Lecture 1

Applications of Integrals

Definite Integral

Definition 1. The definite integral of a continuous function f over an interval from $x = a$ and $x = b$ is the net change of an anti derivative of f over the interval. Symbolically, if $F(x)$ is an anti derivative of $f(x)$, then

$$\int_a^b f(x)\ \mathrm{d}x = F(x)\ |_a^b = F(b) - F(a) \qquad \text{where} \quad F'(x) = f(x)$$

Integrand: $f(x)$ Upper limit: b Lower limit: a

The relationship in the box turns out to be the most important theorem in calculus - the fundamental theorem of calculus.

Remark 1. *Do not confuse a definite integral with an indefinite integral. The definite integral* $\int_a^b f(x)\ \mathrm{d}x$ *is a real number. The indefinite integral* $\int f(x)\ \mathrm{d}x$ *is a whole set of functions - all anti derivatives of* $f(x)$.

Example. Evaluate $\int_{-1}^2 (3x^2 - 2x)\ \mathrm{d}x$.
We choose the simple anti derivative of $(3x^2 - 2x)$, namely $(x^3 - x^2)$, since any anti derivative will do

$$\begin{aligned}\int_{-1}^2 (3x^2 - 2x)\ \mathrm{d}x &= (x^3 - x^2)\ |_{-1}^2 \\ &= (2^3 - 2^2) - ((-1)^3 - (-1)^2) = 4 - (-2) = 6.\end{aligned}$$

Proposition 1 (Definite Integral Properties)**.**

1. $\int_a^a f(x)\ \mathrm{d}x = 0$
2. $\int_a^b f(x)\ \mathrm{d}x = -\int_b^a f(x)\ \mathrm{d}x$
3. $\int_a^b kf(x)\ \mathrm{d}x = k\int_a^b f(x)\ \mathrm{d}x$ *where k is a constant*
4. $\int_a^b f(x) \pm g(x)\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x \pm \int_a^b g(x)\ \mathrm{d}x$
5. $\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x$

Proof. These properties are justified as follows. If $F'(x) = f(x)$, then

1. $$\int_a^a f(x)\ \mathrm{d}x = F(x)\ |_a^a = F(a) - F(a) = 0$$
2. $$\begin{aligned}\int_a^b f(x)\ \mathrm{d}x &= F(x)\ |_a^b = F(b) - F(a) = -[F(a) - F(b)] \\ &= -\int_b^a f(x)\ \mathrm{d}x\end{aligned}$$

and so on.

The evaluation of a definite integral is a two-step process:

- find an anti derivative;
- find the net change in that anti derivative.

Common Errors:

1. $$\int_0^2 e^x \, \mathrm{d}x = e^x \Big|_0^2 = e^2 - 1$$

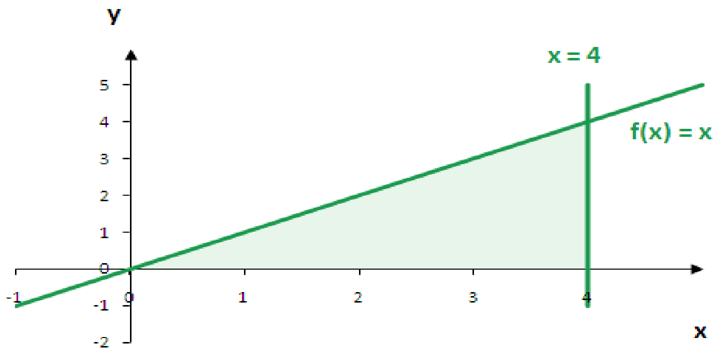
   Do not forget to evaluate the anti derivative at both the upper and lower limits of integration and do not assume that the anti derivative is 0 just because the lower limit is 0.

2. $$\int_0^5 \frac{x}{x^2+10} \, \mathrm{d}x = \frac{1}{2} \int_0^5 \frac{1}{u} \, \mathrm{d}u$$

   If a substitution is made in a definite integral, the limits of integration also must be changed. The new limits are determined by the particular substitution used in the integral.

