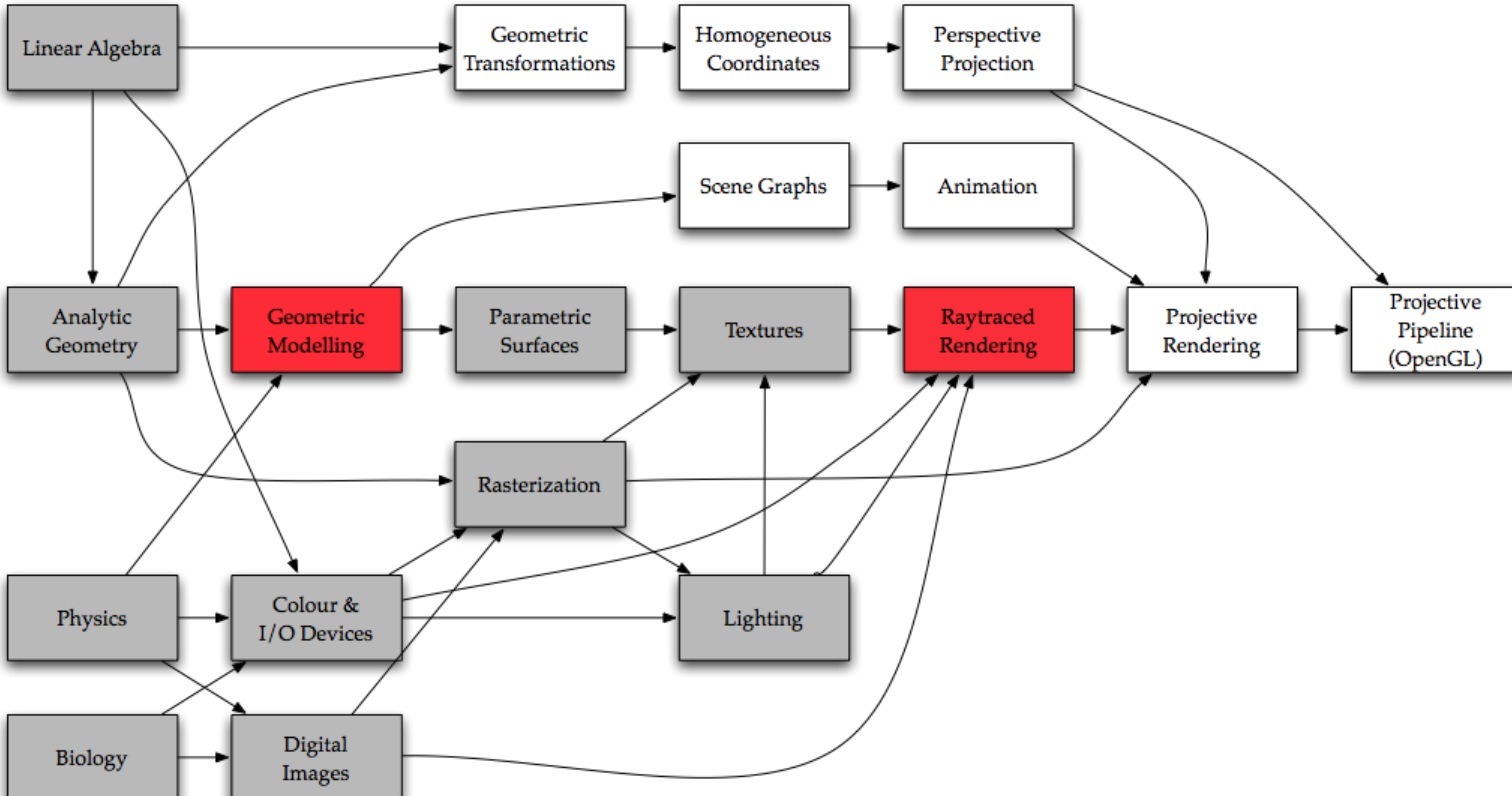
Geometry in 3 Dimensions

Where we Are



Linear Algebra
Geometric Transformations
Homogeneous Coordinates
Perspective Projection
Scene Graphs
Animation
Analytic Geometry
Geometric Modelling
Parametric Surfaces
Textures
Raytraced Rendering
Projective Rendering
Projective Pipeline (OpenGL)
Rasterization
Physics
Colour & I/O Devices
Lighting
Biology
Digital Images

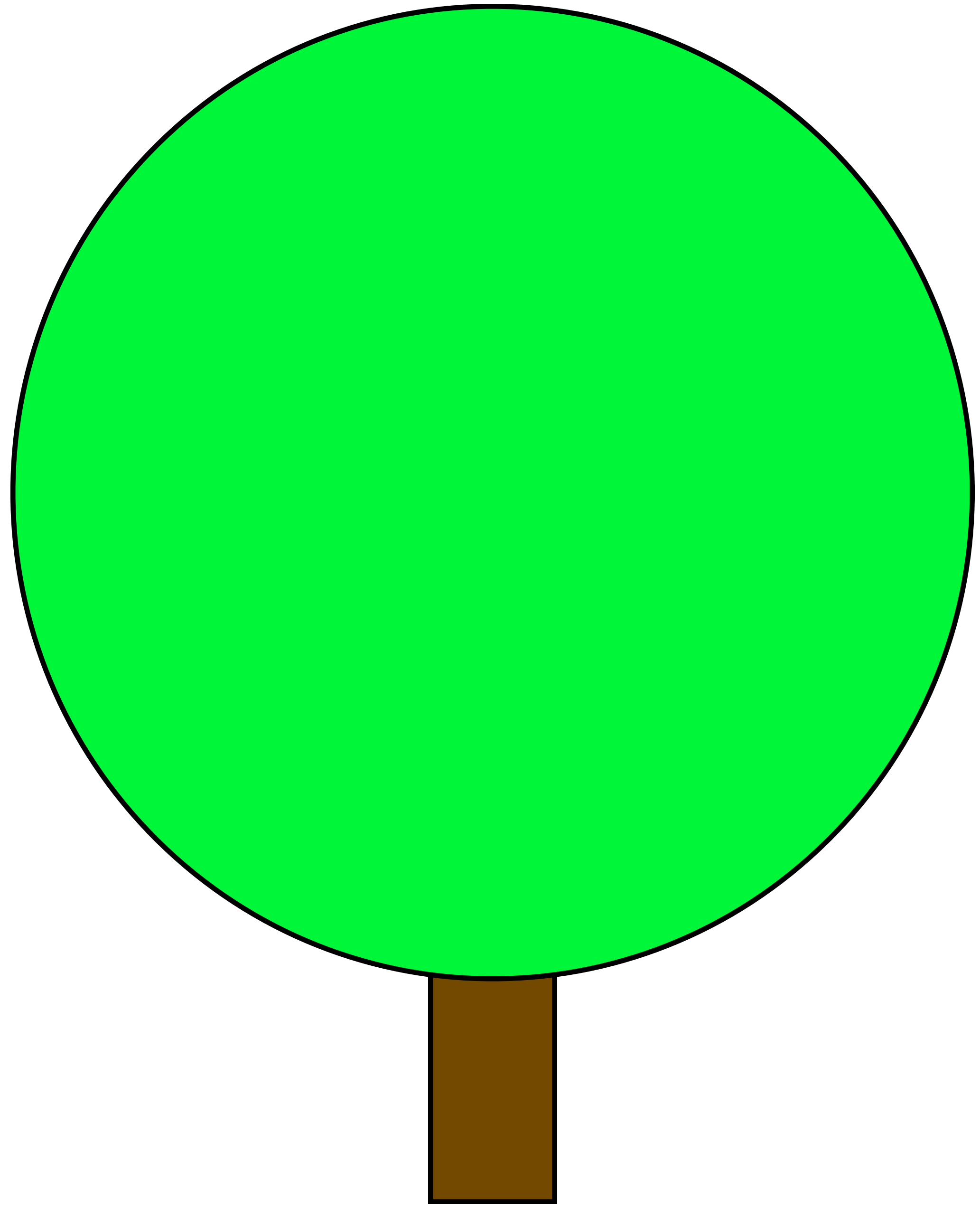
Geometric Modelling

- We need to describe the world
- Most important:
  - where an object is
  - what it is
  - how it behaves

A tree

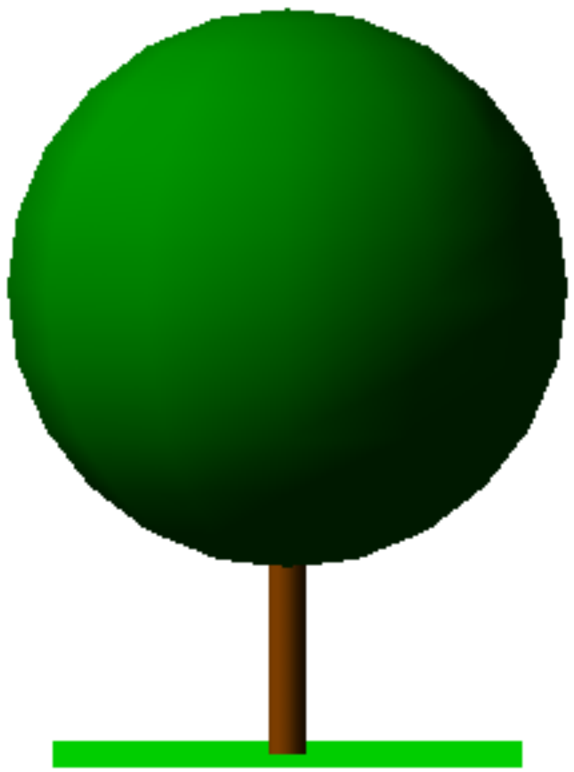
How can we describe this object?

Drawing a Tree



Draw a circle on top of a rectangle

Drawing a 3D Tree



Draw a sphere on top of a cylinder

Complex Geometry

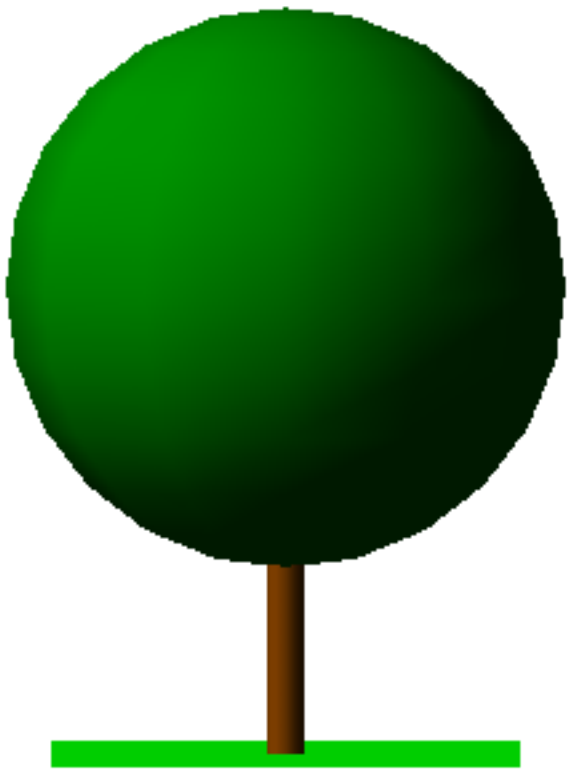
- This isn't a very good tree
- It's too simple geometrically
- A real tree has ?100,000? leaves
- But for now, let's pretend it's OK

Geometric Description

- How to describe a large complex object:
  - break into smaller *primitives*
    - spheres, cylinders, boxes
    - polygons, polyhedra, prisms
  - render each primitive separately

