Automorphism Groups and the Fixing Number of the Dowling Geometry

Gerald Todd II
University of Montana

Abstract

Automorphisms of a matroid are symmetries that preserve, for example, the circuits of the matroid. When representing a matroid geometrically, this corresponds to collineations: mappings of the points that preserve collinearity. The fixing number of a matroid is the fewest number of points that need to be fixed so that the only valid automorphism is the identity, i.e., the fewest number of points that "break" all collineations. The Dowling geometry is a matroid that geometrically represents the multiplication table of a group. It has many interesting symmetries which we will investigate by enumerating the automorphism group and determining the fixing number.

1 Background and Definitions

1.1 Matroids

A *matroid* is a pair $(E, \mathcal{I})$ in which E is a finite set and $\mathcal{I}$ is a family of subsets of E satisfying

1. $\mathcal{I} \neq \emptyset$
2. if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
3. if $I, J \in \mathcal{I}$ with $|I| < |J|$, then there is some element $x \in J - I$ with $I \cup \{x\} \in \mathcal{I}$.

The members of $\mathcal{I}$ are called the *independent sets* and E is called the ground set of the of the matroid. Much of the terminology in matroid theory coincides with terminology in other fields. This is because matroids are generalizations of these structures or properties. For example, the independent sets of our matroid act very much like linear independence in vector spaces.

Consider the ground set $E = \{a, b, c\}$ and $\mathcal{I} = \{\emptyset, a, b, c, ab, bc\}$. The pair $(E, \mathcal{I})$ is indeed a matroid as it satisfies the three conditions above (most notably, the third condition).

The *rank* of our matroid, $r(M)$, is the size of the largest independent set of M.

Throughout our treatment of matroids, we will use a specific geometric representation of matroids in $\mathbb{R}^(r-1)$, where r is the rank of the matroid. The points in $\mathbb{R}^(r-1)$ will correspond to elements in our ground set. Three (or more) points on a lines indicate a dependent set, four points on a plane indicate a dependent set, five points on a hyperplane, and so on. Note that since the rank of a matroid is the size of its largest independent set, no set of points of size greater than r can be independent.

1.2 Automorphisms and the Fixing Number

To characterize the structure of a matroid that needs to be preserved by our automorphisms, we will use the circuits (minimally dependent subsets of E) of a matroid. The set $C \subseteq E$ is a *circuit* if it is not in $\mathcal{I}$, and the removal of any element of C will result in an independent set. Thus, the circuits of M are the minimally dependent subsets of E. We will denote the collection of circuits of a matroid by $\mathcal{C}$.

Given a matroid $M = (E, \mathcal{I})$, a *matroid automorphism* $\phi : E \to E$ is a bijection on E, the ground set of M, that preserves the circuits of M, that is, $\phi(C) \in \mathcal{C}$ if and only if $C \in \mathcal{C}$. Since the circuits of a matroid are preserved under an automorphism, the *collinearities* (sets of collinear points) of the geometric representation of the matroid are preserved as well. A set of elements S of a matroid is a *fixing set* if the only automorphism that fixes S is the identity. The *fixing number* of a matroid is the size of the smallest fixing set of the matroid,

$$\text{fix}(M) = \min\{r | M \text{ has an } r\text{-element fixing set}\}.$$

1.3 The Rank-r Dowling Geometry

The construction of the rank-r Dowling geometry based on a finite group G can be described through the multiplicative group of a finite field F (Dowling) [3], equivalence classes of structures based on the group G (Dowling) [4], graph theory (Doubilet, Rota, and Stanley) [2], or points and lines (Bonin) [1]. For the purposes of this paper, we will use Bonin's construction [1].