Area under a Curve

Consider the graph of $f(x) = x$ from $x = 0$ to $x = 4$



We can easily compute the area of the triangle bounded by $f(x) = x$, the x axis ($y = 0$), and the line $x = 4$, using the formula for the area of a triangle:

$$A = \frac{bh}{2} = \frac{4 \cdot 4}{2} = 8.$$

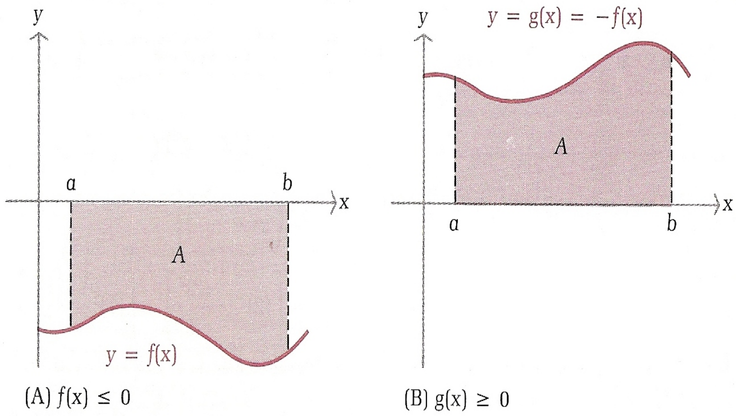
Let us integrate $f(x) = x$ from $x = 0$ to $x = 4$:

$$\int_0^4 x \, \mathrm{d}x = \frac{x^2}{2} \Big|_0^4 = \frac{4^2}{2} - \frac{0}{2} = 8.$$

We get the same result! It turns out that this is not a coincidence.

Proposition 2. *If f is continuous and $f(x) \geqslant 0$ over the interval $[a, b]$, then the area bounded by $y = f(x)$, the x axis $(y = 0)$, and the vertical lines $x = a$ and $x = b$ is given by*

$$A = \int_a^b f(x)\ \mathrm{d}x.$$



Let us see why the definite integral gives us the area exactly. Let $A(x)$ be the area under the graph of $y = f(x)$ from a to x.

$$A(x) = \text{Area from } a \text{ to } x$$
$$A(b) = \text{Area from } a \text{ to } b = A.$$

If we can show that $A(x)$ is anti derivative of $f(x)$, then we can write

$$\int_a^b f(x)\ \mathrm{d}x = A(x)\Big|_a^b = A(b) - A(a)$$
$$= (\text{Area from } a \text{ to } b) \ - (\text{Area from } a \text{ to } a) = A - 0 = A.$$

To show that $A(x)$ is an anti derivative of $f(x)$ - that is $A'(x) = f(x)$ - we use the definition of the derivative and write

$$A'(x) = \lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x}.$$

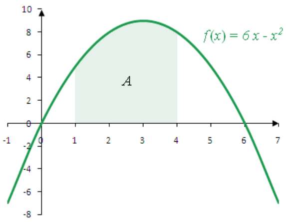
Geometrically, $A(x + \Delta x) - A(x)$ is the area from x to $x + \Delta x$. This is given approximately by the area of the rectangle $\Delta x \cdot f(x)$, and the smaller Δx is, the better the approximation. Using

$$A(x + \Delta x) - A(x) \approx \Delta x \cdot f(x)$$

and dividing both sides by Δx, we obtain

$$\frac{A(x + \Delta x) - A(x)}{\Delta x} \approx f(x).$$

Now, if we let $\Delta x \to 0$, then the left side has $A'(x)$ as a limit, which is equal to the right side. Hence, $A'(x) = f(x)$.
Example. Find the area bounded by $f(x) = 6x - x^2$ and $y = 0$ for $1 \leqslant x \leqslant 4$.



