Lecture 4

Limits of functions

Limits of function - introduction

Once we are acquainted with limits for sequences, we can extend this concept, moving from the discrete variable $n \in \mathbb{N}$ to the continuous variable $x \in \mathbb{R}$. In other words, we want to give a meaning to

$$\lim_{x \to x_0} f(x) = g,$$

where f is a real variable function.

We will translate this by using sequences in the following way

$$\lim_{x \to x_0} f(x) = g \Leftrightarrow \forall_{a_n \to x_0} \quad f(a_n) \to g.$$

The concept of limit is the base for another fundamental concept: that one of continuous function

$$f \quad \text{continuous at } x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0).$$

Definition 1. Suppose $f(x)$ is defined when is near the number x_0. (This means that f is defined on some open interval that contains x_0, except possibly at x_0 itself.) Then we write

$$\lim_{x \to x_0} f(x) = L$$

and say "the limit of $f(x)$, as x approaches x_0, equals L" if we can make the values of arbitrarily close to (as close to L as we like) by taking x to be sufficiently close to x_0 (on either side of x_0) but not equal to x_0.

Definition 2. Let f be a function $f : D_f \to \mathbb{R}$. The function f has a limit L at the point x_0 if and only if for every sequence $\{a_n\}$ such that $\lim\limits_{n \to \infty} a_n = x_0$, the sequence $f(a_n)$ approaches a number L. In other words,

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall_{\{a_n\} \subset D_f \setminus \{x_0\}} a_n \to x_0 \Rightarrow f(a_n) \to L.$$

Definition 3. Let f be a function $f : D_f \to \mathbb{R}$. The function f has a limit L at the point x_0 if and only if

$$\forall_{\varepsilon > 0} \exists_{\delta > 0} |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Remark 1. *We may define also the other limits, namely the finite limit at infinity and the infinite limit at the point x_0, in the following way*

$$\lim_{x \to \infty} f(x) = L \Leftrightarrow \forall_{\varepsilon > 0} \exists_{M > 0} x > M \Rightarrow |f(x) - L| < \varepsilon$$

and

$$\lim_{x \to x_0} f(x) = +\infty \Leftrightarrow \forall_{A > 0} \exists_{\delta > 0} |x - x_0| < \delta \Rightarrow f(x) > A.$$

Example. Prove that $\lim_{x\to 3}(4x-5)=7$.

1. We guess a value of δ Let $\varepsilon>0$. We want to find a number δ such that if $|x-3|<\delta$ then $|(4x-5)-7|<\varepsilon$. But $|(4x-5)-7|=4|x-3|$. Therefore we want δ such that

$$|x-3|<\delta \Rightarrow 4|x-3|<\varepsilon,$$
$$|x-3|<\delta \Rightarrow |x-3|<\frac{\varepsilon}{4}.$$

This suggests that we should choose $\delta=\frac{\varepsilon}{4}$.

2. We show that this δ works. Given ε, choose $\delta=\frac{\varepsilon}{4}$. If $|x-3|<\delta$, then

$$|(4x-5)-7|=4|x-3|<4\delta=4\frac{\varepsilon}{4}=\varepsilon.$$

Therefore, by the definition of a limit,

$$\lim_{x\to 3}(4x-5)=7.$$

Example. Prove that $\lim_{x\to 0}\frac{1}{x^2}=\infty$.
Let $M>0$. We want to find a number δ such that if $|x|<\delta$ then $\frac{1}{x^2}>M$. But

$$\frac{1}{x^2}>M,$$
$$|x|<\frac{1}{\sqrt{M}}.$$

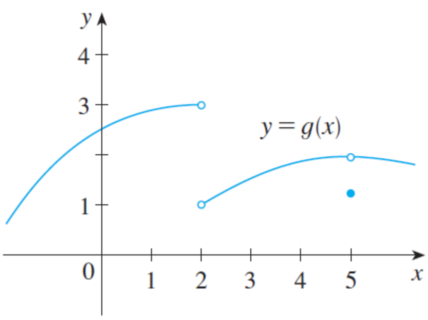
So if we choose $\delta=\frac{1}{\sqrt{M}}$ and $|x|<\delta=\frac{1}{\sqrt{M}}$, then $\frac{1}{x^2}>M$. This shows that $\lim_{x\to 0}\frac{1}{x^2}=\infty$.

