Analysis of Moment Dependent Stochastic Differential Equations

Third Homework of Martingale Theory of Stochastic Integration - Part II

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§1 Introduction, Problem Settings

In the previous homeworks, we were working on classical stochastic differential equations in the following formulation

$$X_t = x_0 + \int_0^t \sigma(s, X_s)\mathrm{d}B_s + \int_0^t b(s, X_s)\mathrm{d}s.$$

However, in this homework, we're going to study a type of generalized SDE. Briefly, the coefficients here are not only related to the path value, but also the moment of path, which means the solution would necessarily dependent on the set-up, because the calculation of the moment is explicitly dependent on the distribution.

Formally, the SDE we're going to study in this homework is of the following form.

$$X_t = x_0 + \int_0^t G(s, X_s, \mathbb{E}_{\mathbb{P}}[X_s^2])\mathrm{d}B_s + \int_0^t b(s, X_s, \mathbb{E}_{\mathbb{P}}[X_s^2])\mathrm{d}s, \quad t \in [0, T]. \tag{1}$$

Here, the function $G, b : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are bounded and continuous. And

$$\mathbb{E}_{\mathbb{P}}[X_s^2] := \int_\Omega X_s^2 \mathrm{d}\mathbb{P}$$

is the second-moment calculated under $\mathbb{P}$.

§2 Definition of Existence and Uniqueness

Since the equation is slightly different from the earlier one, we shall re-clarify the definition of the existence and uniqueness in terms of this special equation.

Like the conventions in the earlier homeworks, by a **set-up**, we mean $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P}, B)$. Here B is a $\mathscr{F}_t$-Brownian motion under probability measure $\mathbb{P}$.

2.1 Strong Scenario

As before, a strong solution should be dependent on the set-up.

Definition 1 (Strong Solution)**.** Given the set-up $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P}, B)$, the strong solution of equation 1 under the set-up is a $\mathscr{F}_t$-adapted process X_t, such that

- $\mathbb{P}$-a.s., the $X(\omega)$ is continuous in t.
- $\mathbb{P}$-a.s., the following equation is true

$$X_t = x_0 + \int_0^t G(s, X_s, \mathbb{E}_{\mathbb{P}}[X_s^2])\mathrm{d}B_s + \int_0^t b(s, X_s, \mathbb{E}_{\mathbb{P}}[X_s^2])\mathrm{d}s, \quad \forall t \in [0, T].$$

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We say that the equation is exact, if for any set-up $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P}, B)$, the equation has exactly one strong solution (up to $\mathbb{P}$-indistinguishable).

The uniqueness under this strong scenario is pathwise uniqueness.

Definition 2 (Pathwise Uniqueness)**.** We shall say that the pathwise uniqueness holds for equation 1, if given any set-up $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P}, B)$, and two continuous semimartingale X and X' such that

$$\mathbb{P}-\text{a.s.} \quad \begin{cases} X_t = x_0 + \int_0^t G(s, X_s, \mathbb{E}_{\mathbb{P}}[X_s^2])\mathrm{d}B_s + \int_0^t b(s, X_s, \mathbb{E}_{\mathbb{P}}[X_s^2])\mathrm{d}s, & \forall t \in [0,T], \\ X'_t = x_0 + \int_0^t G(s, X'_s, \mathbb{E}_{\mathbb{P}}[X_s'^2])\mathrm{d}B_s + \int_0^t b(s, X'_s, \mathbb{E}_{\mathbb{P}}[X_s'^2])\mathrm{d}s, & \forall t \in [0,T]. \end{cases}$$

Then,

$$\mathbb{P}\text{-a.s.} \quad X_t = X'_t, \quad \forall t \in [0,T].$$

2.2 Weak Scenario

In the weak scenario case, the set-up doesn't have to be specified previously. We have

Definition 3 (Weak Solution)**.** A weak solution of the SDE 1 consists of the following

- A filtered space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$.
- A $\mathscr{F}_t$-Brownian motion B.
- An $\mathbb{P}$-a.s. defined continuous semimartingale X, such that

$$\mathbb{P}\text{-a.s.} \quad X_t = x_0 + \int_0^t G(s, X_s, \mathbb{E}_{\mathbb{P}}[X_s^2])\mathrm{d}B_s + \int_0^t b(s, X_s, \mathbb{E}_{\mathbb{P}}[X_s^2])\mathrm{d}s, \quad \forall t \in [0,T].$$

We shall say that weak existence is true for equation 1 if there exists a weak solution of 1.