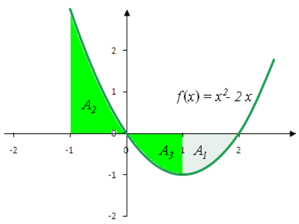
$$A = \int_1^4 (6x - x^2)\,\mathrm{d}x = \left(3x^2 - \frac{x^3}{3}\right)\Big|_1^4 = 48 - \frac{64}{3} - 3 + \frac{1}{3} = 24.$$

Area between a Curve and the x Axis

The condition $f(x) \geqslant 0$ is essential to the relationship between an area under a graph and the definite integral. Suppose $f(x) \leqslant 0$ and A is the area between the graph of f and the x axis for $a \leqslant x \leqslant b$. If we let $g(x) = -f(x)$, then A is also the area between the graph of g and the x axis for $a \leqslant x \leqslant b$. Since $g(x) \geqslant 0$ for $a \leqslant x \leqslant b$, we can use the definite integral of g to find A:

$$A = \int_a^b g(x)\,\mathrm{d}x = \int_a^b [-f(x)]\,\mathrm{d}x$$

Example We consider the function $f(x) = x^2 - 2x$.



a) From the graph we see that $f(x) \leqslant 0$ for $1 \leqslant x \leqslant 2$, so we integrate $-f(x)$

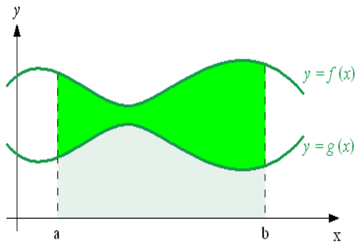
$$\begin{aligned} A_1 &= \int_1^2 [-f(x)]\,\mathrm{d}x = \int_1^2 (2x - x^2)\,\mathrm{d}x \\ &= \left(x^2 - \frac{x^3}{3}\right)\Big|_1^2 = 4 - \tfrac{8}{3} - 1 + \tfrac{1}{3}. \end{aligned}$$

b) Since the graph shows that $f(x) \geqslant 0$ on $[-1, 0]$ and $f(x) \leqslant 0$ on $[0, 1]$, the computation of this area will require two integrals:

$$\begin{aligned} A = A_1 + A_2 &= \int_{-1}^0 f(x)\,\mathrm{d}x + \int_0^1 [-f(x)]\,\mathrm{d}x \\ &= \int_{-1}^0 (x^2 - 2x)\,\mathrm{d}x + \int_0^1 (2x - x^2)\,\mathrm{d}x \\ &= \left(\frac{x^3}{3} - x^2\right)\Big|_{-1}^0 + \left(x^2 - \frac{x^3}{3}\right)\Big|_0^1 \\ &= \tfrac{4}{3} + \tfrac{2}{3} \end{aligned}$$

Area between Two Curves

Consider the area bounded by $y = f(x)$ and $y = g(x)$, $f(x) \geqslant g(x) \geqslant 0$, for $a \leqslant x \leqslant b$

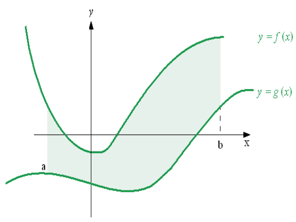


$$\begin{aligned}
&(\text{Area } A \text{ between } f(x) \text{ and } g(x)) \\
&= (\text{Area under } f(x)) - (\text{Area under } g(x)) \\
&= \int_a^b f(x)\ \mathrm{d}x - \int_a^b g(x)\ \mathrm{d}x \\
&= \int_a^b [f(x) - g(x)]\ \mathrm{d}x
\end{aligned}$$

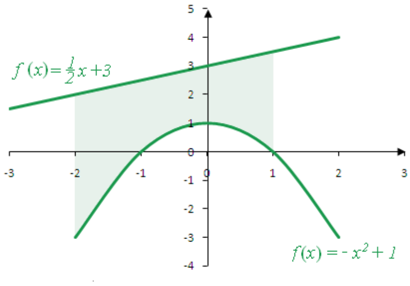
It can be shown that the above result does not require $f(x)$ or $g(x)$ to remain positive over the interval $[a, b]$.

