Lecture 2

Improper Integrals

Improper integrals of Type 1

Definition 1. If $\int_a^t f(x)\ \mathrm{d}x$ exists for every number $t \geqslant a$, then

$$\int\limits_a^\infty f(x)\ \mathrm{d}x = \lim_{T\to\infty} \int\limits_a^T f(x)\ \mathrm{d}x$$

provided this limit exists (as a finite number).
If $\int_t^b f(x)\ \mathrm{d}x$ exists for every number $t \leqslant b$, then

$$\int\limits_{-\infty}^b f(x)\ \mathrm{d}x = \lim_{T\to-\infty} \int\limits_T^b f(x)\ \mathrm{d}x$$

provided this limit exists (as a finite number).

Definition 2. The improper integrals $\int_a^\infty f(x)\ \mathrm{d}x$ and $\int_{-\infty}^b f(x)\ \mathrm{d}x$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

Definition 3. If both $\int_a^\infty f(x)\ \mathrm{d}x$ and $\int_{-\infty}^b f(x)\ \mathrm{d}x$ are convergent, then we define

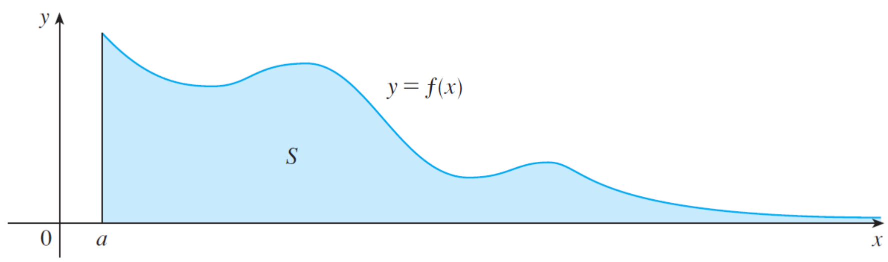
$$\int_{-\infty}^\infty f(x)\ \mathrm{d}x = \int_a^\infty f(x)\ \mathrm{d}x + \int_{-\infty}^a f(x)\ \mathrm{d}x$$

and any real number a can be used.

Any of the improper integrals can be interpreted as an area provided that f is a positive function. For instance, in if $f(x) \geqslant 0$ and the integral $\int_a^\infty f(x)\ \mathrm{d}x$ is convergent, then we define the area of the region $S = \{(x, y) : x \geqslant a, 0 \leqslant y \leqslant f(x)\}$ to be

$$A(S) = \int_a^\infty f(x)\ \mathrm{d}x.$$

This is appropriate because $\int_a^\infty f(x)\ \mathrm{d}x$ is the limit as $t \to \infty$ of the area under the graph of f from a to t.



Example. Determine whether the integral $\int\limits_{1}^{\infty} \frac{\mathrm{d}x}{(x+2)^2}$ is convergent or divergent.
According to definition, we have

$$\int\limits_{1}^{\infty} \frac{\mathrm{d}x}{(x+2)^2} = \lim_{T\to\infty} \int\limits_{1}^{T} \frac{\mathrm{d}x}{(x+2)^2}.$$

We integrate to get

$$\int\limits_{1}^{\infty} \frac{\mathrm{d}x}{(x+2)^2} = \lim_{T\to\infty} \left[-\frac{1}{T+2} + \frac{1}{3}\right] = \frac{1}{3},$$

since $\lim\limits_{T\to\infty} \frac{1}{T+2} = 0$, so the improper integral is convergent.

Example. Determine whether the integral $\int\limits_{\pi}^{\infty} x \sin x \ \mathrm{d}x$ is convergent or divergent.
According to definition, we have

$$\int\limits_{\pi}^{\infty} x \sin x \ \mathrm{d}x = \lim_{T\to\infty} \int\limits_{\pi}^{T} x \sin x \ \mathrm{d}x.$$

Using integration by parts, we obtain

$$\int\limits_{\pi}^{\infty} x \sin x \ \mathrm{d}x = \lim_{T\to\infty} [-T\cos T + \sin T + \pi\cos\pi - \sin\pi].$$

The limit does not exist as a finite number and so the improper integral is divergent.

