Faculty of Power and Aeronautical Engineering

Department of Automatic control & Robotics

Optimization Techniques

LP. Project No.16 (weight-reduction beverage production)

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Table of Contents

1. Instructions

In this project we have three different AMPL files. These files are project1.mod, project1.dat and project1.run. The project1.mod file contain decision variables, objective functions and constraints. The project1.data file contains all the values of the parameters that we declared on the project1.mod files. In order to run the AMPL files we need to copy and paste the three AMPL files on the AMPL folder and write “`ampl: include project1.run;`” on the AMPL command window. The AMPL-file project1.run contains basic commands to run AMPL. Do not forget to write ";" after each command.

To load the model,

Write model project1.mod;

To load the data,

Write model project1.dat;

To choose the solver,

Write option solver "`cplex`"; where . . . has to be replaced by the path to the folder containing the AMPL package.

To obtain the optimal solution,

Write solve;

You may now take a closer look at the solution.

To see the value of a variable use the command display.

As an example to see the number of tablespoons of Strawberry flavouring write display x [1];

You may obtain the reduced costs for these variables by writing display x [1].rc;

In the same fashion, you may get the dual variables corresponding to the constraint M1 by writing

display M1.dual;

You may get the slack in the constraints by writing

display M1.slack;

2. Problem Definition

A company makes a line of nutritionally complete, weight-reduction beverages. One of their products is a strawberry shake which is designed to be a complete meal. The strawberry shake consists of several ingredients. Some information about each of these ingredients is given below.

	Calories from Fat (per tbsp)	Total Calories (per tbsp)	Vitamin Content (mg/ tbsp)	Thickeners (mg/ tbsp)	Cost (€/tbsp)
Strawberry flavouring	1	50	30	3	8
Cream	70	100	0	6	6
Vitamin supplement	0	0	60	1	20
Artificial sweetener	0	120	0	2	13
Thickening agent	20	70	2	20	5

The nutritional requirements are as follows.

- The beverage must total between 370 and 410 calories (inclusive).
- No more than 15% of the total calories should come from fat.
- There must be at least 40 milligrams (mg) of vitamin content.
- For taste reasons, there must be at least two tablespoons (tbsp) of strawberry flavoring for each tbsp of artificial sweetener.
- Finally, to maintain proper thickness, there must be exactly 10 mg of thickeners in the beverage.

Management would like to select the quantity of each ingredient for the beverage which would minimize cost while meeting the above requirements.

3. Model Formulation

3.1. Decision variables

There are five ingredients which might be included as a supplement for the strawberry shake based on the nutritional requirements. And from this problem we can define our decision variables as follows:

X_1 = Number of Tablespoons of strawberry flavoring

X_2 = Number of Tablespoons of cream

X_3= Number of Tablespoons of vitamin supplement

X_4= Number of Tablespoons of artificial sweetener

X_5= Number Tablespoons of thickening agent

3.2. Objective Function

The mathematical model of the linear programing problem is given by:

$$minimize\ z = 8 \cdot X_1 + 6 \cdot X_2 + 20X_3 + 13 \cdot X_4 + 5 \cdot X_5$$

Which can be written as a compact form as:

$$min \sum_{i=1}^{n} P_i X_i$$

Where:

i – The set of ingredients considered.

X_i − The amount of *i-th* ingredients.

P_i −The cost of *i-th* ingredient.

