Time Series Analysis

Lecture 6

Review

1. The Autocovariance Generating Function
2. Sums of ARMA processes
3. Wold's Decomposition and the Box-Jenkins Modeling Philosophy
4. Maximum Likelihood Estimation

Today's Topics

1. Review of AR(1) model
2. Brownian motion and Functional central limit theorem
3. Asymptotic properties of unit root processes and tests for unit root
4. Generalization to processes with serial correlation

Today's Topics

1. Rewiew of AR(1)

2. Brownian Motion and Functional CLT

3. Asymptotic properties of Unit Root processes and tests for Unit Root

Gaussian AR(1) process

Consider a Gaussian AR(1) process,

$$y_t = \rho y_{t-1} + u_t, \tag{1}$$

where $u_t \sim i.i.d. N(0, \sigma^2)$, and $y_0 = 0$.

- if $|\rho| < 1$, y_t is called a stationary process;
- if $|\rho| = 1$, y_t is called the Unit Root (random walk) process;
- if $|\rho| > 1$, y_t is called an explosive(unstable) process.

The OLS estimate of ρ is given by

$$\hat{\rho}_T = \frac{\sum_{t=1}^{T} y_{t-1} y_t}{\sum_{t=1}^{T} y_{t-1}^2}. \tag{2}$$

Asymptotics for stationary AR(1)

We can show that if $|\rho| < 1$, i.e., y_t is stationary, then

$$\sqrt{T}(\hat{\rho}_T - \rho) \xrightarrow{L} N(0, (1 - \rho^2)). \tag{3}$$

To see this, note that

$$\sqrt{T}(\hat{\rho}_T - \rho) \quad = \quad \frac{\frac{1}{\sqrt{T}} \sum_{t=1}^{T} y_{t-1} u_t}{\frac{1}{T} \sum_{t=1}^{T} y_{t-1}^2}.$$

But by the Ergodic theorem, we have

$$\frac{1}{T} \sum_{t=1}^{T} y_{t-1}^2 \xrightarrow{p} E[y_{t-1}^2] = \sigma^2/(1 - \rho^2).$$

By the CLT for MDS, we obtain

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{L} N(0, Var(y_{t-1} u_t)),$$

where $Var(y_{t-1}u_t) = \sigma^2 E[y_{t-1}^2] = \sigma^4/(1 - \rho^2)$. Therefore, an application of the Slutsky Theorem leads to (3).

Asymptotics for Unit Root AR(1)

If (3) were also valid for $\rho = 1$, it would seem to claim that

$$\sqrt{T}(\hat{\rho}_T - \rho) \xrightarrow{p} 0. \tag{4}$$

This is indeed true for unit root processes, but it obviously is not very helpful for hypothesis tests.

Q: How to obtain a nondegenerate asymptotic distribution for $\hat{\rho}_T$ in the unit root case?

It turns out that we need to look at

$$T(\hat{\rho}_T - 1) = \frac{(1/T)\sum_{t=1}^{T} y_{t-1}u_t}{(1/T^2)\sum_{t=1}^{T} y_{t-1}^2}. \tag{5}$$

What is the asymptotic distribution of $T(\hat{\rho}_T - 1)$?

Today's Topics

1. Rewiew of AR(1)

2. Brownian Motion and Functional CLT

3. Asymptotic properties of Unit Root processes and tests for Unit Root

Today's Topics

1. Rewiew of AR(1)

2. Brownian Motion and Functional CLT
    - Brownian Motion
    - Functional CLT

3. Asymptotic properties of Unit Root processes and tests for Unit Root

Random walk

Consider a random walk,

$$y_t = y_{t-1} + \varepsilon_t, \varepsilon_t \sim i.i.d.N(0,1). \tag{6}$$

If $y_0 = 0$, then it follows that

$$\begin{aligned} y_t &= \varepsilon_1 + \ldots + \varepsilon_t, \\ y_t &\sim N(0,t), \end{aligned}$$

and, for $s < t$,

$$\begin{aligned} y_s - y_t &= \varepsilon_{t+1} + \ldots + \varepsilon_s, \\ y_s - y_t &\sim N(0, s-t), \end{aligned}$$

which is independent with $y_q - y_r$, where $t < s < r < q$.

