Homological Stability

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Contents

1 Introduction

1.1 Stability

Given some objects $G_n \to G_{n+1} \to \ldots$, we want to find some invariant, $G \mapsto \text{Inv}(G)$, satisfying *stability*: that for sufficiently large n, $\text{Inv}(G_n) \cong \text{Inv}(G_{n+1})$.

As an example, we can take $\text{Inv}(G) = G^{ab}$, the abelianization, and if for example $G_n = GL_n(k)$, then we have stbility for $n \geq 2$, and similar for the symplectic groups, the special linear groups, etc.

A special case we will discuss in more detail will be the symmetric groups, S_n, the group of bijections from a set of n objects to itself. In particular, the abelianization sends, for $n \geq 2$, $S_n \mapsto \mathbb{Z}/2\mathbb{Z}$; this is appararent from the group representation

$$S_n \cong \left\langle (\sigma_i)_{i=1}^n \mid \sigma_i^2 = 1, \sigma_i\sigma_j = \sigma_i\sigma_i \text{ for } |i-j| > 2, \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1 \right\rangle$$

Similarly, we can look at the braid groups B_n, which are given by

$$B_n \cong \left\langle (\sigma_i)_{i=1}^{n-1} \mid \sigma_i\sigma_j = \sigma_i\sigma_i \text{ for } |i-j| > 2, \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1 \right\rangle$$

as well as the "pure braid groups", given by $\text{Ker}(B_n - > S_n)$; it is clear that the abelianization of B_n is just $\mathbb{Z}$.

Furthermore, it will turn out that the abelianization of a group is the same as the first homology group, $H_1(G)$; the second homology group will be more complication, and will be characterized by

$$H_2(G) = \frac{R \cap [F, F]}{[R, R]}$$

where $G = F/R$, and F is free.

Theorem (Nakaoka, 1960's): For $n \geq 2q$, $H_q(S_n) \cong H_q(S_{n+1})$; this result was generalized to homological stability, which gives it for a lot of other sequences of groups.

A *non-example* will be the pure braid groups; we have already

$$\begin{aligned} B_n &\to \text{Aut}(P_n) \\ t &\mapsto (p \mapsto tpt^{-1}) \\ S_n \cong B_n/P_n &\to \text{Aut}(H_q(P_n)) \end{aligned}$$

so $H_q(P_n$ should be an abelian group with some S_n-action. More explicitly, let $\sigma_{ij} \in B_n$ be the lift of the transposition (i, j), and take

$$P_n = \langle (a_{ij})_{i<j} \mid [a_{ij}, -] = 1 \rangle$$

with the map $a_{ij} \mapsto \sigma_{ij}^2$. It also becomes clear from this representation that the abelianization of P_n will be $\mathbb{Z}^{\oplus \frac{n(n-1)}{2}}$. Furthermore, we have that

$$H_1(P_n) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \text{Sym}^2(St_n)$$

where St_n is the standard representation.

1.2 Representational Stability

This is our first example of representational stability!

In particular, we define $\mathbb{Q}-$ characters of S_n to be the set of class functions of S_n, or all maps to Q invariant by conjugation.

Then, we can see that

$$\chi_{St_n}(\sigma) = \chi_1(\sigma) - 1$$

and so we conclude that since $\chi_{\mathrm{Sym}^2}(\sigma) = \frac{1}{2}(\chi(\sigma^2) + \chi(\sigma)^2)$,

$$\chi_{H_1(P_n)} = -\chi_2 + \binom{\chi_1}{2}$$

Theorem (Church-Farb): For n sufficiently large, $(H_q(P_n))_{\mathbb{Q}}$ is a polynomial sequence of characters of $(S_n)_n$.

1.3 FI Modules

The category FI will have objects indexed by $m \geq 0$, with $\mathrm{Hom}_{FI}(m, n)$ to be injective maps $[m] \to [n]$.

In fact, $H_q(P_n)$ are really functions $FI \to Ab$; FI-modules will be functors $FI \to \mathrm{Vec}_{\mathbb{Q}}$; equivalently, finitely generated FI-modules will be the same as polynomial sqeuences of representations of S_n.

The thing to care about is the Noetherianity of such categories. We will also consider a little bit of topology, in the sense that we may have some FI-spaces, which map $n \mapsto \mathrm{Conf}_n(X)$, the space of injective maps $[n] \to X$.

For example, $\mathrm{Conf}_n\mathbb{C} = \{(z_i)_{i=1}^n \mid z_i \neq z_j\}$, and the fundamental group will be P_n.

Theorem (Church-Eilenberg-Farb): For n sufficiently large, then $(H_q(\mathrm{Conf}_n(X)))_n$ is a polynomial sequence of G_n-representations.

Note that for $n = q = 1$, we have the correspondence

$$H_1(X) = \pi_1^{ab} = H_1(\pi_1)$$

2 Categories

Definition: A category C is the data of a class $\mathrm{Ob}(C)$ of objects and

1. For each $X, Y \in \mathrm{Ob}(C)$ a <u>set</u> $\mathrm{Hom}_C(X, Y)$
2. For each X, Y, Z a map $\mathrm{Hom}_C(X, Y) \times \mathrm{Hom}_C(Y, Z) \mapsto \mathrm{Hom}_C(X, Z)$
3. For all $X \in \mathrm{Ob}(C)$, some identity element id_X in $\mathrm{Hom}_C(X, X)$

where the morphisms satisfy associativity.

Examples:

- *Set*
- *Gp*

- For a fixed (associative, unital) ring R, the category of (right) modules over R, Mod_R

To be more prescise, we make the following definitions:

Definition: A morphism in Mod_R from $M \to N$ is a group homomorphism $f : M \to N$ satisfying that $f(m\lambda) = f(m) \cdot \lambda$.

Example: $R \cong \mathrm{Hom}_{Mod_R}(R, R)$, where $a \mapsto (x \mapsto ax)$.

Definition Functors $C \to D$ are a collection of data, containing

1. For each $X \in \mathrm{Ob}(C)$, an object $F(X) \in \mathrm{Ob}(D)$
2. For each $f \in \mathrm{Hom}_C(X, Y)$, some $F(f) \in \mathrm{Hom}_D(F(X), F(Y))$, which satisfy
   (a) $F(fg) = F(f)F(g)$
   (b) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$

We can also form now a category of categories, i.e. a 2-category, and so on onto n-categories.

OK blah blah blah a bunch of the usual category stuff that I'm not typing - adjoints, equivalences, functor categories, etc.

Example: We call the functor category $\mathrm{Func}(FI, Ab)$, to be the category of FI-modules.

2.1 Finiteness Conditions

Definition: C is a small category if its objects are a set, and essentially small if it admits a small skeleton.

Definition: A subcategory C' of C is a category with objects a subclass of $\mathrm{Ob}(C)$ and morphisms a subset of $\mathrm{Hom}_C(\cdot, \cdot)$. It is called a full subcategory if the inclusion functor $C' \to C$ is fully faithful, or equivalently, if $\mathrm{Hom}'_C = \mathrm{Hom}_C$.

Definition: Finite presentation: See Stacks Project, Tag 00F3.

2.2 Limits/Colimits

Definition: Let I be a (partially) ordered set; an I-diagram on C is a functor $F : I \to C$ when I is considered as a category; a limit of F is an object of C that represents the functor $C \to Set$, $X \mapsto \mathrm{Hom}_{\mathrm{Func}(I,C)}((i \mapsto X), F)$; a colimit is the same, but the Hom arguments are reversed.

Definition A colimit is filted if I has finite supremum.

Definition An object X of C is compact if $\mathrm{Hom}(X, -)$ commites with filtered colimit.

Example: In the categories of $CRing, CAlg, Mod_R, Sets$, etc, compact is the same as finitely presented.

Proof: Let B be a compact object of $CAlg$, and let I the category where the objects $S \subset B$ finite sets and $I \subset A[(x_s)]_{s \in S}$ is a finitely generalted ideal of $A[(x_s)] \to B$. Then, we let the morphisms be $(S, J) \to (S', J')$ where $S \subset S'$, $J \subset J'$; then,

$$\mathrm{colim}_{(S,J) \in I} A[(\alpha_s)_{s \in S}]/J \cong B$$

where we use Yoneda; then use uniervsal property of the colimit and compactness to lift out a finite presentation of B.

Definition Subobject: see nLab, `https://ncatlab.org/nlab/show/subobject`

Definition Noetherian object: see nLab, `https://ncatlab.org/nlab/show/noetherian+object`

Definition We say that a category C is finite if $\mathrm{Ob}(C)$ is finite and the Hom-sets are also finite; similarly, a finite diagram is a functor $D \to C$ where D is finite.

Above, we have a functor from $C \to Sets$, which takes $X \mapsto \mathrm{Hom}_{\mathrm{Func}}(F, (d \mapsto X))$, which is representable iff F has a colimit.

Example: Equalizers: `https://ncatlab.org/nlab/show/equalizer`

Lemma: If a category has all equalizers and finite products, then it has all finite limits.

Proof: Let $F : D \to C$ be a finite diagram, D_0 the finite set of objects of D, and D_1 the finite set of arrows of D; then we should have $t, s : D_1 \to D_0$ that associate each arrow to their target and source.

We want to classify all such $\lambda_d : X \to F_d$, for all $d \in D_0$, such that for all $f \in D_1$, such that the following commutes: Then, we claim that by looking at the following diagram, that $\lim F$ exists, and is the equalizer.

2.3 Abelian Categories

Definition: A category C is preadditive when it has the structures of abelian groups of $\mathrm{Hom}_C(-,-)$, such that composition is bilinear; e.g. in torsion free abelian groups, or in Ab itself. Further, it is additive if it also has finite products and coproducts.

Note that this implies immediately that an additive category has initial and terminal objects.

Lemma: Let C be an additive cateogry; then for any X_1, X_2, we have $X_1 \sqcup X_2 \cong X_1 \times X_2$ functorially in X_1, X_2.

Proof: Let us consider and So by reversing arrows we get a map $q : X_1 \times X_2 \to X_1 \sqcup X_2$, $q = \iota_1\pi_1 + \iota_2\pi_2$, where ι, π are inclusions and projections; then the following diagram can be made to commute: **Definition**: The kernel of $f : X_1 \to X_2$ in an additive category is the equalizer of f_1 and 0, if it exists; the cokernel is the coequalizer.

Lemma: If $\iota : K \to X_1$ is a kernel, then ι is a monomorphism.

Definition: An abelian category is an additive category such that all (co)kernels exist (or equivalently, all equalizers and colimits, and therefore the same as all finite limits + colimits), and all epi/monomorphisms are such (co)kernels.

Lemma: If A is an abelian category amd D is a small category, then $\mathrm{Func}(D, A)$ is an abelian category, where the operation is $F_1 \oplus F_2 = (d \mapsto F_1(d) \oplus F_2(d))$.

Example: $\mathrm{Func}(FI, Mod_R)$ is an abelian category.

Lemma: Full subcategories of abelian categories are themselves abelian, so long as it contains a zero object, is stable by $\oplus$, and contains (co)kernels; further, B preserves (co)kernels.

Disclaimer! We actually need the assumptions; consider $R = A[x, y]$, and $A = Mod_R$, and B the full subcategory of objects M such that $M \cong \mathrm{Ker}(M \oplus M \to M)$, there the first arrow sends $m \mapsto (mx, my)$ and the second $(a, b) \mapsto am - bm$.

3 Homological Algebra

Definition: A sequence $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$ is said to be exact (at X_2) if $X_1 \to \mathrm{Ker}(g)$ is an epi.

Definition: We say a functor $F : A \to B$ is additive if F is $\mathbb{Z}$-linear on Hom-sets and exact if it preserves exact sequences; it also needs to preserve finite products and coproducts, which means that it is equivalent to preserving finite limits and colimits.

Definition: Left/right exact, see Stacks Project Tag 003.

Lemma: If F is a left adjoint of G, and F, G are additive, F commutes with colimits.

Proof: Use Yoneda.

Lemma: Let $X \in \mathrm{Ob}(A)$, A an abelian category; then $\mathrm{Hom}(X, -)$ is a functor $A \to Ab$, and it is left exact.

3.1 Projective Modules

Definition: An object X of A is said to be projective if $\mathrm{Hom}(X, -)$ is exact, and dually it is injective if the contravariant Hom is exact. See Stacks Tag 013A for more details.

Lemma: TFAE for $P \in Mod_R$ projective:

1. P is compact ni Mod_R
2. $\mathrm{Hom}(P, -)$ commutes with direct sums
3. P is finitely presented
4. P is finitely generated
5. P is a direct summand of a finite free module

Theorem (Quillen): If R is commutative Noetherian and any projective R-module is stably free, then the same is true for $R[t]$.

Theorem (Quillen-Suslind): Any finite projective module over $k[x_1, \ldots, x_n]$ is free (also true over PID).

Lemma: $k[x]$ is a PID, and for any commutative PID R, TFAE for M a R-module:

1. M is a submodule of a free module
2. M is free
3. M is projective

Theorem: Same over local rings, without finiteness conditions.

Proof: See new Matsumura, chapter 2.

Lemma: For R commutative, TFAE:

1. Any stably free R-module is free
2. For all m and any $r_1, \ldots, r_n$ generating the unit ideal, there is some $A \in GL_n(R)$ such that $r_1, \ldots, r_n$ form the first column.

For reference, see `https://kconrad.math.uconn.edu/blurbs/linmultialg/stablyfree.pdf`.

Definition We say that an abelian category A has enough projectives, if for all $X \in A$ there is a projective P which surjects onto X.

For example, Mod_R certainly has enough injectives, since you can just take a really big free module.

Lemma: Let A be abelian with small colimits and an object P which is compact, projective, and a generator; then we have an exact functor $A \to Mod_R$, and $X \mapsto \mathrm{Hom}(P, X)$, where $R = \mathrm{End}_A(P)$.

Look carefully: this shows that $Mod_R \cong Mod_{R^{\oplus M}}$ (ref: see Morita equivalence).

3.2 Injective Modules

Definition: $I \in A$ is injective if $\mathrm{Hom}(-, I)$ is left exact; similarly, A has enough injectives if there is always some monic $X \to I$ into an injective object.

For modules, see Baer's criterion, Stacks Tag 05NU.

3.3 Chain Complexes

Definition: $Ch(A)$, the category of chain complexes of objects of an abelian category A, has objects functors $\mathbb{Z} \xrightarrow{C_\bullet} A$

$$\cdots \to C_{n+1} \xrightarrow{d_n} C_n \xrightarrow{d_{n-1}} C_{n+1} \to \cdots$$

such that $d_n \circ d_{n+1} = 0$. Furthermore, a morphism of chain complexes is a natural transformation of functors that commutes with the boundary maps. Cochains are chains with the arrows reversed.