Example. Determine whether the integral $\int\limits_{-\infty}^{0} \left(\frac{\pi}{2} + \arctan x\right) \ \mathrm{d}x$ is convergent or divergent.
According to definition, we have

$$\int\limits_{-\infty}^{0} \frac{\pi}{2} + \arctan x \ \mathrm{d}x = \lim_{T\to-\infty} \int\limits_{T}^{0} \frac{\pi}{2} + \arctan x \ \mathrm{d}x.$$

Using integration by parts, we get

$$\int\limits_{-\infty}^{0} \frac{\pi}{2} + \arctan x \ \mathrm{d}x = \lim_{T\to-\infty} \left[-\frac{\pi}{2}T - T\arctan T + \frac{1}{2}\ln(T^2+1)\right] = \infty,$$

hence the improper integral is divergent.

Example. Determine whether the integral $\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 - 4x + 13}$ is convergent or divergent.
Applying the definition we have

$$\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 - 4x + 13} = \lim_{T\to-\infty} \int\limits_{T}^{2} \frac{\mathrm{d}x}{(x-2)^2 + 9} + \lim_{T\to\infty} \int\limits_{2}^{T} \frac{\mathrm{d}x}{(x-2)^2 + 9}.$$

Using the integration by substitution

$$\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 - 4x + 13} = \lim_{T\to-\infty} -\tfrac{1}{3}\arctan \tfrac{1}{3}(T-2) + \lim_{T\to\infty} \tfrac{1}{3}\arctan \tfrac{1}{3}(T-2)$$
$$= \frac{\pi}{3},$$

so the improper integral is convergent.

Example. Determine whether the integral $\int\limits_{-\infty}^{\infty} e^{|x|}\ \mathrm{d}x$ is convergent or divergent.
According to the definition, we have

$$\int\limits_{-\infty}^{\infty} e^{|x|}\ \mathrm{d}x = \lim_{T\to-\infty} \int\limits_{T}^{0} e^{-x}\ \mathrm{d}x + \lim_{T\to\infty} \int\limits_{0}^{T} e^{x}\ \mathrm{d}x.$$

We integrate to get

$$\int\limits_{-\infty}^{\infty} e^{|x|}\ \mathrm{d}x = \lim_{T\to-\infty} -1 + e^{-T} + \lim_{T\to\infty} e^{T} - 1 = +\infty,$$

hence the improper integral is divergent.

Proposition 1. *The improper integral* $\int\limits_{a}^{\infty} \frac{1}{x^p}\ \mathrm{d}x$*, where* $a > 0$*, is convergent if* $p > 1$ *and divergent if* $p \leqslant 1$.

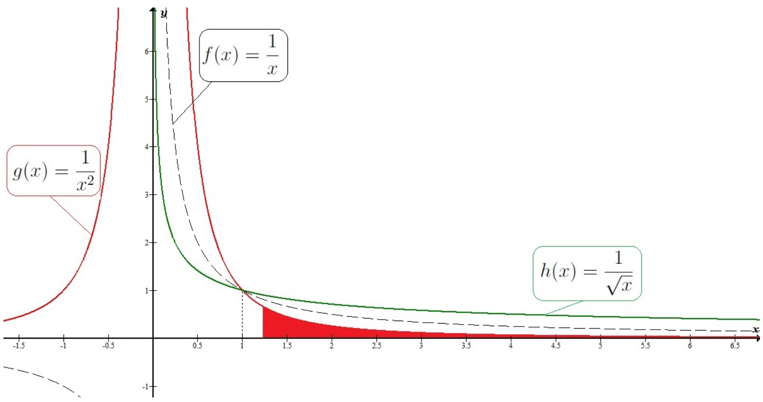
Proof. Assume that $p = 1$. Then

$$\int\limits_{a}^{\infty} \frac{1}{x}\ \mathrm{d}x = \lim_{t\to\infty} \frac{1}{x}\ \mathrm{d}x = \lim_{t\to\infty} (\ln t - \ln a) = +\infty.$$

The limit does not exist as a finite number and so the improper integral is divergent.