Suppose we view ε_t as:

$$\varepsilon_t = e_{1t} + e_{2t}, \tag{7}$$

with $e_{it} \overset{i.i.d}{\sim} N(0, 1/2)$. We might define some interim point $y_{t-(1/2)}$, s.t.,

$$\begin{aligned} y_{t-(1/2)} - y_{t-1} &= e_{1t}, \\ y_t - y_{t-(1/2)} &= e_{2t}. \end{aligned}$$

Similarly, we could imagine partitioning the change between $t-1$ and t into N separate subperiods:

$$y_t - y_{t-1} = e_{1t} + e_{2t} + \cdots + e_{Nt}, \tag{8}$$

with $e_{it} \overset{i.i.d}{\sim} N(0, 1/N)$.
The result would be a process with all the same properties as (6), defined at a finer and finer grid of dates as we increase N.

The limit as $N \to \infty$ is a continuous-time process known as standard Brownian motion. The value of this process at time t is denoted as $W(t)$.

Brownian motion

Definition: Standard Brownian motion $W(\cdot)$ is a continuous-time stochastic process, associating each date $t \in [0, 1]$ with the scalar $W(t)$ such that:

(a) $W(0) = 0$;

(b) For any dates $0 \leq t_1 < t_2 < \cdots < t_k \leq 1$, the changes $[W(t_2) - W(t_1)], [W(t_3) - W(t_2)], \cdots, [W(t_k) - W(t_{k-1})]$ are independent multivariate Gaussian with $[W(s) - W(t)] \sim N(0, s - t)$;

(c) For any given realization, $W(t)$ is continuous in t with probability 1.

- Other continuous-time processes can be generated from standard Brownian motion. For example, the process

$$Z(t) = \sigma \cdot W(t)$$

has independent increment and is distributed $N(0, \sigma^2 t)$ across realizations. Such a process is described as Brownian motion with variance σ^2.

- As another example,

$$Z(t) = [W(t)]^2$$

would be distributed as t times a $\chi^2(1)$ variable.

Today's Topics

1. Rewiew of AR(1)

2. Brownian Motion and Functional CLT
   - Brownian Motion
   - Functional CLT

3. Asymptotic properties of Unit Root processes and tests for Unit Root

Review of conventional CLT

If $u_t \sim i.i.d.$ with zero mean and variance σ^2, then the sample mean $\bar{u}_T = (1/T)\sum_{t=1}^{T} u_t$ satisfies

$$\sqrt{T}\bar{u}_T \xrightarrow{L} N(0, \sigma^2). \tag{9}$$

Consider now an estimator based on only the first half of the sample,

$$\bar{u}_{[T/2]^*} = (1/[T/2]^*) \sum_{t=1}^{[T/2]^*} u_t.$$

Here $[T/2]^*$ denotes the largest integer that is less than or equal to $T/2$. This strange estimator would also satisfy the CLT:

$$\sqrt{[T/2]^*}\bar{u}_{[T/2]^*} \xrightarrow[T\to\infty]{L} N(0, \sigma^2).$$

More generally, we can construct a variable $X_T(r)$ from the sample mean of the first rth fraction of observation, $r \in [0,1]$, defined by

$$X_T(r) = (1/T) \sum_{t=1}^{[Tr]^*} u_t. \tag{10}$$

For any given realization, $X_T(r)$ is a step function in r, with

$$X_T(r) = \begin{cases} 0 & \textit{for } 0 \leq r < 1/T \\ u_1/T & \textit{for } 1/T \leq r < 2/T \\ \vdots & \\ (u_1 + u_2 + \cdots + u_T)/T & \textit{for } r = 1. \end{cases} \tag{11}$$

