时间序列分析

Time Series Analysis

Lecture 4

Review

1. Linear Time Series
2. Simple AR Models
3. Simple MA Models
4. Simple ARMA Models

Today's Topics

1. Linear Time Series
2. Simple AR Models
3. Simple MA Models
4. Simple ARMA Models

3. Simple MA Models

To introduce MA models, two popular approaches are

(1) Extension of White Noise

(2) Infinite Order AR models

We shall focus on (2).

AR Infinity

AR with infinite order writes

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \cdots + a_t.$$

However, such an AR model is not realistic because it has infinite many parameters. One way to make the model practical is to impose certain parameter restrictions, e.g.,

$$\begin{aligned} r_t &= \phi_0 - \theta_1 r_{t-1} - \theta_1^2 r_{t-2} - \theta_1^3 r_{t-3} - \cdots + a_t, \\ \phi_i &= -\theta_1^i \textit{ for } i \geq 1. \end{aligned}$$

MA(1)

Rewrite

$$r_t = \phi_0 - \theta_1 r_{t-1} - \theta_1^2 r_{t-2} - \theta_1^3 r_{t-3} - \cdots + a_t$$

as

$$\begin{aligned} r_t + \theta_1 r_{t-1} + \theta_1^2 r_{t-2} + \cdots &= \phi_0 + a_t, & (1) \\ r_{t-1} + \theta_1 r_{t-2} + \theta_1^2 r_{t-3} + \cdots &= \phi_0 + a_{t-1}. & (2) \end{aligned}$$

$(1) - (2) \times \theta_1 \Rightarrow$

$$r_t = \phi_0(1 - \theta_1) + a_t - \theta_1 a_{t-1}.$$

MA Models

- MA(1)

$$
\begin{aligned}
r_t &= c_0 + a_t - \theta_1 a_{t-1} \\
r_t &= c_0 + (1 - \theta_1 B) a_t
\end{aligned}
$$

- MA(2)

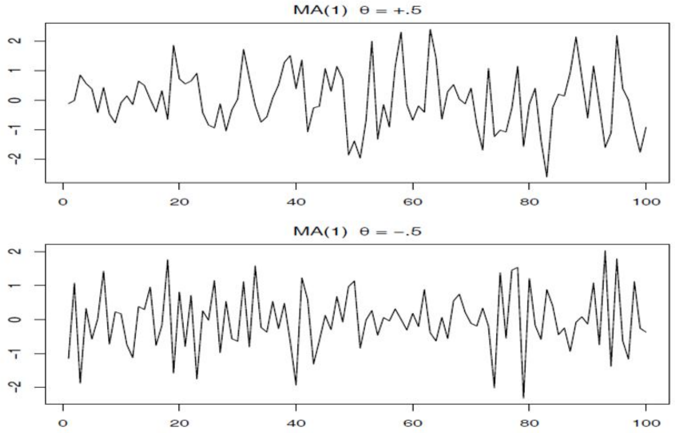
$$r_t = c_0 + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

- MA(q)

$$
\begin{aligned}
r_t &= c_0 + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} \\
r_t &= c_0 + (1 - \theta_1 B - \cdots - \theta_q B^q) a_t, \ \textit{where } q > 0
\end{aligned}
$$

where c_0 is a constant and $\{a_t\}$ is a white noise series.

Simulated MA(1)




```
par(mfrow = c(2,1))
plot(arima.sim(list(order=c(0,0,1), ma=.5), n=100), ylab="x",
    main=(expression(MA(1)~~~theta==+.5)))
plot(arima.sim(list(order=c(0,0,1), ma=-.5), n=100), ylab="x",
    main=(expression(MA(1)~~~theta==-.5)))
```

Properties of MA(1)

- Mean and variance

$$\begin{aligned} E(r_t) &= c_0 \\ Var(r_t) &= \sigma_a^2 + \theta_1^2\sigma_a^2 = (1+\theta_1^2)\sigma_a^2 \end{aligned}$$

- Autocovariance

$$\begin{aligned} \gamma_1 &= -\theta_1\sigma_a^2 \\ \gamma_\ell &= 0, \quad \mathit{for}\ \ell > 1 \end{aligned}$$

- Autocorrelation

$$\begin{aligned} \rho_0 &= 1, \\ \rho_1 &= \frac{-\theta_1}{1+\theta_1^2} \\ \rho_\ell &= 0, \quad \mathit{for}\ \ell > 1. \end{aligned}$$

ACF of an MA(1) model cuts off at lag 1!!!

