Lecture 3

Functions of several variables.

Coordinate systems

Definition 1. The Cartesian product

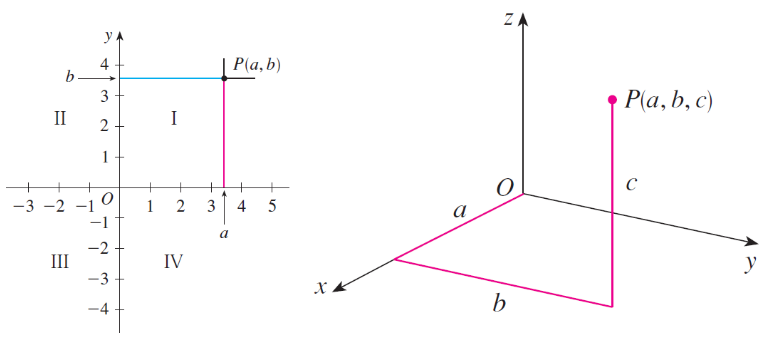
$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

is the set of all ordered pairs of real numbers and is denoted by $\mathbb{R}^2$. We have given a one-to-one correspondence between points P on the plane and ordered pairs (x, y) in $\mathbb{R}^2$. It is called a two-dimensional rectangular coordinate system.

Definition 2. The Cartesian product

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. We have given a one-to-one correspondence between points P in space and ordered triples (x, y, z) in $\mathbb{R}^3$. It is called a three-dimensional rectangular coordinate system.

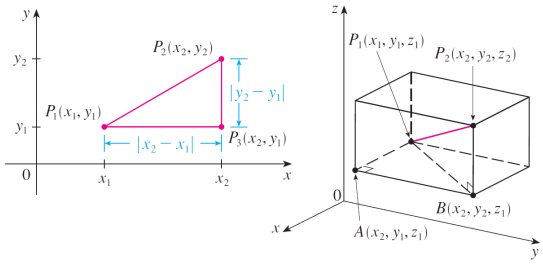


Definition 3. The distance between the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$d(P_1, P_2) = |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Definition 4. The distance between the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is

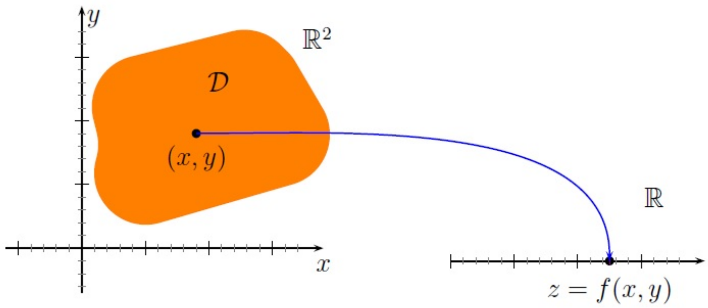
$$d(P_1, P_2) = |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$



Remark 1. *In two-dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional analytic geometry, an equation in x, y and z represents a surface in $\mathbb{R}^3$.*

Function of two variables

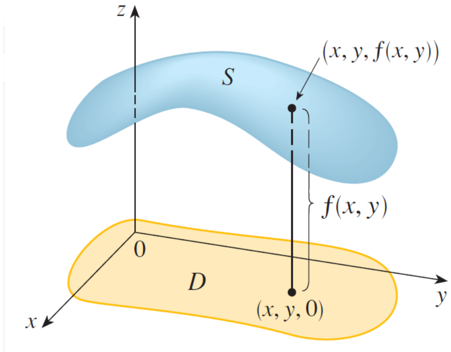
Definition 5. A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set $\mathcal{D}$ a unique real number denoted by $f(x, y)$. The set $\mathcal{D}$ is the domain of and its range is the set of values that f takes on, that is, $\{f(x, y) : (x, y) \in \mathcal{D}\}$.



- We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are independent variables and z is the dependent variable.

- The domain is a subset of $\mathbb{R}^2$, the xy-plane. We can think of the domain as the set of all possible inputs and the range as the set of all possible outputs. If a function is given by a formula and no domain is specified, then the domain of is understood to be the set of all pairs for which the given expression is a well-defined real number.

Definition 6. If f is a function of two variables with domain $\mathcal{D}$, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and is in $\mathcal{D}$.