Theorem: Let A abelian and let C a full subcategory generated by finitely many elements; then there exists a fully faithful functor $C \to Mod_R$ for some R.

OK I was sleepy so a topic list: derived categories, derived functors as factoring through the derived category, homotopy, etc.

4 Group Homology

Definition: An abelian group A is a G-module if it is a $\mathbb{Z}[G]$-module. Note that the category of G-modules is the same as the functor category from G to *Ab*, where G is regarded as a groupoid with one element. A trivial G-module is an abelian group A on which G acts trivially, that is $ga = a$ for all $g \in G$ and $a \in A$. Considering every abelian group as a trivial G-module defines an exact functor from *Ab* to G-mod. For ease of notation we denote $\mathrm{Hom}_{\mathbb{Z}[G]}(X, Y)$ as $\mathrm{Hom}_G(X, Y)$.

Definition: Given a G-module A, we define the invariants to be

$$A^G = \{ga = a \mid g \in G, a \in A\},$$

and coinvariants to be

$$A_G = A/\{ga - a \mid g \in G, a \in A\},$$

and consider them as functors $(-)^G$, and $(-)_G$ from G-mod to *Ab*. $(-)^G$ is right adjoint to the trivial module functor and $(-)_G$ is left adjoint to the trivial module functor, so that $(-)^G$ is left exact and $(-)_G$ is right exact. So, since G-Mod is an abelian category, we have right and left derived functors respectively, which are homology and cohomology functors!

Lemma: $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$, and $A^G \cong \mathrm{Hom}_G(\mathbb{Z}, A)$, and therefore we have left/right derived functors, which are homology and cohomology functors!

Definition: Define the augmentation map $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ by $g \mapsto 1$. The augmentation ideal is the ideal $I_G = \ker(\epsilon)$. In particular we have $\mathbb{Z} \cong \mathbb{Z}[G]/I_G$

Lemma: $A^G \cong \mathrm{Hom}_G(\mathbb{Z}, A)$ and $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$.

Proof: For $a \in A^G$ we define $\varphi_a : \mathbb{Z} \to A$ by $1 \mapsto a$. Note that φ_a is a $\mathbb{Z}[G]$-linear map since for $n \in \mathbb{Z}$ and $g \in G$ we have $\varphi_a(gn) = \varphi_a(n) = n\varphi_a(1) = na = n(ga) = g(na) = g\varphi_a(n)$. Consider the map $A^G \to \mathrm{Hom}_G(\mathbb{Z}, A)$ given by $a \mapsto \phi_a$ where $\phi_a : \mathbb{Z} \to A$ is the map given by $\phi_a(1) = a$. Clearly $\phi_{a+b} = \phi_a + \phi_b$, and the map has the inverse $\mathrm{Hom}_G(\mathbb{Z}, A) \to A^G$, $\phi \mapsto \phi(1)$. For the second result, note that $A_G = A/I_G A$, and hence we have $A_G \cong A/I_G A \cong \mathbb{Z}[G]/I_G \otimes \mathbb{Z}[G] \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$.

By the Lemma above, it then follows that $H^i(G,A) \cong \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z},A)$ and $H_i(G,A) \cong \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z},A)$

4.1 Induced Modules

Definition: Let $H \subset G$, with finite index, and M an H-module, and define the induced G-module

$$\mathrm{Ind}_H^G(N) = \mathbb{Z}[H] \otimes_{\mathbb{Z}[H]} M = \{g \otimes m \mid h \otimes m = 1 \otimes hm\} = \amalg_{s \in G/H} s \otimes M$$

and similarly the coinduced G-module

$$\mathrm{Coind}_H^G(M) = \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M) = \{f : G \to M \mid f(hg) = hf(g)\} = \{f(s)\}_{s \in G/H}$$

If the index is finite, these are isomorphic.

Lemma: $\mathrm{Coind}_H^G(-)$ preserves injectives; $\mathrm{Ind}_H^G(-)$ preserves projectives.

Proof: Let I be an injective H-module. Consider $\mathrm{Hom}_G(-, \mathrm{Coind}_H^G(I)) \cong \mathrm{Hom}_G(-, \mathrm{Hom}_H(\mathbb{Z}[G], I))$. By Tensor-Hom adjunction, we have $\mathrm{Hom}_G(-, \mathrm{Hom}_H(\mathbb{Z}[G], I)) \cong Hom_H(\mathbb{Z}[G] \otimes_{Z[G]} (-), I) \cong Hom_H(-, I)$, which is exact. Hence $\mathrm{Coind}_H^G(I)$ is injective. Also, given a projective module P, since $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-module, have that $\mathrm{Ind}_H^G(P) = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} P$ is projective.

Shapiro's Lemma: For $H \subset G$, M an H-module, then there are canonical isomorphisms

$$H^r(G, \mathrm{Coind}_H^G M) \cong H^r(H, M)$$

$$H_r(G, \mathrm{Ind}_H^G M) \cong H_r(H, M)$$

Proof: We will give the proof for coinduction, the case for induction can be done in similar fashion. Take an injective resolution $0 \to M \to I^\bullet$ of M (as an H-module). Since Coind preserves injectives, we have an injective resolution $0 \to \mathrm{Coind}_H^G M \to \mathrm{Coind}_H^G I^\bullet$. Therefore $H^\bullet(G, \mathrm{Coind}_H^G M)$ can be computed by the complex $(\mathrm{Coind}_H^G I^\bullet)^G$ but note that

$$(\mathrm{Coind}_H^G I^n)^G \cong Hom_G(\mathbb{Z}, \mathrm{Hom}_H(\mathbb{Z}[G], I^n)) \cong \mathrm{Hom}_H(\mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G], I^n) \cong \mathrm{Hom}(\mathbb{Z}, I^n) \cong (I^n)^H.$$

So the complex $(\mathrm{Coind}_H^G I^\bullet)^G$ is $(I^n)^H$ which has cohomology $H^\bullet(H, M)$ giving the result.

Corollary: (Shapiro's Lemma for trivial H) Given an abelian group A, we have

$$H^n(G, \mathrm{Hom}_{Ab}(\mathbb{Z}[G], A)) H_n(G, \mathbb{Z}[G] \oplus_{\mathbb{Z}} A) = \begin{cases} A & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Definition: We have two different restrictions, which take $M \to \mathrm{Coind}_H^G M$ (where $m \mapsto (g \mapsto gm)$ or $M \to \mathrm{Ind}_H^G M$ (where, under $[G:H] < \infty$, sends $m \mapsto \sum_{s \in G/H} s \otimes s^{-1}m$), and two different corestrictions, which take $\mathrm{Coind}_H^G M \to M$ (where, under $[G:H] < \infty$, sends $\varphi^H \mapsto \sum_{s \in G/H} s\varphi(s^{-1})$) and $\mathrm{Ind}_H^G M \to M$ (where $g \otimes m \mapsto gm$).

Now consider the following, and the opposite maps, as above.

Lemma: If $[G:H] = r < \infty$ then the composition $\mathrm{Cores} \circ \mathrm{Res}$ is just multiplication by r. *Proof*:

Theorem: Assume that G is finite, with order m; further, suppose we have $m : A \to A$ is an isomorphism. Then, all higher (co)homology groups vanish, and if $N = \sum_{g \in G}[g]$, then $H_0(G,A) = H^0(G,A) = NA$.

Proof: Take H to be trivial.

4.2 Explicit Resolutions

In the previous section, we computed the group homology $H_i(G,M) = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z},M)$ (resp. $H^i(G,M) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},M)$) through taking a projective (resp. injective) resolution of M, applying the functor $\mathbb{Z}\otimes\mathbb{Z}G(-)$ (resp. $\mathrm{Hom}_G(\mathbb{Z},-)$), and taking the i-th homology (resp. cohomology) in the resulting chain complex. In this section, we will instead take a projective (resp. injective) resolution of $\mathbb{Z}$ to compute $\mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z},M)$ (resp. $\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},M)$). This will allow us to compute arbituary homology and cohomology groups without having to find a projective/injective resolution for M every time.

To start with, we take

$$\cdots \to B_2 \xrightarrow{d_2} B_1 \xrightarrow{d_1} B_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where

$$B_r = \mathbb{Z}[G^{r-1}] \quad , \quad d([g_0,\dots,g_r]) = \sum_{i=0}^{r}(-1)^i[g_0,\dots,\widehat{g_i},\dots,g_r],$$

and ϵ is the augmentation map. One can check that this is a free (and therefore projective) resolution of $\mathbb{Z}$. Applying $\mathrm{Hom}(-,M)$, we get

$$\cdots \to \mathrm{Hom}_G(B_{r-1},M) \to \mathrm{Hom}_G(B_r,M) \to \mathrm{Hom}_G(B_{r+1},M) \to \cdots$$

Note that

$$\mathrm{Hom}_G(B_r,M) = \{\phi : G^{r+1} \to M : \phi(g\cdot(g_0,\cdots,g_r)) = g\phi(g_0,\cdots,g_r))\}.$$

Therefore, ϕ is determined by $\phi(1,g_1,\cdots,g_r)$. For convenience, we take the basis $\{g_1,\cdots,g_2\} := [1,g_1,g_1g_2,\cdots,g_1g_2\cdots g_r]$. It follows that

$$\mathrm{Hom}_{\mathbb{Z}G}(B_r,M) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G^r],M) := C^r(G,M),$$

with the differentials

$$\begin{array}{ccc} C^r(G,M) & \xrightarrow{\quad d_r \quad} & C^{r+1}(G,M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_G(B_r,M) & \xrightarrow{d_r} & \mathrm{Hom}_G(B_{r+1},M) \end{array}$$

where

$$(d_r(\phi))(g_1,\cdots,g_{r+1}) = g_1\phi)(g_2,\cdots,g_{r+1} + \sum_{i=1}^{r}(-1)^i(\phi(g_1,\cdots,g_ig_{i+1},\cdots,g_{r+1}) + (-1)^{r+1}\phi(g_1,\cdots,g_r).$$

We define $C^r(G,M)$ to be the r-cochains of G with values in M, $Z^r(G,M) = \mathrm{Ker}(d^r)$ to be the r-cocyles, and $B^r(G,M) = \mathrm{Im}(d^{r+1})$ to be the r-coboundaries. Then $H^r(G,M) = Z^r(G,M)/B^r(G,M)$.

Proposition 4.1. *If G is a finite group and M is a finitely generated module, then $H^r(G,M)$ is a finite group.*

4.3 Examples

Example 4.2 (First Homology Group)**.** We will use the explicit resolution to calculate $H^1(G, M)$. First, we find the 1−cocyles $Z^1(G, M) = \{\phi : G \to M : d^1\phi = 0\}$. Note that

$$(d^1\phi)(g_1, g_2) = g_1\phi(g_2) - \phi(g_1g_2) + \phi(g_1) = 0 \Longleftrightarrow \phi(g_1g_2) = g_1\phi(g_2) + \phi(g_1)$$

The ϕ's that satisfy this property are called *crossed homomorphisms.*

Next, we find the 1−coboundaries $B^1(G, M) = \mathrm{Im}(d^0)$. Suppose $\phi = d^0\psi$, where $\psi : 1 = G^0 \to M$ is the trivial map that specifies an element m in M. Therefore,

$$\phi(g) = (d^0m)(g) = gm - m.$$

The ϕ's of this form are called *principal crossed homomorphisms.* Thus,

$$H^1(G, M) = Z^1(G, M)/B^1(G, M) = \{\text{crossed homomorphisms}\}/\{\text{principal crossed homomorphisms}\}.$$

Example 4.3 (First Homology Group with Integer Coefficients)**.** We can, of course, use the previous example to calculate $H_1(G, \mathbb{Z})$, but we will present a different method by using the short exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z},$$

which induces a long exact sequence in homology

$$\cdots \to H_1(G, \mathbb{Z}G) \to H_1(G, \mathbb{Z}) \to H_0(G, I_G) \to H_0(G, \mathbb{Z}G) \to H_0(G, \mathbb{Z}) \to 0.$$

Since $\mathbb{Z}G = \mathrm{Ind}_1^G \mathbb{Z}$, by Shapiro's lemma we know that $H_i(G, \mathbb{Z}G) = 0$ for all i. Thus,

$$H_1(G, \mathbb{Z}) \cong H_0(G, I_G) = I_G/I_G^2 \cong G^{ab}$$

The last isomorphism is given by the map $[g] - 1 + I_G^2 \mapsto [g]$.

Example 4.4. Let $G =< t >$, so that $\mathbb{Z}[G]$ is the laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. The augmentation ideal $I_G \subset \mathbb{Z}[G]$ is generated by $t - 1$, so the following sequence is exact:

$$0 \to \mathbb{Z}[G] \xrightarrow{t-1} \mathbb{Z}[G \times G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

Using the induced long exact sequence we can compute

$$H_n(G, A) = \begin{cases} A/(t-1)A, & n = 0 \\ A^G, & n = 1 \\ 0, & n \neq 0, 1 \end{cases}$$

Example 4.5 (finite cyclic group)**.** Let $G =< \sigma : \sigma^m = 1 >$ be the finite cyclic group, and A be a G-module. We will show that

$$H_n(G, A) = \begin{cases} A/(\sigma - 1)A, & n = 0 \\ A^G/NA, & n \text{ is odd} \\ \{a : Na = 0\}/(\sigma - 1)A, & n \text{ is even} \end{cases}$$

where $N = \sum_{i=1}^m [\sigma^m]$ is the norm element.

There is a free resolution of Z as follows

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z} \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z}G \to 0$$

Applying the functor $A \otimes_{\mathbb{Z}G} -$, we obtain the chain complex

$$\cdots \xrightarrow{N} A \xrightarrow{\sigma-1} A \xrightarrow{N} A \xrightarrow{\sigma-1} A \to 0$$

Observing that $\ker(\sigma - 1) = \{a \in A : (\sigma - 1)a = 0\} = A^G$, the result follows.

Example 4.6 (free set).

Example 4.7 (S_n preview).

5 Simplicial Sets

Definition: Let Δ be the category of nonempty finite totally ordered sets with nondecreasing maps as morphisms.

Definition: Let C be a category, and $Simp(C)$ of C simplicial objects is the functor category $\mathrm{Func}(\Delta^{op}, C)$, and $Cosimp(C)$ is $\mathrm{Func}(\Delta, C)$.

Example: Let $C = Top$, so that $\Delta^n = \Delta([m]) = \{f : [m] \to \mathbb{R}_+ \mid \sum_{i=0}^m f(i) = 1\}$. Then, let $\alpha : [n] \to [m]$, and set

$$\Delta(\alpha)(f) = \left(j \in [m] \mapsto \sum_{i \in \alpha^{-1}(j)} f(i) \right)$$

So we obtain $\Delta^\bullet \in Cosimp(Top)$.

