Limits

Exercise 33. Use the definition to prove

$$\text{a)}\quad \lim_{x\to 3}(x-2)^2 = 1, \qquad \text{b)}\quad \lim_{x\to\infty}\frac{2}{x} = 0, \qquad \text{c)}\quad \lim_{x\to 2}\frac{1}{(x-2)^2} = \infty.$$

Exercise 34. Evaluate the limit

$$\begin{array}{llll}
\text{a)}\ \lim\limits_{x\to 1}\dfrac{x^2-1}{x^2-x+1} & \text{b)}\ \lim\limits_{x\to -1}\dfrac{x^2-1}{x^2+4x+3} & \text{c)}\ \lim\limits_{x\to 0^+}\dfrac{x+\sqrt{x}}{\sqrt{x}} & \text{d)}\ \lim\limits_{x\to 2}\dfrac{x^3-8}{x^4-16} \\
\text{e)}\ \lim\limits_{x\to\infty}\dfrac{x^2-5x+4}{x(x-5)} & \text{f)}\ \lim\limits_{x\to 6}\dfrac{\sqrt{x-2}-2}{x-6} & \text{g)}\ \lim\limits_{x\to -\infty}(\sqrt{x^2+1}+x) & \text{h)}\ \lim\limits_{x\to\infty}\dfrac{2^x+1}{3^x+2} \\
\text{i)}\ \lim\limits_{x\to \frac{\pi}{2}^-}\dfrac{\tan^2 x+1}{\tan^2 x+5} & \text{j)}\ \lim\limits_{x\to 0}\dfrac{\sin^2 x}{1-\cos x} & \text{k)}\ \lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+x}+2}{x+1} & \text{l)}\ \lim\limits_{x\to 1}\dfrac{1}{1-x}-\dfrac{3}{1-x^3}.
\end{array}$$

Exercise 35. Find the limit, if it exists. If the limit does not exist, explain why

$$\text{a)}\quad \lim_{x\to 0} x\operatorname{sgn} x, \qquad \text{b)}\quad \lim_{x\to 0} 2^{\frac{1}{x}}, \qquad \text{c)}\quad \lim_{x\to 2}\frac{x^2-4}{|x-2|}, \qquad \text{d)}\quad \lim_{x\to 0} x\arctan\frac{1}{x}.$$

Exercise 36. Use the Squeeze Theorem to show that

$$\text{a)}\quad \lim_{x\to 0^+}\sqrt{x}\cos\frac{1}{x^2} = 0, \qquad \text{b)}\quad \lim_{x\to 0} x^2\arctan\frac{1}{x} = 0, \qquad \text{c)}\quad \lim_{x\to\infty}\frac{2+\sin x}{x^2} = 0.$$

Exercise 37. Evaluate the limit

$$\begin{array}{lll}
\text{a)}\ \lim\limits_{x\to 0}\dfrac{\sin^2 3x}{x^2} & \text{b)}\ \lim\limits_{x\to 4}\dfrac{\sin(x-4)}{\sqrt{x}-2} & \text{c)}\ \lim\limits_{x\to 0}\dfrac{\arcsin 2x}{\arctan x} \\
\text{d)}\ \lim\limits_{x\to\infty} x^2\arctan\dfrac{1}{x} & \text{e)}\ \lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos 5x}{\cos 3x} & \text{f)}\ \lim\limits_{x\to 0}\dfrac{e^{3x}-1}{\sin 2x} \\
\text{g)}\ \lim\limits_{x\to 0}\dfrac{\ln(1+\sqrt[3]{x})}{x} & \text{h)}\ \lim\limits_{x\to -2}\dfrac{\ln(x^2-3)}{x+2} & \text{i)}\ \lim\limits_{x\to 1}\dfrac{x^\pi - x^e}{x-1} \\
\text{j)}\ \lim\limits_{x\to 0}(1+2x)^{\frac{1}{x}} & \text{k)}\ \lim\limits_{x\to 0}(1+\tan 2x)^{\cot x} & \text{l)}\ \lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-\sqrt[6]{1-x}}{x}.
\end{array}$$

Exercise 38. Find the vertical asymptotes and the oblique asymptotes of function

$$\begin{array}{lll}
\text{a)}\ f(x) = \dfrac{x^3+x^2}{x^2-4} & \text{b)}\ f(x) = \dfrac{x^{11}+1}{(x-1)^{10}} & \text{c)}\ f(x) = \dfrac{x-3}{\sqrt{x^2-9}} \\
\text{d)}\ f(x) = \dfrac{x\sqrt{x}+2}{x+1} & \text{e)}\ f(x) = \dfrac{3^x}{3^x-2^x} & \text{f)}\ f(x) = \dfrac{2x^2+\sin x}{x} \\
\text{g)}\ f(x) = \dfrac{\cos x}{e^x-1} & \text{h)}\ f(x) = x-\arctan x & \text{i)}\ f(x) = \dfrac{\sin 2x}{\sin x-1}.
\end{array}$$

Exercise 39. Determine $a, b \in I\!R$ such that the function is continuous on $I\!R$

$$\text{a)}\ f(x) = \begin{cases} -1, & x<0, \\ a+b\sin x, & 0\leqslant x\leqslant \frac{\pi}{2}, \\ 1, & x>\frac{\pi}{2}, \end{cases} \qquad \text{b)}\ f(x) = \begin{cases} \frac{a}{x}+1, & x<-1, \\ b-2x, & x\geqslant -1 \end{cases} \qquad \text{c)}\ f(x) = \begin{cases} ax^2+1, & x<-1, \\ 2x, & -1\leqslant x\leqslant 0, \\ x^3+bx, & x>0. \end{cases}$$

Exercise 40. Locate the discontinuities of the function

a) $$f(x) = \begin{cases} \frac{x+2}{x^2+x+2}, & x \neq 1, 2, \\ 0, & x = 1, \\ 1, & x = 2, \end{cases}$$

b) $$f(x) = \begin{cases} \arctan \frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

c) $$f(x) = \begin{cases} \frac{1}{\ln x^2 - \ln(x^2+1)}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

d) $$f(x) = \begin{cases} 1 - \cos \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Dominika Pilarczyk