Let G be a finite group, written with multiplicative notation. The *rank-r Dowling geometry over* G, denoted $Q_r(G)$, has the following points and lines. There are two kinds of points: *joints* $P_1, P_2, \ldots, P_r$, which form a basis for $Q_r(G)$; and *internal points* g_{ij} for every $g \in G$ and every pair of indices with $1 \leq i < j \leq r$. Hence, $Q_r(G)$ has $r + \binom{r}{2}|G|$ points. There are two types of nontrivial lines (i.e., lines with at least three points): *coordinate lines* $P_i \vee P_j = \{P_i, P_j\} \cup \{g_{ij} | g \in G\}$; and *transversal lines* $\{g_{ij}, h_{jk}, (gh)_{ik}\}$ for each pair $g, h \in G$ and triple of indices with $1 \leq i < j < k \leq r$. Thus the transversal lines are contained in the *coordinate planes* $P_i \vee P_j \vee P_k$, and they encode the group operation.

This construction results in a finite collection of points and lines where any pair of lines meet in at most one point. This guarantees $Q_r(G)$ is a rank-r matroid (Gordon) [7]. Thus, any *collineation* of $Q_r(G)$, automorphisms of the points that preserve collinearity, corresponds to a unique matroid automorphism.

Let us illustrate this construction with an example: $Q_3(C_3)$ where C_3 is the cyclic group of three elements e, x, and x^2. We first draw the joints P_1, P_2, and P_3 and draw our coordinate lines $P_1 \vee P_2$, $P_2 \vee P_3$, and $P_1 \vee P_3$. Since there are only 3 coordinate lines, there are only three indices i, j, k such that $i < j < k$. Then, the internal lines are all of the form $\{g_{12}, h_{23}, (gh)_{13}\}$ (Figure 1). Putting all of these lines together yields the rather messy $Q_3(C_3)$ (Figure 2).

In an attempt to make the figure above more clear, let us draw $Q_3(C_3)$ such that the visible portions of the coordinate lines are parallel. In this way we can "unfold" the coordinate lines making the relationship between the internal points more pronounced (Figure 3).

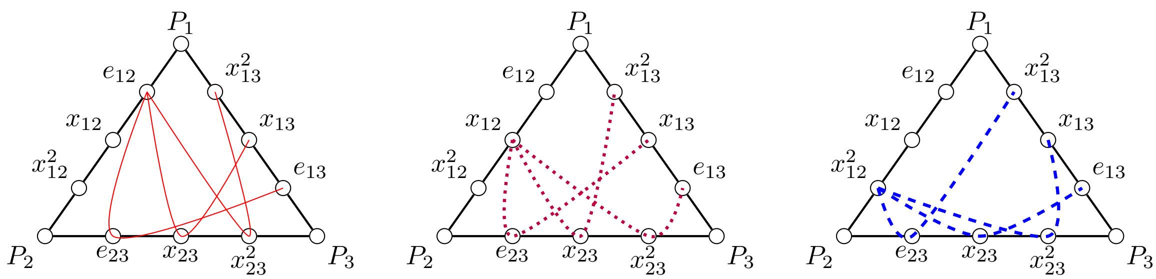

Figure 1: The lines generated by each element on the line $P_1 \vee P_2$.

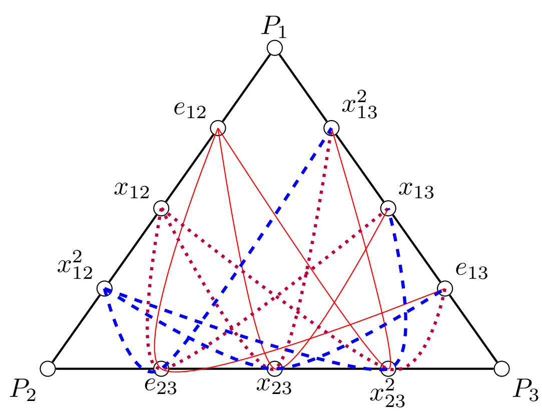

Figure 2: The rank-3 Dowling geometry based on the group C_3.