Remark 2. *In sketching the graphs of functions of two variables, it is often useful to start by determining the shapes of cross-sections (slices) of the graph. For example, if we keep x fixed by putting $x = k$ (a constant) and letting y vary, the result is a function of one variable $z = f(k, y)$, whose graph is the curve that results when we intersect the surface $z = f(x, y)$ with the vertical plane $x = k$. In a similar fashion we can slice the surface with the vertical plane $y = k$ and look at the curves $z = f(x, k)$. We can also slice with horizontal planes $z = k$. All three types of curves are called traces (or cross-sections) of the surface $z = f(x, y)$.*

Another method for visualizing functions, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form contour lines, or level curves.

Definition 7. The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

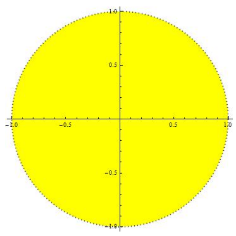
A level curve $f(x, y) = k$ is the set of all points in the domain of f at which takes on a given value k. In other words, it shows where the graph of f has height k.
Example. Find the domain of the function

$$f(x, y) = \ln(1 - x^2 - y^2).$$

The domain of the function f describes the condition

$$1 - x^2 - y^2 > 0,$$

namely $x^2 + y^2 < 1$.



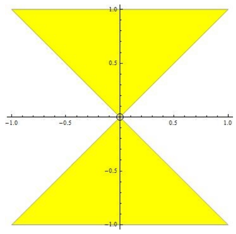
Example. Find the domain of the function

$$g(x, y) = \arcsin \frac{x}{y}.$$

The expression for g makes sense if

$$-1 \leqslant \frac{x}{y} \leqslant 1 \qquad \text{oraz} \qquad y \neq 0,$$

thus $|x| \leqslant |y|$ and $y \neq 0$.

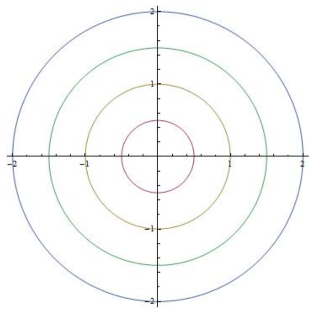


Example. Sketch the level curves of the function

$$h(x,y) = \sqrt{x^2+y^2}.$$

The level curves of the function h are

$$\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = k^2\} \qquad \text{for} \qquad k \geqslant 0.$$



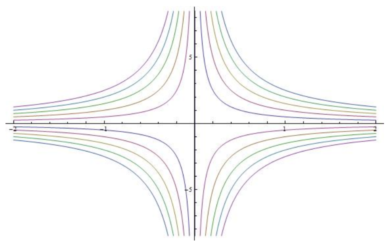
Example. Sketch the level curves of the function

$$i(x,y) = xy.$$

The level curves of i are

$$\left\{(x,y) \in \mathbb{R}^2 : y = \frac{k}{x}\right\} \qquad \text{for} \qquad k \neq 0.$$

For $k = 0$ we have $\{(x,y) \in \mathbb{R}^2 : x = 0 \vee y = 0\}$.

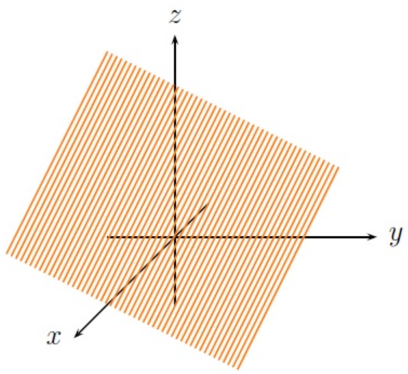


Graphs

The function

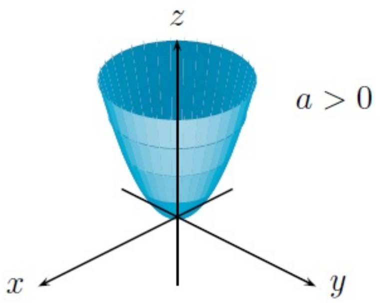
$$f(x, y) = ax + by + c$$

is called a linear function. The graph of such a function has the equation $z = ax+by+c$ it is a plane with a normal vector $[a, b, -1]$.



