# Topological Proof of the Nielsen-Schreier Theorem

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In this paper we give a proof of the Nielsen-Schreier Theorem, which states that a subgroup of a free group is free. This is a result in algebra, and was proved originally through purely algebraic methods by its namesakes Jakob Nielsen (1921) and Otto Schreier (1926). Their proof is long and often regarded as unintuitive, and we will instead give a topological proof this theorem, and in doing so develop the theory of covering spaces. The reader should not worry if they do not know what a free group is as we will briefly cover this material, but we assume the reader to be familiar with the material of our topology (course) up to the discussion of fundamental groups.

## 1 Free Groups

Consider the Dihedral group $D_{10}$, which is generated by elements $r, s$ that satisfy the following relations:

$$r^5 = e, s^2 = e, \text{ and } srs^{-1} = r^{-1}$$

Given some set $S$, we would like to form a group generated by the elements of $S$, which satisfies *no* relations than those demanded by the group axioms. In this sense, we want as "few" relations as possible while still maintaining a group structure. In the proceeding section we will cover, though not with great detail, the construction of free groups and free products.

**Definition 1.1.** Let $S$ be a set. A *word* in $S$ is a finite string consisting of letters $s \in S$ and their inverses $s^{-1} \in S$, that is, strings of the form

$$s_1^{e_1} s_2^{e_2} \cdots s_n^{e_n}, \qquad s_i \in S, e_i \in \{1, -1\},$$

where we also allow for the empty word with no terms. We write $W(S)$ for the set of words in $S$. Note that concatenation of words defines a binary operation on $W(S)$. We say two words $w, w'$ are *elementarily equivalent* if $w$ is obtained from $w'$ by inserting a substring of the form $ss^{-1}$ or $s^{-1}s$. A word that contains no such terms is called a *reduced word*, the set of reduced words in $SS$ is denoted $F(S)$. We say that two words are *equivalent* if one is obtained from the other by a finite sequence of elementary equivalences. It is straightforward to check that equivalence between words is an equivalence relation on $W(S)$, and that relation is compatible with concatenation, i.e that if $w_1 \sim w_1'$ and $w_2 \sim w_2'$ then $w_1 w_2 \sim w_1' w_2'$.

**Example 1.1.** (i) Let $S = \{d, g, o\}$, the following are words in $S$:

$$good \quad dog \quad og^{-1}dgd \quad ddd^{-1}o^{-1}oo^{-1}ggg^{-1}$$

(ii) The word $ddd^{-1}o^{-1}oo^{-1}ggg^{-1}$ is equivalent to $do^{-1}g$ through the following sequence of elementary equivalences:

$$ddd^{-1}o^{-1}oo^{-1}ggg^{-1} \sim do^{-1}oo^{-1}ggg^{-1} \sim do^{-1}ggg^{-1} \sim do^{-1}g$$

Furthermore we have the following result, which should make intuitive sense, but the proof of which is tedious and omitted.

**Proposition 1.1.** *Given a set $S$, every word in $S$ equivalent to a unique reduced word.*

**Definition 1.2.** The *free group* on a set $S$ is the set of reduced words $F(S)$ equipped with concatenation followed by reduction as an operation. The *rank* of the free group on $S$ is the cardinality of its generating set $|S|$. If $|S| = n$, $F(S)$ is often written $F_n$. A familiar example is the free group on one generator, which is an infinite cyclic group, and hence isomorphic to $\mathbb{Z}$.

By construction we can write any element of $F(S)$ as a product of elements in $S$ and their inverses (the generators), using the equivalence we can cancel out elements and their inverses, and moreover our concatenation of strings is associative.

**Theorem 1.1.** *The free group is a group.*

*Proof.* □

We now introduce the notion of a free product of groups $G_1, G_2$, which we write as $G_1 * G_2$. The free product $G_1 * G_2$ generalizes the notion of a free group in the following sense: it is a group containing $G_1$ and $G_2$ as subgroups, and is generated by them with no further relations.