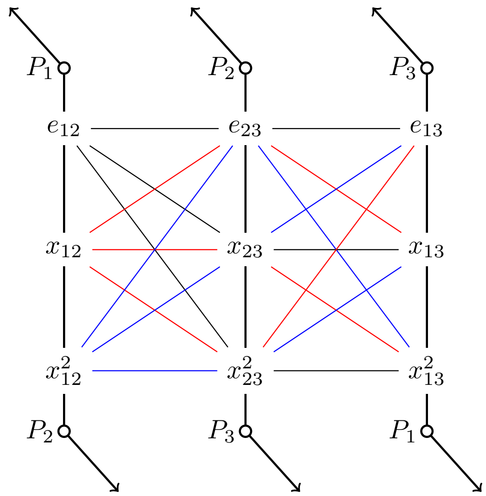

Figure 3: The "unfolded" rank-3 Dowling geometry based on C_3.

1.4 The Regular k-Simplex

A *k-simplex* is the convex hull of $k+1$ linearly independent vectors in $\mathbb{R}^{k+1}$. Simplices generalize triangles to arbitrary dimensions. For example, a 2-simplex is a triangle and a 3-simplex is a tetrahedron. A *regular* simplex is a simplex that is also a regular polygon. To describe a regular k-simplex, we can take the convex hull of the positive unit vectors in $\mathbb{R}^{k+1}$ (Figure 4).

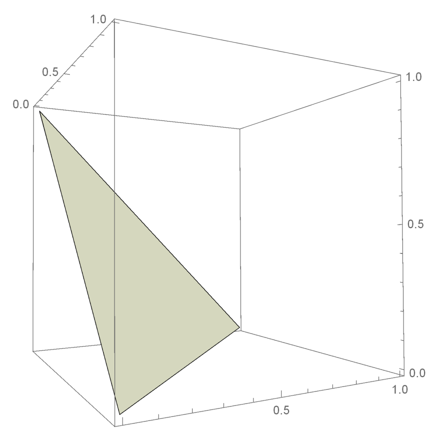

Figure 4: The regular 2-simplex (equilateral triangle) formed from the convex hull of $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in $\mathbb{R}^3$.

By [7], the k-simplex is indeed a rank-$(k+1)$ matroid. Since no 3 points lie on a line and all points lie on a 2-point line with every other point, the autormorphism group is simply permutations of the points. The number of such permutations is $(k+1)!$.

The joints and coordinate lines of a rank-r Dowling geometry can be thought of as an $(r-1)$-simplex. If we let the joints $P_1, P_2, ..., P_r$ be the positive unit vectors in $\mathbb{R}^r$, the Dowling geometry's coordinate lines and planes are formed by taking the convex hull of $\{P_i | 1 \leq i \leq r\}$. If we choose any 3 of the of the joints P_i, P_j, and P_k such that $1 \leq i < j < k \leq r$, the coordinate plane $P_i \vee P_j \vee P_k$ corresponds to the rank-3 Dowling geometry based on G.

It is clear that any automorphisms of the joints must permute all interior points on the coordinate lines incident to the joint since g_{ij} is on $P_i \vee P_j$ for all $g \in G$ and all $1 \leq i < j \leq r$. Also, all joints are collinear with $(r-1)(|G|+1)$ points and internal points are collinear with exactly $|G|+1$ points, so joints may only map to other joints under an automorphism. The set of automorphisms produced by permuting the joints is then simply the automorphism group corresponding to the $(r-1)$-simplex.

1.5 Petrie Polygon Projections

A *Petrie polygon* is an orthogonal projection of a polygon into $\mathbb{R}^2$. Petrie polygons are particularly useful for visualizing higher dimensional simplices in the plane. The Petrie polygon of a simplex is as constructed as follows: the joints of our geometry are arranged on a circle in the plane; all vertices (joints) are connected if there is an edge between them. For a k-simplex, this generates the complete graph K_{k+1}. (Figure 5)

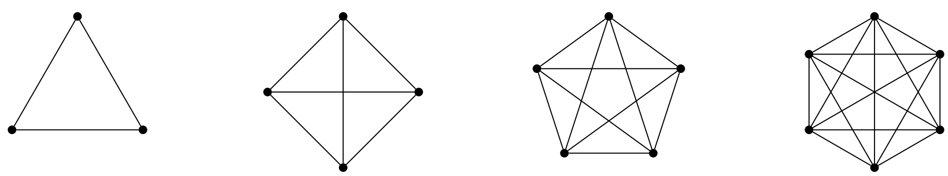

Figure 5: Petrie polygons for a 2, 3, 4, and 5-simplex respectively.

