Lecture 5

Derivative

What is derivative?

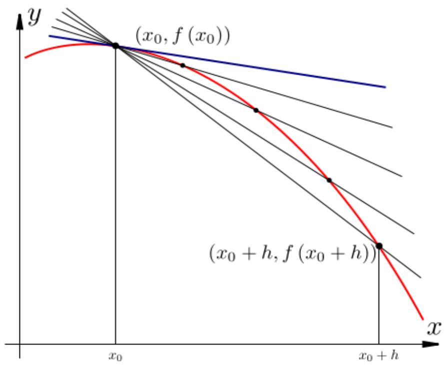
A classical way to introduce the concept of derivative is through the problem to determine the tangent to a plane curve. Imagining the curve as the graph of a certain function $f : D_f \to \mathbb{R}$, we want to find a method to define the tangent to the graph at some point $(x_0, f(x_0))$. A generic (not vertical) straight line passing by $(x_0, f(x_0))$ is described by the equation

$$y = m(x - x_0) + f(x_0) \quad x \in \mathbb{R},$$

So the problem is: how could we compute the angular coefficient m in order that the straight line be tangent to the graph of f? Of course we should define the concept of tangency. We will proceed in the following way: we will fist find a candidate then we will check that in a suitable sense is what we are looking for.

Geometrical idea

The geometrical idea is very easy: to describe a straight line we need two points, so consider a second point along the graph of f, let say $(x_0+h, f(x_0+h))$ with $h \neq 0$ (in such a way we have two points - unique straight line).



Angular coefficient

Then, the angular coefficient for this cord is

$$m = \frac{\text{ordinates variation}}{\text{abscissas variation}} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Such m depends on h and the corresponding line won't be tangent but it will ćut"the graph. However, is natural to think that as $h \to 0$ the straight line will move up to a łimit position", that is should be that one of a tangent line. Therefore, the corresponding slope should be given by

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

This limit, if it exists, is called derivative of f at point x_0 and it is usually denoted by one of the following notations

$$f'(x_0)(\text{Newton}); \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(x_0)(\text{Leibniz})$$

Definition

Definition 1. Let $f : D_f \to \mathbb{R}$. We say that f is differentiable at x_0 if there exists a limit

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \in \mathbb{R}.$$

The number $f'(x_0)$ is called derivative of f at x_0. The straight line of equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is called tangent to f at $(x_0, f(x_0))$.

Theorem 2. *If f is differentiable at x_0, then f is continuous at x_0. The converse to the Theorem is false.*

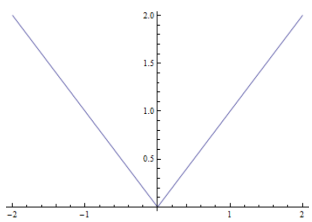
Example

A classical example is the function $f(x) = |x|$. Indeed,

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h},$$

that it does not exist because

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1, \qquad \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

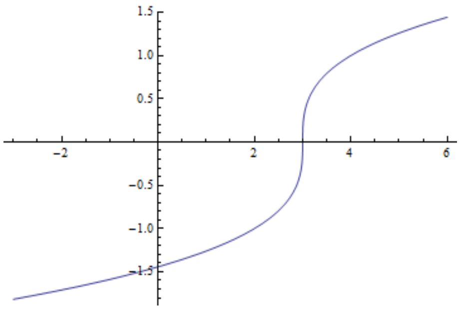


Example

The another example is the function $f(x) = \sqrt[3]{x-3}$. Indeed,

$$\lim_{h\to 0} \frac{f(h)-f(3)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{3+h-3}-\sqrt[3]{3-3}}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{1}{\sqrt[3]{h^2}} = +\infty,$$

which is infinite.



Derivatives of elementary functions

Theorem 3. *We have*

- *if* $f(x) = C$, *then* $f'(x) = 0$;
- *if* $f(x) = e^x$, *then* $f'(x) = e^x$;
- *if* $f(x) = \sin x$, *then* $f'(x) = \cos x$, *while if* $f(x) = \cos x$, *then* $f'(x) = -\sin x$;
- *if* $f(x) = x^a$, *then* $f'(x) = ax^{a-1}$;
- *if* $f(x) = \ln x$, *then* $f'(x) = \frac{1}{x}$.