Properties of MA(q)

- Mean and variance

$$\begin{aligned} E(r_t) &= c_0 \\ Var(r_t) &= (1 + \theta_1^2 + \cdots + \theta_q^2)\sigma_a^2 \end{aligned}$$

- Autocorrelation for MA(2)

$$\begin{aligned} \rho_1 &= \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}, \\ \rho_2 &= \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}, \\ \rho_\ell &= 0, \quad \text{for } \ell > 2. \end{aligned}$$

ACF of an MA(2) model cuts off at lag 2!!!

- Generalize the ACF properties to MA(q)

Stationarity

Moving-average models are always weakly stationary because they are finite linear combinations of a white noise sequence for which the first two moments are time invariant.

Invertibility

Rewriting a zero-mean MA(1) model as

$$a_t = r_t + \theta_1 a_{t-1}.$$

one can use repeatedly substitutions to obtain

$$a_t = r_t + \theta_1 r_{t-1} + \theta_2 r_{t-2} + \theta_3 r_{t-3} + \cdots .$$

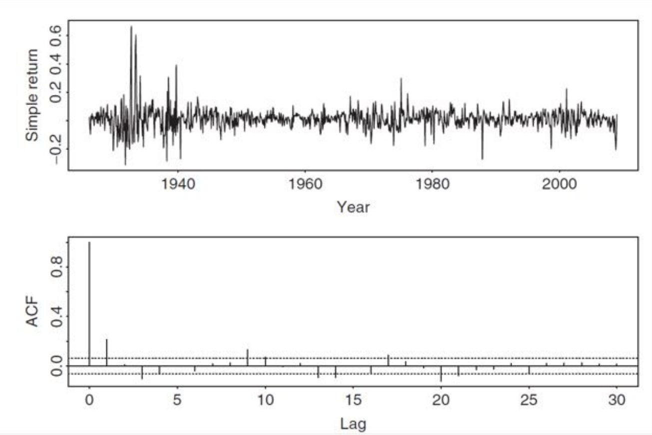
This equation expresses the current shock a_t as a linear combination of the present and past returns. The current shock does not explode only if $|\theta_1| < 1$. We call such a model invertible.

Identifying MA Order

The ACF is useful in identifying the order of an MA model. The basic rule is the following.

For a time series r_t with ACF ρ_ℓ, if $\rho_q \neq 0$, but $\rho_\ell = 0$ for $\ell > q$, then r_t follows an $MA(q)$ model.

Time plot of monthly simple returns of the CRSP equal-weighted index from January 1926 to December 2008 and the sample ACF of the series.



$$r_t = c_0 + a_t - \theta_1 a_{t-1} - \theta_3 a_{t-3} - \theta_9 a_{t-9}$$

Estimation of MA(q)

Maximum likelihood method is often used to estimate MA(q). In particularly, we may use

(a) conditional-likelihood method or

(b) exact-likelihood method

This will be discussed in details later on

Forecasting with MA(q)

For the 1-step-ahead forecast of an MA(1)

$$r_{h+1} = c_0 + a_{h+1} - \theta_1 a_h.$$

Taking the conditional expectation

$$\begin{aligned}
\hat{r}_h(1) &= E(r_{h+1}|F_h) = c_0 - \theta_1 a_h, \\
e_h(1) &= r_{h+1} - \hat{r}_h(1) = a_{h+1}, \\
Var[e_h(1)] &= \sigma_a^2.
\end{aligned}$$

a_h could be obtained via assuming $a_0 = 0$,

$a_1 = r_1 - c_0,\ a_t = r_t - c_0 + \theta_1 a_{t-1}, \quad for\ 2 \leq t \leq h.$

For the 2-step-ahead forecast

$$r_{h+2} = c_0 + a_{h+2} - \theta_1 a_{h+1}.$$

We obtain

$$\begin{aligned}
\hat{r}_h(2) &= E(r_{h+2}|F_h) = c_0, \quad \textit{Unconditional mean} \\
e_h(2) &= r_{h+2} - \hat{r}_h(2) = a_{h+2} - \theta_1 a_{h+1}, \\
Var[e_h(2)] &= (1 + \theta_1^2)\sigma_a^2.
\end{aligned}$$

More generally,

$$\hat{\gamma}_h(\ell) = c_0 \textit{ for } \ell \geq 2.$$

Similarly, for an MA(2) model, we have

$$r_{h+\ell} = c_0 + a_{h+\ell} - \theta_1 a_{h+\ell-1} - \theta_2 a_{h+\ell-2}.$$