Then

$$\sqrt{T} \cdot X_T(r) = (1/\sqrt{T}) \sum_{t=1}^{[Tr]^*} u_t = (\sqrt{[Tr]^*}/\sqrt{T})(1/\sqrt{[Tr]^*}) \sum_{t=1}^{[Tr]^*} u_t.$$

But

$$(1/\sqrt{[Tr]^*}) \sum_{t=1}^{[Tr]^*} u_t \xrightarrow{L} N(0, \sigma^2),$$

by the CLT, while $(\sqrt{[Tr]^*}/\sqrt{T}) \to \sqrt{r}$. Hence,

$$\sqrt{T} \cdot X_T(r) \xrightarrow{L} \sqrt{r} N(0, \sigma^2) = N(0, r\sigma^2),$$

and

$$\sqrt{T} \cdot [X_T(r)/\sigma] \xrightarrow{L} N(0, r).$$

Similarly, for $r_2 > r_1$, we have

$$\sqrt{T} \cdot [X_T(r_2) - X_T(r_1)]/\sigma \xrightarrow{L} N(0, r_2 - r_1).$$

The stochastic functions $\{\sqrt{T} \cdot [X_T(\cdot)/\sigma]\}_{T=1}^{\infty}$ has an asymptotic probability law that is described by standard Brownian motion $W(\cdot)$:

$$\sqrt{T} \cdot [X_T(\cdot)/\sigma] \xrightarrow{L} W(\cdot). \tag{12}$$

Result (12) is known as the functional CLT.

Note that $X_T(\cdot)$ is a random function while $X_T(r)$ is a random variable. The conventional CLT is a special case of functional CLT:

$$\sqrt{T} X_T(1)/\sigma = [1/(\sigma\sqrt{T})] \sum_{t=1}^{T} u_t \xrightarrow{L} W(1) \sim N(0, 1).$$

Continuous Mapping Theorem

- CMT: if $\{x_T\}_{T=1}^{\infty}$ is a sequence of random variables with $x_T \xrightarrow{L} x$ and if $g : \mathbb{R}^1 \to \mathbb{R}^1$ is a continuous function, then $g(x_T) \xrightarrow{L} g(x)$. A similar result holds for sequence of random functions.
- If $\{X_T(\cdot)\}_{T=1}^{\infty}$ is a sequence of stochastic functions with

$$X_T(\cdot) \xrightarrow{L} X(\cdot)$$

and $g(\cdot)$ is a continuous functional, then

$$g(X_T(\cdot)) \xrightarrow{L} g(X(\cdot))$$

.

- **Example:** Since $\sqrt{T} \cdot X_T(\cdot) \xrightarrow{L} \sigma \cdot W(\cdot)$, define $S_T(r) = [\sqrt{T} \cdot X_T(r)]^2$, it follows that

$$S_T(\cdot) \xrightarrow{L} \sigma^2 [W(\cdot)]^2. \tag{13}$$

Applications to Unit Root Processes

Example 1
Consider

$$y_t = y_{t-1} + u_t, \tag{14}$$

where $\{u_t\}$ is an i.i.d. sequence with mean zero and variance σ^2. If $y_0 = 0$, then

$$y_t = u_1 + u_2 + \cdots + u_t. \tag{15}$$

This can be used to express the stochastic function $X_T(r)$ defined in (11) as

$$X_T(r) = \begin{cases} 0 & \text{for } 0 \leq r < 1/T \\ y_1/T & \text{for } 1/T \leq r < 2/T \\ y_2/T & \text{for } 2/T \leq r < 3/T \\ \vdots & \\ y_T/T & \text{for } r = 1. \end{cases} \tag{16}$$