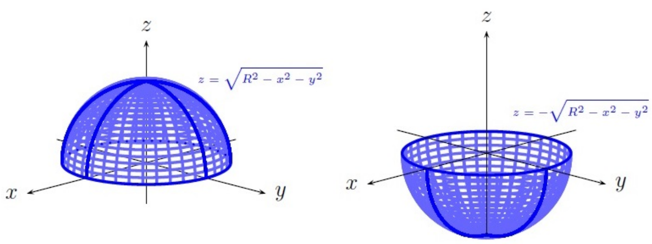
The paraboloid is a quadratic surface which can be specified by the Cartesian equation

$$z = a(x^2 + y^2), \text{ where } a \neq 0.$$



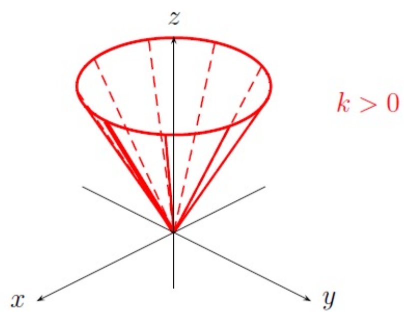
The upper or lower half sphere with the center at the origin and radius $R > 0$ is given by the equation

$$z = \pm\sqrt{R^2 - x^2 - y^2}.$$



The equation of the cone is

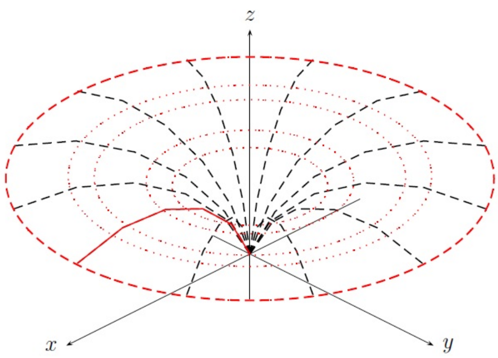
$$z = k\sqrt{x^2 + y^2}, \text{ where } k \neq 0.$$



The equation

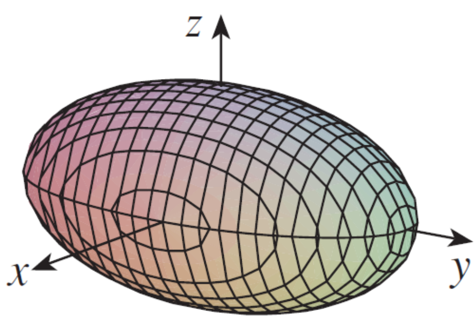
$$z = h\left(\sqrt{x^2 + y^2}\right),$$

describes the solid of revolution obtained by rotating a curve $z = h(x)$, $y = 0$, for $x \geq 0$ around the axis Oz.



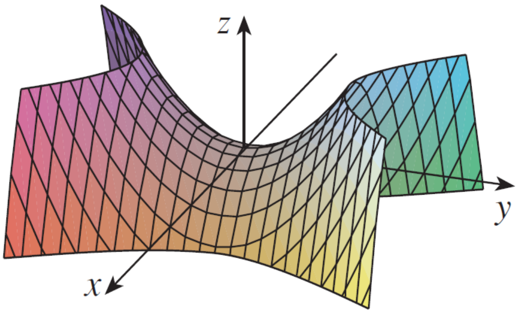
The ellipsoid is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$



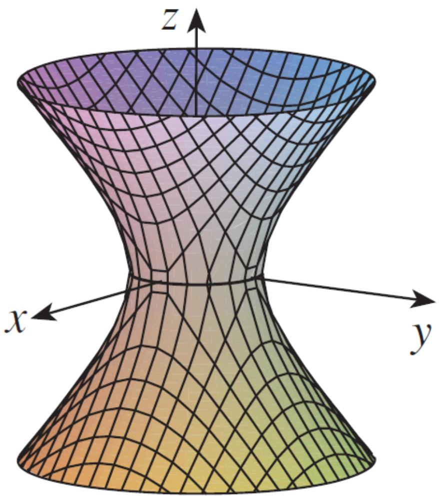
The hyperbolic paraboloid is given by the equation

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$



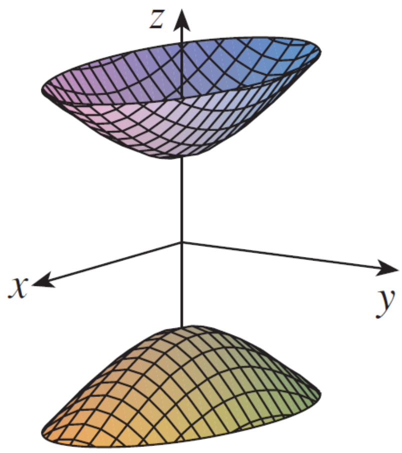
The hyperboloid of one sheet is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$