Not all the information about our k-simplex is preserved by the Petrie polygon projection, but all of the vertices, edges, and 3-faces are. This means that if we view the rank-r Dowling geometry as a subdivided $(r-1)$-simplex, where all the edges are the coordinate lines, the Petrie polygon preserves all coordinate planes $P_i \vee P_j \vee P_k$. Since all transversal lines (lines that are no coordinate lines) are on the coordinate planes, all of the structure of our geometry is preserved.

2 A Motivating Example

Let us enumerate the automorphism group of the matroid $M = Q_3(C_3)$, where C_3 is the cyclic group of three elements e, x, and x^2, by considering the "unfolded" drawing of the Dowling geometry (Figure 3). We claim that an automorphism that fixes the joints is completely determined by its action on one coordinate line and one interior point not on the line. Without loss of generality, we will take the coordinate line $P_1 \vee P_2$ and a single interior point on $P_2 \vee P_3$.

Let ϕ be an automorphism of M that fixes P_1, P_2, and P_3. For each $g_{12} \in M$, ϕ may map g_{12} to any other point on $P_1 \vee P_2$. Thus any permutation of the internal points of the $P_1 \vee P_2$ is allowed. In our example, $\phi(e_{12}) = x_{12}$, $\phi(x_{12}) = e_{12}$, and $\phi(x_{12}^2) = x_{12}^2$ (Figure 6).

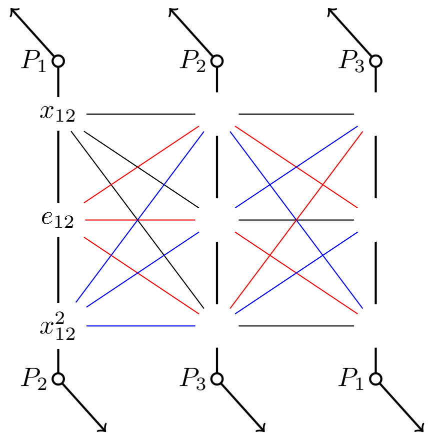

Figure 6: Example of permutation of first coordinate line.

Suppose $\phi(h_{23}) = h'_{23}$ for some $h_{23} \in M$. We claim that determining h'_{23} along with the permutation of the internal points on $P_1 \vee P_2$ determines ϕ. For any x_{13} on $P_1 \vee P_3$, there exists a line $\{g_{12}, h_{12}, x_{13}\}$ for some $g \in G$. If $\phi(g_{12}) = g'_{12}$, then the line $\{g_{12}, h_{23}, (gh)_{13}\}$ has to be mapped to $\{g'_{12}, h'_{12}, (g'h')_{12}\}$ and $x_{13} = (g'h')_{13}$. In our example, knowing that $\phi(e_{12}) = x_{12}$ and $\phi(e_{23}) = e_{23}$ means the line $\{e_{12}, e_{23}, e_{13}\}$ must be mapped to $\{x_{12}, e_{23}, (x \cdot e)_{13}\}$ and so on (Figure 7).

We now know the image of any internal points on $P_1 \vee P_2$ and $P_1 \vee P_3$ so for any $x_{23} \in M$, if $\phi(g_{12}) = g'_{12}$ and $\phi(h_{13}) = h'_{13}$, then $x_{23} = (g'^{-1}h')_{23}$. Thus, all internal points are determined and ϕ is determined.

Considering now the mappings of the joints P_1, P_2, and P_3 under ϕ, as discussed earlier, joints may only be mapped to other joints and all automorphisms of the joints can be counted by automorphisms of the 2-simplex. There are $3! = 6$ such symmetries.

