On the Uniqueness and Existence of Stochastic Differential Equations with Infinite Delay and Lévy Noise

Final Essay of Martingale Theory and Stochastic Integration

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Abstract

In this essay we are going to study one type of **Stochastic Differential Delay Equation (SDDE)**, driven with Brownian motion and Poisson random measure and polynomial growth. In section 2, we will give our formal definition of the uniqueness and existence in sense of this special equation. In section 3 we will introduce some assumptions and the main result of our essay. In section 4 we use the assumptions before to give a formal proof of the main theorem.

§1   Introduction, Problem Setting

The equation we are going to study can be formulated as

$$\mathrm{d}X_t = f(X_t) + \int_{-\infty}^0 X_{t+r} g(r)\mathrm{d}r\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t + \int_Z b(t, X_{t-}, z)\tilde{N}_P(\mathrm{d}z\mathrm{d}t). \tag{1}$$

This equation differs from the classical SDE which we have concerned before, because of involving the delay term and Poisson random noise. Specifically, the equation 1 involves the integral memory term which preserve the whole past information. Consequently this results in more difficult analysis.

Here we present with full problem setting of the equation 1. Assume that we have fixed filtered space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$. Let $B = (B_t)_{t\geqslant 0}$ be the standard Brownian motion on the space defined above. Let the measure space $(Z, \mathscr{B}, \nu)$ be the base of our adapted stationary Poisson point process p with induced Poisson random measure N_P. Denote the compensated Poisson process as

$$\tilde{N}_P := \hat{N}_P(\mathrm{d}z\mathrm{d}t) - \mathrm{d}t\nu(\mathrm{d}z)$$

Moreover, the equation 1 is given with the initial data $X_0 = \xi$, by this we actually means $\xi = \xi_t \in L^2((-\infty, 0], \mathbb{R})$, a deterministic function which is actually the path before $t = 0$.

Additionally, we assume that σ and b are global Lipschitz function, while $f, g : \mathbb{R} \to \mathbb{R}$ are non-random function. And f is assumed to be a polynomial.

Problem   On what additional conditions can we deduce the uniqueness and existence result of the equation 1?

We're going to give a possible solution of this problem in the following sections.

§2   Definition of Uniqueness and Existence

To begin with, we shall clarify the definition of existence and uniqueness for this special equation.

Definition 1 (Pathwise Uniqueness). For fixed set-up $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P}, B, N)$, if there are two semimartingales Y and Y' that solves the equation 1, then they are indistinguishable, i.e., a.s. with same trajectory.

Furthermore, since the equation differs from the usual Brownian-motion driven equation with jumps, we shall introduce some additional definitions of solutions.

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Definition 2 (Global and Local Solution). For fixed set-up $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, \mathbb{P}, B, N)$, we say the $\mathscr{F}_t$-adapted semimartingale X is the **global solution** of the equation 1 if we have

1. $X_t = \xi_t, \forall t \leqslant 0$.

2. $\mathbb{P}$-a.s. X_t is càdlàg.

3. $\forall t \geqslant 0$ we have

$$\mathbb{P}\text{-a.s.} \quad \begin{cases} \int_0^t |\sigma(s, X_s)|^2 \mathrm{d}s < \infty \\ \int_0^t |f(X_s)| \mathrm{d}s < \infty \\ \int_0^t \int_Z |b(s, X_s, z)|^2 \nu(\mathrm{d}z) \mathrm{d}s < \infty \end{cases}$$

4. $\mathbb{P}$-a.s., the following equation is true.

$$X_t = \xi_0 + \int_0^t f(X_s) \mathrm{d}s + \int_0^t \int_{-\infty}^0 X_{s+r} g(r) \mathrm{d}r \mathrm{d}s + \int_0^t \sigma(s, X_s) \mathrm{d}B_s + \int_0^t \int_Z b(s, X_{s-}, z) \tilde{N}_p(\mathrm{d}z \mathrm{d}s), \quad \forall t \geqslant 0.$$