One-sided limits

Definition 4. We define one-sided limits in the following way

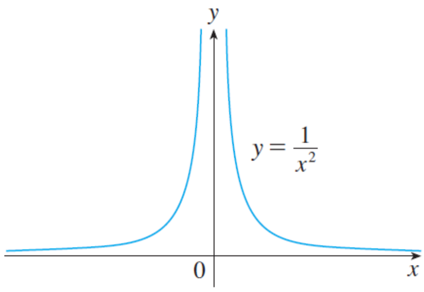
- left-hand limit $\lim_{x\to x_0^-}f(x)=L$,
- right-hand limit $\lim_{x\to x_0^+}f(x)=L$.

Example.



$$\lim_{x\to 2^-} g(x) = 3, \quad \lim_{x\to 2^+} g(x) = 1, \quad \lim_{x\to 5^-} g(x) = 2, \quad \lim_{x\to 5^+} g(x) = 2.$$

Example.



$$\lim_{x\to 0^-} g(x) = +\infty, \quad \lim_{x\to 0^+} g(x) = +\infty.$$

Existence of limit

Theorem 5. *A function f has a limit at x_0 if and only if it has left- and right-hand limits at x_0 and they are equal. More specifically,*

$$\lim_{x\to x_0} f(x) = L$$

if and only if

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = L.$$

Example. Let $f(x) = \frac{|x|}{x}$ for $x \neq 0$. If $x < 0$, then $f(x) = \frac{-x}{x} = -1$, so

$$\lim_{x\to 0^-} f(x) = -1.$$

If $x > 0$, then $f(x) = \frac{x}{x} = 1$, so

$$\lim_{x\to 0^+} f(x) = 1.$$

Therefore the limit of f when x tends to 0 does not exist.

Example. If

$$f(x) = \begin{cases} \sqrt{x-4}, & \text{if} \quad x > 4, \\ 8-2x, & \text{if} \quad x < 4 \end{cases}$$

determine whether $\lim_{x\to 4} f(x)$ exists.

Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have

$$\lim_{x\to 4^+} f(x) = 0.$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x\to 4^-} f(x) = 0.$$

The right- and left-hand limits are equal. Thus the limit exists and $\lim_{x\to 4} f(x) = 0$.

Basic Properties

Theorem 6. *If the functions f and g have finite limits at the point x_0, then*

- $\lim\limits_{x\to x_0} c = c$,
- $\lim\limits_{x\to x_0} \big(f(x) \pm g(x)\big) = \lim\limits_{x\to x_0} f(x) \pm \lim\limits_{x\to x_0} g(x)$,
- $\lim\limits_{x\to x_0} cf(x) = c \lim\limits_{x\to x_0} f(x)$,
- $\lim\limits_{x\to x_0} \big(f(x) \cdot g(x)\big) = \lim\limits_{x\to x_0} f(x) \cdot \lim\limits_{x\to x_0} g(x)$
- $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$ *if* $\lim\limits_{x\to x_0} g(x) \neq 0$
- $\lim\limits_{x\to x_0} [f(x)]^n = \left[\lim\limits_{x\to x_0} f(x)\right]^n$, *where n is a positive integer.*

Theorem 7. *If the functions f has a finite limit at the point x_0, then*

- $\lim\limits_{x\to x_0} x = x_0$,
- $\lim\limits_{x\to x_0} x^n = x_0^n$, *where n is a positive integer,*
- $\lim\limits_{x\to x_0} \sqrt[n]{x} = \sqrt[n]{x_0}$, *where n is a positive integer, (if n is even, we assume that $x_0 > 0$),*
- $\lim\limits_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to x_0} f(x)}$, *where n is a positive integer, (if n is even, we assume that $\lim\limits_{x\to x_0} f(x) > 0$).*

Theorem 8.