§3 A Simple Result on Pathwise Uniqueness

Assumption (Global Lipschitz Continuity)**.** We would always assume that, G and b are global Lipschitz continuous in the following sense

$$|G(s,x,m) - G(s,x',m')| \vee |b(s,x,m) - b(s,x',m')| \leqslant K(|x-x'| + |m-m'|).$$

3.1 Proof of Pathwise Uniqueness

Fix the set-up $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P}, B)$. From now we drop the subscript of expectation symbol. Assume that X and X' both solve the equation. Define the stopping time

$$\tau_n = \inf\{t \geqslant 0 : |X_t| \geqslant n \text{ or } |X'_t| \geqslant n\}.$$

Consider the L^2 difference.

$$
\begin{aligned}
&\mathbb{E}\left[(X_{t\wedge\tau_n} - X'_{t\wedge\tau_n})^2\right] \\
&\leqslant 2\mathbb{E}\left[\left(\int_0^{t\wedge\tau_n} (G(s, X_s, \mathbb{E}\left[X_s^2\right]) - G'(s, X_s, \mathbb{E}\left[X_s^2\right]))\mathrm{d}B_s\right)^2\right] + 2\mathbb{E}\left[\left(\int_0^{t\wedge\tau_n} (b(s, X_s, \mathbb{E}\left[X_s^2\right]) - b'(s, X_s, \mathbb{E}\left[X_s^2\right]))\mathrm{d}s\right)^2\right] \\
&\leqslant 2\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left(G(s, X_s, \mathbb{E}\left[X_s^2\right]) - G'(s, X_s, \mathbb{E}\left[X_s^2\right])\right)^2 \mathrm{d}s\right] + 2\mathbb{E}\left[\left(\int_0^{t\wedge\tau_n} (b(s, X_s, \mathbb{E}\left[X_s^2\right]) - b'(s, X_s, \mathbb{E}\left[X_s^2\right]))\mathrm{d}s\right)^2\right] \\
&\leqslant 2\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left(G(s, X_s, \mathbb{E}\left[X_s^2\right]) - G'(s, X_s, \mathbb{E}\left[X_s^2\right])\right)^2 \mathrm{d}s\right] + 2\mathbb{E}\left[t\wedge\tau_n \int_0^{t\wedge\tau_n} \left(b(s, X_s, \mathbb{E}\left[X_s^2\right]) - b'(s, X_s, \mathbb{E}\left[X_s^2\right])\right)^2 \mathrm{d}s\right] \\
&\leqslant 2K^2(1+T)\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left(\left|X_s - X'_s\right| + \left|\mathbb{E}\left[X_s^2 - X_s'^2\right]\right|\right)^2 \mathrm{d}s\right] \\
&\leqslant 4K^2(1+T)\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left|X_s - X'_s\right|^2 + \mathbb{E}\left[\left|(X_s + X'_s)(X_s - X'_s)\right|^2\right] \mathrm{d}s\right] \\
&\leqslant 4K^2(1+T)\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left|X_s - X'_s\right|^2 + 4n^2\mathbb{E}\left[\left|X_s - X'_s\right|^2\right] \mathrm{d}s\right] \\
&\leqslant 4K^2(1+T)\left(\mathbb{E}\left[\int_0^t \left|X_{s\wedge\tau_n} - X'_{s\wedge\tau_n}\right|^2 \mathrm{d}s\right] + 4n^2 \int_0^t \mathbb{E}\left[\left|X_{s\wedge\tau_n} - X'_{s\wedge\tau_n}\right|^2\right]\right)
\end{aligned}
$$

Define the function

$$h(t) = \mathbb{E}\left[X_{t\wedge\tau_n} - X'_{t\wedge\tau_n}\right].$$

Then we have

$$h(t) \leqslant 4K^2(1+T)(1+4n^2)\int_0^T h(s)\mathrm{d}s.$$

Moreover, h is a non-negative bounded function. By Grownall's inequality

$$h(t) \equiv 0.$$

This implies that,

$$\mathbb{P}\text{-a.s.} \quad X_{t\wedge\tau_n} = X'_{t\wedge\tau_n}, \quad \forall t \in [0, T].$$

By letting $n \to \infty$, the pathwise uniqueness is done.