Proposition 3. *If f and g are continuous and $f(x) \geqslant g(x)$ over the interval $[a, b]$, then the area bounded by $y = f(x)$ and $y = g(x)$ for $a \leqslant x \leqslant b$ is given by*

$$A = \int_a^b [f(x) - g(x)]\ \mathrm{d}x$$



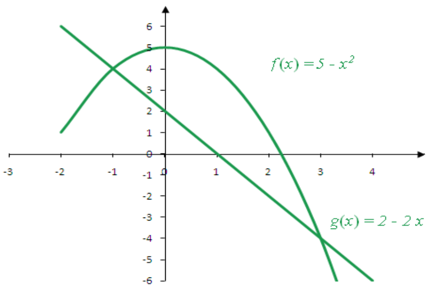
Example We sketch the area bounded by $f(x) = \frac{1}{2}x + 3$, $g(x) = -x^2 + 1$, $x = -2$, and $x = 1$.



We observe from the graph that $f(x) \geqslant g(x)$ for $-2 \leqslant x \leqslant 1$, so

$$
\begin{aligned}
A &= \int_{-2}^{1} [f(x) - g(x)]\ \mathrm{d}x = \int_{-2}^{1} \left[\left(\frac{x}{2} + 3\right) - (-x^2 + 1)\right]\ \mathrm{d}x \\
&= \int_{-2}^{1} \left(x^2 + \frac{x}{2} + 2\right)\ \mathrm{d}x \\
&= \left(\frac{x^3}{3} + \frac{x^2}{4} + 2x\right)\Big|_{-2}^{1} \\
&= \left(\frac{1}{3} + \frac{1}{4} + 2\right) - \left(\frac{-8}{3} + \frac{4}{4} - 4\right) \\
&= \frac{33}{4}.
\end{aligned}
$$

Example We sketch the are bounded by $f(x) = 5 - x^2$ and $g(x) = 2 - 2x$.



To find points of intersection (hence the upper and lower limits of integration), we solve

$$
\begin{aligned}
5 - x^2 &= 2 - 2x \\
x^2 - 2x - 3 &= 0 \\
x_1 = -1 &\qquad x_2 = 3
\end{aligned}
$$

The figure shows that $f(x) \geqslant g(x)$ over the interval $[-1, 3]$, so

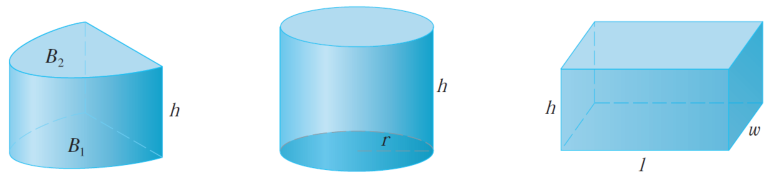
$$\begin{aligned} A &= \int_{-1}^{3} [f(x) - g(x)]\, \mathrm{d}x = \int_{-1}^{3} [5 - x^2 - (2 - 2x)]\, \mathrm{d}x \\ &= \int_{-1}^{3} (3 + 2x - x^2)\, \mathrm{d}x \\ &= \left(3x + x^2 - \frac{x^3}{3}\right)\Big|_{-1}^{3} = \frac{32}{3}. \end{aligned}$$

Volumes

We start with a simple type of solid called a cylinder which is bounded by a plane region B_1, called the base, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments that are perpendicular to the base and join B_1 to B_2. If the area of the base is A and the height of the cylinder (the distance from B_1 to B_2) is h, then the volume V of the cylinder is defined as

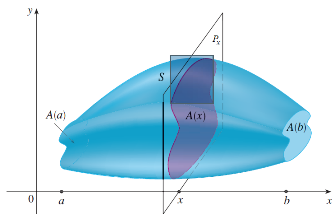
$$V = Ah$$

In particular, if the base is a circle with radius r, then the cylinder is a circular cylinder with volume $V = \pi r^2 h$, and if the base is a rectangle with length l and width w, then the cylinder is a rectangular box (also called a rectangular parallelepiped) with volume $V = lwh$. For a solid S that is not a cylinder we first 'cut' S into pieces and approximate each piece by a cylinder. We estimate the volume of S by adding the volumes of the cylinders. We arrive at the exact volume of S through a limiting process in which the number of pieces becomes large.



Cylinder $V = Ah$ Circular cylinder $V = \pi r^2 h$ Rectangular box $V = lwh$

We start by intersecting S with a plane and obtaining a plane region that is called a cross-section of S. Let $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x, where $a \leqslant x \leqslant b$. The cross-sectional area $A(x)$ will vary as x increases from a to b.