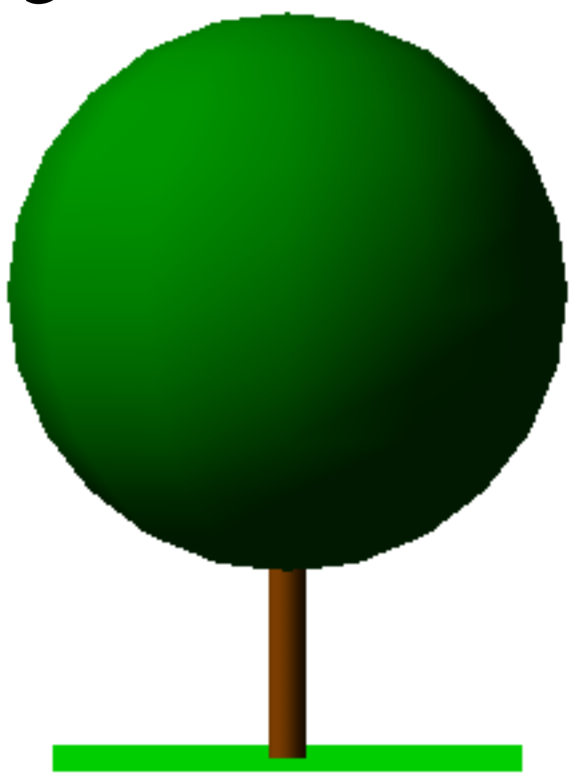
Description

- Draw a sphere
  - radius 10 m
  - centred 7 m above ground
  - colour light green
- Draw a cylinder
  - radius 2 m, height 15 m
  - bottom face on ground
  - colour brown

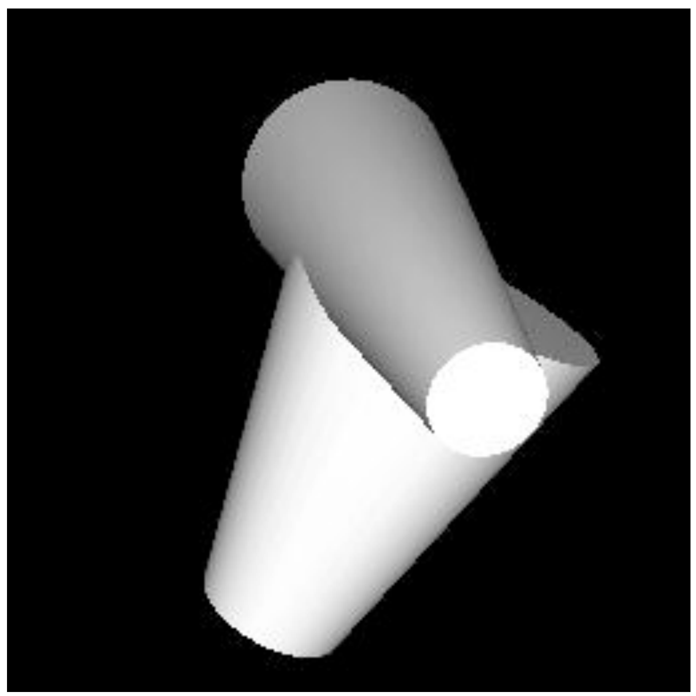


CSG: Constructive Solid Geometry

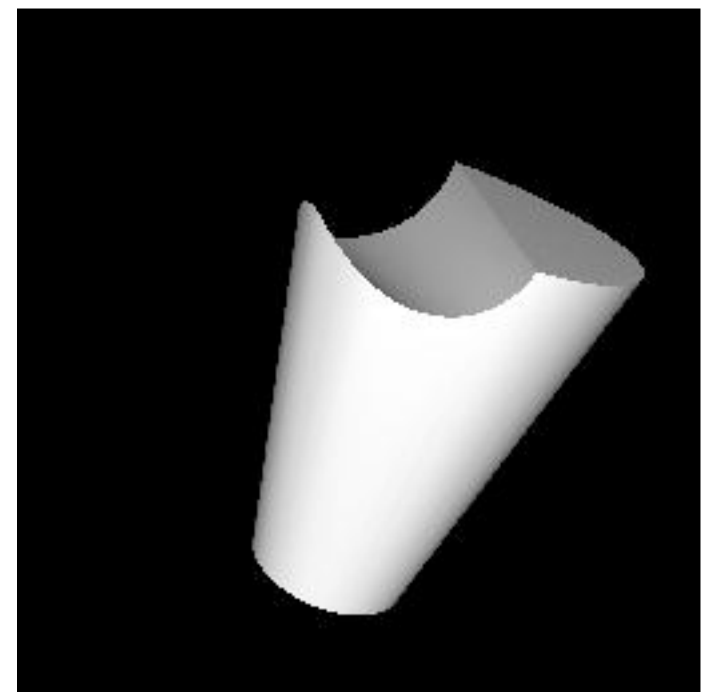
- Tree is built of smaller objects
- Formally, the *union* of them
- Can also do things like:
  - *subtract* objects
  - *xor* objects
  - But not with OpenGL



CSG Example



Union



Subtraction

http://www.cs.unc.edu/~geom/CSG/boole.html

Building Objects

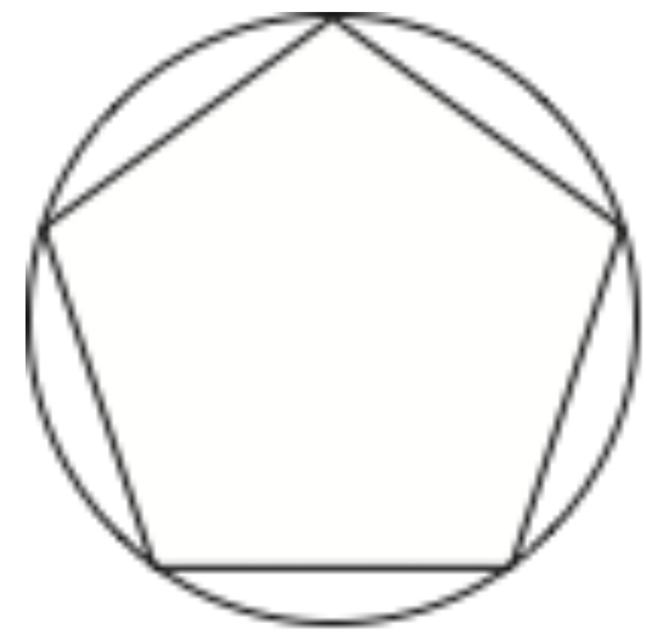
- Objects are *built up* from primitives
  - points, lines, triangles
- We will build the following:
  - *Platonic* solids
  - *Solids* of revolution

Polygons

- A *polygon* is a 2-D shape
  - triangle (3 sides)
  - square / rectangle / quadrilateral (4)
  - pentagon (5)
  - hexagon (6)
  - circle (∞)

Regular Pentagon

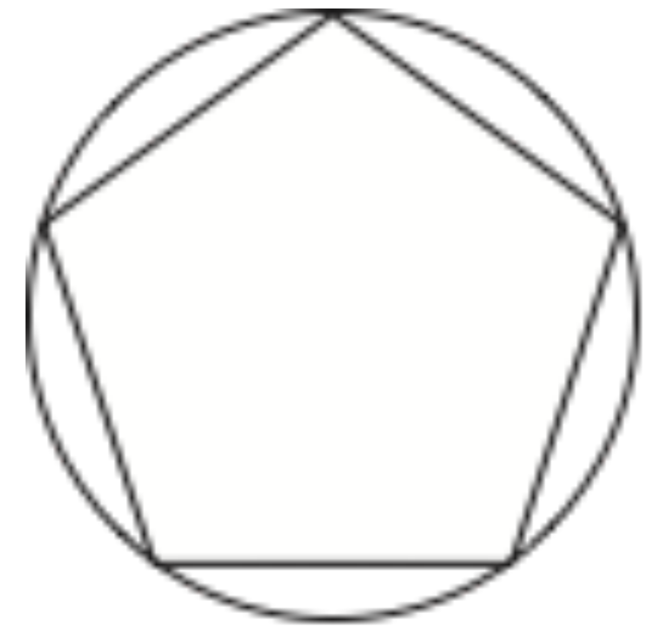
- 5 vertices on a circle
  - spaced 360/5 = 72° apart
- Edges (lines) between them



```
GL11.glBegin(GL_LINES);
for (vertex = 0; vertex < 5; vertex++)
  { // vertex loop
  theta1 = vertex * 72 / 360 * 2 * PI; // in radians
  theta2 = (vertex + 1) * 72 / 360 * 2 * PI;
  GL11.glVertex3f(sin(theta1), cos(theta1),0.0);
  GL11.glVertex3f(sin(theta2), cos(theta2),0.0);
  } // vertex loop
GL11.glEnd();
```

Regular n-gon

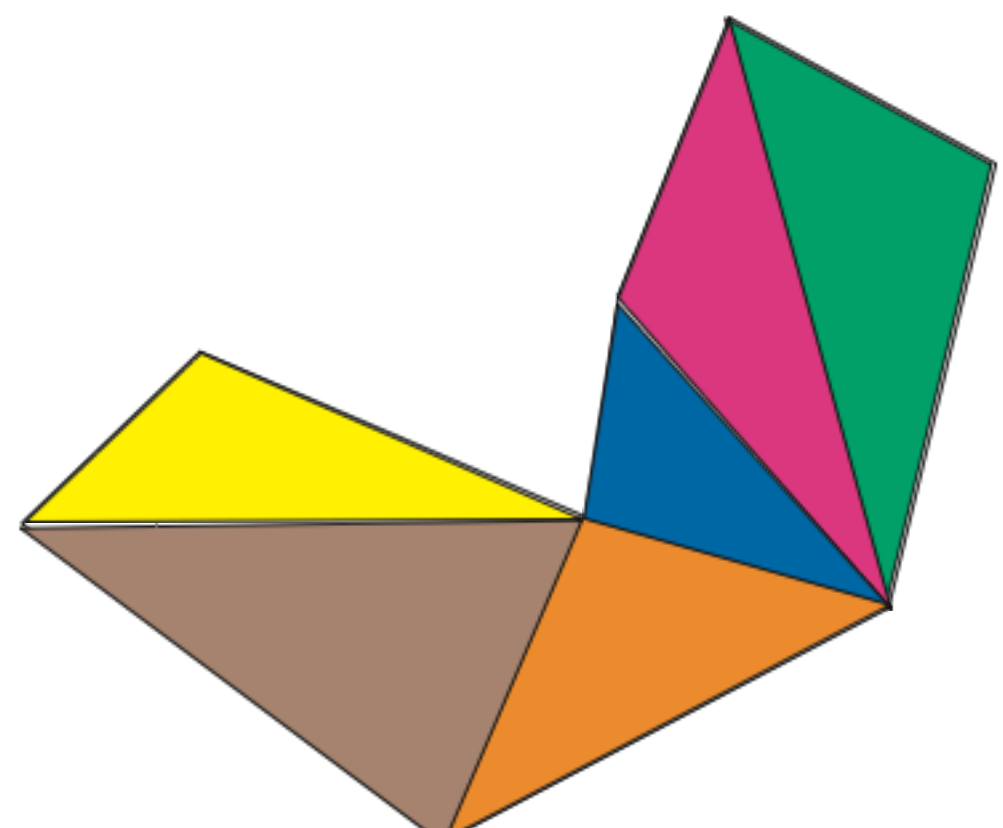
- n vertices on a circle
  - spaced $360/n = 360°/\text{n}$ apart
- Edges (lines) between them



```
GL11.glBegin(GL_LINES);
for (vertex = 0; vertex < n; vertex++)
  { // vertex loop
  theta1 = vertex * 2 * PI / n; // in radians
  theta2 = (vertex + 1) * 2 * PI / n;
  GL11.glVertex3f(sin(theta1), cos(theta1),0.0);
  GL11.glVertex3f(sin(theta2), cos(theta2),0.0);
  } // vertex loop
GL11.glEnd();
```