**Definition 1.3.** Given a family of groups $(G_\alpha)_{\alpha\in A}$ we define the free product of $(G_\alpha)_{\alpha\in A}$ to be the set $\ast_{\alpha\in A} G_\alpha$ consisting of words of the form $g_1 g_2 \cdots g_n$ satisfying the following: each $g_i \in G_{\alpha_i}$ for some $\alpha_i \in A$ with $g_i \neq 1$, and such that adjacent letters $g_i$ and $g_{i+1}$ do not belong to the same group, that is: $\alpha_i \neq \alpha_{i+1}$. In addition, we allow for the empty word, which . We equip $\ast_{\alpha\in A} G_\alpha$ with concatenation followed by a "simplification" as the group operation. Let us elaborate on the "simplification." Concatenation fails to produce an element of the free product in the case when we have elements $g_1 g_2 \cdots g_n$ and $h_1 h_2 \cdots h_m$ such that $g_n, h_1$ are both in the same group $G_\alpha$. We consider two cases, if their product is non-trivial in $G_\alpha$ we resolve this by replacing the term $g_n h_1$ by their product in $G_\alpha$. If their product is the identity in $G_\alpha$, then we remzove the term $g_n h_1$ from the product. In the second case it is again possible for $g_{n-1}, h_2$ to be in the same group $G_{\alpha'}$, so we must repeat the process. The repetition of this procedure yields an element in our free product in a finite number of steps.

**Theorem 1.2.** *The free product is a group.*

We omit the proof of this theorem. While the inverse and identity laws are straightforward to verify, showing associativity is non-obvious. Proving associativity directly is long and complicated, instead there exists an indirect approach detailed in Hatcher section 1.2 [1], which shows associativity through constructing a bijective correspondence between the free product $G = \ast_{\alpha\in A} G_\alpha$ and a subgroup of the permutation group of the free product $P(G)$, such that the product in $G$ corresponds to composition in $P(G)$ under the map.

**Example 1.2.** (i) Given a generating set, and the free group $F(S)$ generated by it, we have

$$\mathop{\ast}_{s\in S} G_s \cong F(S),$$

where $G_s \cong \mathbb{Z}$ for each $s \in S$. That is, the free group on $S$ is the product of $|S|$ copies of infinite cyclic groups.

(ii) While the free product in the above example is free, it is not the case that the free product of *any* groups are free. We consider the following example: $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, which is not free since we have non-identity elements that have finite order, i.e added relations.

One might wonder motivates the definition of the free product when we already have the notion of a direct product, which also contains $G_1$ and $G_2$ as subgroups. A defining property of the free product $G_1 * G_2$ is the following:

**Proposition 1.2.** *Given any group $H$, homomorphisms $f_1 : G_1 \to H$ and $f_2 : G_2 \to H$ there exists a unique homomorphism $F : G_1 \times G_2 \to H$ such that $f_2 = fi_1$ and $f_2 = fi_2$, where $i_1 : G_1 \to G_1 * G_2$ and $i_2 : G_2 \to G_1 * G_2$ is the natural inclusion.*

$$\begin{tikzcd}
& H & \\
G_1 \arrow[ur, "f_1"] \arrow[r, "i_1"] & G_1 * G_2 \arrow[u, "f"] & G_2 \arrow[ul, "f_2"'] \arrow[l, "i_2"']
\end{tikzcd}$$

If the reader finds this property familiar, it is because it is analogous to the universal property of the disjoint union of topological spaces. Just as we can uniquely determine continuous maps from the disjoint union of two spaces by their restrictions to each space, we can uniquely determine homomorphisms from the free product by a map from each factor of the product. Instead, the direct product does not give us a satisfy the "universal property" above: given homomorphisms $f_\alpha : G_\alpha \to H, \alpha \in \{1, 2\}$ there does not exist (except when at most one $G_\alpha$ is non-trivial) a homomorphism $f : G_1 \times G_2 \to H$ with $f_\alpha = fi_\alpha$. Where the direct product differs to the free product is that it imposes the extra relations, namely the commutative relation $(g_1, 1)(1, g_2) = (1, g_2)(g_1, 1)$ for $g_1 \in G_1, g_2 \in G_2$. Taking this to account, we can find many such groups $H$ and homomorphisms $f_\alpha$ for which $f_1(g_1)$ does not commute with $f_2(g_2)$, but we have that $F(\gamma_i(g_1)) = F(g_1, 1)$ and $F(\gamma_i(g_2)) = F(1, g_2))$ commute since $(g_1, 1)$ and $(1, g_2)$ commute, so there cannot exist a homomorphism $F : G_1 \times G_2 \to H$ with $f_\alpha = Fi_\alpha$. As a simple example, consider any nonabelian group $G$, its direct product with itself $G \times G$, set $H = G$ and take both $f_\alpha$ to be the identity map on $G$.
Now that we are familiar with free groups, we conclude this section by stating the theorem which we seek to prove:

