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Use of the program FOURIER for steady waves

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1. Introduction

Throughout coastal and ocean engineering the convenient model of a steadily-progressing periodic wave train is used to give fluid velocities, pressures and surface elevations caused by waves, even in situations where the wave is being slowly modified by effects of viscosity, current, topography and wind or where the wave propagates past a structure with little effect on the wave itself. In these situations the waves do seem to show a surprising coherence of form, and they can be modelled by assuming that they are propagating steadily without change, giving rise to the so-called steady wave problem, which can be uniquely specified and solved in terms of three physical length scales only: water depth, wave length and wave height. In practice, the knowledge of the detailed flow structure under the wave is so important that it is usually considered necessary to solve accurately this otherwise idealised model. In many practical problems it is not the wavelength which is known, but rather the wave period, and in this case, to solve the problem uniquely or to give accurate results for fluid velocities, it is necessary to know the current on which the waves are riding.

The main theories and methods for the steady wave problem which have been used are: Stokes theory, an explicit theory based on an assumption that the waves are not very steep and which is best suited to waves in deeper water; cnoidal theory, an explicit theory for waves in shallower water; and Fourier approximation methods which are capable of high accuracy but which solve the problem numerically. A review and comparison of the methods is given in Sobey *et al.* (1987) and Fenton (1990).

Both the high-order Stokes theories and cnoidal theories suffer from a similar problem, that in the inappropriate limits, shallower water for Stokes theory and deeper water for cnoidal theory, the series become slowly convergent and ultimately do not converge. An approach which overcomes this is the Fourier method, which abandons any attempt to produce series expansions based on a small parameter, and obtains the solution numerically. This is done, not by solving for the flow field numerically, but by using an approach which might well be described as a nonlinear spectral approach, where a series is assumed, each term of which satisfies the field equation, and then the coefficients are found numerically. This is the basis of the theory described below and the accompanying computer program FOURIER. It has been widely used to provide solutions in a number of practical and theoretical applications, providing solutions for fluid velocities and pressures for engineering design. The method provides accurate solutions for waves up to very close to the highest.

The aim of this article is to

- present an introduction to the theory so that input data supplied will be satisfactory
- describe the data format required by the program FOURIER
- describe the output files which are produced and how they might be used, and,
- to describe the basis of the Fourier method and the numerical techniques used.

2. History and critical appraisal

The usual method for periodic waves, suggested by the basic form of the Stokes solution, is to use a Fourier series which is capable of accurately approximating any periodic quantity, provided the coefficients in that series can be found. A reasonable procedure, then, instead of assuming perturbation expansions for the coefficients in the series as is done in Stokes theory, is to calculate the coefficients numerically by solving the full nonlinear equations. This approach would be expected to be more accurate than either of the perturbation expansion approaches, Stokes and cnoidal theory, because its only approximations would be numerical ones, and not the essential analytical ones of the perturbation methods. Also, increasing the order of approximation would be a relatively trivial numerical matter without the need to perform extra mathematical operations.

This approach originated with Chappelear (1961). He assumed a Fourier series in which each term identically satisfied the field equation throughout the fluid and the boundary condition on the bottom. The values of the Fourier coefficients and other variables for a particular wave were then found by numerical solution of the nonlinear equations obtained by substituting the Fourier series into the nonlinear boundary conditions. He used the velocity potential ϕ for the field variable and instead of using surface elevations directly he used a Fourier series for that too. By using instead the stream function ψ for the field variable and point values of the surface elevations Dean (1965) obtained a rather simpler set of equations and called his method ”stream function theory”. Rienecker and Fenton (1981) presented a collocation method which gave somewhat simpler equations, where the nonlinear equations were solved by Newton’s method, such that the equations were satisfied identically at a number of points on the surface rather than minimizing errors there. The presentation emphasized the importance of knowing the current on which the waves travel if the wave period is specified as a parameter.

Results from these numerical methods show that accurate solutions can be obtained with Fourier series of 10-20 terms, even for waves close to the highest, and they seem to be the best way of solving any steady water wave problem where accuracy is important. Sobey Goodwin Thieke and Westberg (1987), made a comparison between different versions of the numerical methods. They concluded that there was little to choose between them.

A simpler method and computer program have been given by Fenton (1988), where the necessary matrix of partial derivatives necessary is obtained numerically. In application of the method to waves which are high, in common with other versions of the Fourier approximation method, it was found that it is sometimes necessary to solve a sequence of lower waves, extrapolating forward in height steps until the desired height is reached. For very long waves all these methods can occasionally converge to the wrong solution, that of a wave one third of the length, which is obvious from the Fourier coefficients which result, as only every third is non-zero. This problem can be avoided by using a sequence of height steps.

