n} = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \ldots \cdot n}{n \cdot n \cdot \ldots \cdot n}\right).$$

Notice that the expression in parentheses is at most 1 because the numerator is less than (or equal to) the denominator. So

$$0 < a_n \leqslant \frac{1}{n}.$$

We know that $\frac{1}{n} \to 0$ as $n \to \infty$. Therefore $a_n \to 0$ as $n \to \infty$ by the Squeeze Theorem.

Example. Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. Clearly $(1+1)^n \geqslant 1 + n > n$, so that $n^{\frac{1}{n}} - 1 < 1$ for $n \geqslant 1$. Also, for $n \geqslant 1$, we observe that $n^{\frac{1}{n}} \geqslant 1$, so that $n^{\frac{1}{n}} - 1 = x_n$ with $x_n \geqslant 0$. In particular, using the binomial theorem, we deduce that

$$n = (1 + x_n)^n \geqslant 1 + nx_n + \frac{n(n-1)}{2}x_n^2 \geqslant 1 + \frac{n(n-1)}{2}x_n^2,$$

which implies that

$$0 \leqslant x_n = n^{\frac{1}{n}} - 1 \leqslant \sqrt{\frac{2}{n}} \quad \text{for} \quad n \geqslant 1.$$

By the Squeeze Theorem, $x_n \to 0$ as $n \to 0$, since $\frac{1}{\sqrt{n}} \to 0$. We conclude that $n^{\frac{1}{n}} \to 1$ as $n \to \infty$, as desired.

Theorem 10. *The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r*

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \textit{if} \quad -1 < r < 1, \\ 1 & \textit{if} \quad r = 1 \end{cases}$$

Definition 11. A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \ldots$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is monotonic if it is either increasing or decreasing.

Example. The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

and so $a_n > a_{n+1}$ for all $n \geqslant 1$.

Example. Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing. We must show that $a_{n+1} < a_n$, that is,

$$\frac{n+1}{(n+1)^2} < \frac{n}{n^2+1}.$$

This inequality is equivalent to the one we get by cross-multiplication

$$\begin{aligned} \frac{n+1}{(n+1)^2} &< \frac{n}{n^2+1}, \\ (n+1)(n^2+1) &< n[(n+1)^2+1], \\ n^3+n^2+n+1 &< n^3+2n^2+2n, \\ 1 &< n^2+n. \end{aligned}$$

Since $n \geqslant 1$, we know that the inequality $n^2+n > 1$ is true. Therefore $a_{n+1} < a_n$ and so $\{a_n\}$ is decreasing.

Definition 12. A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \leqslant M \quad \text{for all} \quad n \geqslant 1.$$

It is bounded below if there is a number m such that

$$m \leqslant a_n \quad \text{for all} \quad n \geqslant 1.$$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Example. The sequence $a_n = n$ is bounded below ($a_n > 0$) but not above. The sequence $a_n = \frac{n}{n+1}$ is bounded because $0 < a_n < 1$ for all n.

Theorem 13 (Monotonic Sequence Theorem)**.** *Every bounded, monotonic sequence is convergent.*

Remark 1. A sequence that is increasing and bounded above is convergent. Likewise, a decreasing sequence that is bounded below is convergent.

Example. Set $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}$. Then $\{a_n\}$ converges. Note that $a_n > 0$ for all $n \geqslant 1$ and

$$a_{n+1} = a_n \left(\frac{2n+1}{2n+2} \right) < a_n$$

Thus, $\{a_n\}$ is decreasing and bounded below by 0. Applying Monotonic Sequence Theorem, we see that $\{a_n\}$ converges. Note also that $a_n < 1$ for $n \geqslant 1$.

