$n, t,$

236379 - Coding and Algorithms for Memories

Homework Assignment 2

Due Date: Thursday 23:59, January 2nd, 2025.

Instructions:

[illegible] ✓

?? , [illegible]

1. The homework assignment will be done **only in pairs**.
2. If you use any result and/or material from books, papers, or online, you need to mention and reference it in every part of your solutions.
3. If you use any programming code in your solutions, you need to include the code you use as part of your solution.

about k [illegible]

$n/3$ $n/3$ $n/3$

Problem 1.

(a) Assume the user writes t messages $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_t$, where t is a multiple of three, to a WOM. After writing the i-th message, for $3 \leq i \leq t$, the decoder needs to be able to recover the messages $\boldsymbol{m}_i$, $\boldsymbol{m}_{i-1}$, and $\boldsymbol{m}_{i-2}$.

why approaching ? ok I get it

  (i) Show that it is possible to achieve WOM codes with sum-rate approaching $\log(t/3+1)$.

  (ii) Show that $\log(t/3+1)$ is an upper bound on the sum-rate.

(b) Assume the user writes t messages $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_t$ to a WOM as follows. After the first write the first message $\boldsymbol{m}_1$ should be decoded, after the second write the first two messages $\boldsymbol{m}_1, \boldsymbol{m}_2$ should be decoded, and after the third write the first three messages $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ should be decoded. Starting the fourth write, the user indicates which message (out of the three messages currently stored in the memory) is chosen to be updated on each write. For example, on the fourth write the user may choose to update the second message with $\boldsymbol{m}_4$, so the three messages $\boldsymbol{m}_1, \boldsymbol{m}_3, \boldsymbol{m}_4$ should be decoded. Then, on the fifth the user may choose to update the message $\boldsymbol{m}_4$ with $\boldsymbol{m}_5$, so the three messages $\boldsymbol{m}_1, \boldsymbol{m}_3, \boldsymbol{m}_5$ should be decoded.

  (i) Design a WOM code construction with the highest possible sum-rate you can find.

  (ii) Find the best upper bound on the sum-rate you can find.

(c) Now assume that the user writes three messages $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$. After the first write, the first message $\boldsymbol{m}_1$ should be recovered. After the second write the first two messages $\boldsymbol{m}_1, \boldsymbol{m}_2$ should be recovered. However, on the third write the user specifies which of the first two messages should be recovered together with $\boldsymbol{m}_3$.

  (i) Prove that the sum-rate in this setup is upper bounded by 1.5.

  (ii) Find the best description you can have to the capacity region of this problem. That is, find all rates triples (R_1, R_2, R_3) in which there is a WOM code satisfying these conditions and all triples (R_1, R_2, R_3) in which there is no such a code.

$\frac{1}{t}$

Problem 2.
In this problem, we consider two-write WOM codes that are not worst case in the sense that their second write does not necessarily always succeed.
Every such a WOM code is characterized by three parameters r_1, r_2, n, where $r_1 < n$ and $r_1 \leq r_2$, and a binary matrix H of size $r_1 \times n$, which is chosen uniformly at random. The code is denoted by $\mathcal{C}(n, r_1, r_2, H)$. On the first write, one of $\sum_{i=0}^{n-r_2} \binom{n}{i}$ messages is written such that at most $n - r_2$ cells are programmed from zero to one. Let $\boldsymbol{c}_1$ be the memory-state vector after the first write. On the second write, the user seeks to write a message $\boldsymbol{s}$ of r_1 bits by choosing a vector $\boldsymbol{c}_2$ such that $H \cdot \boldsymbol{c}_2^T = \boldsymbol{s}^T$ and $\boldsymbol{c}_2 \geq \boldsymbol{c}_1$. Assume the messages on the first and second write are chosen uniformly at random, and we define by $P(n, r_1, r_2, H)$ the probability that the second write succeeds. Prove that for all $p \in (0, 0.5)$ and $\epsilon > 0$ there exists a WOM code $\mathcal{C}(n, r_1, r_2, H)$ such that its rates R_1, R_2 and its success probability $P(n, r_1, r_2, H)$ satisfy the following requirements:

$$R_1 \geq h(p) - \epsilon, R_2 \geq 1 - p - \epsilon, P(n, r_1, r_2, H) \geq 1 - \epsilon.$$

Problem 3.
In this problem we will design a non-binary two-write WOM codes.
Assume there are n q-ary cells, where q is a multiple of 3 and there exists a binary two-write WOM code $[n, 2; 2^{nR_1}, 2^{nR_2}]$.