3.3. Constraints

And this objective function is subjected to many constraints as expresses below:

- The beverage must total between 370 and 410 calories (inclusive).

$$50 \cdot X_1 + 100 \cdot X_2 + 120 \cdot X_4 + 70 \cdot X_5 \geq 370$$

$$50 \cdot X_1 + 100 \cdot X_2 + 120 \cdot X_4 + 70 \cdot X_5 \leq 410$$

- No more than 15% of the total calories should come from fat.

$$X_1 + 70 \cdot X_2 + 20 \cdot X_5 \leq 0.15 \cdot (50 \cdot X_1 + 100 \cdot X_2 + 120 \cdot X_4 + 70 \cdot X_5)$$

- There must be at least 40 milligrams (mg) of vitamin content.

$$30 \cdot X_1 + 60 \cdot X_3 + 2 \cdot X_5 \geq 40$$

- There must be at least two tablespoons (tbsp) of strawberry flavoring for each tbsp of artificial sweetener.

$$X_1 \geq 2 \cdot X_4$$

- There must be exactly 10 mg of thickeners in the beverage.

$$3 \cdot X_1 + 6 \cdot X_2 + X_3 + 2 \cdot X_4 + 20 \cdot X_5 = 10$$

Non-negativity

All decision variables are none negative.

$$X_i \geq 0$$

All those constraints can be written in compact form below (i.e., to make all the inequality consistent or similar, equations with "<= "sign is changed to ">=" with appropriate mathematical calculation). Finally, all the constraint equations can be denoted as:

$$Ax \geq b$$

Where:

X - The vector of all amounts of ingredient.

b - The vector of right-hand coefficient.

A - The matrix describing the contents of ingredients.

4. Methodology

An AMPL translator starts by reading, parsing and interpreting a model. The translator then reads some representation of particular data. The model and data are then processed to determine the linear program that they represent, and the linear program is written out in some appropriate form.

To write our AMPL program, I use the following two steps:

1. Formulation of algebraic model.
2. Read data.

4.1. Formulation of Algebraic Model

The formulation begins with a description of the index sets and numerical parameters that the model requires. Next, the decision variables are defined. Finally the objective and constraints are specified as expressions in the sets, parameters and variables.

1. **Index sets and numerical parameters:**

```
param n; # no of variables
param m; # no of constraint

param C{1..n};
param A{1..m, 1..n};
param B{1..m};
```

2. **Declaration of decision variables:**

```
var x {i in 1..n}  >=0;
```

3. **Objective Function:**

```
minimize z: sum {i in 1..n} C[i] * x[i];  # x is a vector of decision variables
```

4. **Constraints:**

```
s.t. constraint {j in 1..m}: sum{i in 1..n} A[j,i]*x[i] >=B[j];
s.t. M1: 3*x[1] + 6*x[2] + x[3] + 2*x[4] + 20*x[5] = 10;
s.t. M2 {i in 1..n}: x[i] >= 0;
```

4.2. Data

```
param n := 5;   # no of variables

param m := 5;   # no of constraint(note that totally we have 6 constraint but one is with equality, and it is putted as a normal equation)

param C :=     #vector of objective function

    1   8

    2   6

    3   20

    4   13

    5    5;

param  A:    1       2      3      4      5 :=     # matrix of constraint coefficient

    1    50     100     0     120    70
    2   -50    -100     0    -120   -70

    3    6.5    -55     0      18   -9.5

    4    30      0     60       0     2

    5    -2      0      0       1     0;

param  B:=    # vector of right hand coefficient

    1   370

    2  -410

    3   0

    4   40

    5   0;
```

5. Result and Interpretation

5.1. Result

5 variables, all linear

6 constraints, all linear; 22 nonzeros

1 equality constraint

5 inequality constraints

1 linear objective; 5 nonzeros.

CPLEX 20.1.0.0: optimal solution; objective 43.32415902

6 dual simplex iterations (0 in phase I)

x[1] = 1.18043

x[2] = 0.276758

x[3] = 0.0764526

x[4] = 2.36086

x[5] = 0

z = 43.3242

5.2. Interpretation

In this project mathematical linear model of weight-reduction beverages subjected to various nutritional requirement is performed. This nutritional requirement is denoted by a constraint function on the linear programming model. From the result section, it is seen that to produce the strawberry shake with a minimum cost the ingredient should be 1.18043 tbsp of strawberry flavoring, 0.276758 tbsp of cream and 0.0764526 tbsp of vitamin supplement, 2.36086 tbsp of artificial sweetener and with no tbsp thickening agent. With these ingredients all the nutritional requirement/constraints are fulfilled, and the optimal cost function is 43.3242.