- $(\pm\infty) + a = \pm\infty$,
- $(+\infty) + (+\infty) = +\infty$,
- $(-\infty) + (-\infty) = -\infty$,
- $(+\infty) \cdot a = sgn(a)\infty, \quad (a \neq 0)$,
- $(-\infty) \cdot a = sgn(a)(-\infty), \quad (a \neq 0)$,
- $(+\infty) \cdot (+\infty) = +\infty$,
- $(+\infty) \cdot (-\infty) = -\infty$,
- $(-\infty) \cdot (-\infty) = +\infty$,
- $\frac{a}{\pm\infty} = 0, \quad (a \in \mathbb{R})$,
- $\frac{\pm\infty}{a} = sgn(a)\infty, \quad (a \neq 0)$,
- $\frac{-\infty}{a} = sgn(a)(-\infty), \quad (a \neq 0)$,
- $\frac{+\infty}{0^+} = \frac{-\infty}{0^-} = +\infty, \frac{+\infty}{0^-} = \frac{-\infty}{0^+} = -\infty$.

Indeterminate forms

Remark 2. *Indeterminate forms:*

- $(\pm\infty) + (\pm\infty)$ *opposite signs,*
- $(\pm\infty) \cdot 0$,
- $\frac{0}{0}$,
- $\frac{\pm\infty}{\pm\infty}$.

Example. Assume that $\lim\limits_{x\to\infty} f(x) = +\infty$ and $\lim\limits_{x\to\infty} g(x) = +\infty$.

a) let $f(x) = g(x) = x$, then

$$\lim_{x\to+\infty} f(x) - g(x) = \lim_{x\to+\infty} x - x = 0,$$

b) let $f(x) = x^2$ and $g(x) = x$, then

$$\lim_{x\to+\infty} f(x) - g(x) = \lim_{x\to+\infty} x^2 - x = \lim_{x\to+\infty} x^2\,(1 - \frac{1}{x})$$
$$= \lim_{x\to+\infty} x^2 \cdot \lim_{x\to+\infty} (1 - \frac{1}{x}) = +\infty \cdot 1 = +\infty.$$

Example. Assume that $\lim\limits_{x\to0} f(x) = +\infty$ and $\lim\limits_{x\to0} g(x) = 0$.

a) let $f(x) = x$ and $g(x) = \frac{1}{x^2}$, then

$$\lim_{x\to0} f(x)\cdot g(x) = \lim_{x\to0} x \cdot \frac{1}{x^2} = \lim_{x\to0} \frac{1}{x} \quad \text{does not exist,}$$

b) let $f(x) = 2x^2$ and $g(x) = \frac{1}{x^2}$, then

$$\lim_{x\to0} f(x)\cdot g(x) = \lim_{x\to0} 2x^2 \cdot \frac{1}{x^2} = 2,$$

c) let $f(x) = x^2$ and $g(x) = \frac{1}{x^4}$, then

$$\lim_{x\to0} f(x)\cdot g(x) = \lim_{x\to0} x^2 \cdot \frac{1}{x^4} = \lim_{x\to0} \frac{1}{x^2} = +\infty.$$

Example. Assume that $\lim\limits_{x\to0} f(x) = 0$ and $\lim\limits_{x\to0} g(x) = 0$.

a) let $f(x) = x$ and $g(x) = x$, then

$$\lim_{x\to0} \frac{f(x)}{g(x)} = \lim_{x\to0} \frac{x}{x} = 1,$$

b) let $f(x) = x^2$ and $g(x) = x$, then

$$\lim_{x\to0} \frac{f(x)}{g(x)} = \lim_{x\to0} \frac{x^2}{x} = \lim_{x\to0} x = 0.$$

Example. Assume that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} g(x) = +\infty$.

a) let $f(x) = g(x) = x$, then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{x}{x} = 1,$$

b) let $f(x) = 2x^2$ and $g(x) = x$, then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{2x^2}{x} = \lim_{x\to+\infty} 2x = +\infty,$$

