Nielsen-Schreier Theorem

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Free Groups

- A *word* in a set S is a string of elements of $s \in S$ and its inverse s^{-1}. A *reduced word* is a word that contains no adjacent pairs ss^{-1} or $s^{-1}s$. F_S is then the set of reduced words in S.
- The free group F_S over a set S is the set of reduced words in S, it forms a group under juxtaposition. The rank of F_S is $|S|$, when $|S| = n$, we write F_n.
- The *free product* of two groups G and H, denoted $G * H$, is the set consisting of words $g_1 h_1 g_2 h_2 \cdots g_n h_n$ where $g_i \in G$ and $h_i \in H$ are not the identity.

Theorem (Nielsen-Schreier)

Every subgroup of a free group is free.

Fundamental Groups

- Given a space X, a *loop* based at $x_0 \in X$ is continuous map $\gamma : I \to X$ with $\gamma(0) = x_0 = \gamma(1)$.
- The two loops γ_1, γ_2 are equivalent if there exists a *(path) homotopy* between them; that is, if there exists a continuous map $h : I \times I \to X$ with

$$h(t, 0) = \gamma_2(t), \quad h(t, 1) = \gamma_2(t), \quad h(0, t) = h(1, t) = x_0.$$

- Define $\pi_1(X, x_0)$ to be the set of homotopy classes of loops based at x_0.
- $\pi(X, x_0)$ forms a group under concatenation, and is called the *fundamental group*.
- Given a continuous map $p : (X, x_0) \to (Y, y_0)$ with $p(x_0) = y_0$, we define $p_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $p_*([\gamma]) = [p\gamma]$. p_* is a homomorphism.

Fundamental Groups

Theorem

$\pi_1(S^1) \cong \mathbb{Z}$

Van Kampen Theorem (A specialized case)

- We say a space X is *simply-connected* if it is path-connected and has a trivial fundamental group, a space X is *semi-locally simply-connected* if every point $x_0 \in X$ has an open neighborhood such that the homomorphism $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ induced by the inclusion is trivial.

Theorem (Van Kampen)

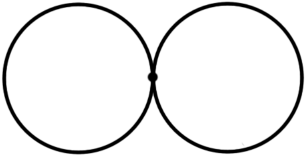
Let X be a space with $x_0 \in X$. If $X = A \cup B$ is the union of path connected open subsets A, B, with $A \cap B$ simply-connected, then

$$\pi_1(X, x_0) = \pi_1(A, x_0) * \pi_2(B, x_0).$$

An application: Wedge Sum of Two Circles

Consider the wedge of two circles $S^1 \vee S^1$ around a point b. By the Van Kampen Theorem we have

$$\pi_1(S^1 \vee S^1, b) \cong \mathbb{Z} * \mathbb{Z} \cong F_2$$

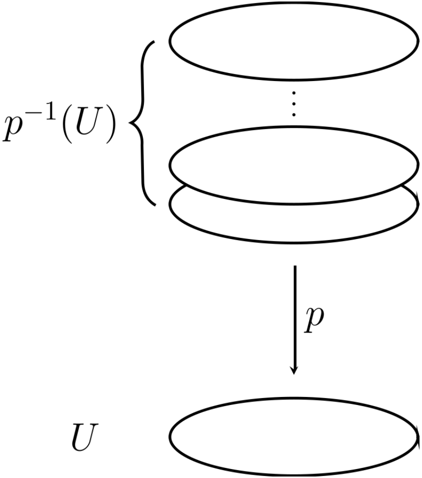


We can similarly show that the fundamental group of the the the wedge of n circles $\bigvee_n S^1$ is free.

Covering Spaces

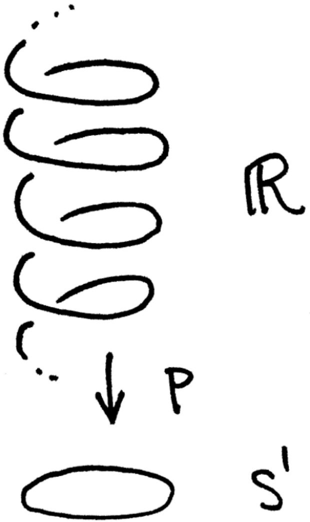
- A *covering space* of a space B is a space E equipped with a continuous surjective map $p : E \to B$ that satisfies the following condition: for every $x \in B$, there exists an open neighborhood $U \subset B$ of x such that $p^{-1}(U) = \bigcup_{i \in I} V_i$ is the union of pairwise disjoint open subsets $(V_i)_{i \in I} \subset E$, and for each $i \in I$ the restriction $p|_{V_i}$ is a homeomorphism onto U.
- Such a U is called a *fundamental neighborhood*, and each V_i is called a *sheet*. We call p a *covering map* (or *cover*), and X the *base space*.

Covering Spaces



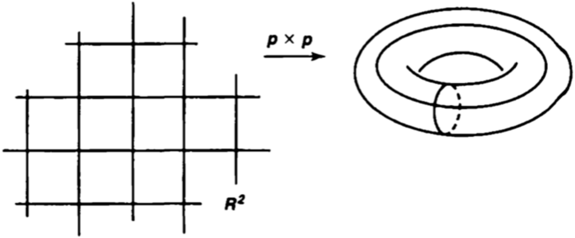
Examples of Covering Spaces

- The map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi i t}$ is a cover.