Definition: The singular simplicial set $S(X)$ attached ti $X \in Top$ is defined as

$$S(X) = \mathrm{Hom}_{Top}(\Delta^\bullet, \underline{X}) \in Simp(Set)$$

Lemma: $X \mapsto S(X)$ has a left adjoint, denoted $E \mapsto |E|$, the "geometric realization."

Proof: Note that $\mathrm{Hom}_{Simp(Sets)}(E, S(X))$ is just the collection of (continuous) maps $f : E_n \to \mathrm{Hom}_{Top}(\Delta^n, X)$ satisfying the appropriate naturality condition, but this is again just the maps $f : E_n \times \Delta^n \to X$, again satisfying the appropriate naturality condition; but lastly, this must be all maps $f : \bigsqcup_{n\in\mathbb{N}} E_n \times \Delta^n \to X$ satisfying that $\forall x \in E_n$ and $\forall s \in \Delta^m$ and $\forall \alpha : [m] \to [n]$, we have that $f(E(\alpha)x, s) = f(x, \Delta(\alpha)s)$.

But since $|E| = \bigsqcup_{n\in\mathbb{N}} E_n \times \Delta^n / \sim$, where $\sim$ is generated by $(E(\alpha)x, s) \sim (x, \Delta(\alpha)s)$, these are exactly the same.

Theorem: For $X \in Top$, TFAE:

1. X is homotopically equivalent to some $|E|$
2. The adjunction $|S(X)| \to X$ is a homotopy equivalence
3. X is Hausdorff and there is some partition $X = \bigsqcup_n \bigsqcup_{i \in C_n} X_i$ into subsets, such that
   (a) A subset $F \subset X$ is closed iff $\forall i, F \cap \overline{X_i}$ is closed in $\overline{X_i}$.

(b) $\forall i \in C_n$, there is a $\Delta^n \xrightarrow{f} X$ continuous such that $f|_{(\Delta^n)^\circ}$ is a homomorphism onto X_i, and $f(\delta\Delta^n) \subset$ a finite union of all cells of degree $< n$.

And if any of the above hold, then X is a CW complex.

Theorem: There is a notion of homotopy for simplicial sets, such that it is equivalent to the homotopy category of CW complexes, where one map is given by $|\cdot|$ and the other by $S(\cdot)$.

Under the idea of simplicial sets, we have that actually represents a category, with E_1 maps, E_0 objects, and E_2 composition! And n categories are just tacking on $E_3, \ldots, E_n$ onto this diagram!

5.1 Classifying Spaces

Let S be a nonempty set, and $\underline{S}_n$ the maps $[n] \to S$, which is now a simplicial set. But its geometric realization is homotopic to a point, so it is not that interesting; but if we consider G a group, then $\underline{G}$ has a right G action, which takes $fg \mapsto (x \mapsto f(x)g$.

Definition: We set $BG = \underline{G}/G$ to be the classifying space of G.

Definition: Let A be an abelian category, and let $X \in Simp(A)$; we construct $C(X) \in Ch(A)$, by setting $C(X)_n = X_n$, and letting the differential map be $\sum_{j=0}^{n}(-1)^j \iota_j^*$ where $\iota_j : [n-1] \to [n]$ is an injection missing j.

Definition: For A abelian, we may now define $H_n(X)$ to be the homology of the chain $H_n(C(X))$.

Definition: We have another complex, the Moore complex, while will be in degree n

$$N(X)_n = \cap_{j=0}^{n-1} \mathrm{Ker}(\iota_j^* : X_n \to X_{m-1}) \subset C(X)_n, \quad \delta_n = (-1)^n \iota_n^*$$

Lemma: The inclusion $N(X) \to C(X)$ induces $H_n(N(X)) \cong H_n(X)$.

Theorem (Dold-Kan Correspondence): The functor $Simp(A) \xrightarrow{N} Ch_{\geq 0}(A)$ is an equivalence of abelian categories.

Definition: We set for $X \in Simp(Sets)$, $H_n(X,R) = H_n(R^{\oplus X})$, and if $X \in Top$, then $H_n(X,R) = H_n(S(X),R)$, where the homology is taken with coefficients in R, a ring.

Theorem: The inclusions $\lambda_0, \lambda_1 : X \to X[0,1]$, given by $\lambda_i : x \mapsto (x,i)$ induce the same homomorphism $H_n(X,R) \to H_n(X \times [0,1], R)$.

Corollary: If $f, g : X \to Y$ are homotopic, then $H_n(f,R) = H_n(g,R)$; further, if $f : X \to Y$ is a homotopy equivlance then $H_n(f,R)$ is ana isomorphism.

Proof: There is some $H : X \times [0,1] \to Y$ which is a homotopy between f, g. Then,

$$H_n(f) = H_n(\lambda_0)H_n(H) = H_n(\lambda_1)H_n(H) = H_n(g)$$

Theorem:

1. If $X \in Simp(Set)$, then $X \to S(|X|)$ ia a homotopy equivalence, and in particular $H_n(X,R) \cong H_n(S(|X|),R)$.

2. If X is a CW complex, then $H_n(|S(X)|,R) \cong H_n(X,R)$.

Example: Let us consider S^1. In particular, S^1 is a line with endpoints identified, so we may let Then, E_n is just generated by formally adding degeneracies of t; let us look at $C(R[E])$, so we have a complex

$$0 \to R[t] \xrightarrow{t \mapsto \iota_0^* t - \iota_1^* t = 0} R[a] \to 0$$

so we see that $H_1(S^1, R) \cong H_0(S^1, R) \cong R$. If we don't glue things together, so it's just an interval I, then we get the complex

$$0 \to R[t] \to R[a_0] \oplus R[a_1] \to 0$$

so we see that $H_1(I, R) = 0$, $H_0(I, R) = R$.

We may also do $S^1 \times S^1$, under the triangulation so we can see the complex will be

$$R[\delta_0] \oplus R[\delta_1] \to R[a] \oplus R[b] \oplus R[c] \to R[e] \to 0$$

so that $H_0 = R, H_1 = R^{\oplus 2}, H_2 = R$.

In general, if $X \in Top$, then $C(S(X), R)$ will end in

$$\text{free modules on paths} \to \text{free modules on points} \to 0$$

so it must be H_0 is the set of free R-modules on $X/(f(0) \sim f(\lambda), f \in C^0([0,1], X))$, or the set of path-connected components of X.

Theorem: $H_1(X, \mathbb{Z}) \cong \pi_1(X)^{ab}$.

Proof: π_1 is just the homotopy classes of $f : \Delta^1 \to X$, such that $f\iota_0 = f\iota_1$; in particular, we have a map $\pi_1 \to Z_1(C(S(X)))$, where $f \mapsto [f]$, where we have that $df = [f\iota_0] - [f\iota_1] = 0$.

Then, we have, since $Z_1(C(S(X)))/B_1 \cong H_1$ is abelian, we have a map that factors $\pi_1^{ab} \to H_1$. Then, just check injective/surjective.

Theorem: Let G be a group; then the $H_n(G, R) \cong H_n(BG, R) \cong H_n(|BG|, R)$. In fact, $\pi_1(BG) = G, \pi_k(BG) = 0$ otherwise; that is, it is a $K(G, 1)$ space.

5.2 Homotopy Groups

Let X be a topological space, and set $\pi_k(X)$ the fundamental group in degree k, e.g. all maps $S^k \to X$ identified up to homotopy. Note that in general, π_0 is just a set, π_1 is a group, and π_k is an abelian group for higher k.

Theorem: The group homology of G is the same as the singular homology of a $K(G, 1)$ space.

We will also take

$$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$$

and show that it is path connected and has π_1 the pure braid group.

Definition: A continuous map $p : E \to B$ is a fiber bundle with typical fiber F if $\forall b \in B$, there is U open containing b such that conmmutes.

Note that if $p : E \to B$ is a fibration with fiber F path-connected, we can make a long exact sequence

$$\cdots \to \pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 0$$

Theorem: $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$ is the pure braid group, and every other homotopy class is trivial.

Let $O_n = \mathrm{Conf}_n(\mathbb{C}_n)$ be the covering space of U_n so that it is a fiber bundle with fiber S_n. In this case, we have a s.e.s.

$$1 \to \pi_1(O_n) \to \pi_1(U_n) \to S_n \to 1$$

Proof: Consider $n = 2$; then, $\mathbb{C}^2 \setminus \Delta \cong S^1$, so $\pi_1(O_2) \cong \pi_1(S^1) \cong \mathbb{Z}$. Then, $U_2 = \{(s_1, s_2) \mid s_1^2 \neq s_2\}$. Then we see that

$$U_2 \to \mathbb{C}^2 \setminus \mathbb{C} \times \{0\} \cong S^1$$

and the s.e.s. must be

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

If you think a little, then $\pi_1(U_n)$ must be the (geometric) braid group, and $\pi_1(O_n)$ must be the (geometric) pure braid group; in fact we will take this by fiat and make this our definitions for these groups.

Furthermore, the higher homotopy groups must be 0, since we have that $O_n \to O_{n-1}$ via forgetting the last coordinate has fiber $\mathbb{C} \setminus \{z_1, \ldots, z_{n-1}\}$, and so we must have that for $k \geq 2$, we have the l.e.s.

$$\cdots \to \pi_k(F) = 0 \to \pi_k(O_n) \to \pi_k(O_{n-1}) = 0 \to \cdots$$

The last thing to do is to check that our geometric groups line up with the group representations

$$\widetilde{B}_n \cong \left\langle (\sigma_i)_{i=1}^{n-1} \mid \sigma_i\sigma_j = \sigma_i\sigma_i \text{ for } |i-j| > 2, \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1 \right\rangle$$

The entire thing is kind of long and annoying! So I will just make the remark that it's really clear if you take σ_i to be the braid that crosses the threads connecting i and $i+1$ once. In fact, it is good enough to check the pure braid group lines up with the representation (I am **not** typing this) But we can just check this for S_{ij} to be (draw a picture, and don't look at this!)

$$S_{ij} = \sigma_{j-1}^{-1}\sigma_{j-2}^{-1} \cdots \sigma_{i+1}^{-1}\sigma_{i^2}\sigma_{i+1} \cdots \sigma_{j-2}\sigma_{j-1}$$

for $i < j$.

6 Spectral Sequences

If we have some s.e.s.

$$1 \to G \to G' \to G'' \to 1$$

how do we relate $H_p(G')$ and $H_p(G'', H_q(G))$?

Let $F : A \to B$ a left exact functor between abelian categories with enough injectives; and $G : B \to C$ also left exact. For $X \in \mathrm{Ob}(A)$, we want to compute $R(G \circ F)(X)$, the right dervied functor of the compositions; in particular if we take an injective resultion $X \to I^\bullet$, then $R(G \circ F)(X)$ is quasiisomorphic to $(G \circ F)(I^\bullet)$, and $RF(X)$ is quasiisomorphic to $F(I^\bullet)$.

Lemma: If $0 \to X \to Y \to Z \to 0$ in B and if $X \to I_X^\bullet, Z \to I_Z^\bullet$ are injective resolutions, then there is an exact sequence

$$0 \to I_X^\bullet \to I_Y^\bullet \to I_Z^\bullet \to 0$$

and we may write $Y \to I_Y^\bullet$ an injective resolution.

Lemma (Cartan-Eilenberg Resolution): Let B be an abelian category with enough injectives, and $C^\bullet$ a bounded below complex. Then, there is a sequence such that

1. For all q,
$$C^q \to I^{q,0} \to I^{q,1}$$
is an injective resolution.
2. For all q,
$$Z^q(C^\bullet) \to Z^q(I^{\bullet,c}) \to Z^q(I^{\bullet,I})$$
is an injective resolution.

3. Same for $B^q = \mathrm{Im}(d^{q-1}) \subset Z^q$

4. Same for H^q

Proof: Take s.e.s.

$$0 \to B^q(C^\bullet) \to Z^q(C^\bullet) \to H^q(C^\bullet) \to 0$$

and

$$0 \to Z^q(C^\bullet) \to C^q \to B^{q+1}(C^\bullet) \to 0$$

Now let $H^{q,\bullet}$ be an injective resolution of $H^q(C^\bullet)$, and $B^{q,\bullet}$ be an injective resolution of $B^q(C^\bullet)$. From the earlier lemma, let $Z^{q,\bullet}$ be an injective resolution of $Z^q(C^\bullet)$ fitting into a diagram Again by the lemma, there is an injective resolution $C^q \to I^{q,\bullet}$ fitting into the diagram We have $C^\bullet \to I^\bullet$ by construction, and we define a differential $I^{q,\bullet} \to I^{q+1,\bullet}$ as the composition given by $\iota' \circ \iota \circ \pi'$, where $\iota : B^{q+1,\bullet} \to Z^{q+1,\bullet}$ is the inclusion.

Then, we have

$$\mathrm{Ker}(d^q) \cong Z^{q,\bullet},\ \mathrm{Im}(d^q) \cong B^{q+1,\bullet}$$

which gives use everything we want. See Stacks Tag 015G for more details.

Now, let $A \xrightarrow{F} B \xrightarrow{G}\to C$ and $X \to I^\bullet$ be as earlier (we technically need B with small coproducts), and let $F(I^\bullet) \to I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution.

Definition: We put

$$\mathrm{Tot}^n(I^{\bullet,\bullet}) = \bigoplus_{p+q=n} I^{p,q}$$

with differential $d : \mathrm{Tot}^m \to \mathrm{Tot}^{m+1}$, given by $d^n|_{I^{p,q}} = d_\uparrow + (-1)^p()d_\to$.

We have $F(I^q) \xrightarrow{\iota} I^{0,q} \subset Tot^q(I^{\bullet,\bullet})$; similarly, let $x \in F(I^q)$; then we have

$$d^q_{\mathrm{Tot}}\iota x = d_\uparrow \iota x + d^0_\to \iota x = d_\uparrow \iota x = \iota \delta^q x$$

We have a few questions:

1. Is $F(I^\bullet) \to \mathrm{Tot}^\bullet(I^{\bullet,\bullet})$ is a quasiisomorphism? If so, then $RGRFX^\bullet$ is quasiisomorphic to $\mathrm{Tot}^\bullet(G(I^{\bullet,\bullet})$.

2. How do we compute the cohomology of $\mathrm{Tot}^\bullet$?

We can do this more generally.

Definition: A filtered complex is a complex $D^\bullet$ alongside $\mathrm{Fil}^q(D^\bullet)$, such that

1. $\mathrm{Fil}^q(D^\bullet) \hookrightarrow D^\bullet$ is an inclusion of a subcomplex.