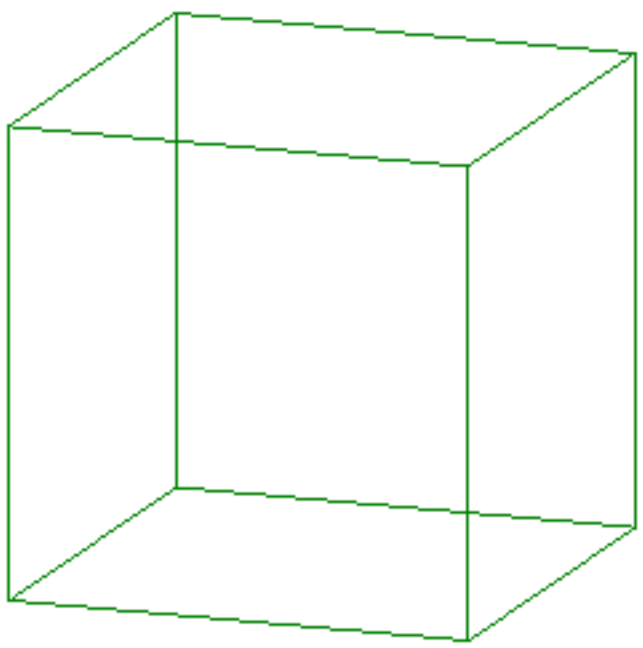
Filling Polygons

- We know how to *rasterize* triangles
  - i.e. how to draw *filled* triangles
- Any polygon can be turned into triangles
  - by cutting vertices off



Polyhedra

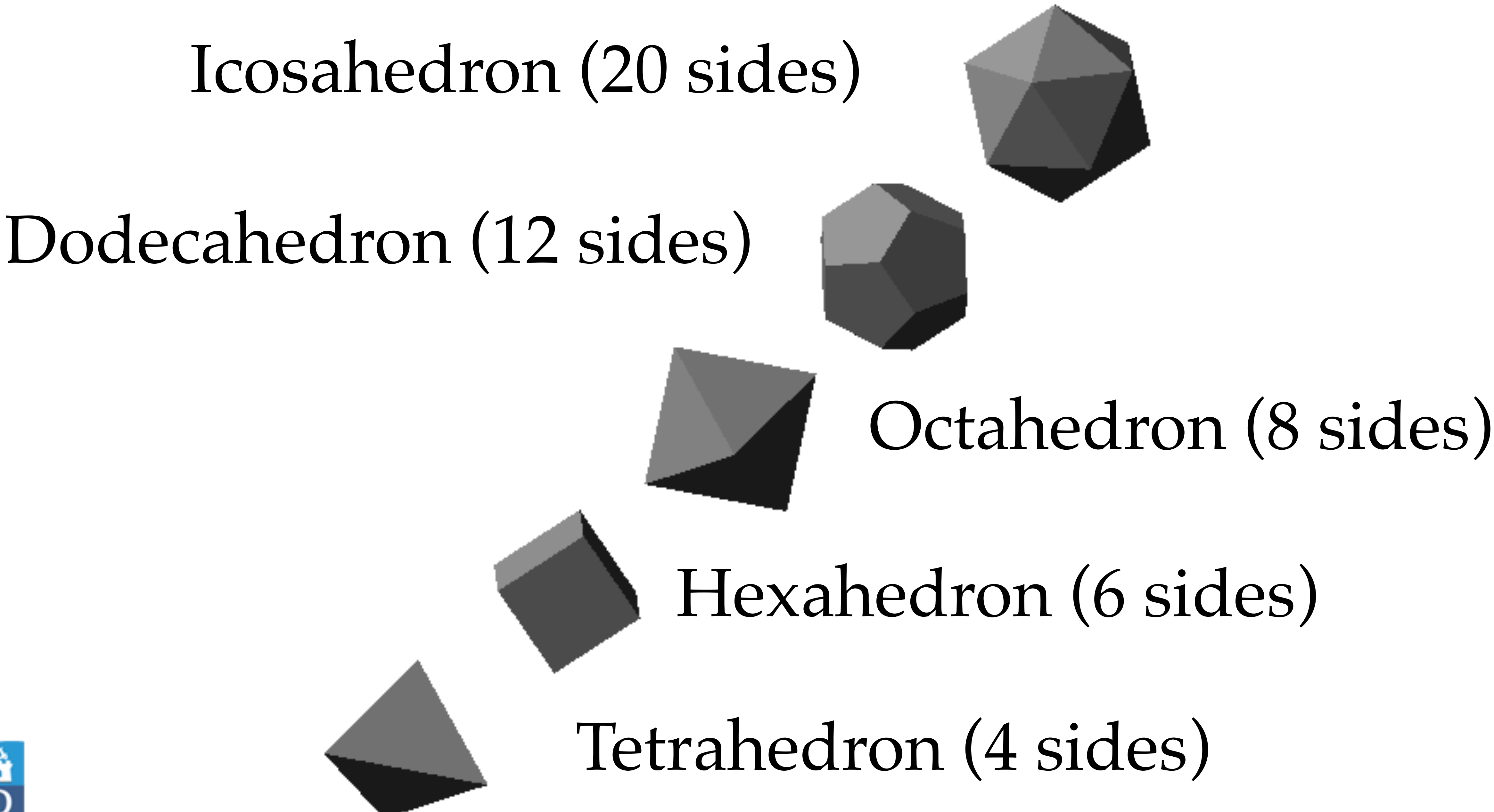
- A *polyhedron* is a 3-D shape
  - of *vertices*
  - connected with *edges*
- Can be rendered as lines
  - this is called *wireframe*



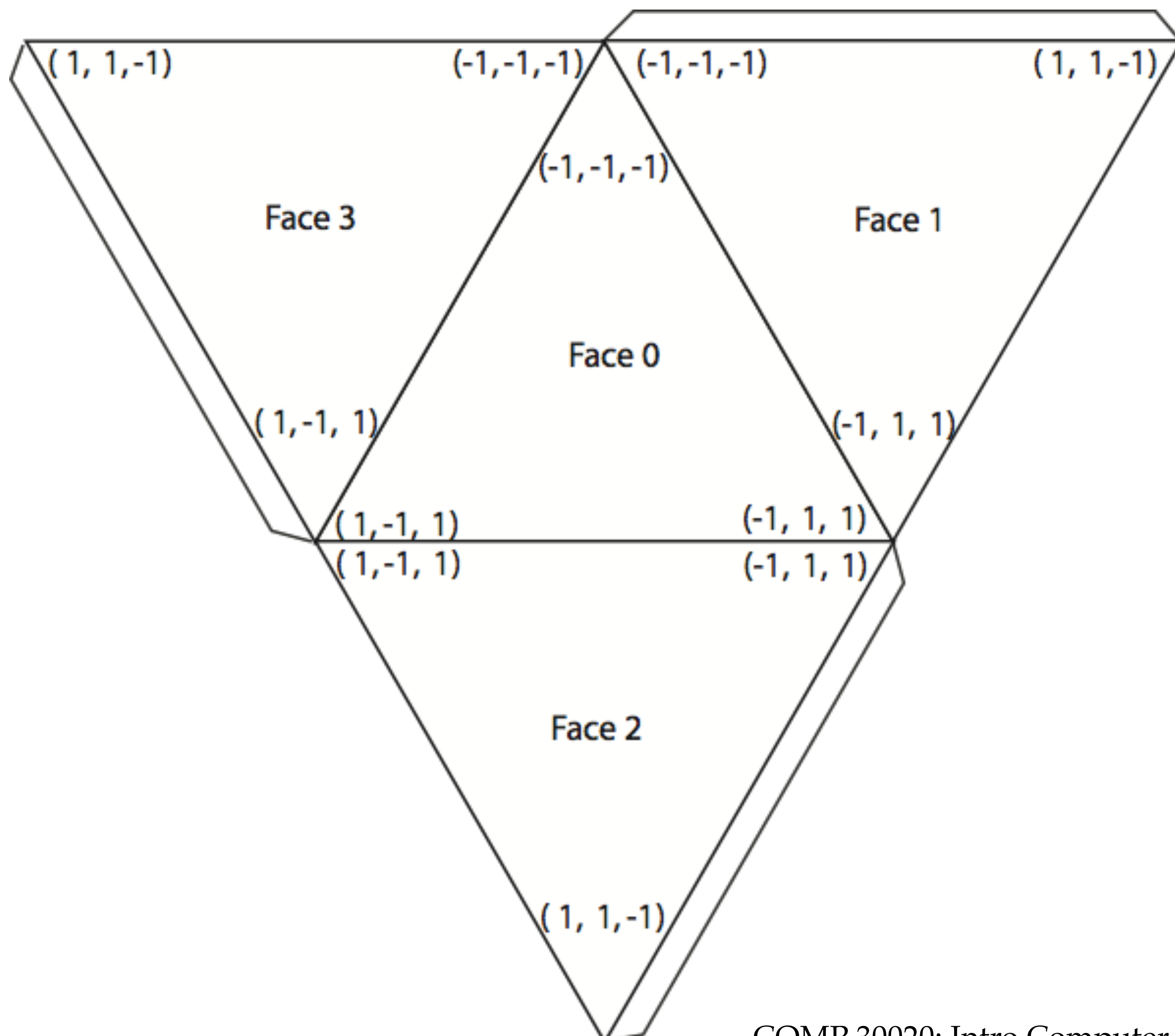
Platonic Solids

- Regular *convex* polyhedra
  - All faces are same size and shape
  - Each face is a regular polygon
  - Each edge is same length
- Convex = no indentations (All angles <180)