**Theorem** (Nielsen-Schreier)**.** *Every subgroup of a free group is free.*

Prima facie, the Nielsen-Schreier theorem may seem to be a trivial result, since if the generators of a free group have no other relations than what is demanded by the axioms, it is natural to expect that the same holds for its subgroups. What is tricky about proving the theorem is that the generators of the free group need not be in the subgroup, and so it becomes difficult to find the free generating set for the subgroup. In this way, it is unclear if this property is passed onto the subgroup, and hence why the result is significant.

# 2 The van Kampen Theorem

So now that we know what a free group is, in our topological approach, we ask the natural question: what is a space that has a free fundamental group? To investigate this matter further, we require the van Kampen theorem, which gives us a method to compute the fundamental group of spaces by decomposing them into fundamental groups of simpler spaces that have a fundamental group known to us, namely $\pi(S^1) \cong \mathbb{Z}$. As a note, we give a specialized version of the van Kampen theorem, as necessary for our result. For the general case, and a complete proof rather than the outline we give, the reader can refer to Hatcher section 1.2 [1].

**Theorem 2.1** (van Kampen)**.** *Let $X$ be a topological space. Suppose $X = A \cup B$ is the union of of path-connected open subsets $A, B$, and $A \cap B$ is simply connected, picking $x_0 \in A \cap B$, we have an isomorphism $\pi(X, x_0) \cong \pi(A, x_0) * \pi(B, x_0)$. (The modification we make to the van Kampen Theorem is the requirement that $A \cap B$ is simply connected).*

*Proof.* The inclusions $A \to X$ and $B \to X$ give us induced homomorphisms $i_1 : \pi(A, x_0) \to \pi(X, x_0)$ and $i_2 : \pi(B, x_0) \to \pi(X, x_0)$. By the universal property of the free product there exists a unique homomorphism $\phi : \pi(A, x_0) * \pi(B, x_0) \to \pi(X, x_0)$. To show surjectivity of $\phi$, given any loop $\gamma$ in $X$ based at $x_0$, we apply the Lebesgue number lemma, to find a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ such that the subpaths $\gamma_i : [t_i, t_{i+1}] \to X$ have an image contained in either $A$ or $B$ (they can also be in $A \cap B$ of course), and also so that $\gamma(t_i) \in A \cap B$. Note that our subpaths are not necessarily loops, and in particular not loops based at $x_0$, however since $A \cap B$ is path connected and the endpoints of our subpaths are in $A \cap B$, for each $i$ there exists a path $\nu_i$ from $\gamma(t_i)$ to $x_0$, and replace each path $\gamma_i$ with $\gamma' = \nu_i^{-1}\gamma_i\nu_{i+1}$ by homotopy. Hence we have a factorization of our loop into the product of loops $\gamma_i'$ such that each loop lies wholly within $A$ or $B$, and hence is an element of the free product. So we have

$$\phi([\gamma_0'] \cdots [\gamma_{n-1}']) = [\gamma],$$

therefore $\phi$ is surjective. Injectivity is more technically difficult to show and we omit the proof, though it is simplified by our condition that $A \cap B$ is simply connected. Suppose $\gamma = [\gamma_1][\gamma_2] \cdots [\gamma_n]$ in the free product, so that each loop $\gamma_i$ is either in $A$ or $B$, and suppose that $\phi(\gamma) = 1$. This means that product of loops $\gamma_1 \cdots \gamma_n$ is homotopic to the constant loop at $x_0$ in $X$. We wish to show $\gamma$ is the identity element in the free product: the empty word. We consider the path homotopy taking $\gamma_1 \cdots \gamma_n$ to the constant loop, and subdivide the unit square into subsquares which are contained wholly in $A$ or $B$, through some work we can carefuly write $\gamma$ as a product of loops that can be reduced to the empty word. $\square$