For any automorphism ϕ, there are $|C_3|! = 6$ permutations of the line $P_1 \vee P_2$, $|C_3| = 3$ choices for a point on $P_2 \vee P_3$, and 6 permutations of the joints. As we have indicated, the automorphisms for all the points are then determined and thus $|\operatorname{Aut}(M)| = 108$.

We will now find the fixing number of M, that is, the minimum number of points that, when fixed, leave only the trivial automorphism. An automorphism ϕ of M may permute the internal points on the coordinate line $P_1 \vee P_2$ without restriction. To break any automorphism that permutes these points, we must fix all but 1 internal point on $P_1 \vee P_2$. We also determined the image of

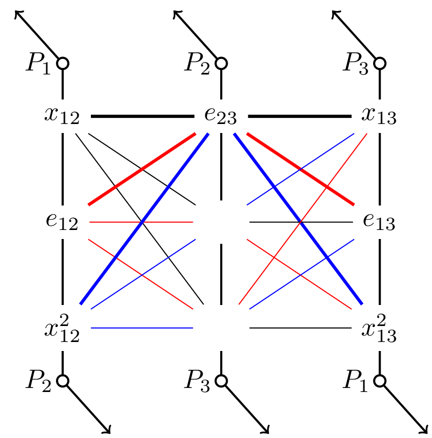

Figure 7: Example of determining $\phi(h_{23})$ for some h_{23} and all internal points on $P_1 \vee P_3$.

any single point on $P_2 \vee P_3$ and then ϕ was completely determined. So we must fix one additional point on $P_2 \vee P_3$. If we fix at least one point on any line $P_i \vee P_j$, we fix the joints of that line except the possibility of mapping P_i to P_j and *vice versa*. Thus we have fixed the placement of all three joints, since permuting any two will move a joint off of its corresponding coordinate line and $\operatorname{fix}(M) = (|C_3| - 1) + 1 = |C_3| = 3$.

3 Results

Lemma. *Given a finite group G on n elements, the matroid, M, associated with the rank-3 Dowling geometry of G has the following properties:*

1. $|\operatorname{Aut}(M)| = 6n!n$
2. $\operatorname{fix}(M) = n$.

Proof. Let G be a finite group of order n and M be the matroid associated with the rank-3 Dowling geometry of G. We claim that an automorphism that fixes the joints is completely determined by its action on one coordinate line and one interior point not on the line. Without loss of generality, we will take the coordinate line $P_1 \vee P_2$ and a single interior point on $P_2 \vee P_3$.

Suppose ϕ is an automorphism of M that fixes P_1, P_2, and P_3. ϕ must permute points on the internal lines since the joints are fixed. Suppose $\phi(h_{23}) = h'_{23}$ for some $h_{23}, h'_{23} \in M$. We claim that determining h'_{23} along with the permutation of the internal points on $P_1 \vee P_2$ determines ϕ. For any x_{13} on $P_1 \vee P_3$, there exists a line $\{g_{12}, h_{12}, x_{13}\}$ for some $g_{12} \in M$. If $\phi(g_{12}) = g'_{12}$, then $x_{13} = (g'h')_{13}$. We now know the image of any internal points on $P_1 \vee P_2$ and $P_1 \vee P_3$ so for any $x_{23} \in M$, if $\phi(g_{12}) = g'_{12}$ and $\phi(h_{13}) = h'_{13}$, then $x_{23} = (g'^{-1}h')_{23}$. Thus, all internal points are determined and ϕ is determined. For any automorphism ϕ, there are $n!$ permutations of the line $P_1 \vee P_2$, n choices for a point on $P_2 \vee P_3$.

Considering now the mappings of the joints P_1, P_2, and P_3 under ϕ, since internal points and joints are collinear with a strictly different number of points, joints may only be mapped to other joints and all automorphisms of the joints can be counted by automorphisms of the 2-simplex. There are $3! = 6$ such automorphisms.

As we have indicated, there are $n!n$ automorphisms of the internal points and 6 permutations of the joints. Thus $|\operatorname{Aut}(M)| = 6n!n$.