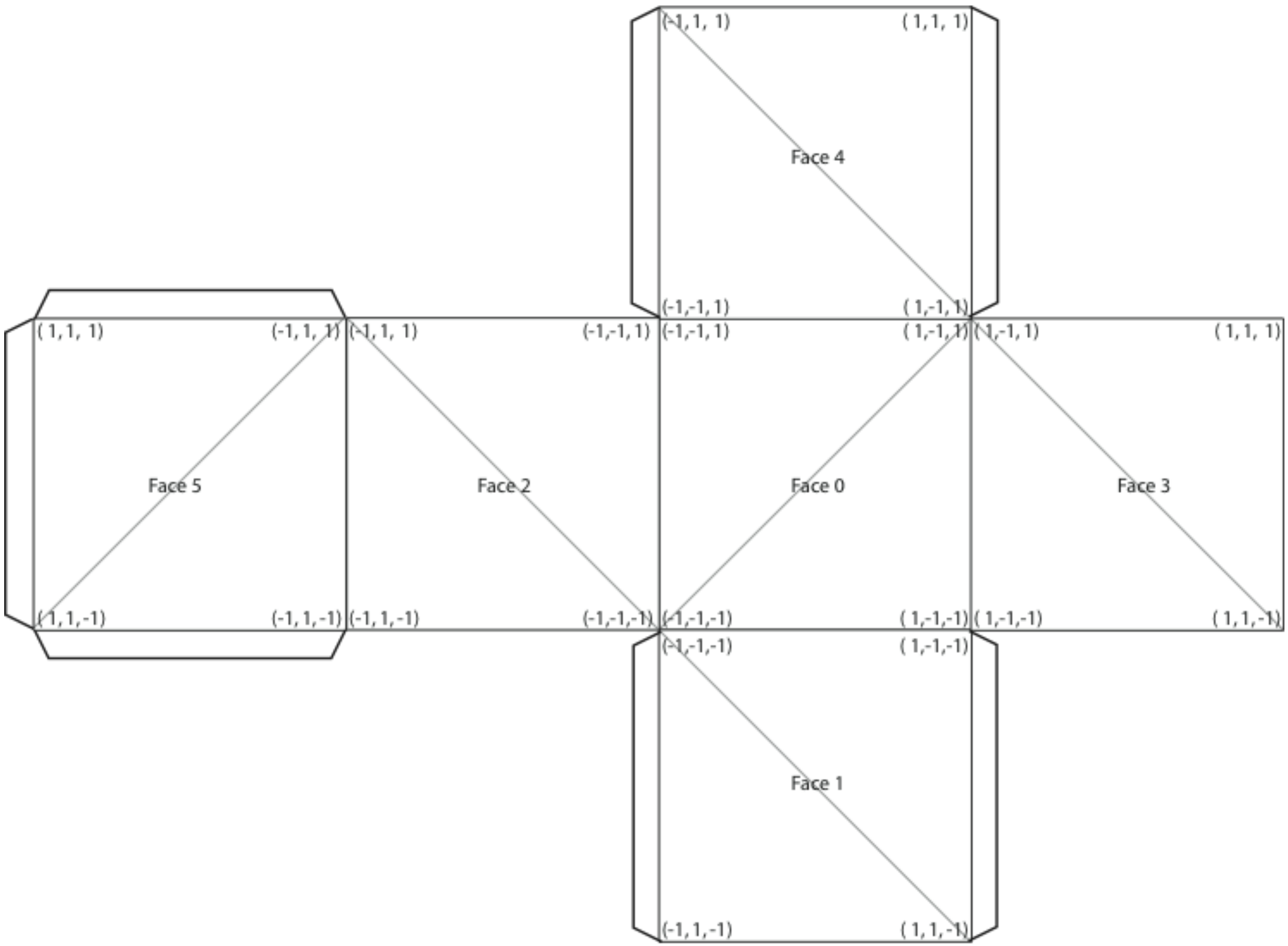
5 Platonic Solids



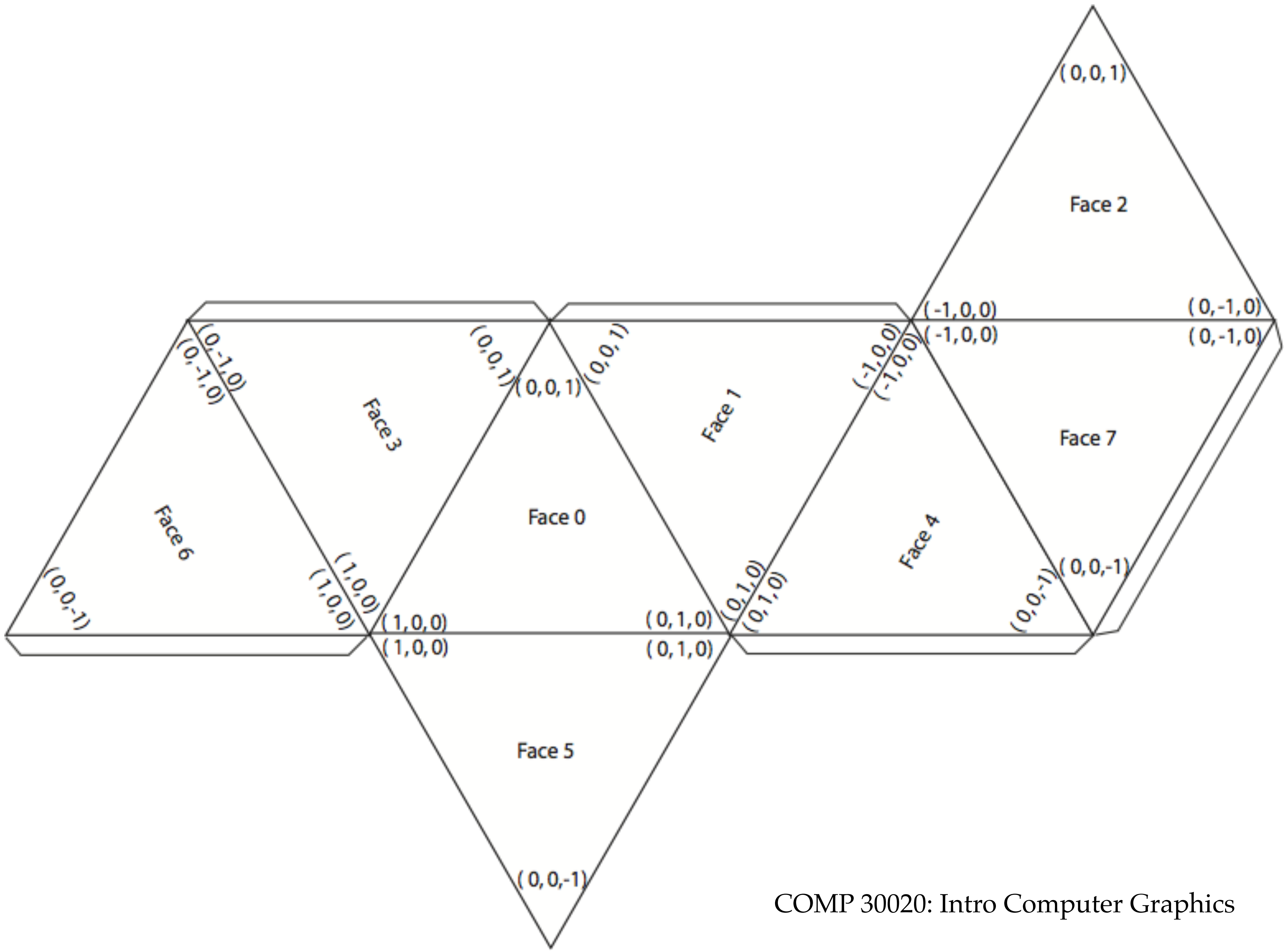
Tetrahedron



Hexahedron (Cube)

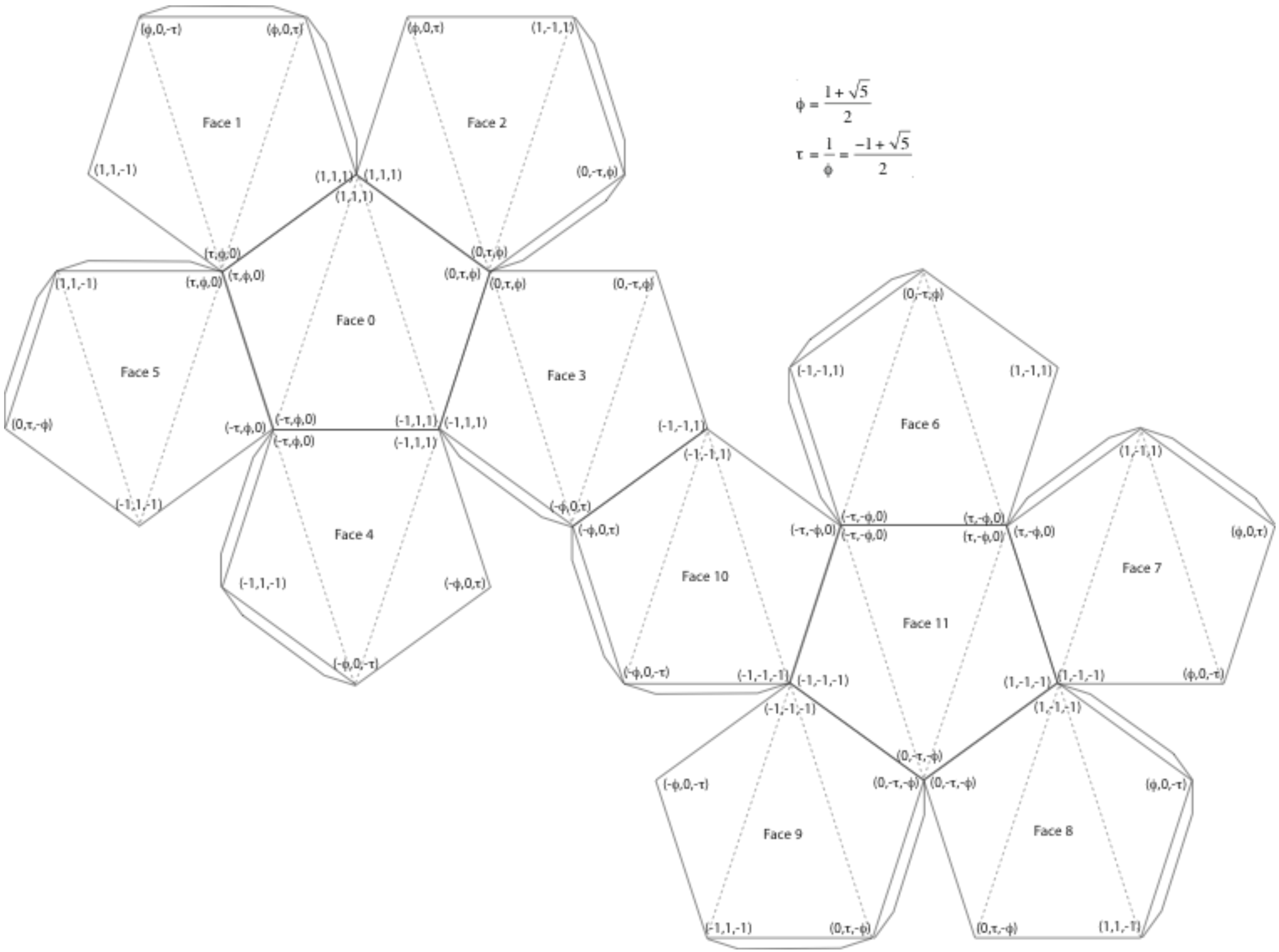


Octahedron



(0,0,1)
Face 2
(-1,0,0)
(0,-1,0)
(-1,0,0)
(0,-1,0)
Face 7
(0,0,-1)
(0,-1,0)
(0,-1,0)
Face 3
(0,0,1)
(0,0,1)
(0,0,1)
Face 1
(-1,0,0)
(-1,0,0)
Face 6
Face 0
Face 4
(0,0,-1)
(1,0,0)
(1,0,0)
(1,0,0)
(0,1,0)
(0,1,0)
(0,1,0)
(0,0,-1)
(1,0,0)
(0,1,0)
Face 5
(0,0,-1)

Dodecahedron



$$\phi = \frac{1+\sqrt{5}}{2}$$

$$\tau = \frac{1}{\phi} = \frac{-1+\sqrt{5}}{2}$$

Icosahedron

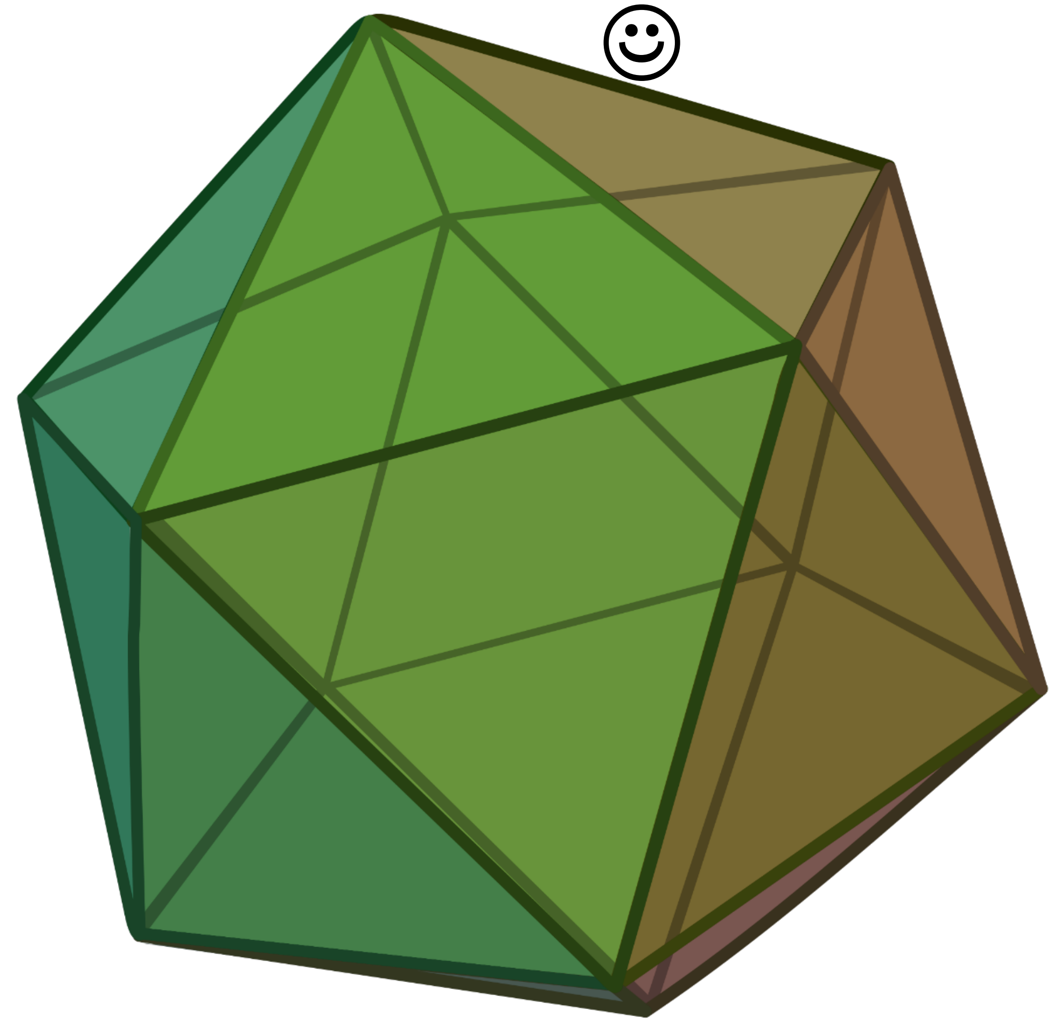
float X = 0.525731112119133606f;

float Z = 0.850650808352039932f;