**Example 2.1.** Recall the wedge sum of two circles $S^1 \vee S^1$, defined as the disjoint union of two copies of $S^1$ with base points identified. We would like to apply the van Kampen Theorem to show that $\pi(S^1 \vee S^1) \cong F_2$. An initial instict may be to choose each copy of the circle as our cover of the wedge, but these sets are not open in $S^1$. Rather instead take a hemipshere $H_1$ of the first circle containing the basepoint together with the second circle, and likewise take a hemisphere $H_2$ of the second circle containing the basepoint together with the second circle. Let $A = H_1 \vee S^1$ and $B = S^1 \vee H_2$. Since the hemispheres deformation retract to the basepoint, we have that $A$ deformation retracts to $S^1 \vee \{1\} \cong S^1$ and hence we have that $\pi(A) \cong \pi(S^1) \cong \mathbb{Z}$. Similarly, we have that $\pi(B) \cong \mathbb{Z}$. Finally $U \cap V = H_1 \vee H_2$ which deformation retracts to the basepoint, so we have that that $\pi(A \cap B) = 0$, i.e $A \cap B$ is simply connected. Hence applying the van Kampen theorem, we conclude

$$\pi(S^1 \vee S^1) \cong \pi(A) * \pi(B) \cong \mathbb{Z} * \mathbb{Z} \cong F_2.$$

By induction we can show that $\pi(\bigvee_{i=1}^n S^1 \cong F_n)$. Applying the Van Kampen theorem for free product of an arbitrary number of groups (which has a very similar proof found in [1]), we similarly have that fundamental group of the wedge sum of $|S|$ copies of circles is isomorphic to $F(S)$.

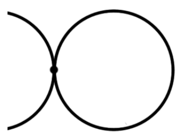

Figure 1: $H_1 \vee S^1$

# 3 Covering Spaces

Given a free group, we now know of a space that has a fundamental group corresponding to it, but we are interested in the subgroups of this free group. In this section we introduce the theory of covering spaces, of particular importance is a result that gives us a bijective correspondence between subgroups of the fundamental group of space and "covering spaces" of it. Let us define what a covering space:

**Definition 3.1.** A *covering space* of a space $B$ is a space $E$ equipped with a continuous surjective map $p : E \to B$ that satisfies the following condition: for every $x \in B$, there exists an open neighborhood $U \subset B$ of $x$ such that $p^{-1}(U) = \bigcup_{i \in I} V_i$ is the union of pairwise disjoint open subsets $V_i \subset E$, and for each $i \in I$ the restriction $p|_{V_i}$ is a homeomorphism onto $U$. Such an open set $U$ is said to be *evenly covered* by $p$ and the $V_i$ are called *sheets*. The map $p$ is often called a *covering* (or *cover*) of $B$. Note that for each $b \in B$, the fiber $p^{-1}(b)$ has the discrete topology since each sheet $V_i$ is open in $E$, and its intersection with $p^{-1}(b)$ is a singleton, which is open in $p^{-1}(b)$ with respect to the subspace topology.

Let us try and make sense of this definition. Intuitively, idea is that every neighbourhood of a point in a covering space resembles the space it covers, and conversely every point in the base space has a neighbourhood that resembles some neighbourhood of the covering space. We can visualize $p^{-1}(U)$, where $U$ is open set that is evenly covered by $p$, as a collection of spaces above $U$, each having the same shape and size as $U$, that are projected onto $U$ by $p$, like a "stack of pancakes" floating above $U$.

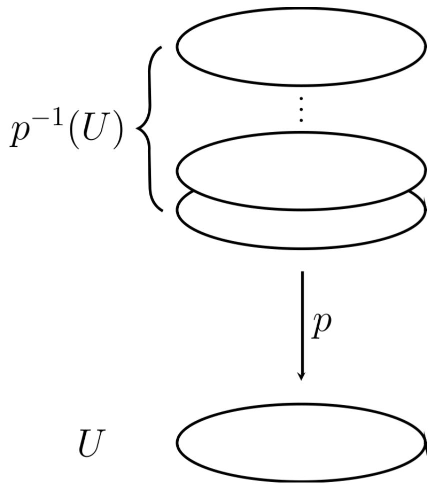


Figure 2: From Wikimedia Commons

**Example 3.1.** (i) Every homeomorphism is a covering map.

(ii) The map $f : \mathbb{R} \to S^1$ given by $f(t) = e^{2\pi i t}$ is a cover. Where the preimage of an open arc on the circle is a collection of open intervals in the real line separated by multiples of $2\pi$. We can visualize $\mathbb{R}$ as an infinite helix projected onto the circle by $f$.