Proof of Theorem

- Let $f(x) = C$, then
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{C-C}{h} = \lim_{h\to 0} = \frac{0}{h} = 0.$$
- Passing to the exponential
$$\lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0} \frac{e^h-1}{h} = e^x,$$
since $\lim\limits_{h\to 0} \frac{e^h-1}{h} = 1$.
- For sine we have
$$\lim_{h\to 0} \frac{\sin(x+h)-\sin h}{h} = \lim_{h\to 0} \frac{\sin x\cos h - \sin h\cos x - \sin x}{h}$$
$$= \lim_{h\to 0} \sin x \frac{\cos h - 1}{h^2} h + \cos x \frac{\sin h}{h} = \cos x,$$
since $\lim\limits_{h\to 0} \frac{\cos h - 1}{h^2} = \lim\limits_{h\to 0} \frac{\sin h}{h} = 1$.

- For powers we assume that $a \in \mathbb{N}$, then

$$\frac{(x+h)^a - x^a}{h} = \frac{1}{h}\left(\sum_{k=0}^{a}\binom{a}{k}x^{a-k}h^k - x^a\right) = \frac{1}{h}\sum_{k=1}^{a}\binom{a}{k}x^{a-1}h^k$$
$$= \sum_{k=1}^{a}\binom{a}{k}x^{a-k}h^{k-1}.$$

  if $k \geqslant 2$, we have $h^{k-1} \to 0$ as $h \to 0$.

- For the logarithm we have

$$\lim_{h\to 0}\frac{\ln(x+h) - \ln x)}{h} = \lim_{h\to 0}\frac{\ln\left(x\left(1+\frac{h}{x}\right)\right) - \ln x}{h}$$
$$= \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\cdot\frac{1}{x} = \frac{1}{x},$$

  since $\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = 1$.

Rules of calculus

We need efficient rules to compute derivatives of complex functions composed either by algebraic operation or compositions by elementary functions.

Theorem 4. *Let f, g be differentiable at x, then*

- $(kf)'(x)kf'(x)$, *where k is a constant,*
- $(f \pm g)'(x) = f'(x) \pm g'(x)$,
- $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$,
- *if $g(x) \neq 0$, we have* $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$.

In particular, we have the formulas

$$(\alpha f + \beta g)(x) = \alpha f'(x) + \beta g'(x) \quad (\textit{linearity}),$$
$$\left(\frac{1}{g}\right)'(x) = \frac{g'(x)}{g^2(x)}, \quad (\textit{if} \quad g(x) \neq 0).$$

Proof of Theorem

- For a constant k, we obtain as $h \to 0$

$$\frac{(kf)(x+h) - (kf)(x)}{h} = \frac{kf(x+h) - kf(x)}{h} = k\frac{f(x+h) - f(x)}{h} \to kf'(x).$$

- For the sum, we have as $h \to 0$

$$\frac{(f+g)(x+h) - (f+g)(x)}{h} = \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$
$$= \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \to f'(x) + g'(x).$$

- For the product, we get as $h \to 0$

$$\begin{aligned}
&\frac{(f \cdot g)(x+h) - (f \cdot g)(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h} \to f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

Examples

- If $f(x) = 115$, then $f'(x) = 0$.
- If $f(x) = x^{15}$, then $f'(x) = 15x^{14}$.
- If $f(x) = 12x^{-6}$, then $f'(x) = 12 \cdot (-6)x^{-7} = -72x^{-7}$.
- If $f(x) = \ln x$ and $g(x) = 2e^x$, then $(f+g)'(x) = \frac{1}{x} + 2e^x$.
- If $f(x) = 2x^4$ and $g(x) = 3\sin x$, then

$$\begin{aligned}
(f \cdot g)'(x) &= (2x^4)' \cdot 3\sin x + 2x^4 \cdot (\sin x)' = 8x^3 \cdot \sin x + 2x^4 \cdot 3\cos x \\
&= 24\sin x + 6x^4 \cos x.
\end{aligned}$$