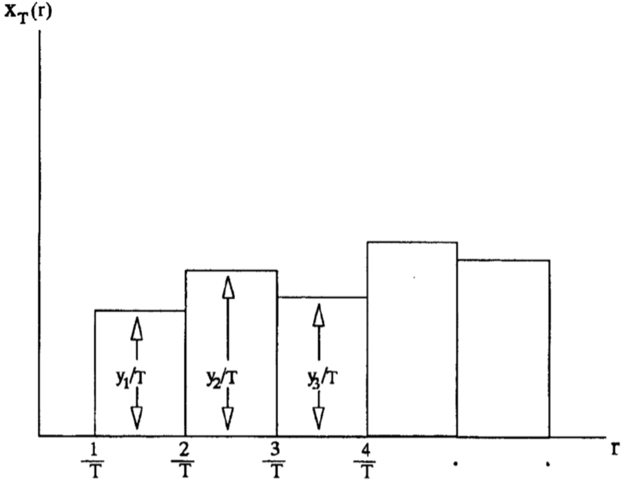

FIGURE 17.1 Plot of $X_T(r)$ as a function of r.

The integral of $X_T(r)$ is thus equivalent to

$$\int_0^1 X_T(r)dr = y_1/T^2 + y_2/T^2 + \cdots + y_{T-1}/T^2. \qquad (17)$$

Multiplying both sides of (17) by $\sqrt{T}$ establishes that

$$\int_0^1 \sqrt{T} \cdot X_T(r)dr = T^{-3/2} \sum_{t=1}^{T} y_{t-1}.$$

We know from the FCLT and the continuous mapping theorem that as $T \to \infty$,

$$\int_0^1 \sqrt{T} \cdot X_T(r)dr \xrightarrow{L} \sigma \cdot \int_0^1 W(r)dr,$$

then we get

$$T^{-3/2} \sum_{t=1}^{T} y_{t-1} \xrightarrow{L} \sigma \cdot \int_0^1 W(r)dr. \tag{18}$$

Example 2

Further consider

$$S_T(r) = T \cdot [X_T(r)]^2,$$

which can be written as

$$S_T(r) = \begin{cases} 0 & \text{for } 0 \le r < 1/T \\ y_1^2/T & \text{for } 1/T \le r < 2/T \\ y_2^2/T & \text{for } 2/T \le r < 3/T \\ \vdots & \\ y_T^2/T & \text{for } r = 1. \end{cases}$$

It follows that

$$\int_0^1 S_T(r)dr = y_1^2/T^2 + y_2^2/T^2 + \cdots + y_{T-1}^2/T^2.$$

Thus from the FCLT and the CMT,

$$T^{-2} \sum_{t=1}^{T} y_{t-1}^2 \xrightarrow{L} \sigma^2 \cdot \int_0^1 [W(r)]^2 dr. \qquad (19)$$

Example 3

Note that for a random walk,

$$y_t^2 = (y_{t-1} + u_t)^2 = y_{t-1}^2 + 2y_{t-1}u_t + u_t^2,$$

implying that

$$y_{t-1}u_t = (1/2)\{y_t^2 - y_{t-1}^2 - u_t^2\}.$$

Then

$$\sum_{t=1}^{T} y_{t-1}u_t = (1/2)\{y_T^2 - y_0^2\} - (1/2)\sum_{t=1}^{T} u_t^2.$$

Recalling that $y_0 = 0$, then

$$\begin{aligned}
(1/T)\sum_{t=1}^{T} y_{t-1}u_t &= (1/2)\cdot y_T^2/T - (1/2)\cdot \sum_{t=1}^{T} u_t^2/T \\
&= (1/2)S_T(1) - (1/2)\cdot \sum_{t=1}^{T} u_t^2/T \\
&\xrightarrow{L} (1/2)\sigma^2[W(1)]^2 - (1/2)\sigma^2.
\end{aligned}$$

Today's Topics

1. Rewiew of AR(1)
2. Brownian Motion and Functional CLT
3. Asymptotic properties of Unit Root processes and tests for Unit Root

Today's Topics

1. Rewiew of AR(1)

2. Brownian Motion and Functional CLT

3. Asymptotic properties of Unit Root processes and tests for Unit Root
   - Asymptotics for unit root processes
   - Dickey-Fuller tests