(a) Design a two-write q-ary $[n, 2; 2^{nR_1} \cdot (q/3)^n, 2^{nR_2} \cdot (q/3)^n]$ WOM code and prove its correctness.

(b) Find the best asymptotic sum-rate which is possible to achieve by this construction and compare it with the capacity upper bound on the sum-rate in this case.

Problem 4.
Your goal in this problem is to design a memory with fast reading according to the following assumptions and requirements:

1. Assume there are n cells, each with q levels $\{0, 1, \ldots, q-1\}$. Then, $q-1$ binary pages (messages) are stored into these cells $p_1, p_2, \ldots, p_{q-1}$, where every page should store the same number of bits, denoted by k.

2. The pages are received *together* and are encoded to the memory by an encoding function $E : (\{0,1\}^k)^{q-1} \rightarrow \{0, \ldots, q-1\}^n$.

3. In order to know the level of each cell there are $q-1$ thresholds, between levels 0 and 1, levels 1 and 2, and so on until levels $q-2$ and $q-1$. When reading the memory cells with the threshold between levels i and $i+1$, a binary length-n vector $\boldsymbol{v}$ is received such that $v_j = 1$ if and only if the value of the jth cell is greater than i.

4. The memory needs to efficiently accommodate reading requests of pages such that every page is read by applying exactly a single threshold. More explicitly, page 1 (p_1) is read by applying the threshold between levels $q-2$ and $q-1$, page 2 (p_2) is read by applying the threshold between levels $q-3$ and $q-2$, and so on page $q-1$ (p_{q-1}) is read by applying the threshold between levels 0 and 1.

Such a coding scheme will be called a *Fast Reading Code* (*FR Code*) and will be denoted by an (n, q, k) FR code, where n is the number of cells, q is the number of levels in each cell, and k is the number of bits in each page. In this question a binary WOM code which stores t messages, each of the same number of bits k will be denoted by an $[n, t, k]$ WOM code.

(a) Show how to construct a $(3, 3, 2)$ FR code.

(b) Prove that if there exists an $[n, t = q - 1, k]$ WOM code, then there exists an (n, q, k) FR code.

(c) Explain in words what the difference between WOM codes and FR codes is, and why FR codes do not imply WOM codes.

(d) Prove that there exists a $(7, 5, 3)$ FR code. Hint: use the coset coding scheme with the length-7 Hamming code.

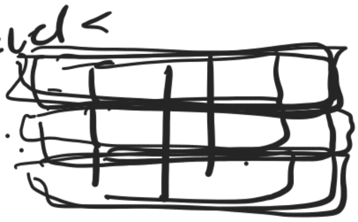
(e) Prove that for all $m \geq 3$ there exists a $(2^m - 1, 2^{m-2} + 3, m)$ FR code.

Problem 5.
The goal in this problem is to understand the connection between the write amplification (or erasure factor) and over-provisioning. You can use all the results and assumptions derived in class. Assume there are U logical pages and T physical pages, so the over-provisioning is $\rho = (T - U)/U$ and the storage rate is $\alpha = U/T$. You can also assume that the number of pages in a block, Z, is large.

(a) The U logical pages are distributed into two groups of hot and cold pages, where the number of hot, cold pages is $H = 0.2U, C = 0.8U$, respectively. First all pages are written to the memory. On each page write, one of the hot, cold pages is written with probability $0.8, 0.2$, respectively. Within these two groups, pages are written uniformly at random. Find the optimal partition of the physical pages in the memory into two parts to write the hot and cold pages separately such that the write amplification is minimized. Solve this problem for the greedy and the least recently used (LRU) garbage collection policies. Write the value of the write amplification you received in each case.