2. We have $\mathrm{Fil}^q(D^\bullet) \hookleftarrow \mathrm{Fil}^{q+1}(D^\bullet)$.

3. $$\bigcup_q \mathrm{Fil}^q(D^\bullet) = D^\bullet$$

4. $$\bigcap_q \mathrm{Fil}^q(D^\bullet) = 0^\bullet$$

In particular, notice that the complex $D^\bullet = \mathrm{Tot}^\bullet(I^{\bullet,\bullet})$ is equipped with a filtration

$$\mathrm{Fil}^q(I^n) = \bigoplus_{p'+q'=n, q'\geq q} I^{p',q'}$$

and we note that $\mathrm{Fil}^q / \mathrm{Fil}^{q+1} \cong I^{n-q,q}$. Similarly, we have the analogy that $I^{p,q}$ should be

$$\mathrm{Gr}^q(D^{p+q}) = \mathrm{Fil}^q(D^{p+q}) / \mathrm{Fil}^{q+1}(D^{p+q})$$

We want to understand the cohomology of a filtered complex $D^\bullet$ in terms of $\mathrm{Gr}^q(D^\bullet)$. Let's work in a module category for now.

Warning! Indicies are *probably* wrong. Check them!

We have then that

$$H^q(C^\bullet) = \frac{\{x \in C^q \mid dx = 0\}}{\{dy \mid y \in C^{q-1}\}}$$

as well as

$$\{x \in C^q \mid dx = 0\} = \bigcap_{r\geq 0} \{x \in \mathrm{Fil}^p(C^q) \mid dx \in \mathrm{Fil}^{p+r}(C^q)\} = \bigcap_{r\geq 0} F_r^{p,q}$$

and

$$\{dy \mid y \in C^{q-1}\} = \bigcup_{r\geq 0} \{y \in \mathrm{Fil}^{p-r}(C^q) \mid dy \in \mathrm{Fil}^p(C^q)\}$$

Now let

$$Z_r^{p,q} = \frac{F_r^{p,q}}{F_{r-1}^{P+1,q-1}}$$

and

$$B_r^{p,q} = \frac{dF_{r-1}^{p+1-r,q+r-2} + \mathrm{Fil}^{p+1} C^{p+q}}{d\,\mathrm{Fil}^{p+1}(C^{p+q})} = \frac{dF_{r-1}^{p+1-r,q+n-2}}{dF_r^{p+1-r,q+r-2}}$$

Let us finally define

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_n^{p,q}}$$

But now, we have a mapping

$$F_r^{p,q} \xrightarrow{d} F_r^{p+r,q+1-r}$$

so we have, for example,

$$E_0^{p,q} = \mathrm{Gr}^p(C^{p+q}) = I^{p,q}$$
$$E_1^{p,q} = \frac{\mathrm{Ker}(\mathrm{Gr}^p(C^{p+1}) \to \mathrm{Gr}^{p+1}(C^{p+q}))}{d\,\mathrm{Gr}^p(C^{p+q-1})} = H^{p+q}(\mathrm{Gr}^p(C^\bullet))$$

and so on.

We examine the differential more closely now; in particular we have a map $F_n^{p,q} \xrightarrow{d_n} E_r^{p+r,q+1-r}$, sending $[x] \mapsto [dx]$. Cruicially, this is well-defined (check!) and factors through $Z_n^{p,q}$; if $x \in F_{r-1}^{p+1,q-1}$ we have $x \in F_{r-1}^{p+r+1-r,q+1-r+r-2}$, hence $[dx] \in B_r^{p+r,q+1-r}$ which is killed in the quotient. And so it must factor through $E_r^{p,q}$ as well, since $d^2 = 0$.

Definition: We call $E_r^{p,q}$ the r^{th} page of the spectral sequence associated to the origiinal filtered complex.

Theorem: We have a natural isomorphism

$$E_{r+1}^{p,q} \cong \frac{\mathrm{Ker}(d_2 : E_r^{p,q} \to E_r^{p+r,q-r+1})}{\mathrm{Im}(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q})}$$

given just by inclusion; we may call this map ι. In other words, we go to the next page of the spectral sequence by taking (co)homology.

Proof: Let $x \in F_r^{p,q}$ with $dx = 0$ in $E_r^{p+r,q+1-r}$; thus $dx = dy + z$ for $y \in F_{r-1}^{p+1,q-1}$ and $z \in F_{r-1}^{p+r+1,q-r}$. Thus, $d(x-y) = z$, so $[x] = [x-y] \in Z_n^{p,q}$; thus $x - y \in F_{n+1}^{p,q}$, and $[x] = \iota[x-y]$. So the

So ι is surjective; to check injectivity, let $x \in F_{r+1}^{p,q}$ such that $\iota[x] = 0$, i.e. such that $[x] = [dy]$ in $E_r^{p,q}$, where $y \in F_r^{p-r,q+r-1}$. Thus, we have $x - dy = dz + t$, where $z \in F_{r-1}^{p+1-r,q+r-2}$ and $t \in F_{r-1}^{p+1,q-1}$, so $x = d(y+z) + t$. Now note that $dt = dx \in \mathrm{Fil}^{p+r+1}(C^{p+q+1})$, so it must be that $t \in F_r^{p+1,q-1}$ in fact. So we win, since $d(y+z) = x - t \implies y + z \in F_r^{p-r,q+r-1}$, and we conclude that $[x] = 0$.

Definition: We say that the spectral sequence degenerates on page r, if for all $r' \geq r$, we have that $d_{r'} = 0$.

Prop: If $E_r^{p,q}$ degenerates at r, the filtration on $H^\bullet(C)$ indueced by $H^n(\mathrm{Fil}^p)$ has p^{th} graded terms isomorphic to $E_r^{p,n-p}$, and we put $E_n^{p,q} \implies H^{p,q}(C)$.

Example: We take the following diagram, where the rows are exact and the left and right columns are exact. We may take the spectral sequence with 0^{th} page $d_\rightarrow$, and we obtain by exactness $E_1^{p,q} = 0$; so it stabilizes at $r = 1$, so $H^n(\mathrm{Tot}) = 0$. However, if we take the spectral sequence with 0^{th} page $d_\uparrow$, we have that the E_1 page looks like this but since either the target or the source of each morphism is 0, we see that the sequence degenerates. So it must be that $H^1(\mathrm{Tot}) = \mathrm{Ker}(\iota) = 0$, $H^2(\mathrm{Tot}) = \frac{\mathrm{Ker}(\pi)}{\mathrm{Im}(\iota)} = 0$, and $H^3(\mathrm{Tot}) = \mathrm{Coker}(\pi) = 0$. This proves the Nine Lemma.

6.1 Grothendieck Spectral Sequences

Theorem: Let $F : A \to B$ be left exact between abelian categories with enough injectives, and let $G : B \to C$ be left exact, such that for all injective $I \in \mathrm{Ob}(A)$, $F(I)$ is G-acyclic; that is, $R^qG(F(I)) = 0$ for $q > 0$. Then, there is a natural exact sequence such that

$$E_2^{p,q} = R^pG(R^qFX) \implies R^{p+q}(G \circ F)(X)$$

for $X \in \mathrm{Ob}(A)$. In particular, we have that $R(GF)(X)$ is quasiisomorphic to $RG \circ RF(X)$.

Proof: Let $X \to I^\bullet$ be an injective resolution, and let $F(I^\bullet) \to I^{\bullet,\bullet}$ be a Cartan-Eilenberg Resolution. Then, there is an induced homomorphism $F(I^\bullet) \to \mathrm{Tot}(I^{\bullet,\bullet})$ of complexes, and it is a quasiisomorphism.

To see this, we note that by construction,

$$E_{1,\rightarrow}^{p,q} = \begin{cases} F(I^q) & p = 0 \\ 0 & p \neq 0 \end{cases}$$

so it must be that this spectral sequence degenerates at the E_1 page, so $F(I^q) \cong H^q(\mathrm{Tot}(I^{\bullet,\bullet}))$. We apply $G(I^{\bullet,\bullet})$, and see that

$$E_{1,\rightarrow}^{p,q} = R^q G(F(I^q))$$
$$E_{1,\uparrow}^{p,q} = G(H^{p,q})$$

since the sequences

$$0 \to B^{p-1,q} \to Z^{p,q} \to H^{p,q} \to 0$$

and

$$0 \to Z^{p,q} \to I^{p,q} \to B^{p+1,q} \to 0$$

split since everything in sight is injective, so they remain exact after applying G. Then, we have that $(H^{p,q})_{p\in\mathbb{Z}}$ is an injective resolution of $H^p(F(I^\bullet)) = R^pF(X)$. Therefore, we have that $E_{2,\uparrow}^{p,q} = R^qG(R^pFX)$. Furthermore, since $\mathrm{Tot}^\bullet(G(I^{\bullet,\bullet})) = G(\mathrm{Tot}^\bullet(I^\bullet,\bullet))$, and since from before $\mathrm{Tot}^\bullet(I^{\bullet,\bullet})$ is quasiisomorphic to $F(I^\bullet)$, we must have

$$\mathrm{Tot}\,\bullet(I^{\bullet,\bullet}) \cong RG(F(I^\bullet)) \cong RG \circ RF(X)$$

Now note that by assumption $R^qG(F(I^q)) = 0$ for $q > 0$, so then

$$E_{1,\rightarrow}^{p,q} = \begin{cases} GF(I^p) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

Thus we have that this spectral sequence degenerates at E_1, and thus $GF(I^\bullet)$ is quasiisomorphic to $\mathrm{Tot}(G(I^{\bullet,\bullet}))$.

6.2 Application to Group Homology

Let G be a group, and $H^q(G,-)$ the right derived functor of the left exact functor $Mod_{R[G]} \to Mod_R$ sending $M \mapsto M^G$, and $H_q(G,-)$ the left dervied functors of $M \mapsto M_G$.

Let us consider an exact sequence

$$1 \to P \to G \to Q \to 1$$

Then for M a G-module, we have that $M^G = (M^P)^Q$, i.e. we have Then, if $(-)^P$ sends $R[G]$-injectives to $H_r^\bullet(P,-)$ acyclic then we obtain a natural spectral sequence

$$E_2^{p,q} = H^q(Q, H^p(P,M)) \implies H^{p+q}(G,M)$$

Lemma (Five-term exact sequence): Let $M^{p,q}$ be a double complex in the first quadrant, A its total complex, and $E_2^{p,q} \implies H^n(A)$ be the second page of its spectral sequence. Then there is an exact sequence

$$0 \to E_2^{1,0} \to H^1(A) \to E_2^{0,1} \to E_2^{2,0} \to H^2(A)$$

7 Representation Theory

Our interest in homological stability extends past S_n itself; in fact, we are also interested in the homological stability of *representations* of S_n. To start, we should define a group representation:

Definition: A representation of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho : G \to GL(V)$ of G to the group of automorphisms of V. If there is little ambiguity to the mapping rho, we often say that V itself is a representation of G. A subrepresentation of a representation V is a vector subspace W of V which is invariant under G. A representation V is called irreducible if there is no proper nonzero invariant subspace W of V.

For example, given a field k of characteristic 0, there are three irreducible representations of the three element permutation group S_3:

1. The trivial representation $V_0 = k, \rho : G \to GL_1(k), \rho(\sigma) = 1$.
2. The sign representation $V_1 = k, \rho : G \to GL_1(k), \rho(\sigma) = \epsilon(\sigma) \cdot 1$.
3. The standard representation $V = \{(x_1, x_2, x_3) \in k^3 | \sum_i x_i = 0\}, \rho : G \to GL_3(k), \rho(\sigma) = P_\sigma$, where P_σ permutes the three basis vectors of V according to σ.

Any other representation of S_3 is isomorphic to a product of these irreducible representations.

It will be important for us to enumerate irreducible representations of S_n so that we have representations to examine for homological stability. However, it is quite difficult to directly find irreducible representations of S_n. It turns out that there is a bijection between Young tableaux of size n and irreducible representations of S_n; we can leverage this bijection to produce S_n representations.

7.1 Young Tableaux and Irreducible Representations of S_n

Observe that the conjugacy classes of S_n are parameterized by partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n \in \mathbb{N}$, such that $n = \lambda_1 + \ldots + \lambda_k$ and $\lambda_1 \geq \ldots \lambda_k \geq 1$. This is because cycle type determines conjugacy class, and each partition of this form describes a cycle type. Indeed, the conjugation operation gag^{-1} for $g, a \in S_n$ preserves the cycle type of a. These partitions can also be viewed in a different way:

Definition: Let $\lambda = (\lambda_1, \cdots, \lambda_k)$ be a partition of $d \in \mathbb{N}$, as described above. We associate a Young diagram with λ_i boxes in the i^{th} row and rows of boxes aligned on the left. For example, the partition $\lambda = (3, 2, 2, 1)$ would produce the following Young diagram:

<table>
<tr><td> </td><td> </td><td> </td></tr>
<tr><td> </td><td> </td></tr>
<tr><td> </td><td> </td></tr>
<tr><td> </td></tr>
</table>

A Young tableau is a numbering of the boxes of a Young diagram by the integers $1 \ldots d$. Using the Young diagram above, we could produce the following Young tableau:

<table>
<tr><td>1</td><td>2</td><td>3</td></tr>
<tr><td>4</td><td>5</td></tr>
<tr><td>6</td><td>7</td></tr>
<tr><td>8</td></tr>
</table>

Clearly, there is a bijection between conjugacy classes of S_n and Young diagrams with n boxes. The key result of this section will be to extrapolate to a bijection between irreducible representations of S_n and Young diagrams with n boxes. To do so, we must define further features of Young tableaux.

Given a tableau with diagram λ, let P be the subgroup of S_n that preserves each row of the tableau, and let Q be the subgroup of S_n that preserves each column of the tableau; that is,

$$P = P_\lambda = \{g \in S_n \mid g \text{ preserves each row}\}$$

$$Q = Q_\lambda = \{g \in S_n \mid g \text{ preserves each column}\}$$

Then, P and Q have corresponding elements in the group algebra $\mathbb{C}S_n$:

$$a_\lambda = \sum_{g \in P} e_g$$

$$b_\lambda = \sum_{g \in Q} \epsilon(g) e_g$$

From a_λ and b_λ, we create the Young symmetrizer c_λ:

$$c_\lambda = a_\lambda b_\lambda$$

The claim is that the image of c_λ by right multiplication on $\mathbb{C}S_n$ is an irreducible representation V_λ of S_n, and that every irreducible representation of S_n can be derived in this way from a partition.

Lemma:

1. For all $p \in P$, $p \cdot a = a \cdot p = p$.

2. For all $q \in Q$, $(\epsilon(q)q) \cdot b = b \cdot (\epsilon(q)q) = b$.

3. For all $p \in P$, $q \in Q$, $p \cdot c \cdot (\epsilon(q)q) = (\epsilon(q)q) \cdot c \cdot p = c$, and, up to multiplication by a scalar, c_λ is the only such element in $\mathbb{C}S_n$.