Asymptotics for unit root processes

Proposition 1 Suppose that y_t follows a random walk without drift,

$$y_t = y_{t-1} + u_t,$$

where $y_0 = 0$ and $\{u_t\}$ is an $i.i.d.$ sequence with mean zero and variance σ^2. Then

(a) $T^{-1/2} \sum_{t=1}^{T} u_t \xrightarrow{L} \sigma \cdot W(1)$;

(b) $T^{-1} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{L} (1/2)\sigma^2\{[W(1)]^2 - 1\}$; (**Example 3**)

(c) $T^{-3/2} \sum_{t=1}^{T} t u_t \xrightarrow{L} \sigma \cdot W(1) - \sigma \cdot \int_0^1 W(r)dr$;

(d) $T^{-3/2} \sum_{t=1}^{T} y_{t-1} \xrightarrow{L} \sigma \cdot \int_0^1 W(r)dr$; (**Example 1**)

(e) $T^{-2} \sum_{t=1}^{T} y_{t-1}^2 \xrightarrow{L} \sigma^2 \cdot \int_0^1 [W(r)]^2 dr$; (**Example 2**)

(f) $T^{-5/2} \sum_{t=1}^{T} t y_{t-1} \xrightarrow{L} \sigma \cdot \int_0^1 rW(r)dr$;

(g) $T^{-3} \sum_{t=1}^{T} t y_{t-1}^2 \xrightarrow{L} \sigma^2 \cdot \int_0^1 r \cdot [W(r)]^2 dr$;

(h) $T^{(\nu+1)} \sum_{t=1}^{T} t^{\nu} \to 1/(\nu+1)$ for $\nu = 0, 1, \cdots$.

The asymptotics of $T(\hat{\rho}_T - 1)$

Now we can apply Proposition 1 (b) and (e) to show that

$$\begin{aligned} T(\hat{\rho}_T - 1) &= \frac{(1/T)\sum_{t=1}^{T} y_{t-1}u_t}{(1/T^2)\sum_{t=1}^{T} y_{t-1}^2} \\ &\xrightarrow{L} \frac{(1/2)\{[W(1)]^2 - 1\}}{\int_0^1 [W(r)]^2 dr}. \end{aligned}$$

This answers the question raised at the beginning of this lecture. It is worth noting that such a limiting result involves:

(i) a convergence rate T (super-consistent), and

(ii) a nonstandard limiting distribution.

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1. Rewiew of AR(1)
2. Brownian Motion and Functional CLT
3. Asymptotic properties of Unit Root processes and tests for Unit Root
   - Asymptotics for unit root processes
   - Dickey-Fuller tests

Dickey-Fuller tests of Unit Root

We shall consider four cases in sequence,

Case 1 Estimated regression: $y_t = \rho y_{t-1} + u_t$
True process: $y_t = y_{t-1} + u_t \; u_t \sim i.i.d.N(0, \sigma^2)$

Case 2 Estimated regression: $y_t = \alpha + \rho y_{t-1} + u_t$
True process: $y_t = y_{t-1} + u_t \; u_t \sim i.i.d.N(0, \sigma^2)$

Case 3 Estimated regression: $y_t = \alpha + \rho y_{t-1} + u_t$
True process: $y_t = \alpha + y_{t-1} + u_t \; \alpha \neq 0, u_t \sim i.i.d.(0, \sigma^2)$

Case 4 Estimated regression: $y_t = \alpha + \delta t + \rho y_{t-1} + u_t$
True process: $y_t = \alpha + y_{t-1} + u_t \; \alpha$ any, $u_t \sim i.i.d.N(0, \sigma^2)$

Dickey-Fuller tests of Unit Root: Case 1

We first consider Case 1

Case 1 Estimated regression: $y_t = \rho y_{t-1} + u_t$
True process: $y_t = y_{t-1} + u_t \; u_t \sim i.i.d.N(0, \sigma^2)$