Let us divide S into n 'slabs' of equal width Δx by using the planes $P_{x_1}, P_{x_2}, \dots$ to slice the solid. If we choose sample points x_i^* in $[x_{i-1}, x_i]$, we can approximate the ith slab S_i (the part of S that lies between the planes $P_{x_{i-1}}$ and P_{x_i}) by a cylinder with base area $A(x_i^*)$ and 'height' Δx. The volume of this cylinder is $A(x_i^*)\Delta x$, so an approximation to our intuitive conception of the volume of the ith slab S_i is

$$V(S_i) \approx A(x_i^*)\Delta x$$

Adding the volumes of these slabs, we get an approximation to the total volume

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$$

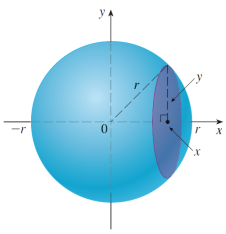
This approximation appears to become better and better as $n \to \infty$. Therefore we define the volume as the limit of these sums as $n \to \infty$. But we recognize the limit of Riemann sums as a definite integral and so we have the following definition.

Definition 2. Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where is a continuous function, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\ \mathrm{d}x.$$

Remark 2. *When we use the volume formula $V = \int_a^b A(x)\ \mathrm{d}x$, it is important to remember that $A(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis. Notice that, for a cylinder, the cross-sectional area is constant: $A(x) = A$ for all x. So our definition of volume gives $V = \int_a^b A = A(b-a)$; this agrees with the formula $V = Ah$.*

Example. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.



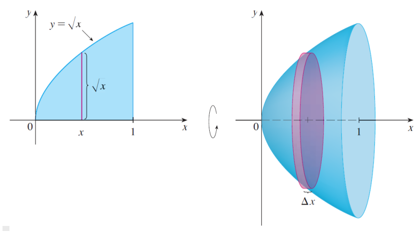
If we place the sphere so that its center is at the origin, then the plane P_x intersects the sphere in a circle whose radius (from the Pythagorean Theorem) $y = \sqrt{r^2 - x^2}$. So the cross-sectional area is

$$A(x) = \pi(r^2 - x^2).$$

Using the definition of volume with $a = -r$ and $b = r$, we have

$$V = \int_{-r}^{r} \pi(r^2 - x^2)\ \mathrm{d}x = \frac{4}{3}\pi r^3.$$

Example. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



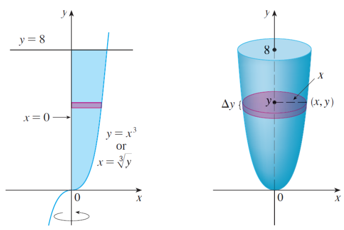
When we slice through the point x, we get a disk with radius $\sqrt{x}$. The area of this cross-section is

$$A(x) = \pi x.$$

The solid lies between $x = 0$ and $x = 1$, so its volume is

$$V = \int_0^1 \pi x \, \mathrm{d}x = \frac{\pi}{2}.$$

Example. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.



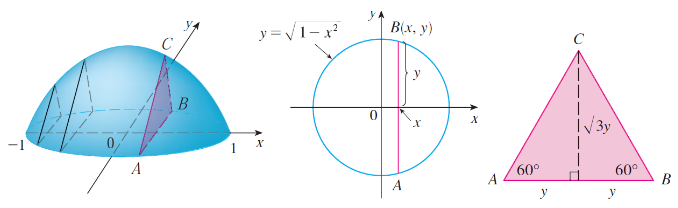
Because the region is rotated about the y-axis, it makes sense to slice the solid perpendicular to the y-axis and therefore to integrate with respect to y. If we slice at height y, we get a circular disk with radius x, where $x = \sqrt[3]{y}$. So the area of a cross-section through y is

$$A(y) = \pi y^{\frac{2}{3}}.$$

The solid lies between $y = 0$ and $y = 1$, so its volume is

$$V = \int_0^8 \pi y^{\frac{2}{3}} \, \mathrm{d}y = \frac{96\pi}{5}.$$

Example. We have a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



The solid — Its base — A cross-section

Let us take the circle to be $x^2 + y^2 = 1$. Since B lies on the circle, we have $y = \sqrt{1 - x^2}$ and so the base of the triangle ABC is $|AB| = 2\sqrt{1 - x^2}$. Since the triangle is equilateral, the cross-sectional area is