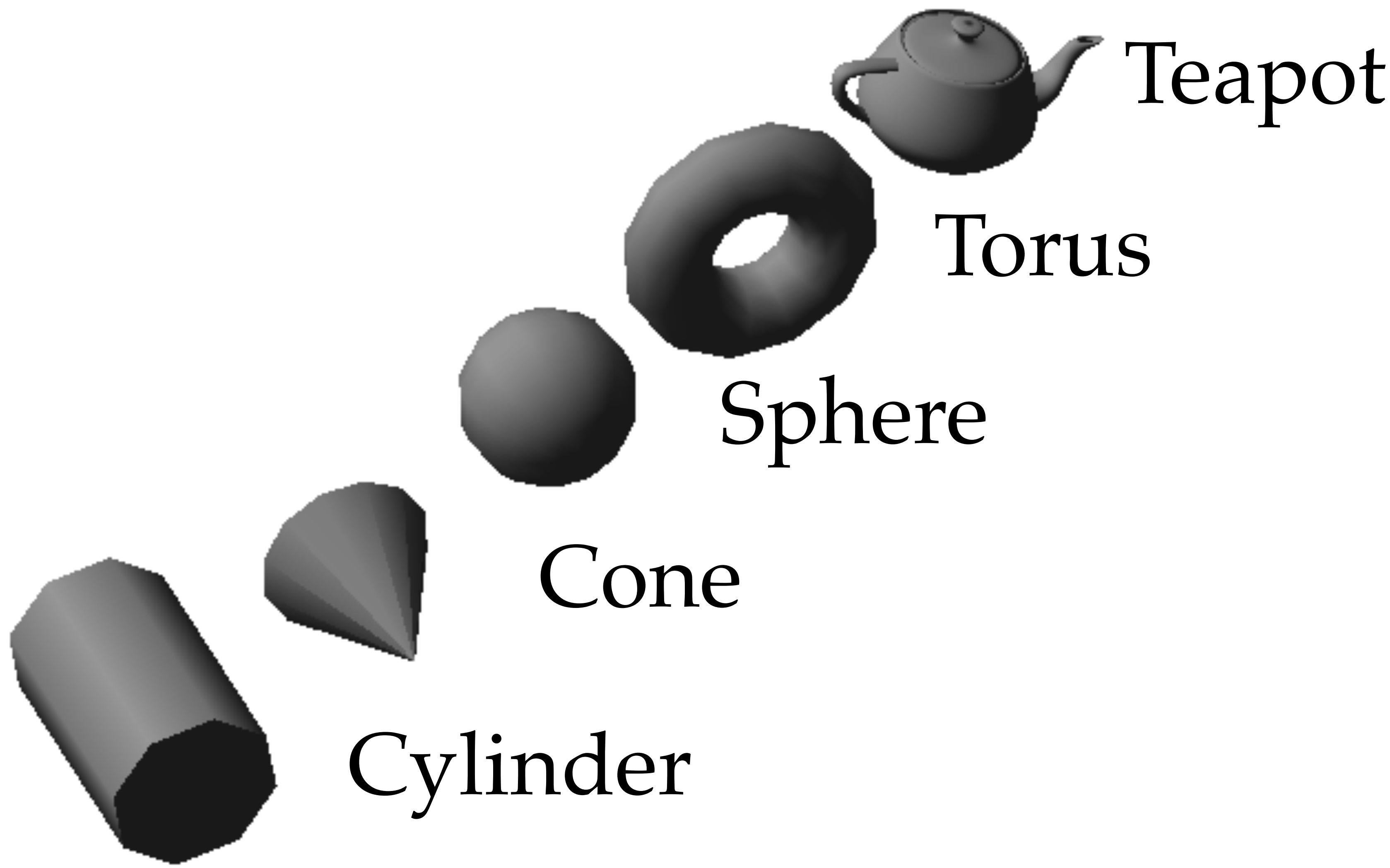
```
Point4f vertices[] = {    (-X, 0.0f, Z ,0.0f) ,     (X, 0.0f, Z,0.0f),
          (-X, 0.0f, -Z,0.0f),     (X, 0.0f, -Z,0.0f),
          (0.0f, Z, X,0.0f),    (0.0f, Z, -X,0.0f),
          (0.0f, -Z, X,0.0f),    (0.0f, -Z, -X,0.0f),
          (Z, X, 0.0f,0.0f),    (-Z, X, 0.0f,0.0f),
          (Z, -X, 0.0f,0.0f),     (-Z, -X, 0.0f,0.0f)
                        };
```

```
int faces[][] = {  {0,4,1},   {0,9,4},
          {9,5,4},  {4,5,8},
          {4,8,1}, {8,10,1},
          {8,3,10},{5,3,8},
          {5,2,3}, {2,7,3},
          {7,10,3}, {7,6,10},
          {7,11,6}, {11,0,6},
          {0,1,6},{6,1,10},
          {9,0,11}, {9,11,2},
          {9,2,5}, {7,2,11}};
```

Gets to difficult to make a paper model for a D20 ☺

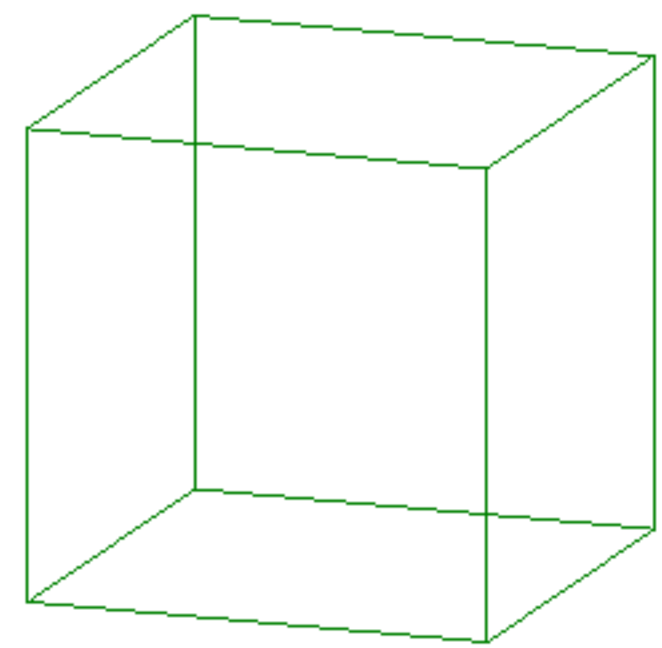


Some Round Objects



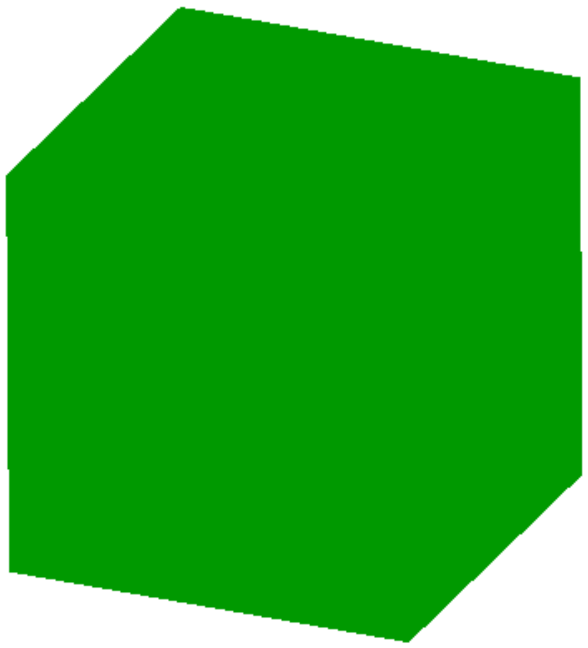
Example: A Cube

```
(-1.0, -1.0, -1.0); (-1.0, -1.0,  1.0);
(-1.0, -1.0,  1.0); (-1.0,  1.0,  1.0);
(-1.0,  1.0,  1.0); (-1.0,  1.0, -1.0);
(-1.0,  1.0, -1.0); (-1.0, -1.0, -1.0);

(-1.0, -1.0, -1.0); ( 1.0, -1.0, -1.0);
(-1.0, -1.0,  1.0); ( 1.0, -1.0,  1.0);
(-1.0,  1.0,  1.0); ( 1.0,  1.0,  1.0);
(-1.0,  1.0, -1.0); ( 1.0,  1.0, -1.0);

( 1.0, -1.0, -1.0); ( 1.0, -1.0,  1.0);
( 1.0, -1.0,  1.0); ( 1.0,  1.0,  1.0);
( 1.0,  1.0,  1.0); ( 1.0,  1.0, -1.0);
( 1.0,  1.0, -1.0); ( 1.0, -1.0, -1.0);
```



Solid Polyhedra

- Polyhedra also have *surfaces*
  - each *face* is a polygon
  - broken into triangles
  - specified in CCW order

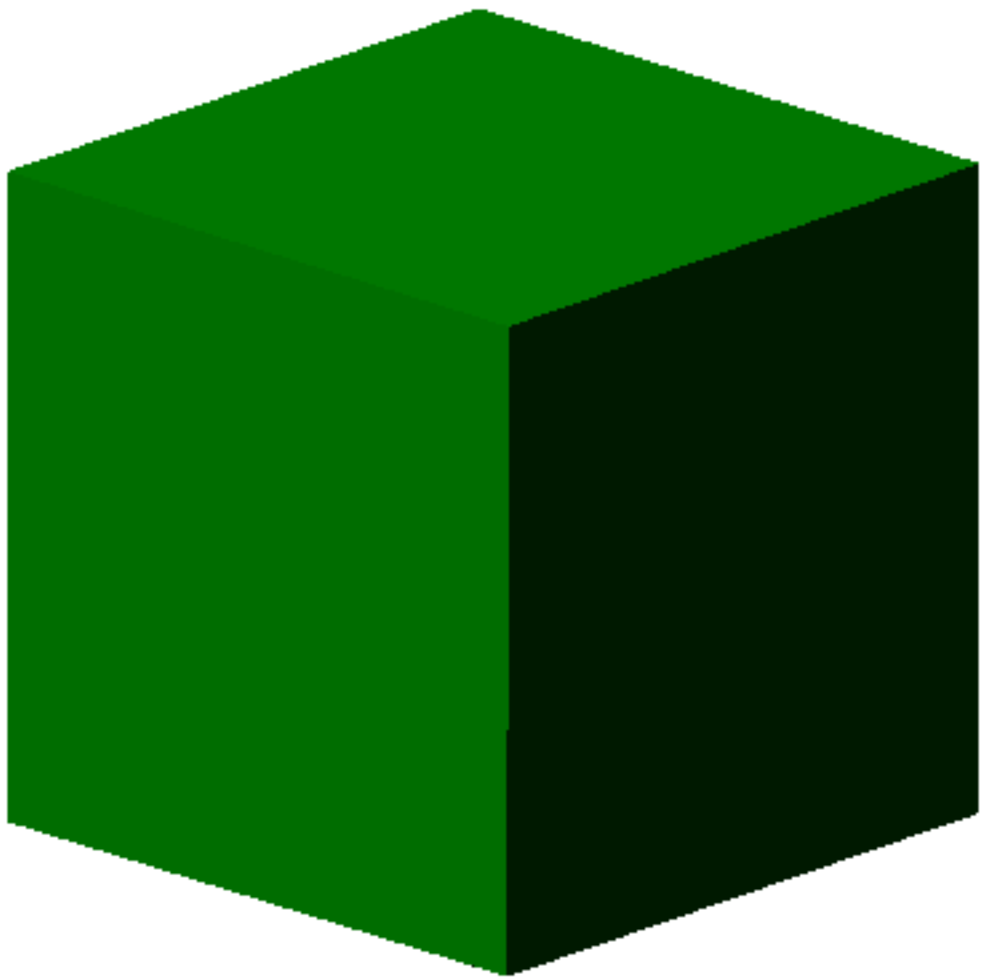


Example: A Cube

```
int vertices[8][3] =
  {-1, -1, -1,   -1, -1,  1,   -1,  1,  -1,   -1,  1,  1,
    1, -1, -1,    1, -1,  1,    1,  1,  -1,    1,  1,  1};

int triangles[12][3] = {
   0,  1,  3,                         0,  3,  2,
   0,  2,  4,                         2,  6,  4,
   0,  1,  4,                         1,  4,  5,
   1,  5,  7,                         1,  7,  3,
   5,  4,  6,                         5,  6,  7,
   2,  3,  7,                         2,  7,  6
   };
```