Proof:

1. $p \cdot a_\lambda = \; p \ldots \sum_{\sigma \in P_\lambda} \sigma = \; \sum_{\sigma \in P_\lambda} p\sigma = \; \sum_{\sigma' \in P_\lambda} \sigma' = \; a_\lambda$

2. $\epsilon(q) \cdot q \cdot b_\lambda = \; q \cdot \sum_{\sigma \in Q_\lambda} \epsilon(\sigma)\sigma = \; \sum_{\sigma \in Q_\lambda} \epsilon(q\sigma) q\sigma = \; \sum_{\sigma' \in Q_\lambda} \epsilon(\sigma')\sigma' = \; b_\lambda$

3. Let $c_\lambda = \sum_{g \in S_n} n_g g$, where n_g are coefficients in $\mathbb{Q}$. Observe that for the lemma to hold, then $p \cdot c_\lambda \cdot \epsilon(q)q = \sum_{g \in S_n} \epsilon(q) n_g pgq = \sum_{g \in S_n} n_g g$. Thus, $n_{pgq} = \epsilon(q) n_g$; in particular, $n_{pq} = \epsilon(q) n_1 = \epsilon(q)$ for all $p \in P_\lambda$ and all $q \in Q_\lambda$. Recall that we defined c_λ to be $c_\lambda := \sum_{\substack{p \in P_\lambda \\ q \in Q_\lambda}} \epsilon(q) pq$; based on this definition, it is indeed the case that $n_{pq} = \epsilon(q)$ Therefore, it remains to be shown that $n_g = 0$ if $g \notin PQ$. For any such g, it is sufficient to find a transposition t such that $p = t \in P$ and $q = g^{-1} t g$. Indeed, if such a transposition exists, then $pgq = tgg^{-1}tg = ttg = g$. Observe also that $\epsilon(q) = \epsilon(g^{-1}tg) = \epsilon(t) = -1$. Thus, $n_{pgq} = n_g = \epsilon(q) n_g = -n_g$, which can only be the case if $n_g = 0$. Let T be a Young tableau, and let T' be the Young tableau acted on by g, such that each integer i is replaced by $g(i)$. Observe that if $g = pgq$, T' could be produced by first applying p to T to swap integers in rows, then applying g, and finally

applying q to swap integers in columns; that is, the integers permuted by p must be carried to the same column by g. Therefore, to show the existence of t, we must demonstrate that g carries the pair of integers transposed by t to the same column of T'. To do so, we will verify that if no such pair of integers exists, then $g \in PQ$. Assume that no such transposition t exists; then, the integers in the first row of T must appear in distinct columns of T'. Therefore, there exists a $p \in P$ such that the integers in the first row of pT appear in the same left-to-right order of T', and so there exists a $q \in Q$ such that the first rows of pT and qT' are the same. By induction on the rows of T, there exists a $p \in P$ and a $q \in Q$ such that $pT = qT'$. It follows that $pT = qgT$, and so $pq^{-1} = g$ so that that $g \in PQ$. Since t must exist if $g \notin PQ$, $n_g = 0$, and we may conclude that $p \cdot c_\lambda \cdot (\epsilon(q)q) = c_\lambda$.

Next, we define a lexographic order on partitions; given two partitions λ and μ, $\lambda > \mu$ if the first nonvanishing $\lambda_i - \mu_i$ is positive.

Lemma: Let λ be the partition associated to tableau T and μ be the partition associated to tableau T'. If $\lambda > \mu$ and $|T| = |T'|$, then there exist two integers on the same row of T and the same column of T'.

Proof: We will prove this using the pigeonhole principle. For any index j, if the integers in row j of T lie in different columns in T', then there must exist a row of length at least λ_j in T'. Let i be the first row index where $\lambda_i > \mu_i$. Since $\lambda_j = \mu_j$ for all $j < i$ and $\mu_j \leq \mu_i$ for all $j > i$, there doe not exist enough rows of T' of sufficient length such that the integers in each of the first i rows of T can lie in different columns of T'. We conclude that there must be at least two integers in the same row of T and the same column of T'.

We continue to uncover useful information about the Young symmetrizer using this new ordering along with the previous lemma:

Lemma:

1. Let λ be the partition associated to tableau T and μ be the partition associated to tableau T'. If $\lambda > \mu$, then for all $x \in \mathbb{C}S_n$, $a_\lambda \cdot x \cdot b_\lambda = 0$. In particular, if $\lambda > \mu$, then $c_\lambda \cdot c_\mu = 0$.

2. For all $x \in \mathbb{C}S_n$, $c_\lambda \cdot x \cdot c_\lambda$ is a scalar multiple of c_λ. In particular, $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$.

Proof:

1. We may take $x = g \in S_n$. Since $g \cdot b_\mu \cdot g^{-1}$ is the element constructed from gT', where T' is the tableau used to construct b_μ, it suffices to show that $a_\lambda \cdot b_\mu = 0$. By the previous lemma, there must be two integers in the same row of T and the same column of T'. Let t between the transposition between these two integers; then, $a_\lambda \cdot t = a_\lambda$ and $t \cdot b_\mu = -b_\mu$, since $\epsilon(t) = -1$. Thus, $a_\lambda \cdot b_\mu = a_\lambda \cdot t \cdot t \cdot b_\mu = -a_\lambda \cdot b_\mu$, and so $a_\lambda \cdot b_\mu = 0$.

2. For all $p \in P_\lambda$ and $q \in Q_\lambda$, we have that $p \cdot (c_\lambda \cdot x \cdot c_\lambda) \cdot (\epsilon(q) \cdot q) = (p \cdot c_\lambda) \cdot x \cdot (c_\lambda \cdot \epsilon(q) \cdot q) = c_\lambda \cdot x \cdot c_\lambda$. By the previous lemma, c_λ is the only element of $\mathbb{C}S_n$ with this property, up to scalar multiplication. Thus, we conclude that $c_\lambda \cdot x \cdot c_\lambda = n_\lambda c_\lambda$.

Lemma:

1. $V_\lambda = \mathbb{C}S_n$ is an irreducible representation as a left $\mathbb{C}S_n$- module.

2. If $\lambda \neq \mu$, then $V_\lambda \neq V_\mu$.

Proof:

1. First, observe that $V_\lambda \neq 0$ since $c_\lambda \in V_\lambda$. Let $W \subset V_\lambda$ be a subrepresentation; then, $c_\lambda W \subset c_\lambda V_\lambda = c_\lambda \mathbb{C}S_n c_\lambda = \mathbb{C}c_\lambda$, by the previous lemma. Observe that $\mathbb{C}c_\lambda$ is a 1-dimensional vector space. It follows that $c_\lambda W$ is either $\mathbb{C}c_\lambda$ or 0. If $c_\lambda W = 0$, then, since W is a subset of V_λ, $W \cdot W \subset \mathbb{C}S_n c_\lambda \cdot W = 0$. It follows that $W = 0$. Indeed, let $p : V_\lambda \to W$ be a projection given by right multiplication by some $\phi \in CS_n$. Since projections are idempotent, $\phi = \phi^2$, and so it follows that $\phi = \phi^2 \in W \cdot W = 0$. But if $\phi = 0$, then the projection must be the trivial map, and we conclude that $W = 0$. Otherwise, if $c_\lambda W = \mathbb{C}c_\lambda$, then $c_\lambda \in c_\lambda W \subset W$, since c_λ is a linear combination of group elements and each group element acts on the subrepresentation W as an automorphism of W. Thus, $V_\lambda = \mathbb{C}S_n c_\lambda \subset W$, since $c_\lambda \subset W$ and again, W is a subrepresentation. We conclude that $V = W$.

2. Without loss of generality, we assume that $\lambda > \mu$. Observe that $c_\lambda V_\lambda = \mathbb{C}c_\lambda$; however, if $\lambda > \mu$, then $c_\lambda V_\mu = a_\lambda b_\lambda \mathbb{C}S_n a_\mu b_\mu = 0$, by the previous lemma. Thus, $V_\lambda \neq V_\mu$.

We can use the above lemmas in practice to produce an irreducible representation of S_n from a Young diagram. For example, consider Young tableaux of the following form:

1	2	3	$\dots$	$n-1$
n				

We begin by calculating the Young symmetrizer c_λ:

$$P_\lambda = \{\sigma \in S_n | \sigma(n) = n\}$$
$$Q_\lambda = \{1, (1n)\}$$
$$a_\lambda = \sum_{\sigma:\sigma(n)=n} \sigma$$
$$b_\lambda = 1 - (1n)$$
$$c_\lambda = a_\lambda b_\lambda = \sum_{\sigma(n)=n} \sigma - \sum_{\sigma(n)=n} \sigma \cdot (1n) = \sum_{\sigma(n)=n} \sigma - \sum_{\sigma(1)=n} \sigma$$

Recall that $V_\lambda = \mathbb{C}S_n c_\lambda$, and that $\mathbb{C}S_n$ is generated by $\tau \in S_n$. Observe that $\tau c_\lambda = \sum_{\sigma(n)=\tau(n)} \sigma - \sum_{\sigma(1)=\tau(n)} \sigma$. Define γ_i to be $\sum_{\sigma(n)=i} \sigma - \sum_{\sigma(1)=i} \sigma$; then, $\tau c_\lambda = \gamma_{\tau(n)}$. Then, $\gamma_1, \dots, \gamma_n$ generate V_λ as a $\mathbb{C}$ vector space.

Consider when $V_\lambda = 0$; that is, when $\sum_{i=1}^n \alpha_i \gamma_i = 0$, where $\alpha_i \in \mathbb{C}$. Observe that $\sum_{i=1}^n \alpha_i \gamma_i = \sum_\sigma \alpha_{\sigma(n)} - \alpha_{\sigma(1)}$. Then, $V_\lambda = 0$ if and only if $\forall \sigma, \alpha_{\sigma(n)} - \alpha_{\sigma(1)} = 0$. This is the case if and only if $\forall i \neq j, \alpha(i) = \alpha(j)$, since for all pairs i, j there exists a $\sigma \in S_n$ such that $(i, j) = \sigma(1, n)$. This in turn is the case if and only if $\exists \alpha$ such that $\forall i, \alpha_i = \alpha$.

Let $X = \{(x_1, \dots, x_n) \in \mathbb{C}^n | \sum_i x_i = 0\}$, and consider the map $\rho : X \to V_\lambda$ such that $(0, \dots, 0, x_i, 0, \dots, 0) \mapsto \gamma_i$. Then, (???). Now, observe that for $\tau \in S_n$, $\tau\gamma_i = \gamma_{\tau(i)}$; thus, S_n acts on V_λ by permuting coordinates, and we can conclude that V_λ is the standard representation of S_n.

7.2 Calculating the Dimension of V_λ

Definition: Let V be a representation of G. The character $\chi_V : \sigma \to \mathbb{C}$ is the complex-valued function on the group defined by

$$\chi_V(\sigma) = \mathrm{Tr}(\rho(\sigma))$$

where Tr is the trace function on the matrix representation of $\rho(\sigma)$).

Observe that the matrix representation for the identity element 1 of a group is the identity matrix; thus, $\mathrm{Tr}(\rho(\sigma)) = \dim(V_\lambda) \cdot 1 = dim(V_\lambda)$. So, $\chi_{V_\lambda}(1)$ will produce the dimension of V_λ.

We must define several invariants to write out an explicit formula for the character of an irreducible representation. For the remainder of this section, we will work with irreducible representation V_lambda which has an associated Young diagram λ with row lengths $\lambda_1, \dots, \lambda_k$.

First, we define a list of integers $i_1, \dots, i_n$, where i_j represents the number of permutations in σ of length j. For example, for the element $\tau = (123)(45)(57)(8) \in S_8$, $i_1 = 1, i_2 = 2$, and $i_3 = 3$. Observe that if $\sigma \in S_n$, then $i_j = 0$ for all $j > n$, since S_n can have a cycle of length at most n.

Next, we define a list of integers $l_1, \dots, l_k$, where $l_i = \lambda_i + k - i$. l_i can be interpreted as the hook length of the cell in row i and column 1 of the Young diagram:

Definition: Given a cell in position (i, j) of a Young diagram, let the hook of that cell be the set of cells (a, b) such that $a \geq i$ and $b = j$ or $a = i$ and $b \geq j$. Then the hook length of (i, j) is the number of cells in the hook of (i, j).

Finally, we define two polynomials:

$$P_j(x_1, \dots, x_k) = x_1^j + \dots + x_k^j$$
$$\Delta(x_1, \dots, x_k) = \prod_{i<j}(x_i - x_j)$$

Observe that these two polynomials do not depend on σ; they only depend on the number of rows in λ.

We use these invariants to define a polynomial:

$$f_\sigma(\bar{x}) = f_\sigma(x_1, \dots, x_k) = \Delta(\bar{x}) \prod_{j=1}^{n} P_j(\bar{x})^{i_j}$$

$\chi_V(\sigma)$ is the coefficient of $x_1^{l_1} \dots x_k^{l_k}$ in $f_\sigma(\bar{x})$.

Add a reference to Fulton Harris here.