(b) Now assume that the number of hot, cold pages is $H = 0.5U, C = 0.5U$, respectively. First all pages are written to the memory and assume that $\alpha = 2/3$. On each page write, one of the *hot pages* is written uniformly at random (the cold pages are *not* rewritten at all). Find the best garbage collection policy you can find and analyze its erasure factor and write amplification.

level< level<



P2 oo

P1

aoo
1 1 1
1 1 0

Problem 1.

(a) Assume the user writes t messages $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_t$, where t is a multiple of three, to a WOM. After writing the i-th message, for $3 \leq i \leq t$, the decoder needs to be able to recover the messages $\boldsymbol{m}_i$, $\boldsymbol{m}_{i-1}$, and $\boldsymbol{m}_{i-2}$.

(i) Show that it is possible to achieve WOM codes with sum-rate approaching $\log(t/3+1)$.

(ii) Show that $\log(t/3+1)$ is an upper bound on the sum-rate.

(b) Assume the user writes t messages $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_t$ to a WOM as follows. After the first write the first message $\boldsymbol{m}_1$ should be decoded, after the second write the first two messages $\boldsymbol{m}_1, \boldsymbol{m}_2$ should be decoded, and after the third write the first three messages $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ should be decoded. Starting the fourth write, the user indicates which message (out of the three messages currently stored in the memory) is chosen to be updated on each write. For example, on the fourth write the user may choose to update the second message with $\boldsymbol{m}_4$, so the three messages $\boldsymbol{m}_1, \boldsymbol{m}_3, \boldsymbol{m}_4$ should be decoded. Then, on the fifth the user may choose to update the message $\boldsymbol{m}_4$ with $\boldsymbol{m}_5$, so the three messages $\boldsymbol{m}_1, \boldsymbol{m}_3, \boldsymbol{m}_5$ should be decoded.

(i) Design a WOM code construction with the highest possible sum-rate you can find.

(ii) Find the best upper bound on the sum-rate you can find.

(c) Now assume that the user writes three messages $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$. After the first write, the first message $\boldsymbol{m}_1$ should be recovered. After the second write the first two messages $\boldsymbol{m}_1, \boldsymbol{m}_2$ should be recovered. However, on the third write the user specifies which of the first two messages should be recovered together with $\boldsymbol{m}_3$.

(i) Prove that the sum-rate in this setup is upper bounded by 1.5.

(ii) Find the best description you can have to the capacity region of this problem. That is, find all rates triples (R_1, R_2, R_3) in which there is a WOM code satisfying these conditions and all triples (R_1, R_2, R_3) in which there is no such a code.

(a) Show that it is possible to achieve WOM codes with sum-rate approaching $\log(t/3+1)$.

Show that $\log(t/3+1)$ is an upper bound on the sum-rate

(i)

General Concept

Given n bits of memory, consider the following:

1) Divide the n bits into 3 blocks of memory

2) For the ith write do as follows:

   1) if $(i \% 3 = 0)$: write m_i to the first block.
   2) if $(i \% 3 = 1)$: write m_i to the second block
   3) if $(i \% 3 = 2)$: write m_i to the third block.

Why this will work: Effectively, we will have 3 blocks of WOM-code memory: $[\alpha n, t/3; M_K, M_{K+3}, M_{K+6}, \dots, M_{K+\frac{t}{3}}]$. As we've seen in lectures, the max sum-rate for each of these approaches $\log(t/3+1)$ for very large n's. Which satisfies the requirement.