Shaded Polyhedra

- Each *face* lies on a plane
    - we can compute a *normal*
    - for reflecting light



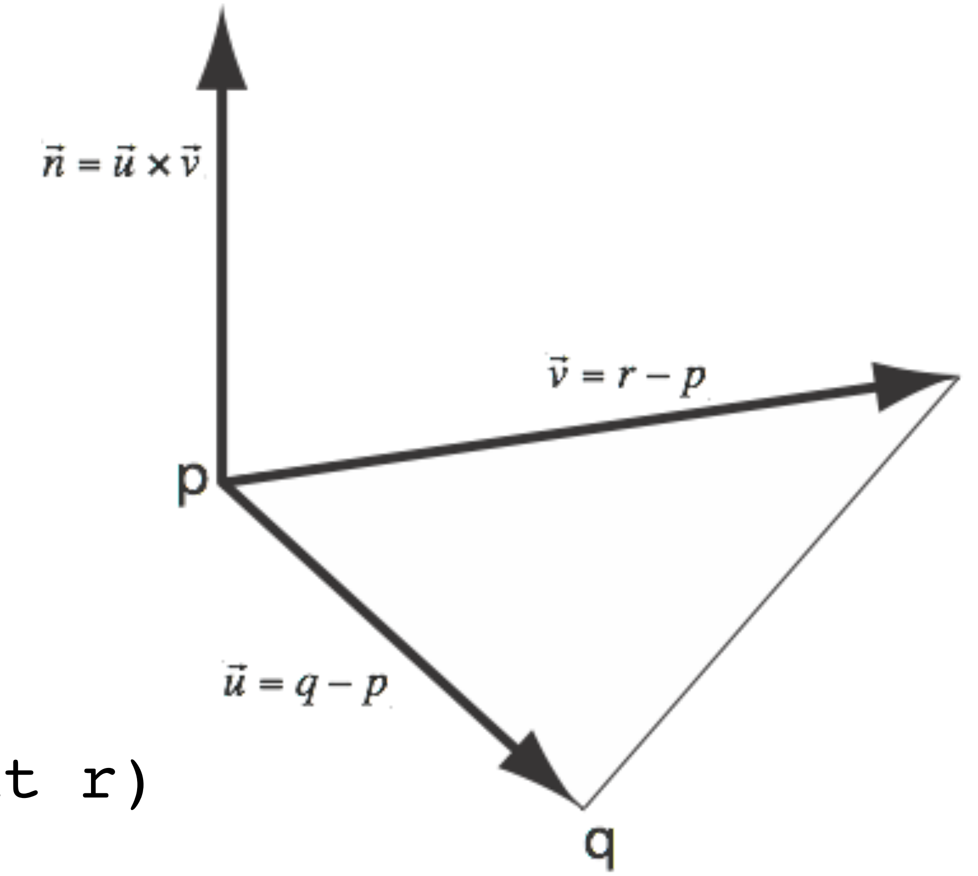
Normal for a Triangle

For each triangle pqr,

let $\vec{u} = q - p$ and $\vec{v} = r - p$

Then

$\vec{n} = \vec{u} \times \vec{v}$ is normal to pqr




```
Vector Normal(Point p, Point q, Point r)
  { // Normal()
  Vector u = q - p;
  Vector v = r - p;
  Vector n = u.Cross(v);
  // optional - make vector unit length
  n = n / n.Length();

  return n;
  } // Normal()
```

Example: A Cube

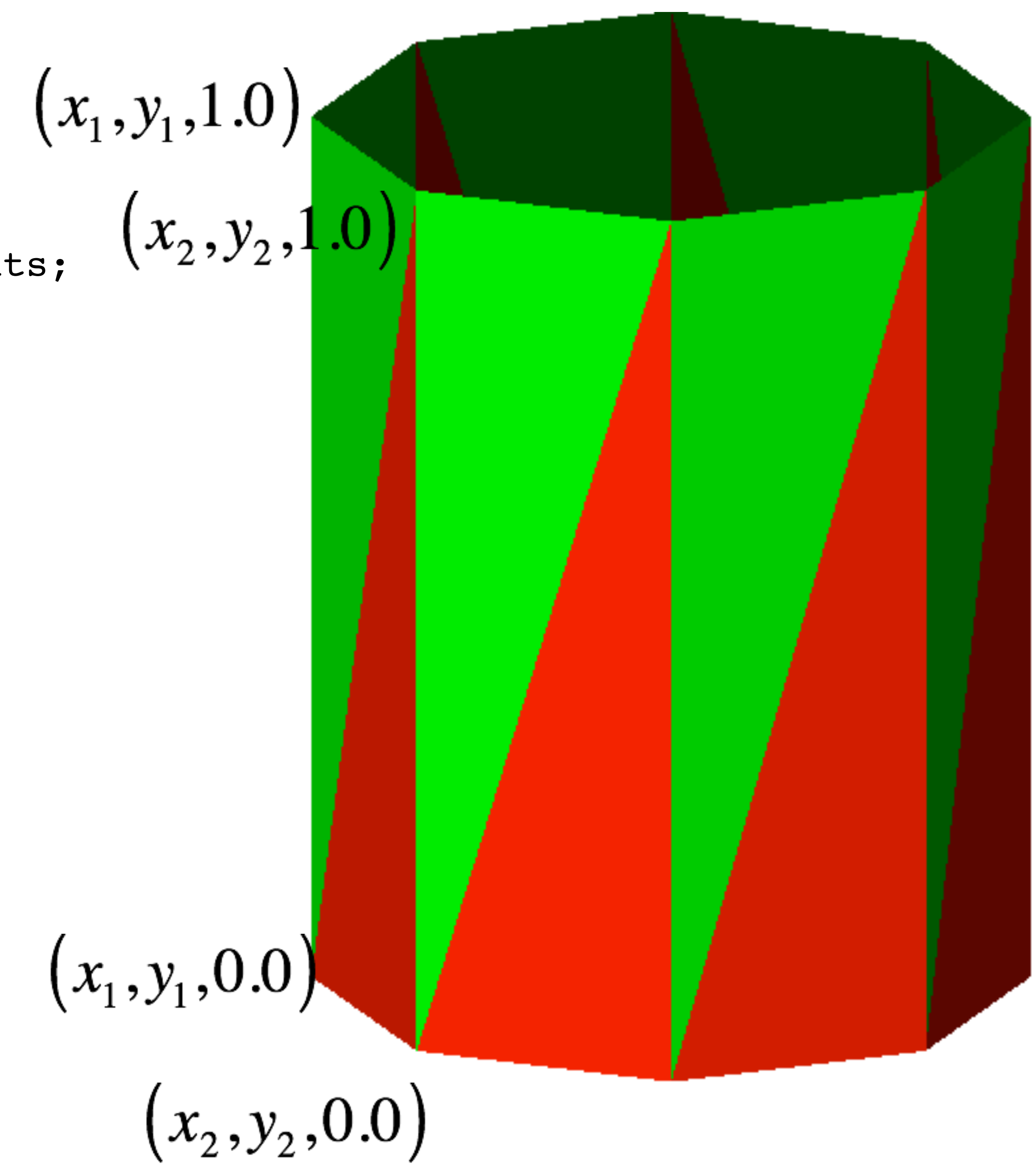
```
int vertices[8][3] =
  {-1, -1, -1,   -1, -1,  1,   -1,  1,  -1,   -1,  1,  1,
    1, -1, -1,    1, -1,  1,    1,  1,  -1,    1,  1,  1};

int triangles[12][3] = {
       0,  1,  3,                         0,  3,  2,
       0,  1,  4,                         1,  4,  5,
       0,  2,  4,                         2,  6,  4,
       5,  4,  6,                         5,  6,  7,
       1,  5,  7,                         1,  7,  3,
       2,  3,  7,                         2,  7,  6
       };

int normals[12][3] = {
      -1,  0,  0,                        -1,  0,  0,
       0, -1,  0,                         0, -1,  0,
       0,  0, -1,                         0,  0, -1,
       1,  0,  0,                         1,  0,  0,
       0,  1,  0,                         0,  1,  0,
       0,  0,  1,                         0,  0,  1
      };
```

Round Objects

- We *approximate* round objects
  - with polyhedra
  - possibly with *lots* of faces
- How can we approximate a cylinder?
  - as an *extruded* polygon
  - turn each edge into a vertical strip

A Vertical Cylinder

```
for (float i = 0.0; i < nSegments; i += 1.0)
  { /* a loop around circumference of a tube */
  float angle = PI * i * 2.0 / nSegments ;
  float nextAngle = PI * (i + 1.0) * 2.0 / nSegments;

  /* compute sin & cosine */
  float x1 = sin(angle), y1 = cos(angle);
  float x2 = sin(angle), y2 = cos(angle);

  /* draw top (green) triangle */
  VertexAt(x1, y1, 0.0);
  VertexAt(x2, y2, 1.0);
  VertexAt(x1, y1, 1.0);

  /* draw bottom (red) triangle */
  VertexAt(x1, y1, 0.0);
  VertexAt(x2, y2, 0.0);
  VertexAt(x2, y2, 1.0);

  } /* a loop around circumference of a tube */
```



Smoother Cylinders

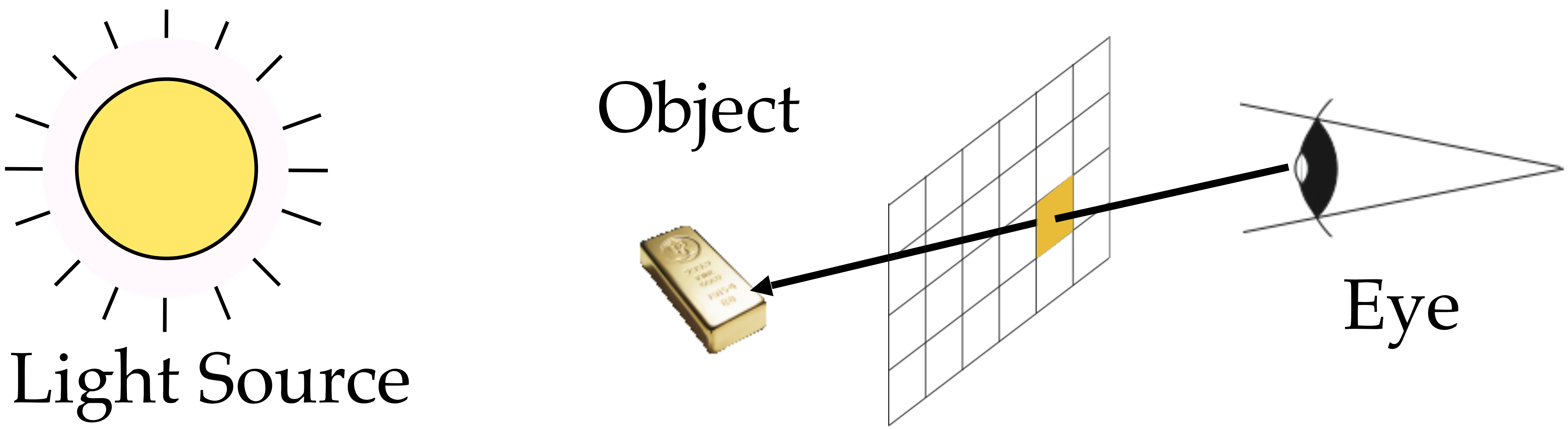
- How can we make the cylinder *smoother?*
  - provide normals for *each* vertex
  - based on the *tangent plane*
- Cylinder is vertical, so normal is horizontal
  - sticking straight out through each vertex