- If $f(x) = \cos x$ and $g(x) = 7x^{-2}$, then

$$\begin{aligned}
\left(\frac{f}{g}\right)(x) &= \frac{(\cos x)' \cdot 7x^{-2} - \cos x \cdot (7x^{-2})'}{(7x^{-2})^2} \\
&= \frac{-\sin x \cdot 7x^{-2} - \cos x \cdot (-14x^{-3})}{49x^{-4}} = \frac{-x^2 \sin x + 2x \cos x}{7}.
\end{aligned}$$

Chain Rule

Theorem 5. *Assume that there exist $g'(x)$ and $f'(g(x))$. Then*

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

Proof of Theorem

A natural computation gives us

$$\begin{aligned}
\frac{f \circ g(x+h) - f \circ g(x)}{h} &= \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}.
\end{aligned}$$

Remark 1. The chain rule is a sufficient condition in order that $f \circ g$ is differentiable! The composition may be differentiable even if the components are NOT!

Example

We consider the composition

$$f(x) = x^2 \qquad \text{and} \qquad g(x) = |x|.$$

Then

$$f \circ g(x) = f(g(x)) = |x|^2 = x^2$$

which is clearly differentiable on $\mathbb{R}$. Nevertheless, hypotheses for the chain rule are not fulfilled. Indeed, g is not differentiable at 0.

Examples

- Let $f(x) = e^x$ and $g(x) = x^2 - 2$. Then $f \circ g(x) = e^{x^2-2}$, hence, using the chain rule, we get

$$(f \circ g)(x) = \left(e^{x^2-2}\right)' = e^{x^2-2} \cdot (2x) = 2xe^{x^2-2}.$$

- Let $f(x) = \ln x$ and $g(x) = x^4 + 4$. Then $f \circ g(x) = \ln(x^4 + 4)$, hence, using the chain rule, we get

$$(f \circ g)(x) = \left(\ln(x^4 + 4)\right)' = \frac{1}{x^4 + 4} \cdot 4x = \frac{4x}{x^4 + 4}.$$

- Let $f(x) = \sin x$ and $g(x) = 2x + 1$. Then $f \circ g(x) = \sin(2x + 1)$, hence, using the chain rule, we get

$$(f \circ g)(x) = (\sin(2x + 1))' = \cos(2x + 1) \cdot 2 = 2\cos(2x + 1).$$

Differentiable inverse mapping theorem

Theorem 6. *Let f be continuous with continuous f' and $f' \neq 0$. Then f is invertible and the following formula holds true*

$$\left(f^{-1}\right)'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}.$$

Example

Let $f(x) = e^x$. then f is continuous and its derivative $f'(x) = e^x$ is also continuous. Moreover, we know that $f^{-1}(y) = \ln y$. Hence, applying Theorem, we obtain

$$(\ln y)' = \frac{1}{e^{\ln y}} = \frac{1}{y}.$$

Derivatives of inverse of elementary functions

Let $f(x) = \sin x$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We have $(\sin x)' = \cos x$ and $\cos x > 0$ for x in this interval. Therefore, by the differentiable inverse mapping theorem the inverse - the function $\arcsin y$ - is differentiable and

$$(\arcsin y)' = \frac{1}{\cos(\arcsin y)}.$$

But, by the remarkable identity for sine and cosine

$$\cos x = \sqrt{1 - \sin^2 x},$$

we have

$$(\arcsin y)' = \frac{1}{\sqrt{1 - (\sin(\arcsin y))^2}} = \frac{1}{\sqrt{1 - y^2}}.$$