Formal Solution

Give n bits of memory, let $\alpha, \beta \in [0,1]$ such that $\alpha+\beta<1$, define $\gamma = (1-\alpha-\beta)$.
Consider the three following WOM codes with their respective encoders:
$[\alpha n, t/3; 2^{\alpha n R_i^1}, 2^{\alpha n R_{i+1}^1}, \dots, 2^{\alpha n R_{i+t/3}^1}] = \mathcal{C}_1$

$$[\beta n, t/3; 2^{\beta n R_i^2}, 2^{\beta n R_{i+1}^2}, \ldots, 2^{\beta n R_{i+t/3}^2}] = \mathcal{C}_2$$

$$[\gamma n, t/3; 2^{\gamma n R_i^3}, 2^{\gamma n R_{i+1}^3}, \ldots, 2^{\gamma n R_{i+t/3}^3}] = \mathcal{C}_3.$$

$$E_{C_1}(m_i, c_{i-1}^1), E_{C_2}(m_i, c_{i-1}^2), E_{C_3}(m_i, c_{i-1}^3),$$

Where c_i^1, c_i^2, c_i^3 for $i \in \{1, 2, \ldots t\}$ are the the state vectors after the ith write.
And $\alpha R_i^1 = R_{2i}$, $\beta R_i^2 = R_{2i+1}$, $\gamma R_i^3 = R_{2i+2}$ for all $i \in \{1, 2, \ldots, t\}$. Let $c_i = (c_i^1, c_i^2, c_i^3)$ for all $i \in \{1, 2, \ldots t\}$.
Given all of the above, we construct the following WOM-code: (p stands for problem)

$\mathcal{C}_p = [n, t; 2^{nR_1}, 2^{nR_2}, \ldots, 2^{nR_t}]$, where the encoder is defined as follows

$$E(m_i, c_{i-1}) = \begin{cases} (E_{C_1}(m_i, c_{i-1}^1), c_{i-1}^2, c_{i-1}^3) & i \% 3 = 0 \\ (c_{i-1}^1, E_{C_2}(m_i, c_{i-1}^2), c_{i-1}^3) & i \% 3 = 1 \\ (c_{i-1}^1, c_{i-1}^2, E_{C_3}(m_i, c_{i-1}^3)) & i \% 3 = 2 \end{cases}$$

as previously defined

And the n bits are partitioned such that the code $\mathcal{C}_1$ is used for the first αn bits, $\mathcal{C}_2$ is used for the $\{\alpha n, \ldots, \alpha n + \beta n - 1\}$ bits and $\mathcal{C}_3$ is used for the $\{\alpha n + \beta n, \ldots, \alpha n + \beta n + \gamma n - 1\}$ bits.

As can be seen based off the construction, The code $\mathcal{C}_P$ does, in fact, satisfy that the decoder can recover the writes m_{i-2}, m_{i-1}, m_i for $3 \leq i \leq t$, because m_{i-2}, m_{i-1}, m_i are all in the memory, in the blocks $[0, \ldots, \alpha n - 1], [\alpha n, \ldots, \alpha n + \beta n - 1], [\alpha n + \beta n, \ldots, \alpha n + \beta n + \gamma n - 1]$ respectively.

If we assume that the blocks are divided evenly, meaning $\alpha = 1/3, \beta = 1/3, \gamma = 1/3$, then we will have that $R_i^1 = R_i^2 = R_i^3$ and thus the sum-rate is given by:

$$\sum_{i=1}^{t} R_i = \alpha \sum_{i=1}^{t/3} R_i^1 + \beta \sum_{i=1}^{t/3} R_i^2 + \gamma \sum_{i=1}^{t/3} R_i^3 = \frac{1}{3} \sum_{i=1}^{t/3} R_i^1 + \frac{1}{3} \sum_{i=1}^{t/3} R_i^1 + \frac{1}{3} \sum_{i=1}^{t/3} R_i^1$$

$$= \sum_{i=1}^{t/3} R_i^1 \text{ which approaches } \log(t/3 + 1)$$ as we've seen in class.

As requested.

(ii) Show that $\log(t/3 + 1)$ is an upper bound on the sum-rate.

Then the construction in (i) we will get that the upper-bound has to be greater or equal to $\log(t/3+1)$.

Suppose, for the sake of contradiction, that the upper bound is greater than $\log(t/3+1)$.

We will now construct a $\frac{t}{3}$-write WOM-code, for a $t = 3k$, where $k \in \mathbb{N}$:

Assume we have a t-write WOM like the one we constructed in (i), denote it by M.