We will now find the fixing number of M, that is, the minimum number of points that, when fixed, leave only the trivial automorphism. An automorphism ϕ of M may permute the internal points on the coordinate line $P_1 \vee P_2$ without restriction. To break any automorphism that permutes these points, we must fix all but 1 internal point on $P_1 \vee P_2$. We also determined the image of any single point on $P_2 \vee P_3$ and then ϕ was completely determined. So we must fix one additional point on $P_2 \vee P_3$. If we fix at least one point on any line $P_i \vee P_j$, we fix the joints of that line except the possibility of mapping P_i to P_j and *vice versa*. Thus we have fixed the placement of all three joints, since permuting any two will move a joint off of its corresponding coordinate line. Fixing any fewer points would have resulted in a non-trivial automorphism and we have a fixing set of minimum size. Thus $\operatorname{fix}(M) = (n-1) + 1 = n$. □

Theorem. *Given a finite group G on n elements, the matroid M associated with the rank-r Dowling geometry on G, for $r \geq 3$, has the following properties:*

1. $|\operatorname{Aut}(M)| = r!n!n^{r-2}$
2. $\operatorname{fix}(M) = n + r - 3$

Proof. Let G be a finite group of order n and M be the matroid corresponding to the rank-r Dowling geometry based on G. We claim that an automorphism that fixes the joints is completely determined by its action on one coordinate line and one interior point not on the line.

Let ϕ be an automorphism of M that fixes the joints $P_1, P_2, \ldots, P_r$. ϕ must permute points on the internal lines since the joints are fixed. Consider the coordinate line $P_1 \vee P_r$. ϕ may permute the internal points of $P_1 \vee P_r$ without restriction. Consider the coordinate planes $P_1 \vee P_i \vee P_r$ for $1 < i < r$ (Figure 8). Each plane contains the line $P_1 \vee P_r$ with all mappings of the internal points determined. As shown in the proof of the lemma, fixing one more point on $P_1 \vee P_i$ will determine all internal points of the coordinate plane $P_1 \vee P_i \vee P_r$. There are $r-2$ many such planes and n choices of $\phi(g_{1i})$ for some g_{1i} on $P_1 \vee P_i$ so this results in n^{r-2} many choices. Thus, there are at least $n!n^{r-2}$ automorphisms of M.

For any coordinate plane $P_1 \vee P_i \vee P_j$, the coordinate lines $P_1 \vee P_i$ and $P_1 \vee P_j$ are determined and thus all internal points of the plane are determined by ϕ. This implies that the internal points of $P_i \vee P_j$ are determined for all $1 < i < j < r$. Since $P_1 \vee P_r$ is also determined, all internal points M have been determined. Thus, there are exactly $n!n^{r-2}$ automorphisms of M that fix $P_1, P_2, \ldots, P_r$.

Considering automorphisms that permute the joints $P_1, P_2, \ldots, P_r$, as shown before, these are the automorphisms of the $(r-1)$-simplex. The number of automorphisms of a $(r-1)$-simplex is $r!$.

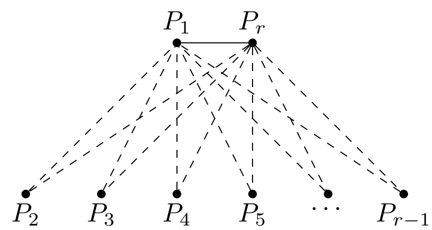

Figure 8

As we have indicated, there are $n!n^{r-2}$ automorphisms of the internal points and $r!$ permutation of the joints. Thus $|\operatorname{Aut}(M)| = r!n!n^{r-2}$.