Dickey-Fuller ρ test for Case 1

Under the null hypothesis that $\rho = 1$, the Dickey-Fuller ρ statistic

$$T(\hat{\rho}_T - 1) \xrightarrow{L} \frac{(1/2)\{[W(1)]^2 - 1\}}{\int_0^1 [W(r)]^2 dr} \triangleq DF_{\rho,case1}. \qquad (20)$$

Sample size T	Probability that $T(\hat{\rho} - 1)$ is less than entry							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
				Case 1				
25	−11.9	−9.3	−7.3	−5.3	1.01	1.40	1.79	2.28
50	−12.9	−9.9	−7.7	−5.5	0.97	1.35	1.70	2.16
100	−13.3	−10.2	−7.9	−5.6	0.95	1.31	1.65	2.09
250	−13.6	−10.3	−8.0	−5.7	0.93	1.28	1.62	2.04
500	−13.7	−10.4	−8.0	−5.7	0.93	1.28	1.61	2.04
∞	−13.8	−10.5	−8.1	−5.7	0.93	1.28	1.60	2.03

Figure 1 : Table 1 Critical values for the Dickey-Fuller ρ test

Skewed to the left! For finite T, these are exact only under the assumption of Gaussian innovations. As T becomes large, these values are also valid for non-Gaussian innovations.

Example 1 (Nominal Interest Rate)

Data: nominal three-month U.S. Treasury bill rate, quarterly, from 1947:2 to 1989:1, $T = 168$
Model: AR(1) by OLS estimation

$$i_t = \underset{(0.010592)}{0.99694} \quad i_{t-1},$$

The Dickey-Fuller ρ test of $\rho = 1$ is

$$T(\hat{\rho}_T - 1) = 168(0.99694 - 1) = -0.51,$$

This is well above the critical value -7.9 (T=100). So the null is accepted at the 5% level.

Dickey-Fuller t test for Case 1

Another popular statistics for testing the null hypothesis that $\rho = 1$ is based on the usual OLS t test of this hypothesis,

$$t_T = \frac{(\hat{\rho}_T - 1)}{\hat{\sigma}_{\hat{\rho}_T}}, \tag{21}$$

where $\hat{\sigma}_{\hat{\rho}_T}$ is the usual OLS standard error for the estimated coefficient,

$$\hat{\sigma}_{\hat{\rho}_T} = \left\{ s_T^2 \div \sum_{t=1}^{T} y_{t-1}^2 \right\}^{1/2},$$

and s_T^2 denotes the OLS estimate of the residual variance:

$$s_T^2 = \sum_{t=1}^{T} (y_t - \hat{\rho}_T y_{t-1})^2 / (T - 1).$$

As $T \to \infty$,

$$t_T \xrightarrow{L} \frac{(1/2)\{[W(1)]^2 - 1\}}{\left\{\int_0^1 [W(r)]^2 dr\right\}^{1/2}} \triangleq DF_{t,case1}. \tag{22}$$

Sample size T	Probability that $(\hat{\rho} - 1)/\hat{\sigma}_{\hat{\rho}}$ is less than entry							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
				Case 1				
25	−2.66	−2.26	−1.95	−1.60	0.92	1.33	1.70	2.16
50	−2.62	−2.25	−1.95	−1.61	0.91	1.31	1.66	2.08
100	−2.60	−2.24	−1.95	−1.61	0.90	1.29	1.64	2.03
250	−2.58	−2.23	−1.95	−1.62	0.89	1.29	1.63	2.01
500	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
∞	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00

Figure 2 : Table 2 Critical values for the Dickey-Fuller t test

Example 1 (Nominal Interest Rate)

$$i_t = \underset{(0.010592)}{0.99694} \quad i_{t-1},$$

The Dickey-Fuller t test of $\rho = 1$ is

$$t = (0.99694 - 1)/0.010592 = -0.29,$$

This is well above the 5% critical value of -1.95 (T=100). So the null is again accepted.