Now, let us assume that $p \neq 1$. Then

$$\begin{aligned}
\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x &= \lim_{T\to\infty}\int_a^T \frac{1}{x^p}\,\mathrm{d}x \\
&= \lim_{T\to\infty}\left(\frac{1}{1-p}\left(\frac{1}{T}\right)^{p-1} - \frac{1}{1-p}\frac{1}{a^{p-1}}\right) \\
&= \begin{cases} \frac{1}{p-1}a^{1-p}, & \text{if} \quad p>1, \\ +\infty, & \text{if} \quad p<1. \end{cases}
\end{aligned}$$



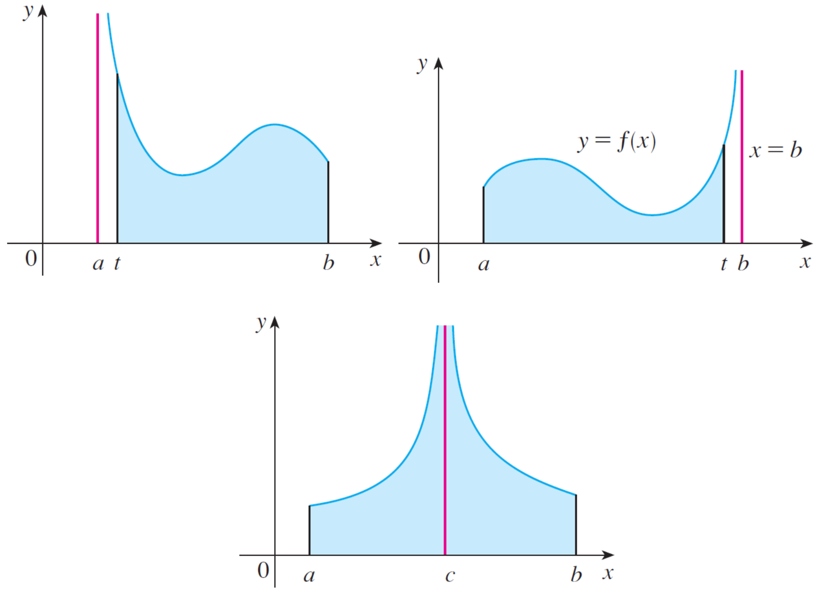
Improper integrals Type 2

Suppose that f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b. Let S be the unbounded region under the graph of f and above the x-axis between a and b. (For Type 1 integrals, the regions extended indefinitely in a horizontal direction. Here the region is infinite in a vertical direction.) The area of the part of S between a and t is

$$A(t) = \int_a^t f(x)\,\mathrm{d}x.$$

If it happens that $A(t)$ approaches a definite number A as $t \to b^-$, then we say that the area of the region S is A and we write

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-} f(x)\,\mathrm{d}x.$$



Definition 4. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{\varepsilon\to b^-} \int_a^\varepsilon f(x)\ \mathrm{d}x$$

if this limit exists (as a finite number).
If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{\varepsilon\to a^+} \int_\varepsilon^b f(x)\ \mathrm{d}x.$$

if this limit exists (as a finite number).
The improper integral $\int_a^b f(x)\ \mathrm{d}x$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

Definition 5. If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\ \mathrm{d}x$ and $\int_c^b f(x)\ \mathrm{d}x$ are convergent, then we define

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x.$$

Example. Find $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{1}{\sin x}\ \mathrm{d}x.$

Since the infinite discontinuity occurs at the right endpoint of $\left[\frac{\pi}{2}, \pi\right]$, we use the definition to get

$$\int\limits_{\frac{\pi}{2}}^{\pi} \frac{\mathrm{d}x}{\sin x} = \lim_{b \to \pi^-} \int\limits_{\frac{\pi}{2}}^{b} \frac{\mathrm{d}x}{\sin x} = \lim_{b \to \pi^-} \left(\ln \tan \frac{b}{2} - \ln \tan \frac{\pi}{4} \right) = \infty.$$

Thus the given improper integral is divergent.

Example. Find $\int\limits_{0}^{e} \frac{\ln x}{x} \, \mathrm{d}x$.

Since the infinite discontinuity occurs at the left endpoint of $[0, e]$, we use the definition to get

$$\int\limits_{0}^{e} \frac{\ln x}{x} \, \mathrm{d}x = \lim_{a \to 0+} \int\limits_{a}^{e} \frac{\ln x}{x} \, \mathrm{d}x = \lim_{a \to 0+} \left(\frac{1}{2} (\ln e)^2 - \frac{1}{2} (\ln a)^2 \right) = -\infty,$$

Thus the given improper integral is divergent.