Example. We use the properties of limits to evaluate each limit:

a) the function $f(x) = 3x^5 - 2x^2 + 3x - 1$ is continuous, thus

$$\lim_{x\to 2}(3x^5 - 2x^2 + 3x - 1) = 3\cdot 2^5 - 2\cdot 2^2 + 3\cdot 2 - 1 = 93,$$

b) the function $f(x) = \sqrt{x^2 - 4}$ is continuous, thus

$$\lim_{x\to 4} \sqrt{x^2-4} = \sqrt{4^2-4} = 2\sqrt{3},$$

c) note that $\lim_{x\to -1} x + 4 = 3 \neq 0$, thus

$$\lim_{x\to -1} \tfrac{x^2+1}{x+4} = \tfrac{\lim_{x\to -1}(x^2+1)}{\lim_{x\to -1}(x+4)} = \tfrac{2}{3}.$$

Example. We use properties of limits and algebraic manipulations to find the limit $\lim_{x\to 3} \frac{x^2-x-6}{x-3}$. Since $\lim_{x\to 3}(x-3) = 0$, we factor the numerator to see if we can simplify the function

$$\frac{x^2-x-6}{x-3} = \frac{(x-3)(x+2)}{x-3} = x+2.$$

The left and right sides are equal for all values of x except $x = 3$. Since the limit process involves functional values for x near 3 but not equal to 3, we can write

$$\begin{aligned}\lim_{x\to 3} \frac{x^2-x-6}{x-3} &= \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} \\ &= \lim_{x\to 3}(x+2) = 5.\end{aligned}$$

Example. We use properties of limits and algebraic manipulations to find the limit $\lim_{x\to -1} \frac{x-1}{x^2-1}$. Since $\lim_{x\to -1}(x^2-1) = 0$, we factor the denominator to see if we can simplify the function

$$\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}.$$

The left and right sides are equal for all values of x except $x = -1$. Since the limit process involves functional values for x near -1 but not equal to -1, we can write

$$\lim_{x \to -1} \frac{x-1}{x^2-1} = \lim_{x \to -1} \frac{1}{x+1}.$$

This limit does not exist, since

$$\lim_{x \to -1^-} \frac{1}{x+1} = -\infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{x+1} = +\infty.$$

Example. We use properties of limits and algebraic manipulations to find the limit $\lim\limits_{x \to 9} \frac{\sqrt{x}-3}{x-9}$. We can simplify the function

$$\begin{aligned} \frac{\sqrt{x}-3}{x-9} &= \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \frac{x-9}{(x-9)(\sqrt{x}+3)} \\ &= \frac{1}{\sqrt{x}+3}. \end{aligned}$$

The left and right sides are equal for all values of x except $x = 9$. Since the limit process involves functional values for x near 9 but not equal to 9, we can write

$$\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x \to 9} \frac{1}{\sqrt{x}+3} = \tfrac{1}{6}.$$

Example. We compute the limit $\lim\limits_{x \to \infty} \frac{5x+4}{2x+3}$. Since $\lim\limits_{x \to \infty}(5x+4)$ and $\lim\limits_{x \to \infty}(2x+3)$ are equal to ∞, we divide numerator and denominator by x in order to express the fraction in a form where the limits of the numerator and denominator are finite

$$\lim_{x \to \infty} \frac{5x+4}{2x+3} = \lim_{x \to \infty} \frac{\frac{5x+4}{x}}{\frac{2x+3}{x}} = \lim_{x \to \infty} \frac{5+\frac{4}{x}}{2+\frac{3}{x}} = \frac{5}{2}.$$

Example. We compute the limit $\lim\limits_{x \to \infty} \frac{4x^2+3x+2}{2x^3+5}$. We divide numerator and denominator by x^3, the highest power of x that occurs in the denominator

$$\lim_{x \to \infty} \frac{4x^2+3x+2}{2x^3+5} = \lim_{x \to \infty} \frac{\frac{4x^2+3x+2}{x^3}}{\frac{2x^3+5}{x^3}} = \lim_{x \to \infty} \frac{\frac{4}{x}+\frac{3}{x^2}+\frac{2}{x^3}}{2+\frac{5}{x^3}} = \frac{0}{2} = 0.$$