We apply this formula to find $\chi_{V_\lambda}(1)$. Since we are working with $\sigma = 1$, we have that $l_1 = n$ and $l_j = 0$ for all $j > 1$.Thus, $f_1(\bar{x})$ decomposes to:

$$f_1(\bar{x}) = \Delta(\bar{x}) P_1(\bar{x})^n$$

Next, we apply the multinomial theorem to further simplify $P_1(\bar{x})^n$:

$$f_1(\bar{x}) = \Delta(\bar{x}) \sum_{r_1 + \dots + r_k = n} \binom{n}{r_1, \dots, r_n} x_1^{r_1} \dots x_k^{r_k}$$

Observe that $\Delta(\bar{x})$ can be expressed as a Vandermonde determinant; that is:

$$\Delta(\bar{x}) = \det \begin{pmatrix} 1 & x_k & \dots & x_k^{k-1} \\ \vdots & & & \\ 1 & x_1 & \dots & x_1^{k-1} \end{pmatrix} = \sum_{\sigma \in S_k} \epsilon(\sigma) x_k^{\sigma(1)-1} x_{k-1}^{\sigma(2)-1} \dots x_1^{\sigma(k)-1}$$

Using this, we can rewrite $f_1\bar{(}x)$ as:

$$f_1(\bar{x}) = \sum_{\sigma \in S_k} \epsilon(\sigma) x_k^{\sigma(1)-1} x_{k-1}^{\sigma(2)-1} \dots x_1^{\sigma(k)-1} \sum_{r_1+\ldots+r_k=n} \binom{n}{r_1, \ldots, r_n} x_1^{r_1} \dots x_k^{r_k}$$

Since we are only interested in the coefficient for $x_1^{l_1} \dots x_k^{l_k}$ in $f_1(\bar{x})$, we need only consider the value of

$$\chi_{V_\lambda}(1) = \sum_{\sigma} \epsilon(\sigma) \frac{n!}{(l_1 - \sigma(k) + 1)! \dots (l_k - \sigma(1) + 1)!}$$

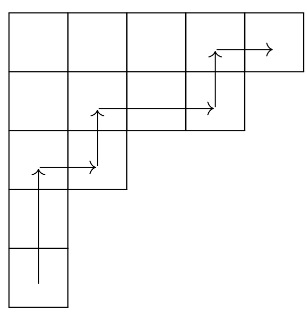
Through algebraic manipulations which can be found in Fulton and Harris (should I show these explicitly?), this constant can be written as:

$$\chi_{V_\lambda}(1) = \frac{n!}{l_1! \dots l_k!} \prod_{i<j} (l_i - l_j)$$

We claim that this is equal to:

$$\chi_{V_\lambda}(1) = \frac{n!}{\prod(\text{hook lengths})}$$

where the denominator represents the product of the hook lengths of each cell in the Young tableau. To see why this is the case, consider the following path traversal across the Young diagram λ:



For each cell (i, j) in the path, let $h'(i, j)$ be the number of cells (a, b) such that either $b = j$ and $1 \leq a \leq i$ or $a = 1$ and $b > j$. Observe that $h'(i, j)$ can be written in terms of the hook length from cell $(1, 1)$, i.e. l_1. Below, $h'(i, j)$ is written in terms of i_1 for every cell in the path:

			l_1-7	l_1-8
	l_1-4	l_1-5	l_1-6	
l_1-2	l_1-3			
l_1-1				
l_1				

Observe that the hook length for every cell in the first row of the Young diagram is represented by $h'(i,j)$ for some cell (i,j) in the path. The cells along the path which represent hook lengths are highlighted in blue:

			l_1-7	l_1-8
	l_1-4	l_1-5	l_1-6	
l_1-2	l_1-3			
l_1-1				
l_1				

The key observation here is that $h'(i,j)$ for the cells (i,j) which do not represent hook lengths can be written as a difference between l_1 and l_j for each $j > 1$:

			l_1-l_2	l_1-8
	l_1-l_3	l_1-5	l_1-6	
l_1-l_4	l_1-3			
l_1-l_5				
l_1				

We could thus represent the product of the hook lengths of the first row of the Young diagram as the product of $h'(i,j)$ for the cells (i,j) divided by the product of those $h'(i,j)$ which do not represent a hook length. But this is precisely

$$\frac{l_1!}{\prod_{j>1}(l_1-l_j)}$$

This argument could be applied to any row r of the diagram by taking the subdiagram λ' of rows with index r or greater. Thus, by induction on the row of the Young diagram, we conclude that the product of hook lengths in the diagram is:

$$\frac{l_1!\dots l_k!}{\prod_{i<j}(l_i-l_j)}$$

so that

$$\dim(V_\lambda) = \chi_{V_\lambda}(1) = \frac{n!}{\prod(\text{hook lengths})}$$

8 Homological Stability of Symmetry Groups

Given a fix q, we are interested in how $H_q(S_n)$ behaves as n increases. To start off, we will calculate $H_q(S_n)$ for some small n's.

- $n = 1$: S_n is the trivial group, and we have

$$H_q(S_1) \begin{cases} \mathbb{Z}, & \text{if } q = 0 \\ , & \text{otherwise.} \end{cases}$$

- $n = 2$: $S_2 = C_2$ is cyclic. By Example 4.5, we obtain

$$H_q(S_2) \begin{cases} \mathbb{Z}, & \text{if } q = 0 \\ \mathbb{Z}/2, & \text{if } q \text{ is odd} \\ 0, & \text{if } q \text{ is even.} \end{cases}$$

- $n = 3$: Given that $S_3/A_3 \cong \mathbb{Z}/2$ and $A_3 \cong \mathbb{Z}_3$, there is the short exact sequence

$$1 \to \mathbb{Z}/3 \to S_3 \to \mathbb{Z}/2 \to 1.$$

By the Lydon-Serre spectral sequence, we get

$$E_2^{s,t} = H_s(\mathbb{Z}_2, H_t(\mathbb{Z}/3, \mathbb{Z})) \implies H_{s+t}(S_3, \mathbb{Z}).$$

Recall from Example 4.5 that

$$H_n(G, A) = \begin{cases} A/(\sigma - 1)A, & n = 0 \\ A^G/NA, & n \text{ is odd} \\ \{a : Na = 0\}/(\sigma - 1)A, & n \text{ is even} \end{cases}$$

Setting $G = \mathbb{Z}/3$ and $A = \mathbb{Z}$, we get

$$H_t(\mathbb{Z}/3, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & t = 0 \\ \mathbb{Z}/3, & t \text{ is odd} \\ 0, & t \text{ is even} \end{cases}$$

For simplicity of notation, denote $H_t := H_t(\mathbb{Z}/3, \mathbb{Z})$. Observe that $Z_2 \curvearrowright H_t$ via $\mathbb{Z}/2 \curvearrowright Z/3$, where $\sigma \cdot \tau = \tau^{-1}$. We claim that if $t = 2r - 1$, then $\sigma \curvearrowright H_t$ via $(-1)^r$. (The proof of this claim is a little involved, so we will skip it. For interested readers, see xxx) We now calculate $H_s(\mathbb{Z}/2, H_t)$ through casework on t.

(i) $t \equiv 3 \pmod 4$: r is even, and $\mathbb{Z}/2 \curvearrowright H_t$ trivially. Then

$$H_s(\mathbb{Z}/2, H_t) = \begin{cases} H_t, & s = 0 \\ H_t/(1+\sigma)H_t = H_t/2H_t = 0, & s \text{ is odd} \\ 0, & s \text{ is even} \end{cases}$$

(ii) $t \equiv 1 \pmod 4$: r is odd, and $\mathbb{Z}/2 \curvearrowright H_t$ via -1. Then

$$H_s(\mathbb{Z}/2, H_t) = \begin{cases} 0, & s = 0 \\ 0, & s \text{ is odd} \\ 0, & s \text{ is even} \end{cases}$$

(iii) t is even: $H_t = 0$, so $H_s(\mathbb{Z}/2, H_t) = 0$ for all s.

The E_2 page of the Lydon-Serre spectral sequence looks like

$s=4$	0	0	0	0	0	0	0	0
$s=3$	$\mathbb{Z}/2$	0	0	0	0	0	0	0
$s=2$	0	0	0	0	0	0	0	0
$s=1$	$\mathbb{Z}/2$	0	0	0	0	0	0	0
$s=0$	$\mathbb{Z}$	0	0	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$
	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$

Stabilizing at the E_2 page, the spectral sequence converges to $H_{s+t}(S_3)$. Thus

$$H_q(S_3) \begin{cases} \mathbb{Z}, & \text{if } q = 0 \\ 0, & \text{if } q \text{ is even} \\ \mathbb{Z}/2, & \text{if } q \equiv 1 \pmod 4 \\ \mathbb{Z}/6, & \text{if } q \equiv 3 \pmod 4 \end{cases}$$

The examples above and further calculations of small cases yield the following table of homology: Then we can observe a few patterns, which make up Nakaoka's Theorem.

Theorem 8.1 (Nakaoka)**.**

1. *$\forall n \geq 2q$, $H_q S_n \to H_q S_{n+1}$ is an isomorphism*
2. *$\forall n$, $H_q S_{n-1} \to H_q S_n$ is an injection*
3. *$\forall n$, $H_q S_n / H_q S_{n-1}$ is killed by n.*

The last one is actually easy to prove; just use restriction/corestriction.

For the first two statements, recall the earlier discussion of contravariant functors $FI \to Sets$. Let

$$W^{(n)} = C_\bullet(\mathrm{Hom}_{FI}(-, \{1, \cdots, n\})) \in Ch(Mod_{\mathbb{Z}[S_n]}).$$

We will briefly sketch out an outline of the proof before delving in.

1. "Replace" $H_q(S_n, \mathbb{Z})$ with $H_q(S_n, W^{(n)})$ for small q.
2. For this to make sense, we want $W_*^{(n)} \to \mathbb{Z}$ to be acyclic in low degrees.
3. Instead, we study the homology of $G_*^{(n)} = W_*^{(n)}/W_*^{(n-1)}$.
4. Relate $H_*(G^{(n)})$ and $H_*(W^{(n)})$.
5. Use spectral sequence to compute $H_*(S_n, \mathbb{Z})$ from $H_*(S_n, W^{(n)})$.
6. Use Shapiro's lemma and induction.

8.1 Proof of Nakaoka's Theorem

Let $[q] = \{0, 1, ..., q\}$. Define a chain complex

$$W_*^{(n)} = \cdots \xrightarrow{d_{q+1}} W_q^{(n)} \xrightarrow{d_q} ... \xrightarrow{d_2} W_1^{(n)} \xrightarrow{d_1} W_0^{(n)},$$

where $W_q^{(n)} = \mathbb{Z}[\mathrm{Hom}_{FI}([q], \{1, 2, \cdots, n\})]$, and $d_q[\sigma] = \sum_{j=0}^{q}(-1)^j[\sigma \circ e_j]$. Here $e_j : [q] \to q+1$ is the function that skips the j-th index. Note that $W_q^{(n)}$ is a free S_n-module via $\tau \cdot [\sigma] = [\tau \circ \sigma]$, $\tau \in S_n$.

Proposition 8.2. *$W_*^{(n)} \to \mathbb{Z}$ is exact up to $W_{n-1}^{(n)}$.*

Proof. Let $G_q^{(n)} = W_q^{(n)}/W_q^{(n-1)}$, and define a chain complex

$$G_*^{(n)} = ... \to G_q^{(n)} \to \cdots \to G_1^{(n)} \to G_0^{(n)},$$

where the differentials descend from the differentials in $W_*^{(n)}$. Note that

$$\begin{aligned} G_q^{(n)} &= \mathbb{Z}[\{\sigma : [q] \to \{1, 2, \cdots n\} | \exists 0 \leq i \leq q-1 \text{ such that } \sigma(i) = n\}] \\ &= \bigoplus_{j=0}^{q} D_j(W_{q-1}^{(n-1)}), \end{aligned}$$

where D_j is the function that inserts n in the j-th position,

$$D_j[\sigma] = \left[i \mapsto \begin{cases} \sigma(i) & i < j \\ n & i = j \\ \sigma(i-1) & i > j \end{cases} \right].$$

Thus, $G_q^{(n)}$ has a natural filtration

$$\operatorname{Fil}_k G_q^{(n)} = \bigoplus_{j=0}^{k} D_j(W_{q-1}^{(n-1)}) = \mathbb{Z}[\{\sigma : [q] \to \{1, 2, \cdots n\} | \exists j \le k \text{ such that } \sigma(j) = n\}],$$

with graded parts

$$\begin{aligned}
\operatorname{Gr}_p(G_q^{(n)}) &= \operatorname{Fil}_p(G_q^{(n)}) / \operatorname{Fil}_{p-1}(G_q^{(n)}) \\
&= D_p(W_{q-1}^{(n-1)}) \\
&= \mathbb{Z}[\{\sigma : [q] \to \{1, 2, \cdots n\} | \sigma(p) = n, \sigma(i) \ne n \text{ for } 0 \le i \le p-1\}].
\end{aligned}$$

Are $G_*^{(n)}$ and $\operatorname{Gr}_p(G_*^{(n)})$ subcomplexes of $W_*^{(n)}$? We compute the differential

$$\begin{aligned}
dD_p[\sigma] &= \sum_{i=0}^{q} (-1)^i (D_p[\sigma]) e_i \\
&= \sum_{i=0}^{p-1} (-1)^i (D_{p-1}[\sigma \circ e_i]) + \sum_{i=p+1}^{q} (-1)^i (D_p[\sigma \circ e_{i-1}]) \\
&= \sum_{0 \le i < p} (-1)^i D_{p-1}([\sigma \circ e_i]) - \sum_{p \le i \le q-1} (-1)^i D_p([\sigma \circ e_i])
\end{aligned}$$

Note that since σ misses n, the $i = p$ term vanishes. (??) So unfortunately the D_j symbols don't form a subcomplex. But we do have that $dD_0[f] = -D_0 d[f]$! And everything gets messy afterwards pretty quickly. Further, we do have that $G_*^{(n)}$ and $\operatorname{Gr}_p(G_*^{(n)})$ are subcomplexes of $W_*^{(n)}$.

What is the relationship between $\operatorname{Gr}_p(G_*^{(n)})$ and $W^{(n)}$? Fix a function $\tau : [p] \to \{1, \cdots, n\}$ such that $\tau(p) = n$, and $\tau(i) \ne n$ for $0 \le i \le p-1$. We collect all the $\sigma \in \operatorname{Gr}_p(G_*^{(n)})$ such that $\sigma|_{[p]} = \tau$. Thus,

$$\begin{aligned}
\operatorname{Gr}_p G_q^{(n)} &= \bigoplus_{\substack{\tau:[p]\to\{1,\cdots,n\}, \\ \tau(p)=n, \\ \tau(i)\ne n \text{ for } 0\le i\le p-1}} \mathbb{Z}[\{\sigma : [q] \to \{1, \cdots n\} | \sigma|_{[p]} = \tau\}], \\
&\cong \bigoplus_{\tau} W_{q-p}^{(n-p-1)}
\end{aligned}$$

where the isomorphism is between complexes. Consequently, $\operatorname{Gr}_p(G_*^{(n)}) \cong \bigoplus_\tau W_*^{(n-k-1)}[-p]$, where the latter is the shifted complex, with $W_q^{(n-k-1)}[-p] = W_{q-p}^{(n-p-1)}$.

Now, we replace $W^{(n)}$ with the augmented complex

$$\widetilde{W_*^{(n)}} = \ldots \to W_q^{(n)} \to \cdots \to W_1^{(n)} \to W_0^{(n)} \xrightarrow{\epsilon} \mathbb{Z},$$

where is the augmentation map. In other words, we are shifting everything to the right by degree one. To show that $W_*^{(n)}$ is acyclic up to $n-1$ is equivalent to showing that $H_p W_q^{(n)} = 0$ for $q < n$. Similarly, replace $G^{(n)}$ by

$$\widetilde{G_*^{(n)}} = \ldots \to G_q^{(n)} \to \cdots \to G_1^{(n)} \to G_0^{(n)} \to 0,$$

and replace $\mathrm{Gr}_p(G_*^{(n)})$ by $\mathrm{Gr}_p(\widetilde{G_*^{(n)}})$.