For every message that we recieve, m_i, we will divide it into 3 parts m_i^1, m_i^2, m_i^3, where $m_i^1 = m[0 \ldots i]$, $m_i^2 = [i+1, \ldots, j]$, $m_i^3 = [j+1, \ldots n]$ and $n = |m_i^1| + |m_i^2| + |m_i^3| = m$. We will then write m_i^1, m_i^2, m_i^3 to M, in that order.

By the end of all writes, we will end up with $m^1_{t/3}, m^2_{t/3}, m^3_{t/3}$ in the memory M. And using the decoder we can decode the message written in M, which is the $(t/3)$th message that we received.

We will prove that this setup constucts a $t/3$-write WOM code, by proving that for all $1 \le i \le t/3$, the E_i, D_i encoder & decoder of this setup are valid WOM-code encoders & decorders.

Let

(b) Assume the user writes t messages $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_t$ to a WOM as follows. After the first write the first message $\boldsymbol{m}_1$ should be decoded, after the second write the first two messages $\boldsymbol{m}_1, \boldsymbol{m}_2$ should be decoded, and after the third write the first three messages $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$ should be decoded. Starting the fourth write, the user indicates which message (out of the three messages currently stored in the memory) is chosen to be updated on each write. For example, on the fourth write the user may choose to update the second message with $\boldsymbol{m}_4$, so the three messages $\boldsymbol{m}_1, \boldsymbol{m}_3, \boldsymbol{m}_4$ should be decoded. Then, on the fifth the user may choose to update the message $\boldsymbol{m}_4$ with $\boldsymbol{m}_5$, so the three messages $\boldsymbol{m}_1, \boldsymbol{m}_3, \boldsymbol{m}_5$ should be decoded.

(i) Design a WOM code construction with the highest possible sum-rate you can find.

(ii) Find the best upper bound on the sum-rate you can find.

(i) **General Concept**

Given n bits of memory, divide the n bits into 3 blocks. For the $i \in \{1,2,3\}$th write, write m_i in the ith block. For the $i \in \{4,5,\ldots,t\}$th write, write m_i in the user-specified block $(1,2,3)$.

Formal solution

Give n bits of memory, let $\alpha, \beta \in [0,1]$ such that $\alpha + \beta < 1$, define $\gamma = (1-\alpha-\beta)$.
Consider the three following WOM codes with their respective encoders:

$$[\alpha n, t/3; 2^{\alpha n R_i^1}, 2^{\alpha n R_{i+1}^1}, \ldots, 2^{\alpha n R_{i+t/3}^1}] = \mathcal{C}_1$$

$$[\beta n, t/3; 2^{\beta n R_i^2}, 2^{\beta n R_{i+1}^2}, \ldots, 2^{\beta n R_{i+t/3}^2}] = \mathcal{C}_2$$

$$[\gamma n, t/3; 2^{\gamma n R_i^3}, 2^{\gamma n R_{i+1}^3}, \ldots, 2^{\gamma n R_{i+t/3}^3}] = \mathcal{C}_3.$$

$$E_{\mathcal{C}_1}(m_i, c_{i-1}^1), \; E_{\mathcal{C}_2}(m_i, c_{i-1}^2), \; E_{\mathcal{C}_3}(m_i, c_{i-1}^3),$$

Where c_i^1, c_i^2, c_i^3 for $i \in \{1, 2, \dots t\}$ are the the state vectors after the ith write.
And $\alpha R_i^1 = R_{2i}$, $\beta R_i^2 = R_{2i+1}$, $\gamma R_i^3 = R_{2i+2}$ for all $i \in \{1, 2, \dots, t\}$. Let $c_i = (c_i^1, c_i^2, c_i^3)$ for all $i \in \{1, 2, \dots t\}$.
Given all of the above, we construct the following WOM-code: (p stands for problem)