It is possible to obtain nonlinear solutions for waves on shear flows for special cases of the vorticity distribution. For waves on a constant shear flow, Dalrymple (1974), and a bi-linear shear distribution (Dalrymple, 1974b) used a Fourier method based on the approach of Dean (1965). The ambiguity caused by the specification of wave period without current seems to have been ignored, however.

3. The physical problem

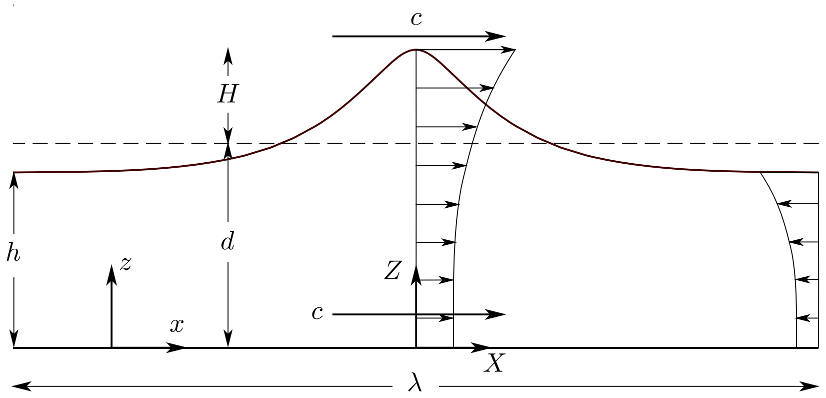

Figure 3-1. One wave of a steady train, showing principal dimensions, co-ordinates and velocities

The problem considered is that of two-dimensional periodic waves propagating without change of form over a layer of fluid on a horizontal bed, as shown in Figure 3-1. A co-ordinate system (x, y) has its origin on the bed, and waves pass through this frame with a velocity c in the positive x direction. It is this stationary frame which is the usual one of interest for engineering and geophysical applications. Consider also a frame of reference (X, Y) moving with the waves at velocity c, such that $x = X + ct$, where t is time, and $y = Y$. The fluid velocity in the (x, y) frame is (u, v), and that in the (X, Y) frame is (U, V). The velocities are related by $u = U + c$ and $v = V$. It is easier to solve the problem in this moving frame in which all motion is steady and then to compute the unsteady velocities.

4. Input data

It is well-known that a steadily-progressing periodic wave train is uniquely specified by three length scales, the water depth d, the wave height H, and the wavelength λ, or, in terms of only two dimensionless quantities involving these, such as dimensionless wave height H/d and dimensionless wavelength λ/d. The program allows for the specification of these, however in many practical situations it is not the wavelength which is known, but the wave period τ. If this is the case, it is not enough to uniquely specify the wave problem, as if there is a current, any current, then the period will be Doppler-shifted. Hence, it is necessary also to specify the current in such cases. The value of this current will also affect the horizontal velocity components, and users of the program should be aware of this and if it is unknown, some maximum and minimum values might be tried and their effects evaluated.

All input data and output results are given in terms non-dimensionalised with respect to gravitational acceleration g and mean depth d.

4.1 Files necessary

4.1.1 Data.txt

The data is to be given in a file DATA.TXT (upper and/or lower case letters), and is of the form as given in the first column of Table 4-1. Any other information can be placed after that on each line, such as we have done here, to label each line.

Test wave (A title line to identify each wave)	
0.6	H/d
Wavelength	Measure of length: ”Wavelength” or ”Period”
10.	Value of that length: L/d or $T\sqrt{g/d}$ respectively
1	Current criterion (1 or 2)
0.	Current magnitude, $\bar{u}_1/\sqrt{gd}$ or $\bar{u}_2/\sqrt{gd}$
20	Number of Fourier components (max. 32)
4	Number of height steps to reach H/d
Any number of other wave data can be placed here, each occupying 8 lines as above	
FINISH	Must be used to tell the program to stop - the file can continue after this

Table 4-1. Form of data to be supplied for each wave

4.1.2 Convergence.dat

A three-line file which controls convergence of the iteration procedure, for example:

Control file to control convergence and output of results	
20	Maximum number of iterations for each height step; 10 OK for ordinary waves, 40 for highest
1.e-4	Criterion for convergence, typically 1.e-4, or 1.e-5 for highest waves

4.2 Maximum wave height possible for a given length

The range over which periodic solutions for waves can occur is given in Figure 4-1, which shows limits to the existence of waves determined by computational studies. The highest waves possible, $H = H_m$, are shown by the thick line, which is the approximation to the results of Williams (1981), presented as equation (32) in Fenton (1990):

$$\frac{H_m}{d} = \frac{0.141063\,\frac{\lambda}{d} + 0.0095721\left(\frac{\lambda}{d}\right)^2 + 0.0077829\left(\frac{\lambda}{d}\right)^3}{1 + 0.0788340\,\frac{\lambda}{d} + 0.0317567\left(\frac{\lambda}{d}\right)^2 + 0.0093407\left(\frac{\lambda}{d}\right)^3}. \tag{4.1}$$

Nelson (1987 and 1994), has shown from many experiments in laboratories and the field, that the maximum wave height achievable in practice is actually only $H_m/d = 0.55$. Further evidence for this conclusion is provided by the results of Le Méhauté *et al.* (1968), whose maximum wave height tested was $H/d = 0.548$, described as ”just below breaking”. It seems that there may be enough instabilities at work that real waves propagating over a flat bed cannot approach the theoretical limit given by equation (4.1).