Define the spectral sequence with $E_0^{p,q} = \mathrm{Gr}_p(\widetilde{G_{p+q}^{(n)}})$, as shown below:

$$\begin{array}{ccc}
\mathrm{Gr}_2(\widetilde{G_2^{(n)}}) & \mathrm{Gr}_1(\widetilde{G_1^{(n)}}) & \mathrm{Gr}_0(\widetilde{G_0^{(n)}}) \\
\uparrow & \uparrow & \uparrow \\
\vdots & \mathrm{Gr}_1(\widetilde{G_2^{(n)}}) & \mathrm{Gr}_0(\widetilde{G_1^{(n)}}) \\
 & \uparrow & \uparrow \\
 & \vdots & \mathrm{Gr}_0(\widetilde{G_2^{(n)}})
\end{array}$$

Taking the vertical differentials, we get $E_{1,\uparrow}^{p,q} = H_{p+q}(\mathrm{Gr}_p\, \widetilde{G_*^{(n)}})$:

$$\begin{array}{lll}
H_2(\mathrm{Gr}_2(\widetilde{G_*^{(n)}})) \longrightarrow & H_1(\mathrm{Gr}_1(\widetilde{G_*^{(n)}})) \longrightarrow & H_0(\mathrm{Gr}_0(\widetilde{G_*^{(n)}})) \\
\cdots \qquad\qquad \longrightarrow & H_2(\mathrm{Gr}_1(\widetilde{G_*^{(n)}})) \longrightarrow & H_1(\mathrm{Gr}_0(\widetilde{G_*^{(n)}})) \\
 & \cdots \qquad\qquad \longrightarrow & H_2(\mathrm{Gr}_0(\widetilde{G_*^{(n)}}))
\end{array}$$

Noting that direct sum commutes with homology, we have

$$H_{p+q}(\mathrm{Gr}_p\, \widetilde{G_*^{(n)}}) = \bigoplus_\tau H_{p+q}(\widetilde{W_*^{(n-p-1)}}[-p]) = \bigoplus_\tau H_q(\widetilde{W_*^{(n-p-1)}}).$$

Therefore, to show $H_i(\widetilde{G_*^{(n)}}) = 0$ for $i < n$, it suffices to show that $H_q(\widetilde{W_*^{(n-p-1)}}) = 0$ for $p+q < n$. We will use induction on n to complete the proof.

Suppose $H_q(\widetilde{W^{(m)}}) = 0$ for $q < m$ and $m < n$. Then $H_i(\widetilde{G_*^{(n)}}) = 0$ for $i < n$. By the definition of $G^{(n)}$, there are short exact sequences

$$0 \to W_*^{(n-1)} \to W_*^{(n)} \to G_*^{(n)} \to 0,$$

and similarly,

$$0 \to \widetilde{W_*^{(n-1)}} \xrightarrow{\iota} \widetilde{W_*^{(n)}} \xrightarrow{\pi} \widetilde{G_*^{(n)}} \to 0,$$

which induces a long exact sequence

$$\begin{array}{l}
\cdots \longrightarrow H_{n-1}(\widetilde{W^{(n-1)}}) \longrightarrow H_{n-1}(\widetilde{W^{(n)}}) \longrightarrow H_{n-1}(\widetilde{G^{(n)}}) \\
\longrightarrow H_{n-2}(\widetilde{W^{(n-1)}}) \longrightarrow H_{n-2}(\widetilde{W^{(n)}}) \longrightarrow H_{n-2}(\widetilde{G^{(n)}}) \\
\longrightarrow \cdots \longrightarrow H_0(\widetilde{W^{(n)}}) \longrightarrow H_0(\widetilde{G^{(n)}}) \longrightarrow 0
\end{array}$$

We already know that $H_i(\widetilde{G_*^{(n)}}) = 0$ for $i < n$. By the following lemma, we get the desired results. $\square$

Lemma 8.3. *The homomorphism* $H_q(\widetilde{W^{(n-1)}}) \xrightarrow{H_q(\iota)} H_q(\widetilde{W^{(n)}})$ *is 0.*

Proof. Take $[\sigma] \in W_{q-1}^{(n-1)}$, and compute

$$\begin{aligned} dD_q[\sigma] &= \sum_{j=0}^{q} (-1)^j (D_q[\sigma] \circ e_j) \\ &= (-1)^q \iota(\sigma) + \sum_{j=0}^{q-1} (-1)^j (D_q[\sigma] \circ e_j) \\ &= (-1)^q \iota(\sigma) + D_{q-1}(d\sigma) \end{aligned}$$

Thus, $H_q(\iota) = 0$ for all q. $\square$

Proposition 8.4. *There is a spectral sequence with* $E_1^{p,q} = H_p(S_n, W_q^{(n)})$ *converging to* $H_{p+q}(S_n, \mathbb{Z})$ *for* $p + q < n - 1$.

Proof. Let $P. \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$, and consider the spectral sequence with $E_0^{p,q} = P_p \otimes_{\mathbb{Z}[S_n]} W_q^{(n)}$. Taking the vertical differential, we get $E_{1,\uparrow}^{p,q} = \operatorname{Tor}_p^{S_n}(\mathbb{Z}, W_q^{(n)} = H_p(S_n, W_q^{(n)})$, as desired. We can also take the horizontal differentials, giving us $E_{1,\to}^{p,q} = P_p \otimes_{\mathbb{Z}[S_n]} H_q(W_*^{(n)})$. (Note that P_p is a projective module, and therefore it is flat, so $P_p \otimes H_q(W_*^{(n)}) = H_q(P_p \otimes W_*^{(n)})$). The E_2 page is

$$E_{2,\to}^{p,q} = \operatorname{Tor}_p^{S_n}(\mathbb{Z}, H_q(W^{(n)})) = H_p(S_n, H_q(W_*^{(n)})) = \begin{cases} H_p(S_n, \mathbb{Z}) & q = 0 \\ 0 & 0 < q < n \\ * & q \geq n \end{cases}$$

Thus, $E_{1,\uparrow}^{p,q} = \operatorname{Tor}_p^{S_n}(\mathbb{Z}, W_q^{(n)}) = H_p(S_n, W_q^{(n)}) \implies H_{p+q}(S_n, \mathbb{Z})$ for $p + q < n - 1$. $\square$

Next, we compute $H_p(S_n, W_q^{(n)})$ for $p + q < n - 1$. Recall that $W_q^{(n)} = \mathbb{Z}[\operatorname{Hom}_{FI}([q], \{1, ..., n\})]$, and fix $[\sigma] \in W_q^{(n)}$. If $q \leq n - 1$, S_n acts on $\operatorname{Hom}_{FI}([q], \{1, ..., n\})$ transitively. The stabilizer of σ is

$$\operatorname{Stab}(\sigma) \cong \operatorname{Bij}(\{1, ..., n\} \backslash \operatorname{Im} \sigma) \cong S_n / S_{n-q-1}$$

It follows that

$$W_q^{(n)} \cong \mathbb{Z}[S_n / S_{n-q-1}] \cong \operatorname{Ind}_{S_{n-q-1}}^{S_n} \mathbb{Z},$$

and by Shapiro's lemma,

$$H_p(S_n, W_q^{(n)}) \cong H_p(S_{n-q-1}, \mathbb{Z}). \tag{1}$$

Blabla

Thus, $d_{1,\uparrow}^{p,q}$ corresponds to

$$\sum_{j=0}^{q}(-1)^j H_*(\iota_0) = \begin{cases} 0 & \text{q is odd} \\ H_*(\iota_0) & \text{q is even} \end{cases} \tag{2}$$

We will use induction on n to conclude the proof of Nakaoka's stability. We assume that $H_p(S_{m-1}) \to H_p(S_m)$ is an isomorphism for $m \geq 2p$ and $m < n$. We will show that $H_p(S_{n-1}) \to H_p(S_n)$ for $n \geq 2p$. To do so, we induct on p. Assume that $H_p(S_{n-1}) \to H_p(S_n)$ is an isomorphism for $n > n_0(p)$ $(0 \leq p < p_0)$ and that $n_0(p) = 2p$ for $0 \leq p < p_0$.

Now we compute the E_2 page of the spectral sequence in Proposition 8.4. With the identifications made in (1) and (2), the E_1 page of the spectral sequence looks like

$$\begin{array}{ccccccccccc}
 & q=n-1 & & q=n-2 & & \cdots & & q=1 & & q=0 \\
p=0 & H_0(S_0) & \longrightarrow & H_0(S_1) & \longrightarrow & \cdots & \longrightarrow & H_0(S_{n-2}) & \longrightarrow & H_0(S_{n-1}) \\
p=1 & H_1(S_0) & \longrightarrow & H_1(S_1) & \longrightarrow & \cdots & \longrightarrow & H_1(S_{n-2}) & \longrightarrow & H_0(S_{n-1}) \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

When $q = 0$, we have the chain in the E_1 page:

$$\cdots \to H_p(S_{n-2}, \mathbb{Z}) \xrightarrow{0} H_p(S_{n-1}, \mathbb{Z}) \to 0.$$

Taking the homology, we get $E_{2,\to}^{p,0} = H_p(S_{n-1}, \mathbb{Z})$.

Otherwise, when q is even, we have the chain

$$\cdots \xrightarrow{0} H_p(S_{n-q-1}, \mathbb{Z}) \xrightarrow{H_*(\iota_0)} H_p(S_{n-q}, \mathbb{Z}).$$

Using the induction hypothesis, we have $E_{2,\to}^{p,q} = 0$, if $n - q \geq 2p$, or equivalently, $2p + q \leq n$.

When q is odd, the chain becomes

$$\cdots \xrightarrow{H_*(\iota_0)} H_p(S_{n-q-1}, \mathbb{Z}) \xrightarrow{0} H_p(S_{n-q}, \mathbb{Z}).$$

We then get $E_{2,\to}^{p,q} = 0$, if $n - q - 1 \geq 2p$, or equivalently, $2p + q \leq n - 1$.

Without loss of generality, assume that n is odd (if n is even, the argument runs analogously.) The E_2 page of the spectral sequence looks as follows

	$q=n$	$q=n-1$	$q=n-2$	$q=n-3$	$\cdots$	$q=2$	$q=1$	$q=0$
$p=0$	$*$	0	0	0	$\cdots$	0	0	$H_0(S_{n-1})$
$p=1$	$*$	$*$	$*$	0	$\cdots$	0	0	$H_1(S_{n-1})$
$\vdots$					$\cdots$			
$p=p_0-1$	$*$	$*$	$*$	$*$	$\cdots$	0	0	$H_{p_0-1}(S_{n-1})$
$p=p_0$	$*$	$*$	$*$	$*$	$\cdots$	0	0	$H_{p_0}(S_{n-1})$

Recall that the spectral sequence converges to $H_*(S_n, \mathbb{Z})$, so $H_{p_0}(S_{n-1}, \mathbb{Z})$ must be isomorphic to $H_{p_0}(S_n, \mathbb{Z})$. Therefore, $n_0(p_0) = 2p_0$.

9 FI-Modules

Again, let FI be the category of finite sets with injective maps. First, we fix a ring k, and let a FI-module (over k) be a functor $FI \to Mod_k$. For example, we may take

$$F^{(d)} : S \mapsto k[\mathrm{Hom}_{FI}(d, S)]$$

In particular, we have for V any FI-module, we have that

$$\mathrm{Hom}_{Mod_{FI}}(F^{(d)}, V) = \mathrm{Hom}(\mathrm{Hom}_{FI}(d, -), V) = V_d$$

In particular, any FI-module is a quotient of a direct sum of copies of $F^{(d)}$: if $M_d \to V_d$ is surjective with M_d free, then

$$\bigoplus_{d \geq 0} M_d \otimes_k F^{(d)} \to V$$

is surjective.

Definition: A FI-module V is a finitely generated (in degrees $\leq D$) if for all $d \leq D$, V_d is a finitely generated k-module, and if

$$\bigoplus_{d \leq D} V_d \otimes_k F^{(d)} \to V$$

is surjective.

Theorem: If k is Noetherian commutative, then the category of finitely generated FI-modules over k is Noetherian.

For $\lambda = (\lambda_j)_j$ with $\lambda_j \geq \lambda_{j+1}$, we may define

$$V_n(\lambda) = \begin{cases} V_{(n - \sum \lambda_j, \lambda_1, \lambda_2, \dots)} & n \geq \lambda_1 + \sum_{j \geq 1} \lambda_j \\ 0 & \text{otherwise} \end{cases}$$

For example, we see that $V_n(0)$ is the trivial representation of k, $V_n(1)$ is the standard representation, and so on. (In this case we need k a field.)

Prop: There is a f.g. FI-module $V_\bullet(\lambda)$ such that in degree d it is simply $V_d(\lambda)$.

Proof: Take

$$V(\lambda)_d = \mathrm{colim}_T\, k[S_d] c_T$$

where the colimit is in the category with objects Young tabelaus and the indexing set is tabelaus with shape $(d - |\lambda|, \lambda)$.

Theorem: Let k be a field of characteristic 0. Then, a FI-module W is finitely generated if and only if

1. For all $d \to d+1$, the induced map $W_d \to W_{d+1} = \mathrm{Res}^{G_{d+1}}_{G_d} W_{d+1}$ is (eventually) injective.

2. For all $d \to d+1$, the induced map $\mathrm{Ind}^{G_{d+1}}_{G_d} W_d \to W_{d+1}$ is (eventually) surjective.

3. $c_{\lambda,n}$, a multiple of $W_n(\lambda)$ in W_n is independent of n for $n > N$ (uniform representation stability).

When combined with the Noetherianity conditions, it becomes really easy to show representational stability!

Proof: We start with the easier direction, which is $\Longleftarrow$. Let N such that for all $d \geq N$, $k[S_{d+1}] \otimes_{k[S_d]} W_d \to W_{d+1}$ is surjective. Then, $k[S_d] \otimes_{k[S_N]} W_N \to W_{d+1}$. Thus $F^{(N)} \otimes W_N \to W$ is surjective in degree $\geq N$, and so W is f.g. in degree $\leq N$.

To do the other direction, we need more setup.

Definition: The injectivity degree (resp. surjectivity degree) is the smallest $s \geq 0$ (if it exists), such that $(W_n)_{S_{n-r}} \to (W_{n+1})_{S_{n+1-r}}$ is injective (resp. surjective) for $n \geq s + r$.

Lemma: If $V \to W$ is surjective, then the surjectivity degree of W is at most the surjectivity degree of V. The opposite result holds for the injectivity degree.

Lemma: The FI-module $F^{(d)} = k[\mathrm{Hom}_{FI}(d, -)]$ has injectivity degree 0 and surjectivity degree d.

Now let W be a finitely generated FI-module in degree $\leq N$; let F be a finite direct sum of $F^{(d)}$ with $d \leq N$ such that $F \to W$ is a surjection; we have that surj $\deg(W) \leq$ surj $\deg(F) \leq N$.