$\mathcal{C}_p = [n, t; 2^{nR_1}, 2^{nR_2}, \dots, 2^{nR_t}]$, where the encoder is defined as follows for $i \in \{4, 5, \dots t\}$

$$E(m_i, c_{i-1}) = \begin{cases} (E_{C_1}(m_i, c_{i-1}^1), c_{i-1}^2, c_{i-1}^3) & \text{user requests to update first message in memory} \\ (c_{i-1}^1, E_{C_2}(m_i, c_{i-1}^2), c_{i-1}^3) & \text{user requests to update second message in mem} \\ (c_{i-1}^1, c_{i-1}^2, E_{C_3}(m_i, c_{i-1}^3)) & \text{user requests to update third message in mem} \end{cases}$$

as previously defined

And for $i \in \{1, 2, 3\}$ it is defined as follows:

$$E(m_i, c_{i-1}) = \begin{cases} (E_{C_1}(m_i, c_{i-1}^1), c_{i-1}^2, c_{i-1}^3) & i = 1 \\ (c_{i-1}^1, E_{C_2}(m_i, c_{i-1}^2), c_{i-1}^3) & i = 2 \\ (c_{i-1}^1, c_{i-1}^2, E_{C_3}(m_i, c_{i-1}^3)) & i = 3 \end{cases}$$

And the n bits are partitioned such that the code $\mathcal{C}_1$ is used for the first αn bits, $\mathcal{C}_2$ is used for the $\{\alpha n, \dots, \alpha n + \beta n - 1\}$ bits and $\mathcal{C}_3$ is used for the $\{\alpha n + \beta n, \dots, \alpha n + \beta n + \gamma n - 1\}$ bits.

As can be seen based off the construction, The code $\mathcal{C}_P$ does, in fact, satisfy that the decoder can recover the three messages in memory for all $i \in \{1, 2, \dots, t\}$, because they are all stored in our memory ... in the blocks $[0, \dots, \alpha n - 1]$, $[\alpha n, \dots, \alpha n + \beta n - 1]$, $[\alpha n + \beta n, \dots, \alpha n + \beta n + \gamma n - 1]$ respectively.

(ii) If

(c) Now assume that the user writes three messages $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$. After the first write, the first message $\boldsymbol{m}_1$ should be recovered. After the second write the first two messages $\boldsymbol{m}_1, \boldsymbol{m}_2$ should be recovered. However, on the third write the user specifies which of the first two messages should be recovered together with $\boldsymbol{m}_3$.

(i) Prove that the sum-rate in this setup is upper bounded by 1.5.

(ii) Find the best description you can have to the capacity region of this problem. That is, find all rates triples (R_1, R_2, R_3) in which there is a WOM code satisfying these conditions and all triples (R_1, R_2, R_3) in which there is no such a code.

(i) We will construct a coding scheme that has a sum-rate that is equal to 1.5.

$\log(4) = 2$

Problem 4.
Your goal in this problem is to design a memory with fast reading according to the following assumptions and requirements:

1. Assume there are n cells, each with q levels $\{0, 1, \ldots, q-1\}$. Then, $q-1$ binary pages (messages) are stored into these cells $p_1, p_2, \ldots, p_{q-1}$, where every page should store the same number of bits, denoted by k.

2. The pages are received *together* and are encoded to the memory by an encoding function $E : (\{0,1\}^k)^{q-1} \to \{0, \ldots, q-1\}^n$.

3. In order to know the level of each cell there are $q-1$ thresholds, between levels 0 and 1, levels 1 and 2, and so on until levels $q-2$ and $q-1$. When reading the memory cells with the threshold between levels i and $i+1$, a binary length-n vector $\boldsymbol{v}$ is received such that $v_j = 1$ if and only if the value of the jth cell is greater than i.

4. The memory needs to efficiently accommodate reading requests of pages such that every page is read by applying exactly a single threshold. More explicitly, page 1 (p_1) is read by applying the threshold between levels $q-2$ and $q-1$, page 2 (p_2) is read by applying the threshold between levels $q-3$ and $q-2$, and so on page $q-1$ (p_{q-1}) is read by applying the threshold between levels 0 and 1.

Such a coding scheme will be called a *Fast Reading Code* (*FR Code*) and will be denoted by an (n, q, k) FR code, where n is the number of cells, q is the number of levels in each cell, and k is the number of bits in each page. In this question a binary WOM code which stores t messages, each of the same number of bits k will be denoted by an $[n, t, k]$ WOM code.