X_1 = Number of Tablespoons of strawberry flavoring

X_2 = Number of Tablespoons of cream

X_3= Number of Tablespoons of vitamin supplement

X_4= Number of Tablespoons of artificial sweetener

X_5= Number Tablespoons of thickening agent

6. Integrity Check

Finally, we have to check the integrality in such a way that the result obtained in the continuous time domain being discrete. And to determine this property, the condition below should be fulfilled. Under these conditions on *A* and *b* all vertices of the admissible set

$\{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\}$ or $\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ (a polyhedron) are integer:

- A is totally unimodular.
- b has integer coordinates.

A totally unimodular matrix is a matrix for which every square submatrix is unimodular, that is, it has determinant 0, 1 or -1.

To check whether A is totally unimodular or not, the Hoffman Gale theorem is used which states that:

Let A be a matrix with elements from the set $\{0; 1; -1\}$, whose rows can be partitioned into two sets K and L. The following two conditions together are sufficient for A to be totally unimodular:

- Every column of A contains at most two nonzero elements.
- If two nonzero elements in a column of A have the same sign, then the row of one is in K, and the other in L. If two nonzero elements in a column of A have different signs, then the rows of both are in K, or both in L.

From our problem the matrix A (matrix of constrained coefficient) and B (vector of the right-hand coefficient) can be defined as:

$$A = \begin{matrix} 50 & 100 & 0 & 120 & 70 \\ -50 & -100 & 0 & -120 & -70 \\ 6.5 & -55 & 0 & 18 & -9.5 \\ 30 & 0 & 60 & 0 & 2 \\ -2 & 0 & 0 & 1 & 0 \end{matrix} \qquad B = \begin{matrix} 370 \\ -410 \\ 0 \\ 40 \\ 0 \end{matrix}$$

In order to have the values in the range of [-1, 1], both A and B are devided by the highest number from A. and the modified matrix will be:

$$A = \begin{matrix} -0.4167 & 0.8333 & 0 & 1 & 0.5833 \\ -0.4167 & -0.8333 & 0 & -1 & -0.5833 \\ 0.0542 & -0.4583 & 0 & 0.15 & -0.0792 \\ 0.2500 & 0 & 0.5 & 0 & 0.0167 \\ -0.0167 & 0 & 0 & 0.0083 & 0 \end{matrix} \qquad B = \begin{matrix} 3.0833 \\ -3.4167 \\ 0 \\ 0.3333 \\ 0 \end{matrix}$$

For matrix A, the determinant is 0 which satisfyies one condition but, It can be seen that there are above two non-zero elementes in all columns except column 3. Since the first conditions from Hoffman Gale theorem is not satisfied, it can be concluded that A is not totally unimodular. Thus the LP problem does not satisfay the integrality property and the solution will be in continues range instead of being descrete. And from AMPL code implementation after runing the code with integer command (`var x {i in 1..n} integer >=0`), the result for the objective function (z = 67) and decision vaiables are not the same too. Which in turn indicates the integrality property is not fulfield.

Bibliography

1. Dantzig, G.B., Thapa, M.N., Linear Programming:
2. Introduction, Springer, 1997. Nocedal, J., Wright, S.J., Numerical Optimization, Springer, 1999.
3. H.P. Williams, Model Building in Mathematical Programming, 5th Ed, Wiley 2013. M.S. Bazaraa,
4. J.J. Jarvis, H.D. Sherali, Linear Programming and Network Flows, 4th Ed, Wiley, 2010. M.S. Bazaraa,
5. Fourer R., Gay D.M., Kernighan B.W, A Modeling Language for Mathematical Programming, 2nd Ed, Duxbury Press, Year: 2002