Example. Find $\int\limits_{-1}^{0} \frac{\mathrm{d}x}{\sqrt[5]{x^2}}$.

Since the infinite discontinuity occurs at the right endpoint of $[-1, 0]$, we use the definition to get

$$\int\limits_{-1}^{0} \frac{\mathrm{d}x}{\sqrt[5]{x^2}} = \lim_{b \to 0^-} \int\limits_{-1}^{b} \frac{\mathrm{d}x}{\sqrt[5]{x^2}} = \lim_{b \to 0^-} \left(\tfrac{5}{3} b^{\frac{3}{5}} + \tfrac{5}{3} \right) = \tfrac{5}{3},$$

Thus the given improper integral is convergent.

Example. Find $\int\limits_{0}^{3} \frac{1}{x-1} \, \mathrm{d}x$.

Observe that the line $x = 1$ is a vertical asymptote of the integrand. Since it occurs in the middle of the interval $[0, 3]$, we must use the definition with $c = 1$

$$\begin{aligned} \int\limits_{0}^{3} \frac{\mathrm{d}x}{x-1} &= \int_0^1 \frac{\mathrm{d}x}{x-1} + \int_1^3 \frac{\mathrm{d}x}{x-1} \\ &= \lim_{b \to 1^-} \int\limits_{0}^{b} \frac{\mathrm{d}x}{x-1} + \lim_{a \to 1^+} \int_a^3 \frac{\mathrm{d}x}{x-1} \\ &= \lim_{b \to 1^-} \ln |b-1| + \lim_{a \to 1^+} (\ln 2 - \ln |a-1|) = -\infty + \infty. \end{aligned}$$

This implies that the given improper integral is divergent.

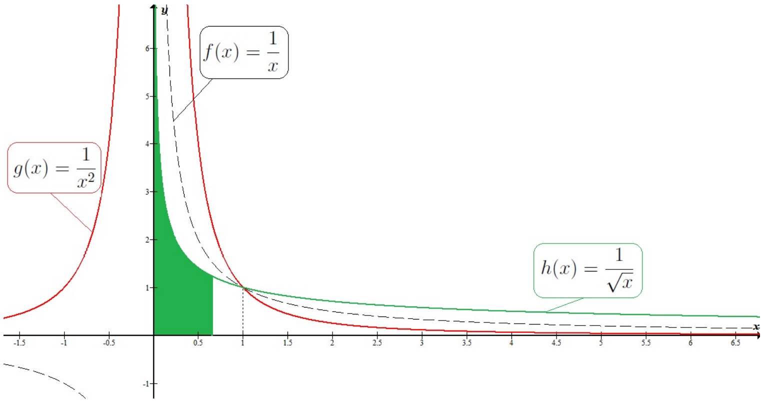
Remark 1. *WARNING: If we had not noticed the asymptote $x = 1$ in the previous example and had instead confused the integral with an ordinary integral, then we might have made the following erroneous calculation:*

$$\int_0^3 \frac{\mathrm{d}x}{x-1} = \ln 2 - \ln 1 = \ln 2.$$

This is wrong because the integral is improper and must be calculated in terms of limits.

Remark 2. *From now on, whenever you meet the symbol* $\int\limits_a^b f(x)\,\mathrm{d}x$ *you must decide, by looking at the function* f *on* $[a, b]$, *whether it is an ordinary definite integral or an improper integral.*

Proposition 2. *The improper integral of Type 2* $\int\limits_0^c \frac{1}{x^p}\,\mathrm{d}x$, *where* $c > 0$, *is convergent if* $p < 1$ *and divergent if* $p \geq 1$.



Proof.