Limits at infinity for rational functions

Theorem 9. *If*

$$f(x) = \frac{p(x)}{q(x)},$$

where

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1} x^{n-1} + \dots + a_0, \quad \text{and} \quad a_n \neq 0, \\ q(x) &= b_m x^m + b_{m-1} x^{m-1} + \dots + b_0, \quad \text{and} \quad b_m \neq 0, \end{aligned}$$

then

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} 0 & \text{if} \quad n < m, \\ \frac{a_n}{b_m} & \text{if} \quad n = m, \\ \pm\infty & \text{if} \quad n > m. \end{cases}$$

Comparison properties - two policemen

Theorem 10 (The Squeeze Theorem)**.** *Let* $f, g, h : D \to \mathbb{R}$*. Suppose that*

- $f(x) \leqslant g(x) \leqslant h(x)$,
- $\lim\limits_{x\to x_0} f(x) = g$ *and* $\lim_{x\to x_0} h(x) = g$,

then $\lim\limits_{x\to x_0} g(x) = g$*.*

Example. Show that $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0$. We have $-1 \leqslant \sin x \leqslant 1$ for any x, therefore

$$-\frac{1}{x} \leqslant \frac{\sin x}{x} \leqslant \frac{1}{x}.$$

Now, $-\frac{1}{x}$ and $\frac{1}{x}$ are two policemen going to 0 as $x \to \infty$. Hence, $\frac{\sin x}{x} \to 0$ as $x \to \infty$.

Important limits

Theorem 11.

$$\lim_{x\to+\infty} a^x = \begin{cases} 0, & \textit{for} \quad 0 \leqslant a < 1 \\ 1, & \textit{for} \quad a = 1 \\ +\infty, & \textit{for} \quad a > 1 \end{cases}$$

$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$$

$$\lim_{x\to 0} \frac{x}{\log_a(x+1)} = \ln a$$

Vertical asymptotes

Definition 12. The line $x = x_0$ is a vertical asymptote of the graph of the function $y = f(x)$ if at least one of the following statements is true:

- $\lim\limits_{x\to x_0^-} f(x) = \pm\infty$,
- $\lim\limits_{x\to x_0^+} f(x) = \pm\infty$.

The function $f(x)$ may or may not be defined at the point x_0, and its precise value at the point $x = x_0$ does not affect the asymptote. For example, for the function

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0, \\ 2 & \text{if } x \leqslant 0. \end{cases}$$

has a limit of $+\infty$ as $x \to 0^+$, $f(x)$ has the vertical asymptote $x = 0$, even though $f(0) = 2$. The graph of this function intersect the vertical asymptote once, at $(0, 2)$. It is impossible for the graph of a function to intersect a vertical asymptote (or a vertical line in general) in more than one point.

Horizontal asymptotes

Definition 13. Horizontal asymptotes are horizontal lines that the graph of the function approaches as $x \to \pm\infty$. The horizontal line $y = c$ is a horizontal asymptote of the function $y = f(x)$ if

$$\lim_{x\to-\infty} f(x) = c \quad \text{or} \quad \lim_{x\to+\infty} f(x) = c.$$

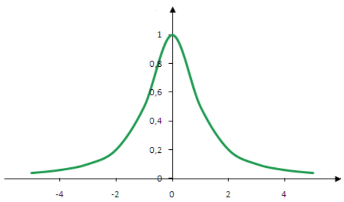
In the first case, $f(x)$ has $y = c$ as asymptote when x tends to $-\infty$, and in the second that $f(x)$ has $y = c$ as an asymptote as x tends to $+\infty$.