Moreover, if this equation is true only before the stopping time τ, i.e.,

$$X_{t \wedge \tau} = \xi_0 + \int_0^{t \wedge \tau} f(X_s) \mathrm{d}s + \int_0^{t \wedge \tau} \int_{-\infty}^0 X_{s+r} g(r) \mathrm{d}r \mathrm{d}s + \int_0^{t \wedge \tau} \sigma(s, X_s) \mathrm{d}B_s + \int_0^{t \wedge \tau} \int_Z b(s, X_{s-}, z) \tilde{N}_p(\mathrm{d}z \mathrm{d}s), \quad \forall t \geqslant 0.$$

Then we say that the couple (X, τ) is said to be the local solution of the equation 1. If τ is the explosion time of X and there is a increasing sequence of stopping time $\tau_n \nearrow \tau$, and (X, τ_n) are all the local solutions, then (X, τ) is said to be the maximal local solution.

§3 Main Result

Assumption. The following assumptions may be critical in our proof.

A.1 (Global Lipschitz of σ and b) In the following context, we always regard σ and b to be globally Lipschitz continuous, i.e.

$$|\sigma(s, x) - \sigma(s, y)| \wedge |b(s, x, z) - b(s, y, z)| \leqslant L|x - y|, \quad \forall x, y \in \mathbb{R}, z \in Z, s \in \mathbb{R}.$$

A.2 (Linear Growth) f is a linear function, $f(x) = a_0 + a_1 x$, and without loss of generality $|a_1| \leqslant L$. This implies that f is also L Lipschitz function.

A.3 (Nonlinear Polynomial Growth) The order of f is no less than 2, this means f is locally Lipschitz.

A.4 (Integrability) We assume $g \in L^2((-\infty, 0], \mathbb{R})$.

Based on the aforementioned assumptions, we present our main results in this essay.

Theorem 1 (Global). Under assumptions A.1, A.2 and A.4, the equation 1 exists a unique global solution(modulo indistinguishable).

Theorem 2 (Local). Under assumptions A.1, A.3 and A.4, the equation 1 exists a unique maximal local solution.

§4 Proof of Main Result

4.1 Uniqueness

The first part concerns with the uniqueness. Let X and Y be the τ-localized solution, on the finite interval $[0, T]$. Moreover, define the stopping time

$$\tau_n = \tau \wedge \{s : |X_s| \vee |Y_s| \geqslant n\}.$$

Define $X_t^n = X_{t \wedge \tau_n}$ and $Y_t^n = Y_{t \wedge \tau_n}$. Consider the square expectation of the difference

$$\mathbb{E}\left[(X_t^n - Y_t^n)^2\right] = \mathbb{E}\left[\left(\int_0^t (f(X_s^N n) - f(Y_s^n)) \mathrm{d}s + \int_0^t \int_{-\infty}^0 (X_{s+r}^n - Y_{s+r}^n) g(r) \mathrm{d}r \mathrm{d}s + \int_0^t (\sigma(s, X_s^n) - \sigma(s, Y_s^n)) \mathrm{d}B_s\right.\right.$$
$$\left.\left. + \int_0^t \int_Z (b(s, X_{s-}^n, z) - b(s, Y_{s-}^n, z)) \tilde{N}_p(\mathrm{d}z \mathrm{d}s)\right)^2\right]$$

By the basic arithmetic-square mean value inequality,

$$\text{RHS} \leqslant 4\mathbb{E}\left[\left(\int_0^t (f(X_s^n) - f(Y_s^n)) \mathrm{d}s\right)^2\right] + 4\mathbb{E}\left[\left(\int_0^t \int_{-\infty}^0 (X_{s+r}^n - Y_{s+r}^n) g(r) \mathrm{d}r \mathrm{d}s\right)^2\right]$$
$$+ 4\mathbb{E}\left[\left(\int_0^t (\sigma(s, X_s^n) - \sigma(s, Y_s^n)) \mathrm{d}B_s\right)^2\right] + 4\mathbb{E}\left[\left(\int_0^t \int_Z (b(s, X_{s-}^n, z) - b(s, Y_{s-}^n, z)) \tilde{N}_p(\mathrm{d}z \mathrm{d}s)\right)^2\right].$$