- Let us assume that $p \neq 1$ and $c > 0$. Using the definition we have

$$\int\limits_0^c \frac{\mathrm{d}x}{x^p} = \lim_{a \to 0^+} \int\limits_a^c \frac{\mathrm{d}x}{x^p} = \lim_{a \to 0^+} \left(\frac{1}{1-p} c^{-p+1} - \frac{1}{1-p} a^{-p+1} \right)$$

$$= \begin{cases} \frac{1}{1-p} c^{-p+1} & \text{if} \quad p < 1, \\ +\infty & \text{if} \quad p > 1. \end{cases}$$

- Let $p = 1$ and $c > 0$. Applying the definition we get

$$\int\limits_0^c \frac{\mathrm{d}x}{x^p} = \lim_{a \to 0^+} \int\limits_a^c \frac{\mathrm{d}x}{x^p} = \lim_{a \to 0^+} (\ln c - \ln a) = +\infty.$$

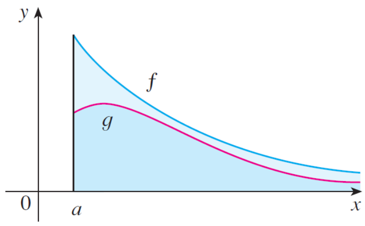
  Thus the improper integral is divergent.

Comparison Test for Improper Integrals

Theorem 6. *Suppose that* f *and* g *are continuous functions with* $0 \leqslant f(x) \leqslant g(x)$ *for* $x \geqslant a$

- *if* $\int\limits_a^\infty g(x)\ \mathrm{d}x$ *is convergent, then* $\int\limits_a^\infty f(x)\ \mathrm{d}x$ *is convergent;*
- *if* $\int\limits_a^\infty f(x)\ \mathrm{d}x$ *is divergent, then* $\int\limits_a^\infty g(x)\ \mathrm{d}x$ *is divergent.*

If the area under the top curve $y = f(x)$ is finite, then so is the area under the bottom curve $y = g(x)$. And if the area under $y = g(x)$ is infinite, then so is the area under $y = f(x)$. Note that the reverse is not necessarily true: If $\int\limits_a^\infty g(x)\ \mathrm{d}x$ is convergent, $\int\limits_a^\infty f(x)\ \mathrm{d}x$ may or may not be convergent, and if $\int\limits_a^\infty f(x)\ \mathrm{d}x$ is divergent, $\int\limits_a^\infty g(x)\ \mathrm{d}x$ may or may not be divergent.



Proof. Consider the functions

$$F(t) = \int_a^t f(x)\ \mathrm{d}x, \quad G(t) = \int_a^t g(x)\ \mathrm{d}x.$$

They are defined for $t > a$. Since $f(x) \geqslant 0$ and $g(x) \geqslant 0$, both $F(t)$ and $G(t)$ are increasing. Furthermore, $g(x) \leqslant f(x)$ for all $x \geqslant a$ and therefore,

$$G(t) \leqslant F(t) \quad \text{for all} \quad t > a.$$

Our assumption is that the following improper integral converges:

$$M = \int_a^\infty f(x)\ \mathrm{d}x.$$

By definition, $M = \lim\limits_{t\to\infty} F(t)$. Since $F(t)$ is increasing, $F(t) \leqslant M$ for all $t > a$. It follows that $G(t) \leqslant M$ for all $t > a$. Since we have shown that $G(t)$ is increasing and bounded by M, $\lim\limits_{t\to\infty} G(t)$ exists. But this limit is equal to the desired improper integral, which proves theorem. A similar theorem is true for Type 2 integrals.

Theorem 7. *Suppose that f and g are continuous on $[a, b)$ and is discontinuous at a with $0 \leq f(x) \leq g(x)$ for $x \in (a, b)$.*

- *If* $\int\limits_a^b g(x)\ \mathrm{d}x$ *is convergent, then* $\int\limits_a^b f(x)\ \mathrm{d}x$ *is convergent.*

- *If* $\int\limits_a^b f(x)\ \mathrm{d}x$ *is divergent, then* $\int\limits_a^b g(x)\ \mathrm{d}x$ *is divergent.*

Example. Determine whether $\int\limits_1^\infty \frac{(x^2+1)\ \mathrm{d}x}{x^4+x^2+1}$ converges or diverges.

By the Comparison Theorem since $0 \leqslant \frac{(x^2+1)}{x^4+x^2+1} \leqslant \frac{1}{x^2}$ and

$$\int\limits_1^\infty \frac{1}{x^2}\ \mathrm{d}x$$

is convergent, thus the integral $\int\limits_1^\infty \frac{(x^2+1)}{x^4+x^2+1}\ \mathrm{d}x$ is also convergent.

Example. Determine whether $\int\limits_2^\infty \frac{(\sqrt{2}+\cos x)}{\sqrt{x}-1}\ \mathrm{d}x$ converges or diverges.