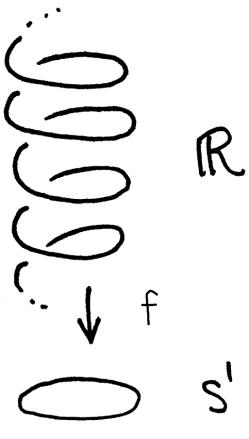


Figure 3: From Math3ma.com [4]

(iii) The product of the above map, $f \times f$, is a cover of Torus , where each square in the grid gets wrapped around the torus by the map.

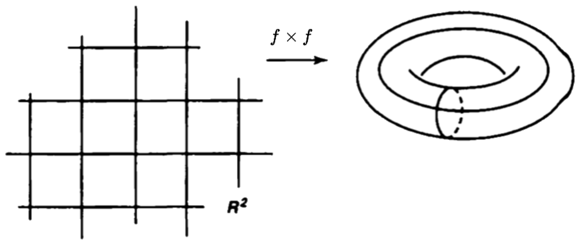


Figure 4: From Munkres [3]

We now give some definitions of properties for which covering space theory has important results for spaces satisfying them.

**Definition 3.2.** A space $X$ is *locally path-connected* if, for each $x \in X$ and each open neighborhood $V$ of $x$, there exists a path connected open neighborhood $U$ of $x$ such that $U \subset V$ (this is a stronger version of "having path-connected neighborhoods", in particular if $X$ is locally path-connected and connected it is path-connected). A space $X$ is *semi-locally simply-connected* if every point $x_0 \in X$ has an open neighborhood such that the homomorphism $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ induced by the inclusion is trivial.

Semi-locally simple-connectedness is worth elaborating on, as it may seem like a confusing property. We are familiar with simply-connected spaces which have no "holes", but it is not a property that many spaces have. Semi-locally simple-connected spaces are instead quite common, in fact most of the "nice" spaces we encounter are semi-locally simply-connected, the idea is that there is some lower bound on the size of "holes" in our space.
To further develop our understanding of the relation between the covering space and the fundamental group of a space, we introduce lifts, which are a incredibly useful way to connect the base space and the covering space through the covering.

**Definition 3.3.** A *lift* of a continuous map $f : X \to B$ to the covering space $p : E \to B$ is a map $\tilde{f} : X \to E$ such that $p\tilde{f} = f$:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

The two lemmas below, give us unique lifts from paths in $B$ to paths in $E$ and unique lifts from path homotopies in $B$ to path homotopies in $E$, they combine to help prove the result we are after: given paths which are homotopic in $B$, we would like their lifts to be homotopic, with the respective homotopy given by the lift of the homotopy of the paths in $B$.

**Lemma 3.1** (The path lifting lemma)**.** *Let $p : (E, e_0) \to (B, b_0)$ be a based covering map. Any path $\gamma : [0, 1] \to B$ with $\gamma(0) = b_0$ has a unique lift to a path $\tilde{\gamma} : [0, 1] \to E$ with $\tilde{\gamma}(0) = e_0$.*

*Proof.* We first cover $B$ by evenly covered open neighborhoods $U_\alpha$. By Lebegsue number lemma, we find a subdivision of $[0, 1]$ where $0 = t_0 < t_1 \ldots < t_n = 1$ such that the image of intervals $[t_i, t_{i+1}]$ under $\gamma$ lies in an open set $U_i$ which is evenly covered by $p$. We define our lift $\tilde{\gamma}$ inductively. Firstly, we set $\tilde{\gamma}(0) = e_0$. Then assuming $\tilde{\gamma}(t)$ is defined for $0 \leqslant t \leqslant t_i$, we define $\tilde{\gamma}$ on $[t_i, t_{i+1}]$ as follows:

let $(V_j)_{j\in J}$ be the partition of the set evenly covered set $U_i$ which contains the image of $\gamma([t_i, t_{i+1})$. Since each $V_i$ is mapped homomorphically to $U$ by $p$, we have that $\tilde{\gamma(t_i)}$ lies in one of these sets, say $V_\beta$ for some $\beta \in J$. For $t \in [t_i, t_{i+1}]$ we define