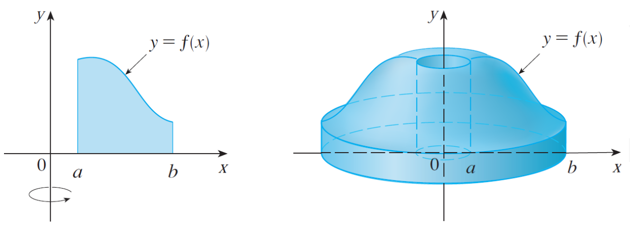
$$A(x) = \sqrt{3}(1 - x^2).$$

and the volume of the solid is

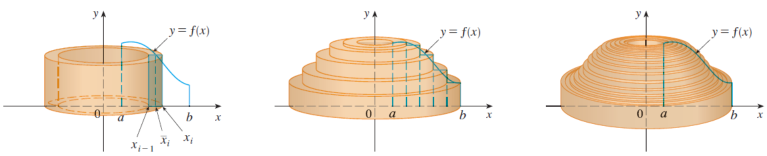
$$V = \int_{-1}^{1} \sqrt{3}(1 - x^2)\ \mathrm{d}x = \frac{4\sqrt{3}}{3}.$$

Volumes by Cylindrical Shells

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where $f(x) \geqslant 0$], $y = 0$, $x = a$ and $x = b$, where $b > a \geqslant 0$.



We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and let $\bar{x}_i$ be the midpoint of the ith subinterval. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\bar{x}_i)$ is rotated about the y-axis, then the result is a cylindrical shell with average radius $\bar{x}_i$, height $f(\bar{x}_i)$, and thickness Δx, so its volume is



$$V_i = (2\pi\bar{x}_i)[f(\bar{x}_i)]\Delta x.$$

Therefore an approximation to the volume V of S is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} (2\pi\bar{x}_i)[f(\bar{x}_i)]\Delta x.$$

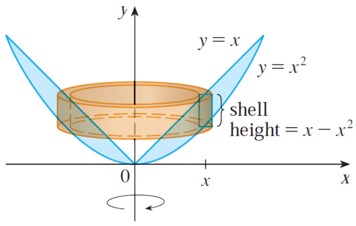
This approximation appears to become better as $n \to \infty$. But, from the definition of an integral, we know that

$$\lim_{n\to\infty} \sum_{i=1}^{n} (2\pi\bar{x}_i)[f(\bar{x}_i)]\Delta x = \int_a^b 2\pi x f(x)\ \mathrm{d}x.$$

Definition 3. The volume of the solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b \underbrace{2\pi x}_{circumference} \underbrace{f(x)}_{height} \underbrace{\mathrm{d}x}_{thickness} \quad where \quad 0 \leqslant a < b.$$

Example. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.



We see that the shell has radius x, circumference $2\pi x$, and height $x - x^2$. So the volume is

$$V = \int_0^1 (2\pi x)(x - x^2)\ \mathrm{d}x = \frac{\pi}{6}.$$

Average Value of Continuous Function

We know that the average of a finite number of values

$$a_1, a_2, ..., a_n$$

is given by

$$\text{Average} = \frac{a_1 + a_2 + ... + a_n}{n}.$$

How can we handle a continuous function with infinitely many values?

- we divide the time interval $[a, b]$ into n equal subintervals,
- we compute the value of the function at a sample points in each subinterval,
- we use the average of these values as an approximation of the average value of the continuous function f over $[a, b]$.

We would expect the approximations to improve as n increases. In fact, we would be inclined to define the limit of the average for n values as $n \to \infty$ as the average value of f over $[a, b]$, if the limit exists.