Let

$$0 \to K \to F \to W \to 0$$

be exact; then K is a submodule of F and is thus finitely generated. Then, in particular we have from the snake lemma in the diagram

that inj $\deg(W) \leq$ inj $\deg(K) \leq M$.

Prop: Let W be a FI-module over k with characteristic 0; then $(W_n)_n$ is uniformly representation stable with stable range $n \geq \max\{\text{inj deg}(W), \text{surj deg}(W)\} + \text{weight}(W)$, where

$$\text{weight}(V) = \max\{|\lambda| \mid V_n(\lambda) \text{ appears in some } W_n\}$$

Theorem: The FI-module $n \mapsto H^q(P_n, \mathbb{Q})$ is uniformly representation stable for $n \geq 4q$.

In particular, we may compute that

$$H^1(P_n, \mathbb{Q}) \cong V_n(0) \oplus V_n(1) \oplus V_n(2)$$

for $n \geq 4$ and

$$H^2(P_n, \mathbb{Q}) \cong V_n(1)^{\oplus 2} \oplus V_n(1,1)^{\oplus 2} \oplus V_n(2)^{\oplus 2} \oplus V(2,1)^{\oplus 2} \oplus V(3) \oplus V(3,1)$$

for $n \geq 7$ (so the bound is not *always* sharp).

It remains to prove the Noetherianity proposition. In spirit, it will be similar to the Hilbert basis thoerem.

Consider now the category of $FI - Mod_R$; if W is such a FI-module, consider the grading of W given by

$$R(W) = \bigoplus_{n \geq 0} (W_n)_{S_n}$$

where $(W_n)_{S_n} \xrightarrow{T} (W_{n+1})_{S_{n+1}}$, where T is induced by the cannonical injection. In particular, we already require R to be a $\mathbb{Q}$-algebra since we need left exactness of coinvariants. If $W \subset F^{(d)}$, then

$R(W) \subset R(F^{(d)})$; then, $R(W)$ is f.g. as a $R[T]$ module. Next, we also have that

$$W_n \cong \bigoplus_\lambda (W_n \otimes V_n(\lambda)^*) \otimes V_n(\lambda)$$

Then, if $W \subset F^{(d)}$, then we have a bound m, namely the weight and each $R(W \otimes V(\lambda))$ is finitely generated; hence, W is finitely generated.

Prop: Let R be Noetherian, W a sub $FI - Mod_R$ of $F^{(d)}$. Then W is f.g.

Proof: Recall that

$$F^{(d)} = S \mapsto R[\mathrm{Hom}_{FI}(d, S)]$$

We may induct on d), starting with $d = -1$, $F^{(-1)} = 0$; as mimick the proof of the Hilbert basis theorem.

Let A be a finite set; we consider the shift

$$W[A] : S \in FI \mapsto W_{S \sqcup A}$$

In particular, we can map $(F^d[A])_s \to (F^{(d)})_s$ surjectively, just by taking

$$[f : d \to S \sqcup A] \mapsto \begin{cases} [f] & \mathrm{Im}(f) \subset S \\ 0 & \text{otherwise} \end{cases}$$

and so we discover a s.e.s

$$0 \to \bigoplus_{\substack{A' \subset A \\ A' \neq \emptyset \\ \text{bij } d \setminus (d - |A'|) \to A'}} (F^{d-|A'|})_s \to (F^d[A])_s \to (F^{(d)})_s \to 0$$

Correspondingly,

$$0 \to K \cap W[A] \to W[A] \to W^{(A)} \to 0$$

By induction, $K \cap W[A]$ is finitely generated, and W is finitely generated $\iff$ $W[a]$ is finitely generated, and we can prove that $W^{(A)}$ is finitely generated. (Skipped)

Theorem: Fix R Noetherian, and let $N \subset M$ FI-modules, with M f.g. Then N is f.g. as well.

Proof: Induct on the number of nonzero graded pieces in M.

Theorem: Let k be a field, and let W be a f.g. $FI - Mod_R$; then $\dim_k(W_n)$ is eventually polynomial in n.

Theorem: If W is a f.g. FI-module, then

$$H_q(S_n, W_n) \to H_q(S_{n+1}, W_{n+1})$$

induced by the image of any inclusion, will be an isomorphism.

Proof: There are just two ingredients: that f.g. FI-modules are Noetherian, and homological stability for $W_n = \mathbb{Z}$.

For $W = F^{(d)}$, we have that $H_q(S_n, W_n) = H_q(S_n, \mathbb{Z}[\mathrm{Hom}_{FI}(d,n)]) = H_q(S_n, \mathrm{Ind}_{S_{n-d}}^{S_n} \mathbb{Z})$. Via Schapiro, this becomes just $H_q(S_{n-d}, \mathbb{Z})$, and by homological stability for $W_n = \mathbb{Z}$ gives us what we want for W free.

For a general FI-module, we take a resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to W \to 0$$

and by the Noetherianity, we may let F_p be a finite free FI-module. Thus, we have a spectral sequence $H_p(S_n, (F_q)_n) \implies H_{p+q}(S_n, W_n)$. Since there is an isomorphism $H_p(S_{n+1}), (F_q)_{n+1}) \implies H_{p+q}(S_{n+1}, W_{n+1})$, there must be a isomorphism $H_p(S_n, (F_q)_n) \implies H_{p+q}(S_n, W_n)$ as well.

We can give this result with greater generality.

Definition: A complemented category is a category C with a functor $* : C \times C \to C$, with natural isomorphisms $X * (Y * Z) \cong (X * Y) * Z$ and $X * Y \cong Y * X$ with natural compatibility, such that

1. C has initial object 1 such that $X * 1$ naturally.
2. Any arrow in C is monic and $\mathrm{Hom}(X_1 * X_2, Y) \to \mathrm{Hom}(X_1, Y) \times \mathrm{Hom}(X_2, Y)$.
3. For every $f : X \to Y$ there exists a unique, up to isomorphism, complment $X' \to Y$ such that $X * X' \cong Y$.

For example, $C = FI$ and $* = \sqcup$ is a complemented category. However, finite dimensional vector spaces do not! (Check that uniqueness in 3 does not hold). But you can just declare by fiat that the Hom-sets have a specific complement and it is OK, and we call this VIC.

Theorem: Let C be complemented, and assume there is X such that $\forall Y \in C$, there exists i such that $Y \cong X^{*i}$, and we set $\mathrm{rank}(Y) = i$. Assume that the category of functors $[C, Mod_R]$ (k Noetherian), has the Noetherianity property; then for $W : C \to Mod_k$ f.g., we have

1. Injectivity: for $f : X \to Y$ injective in C we have that $W(X) \to W(Y)$ is injective if the ranks are $\gg 1$.
2. Surjectivity: for $f : X \to Y$ injective in C we have that

   $$\mathrm{Ind}_{\mathrm{Aut}(X)}^{\mathrm{Aut}(Y)} W(X) \to W(Y)$$

   is surjective for ranks $\gg 1$.
3. For N large enough, for all X,

   $$\operatorname*{colim}_{\substack{Y \to X \\ \mathrm{rank}(Y) \leq \mathrm{rank}(X)}} W(Y) \cong W(X)$$

Lemma: Let C be a complemented category with generator X; then for all $n \geq r$,

$$k[\mathrm{Hom}_C(X^{*r}, X^{*n})] \cong \mathrm{Ind}_{X^{*(n-r)}}^{X^{*n}} k$$

Proof: The action of $\mathrm{Aut}_C(X^{*n}) \, \mathrm{Hom}_C(X^{*r}, X^{*n})$ is transitive (check!).

Theorem: Let C be complemented with generator X, and assume

1. f.g. $C - Mod_k$ are Noetherian

2. $\forall p \geq 0$ we have $f \mapsto f \times \mathrm{id}_X$,

$$H_p(\mathrm{Aut}_C(X^{*n}, k)) \to H_p(\mathrm{Aut}_C(X^{*+1}), k)$$

   is an isomorphism for $n \gg 1$.

Proof: Same as for S_n.

As an example, VIC_F works when F is a finite field. Alternatively, SI_F (symplectic vector space) works as well (again, F is finite).

Fix a Noetherian ring k.

Definition: If C is a cateogry, a C-module is a functor $C \to Mod_k$, and for $X \in C$ the corresponding free module is

$$F^{(X)} = k[\mathrm{Hom}_C(X, -)]$$

and we set $\mathrm{Hom}_{C-Mod_k}(F^{(X)}, W) = W(X$, and we say that W is f.g. if it is a quotient of a free module.

Lemma: f.g. $C - Mod_k$ are Noetherian iff for all X, and submodule of $F^{(X)}$ is f.g.

If we have a functor $\varphi : C \to D$ and set $\varphi* : C - Mod_k \to D - Mod_k$ taking $W \mapsto W \circ \phi$, we have the following:

Lemma: If $C - Mod_k$ is Noetherian, and if φ is essentially surjective, and if for all x, $\varphi^* F^{(X)}$ is f.g. then $D - Mod_k$ is Noetherian. For example, $VIC \to VI$ and $OFI \to FI$ (OFI is ordered finite sets).

Let $P = \coprod_{Y \in C} \mathrm{Hom}_C(X, Y)$; we assume that P is endowed with a well partial order $\preceq$ and a total order $\leq$ refining $\preceq$ such that $\forall f \preceq g$, $\exists \varphi : Y \to Y'$ such that $g = \varphi f$.

Then, an element of $F^{(X)}(Y)$ is of the form

$$\sum_{f : X \to Y} \alpha_f [f]$$

where $\alpha_f \in k$. Let f_0 be the f hwich is maximal for the property $\alpha_f \neq 0$. We set $\mathrm{Init}(\alpha) = \alpha_{f_0}[f_0]$ (and 0 if $\alpha = 0$). Now if $W \subset F^{(X)}$, define the initial module $\mathrm{Init}(W)_Y$ as the k-module generated by $\mathrm{Init}(\alpha)$, $\alpha \in W(Y)$.

Lemma: If $W, W' \subset F^{(X)}$ and $W \subset W'$ then $W = W'$ iff $\mathrm{Init}(W) = \mathrm{Init}(W')$.

And if $(W_j)_j$ be a strictly increasing sequence of submodules of $F^{(X)}$; we have that $\mathrm{Init}(W_j) \neq \mathrm{Init}(W_{j+1})$ by the above, so there must be some $\alpha_i[f_i] \in \mathrm{Init}(W_j) \setminus \mathrm{Init}(W_{j+1})$.

Then you do some stuff and conclude that f.g. $C - Mod_k$ are Noetherian.

10 Applications

Some examples of groups that do/do not satisfy homological stability:

1. P_n *does not* satisfy homological stability.

2. $GL_n(\mathbb{Z})$ *does* satisfy stability.

3. $\Gamma_n(p)$, which is defined via

$$1 \to \Gamma_n(p) \to GL_n(\mathbb{Z}) \to GL_n(\mathbb{F}_p) \to 1$$

does not satisfy stability. In fact, it is known that

$$\dim_{\mathbb{F}_p} H_q(\Gamma_n(p), \mathbb{F}_p) = \text{ polynomial in } n \text{ with leading term } \frac{m^{2q}}{q!}$$

And yet,
Theorems: Set R Noetherian. Then,

1. $S \mapsto H_q(P_s, R)$ is a f.g. FI-module over R.

2. If X is a connected manifold of dimension ≥ 2, oriented, with the homotopy type of a finite CW complex, then $S \mapsto H^q(\mathrm{Conf}_S(X), R)$ is a f.g. FI module over R.

3. $S \mapsto H_q(\Gamma_s(p), R)$ is f.g. FI-module over R.

Proof: For 3, look at the spectral sequence

$$(E^1_{pq})_S = \bigoplus_{\substack{T \subset S \\ |T| = |S| - p - 1}} H_q(\Gamma_T(p), R) \implies H_{p+q}(\Gamma_s(p), R)$$

and use the boundary maps being eventually surjective.

For 2, consider first that if X) is a topological space with the homotopy type of a CW complex with finitely many cells in each degree, then $S \mapsto H^q(X^S, R)$ is generated in finite degree. To see this, consider that if $R = k$ is a field, then the Kunneth formula yields that

$$H^q(X^s, R) \cong \bigoplus_{\substack{\epsilon q_s = q \\ s \in S}} \bigotimes_{s \in S} H^{q_s}(X, R)$$

If $|S| > q$, then we have that there is some s_0 such that $q_{s_0} = 0$, and hence any class in $\bigotimes_{s \in S} H^{q_s}(X, R)$ comes from $H^q(X^{S \setminus \{s_0\}}, R)$, so $S \mapsto H^q(X^S, R)$ is generated in degree $\leq q$.

In the general case, let $\Sigma : \Delta^{op} \to FinSet$ be a CW complex structure such that X is homotopic to $|\Sigma|$, and in particular $C_\bullet(S_\bullet(X), R)$ is quasiisomorphic to $C_\bullet(\Sigma, R)$. We then have $C_\bullet(S_\bullet(X^T), R)$ is quasiisomorphic to $C_\bullet(S_\bullet(X), R)^{\otimes T}$, which is quasiisomorphic to $C_\bullet(\Sigma)^{\otimes T}$.

In particular, by Ellenberg-Church-Farb we see that

$$H^q(X^T, R) \cong H^q(\mathrm{Hom}_R(C_\bullet(\Sigma)^{\otimes T}, R))$$

where the latter is a FI-cochain. Then, note that

$$(\mathrm{Hom}_R(C_\bullet^{\otimes T}, R))^q = \bigoplus_{\substack{(q_t)_{t \in T} \\ \Sigma q_t = q}} = \mathrm{Hom}_R(\bigotimes_t \in TC_{q_t}), R$$

and conclude as before.

But for configuration space, we will need to invoke sheaves. Consdier $i : \mathrm{Conf}_S(X) \to X^S$, so that we have the Leray spectral sequence

$$E_2^{p,q} = H^p(X^S, R^q i_* R) \implies H^{p+q}(\mathrm{Conf}_S(X), R)$$

Now fix X a manifold, so that $R^q i_* R$ should only depend on the dimension of X (???)/

Skipping the computation, it is known that $E_2^{\bullet,\bullet}$ is generated (by product) by $E_2^{\bullet,0}$ and $E_2^{0,\bullet}$. Even better $E_2^{0,\bullet}$ is generated by $E_2^{0,d-1}$. We have that $(E_2^{p,0})_S = H^p(X^S, R)$, which is ok by earlier, and $E_2^{0,q} = H^0(X^S, R^q i_* R)$, which is a f.g. FI-module, and so $E_2^{p,q}$ is a quotient of finitely many tensor products of finitely generated FI-modules, and this is a f.g. FI-module by Noetherianity.

Then, we see that $H^q(\mathrm{Conf}_\bullet(X), R)$ has a finite filtration by f.g. things, and thus is f.g. as well.