Example. Let $a_n = \left(1+\frac{1}{n}\right)^n$, $n \geqslant 1$. The sequence $\{a_n\}$ is called Euler's sequence. Note that using Bernoulli's inequality $(1+x)^n \geqslant 1+nx$ for $x \geqslant 0$ and $n \geqslant 1$ for $x = \frac{1}{n}$ we get

$$\left(1+\frac{1}{n}\right)^n \geqslant 2 \quad \text{for} \quad n \geqslant 1.$$

We show that the sequence is increasing. We have applying Bernoulli's inequality

$$\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{n+1}{n}\left(1-\frac{1}{(n+1)^2}\right)^{n+1} > \frac{n+1}{n}\left[1-\frac{n+1}{(n+1)^2}\right] = 1.$$

We show that the sequence is bounded. Since $k! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot k \geqslant 1 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^{k-1}$ for $k \geqslant 2$, we have

$$2 < a_n < 1 + \sum_{k=1}^{n} \frac{1}{k!} < 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} < 3.$$

It follows that the sequence $\{a_n\}$ converges to a real number that is at most 3.

Definition 14.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Remark 2.

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \quad x \in \mathbb{R}\setminus\{0\}.$$

Example.

a)

$$\lim_{n\to\infty}\left(1-\frac{3}{n}\right)^{n+2} = \lim_{n\to\infty}\left[\left(1-\frac{1}{3n}\right)^{3n}\right]^{\frac{n+2}{3n}} = e^{-\frac{1}{3}},$$

since $\lim\limits_{n\to\infty}\frac{n+2}{3n} = \frac{1}{3}$.

b)

$$\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n = e^5.$$

Theorem 15. *For $p > 0$, we have*

$$\lim_{n\to\infty}\frac{r^n}{n^p} = \begin{cases} 0 & \textit{if} \quad |r| \leqslant 1, \\ \infty & \textit{if} \quad r > 1, \\ \textit{does not exist} & \textit{if} \quad r < -1. \end{cases}$$

Theorem 16.
- $(\pm\infty) + a = \pm\infty$, $a \in \mathbb{R}$
- $(+\infty) + (+\infty) = +\infty$,
- $(-\infty) + (-\infty) = -\infty$,
- $(+\infty)\cdot a = sgn(a)\infty, \quad (a \neq 0)$,
- $(-\infty)\cdot a = sgn(a)(-\infty), \quad (a \neq 0)$,
- $(+\infty)\cdot(+\infty) = +\infty$,
- $(+\infty)\cdot(-\infty) = -\infty$,
- $(-\infty)\cdot(-\infty) = +\infty$,
- $\frac{a}{\pm\infty} = 0, \quad (a \in \mathbb{R})$,
- $\frac{\pm\infty}{a} = sgn(a)\infty, \quad (a \neq 0)$,
- $\frac{-\infty}{a} = sgn(a)(-\infty), \quad (a \neq 0)$,
- $\frac{+\infty}{0^+} = \frac{-\infty}{0^-} = +\infty, \frac{+\infty}{0^-} = \frac{-\infty}{0^+} = -\infty$.

Remark 3. Indeterminate forms:

- $(\pm\infty) + (\pm\infty)$ opposite signs,
- $(\pm\infty)\cdot 0$,
- $\frac{0}{0}$,

- $\frac{\pm\infty}{\pm\infty}$.

Example. Assume that $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} b_n = +\infty$.

a) let $a_n = b_n = n$, then

$$\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} n - n = 0,$$

b) let $a_n = n^2$ and $b_n = n$, then

$$\begin{aligned}\lim_{n\to\infty} a_n - b_n &= \lim_{n\to\infty} n^2 - n = \lim_{n\to\infty} n^2\,(1 - \frac{1}{n})\\ &= \lim_{n\to\infty} n^2 \cdot \lim_{n\to\infty} (1 - \frac{1}{n}) = \infty \cdot 1 = \infty.\end{aligned}$$

Example. Assume that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = \infty$.

a) let $a_n = b_n = n$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n} = 1,$$

b) let $a_n = 2n^2$ and $b_n = n$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2}{n} = \lim_{n\to\infty} 2n = \infty,$$