We obtain

$$\begin{aligned} \hat{r}_h(1) &= c_0 - \theta_1 a_h - \theta_2 a_{h-1}, \\ \hat{r}_h(2) &= c_0 - \theta_2 a_h, \\ \hat{r}_h(\ell) &= c_0, \quad \text{for } \ell > 2. \end{aligned}$$

Summary of AR and MA

- For MA models, ACF is useful in specifying the order because ACF cuts off at lag q for an MA(q) series.
- For AR models, PACF is useful in order determination because PACF cuts off at lag p for an AR(p) process.
- An MA series is always stationary, but for an AR series to be stationary, all of its characteristic roots must be less than 1 in modulus.

4. Simple ARMA Models

- An ARMA model combines the ideas of AR and MA models into a compact form so that the number of parameters used is kept small, achieving parsimony in parameterization.
- We study the simplest ARMA(1,1) model first, then turn to general ARMA(p,q).

ARMA(1,1)

An ARMA(1,1) model takes the form of

$$\begin{array}{ccc} r_t - \phi_1 r_{t-1} & = & \phi_0 + a_t - \theta_1 a_{t-1}, \\ \Downarrow & & \Downarrow \\ \textit{AR component} & & \textit{MA component} \end{array}$$

a_t is a white noise series and $\phi_1 \neq \theta_1$.

Properties of ARMA(1,1)

First, we assume the stationarity condition. We obtain

$$
\begin{aligned}
E(r_t) - \phi_1 E(r_{t-1}) &= \phi_0 + E(a_t) - \theta_1 E(a_{t-1}), \\
E(r_t) &= \mu = \frac{\phi_0}{1 - \phi_1}.
\end{aligned}
$$

This is the same as that of AR(1). Assuming $\phi_0 = 0$, we get

$$
\begin{aligned}
E(r_t a_t) &= E(a_t^2) - \theta_1 E(a_t a_{t-1}) = E(a_t^2) = \sigma_a^2. \\
Var(r_t) &= \phi_1^2 Var(r_{t-1}) + \sigma_a^2 + \theta_1^2 \sigma_a^2 - 2\phi_1 \theta_1 E(r_{t-1} a_{t-1}).
\end{aligned}
$$

Hence

$$Var(r_t) = \frac{1 - 2\phi_1\theta_1 + \theta_1^2}{1 - \phi_1^2}, \quad |\phi_1| < 1.$$

Note that this is again the same stationarity condition for AR(1).

To obtain ACF, first assume $\phi_0 = 0$.

$$\begin{aligned}
r_t r_{t-\ell} - \phi_1 r_{t-1} r_{t-\ell} &= a_t r_{t-\ell} - \theta_1 a_{t-1} r_{t-\ell}. \\
\gamma_1 - \phi_1 \gamma_0 &= \theta_1 \sigma_a^2, \\
\gamma_2 - \phi_1 \gamma_1 &= 0, \\
\gamma_\ell - \phi_1 \gamma_{\ell-1} &= 0, \quad \textit{for } \ell > 1.
\end{aligned}$$

We therefore obtain

$$\rho_1 = \phi_1 - \frac{\theta_1 \sigma_a^2}{\gamma_0}, \quad \rho_\ell = \phi_1 \rho_{\ell-1}, \text{ for } \ell > 1.$$

We conclude that the ACF of an ARMA(1,1) model behaves very much like that of an AR(1) model except that the exponential decay starts with lag 2.

Consequently, the ACF of an ARMA(1,1) model does not cut off at any finite lag.

PACF of ARMA(1,1)

Turning to PACF, one can show that the PACF of an ARMA(1,1) model does not cut off at any finite lag either. It behaves very much like that of an MA(1) model except that the exponential decay starts with lag 2 instead of lag 1.

General ARMA Models

A general ARMA(p,q) model is in the form

$$r_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + a_t - \sum_{i=1}^{q} \theta_i a_{t-i}$$

It can be written as

$$(1 - \phi_1 B - \cdots - \phi_p B^p) r_t = \phi_0 + (1 - \theta_1 B - \cdots - \theta_q B^q) a_t.$$

We require that there is no common roots in the AR and MA polynomials.

Identifying ARMA Order

- The ACF and PACF are not informative in determining the order of an ARMA model.
- Tsay and Tiao (1984) propose a new approach that uses the extended autocorrelation function (EACF) to specify the order of an ARMA process.
- The derivation of EACF is relatively involved; see Tsay and Tiao (1984) for details. Yet the function is easy to use.