A Smoother Cylinder

```
for (float i = 0.0; i < nSegments; i += 1.0)
  { /* a loop around circumference of a tube */
  float angle = PI * i * 2.0 / nSegments ;
  float nextAngle = PI * (i + 1.0) * 2.0 / nSegments;

  /* compute sin & cosine */
  float x1 = sin(angle), y1 = cos(angle);
  float x2 = sin(nextAngle), y2 = cos(nextAngle);

  /* draw top (green) triangle */
  NormalIs(x1, y1, 0.0); VertexAt(x1, y1, 0.0);
  NormalIs(x2, y2, 0.0); VertexAt(x2, y2, 1.0);
  NormalIs(x1, y1, 0.0); VertexAt(x1, y1, 1.0);

  /* draw bottom (red) triangle */
  NormalIs(x1, y1, 0.0); VertexAt(x1, y1, 0.0);
  NormalIs(x2, y2, 0.0); VertexAt(x2, y2, 0.0);
  NormalIs(x2, y2, 0.0); VertexAt(x2, y2, 1.0);

  } /* a loop around circumference of a tube */
```

Raytracing

- For each pixel
  - Start at eye
  - Trace a ray through image plane
  - Compute colour of object it hits



Rendering Triangles

- Each triangle lies on a plane
- So compute where ray intersects plane
- Find barycentric coordinates at intersection
- Compute shading
- Interpolate texture coordinates

Point-Plane Intersection

Given a triangle Δabc and a line $\overleftrightarrow{l} = q + \vec{w}t$

Let $\vec{u} = b - a$ and let $\vec{v} = c - a$

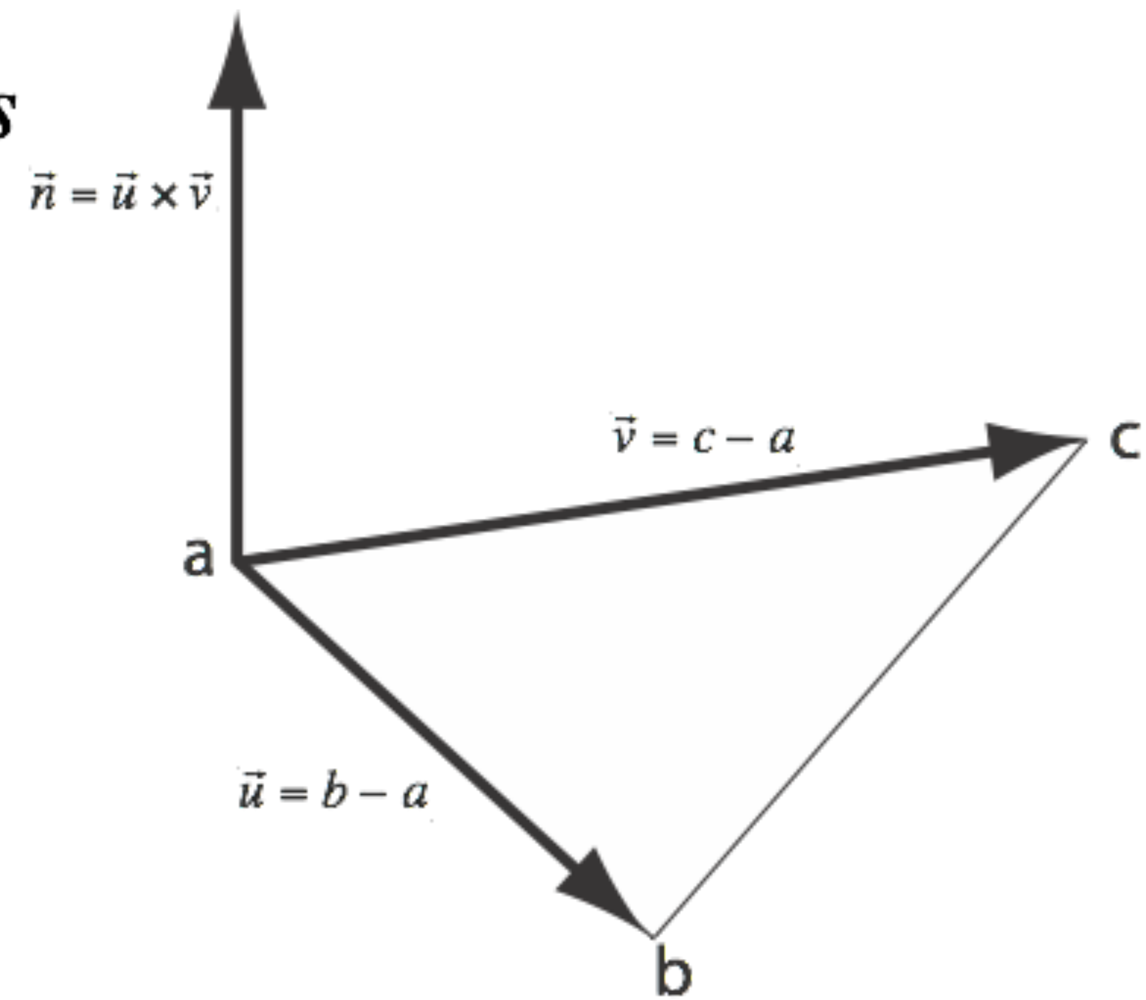
Then Δabc is on plane $\Pi = a + \vec{u}r + \vec{v}s$

Find the intersection of these two:

$q + \vec{w}t = a + \beta\vec{u} + \gamma\vec{v}$

Solve for t, β and γ:

$$\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} + \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} t = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \beta \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} + \gamma \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$



Continued . . .

$$\beta \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} + \gamma \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} t = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} - \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$$

$$\begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} a_x - q_x \\ a_y - q_y \\ a_z - q_z \end{bmatrix}$$

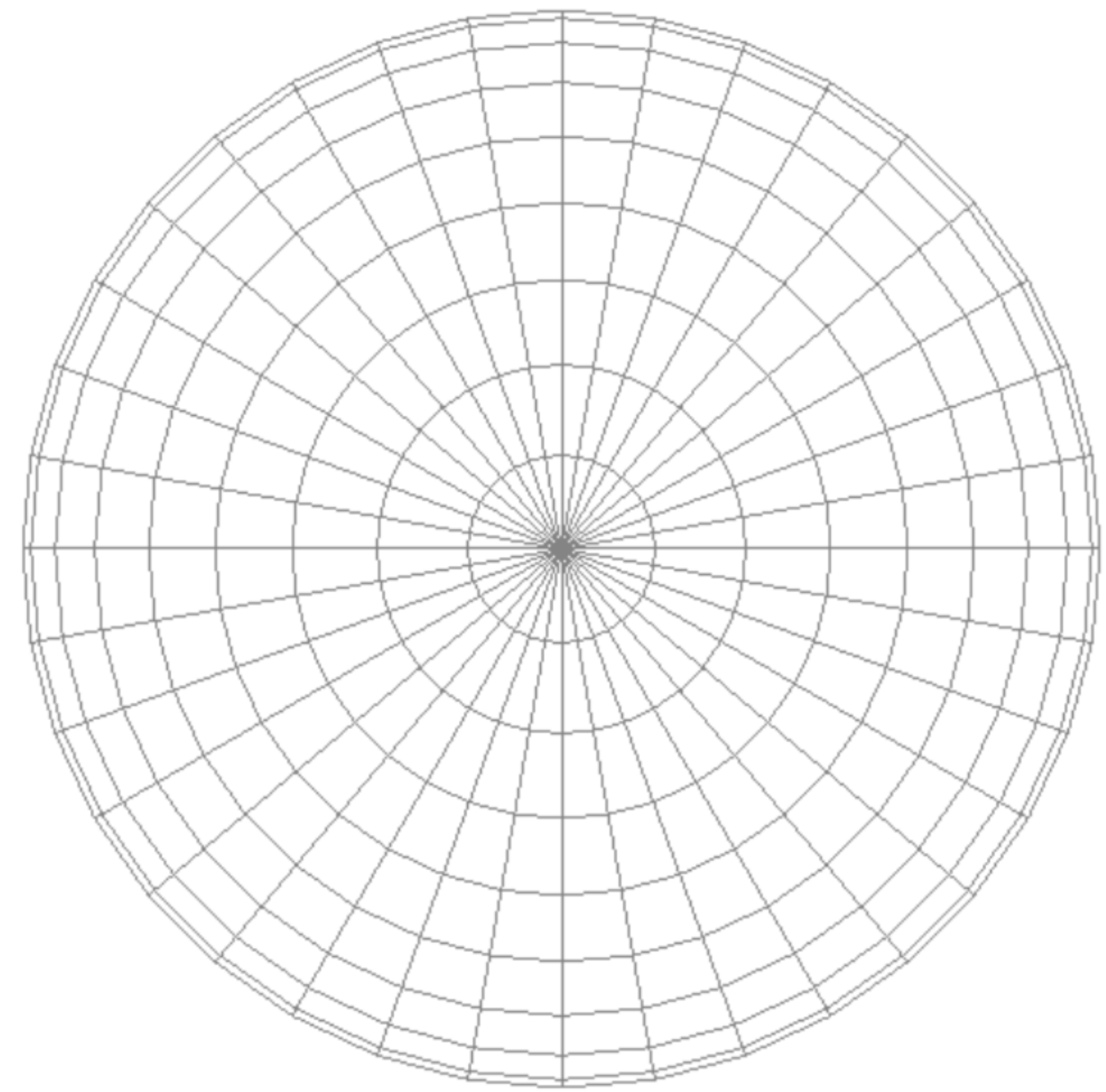
$$\begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}^{-1} \begin{bmatrix} a_x - q_x \\ a_y - q_y \\ a_z - q_z \end{bmatrix}$$

Ugggh!

- This is bad enough
  - it gets *worse* for curved surfaces
  - and it's slow
- We want a simpler method for rendering
  - projective rendering(next weeks topic)

Sphere in Parametric Form

- Lets say we want a perfect sphere or as close as we can get to it



- We will use two parameters:
- • latitude (ϕ)
- • longitude (θ)