The hyperboloid of two sheets is given by the equation

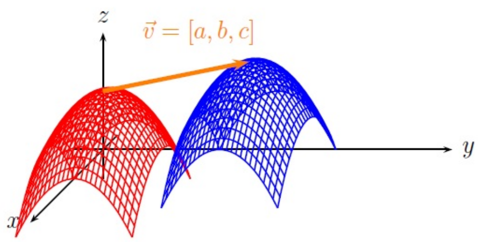
$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$



The equation

$$z = f(x-a, y-b) + c$$

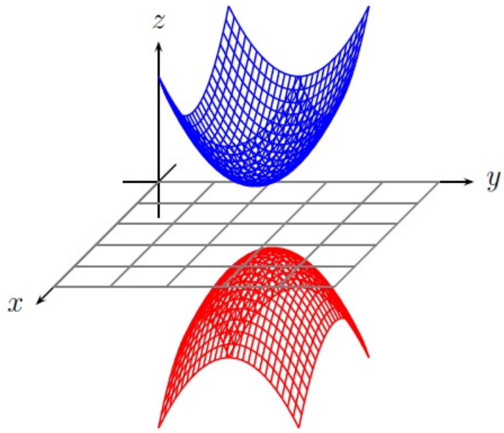
represents a translation of the graph of the function $z = f(x,y)$ by the vector $\vec{v} = [a,b,c]$.



The graph of the function

$$z = -f(x,y)$$

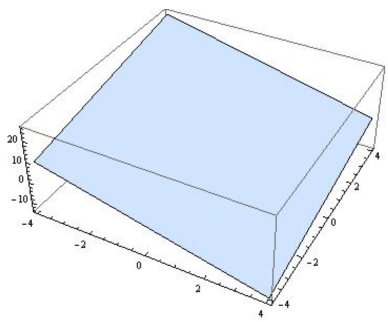
is symmetric to the graph of the function $z = f(x,y)$ about the plane xOy .



Example. Sketch the graph of the function

$$j(x,y) = 6 - 3x + 2y.$$

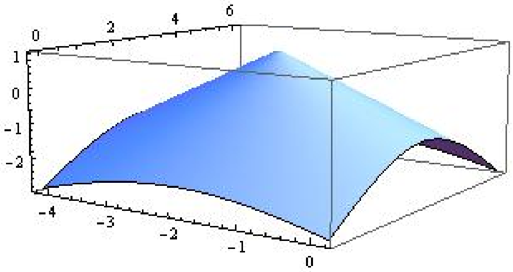
The plane passes through the points $(2,0,0)$, $(0,3,0)$, $(0,0,6)$.



Example. Sketch the graph of the function

$$k(x, y) = 1 - \sqrt{(x+2)^2 + (y-3)^2}.$$

We get the cone with the apex at the point $(-2, 3, 1)$.



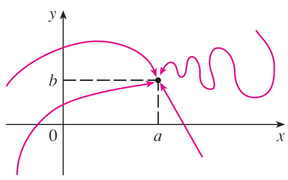
Limits and continuity

Definition 8. We write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

and say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L if we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b).

Remark 3. *For functions of two variables we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever as long as (x, y) stays within the domain of f. The distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the distance between (x, y) and (a, b). It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.*



Definition 9. If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2 where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exists.

Theorem 10 (Limit Laws)**.** *Suppose that the functions f and g have a finite limit at the point (x_0, y_0), then*

$$\lim_{(x,y)\to(x_0,y_0)} [f(x, y) + g(x, y)] = \lim_{(x,y)\to(x_0,y_0)} f(x, y) + \lim_{(x,y)\to(x_0,y_0)} g(x, y)$$

$$\lim_{(x,y)\to(x_0,y_0)}[f(x,y)\cdot g(x,y)]=\left[\lim_{(x,y)\to(x_0,y_0)}f(x,y)\right]\cdot\left[\lim_{(x,y)\to(x_0,y_0)}g(x,y)\right]$$

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)}{g(x,y)}=\frac{\lim_{(x,y)\to(x_0,y_0)}f(x,y)}{\lim_{(x,y)\to(x_0,y_0)}g(x,y)},\textit{ if }\quad\lim_{(x,y)\to(x_0,y_0)}g(x,y)\neq 0$$

Theorem 11 (Squeeze Theorem)**.** *Suppose that* $f(x,y)\leqslant g(x,y)\leqslant h(x,y)$ *for* (x,y) *near* (a,b)*. If* $\lim_{(x,y)\to(a,b)}f(x,y)=L=\lim_{(x,y)\to(a,b)}h(x,y)$ *then* $\lim_{(x,y)\to(a,b)}g(x,y)=L$.