Since $0 \leqslant \frac{(\sqrt{2}-1)}{\sqrt{x}} \leqslant \frac{(\sqrt{2}+\cos x)}{\sqrt{x}-1}$ and

$$\int\limits_2^\infty \frac{\sqrt{2}-1}{\sqrt{x}}\ \mathrm{d}x$$

is divergent, thus by the Comparison Theorem the given integral is also divergent.

Example. Determine whether $\int\limits_0^{\sqrt{2}} \frac{1}{\sqrt{x}} \arctan \frac{1}{x}\ \mathrm{d}x$ converges or diverges.

Since $0 \leqslant \frac{1}{\sqrt{x}} \arctan \frac{1}{x} \leqslant \frac{\pi}{2} \cdot \frac{1}{\sqrt{x}}$ and

$$\int\limits_0^{\sqrt{2}} \frac{1}{\sqrt{x}}\ \mathrm{d}x$$

is convergent, thus by the Comparison Theorem the given integral is also convergent.

Example. Determine whether $\int\limits_0^2 \frac{e^x}{x^3}\ \mathrm{d}x$ converges or diverges.

Since $0 \leqslant \frac{1}{x^3} \leqslant \frac{e^x}{x^3}$ and

$$\int\limits_0^2 \frac{e^2}{x^3}\ \mathrm{d}x$$

is divergent, thus by the Comparison Theorem the given integral is also divergent.

Limit Comparison Test

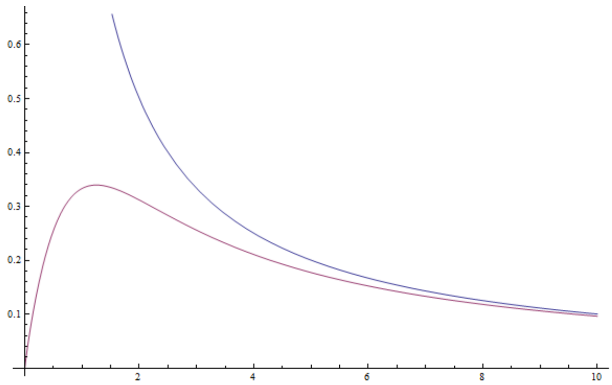
A useful method for demonstrating the convergence or divergence of an improper integral is comparison to an improper integral with a simpler integrand. However, often a direct comparison to a simple function does not yield the inequality we need. For example, consider the following improper integral:

$$\int_1^\infty \frac{x}{x^2+\sqrt{x}+1}\,\mathrm{d}x.$$

Estimating the degree, we see that $\frac{x}{x^2+\sqrt{x}+1} \approx \frac{1}{x}$ and we expect the improper integral to diverge. If we plot the functions, we find that

$$\frac{x}{x^2+\sqrt{x}+1} \leqslant \frac{1}{x} \quad x \geqslant 1$$

so that we cannot directly compare our integral to that of $\frac{1}{x}$ to show it diverges.



One trick is to find some constant C so that

$$C\frac{1}{x} \leqslant \frac{x}{x^2+\sqrt{x}+1} \quad x \geqslant 1.$$

The value of C, in practice, has no effect on our conclusion and takes work to find. The limit comparison test is a result which makes precise the notion of two functions growing at the same rate and reduces the process of finding some constant C to the computation of a single, often easy limit.

Theorem 8. *Suppose that f and g are continuous and positive functions defined on $[a,\infty)$ such that*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = k, \quad \textit{where } k \in (0,\infty),$$

then $\int\limits_a^\infty g(x)\ \mathrm{d}x$ converges if and only if $\int\limits_a^\infty f(x)\ \mathrm{d}x$ converges.

A similar theorem is true for Type 2 integrals.

Theorem 9. *Suppose that f and g are continuous and positive functions defined on $(a,b]$ such that*

$$\lim_{x\to a^+}\frac{f(x)}{g(x)}=k, \quad \textit{gdzie } k\in(0,\infty),$$

then $\int\limits_a^b g(x)\ \mathrm{d}x$ converges if and only if $\int\limits_a^b f(x)\ \mathrm{d}x$ converges.