(a) Show how to construct a $(3, 3, 2)$ FR code.

(b) Prove that if there exists an $[n, t = q-1, k]$ WOM code, then there exists an (n, q, k) FR code.
הודעות q-1 כתיבות k ביטים

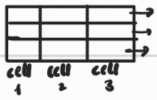
(c) Explain in words what the difference between WOM codes and FR codes is, and why FR codes do not imply WOM codes.

(d) Prove that there exists a $(7, 5, 3)$ FR code. Hint: use the coset coding scheme with the length-7 Hamming code. ?

(e) Prove that for all $m \geq 3$ there exists a $(2^m - 1, 2^{m-2} + 3, m)$ FR code.

(a) $\boxed{\text{Intuition: Rivest \& Shamir}}$: Every "layer" (level) of the cells represents a page. Last level (3rd level) will store the second page and the page will be written according to the first-write table in the Rivest & Shamir coding scheme. Second level will store the 1st page and will be written to according to the second-write table in the Rivest & Shamir coding scheme. The first level might have a degree of freedom in some cases, and exists to accommodate the second and third layers (does not have any value by itself).

for the sake of intuition, think of the cells in terms of the following visual:



→ 3rd level - first write = second page
→ 2nd level - second write = first page
→ 1st level - facilitator and will be equal to the second write in this specific construction but doesn't have to (there is a freedom degree)

Formal Solution:

Let $E: (\{0,1\}^k)^2 \rightarrow \{0,1,2\}^3$ be the encoder of the FR code such that $\forall g = (p_1, p_2) \in (\{0,1\}^k)^2$, $E(\underbrace{(p_1, p_2)}_{g}) = (q_1 q_2 q_3)$ where: q_i is the number encoded in cell i.

Here's a table for reference:

* in the visuals, if a level is red, it means it's coded, and o.w. it's not coded.

p_1 \ p_2	00	01	10	11
00	$(q_1 q_2 q_3) =$ (000) visual	$(q_1 q_2 q_3) =$ (112) visual	$(q_1 q_2 q_3) =$ (121) visual	$(q_1 q_2 q_3) =$ (211) visual
01	$(q_1 q_2 q_3) =$ (110) visual	$(q_1 q_2 q_3) =$ (002) visual	$(q_1 q_2 q_3) =$ (120) visual	$(q_1 q_2 q_3) =$ (210) visual
10	$(q_1 q_2 q_3) =$ (101) visual	$(q_1 q_2 q_3) =$ (102) visual	$(q_1 q_2 q_3) =$ (020) visual	$(q_1 q_2 q_3) =$ (201) visual
11	$(q_1 q_2 q_3) =$ (011) visual	$(q_1 q_2 q_3) =$ (012) visual	$(q_1 q_2 q_3) =$ (021) visual	$(q_1 q_2 q_3) =$ (200) visual

We will now prove that this code is a FR code, and for that, we need to prove that it satisfies 1,2,3,4 as described in the instructions.

1 and 2 follow immediately from the construction.

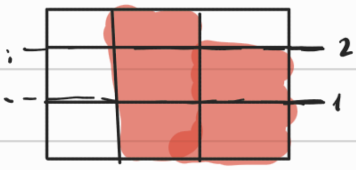
3,4: we want to show that by applying the threshold between 0 and 1 we will be able to read the first page, and by applying the threshold between 1 and 2, we will be able to read the second page.

Since we encoded the pages according to Rivest Shamir, if we want when we apply a threshold, the vector that we will read will be a three-bit vector that will correspond to the Rivest & Shamir coding scheme.

We can, then, use the Rivest and Shamir decoder, while keeping in

mind that for the threshold 0-1 we used second-write rules
and will be able to read the first page.
And for the 1-2 threshold, we used first-write rules, and will be able
to read the second page.
(Because of the way we constructed the encoder & the code)

For example, given the memory state 021:

visual:  ⇒

reading the 1st threshold yields the vector 011, we can decode this using the second-write rules for Rivest & Shamir and we will get that the first page has the bits 01 in it.
reading the 2nd threshold yields the vector 010, and similarly we can decode using the first-write rules, and will get that the second page has the bits 10 in it.