We bound it by term. Under assumption A.1, the linear growth and global Lipschitz property is true and therefore the first term can be written as

$$4\mathbb{E}\left[\left(\int_0^t (f(X_s^n) - f(Y_s^n)) \mathrm{d}s\right)^2\right] \leqslant 4\mathbb{E}\left[t \int_0^t (f(X_s^N) - f(Y_s^n))^2 \mathrm{d}s\right]$$
$$\leqslant 4TL^2 \mathbb{E}\left[\int_0^t (X_s^n - Y_s^n)^2 \mathrm{d}s\right].$$

Now turn to the second term, which involves the past memory. Since both X, Y solve the equation, they have identical past data before $t = 0$. So

$$4\mathbb{E}\left[\left(\int_0^t \int_{-\infty}^0 (X_{s+r}^n - Y_{s+r}^n) g(r) \mathrm{d}r \mathrm{d}s\right)^2\right] = 4\mathbb{E}\left[\left(\int_0^t \int_{-s}^0 (X_{s+r}^n - Y_{s+r}^n) g(r) \mathrm{d}r \mathrm{d}s\right)^2\right]$$
$$\leqslant 4\mathbb{E}\left[\left(\int_0^t \int_0^s (X_r^n - Y_r^n) g(r - s) \mathrm{d}r \mathrm{d}s\right)^2\right]$$
$$\leqslant 4\mathbb{E}\left[\left(\int_0^t \left(\int_0^s (X_r^n - Y_r^n)^2 \mathrm{d}r\right)^{1/2} \left(\int_0^s g(r - s)^2 \mathrm{d}r\right)^{1/2} \mathrm{d}s\right)^2\right]$$
$$\leqslant 4\mathbb{E}\left[\left(\int_0^t \left(\int_0^s (X_r^n - Y_r^n)^2 \mathrm{d}r\right)^{1/2} \|g\|_{L^2((-\infty, 0])} \mathrm{d}s\right)^2\right]$$
$$\leqslant 4 \|g\|_{L^2((-\infty, 0])}^2 T^2 \mathbb{E}\left[\int_0^t (X_r^n - Y_r^n)^2 \mathrm{d}r\right].$$

For the third drift term, global Lipschitz continuity and standard Ito's isometry gives

$$4\mathbb{E}\left[\left(\int_0^t (\sigma(s, X_s^n) - \sigma(s, Y_s^n)) \mathrm{d}B_s\right)^2\right] \leqslant 4\mathbb{E}\left[\int_0^t (\sigma(s, X_s^n) - \sigma(s, Y_s^n))^2 \mathrm{d}s\right]$$
$$\leqslant 4\mathbb{E}\left[\int_0^t L^2 (X_s^n - Y_s^n)^2 \mathrm{d}s\right].$$

And then comes to the jump term,

$$4\mathbb{E}\left[\left(\int_0^t \int_Z (b(s, X_{s-}^n, z) - b(s, Y_{s-}^n, z)) \tilde{N}_p(\mathrm{d}z \mathrm{d}s)\right)^2\right] \leqslant 4L^2 \mathbb{E}\left[\int_0^t \int_Z (X_s^n - Y_s^n)^2 \nu(\mathrm{d}z) \mathrm{d}s\right]$$
$$\leqslant 4L^2 \nu(Z) \mathbb{E}\left[\int_Z (X_s^n - Y_s^n)^2 \nu(\mathrm{d}z) \mathrm{d}s\right].$$

Let $h(t) = \mathbb{E}[(X_t^n - Y_t^n)^2]$, then combine the last four bounds to obtain

$$h(t) \leqslant C_{T,g,L} \int_0^t h(s) \mathrm{d}s, \quad t \leqslant T.$$

By the Grownall's inequality, we now have

$$h(t) = 0, \quad \forall t \leqslant T.$$

Thus, any two solutions of the equation 1 (could be both global or local) must be a.s. indistinguishable, thus we're done.