Let $f(x) = \tan x$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We have

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x + \sin x \cdot \sin x}{(\cos x)^2} = \frac{1}{\cos^2 x} = 1 + \tan^2 x > 0$$

for x in this interval. Therefore, by the differentiable inverse mapping theorem the inverse - the function $\arctan y$ - is differentiable and

$$(\arctan y)' = \frac{1}{1 + (\tan(\arctan y))^2} = \frac{1}{1 + y^2}.$$

Logarithmic derivative

Let $f(x)$ be a positive function. Then we may define a function $g(x)$ in the following way

$$g(x) = \ln f(x).$$

Using the chain rule, the derivative of g is equal

$$g'(x) = (\ln f(x))' = \frac{f'(x)}{f(x)}.$$

Now, let $f(x) = a^x$, then we obtain

$$g(x) = \ln a^x = x \ln a$$

and it is easy to see that

$$g'(x) = \ln a.$$

On the other hand, we have

$$g'(x) = \frac{(a^x)'}{a^x}.$$

Hence,

$$(a^x)' = a^x \ln a.$$

l'Hospital's Rules

A typical problem computing limits is to compute

$$\lim_{x\to x_0} \frac{f(x)}{g(x)}$$

with indeterminate from $\frac{0}{0}$ or $\frac{\infty}{\infty}$. L'Hôpital's rules are very useful tool to treat this type of limits, transforming then into easier ones. There are four rules (basically with the same philosophy)

- $\frac{0}{0}$ at finite,
- $\frac{0}{0}$ at infinity,
- $\frac{\infty}{\infty}$ at finite,
- $\frac{\infty}{\infty}$ at infinity.

$\frac{0}{0}$ at finite

Theorem 7. *Let $f, g : D \to \mathbb{R}$, null at x_0. Suppose moreover that*

a) f and g are differentiable,

b) there exists

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L.$$

Then there exists

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = L.$$

Example

We compute the limit

$$\lim_{x\to 0} \frac{x^2}{\sin x}.$$

We recognize a form $\frac{0}{0}$ and we use L'Hôspital rule to get

$$\lim_{x\to 0} \frac{2x}{\cos x} = 0.$$

$\frac{0}{0}$ at ∞

Theorem 8. *Let $f, g : D \to \mathbb{R}$, null at ∞. Suppose moreover that*

a) f and g are differentiable,

b) there exists

$$\lim_{x\to \infty} \frac{f'(x)}{g'(x)} = L.$$

Then there exists

$$\lim_{x\to \infty} \frac{f(x)}{g(x)} = L.$$

Example
We compute the limit

$$\lim_{x\to+\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}.$$

We recognize a form $\frac{0}{0}$ and we use L'Hôspital rule to have

$$\lim_{x\to+\infty} \frac{-x^{-2}\cos\frac{1}{x}}{-x^{-2}} = \lim_{x\to+\infty} \cos\frac{1}{x} = 1.$$

$\frac{\infty}{\infty}$ at x_0

Theorem 9. *Let $f, g : D \to \mathbb{R}$ be infinite at x_0. Suppose moreover that*

a) f and g are differentiable,

b) there exists

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L.$$

Then there exists

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = L.$$

Example
We compute the limit

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x}.$$

We recognize a form $\frac{\infty}{\infty}$ and we use L'Hôspital rule to get

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = \lim_{x\to 0^+} \left(-\sin x \cdot \frac{\sin x}{x}\right) = 0.$$

$\frac{\infty}{\infty}$ at ∞

Theorem 10. *Let $f, g : D \to \mathbb{R}$ be infinite at ∞. Suppose moreover that*

a) f and g are differentiable,

b) there exists

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L.$$

Then there exists

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

Example

We compute the limit

$$\lim_{x\to+\infty} \frac{x}{e^x}.$$

We recognize a form $\frac{\infty}{\infty}$ and we use L'Hôspital rule we have

$$\lim_{x\to+\infty} \frac{1}{e^x} = 0.$$

Remark 2. We use the following notation

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} \overset{\text{H}}{=} \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

This means the equality holds if and only if hypotheses of the rule holds.

Remark 3. The existence of the limit $\frac{f'}{g'}$ implies the existence (with the same value) of the limit $\frac{f}{g}$ but not vice versa. The limit $\frac{f'}{g'}$ could not exist while the limit $\frac{f}{g}$ exists.

Example

It is known that $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0$. However,

$$\lim_{x\to\infty} \tfrac{(\sin x)'}{x'} = \lim_{x\to\infty} \tfrac{\cos x}{1}$$

does not exist!