$$\tilde{\gamma(t)} = p|_{V_\beta}^{-1}(f(t)).$$

The continuity of $\tilde{\gamma}$ on $[t_i, t_{i+1}]$ follows from the fact that $p|_{V_\beta}$ is a homeomorphism. Following this way, we can define $\tilde{\gamma}$ on $[0, 1]$, and the continuity of $\tilde{\gamma}$ on $[0, 1]$ follows from the pasting lemma. Finally, we have $p\tilde{\gamma} = \gamma$ by construction. Uniqueness is shown in a similar inductive manner, and can be found in Munkres Lemma 54.1 [3]. □

**Lemma 3.2** (Path Homotopy Lifting Property). *Let $p : (E, e_0) \to (B, b_0)$ be a based covering map. Let $F : [0, 1] \times [0, 1] \to B$ be continuous with $F(0, 0) = b_0$. There is a unique lifting of $F$ to a continuous map $\tilde{F} : [0, 1] \to E$ with $\tilde{F}(0, 0) = e_0$. Moreover, if $F$ is a path homotopy, then $\tilde{F}$ is a path homotopy.*

*Proof.* We give an outline of this proof, and it can be found in Munkres Lemma 54.2 [3]) Let $\tilde{F}(0, 0) = e_0$. The previous lemma is applied to extend $\tilde{F}$ to $[0, 1] \times \{0\}$ and $\{0\} \times [0, 1]$, then extend it onto all of $[0, 1] \times [0, 1]$ by subdividing $[0, 1] \times [0, 1]$ into subsquares and defining it inductively. If $F$ is a path homotopy, we note that $F$ maps $\{0\} \times [0, 1]$ to $b_0$, and hence $\tilde{F}$ maps $\{0\} \times [0, 1]$ to $p^{-1}(b_0)$, uniqueness follows similarly by construction. Since fibers are discrete and $\{0\} \times [0, 1]$ is connected, it follows that $\tilde{F}$ is constant on $\{0\} \times [0, 1]$ and since $\tilde{F}(0, 0) = e_0$ we have $F(\{0\} \times [0, 1]) = \{e_0\}$. Similarly $\tilde{F}$ is constant on $\{1\} \times [0, 1]$, where $F(\{1\} \times [0, 1]) = \{e_1\}$ for some $e_1 \in E$. Thus $\tilde{F}$ is a path homotopy. □

**Theorem 3.1.** *Let $p : (E, e_0) \to (B, b_0)$ be a based covering map, and let $\gamma, \eta : [0, 1] \to B$ be paths from $b_0$ to $b_1$, with unique path liftings $\tilde{\gamma}, \tilde{\eta} : [0, 1] \to E$ that start at $e_0$. If $\gamma$ and $\eta$ are path homotopic, then $\tilde{\gamma}$ and $\tilde{\eta}$ end at the same point, and are path homotopic.*

*Proof.* Let $F : [0, 1] \times [0, 1] \to B$ be the path homotopy from $\gamma$ to $\eta$. We have $F(0, 0) = b_0$. By our lemma above, there exists a unique lift $\tilde{F} : [0, 1] \times [0, 1] \to E$ such that $\tilde{F}(0, 0) = e_0$, and $\tilde{F}$ is a path homotopy, with $F(\{0\} \times [0, 1]) = \{e_0\}$ and $F(\{1\} \times [0, 1]) = \{e_1\}$. The restrictions $\tilde{F}|_{[0,1]\times\{0\}}$ and $\tilde{F}|_{[0,1]\times\{1\}}$ are paths on $E$ which start on $e_0$, and they are a liftings of $\tilde{F}|_{[0,1]\times\{0\}} = \gamma$ and $\tilde{F}|_{[0,1]\times\{1\}} = \eta$ respectively. By the uniqueness of path liftings, it must be that $\tilde{F}(t, 1) = \tilde{\gamma}(t)$ and $\tilde{F}(t, 1) = \tilde{\eta}(t)$. In particular $\tilde{\gamma}(1) = \tilde{\eta}(1) = \tilde{F}(1, 1) = e_1$ so that $\tilde{\gamma}$ and $\tilde{\eta}$ end at the same point, and $\tilde{F}$ is a path homotopy between them. □