Example. The arctangent function satisfies

$$\lim_{x\to-\infty} \arctan(x) = -\tfrac{\pi}{2} \quad \text{and} \quad \lim_{x\to+\infty} \arctan(x) = \tfrac{\pi}{2}.$$

So the line $y = -\frac{\pi}{2}$ is a horizontal asymptote for the arctangent when x tends to $-\infty$, and $y = \frac{\pi}{2}$ is a horizontal asymptote for the arctangent when x tends to $+\infty$.
Example. Functions may lack horizontal asymptotes on either or both sides, or may have one horizontal asymptote that is the same in both directions. For instance, the function $f(x) = \frac{1}{x^2+1}$ has a horizontal asymptote at $y = 0$ when x tends both to $-\infty$ and $+\infty$ because, respectively,

$$\lim_{x\to-\infty} \frac{1}{x^2+1} = \lim_{x\to+\infty} \frac{1}{x^2+1} = 0.$$



Oblique asymptotes

Definition 14. When a linear asymptote is not parallel to the $x-$ or $y-$axis, it is called an oblique asymptote or slant asymptote. A function $f(x)$ is asymptotic to the straight line $y = ax + b (a \neq 0)$ if

$$\lim_{x\to+\infty} [f(x) - (ax + b)] = 0 \;\text{ or }\; \lim_{x\to-\infty} [f(x) - (ax + b)] = 0.$$

In the first case the line $y = ax + b$ is an oblique asymptote of $f(x)$ when x tends to $+\infty$, and in the second case the line $y = ax + b$ is an oblique asymptote of $f(x)$ when x tends to $-\infty$.

The value for a is computed first and is given by

$$a = \lim_{x\to s} \frac{f(x)}{x}$$

where s is either $-\infty$ or $+\infty$ depending on the case being studied. If this limit does not exist then there is no oblique asymptote in that direction.

Having a then the value for b can be computed by

$$b = \lim_{x \to s} (f(x) - ax)$$

where s should be the same value used before. If this limit fails to exist then there is no oblique asymptote in that direction, even should the limit defining a exist.

Example. The function $f(x) = \frac{2x^2+3x+1}{x}$ has

$$a = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{2x^2 + 3x + 1}{x^2} = 2$$

and then

$$b = \lim_{x \to +\infty} (f(x) - ax) = \lim_{x \to +\infty} \left(\frac{2x^2 + 3x + 1}{x} - 2x \right) = 3$$

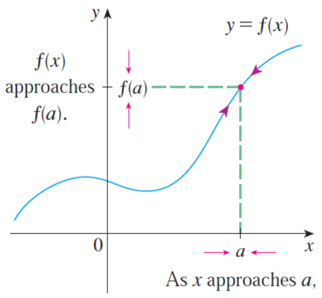
so that $y = 2x + 3$ is the asymptote of $f(x)$ when x tends to $+\infty$.

Continuity

Definition 15. A function f is continuous at the point a if

- $\lim_{x \to a} f(x)$ exists,
- $f(a)$ exists,
- $\lim_{x \to a} f(x) = f(a)$.

A function is continuous on the open interval I if it is continuous at each point on the interval.



Definition 16. A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

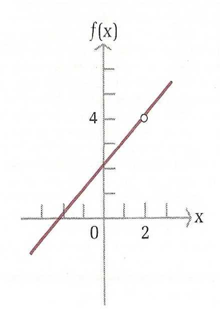
$$\lim_{x \to a^-} f(x) = f(a).$$

Theorem 17. *If f and g are continuous at a and c is a constant, then the following functions are also continuous at a*

- $f \pm g$,
- fg,
- $\frac{f}{g}$ *if* $g(a) \neq 0$,
- cf.

Theorem 18.
- *Any polynomial is continuous everywhere; that is, it is continuous on* $\mathbb{R}$.
- *Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.*
- *Root functions are continuous at every number in their domains.*
- *Trigonometric functions are continuous at every number in their domains.*

Example. Using the definition of continuity, we discuss the continuity of the function $f(x) = \frac{x^2-4}{x-2}$ at points $a = 1$ and $a = 2$.



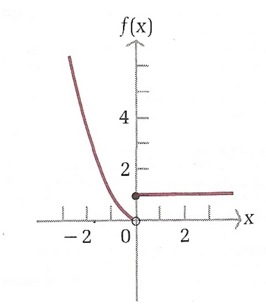
We have $\lim_{x \to 1} f(x) = 3 = f(1)$, thus all three conditions in the definition are satisfied and f is continuous at $a = 1$.