4.2 Global Existence

The second part concerns with the existence. It's based on the standard Picard approximation. Consider the class

$$H_T = \{h : \ h \text{ is } \mathscr{F}_t\text{-adapted and } \mathbb{P}\text{-a.s. càdlàg, with } h(0) = \xi\}.$$

And define the norm on H_T as

$$\|h\| := \mathbb{E}\big[\sup_{s \leqslant T} |h_s|^2 \big].$$

Consider the linear operator $J : H_T \to H_T$,

$$h \mapsto Jh := (Jh)_t = \xi_0 + \int_0^t f(h_s)\mathrm{d}s + \int_0^t \int_{-\infty}^0 h_{s+r} g(r)\mathrm{d}r\mathrm{d}s + \int_0^t \sigma(s, h_s)\mathrm{d}B_s + \int_0^t \int_Z b(s, h_{s-}, z)\tilde{N}_p(\mathrm{d}z\mathrm{d}s), \quad \forall T \geqslant t \geqslant 0.$$

Then, if $\|h\|_{H_T} < \infty$, then $\|Jh\|_{H_T} < \infty$.
On the other hand, if $h, h' \in H_T$, then we shall bound $\|Jh - Jh'\|$ in the following sense.

$$\mathbb{E}\big[\sup_{t \leqslant T}(Jh_t - Jh_t')^2 \big] \leqslant 4\mathbb{E}\bigg[\sup_{t \leqslant T} \Big(\int_0^t \big(f(h_s) - f(h_s')\big)\mathrm{d}s \Big)^2 + \sup_{t \leqslant T} \Big(\int_0^t \int_{-\infty}^0 (h_{s+r} - h_{s+r}')g(r)\mathrm{d}r\mathrm{d}s \Big)^2$$

$$+ \sup_{t \leqslant T} \Big(\int_0^t \big(\sigma(s, h_s) - \sigma(s, h_s')\big)\mathrm{d}B_s \Big) + \sup_{t \leqslant T} \Big(\int_0^t \int_Z \big(b(s, h_{s-}, z) - b(s, h_{s-}', z)\big)\tilde{N}_p(\mathrm{d}z\mathrm{d}s) \Big)^2 \bigg].$$

Again we bound each term separately. For the first term

$$4\mathbb{E}\bigg[\sup_{t \leqslant T} \Big(\int_0^t \big(f(h_s) - f(h_s')\big)\mathrm{d}s \Big)^2 \bigg] \leqslant 4T\mathbb{E}\bigg[\int_0^T \big(f(h_s) - f(h_s')\big)^2 \mathrm{d}s \bigg]$$

$$\leqslant 4T\mathbb{E}\bigg[\int_0^T L^2 \sup_{u \leqslant s}\big(h_u' - h_u\big)^2 \mathrm{d}s \bigg].$$

For the memory term.

$$4\mathbb{E}\bigg[\sup_{t \leqslant T} \Big(\int_0^t \int_{-\infty}^0 (h_{r+s} - h_{r+s}')g(r)\mathrm{d}r\mathrm{d}s \Big)^2 \bigg] \leqslant 4\mathbb{E}\bigg[\sup_{t \leqslant T} \Big(\int_0^t \int_0^s (h_r - h_r')g(r - s)\mathrm{d}r\mathrm{d}s \Big)^2 \bigg]$$

$$\leqslant 4\mathbb{E}\bigg[\sup_{t \leqslant T} \Big(\int_0^t \big(\int_0^s (h_r - h_r')^2\mathrm{d}r \big)^{1/2} \big(\int_0^s g^2(r - s)\mathrm{d}r \big)^{1/2}\mathrm{d}s \Big)^2 \bigg]$$

$$\leqslant 4\|g\|_{L^2((-\infty,0])} \mathbb{E}\bigg[\sup_{t \leqslant T} \int_0^t \Big(\int_0^s (h_r - h_r')^2\mathrm{d}r \Big)\mathrm{d}s \bigg]$$

$$\leqslant 4\|g\|_{L^2((-\infty,0])} \mathbb{E}\bigg[\int_0^T T \sup_{r \leqslant s}(h_r - h_r')^2\mathrm{d}s \bigg].$$