EACF

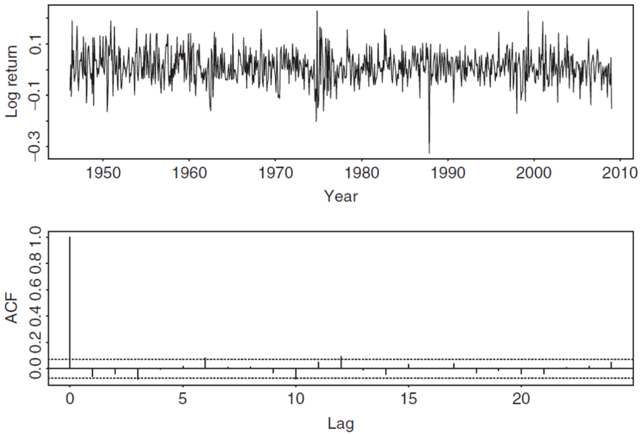
- The output of EACF is a two-way table, where the rows correspond to AR order p and the columns to MA order q.
- For ARMA(1,1), the key feature of the table is that it contains a triangle of O with the upper left vertex located at the order (1,1). This is the characteristic we use to identify the order of an ARMA process.

Theoretical EACF Table for an ARMA(1,1) Model, Where X Denotes Nonzero, O Denotes Zero, and * Denotes Either Zero or Nonzero

	MA							
AR	0	1	2	3	4	5	6	7
0	X	X	X	X	X	X	X	X
1	X	O	O	O	O	O	O	O
2	*	X	O	O	O	O	O	O
3	*	*	X	O	O	O	O	O
4	*	*	*	X	O	O	O	O
5	*	*	*	*	X	O	O	O

In general, for an ARMA(p,q) model, the triangle of O will have its upper left vertex at the (p,q) position.

Time plot and sample autocorrelation function of monthly log stock returns of 3M Company from February 1946 to December 2008.



Sample Extended Autocorrelation Function													
MA Order: q													
p	0	1	2	3	4	5	6	7	8	9	10	11	12
0	−0.06	−0.04	−0.08	−0.00	0.02	0.08	0.01	0.01	−0.03	−0.08	0.05	0.09	−0.01
1	−0.47	0.01	−0.07	−0.02	0.00	0.08	−0.03	0.00	−0.01	−0.07	0.04	0.09	−0.02
2	−0.38	−0.35	−0.07	0.02	−0.01	0.08	0.03	0.01	0.00	−0.03	0.02	0.04	0.04
3	−0.18	0.14	0.38	−0.02	0.00	0.04	−0.02	0.02	−0.00	−0.03	0.02	0.01	0.04
4	0.42	0.03	0.45	−0.01	0.00	0.00	−0.01	0.03	0.01	0.00	0.02	−0.00	0.01
5	−0.11	0.21	0.45	0.01	0.20	−0.01	−0.00	0.04	−0.01	−0.01	0.03	0.01	0.03
6	−0.21	−0.25	0.24	0.31	0.17	−0.04	−0.00	0.04	−0.01	−0.03	0.01	0.01	0.04

Simplified EACF Table													
MA Order: q													
p	0	1	2	3	4	5	6	7	8	9	10	11	12
0	O	O	X	O	O	X	O	O	O	X	O	X	O
1	X	O	O	O	O	X	O	O	O	O	O	X	O
2	X	X	O	O	O	X	O	O	O	O	O	O	O
3	X	X	X	O	O	O	O	O	O	O	O	O	O
4	X	O	X	O	O	O	O	O	O	O	O	O	O
5	X	X	X	O	X	O	O	O	O	O	O	O	O
6	X	X	X	X	X	O	O	O	O	O	O	O	O

Simulation with EACF

```
>install.packages('TSA')
> library('TSA')
>z=arima.sim(list(order=c(1,0,1),ma=0.5,ar
=0.5),n=100)
> eacf(z,ar.max=8,ma.max=8)
```

AR/MA

	0	1	2	3	4	5	6	7	8
0	x	x	x	o	o	o	o	o	o
1	x	o	o	o	o	o	o	o	o
2	x	o	o	o	o	o	o	o	o
3	x	o	o	o	o	o	o	o	o
4	x	o	o	o	o	o	o	o	o
5	x	o	x	x	o	o	o	o	o
6	o	o	x	x	o	o	o	o	o
7	x	o	x	o	o	x	o	o	o
8	o	o	o	o	o	x	o	o	o

Simplified Sample EACF

The simplified table is constructed by using the following notation:

(1) X denotes that the absolute value of the corresponding EACF is greater than or equal to $2/\sqrt{T}$, which is twice of the asymptotic standard error of the EACF.