Therefore, the constructed code is a (3,3,2) FR code.

(b) Prove that if there exists an $[n, t = q-1, k]$ WOM code, then there exists an (n, q, k) FR code.

Intuition: the construction in section (a). (Rivest & Shamir is a $[3,2,2]$ Wom-code), but we want to generalize now!

Formal Solution: Let $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_t, D_1, D_2, \dots D_t$ be the encoders and decoders of the given (n,t,k) WOM-code, respectively.

We will construct an FR code:

Let $E: (\{0,1\}^k)^{q-1} \to \{0,1,\dots q-1\}^n$ be the encoder of the FR code.

Then $\forall (p_1, p_2, \dots, p_{q-1}) \in (\{0,1\}^k)^{q-1}$, store the pages in the n cells as follows:

The (t)th cell level, will store the (t)th page, such that for a cell $1 \le j \le n$ the level t is coded according to the jth bit in the vector $\mathcal{E}_1(p_t, \bar{0})$: if that bit is 0 then the (t)th level in the cell j in not coded. Otherwise, it is coded.

The ith cell level, for $1 \le i \le t-1$, will store the ith page, and will be coded using the $(t-i+1)$th encoder such that c_{i-1} = the vector $c_{i-1}^1 c_{i-1}^2 c_{i-1}^3 \cdots c_{i-1}^n$ when c_{i-1}^j for $1 \le j \le n$ is the $(i-1)$th level in the $(i-1)$th cell such that if the level is coded $c_{i-1}^j = 1$ o.w. $c_{i-1}^j = 0$ and c_i^j for $1 \le j \le n$ is coded according to the jth bit in the vector $\mathcal{E}_{(t-i+1)}(p_i, c_{i-1})$.

The $0$the cell level will store the vector $c_i^1 c_i^2 c_i^3 \cdots c_i^n$ where c_i^j is 1 if the level j in the cell i is coded. o.w. it's 0.

Now, define that, $E((p_1, p_2, \ldots, p_{q-1})) = (q_1 q_2 q_3 \ldots q_n)$
such that $\forall\ 1 \le i \le n$, q_i is the q-nary value of the ith cell.

We want to prove that the above construction indeed describes an FR code. For that, we need to prove 1,2,3,4 as described in the instructions.

1,2: follow immediately by construction.

3,4:

(c) Explain in words what the difference between WOM codes and FR codes is, and why FR codes do not imply WOM codes.

FR codes differ from WOM codes. Most importantly by the fact that when we encode the ith write The WOM-encoder needs to satisfy that $\mathcal{E}_i(m_i, c_{i-1}) \geq c_{i-1}$, wheras FR does not need to satisfy that.

This is why FR-codes do not imply WOM-code: we can't safely encode to a WOM code using an FR-code encoder.

An example of this is the $(3, 2, 2)$ FR-code that we constructed in first section. We would want to write the pages p_1, p_2 (in this order) to the WOM. If we try to read the pages p_1 and p_2 from the FR memory block, we'll get two vectors v_1, v_2 which are p_1 and p_2's contents respectively. However, $v_2 \leq v_1$, for all possible cases because we constructed the encoding to ensure that $v_2 \leq v_1$ for all possible cases by having p_1 coded using second-write rules and p_2 using first-write rules.
And so trying to write v_1 then v_2 to the WOM will FAIL.

(d) Prove that there exists a $\overset{n}{7}, \overset{q}{5}, \overset{k}{3}$ FR code. Hint: use the coset coding scheme with the length-7 Hamming code.

We can prove this by showing that there exists a $(\underset{n}{7}, \underset{q-1}{4}, \underset{k}{3})$ WOM-code, as per section (b).

We will show that there exists a hamming code with 3 bits per write, 4 write, 7 cells, gives the desi

(e) Prove that for all $m \geq 3$ there exists a $(2^m - 1, 2^{m-2} + 3, m)$ FR code.