For the drift term, we shall use the Burkholder-Davis-Gundy inequality to bound

$$4\mathbb{E}\bigg[\sup_{t \leqslant T} \Big(\int_0^t \big(\sigma(s, h_s) - \sigma(s, h_s')\big)\mathrm{d}B_s \Big)^2 \bigg] \leqslant 4C_p\mathbb{E}\bigg[\int_0^T \big(\sigma(s, h_s) - \sigma(s, h_s')\big)^2\mathrm{d}s \bigg]$$

$$\leqslant 4C_p L^2\mathbb{E}\bigg[\int_0^T \sup_{u \leqslant s}(h_u' - h_u)^2\mathrm{d}s \bigg].$$

And finally to the jump term, by the generalized Burkholder-Davis-Gundy inequality, we have

$$4\mathbb{E}\bigg[\sup_{t \leqslant T} \Big(\int_0^t \int_Z \big(b(s, h_{s-}, z) - b(s, h_{s-}', z)\big)\tilde{N}_p(\mathrm{d}z\mathrm{d}s) \Big)^2 \bigg] \leqslant 4C_p'\mathbb{E}\bigg[\int_0^{T+} \int_Z \big(b(s, h_{s-}, z) - b(s, h_{s-}', z)\big)^2 N_p(\mathrm{d}z\mathrm{d}s) \bigg]$$

$$\leqslant 4C_p'\bigg[\int_0^T \int_Z L^2\big(h_s - h_s'\big)^2 \nu(\mathrm{d}z)\mathrm{d}s \bigg] \leqslant 4C_p' L^2 \nu(Z)\mathbb{E}\bigg[\int_0^T \big(h_s - h_s'\big)^2\mathrm{d}s \bigg].$$

Combine all the bounds above to get

$$\mathbb{E}\left[\sup_{t\leqslant T}(Jh_t - Jh'_t)^2\right] \leqslant C_{p,T,Z,L}\mathbb{E}\left[\int_0^T \sup_{u\leqslant t}(h_u - h'_u)^2 \mathrm{d}t\right]. \tag{2}$$

Now we construct a sequence, $h^n = Jh^{n-1}$, define

$$g_n(t) = \mathbb{E}\left[\sup_{s\leqslant t}(h_s^n - h_s^{n-1})^2\right].$$

By the estimation 2, we have

$$g_n(t) \leqslant C\int_0^t g_{n-1}(s)\mathrm{d}s.$$

This inequality implies that

$$g_n(t) \leqslant C_0 C^{n-1}\frac{t^{n-1}}{(n-1)!}.$$

In particular, we have $\sum_n g_n(T)^{1/2} < \infty$, which implies that

$$\mathbb{P}\text{-a.s.}\quad \sum_n \sup_{s\leqslant T}|h_s^n - h_s^{n-1}| < \infty.$$

This means, h^n converges uniformly to a limit h. Theorem 3 implies that the limit process is (a.s.) càdlàg. And by a limit process we can prove that the limit is exactly the solution.

4.3 Localization

The following part is based on assumption A.1, A.3 and A.4, therefore the f is only local Lipschitz. We shall prove the local result with a truncation technique.