Rendering a Sphere

- At a given ϕ, the sphere is just a circle of radius

$$r_\phi = r \cos\phi$$

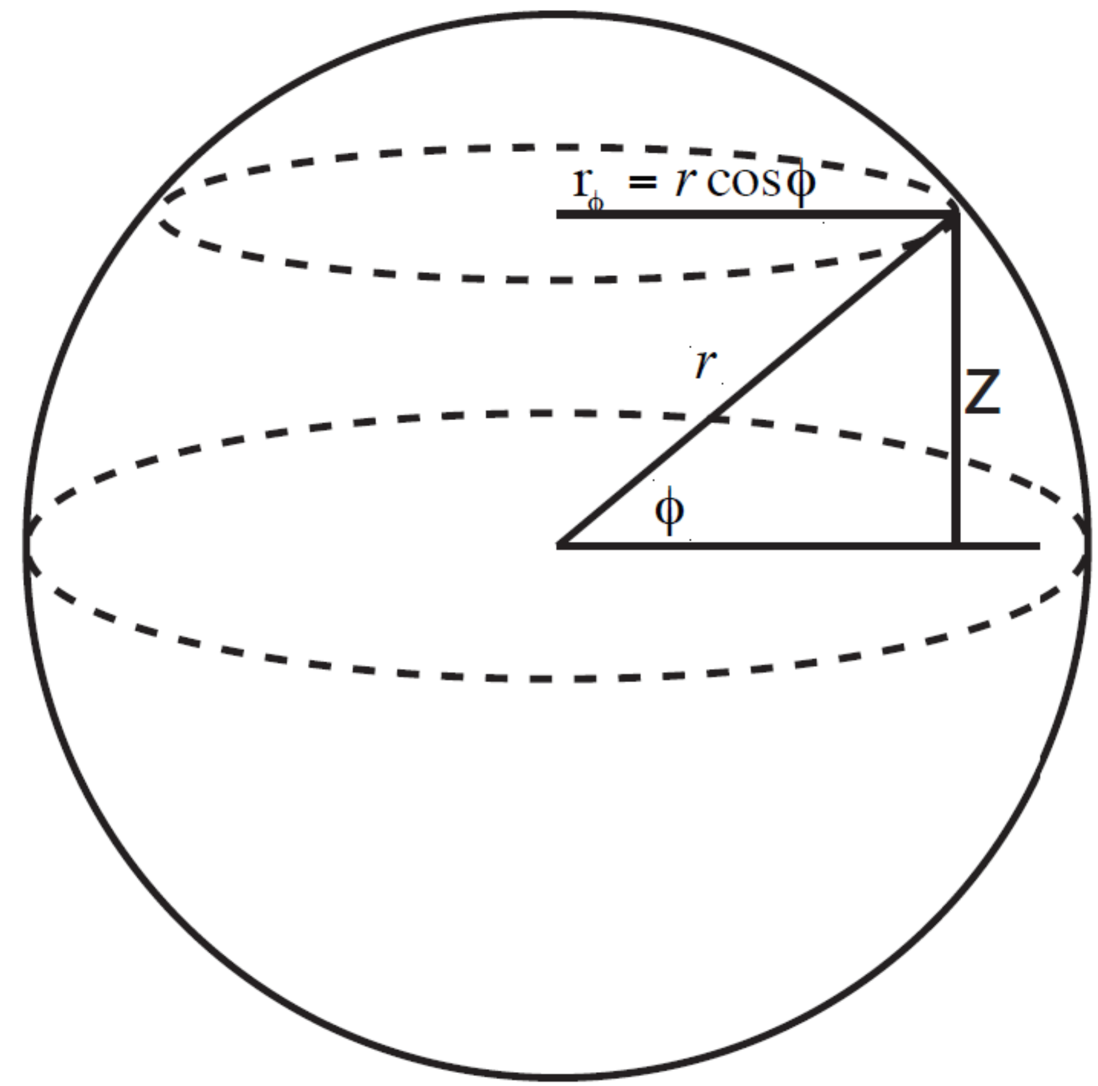
- and the z-value of all points on this circle is

$$z = r \sin\phi$$

- But we know how to find points on a circle, so

$$x = r_\phi \cos\theta = r \cos\phi \cos\theta$$

$$y = r_\phi \sin\theta = r \cos\phi \sin\theta$$



Segments and Slices

float inctheta = (2.0f*pi)/ float(nSlices);

float incphi = pi/ float(nSegments);

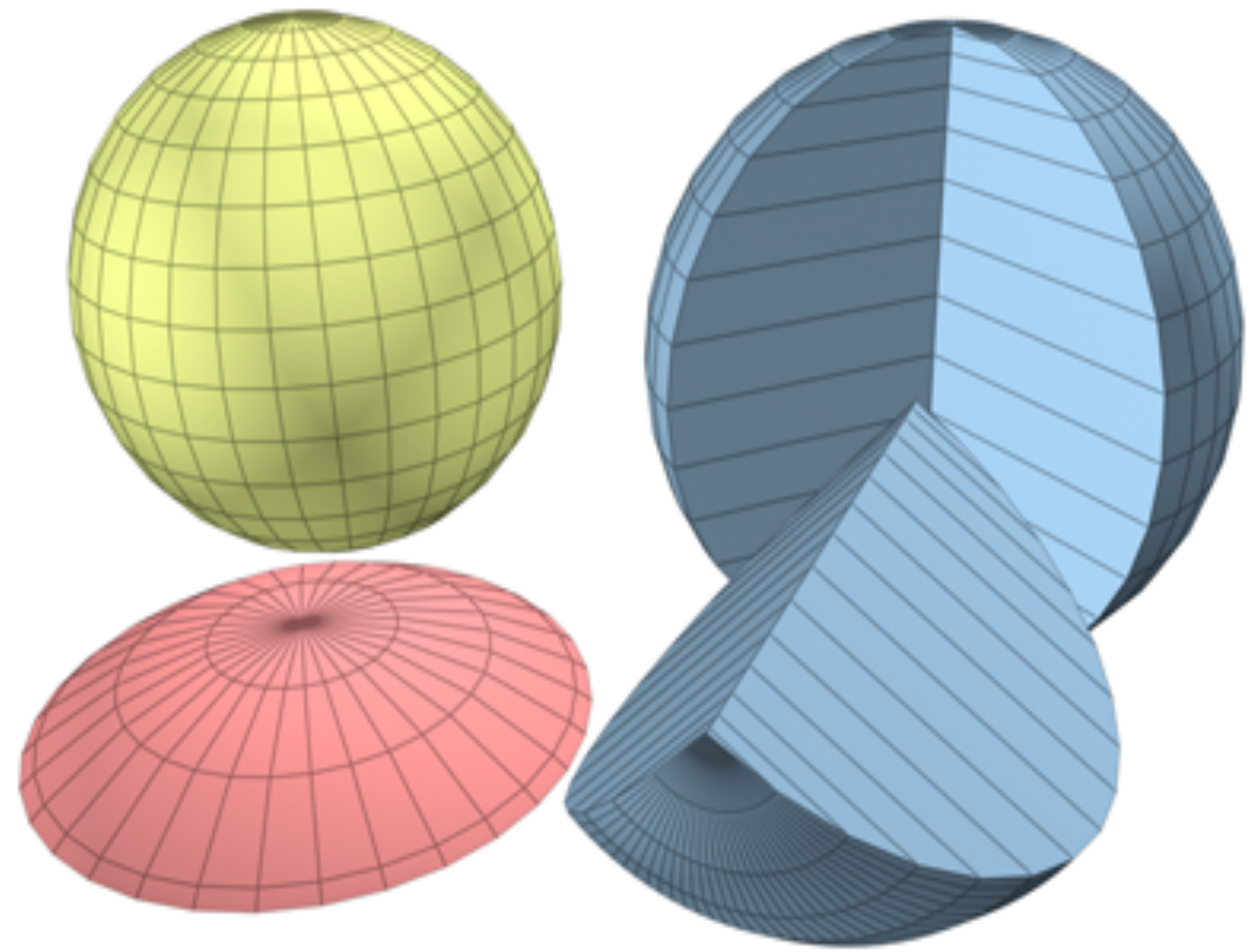
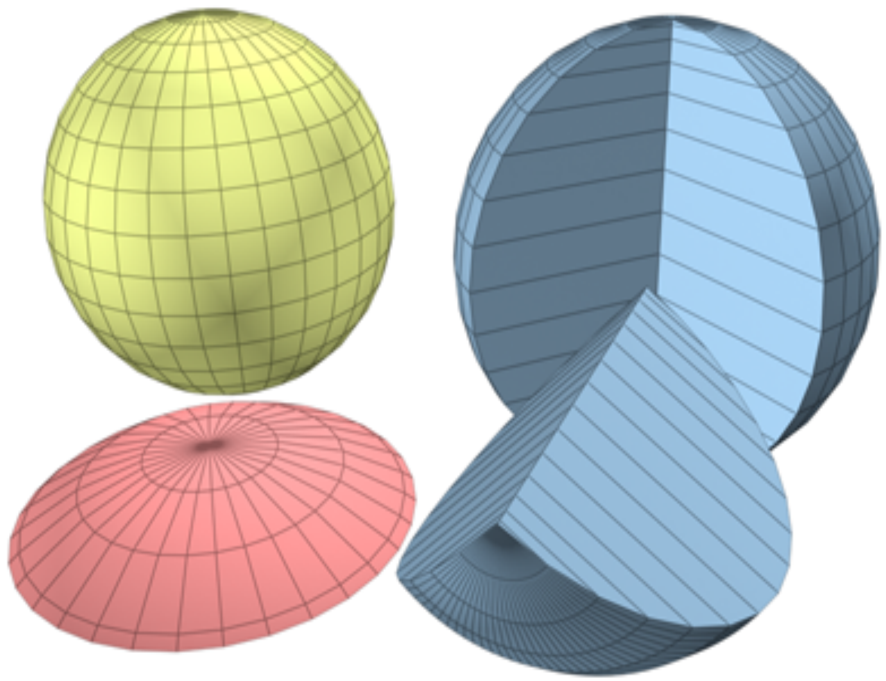


Image from Autodesk

Then we just need to loop



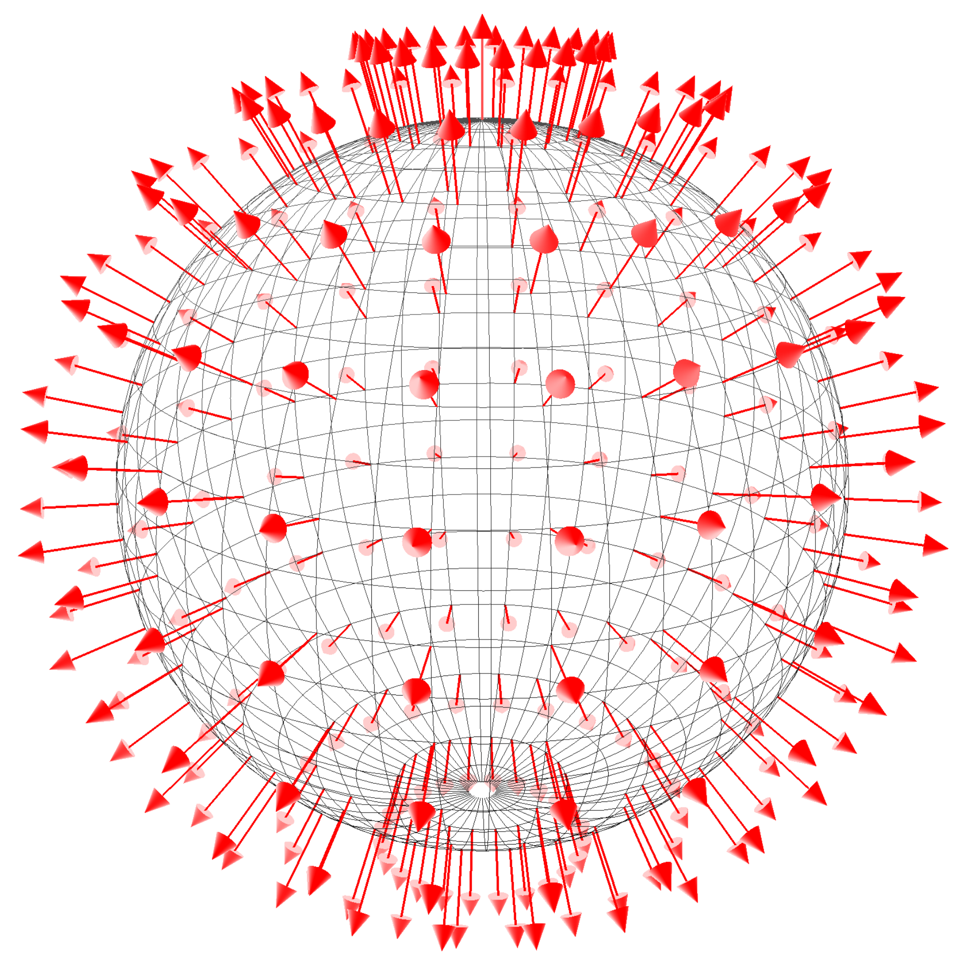
- Using two loops , we need to build up our sphere

```
for(float theta=-pi; theta<pi; theta+=inctheta)
    {
        for(float phi=-(pi/2.0f); phi<(pi/2.0f); phi+=incphi)
            {
                .......
            }
    }
```

What about our Normals?

- Well , if we have our origin at 0 ,0
- Then every point on our sphere is also if changed to a vector is own normal e.g

```
glNormal3f(x,y,z);
glVertex3f(x,y,z);
```



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