We have the following corollary:

**Corollary 3.1.** *Let $p : (E, e_0) \to (B, b_0)$ be a based covering map.*

(i) *The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.*

(ii) *The elements of $p_*(\pi(E, e_0))$ are precisely of all homotopy classes of loops in $B$ based at $b_0$ which lift to loops in $E$ based at $e_0$.*

*Proof.* (i) We will show that the kernel of $p_*$ is trivial. Suppose $[\tilde{\gamma}] \in \ker p_*$, then $[\tilde{\gamma}]$ is the homotopy class of loops such that $p\tilde{\gamma} \sim c_{x_0}$, the constant loop at $x_0$. Clearly $\tilde{\gamma}$ is the lift of $p\tilde{\gamma}$ and $c_{e_0} = pc_{x_0}$ is the lift of $c_{x_0}$. Hence, by Theorem 3.1, this homotopy lifts to a homotopy between $\tilde{\gamma}$ and $c_{e_0}$. Thus $[\tilde{\gamma}] = [c_{e_0}]$, the identity element in $\pi_1(E, e_0)$.

(ii) We note that an element of $p_*(\pi_1(E, e_0))$ is a class of loops in $B$ that contains at least one loop $\gamma$ which is the image of a loop in $\tilde{\gamma}$ in $E$ so that $p\tilde{\gamma} = \gamma$ and thus has a lift to a loop in $E$. By Theorem 3.1, it follows that any other loop in the same class can be lifted to path which starts and ends at the same points of $\tilde{\gamma}$, i.e it can be lifted to a loop. Conversely, if class containing a loop $\gamma$ in $B$ lifts to a loop $\tilde{\gamma}$, then explicitly we have $p_*([\tilde{\gamma}]) = [p\tilde{\gamma}] = [\gamma]$ so $[\gamma]$ is in the image of $p_*$.

□

The following two results we omit the proof of but they can be found in Hatcher [1]. Of particular importance to our result is the Galois correspondence of covering spaces.

**Theorem 3.2** (Existence of Universal Cover). *A connected, locally path-connected, and semi-locally simply-connected space $B$ with a basepoint $b_0$ has a simply connected covering space $p : E \to B$, where $E = \{[\gamma] : \gamma \text{ is a path in } B \text{ starting at } b_0\}$ with $p([\gamma]) = \gamma(1)$.*

*Proof.* □

**Theorem 3.3** (Galois correspondence of covering spaces). *Let $(B, b_0)$ be a connected, locally path-connected, and semi-locally simply-connected based space. Then for every subgroup $H \leqslant \pi_1(B, b_0)$, there exists a path-connected covering space $p : (E_H, e_0) \to (B, b_0)$ such that $p_*(\pi_1(E_H, e_0)) = H$.*

# 4 Graphs

**Definition 4.1.** A *graph* $X$ is a topological space obtained from a discrete set $X^0$ of points and a collection of copies of the unit interval $I_\alpha = [0, 1]$, as the quotient space of the disjoint union of $X^0$ and the intervals, where the endpoints of each $I_\alpha$ is identified with points in $X$. The elements of $X^0$ are called vertices, with edges being the intervals $I_\alpha := e_\alpha$. Note that a subset $U$ of $X$ is open if and only if $U \cap e_\alpha$ is open in $e_\alpha$ for each edge. A *subgraph* $A$ of $X$ is a graph $A \subset X$ with $A^0 \subset X^0$.

This definition matches up with visual interpretation of the graph. We note that a graph is connected topologically if and only if it is connected graph theoretically. The following is a relative straightforward result and in fact Graphs belong to a larger collection of "nice" spaces that have similar properties.

**Proposition 4.1.** *A graph is locally path-connected and semi-locally simply-connected.*

**Definition 4.2.** A *tree* $T$ is a graph in which every two vertices are connected by a unique path, or equivalently a connected graph with no loops (cycles in graph theoretic language). A *maximal tree* $T$ of a graph $X$ is a subgraph of a graph $X$ which is a tree and contains all vertices of $X$.