Moreover, $\lim_{x \to 2} f(x) = 4$, but $f(2)$ is not defined. The second condition in the definition is not satisfied and f is not continuous at the point $a = 2$.

Example. Using the definition of continuity, we discuss the continuity of the function

$$f(x) = \begin{cases} x^2, & \text{if} \quad x < 0, \\ 1, & \text{if} \quad x \geqslant 0 \end{cases}$$

at the point $a = 0$.

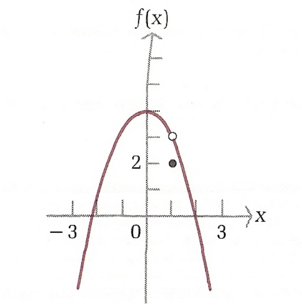


We have $f(0) = 1$, but $\lim_{x\to 0} f(x)$ does not exist. The first condition in the definition is not satisfied and f is not continuous at $a = 0$.

Example. Using the definition of continuity, we discuss the continuity of the function

$$f(x) = \begin{cases} 4 - x^2, & \text{if} \quad x \neq 1, \\ 2, & \text{if} \quad x = 1 \end{cases}$$

at the point $a = 1$.



We have $\lim\limits_{x\to 1} f(x) = 3$ and $f(1) = 2$. But $\lim\limits_{x\to 1} f(x) \neq f(1)$. The third condition in the definition is not satisfied and f is not continuous at $a = 1$.

Composite Functions

Theorem 19. *Suppose that g is continuous at x_0 and $g(x_0)$ is an interior point of D_f and f is continuous at $g(x_0)$. Then $f \circ g$ is continuous at x_0.*

Example. The function

$$f(x) = \sqrt{x}$$

is continuous for $x \geqslant 0$, and the function

$$g(x) = \frac{9 - x^2}{x + 1}$$

is continuous for $x \neq -1$. Since $g(x) > 0$ if $x < -3$ or $-1 < x < 3$, Theorem implies that the function

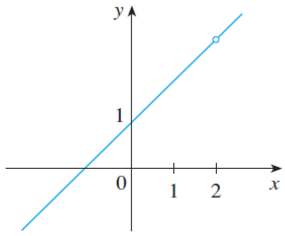
$$(f \circ g)(x) = \sqrt{\frac{9 - x^2}{x + 1}}$$

is continuous on $(-\infty, -3) \cup (-1, 3)$. It is also continuous from the left at -3 and 3.

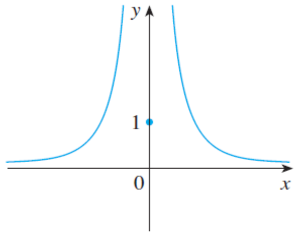
Discontinuities

- The kind of discontinuity is called removable if we can remove the discontinuity by redefining f at just the single number a.
- The discontinuity is called an infinite discontinuity if the function does not approach a particular finite value, the limit does not exist.

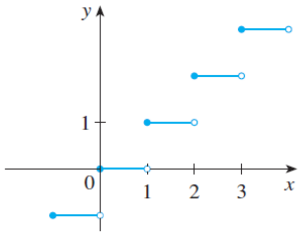
- The discontinuities are called jump discontinuities if the function "jumps" from one value to another.



$f(x) = \dfrac{x^2 - x - 2}{x - 2}$



$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



$f(x) = [\![x]\!]$

Removable Discontinuities

Definition 20. Let f be defined on a deleted neighborhood of x_0 and discontinuous (perhaps even undefined) at x_0. We say that f has a removable discontinuity at x_0 if $\lim\limits_{x \to x_0} f(x)$ exists. In this case, the function

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in D_f \setminus \{x_0\}, \\ \lim_{x \to x_0} f(x) & \text{if} \quad x = x_0 \end{cases}$$

is continuous at x_0.

Example. The function

$$f(x) = x \sin \frac{1}{x}$$

is not defined at $x_0 = 0$, and therefore certainly not continuous there, but $\lim\limits_{x \to 0} f(x) = 0$. Therefore, f has a removable discontinuity at 0.