Examples of Covering Spaces

- The map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$. is a cover of the torus.



Path Lifting

A *lift* of a continuous map $f : X \to B$ to the covering space $p : E \to B$ is a map $\tilde{f} : X \to E$ such that $p\tilde{f} = f$:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

Lemma

Let $p : E \to B$ be a covering with $p(e_0) = b_0$. Any path $\gamma : I \to B$ with $\gamma(0) = b_0$ has a unique lift to a path $\tilde{\gamma} : I \to E$ with $\tilde{\gamma}(0) = e_0$.

(Path) Homotopy Lifting

Lemma

Let $p : E \to B$ *be a covering with* $p(e_0) = b_0$*. Let* $F : I \times I \to B$ *be continuous with* $F(0,0) = b_0$*. There is a unique lifting of* F *to a continuous map* $\tilde{F} : I \times I \to E$ *with* $\tilde{F}(0,0) = e_0$*. Moreover, if* F *is a path homotopy, then* $\tilde{F}$ *is a path homotopy.*

(Path) Homotopy Lifting

Theorem

Let $p : E \to B$ *be a covering with* $p(e_0) = b_0$*, and let* $\gamma, \eta : I \to B$ *be paths from* b_0 *to* b_1*, with unique path liftings* $\tilde{\gamma}, \tilde{\eta} : I \to E$ *that start at* e_0*. If* γ *and* η *are path homotopic, then* $\tilde{\gamma}$ *and* $\tilde{\eta}$ *end at the same point, and are path homotopic.*

Proof.

We have a unique lift $\tilde{F}$ of our homotopy F from γ to η. The restrictions of $\tilde{F}$ to $I \times \{0\}$ and $I \times \{1\}$ are paths lifts of γ and η respectively, so by uniqueness of path lifting they are equal to $\tilde{\gamma}$ and $\tilde{\eta}$. In particular $\tilde{\gamma}(1) = \tilde{\eta}(1) = \tilde{F}(1, 1)$. □

Corollary

The induced homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ *is injective.*

Universal Cover

Theorem

A connected, locally path-connected, and semi-locally simply-connected space B with a basepoint b_0 has a simply connected covering space $E == \{[\gamma] : \gamma$ is a path in B starting at $b_0\}$, with $p : E \to B$ given by $p([\gamma]) = \gamma(1)$.

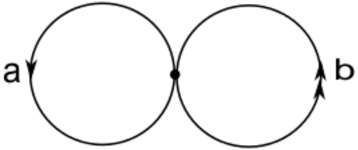
Galois correspondence of covering spaces

Theorem

Let (B, b_0) *be a connected, locally path-connected, and semi-locally simply-connected based space. Then for every subgroup* $H \leq \pi_1(B, b_0)$*, there exists a path-connected covering space* $p : (E_H, e_0) \to (B, b_0)$ *such that* $p_*(\pi_1(E_H, e_0)) = H$*.*

Graphs

A *graph* X is a topological space obtained from a discrete set X^0 of points and a collection of copies of the unit interval $I_\alpha = [0, 1]$, as the quotient space of the disjoint union of X^0 and the intervals, where the endpoints of each I_α is identified with points in X.



Trees

- A *tree* is a graph such that every two vertices are connected by a unique path. A *maximal tree* T of a graph X is a subgraph of X which is a tree and contains all vertices of T.
- If X is a connected graph which admits a maximal tree, then X/T is the wedge of $|K|$ circles where K consists of edges in X that are not in T.

Theorem

Every connected graph X has a maximal tree T.

Proposition

Every Tree is contractible.

Fundamental Group and Covering Spaces of a Graph

Theorem

The fundamental group of a connected graph X is free.

Proof.

There exists a maximal tree T of X. Since T is contractible, we have a homotopy equivalence $X \to X/T$. So $\pi_1(X, x_0) \cong \pi_1(X/T, x_0)$. Since X/T is the wedge of $|K|$ circles, its fundamental group is free, so it follows that the fundamental group of X is also free. □

Lemma

Every covering space of a graph is also a graph.

Proof.

Let $p : E \to X$ be a covering space of a graph $X = X^0 \bigsqcup_\alpha I_\alpha / \sim$. We can realize E as a graph with a vertex-set $E^0 = p^{-1}(X^0)$ and a edges corresponding to the unique path lifts of maps $I_\alpha \to X$. □

Proof of the Nielsen-Schreier Theorem

Proof.

Let F be a free group with rank $|K|$, and let X be a wedge of $|K|$ circles, so that $\pi_1(X, x_0) \cong F$. Given a subgroup H of $\pi_1(X, x_0)$, there exists a covering space $p : (X_H, x_H) \to (X, x_0)$ such that $p_*(\pi_1(X_H, x_H)) = H$, p_* is injective so it is an isomorphism onto H. Since X is a graph, it follows that X_H is also a graph, so $H \cong \pi(X_H, x_H)$ is free. □