Example. Does the limit

$$\lim_{(x,y)\to(0,0)}\frac{2xy}{x^2+y^2}$$

exist?
Let x_n be a sequence such that $\lim_{n\to\infty}x_n=0$, and $y_n=\alpha x_n$ for $\alpha\in\mathbb{R}\setminus\{0\}$. For $x_n\neq 0$ we have

$$\lim_{(x,y)\to(0,0)}\frac{2x_ny_n}{x_n^2+y_n^2}=\lim_{(x,y)\to(0,0)}\frac{2\alpha x_n^2}{x_n^2+\alpha^2x_n^2}=\frac{2\alpha}{\alpha^2+1}.$$

Since different paths lead to different limiting values, the given limit does not exist.
Example. Does the limit

$$\lim_{(x,y)\to(1,1)}\frac{x^2-y^2}{(x-1)^2+(y-1)^2}$$

exist?
We consider two cases. Let $x_n=1$ and $y_n=1+\frac{1}{n}$. Then we have

$$\lim_{n\to\infty}\frac{1-\left(1+\frac{1}{n}\right)^2}{\frac{1}{n^2}}=-\infty.$$

Let $x_n=y_n=1+\frac{1}{n}$. Then

$$\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2-\left(1+\frac{1}{n}\right)^2}{\frac{2}{n^2}}=0.$$

Since different paths lead to different limiting values, the given limit does not exist.
Example. Find the limit

$$\lim_{(x,y)\to(0,0)}\frac{\sqrt{1+x^2+y^2}-1}{x^2+y^2}.$$

We substitute $u=x^2+y^2$. Notice that $u\to 0$ as $(x,y)\to(0,0)$. Thus, we have

$$\lim_{u\to 0}\frac{\sqrt{1+u}-1}{u}.$$

Using l'Hospital's Rule we obtain

$$\lim_{u\to 0}\frac{\sqrt{1+u}-1}{u}\overset{H}{=}\lim_{u\to 0}\frac{1}{2\sqrt{1+u}}=\frac{1}{2}.$$

Example. Find the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\cos\frac{1}{|x|+|y|}.$$

We apply the Squeeze Theorem. For every $(x,y) \neq (0,0)$ the following inequality holds true

$$\left|(x^2+y^2)\cos\tfrac{1}{|x|+|y|}\right| \leqslant x^2+y^2.$$

Since

$$\lim_{(x,y)\to(0,0)} (x^2+y^2) = 0,$$

we conclude that $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\cos\frac{1}{|x|+|y|} = 0$

Definition 12. A function f of two variables is called continuous at (x_0,y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

We say that f is continuous on $\mathcal{D}$ if f is continuous at every point (x_0,y_0) in $\mathcal{D}$.

Theorem 13. *If the functions f and g are continuous at the point (x_0,y_0), then the functions*

$$f+g,\ \ f\cdot g,\ \ \frac{f}{g}\ \ (\textit{if}\quad g\neq 0)$$

are continuous.

Example. Evaluate $\lim\limits_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

Since $x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 11.$$

Example. Where is the function

$$l(x,y) = \begin{cases} x+y & \text{if} \quad x>0, y\in\mathbb{R} \\ \sqrt{x^2+y^2} & \text{if} \quad x<0, y\in\mathbb{R}. \end{cases}$$

continuous?

Notice that for $y\in\mathbb{R}$ we have

$$\lim_{x\to 0^-} l(x,y) = |y|,$$

$$\lim_{x\to 0^+} l(x,y) = y.$$

Since the limits are equal only for $y>0$, the function is continuous on

$$\mathbb{R}^2 \setminus \{(x,y)\in\mathbb{R}^2 : x=0, y<0\}.$$

Example. Where is the function

$$h(x,y) = \arctan\frac{y}{x}$$

continuous?

The function $f(x, y) = \frac{y}{x}$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function

$$g\left(f(x, y)\right) = \arctan \frac{y}{x} = h(x, y)$$

is continuous except where $x = 0$.