(2) O denotes that the corresponding EACF is less than $2/\sqrt{T}$ in modulus.

Information Criteria

The information criteria discussed earlier can also be used to select ARMA(p,q) models. Typically, for some prespecified positive integers P and Q, one computes AIC (or BIC) for ARMA(p,q) models, where $0 \leq p \leq P$ and $0 \leq q \leq Q$, and selects the model that gives the minimum AIC (or BIC).

Other Issues

ARMA(p,q) model could be estimated by either the conditional or exact-likelihood method.

The Ljung-Box statistics of the residuals can be used to check the adequacy of a fitted model, as have been discussed in AR models.

Forecasting Using ARMA

The 1-step-ahead forecast

$$
\begin{aligned}
\hat{r}_h(1) &= E(r_{h+1}|F_h) = \phi_0 + \sum_{i=1}^{p} \phi_i r_{h+1-i} - \sum_{i=1}^{q} \theta_i a_{h+1-i}, \\
e_h(1) &= r_{h+1} - \hat{r}_h(1) = a_{h+1}, \\
Var[e_h(1)] &= \sigma_a^2.
\end{aligned}
$$

The ℓ-step-ahead forecast

$$
\hat{r}_h(\ell) = E(r_{h+\ell}|F_h) = \phi_0 + \sum_{i=1}^{p} \phi_i \hat{r}_h(\ell - i) - \sum_{i=1}^{q} \theta_i a_h(\ell - i)
$$

Three Representations of ARMA

(1) The ARMA(p,q) model was written as

$$(1 - \phi_1 B - \cdots - \phi_p B^p) r_t = \phi_0 + (1 - \theta_1 B - \cdots - \theta_q B^q) a_t.$$

Define the following polynomials

$$\begin{aligned}
\phi(B) &= 1 - \sum_{i=1}^{p} \phi_i B^i, \quad \theta(B) = 1 - \sum_{i=1}^{q} \theta_i B^i \\
\frac{\theta(B)}{\phi(B)} &= 1 + \psi_1 B + \psi_2 B^2 + \cdots \equiv \psi(B) \\
\frac{\phi(B)}{\theta(B)} &= 1 - \pi_1 B - \pi_2 B^2 - \cdots \equiv \pi(B)
\end{aligned}$$

For instance, if $\phi(B) = 1 - \phi_1 B$ and $\theta(B) = 1 - \theta_1 B$, then

$$
\begin{aligned}
\phi(B) &= \frac{1 - \theta_1 B}{1 - \phi_1 B} = 1 + (\phi_1 - \theta_1)B + \phi_1(\phi_1 - \theta_1)B^2 + \phi_1^2(\phi_1 - \theta_1)B^3 + \cdots \\
\pi(B) &= \frac{1 - \phi_1 B}{1 - \theta_1 B} = 1 - (\phi_1 - \theta_1)B - \theta_1(\phi_1 - \theta_1)B^2 - \theta_1^2(\phi_1 - \theta_1)B^3 + \cdots
\end{aligned}
$$

From the definition, $\phi(B)\pi(B) = 1$. Making use of the fact that $Bc = c$ for any constant (because the value of a constant is time invariant), we have

$$
\frac{\phi_0}{\theta(1)} = \frac{\phi_0}{1 - \theta_1 - \cdots - \theta_q} \qquad \frac{\phi_0}{\phi(1)} = \frac{\phi_0}{1 - \phi_1 - \cdots - \phi_p}
$$

AR Representation

ARMA(p,q) could have the AR form

$$r_t = \frac{\phi_0}{1 - \theta_1 - \cdots - \theta_1} + \pi_1 r_{t-1} + \pi_2 r_{t-2} + \pi_3 r_{t-3} + \cdots + a_t$$

(π weights)

An ARMA(p,q) model that has this property is said to be invertible. A sufficient condition for invertibility is that all the zeros of the polynomial $\theta(B)$ are greater than unity in modulus.

MA Representation

ARMA(p,q) model can also be written as

$$\begin{aligned} r_t &= \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots = \mu + \psi(B) a_t \\ \mu &= E(r_t) = \phi_0 / (1 - \phi_1 - \cdots - \phi_p) \end{aligned}$$

The coefficients ψ_i are referred to as the impulse response function of the ARMA model.

The stationarity implies that ψ_i approaches zero as $i \to \infty$.