Define the truncated growth function

$$f_N(x) = f\left(\frac{|x|\wedge N}{|x|}x\right).$$

Since f is a polynomial, f_N is globally Lipschitz continuous with constant L_N. So, the global scenario implies that for fixed set-up, following equation

$$\mathrm{d}X_t = f_N(X_t) + \int_{-\infty}^0 X_{t+r}g(r)\mathrm{d}r\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t + \int_Z b(t,X_{t-},z)\tilde{N}_p(\mathrm{d}z\mathrm{d}t).$$

has unique global solution (up to indistinguishable). Note that before the stopping time $\tau_N := \inf\{t : |X_t| \geqslant N\}$, the process $X_{t\wedge\tau_N}$ solves the equation 1. Hence (X, τ_N) is a local solution of equation 1.

Furthermore, $\tau = \lim_N \tau_N$, is the explosion time. From the uniqueness part we know that (X, τ_N) and (X, τ_{N-1}) are identical before τ_{N-1}. This implies that we can extend the unique local solution to the unique maximal local solution. The existence and uniqueness is inherited.

§5 Conclusion

In this essay, we studied stochastic differential equations with infinite delay and Lévy noise. We proved the global results under linear growth condition, and local result under polynomial nonlinear growth condition.

This essay mainly referred from [Wei and Wang, 2007] and the proof in [Gall, 2016, Theorem 8.3].

Appendix

The following theorem is usually abbreviated as BDG inequality, [Gall, 2016, Theorem 5.16]

Theorem 3 (Burkholder-Davis-Gundy Inequality)**.** For every real $p > 0$, there exist two constants $c_p, C_p > 0$ depending only on p such that, for every continuous local martingale M with $M_0 = 0$, and every stopping time T,

$$c_p\mathbb{E}\left[\langle M, M\rangle_T^{p/2}\right] \leqslant \mathbb{E}\left[\sup_{s\leqslant T}|M_s|^p\right] \leqslant C_p\mathbb{E}\left[\langle M, M\rangle_T^{p/2}\right].$$

The following inequality generalize the BDG inequality to the càdlàg processes

Theorem 4 (Bukholder-Davis-Gundy, Generalization). For every real $p \geqslant 2$, there exist two constants $c_p, C_p > 0$ depending only on p such that, for every càdlàg martingale M with $M_0 = 0$, and every stopping time T,

$$c_p \mathbb{E}\big[[M, M]_T^{p/2}\big] \leqslant \mathbb{E}\big[\sup_{s \leqslant T} |M_s|^p\big] \leqslant C_p \mathbb{E}\big[[M, M]_T^{p/2}\big].$$

Theorem 5 (Uniform Limit Preserve Càdlàg Property). Let $f_n : [0, T] \to \mathbb{R}$ be a uniformly convergent sequence with limit f. Assume that f_n are all càdlàg, then f is càdlàg.

Proof. The proof of this result is purely a mathematical analysis stuff.

Let x be the discontinuous point of f. For $\varepsilon > 0$, exists $N > 0$ such that for every $m, n > N$ we have

$$|f_m(t) - f_n(t)| \leqslant \varepsilon/3, \quad \forall t \in [0, T].$$

For every m, n, we can choose t_0 near x such that

$$|f_m(t_0) - f_m(x-)| \wedge |f_n(t_0) - f_n(x-)| \leqslant \varepsilon/3.$$

Hence,

$$|f_n(x-) - f_m(x-)| \leqslant \varepsilon, \quad \forall m, n \geqslant N.$$

This implies that $f_n(x-)$ is Cauchy, hence limits in $\mathbb{R}$.

Now consider

$$|f(t) - \lim_n f_n(x-)| \leqslant |f(t) - f_n(t)| + |f_n(t) - f_n(x-)| + |f_n(x-) - \lim_n f_n(x-)|.$$

For fixed $\varepsilon > 0$, the first and third term are less than $\varepsilon/3$ when $n > N$. Fix $n > N$, the second term is less than $\varepsilon/3$ when $x - \delta < t < x$. This implies that $\lim_n f_n(x-) = f(x-)$.

For the second part, basically we have

$$\lim_n f_n(x+) = f(x+).$$

On the other hand,

$$f(x) = \lim f_n(x) = \lim f_n(x+) = f(x+).$$

So f is càdlàg. We are done. ∎

References

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