**Proposition 4.2.** *Every connected graph $X$ has a maximal subtree, and any tree in the graph is contained in a maximal tree.*

*Proof.* This theorem requires Zorn's Lemma (for the infinite case). Let $X$ be any connected graph. We consider the collection of all trees of $T_\alpha \subset X$ ordered by subgraph relation. Any edge of the union $\cup_{\alpha \in A} T_\alpha$ is contained in some $T_{\alpha'}$ with $\alpha' \in A$. In particular, suppose for the sake of contradiction that there is a loop in the union, then there is a tree $T_{\alpha'}$ which contains the loop, and is hence not a tree. Therefore the union $\cup_{\alpha \in A} T_\alpha$ no loops. Likewise, if $\cup_{\alpha \in A} T_\alpha$ was not connected, then there would be two vertices which are not joined by a path, and hence some tree $T_{\alpha''}$ which contained both vertices but was not connected, which is a contradiction since $T_{\alpha''}$ is a tree. Hence $T_{\alpha''}$ is a connected graph with no loops, so it is a tree. Since $T_{\alpha''}$ it contains every tree in the collection, it serves as an upper bound for every chain. By Zorn's lemma, there exists a maximal element $T$ in the set of all trees of $X$. Since $X$ is connected, $T$ must contain every vertex of $X$ since otherwise we add a vertex in $T$ to a vertex in $X - T$ which yields a larger tree than $T$ contradicting its minimality. Hence $T$ is a maximal subtree of $X$. □

**Theorem 4.1.** *Every tree is contractible.*

*Proof.* We prove this in the finite case. We induct on the number of edges. If $T$ has one edge, it is homeomorphic to the unit interval and hence contractible. Assume that a tree with $n$ edges deformation retracts onto some vertex, let $T$ be a tree with $n + 1$ edges. Let $v \in T$ and choose $v' \in T$ such that it is the furthest vertex away from $v$ by edges traversed to reach. Then $v'$ is the endpoint of some unique edge , so we can deformation retract $v'$ onto the other endpoint, and hence we are left with a tree with $n$ edges, which is contractible by our assumption. For a proof in the infinite case see May [2]. □

**Theorem 4.2.** *Given a connected graph $X$ with a maximal tree $T$ the quotient map $X \to X/T$ is a homotopy equivalence.*

*Proof.* This follows as an application of a more general theorem that is outlined in [1], but essentially graphs are among the collection of spaces that have the property that the quotient map $X \to X/T$ is a homotopy equivalence if $T$ is contractible. □

The above result gives us the immediate corollary

**Corollary 4.1.** *Given a maximal tree $T$ of a connected graph $X$, the space $X/T$ is a graph with one vertex and homotopically equivalent to the wedge sum of circles, and hence The fundamental group of a connected graph is free.*

*Proof.* There exists a maximal tree $T$ of $X$. Since $T$ is contractible, we have a homotopy equivalence $X \to X/T$. So $\pi_1(X, x_0) \cong \pi_1(X/T, x_0)$. Since $X/T$ is the wedge of $|K|$ circles, its fundamental group is free, so it follows that the fundamental group of $X$ is also free. □

Now the pieces we have set up are falling into place, we continue with the final piece to our puzzle which will allow us to use the Galois correspondence of covering spaces to prove the the Nielsen-Schreier Theorem

**Lemma 4.1.** *A covering space of a graph is a graph.*

*Proof.* Let $p : E \to X$ be a covering space of a graph $X = X^0 \bigsqcup_\alpha I_\alpha / \sim$. We can realize $E$ as a graph with a vertex-set $E^0 = p^{-1}(X^0)$ and a edges corresponding to the unique path lifts of maps $I_\alpha \to X$. □

# 5 The Nielsen-Schreier Theorem

**Theorem** (Nielsen-Schreier)**.** *Every subgroup of a free group is free.*

*Proof.* Let $F(S)$ be a free group over $S$, and let $X$ be a wedge of $|S|$ circles, so that $\pi_1(X, x_0) \cong F(S)$. Given a subgroup $H$ of $\pi_1(X, x_0)$, there exists a covering space $p : (X_H, x_H) \to (X, x_0)$ such that $p_*(\pi_1(X_H, x_H)) = H$, $p_*$ is injective so it is an isomorphism onto $H$. Since $X$ is a graph, it follows that $X_H$ is also a graph, hence $H \cong \pi(X_H, x_H)$ is free. □

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