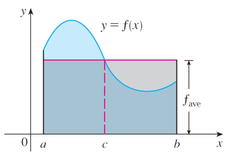
$$(\text{Average value for } n \text{ values}) = \frac{1}{n}[f(x_1) + f(x_2) + ... + f(x_n)]$$

where x_k is a sample point in the kth subinterval. We have

$$\begin{aligned}
&\frac{b-a}{b-a} \cdot \frac{1}{n}[f(x_1) + f(x_2) + \ldots + f(x_n)] \\
&= \frac{1}{b-a} \cdot \frac{b-a}{n}[f(x_1) + f(x_2) + \ldots + f(x_n)] \\
&= \frac{1}{b-a} \cdot \left[f(x_1)\frac{b-a}{n} + f(x_2)\frac{b-a}{n} + \ldots + f(x_n)\frac{b-a}{n}\right] \\
&= \frac{1}{b-a} \cdot [f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x].
\end{aligned}$$

Thus,

$$(\text{Average value over } [a,b]) = \lim_{n\to\infty} \frac{1}{b-a} \sum_{k=1}^{n} f(x_k)\Delta x.$$



Now the part in the braces is a definite integral. Thus,

$$\begin{aligned}
&(\text{Average value over } [a,b]) \\
&= \frac{1}{b-a} \int_a^b f(x)\ \mathrm{d}x.
\end{aligned}$$

Proposition 4. *Average value of a continuous function f over $[a,b]$*

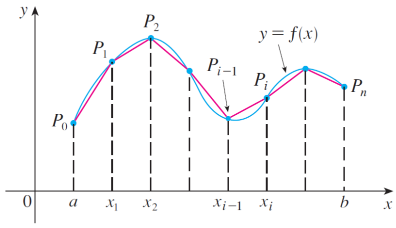
$$\frac{1}{b-a} \int_a^b f(x)\ \mathrm{d}x.$$

Example. We will find the average value of $f(x) = x - 3x^2$ over the interval $[-1, 2]$.

$$\begin{aligned}
\frac{1}{b-a} \int_a^b f(x)\ \mathrm{d}x &= \frac{1}{2-(-1)} \int_{-1}^{2} (x - 3x^2)\ \mathrm{d}x \\
&= \frac{1}{3} \left(\frac{x^2}{2} - x^3\right) \Big|_{-1}^{2} = -\frac{5}{2}.
\end{aligned}$$

Arc Length

Suppose that a curve C is defined by the equation $y = f(x)$, where f is continuous and $a \leqslant x \leqslant b$. We obtain a polygonal approximation to C by dividing the interval $[a,b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. If $y_i = f(x_i)$, then the point $P_i = (x_i, y_i)$ lies on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ illustrated is an approximation to C.



The length L of C is approximately the length of this polygon and the approximation gets better as we let n increase. Therefore we define the length L of the curve C with equation $y = f(x)$, $a \leqslant x \leqslant b$, as the limit of the lengths of these inscribed polygons (if the limit exists):

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|.$$

If we let $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

By applying the Mean Value Theorem to on the interval $[x_{i-1}, x_i]$, we find that there is a number x_i^* between x_{i-1} and x_i such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}),$$
$$\Delta y_i = f'(x_i^*)\Delta x.$$

Thus we have

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

Therefore

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

Theorem 4. *If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, is*

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ \mathrm{d}x.$$

Remark 3. *If a curve has the equation $x = g(y)$, $c \leqslant y \leqslant d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y, we obtain the following formula for its length:*

$$L = \int_c^d \sqrt{1 + [g'(x)]^2}\ \mathrm{d}y.$$

Example. Find the length of the arc of the semi-cubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.
For the top half of the curve we have $y = x^{\frac{3}{2}}$ and $y' = \frac{3}{2}x^{\frac{1}{2}}$ and so the arc length formula gives

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\ \mathrm{d}x = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13}).$$

Example. Find the length of the arc of the parabola $y^2 = x$ between the points $(0, 0)$ and $(1, 1)$.
Since $x = y^2$, we have $y' = 2y$ and so the arc length formula gives

$$L = \int_0^1 \sqrt{1 + 4y^2}\, \mathrm{d}x = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5} + 2)}{4}.$$