3.2 Proof of Strong Existence

§4 Exactness from Yamada-Watanabe Theorem

Assume that for the fixed set-up $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, B)$, we have a continuous semimartingale that solves the equation 1. Then, we fix

$$f_t = \mathbb{E}_{\mathbb{P}}\left[X_t^2\right].$$

And then change the coefficients into

$$G^f(s, \cdot) := G(s, \cdot, f_s), \quad b^f(s, \cdot) := b(s, \cdot, f_s).$$

And consider the equation

$$Y_t = x_0 + \int_0^t G^f(s, Y_s)\mathrm{d}B_s + \int_0^t b^f(s, X_s)\mathrm{d}s. \tag{2}$$

This equation is of the classical form. And we can therefore use the Yamada-Watanabe theorem to obtain

Theorem 1. If the equation 2 is of pathwise uniqueness and exists a weak solution, then it's exact.

§5 Perturbation on Noise

In this section, we assume the following proposition to be true.

Proposition. There exists a function $F : C([0,T]) \to C([0,T])$, such that, for given set-up $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P}, B)$, we have $X = F(B)$ that solves the equation, i.e., the semimartingale $X = F(B)$ satisfy

$$\mathbb{P}\text{-a.s.} \quad \begin{cases} X(\omega) \text{ is continuous in } t, \\ X_t = x_0 + \int_0^t G(s, X_s, \mathbb{E}_{\mathbb{P}}[X_s^2])\mathrm{d}B_s + \int_0^t b(s, X_s, \mathbb{E}_{\mathbb{P}}[X_s^2])\mathrm{d}s, \quad \forall t \in [0,T]. \end{cases}$$

Based on this, we are going to study the semimartingale

$$X = F(B + \int_0^{\cdot} h(s)\mathrm{d}s).$$

We begin with recalling some results in homework 3-1. On the fixed set-up $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P}, B)$. Let

$$L_t = -\int_0^t h(s)\mathrm{d}B_s.$$

And define the measure $\mathrm{d}\mathbb{Q} = \mathscr{E}(L)_T \mathrm{d}\mathbb{P}$. Then we have proved that

$$\tilde{B}_t := B_t + \int_0^t h(s)\mathrm{d}s,$$

defines a $(\mathscr{F}_t, \mathbb{Q})$-Brownian motion. Now under the set-up $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P}, \tilde{B})$, the semimartingale $\tilde{X} = F(\tilde{B})$ is a strong solution of the equation

$$\mathbb{Q}\text{-a.s.} \quad \tilde{X}_t = x_0 + \int_0^t G(s, \tilde{X}_s, \mathbb{E}_{\mathbb{Q}}[\tilde{X}_s^2])\mathrm{d}\tilde{B}_s + \int_0^t b(s, \tilde{X}_s, \mathbb{E}_{\mathbb{Q}}[\tilde{X}_s^2])\mathrm{d}s, \quad \forall t \in [0,T].$$

Recall that the stochastic integral of progressive process with semimartingales are stable under the mutually absolutely continuous measure, we have

$$\mathbb{P}\text{-a.s.} \quad \tilde{X}_t = x_0 + \int_0^t G(s, \tilde{X}_s, \mathbb{E}_{\mathbb{P}}[\tilde{X}_s^2 \cdot D_T])\mathrm{d}B_s + \int_0^t G(s, \tilde{X}_s, \mathbb{E}_{\mathbb{P}}[\tilde{X}_s^2 \cdot D_T])h(s)\mathrm{d}s + \int_0^t b(s, \tilde{X}_s, \mathbb{E}_{\mathbb{P}}[\tilde{X}_s^2 \cdot D_T])\mathrm{d}s, \quad \forall t \in [0,T].$$

Here

$$D_T = \exp\{L_t - \frac{1}{2}\langle L, L\rangle_T\} = \exp\{-\int_0^T h(s)\mathrm{d}B_s - \frac{1}{2}\int_0^T h^2(s)\mathrm{d}s\}.$$