We will now find the fixing number of M. For any automorphism ϕ, it may permute the internal points on the coordinate line $P_1 \vee P_r$ without restriction. To allow only the trivial automorphism, we must fix all but 1 internal point on $P_1 \vee P_r$. We determined the image of one more point on each line $P_1 \vee P_i$ for $1 < i < r$ to determine all internal points, and so we must fix one additional point on each of these $r-2$ lines. We have now fixed a point on every line $P_i \vee P_j$ and thus we have fixed the joints of that line except the possibility of mapping P_i to P_j and *vice versa*. Thus we have fixed the placement of all r joints, since permuting any two will move a joint off of its corresponding coordinate line. Fixing any fewer points would have resulted in a non-trivial automorphism and we have a fixing set of minimum size. Thus $\operatorname{fix}(M) = (n-1) + (r-2) = n + r - 3$. $\square$

4 Notes

It may be striking that the size of the automorphism group and the fixing number do not depend on the structure of the group, rather only the size of the group. This begs the question of how different are two Dowling geometries based on two non-isomorphic groups of the same size. Indeed, the structure of the transversal lines depend directly on the structure of G since every coordinate plane is encoding the group action. Two Dowling geometries based on groups of the same size can be very different from each other in structure while having automorphism groups of the same size.

Let us look at the groups $C_4 = \langle x | x^4 = e \rangle$ and $C_2 \times C_2 = \langle x, y | x^2 = y^2 = e; xy = yx \rangle$. In order to compare the lines in $Q_3(G)$, for C_4 we will write the elements x, x^2, and x^3 as a, b, and c respectively. For $C_2 \times C_2$, we will write the elements x, y, and xy as a, b, and c respectively. Not writing the subscript since it is implicit in a rank-3 Dowling geometry, Figure 9 lists all transversal lines of $Q_3(G)$ for both groups. The lines that differ between the groups are in bold.

C_4	$C_2 \times C_2$
(e, e, e)	(e, e, e)
(e, a, a)	(e, a, a)
(e, b, b)	(e, b, b)
(e, c, c)	(e, c, c)
(a, e, a)	(a, e, a)
$(\mathbf{a}, \mathbf{a}, \mathbf{b})$	$(\mathbf{a}, \mathbf{a}, \mathbf{e})$
(a, b, c)	(a, b, c)
$(\mathbf{a}, \mathbf{c}, \mathbf{e})$	$(\mathbf{a}, \mathbf{c}, \mathbf{b})$

C_4	$C_2 \times C_2$
(b, e, b)	(b, e, b)
(b, a, c)	(b, a, c)
(b, b, e)	(b, b, e)
(b, c, a)	(b, c, a)
(c, e, c)	(c, e, c)
$(\mathbf{c}, \mathbf{a}, \mathbf{e})$	$(\mathbf{c}, \mathbf{a}, \mathbf{b})$
(c, b, a)	(c, b, a)
$(\mathbf{c}, \mathbf{c}, \mathbf{b})$	$(\mathbf{c}, \mathbf{c}, \mathbf{e})$

Figure 9: Comparison of the lines of $Q_3(G)$ for two non-isomorphic groups of order 4.

The rank-r Dowling geometry of a group G is sometimes thought of as a sequence of geometries over r (Kahn and Kung) [7]. It may be useful to investigate the asymptotic growth of the automorphism groups of the sequence as r grows large. For some group G of fixed size n and the matroid $M = Q_r(G)$,

$$\begin{aligned} |\operatorname{Aut}(M)| &= r!n!n^{r-2} \\ &= \mathcal{O}(r!n^{r-2}) \\ &= \mathcal{O}(r!n^r) \\ &= \mathcal{O}(r^r n^r) \\ &= \mathcal{O}((nr)^r) \end{aligned}$$

as $r \to \infty$. We can improve this upper bound by using Stirling's approximation:

$$\begin{aligned} |\operatorname{Aut}(M)| &= r!n!n^{r-2} \\ &\sim \sqrt{2\pi r}\left(\frac{r}{e}\right)^r n!n^{r-2} \\ &= \sqrt{2\pi}n!\frac{1}{e^2}r^{r+1/2}\left(\frac{n}{e}\right)^{r-2} \\ &= \mathcal{O}\left(r^{r+1/2}\left(\frac{n}{e}\right)^{r-2}\right) \\ &= \mathcal{O}\left(\frac{nr}{e}\right)^{r+1/2} \end{aligned}$$

as $r \to \infty$.