Proof. Since our functions are both positive, the limit c must be positive. We may choose x_0 close to b such that for $x>x_0$, we have

$$\frac{c}{2}<\frac{f(x)}{g(x)}<\frac{3c}{2}.$$

For any y in the interval $[x_0,b)$ we have

$$\int_{x_0}^{y} f(x)\ \mathrm{d}x=\int_{x_0}^{y}\frac{f(x)}{g(x)}g(x)\ \mathrm{d}x,$$

and then we deduce

$$0<\frac{c}{2}\int_{x_0}^{y} g(x)\ \mathrm{d}x<\int_{x_0}^{y}\frac{f(x)}{g(x)}g(x)\ \mathrm{d}x<\frac{3c}{2}\int_{x_0}^{y} g(x)\ \mathrm{d}x.$$

Taking limits as $y\to\infty$ we have

$$0<\frac{c}{2}\int_{x_0}^{\infty} g(x)\ \mathrm{d}x<\int_{x_0}^{\infty}\frac{f(x)}{g(x)}g(x)\ \mathrm{d}x<\frac{3c}{2}\int_{x_0}^{\infty} g(x)\ \mathrm{d}x.$$

Now we apply comparison: if $\int\limits_{x_0}^{\infty} g(x)\ \mathrm{d}x$ converges, then the above inequalities show that $\int\limits_{x_0}^{\infty} f(x)\ \mathrm{d}x$ does as well (here we are using $f(x)>0$ and $g(x)>0$). If $\int\limits_{x_0}^{\infty} g(x)\ \mathrm{d}x$ diverges, then the second inequality shows $\int\limits_{x_0}^{\infty} f(x)\ \mathrm{d}x$ diverges as well. We assumed both functions were continuous on $[a,\infty)$ so integrating from x_0 instead of a does not affect convergence. Hence, $\int\limits_{a}^{\infty} f(x)\ \mathrm{d}x$ and $\int\limits_{x_0}^{\infty} g(x)\ \mathrm{d}x$ either both converge or both diverge.

Example. We want to determine the convergence of $\displaystyle\int\limits_5^{\infty}\frac{x^2}{\sqrt{x^5-3}}\ \mathrm{d}x$.

Let $f(x)=\frac{x^2}{\sqrt{x^5-3}}$ and $g(x)=x^{-\frac{1}{2}}$. Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^{\frac{5}{2}}}{\sqrt{x^5-3}}=1.$$

Moreover, $\displaystyle\int\limits_5^{\infty} x^{-\frac{1}{2}}\ \mathrm{d}x$ is divergent, thus $\displaystyle\int\limits_5^{\infty}\frac{x^2}{\sqrt{x^5-3}}\ \mathrm{d}x$ is also divergent.

Example. We want to determine the convergence of $\displaystyle\int\limits_0^1 \frac{\mathrm{d}x}{\arcsin^2 x}$.

Let $f(x) = \frac{1}{\arcsin^2 x}$ and $g(x) = \frac{1}{x^2}$. Then

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{x^2}{\arcsin^2 x} = 1.$$

Moreover, $\displaystyle\int\limits_0^1 \frac{1}{x^2}\,\mathrm{d}x$ is divergent, hence $\displaystyle\int\limits_0^1 \frac{\mathrm{d}x}{\arcsin^2 x}$ is also divergent.

Example. We want to determine the convergence of $\displaystyle\int\limits_{\frac{\pi}{2}}^{\pi} \frac{\mathrm{d}x}{\sqrt[3]{\cos x}}$.

Let $f(x) = \frac{1}{\sqrt[3]{\cos x}}$ and $g(x) = \frac{1}{\sqrt[3]{x - \frac{\pi}{2}}}$. Then

$$\lim_{x \to \frac{\pi}{2}^+} \frac{f(x)}{g(x)} = \lim_{x \to \frac{\pi}{2}^+} \frac{\sqrt[3]{x - \frac{\pi}{2}}}{\sqrt[3]{\cos x}} = 1.$$

Moreover, $\displaystyle\int\limits_{\frac{\pi}{2}}^{\pi} \frac{\mathrm{d}x}{\sqrt[3]{x - \frac{\pi}{2}}}$ is convergent, thus $\displaystyle\int\limits_{\frac{\pi}{2}}^{\pi} \frac{\mathrm{d}x}{\sqrt[3]{\cos x}}$ is also convergent.