Let us illustrate this growth by fixing $n = 3$ and looking at the values of $|\operatorname{Aut}(M)|$ and $\operatorname{fix}(M)$ for $r = 1, 2, \ldots, 10$ (Figure 12).

size of group

rank	1	2	3	4	5	6	7
1	1	1	2	6	24	120	720
2	2	4	12	48	240	1 440	10 080
3	6	24	108	576	3 600	25 920	211 680
4	24	192	1 296	9 216	72 000	622 080	5 927 040
5	120	1 920	19 440	184 320	1 800 000	1 8662 400	207 446 400
6	720	23 040	349 920	4 423 680	54 000 000	671 846 400	8 712 748 800
7	5 040	322 560	7 348 320	123 863 040	1 890 000 000	28 217 548 800	426 924 691 200

Figure 11: Values of $|\operatorname{Aut}(Q_r(G))|$ for various ranks r and sizes of groups.

r	$\lvert\operatorname{Aut}(M)\rvert$	$\operatorname{fix}(M)$
1	2	1
2	12	2
3	108	3
4	1 296	4
5	19 440	5
6	349 920	6
7	7 348 320	7
8	176 359 680	8
9	4 761 711 360	9
10	142 851 340 800	10

Figure 10: Size of automorphism group and fixing number for rank-r Dowling geometries based on a group of size 3.

It is interesting that the size of the automorphism group grows very quickly [$\mathcal{O}((3r)^r)$], but the fixing number, the fewest number of points to break all of those automorphisms, grows linearly. For example, when $r = 10$ and $n = 3$, there are 142 851 340 800 automorphisms and it only takes 10 fixed points to break them all. Figure 11 shows values of $|\operatorname{Aut}(Q_r(G))|$ for each $r, n \in \{1, \ldots, 7\}$.

5 Further Research

The importance of the the Dowling geometry was established by Khan and Kung [7]. There are four more geometries mentioned involved in the theorem. Enumerating their automorphism groups and finding the fixing number would be interesting.

The structure of the automorphism groups of the Dowling geometry would be interesting to determine. For example, is $\operatorname{Aut}(Q_3(C_4)) \cong \operatorname{Aut}(Q_3(C_2 \times C_2))$?

The fixing number of a matroid is a relatively new concept and only recently has been studied (Gordon, McNulty, and Neudauer, 2013) [6]. There are many classes of matroids yet to be considered.

6 Acknowledgments

I would firstly like to thank my research adviser Dr. Jenny McNulty for literally writing the book on matroids and motivating the creation of this paper. I would also like to acknowledge my colleagues Roger Madplume, Scott Davis, and Erik Borke for showing me just how many times I can be wrong.

References

[1] Joseph E. Bonin. *Characterizing Combinatorial Geometries by Numerical Invariants.* European Journal of Combinatorics, Vol. 20, Issue 8 (1999), pp. 713-724

[2] Peter Doubilet, Gian-Carlo Rota, and Richard P. Stanley. *On the foundations of combinatorial theory (VI): The idea of generating function.* In: *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (Berkeley, Calif., 1970/71), Vol. II: *Probability Theory*, pp. 267-318. University of California Press, Berkeley, Calif., 1972.

[3] T.A. Dowling. *A q-analog of the partition lattice.* Chapter 11 in: J.N. Srivastava et al., eds., *A Survey of Combinatorial Theory* (Proceedings of an International Symposium, Ft. Collins, Colo., 1971), pp. 101-115. North-Holland, Amsterdam, 1973.

[4] T.A. Dowling. *A class of geometric lattices based on finite groups.* Journal of Combinatorial Theory Series B, Vol. 14 (1973), pp. 61-86.

[5] Gary Gordon. *Matroid Representation, Geometry and Matrices.* 2001. Web. Accessed 5 May 2016.

[6] Gary Gordon, Jennifer McNulty, & Nancy Ann Neudauer. *Fixing numbers for matroids.* 2013, arXiv:1307.7460

[7] J. Kahn and J. P. S. Kung. *Varieties of combinatorial geometries.* Trans. Amer. Math. Soc. 271 (1982), 485-499.