Also shown on the figure, although not so important for applications of the FOURIER program is the boundary between regions where Stokes and cnoidal theories can be applied, as suggested by Hedges (1995):

$$\mathbf{U} = \frac{H\lambda^2}{d^3} = 40, \tag{4.2}$$

where **U** is the Ursell number. The FOURIER program can be used over almost the whole region of possible waves, close to the boundary given by equation (4.1).

4.3 Wavelength or Period

If ”Wavelength” is chosen, then a value of λ/d is then specified; if ”Period” then a value of dimensionless period $\tau\sqrt{g/d}$ is to be given.

4.4 Type of current specified

This is described in more detail in Appendix B below. There are actually two main types of currents, the first is the ”Eulerian mean current”, the time-mean horizontal fluid velocity at any point denoted by $\bar{u}_1$, the mean current which a stationary meter would measure. A second type of mean current is the depth-integrated mean current, the ”mass-transport velocity”, which we denote by $\bar{u}_2$. If there is no mass transport, such as in a closed wave

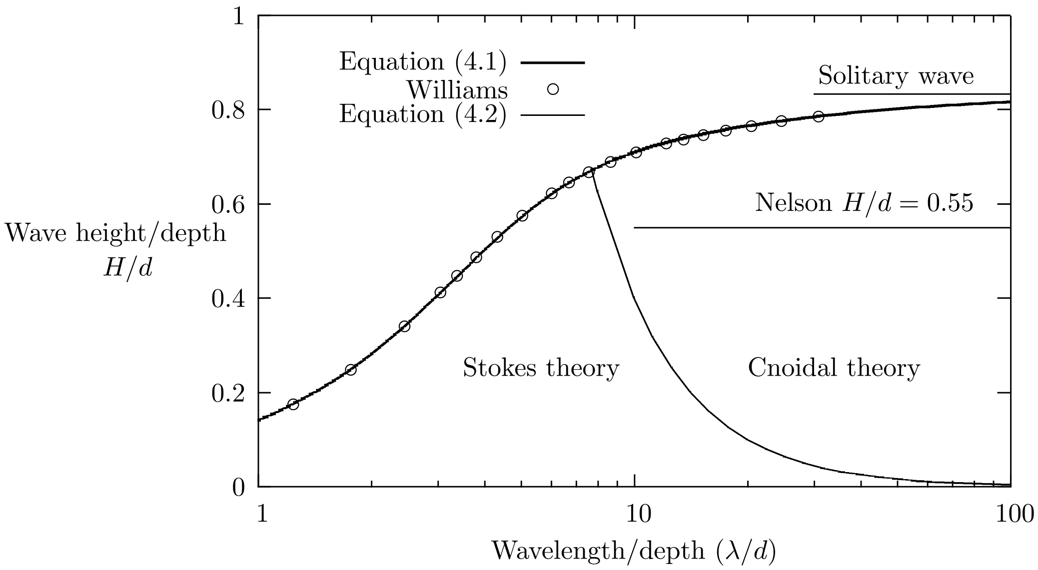

Figure 4-1. The region of possible steady waves, showing the theoretical highest waves (Williams) and the fitted equation (1), and the highest long waves in the field (Nelson).

tank, $\bar{u}_2 = 0$. Usually the overall physical problem will impose a certain value of current on the wave field, thus determining the wave speed. To apply the methods of this theory where wave period is known, to obtain a unique solution it is necessary to specify both the nature (1 or 2) and magnitude of that current. If the current is unknown, any horizontal velocity components calculated are approximate only.

4.5 Number of Fourier components

This is the primary computational parameter in the program, which we denote by N. The program can process up to $N = 32$, but for many problems, $N = 10$ is enough. Results show that accurate solutions can be obtained with Fourier series of 10-20 terms, even for waves close to the highest, although for longer and higher waves it may be necessary to increase N. The adequacy of the particular value of N used can be monitored by examining the output file SOLUTION.TXT, where the spectra of Fourier coefficients obtained as part of the solution is presented, the B_j which are at the core of the method, as presented in equation (A-5) for $j = 1, \ldots, N$, and the Fourier coefficients of the computed free surface, the E_j as presented in equation (D-4). The value of E_N must be sufficiently small (less than 10^{-4} say) that there would be no identifiable high-frequency wave apparent on the surface.

4.6 Number of height steps

In application of the method to waves which are high, in common with other versions of the Fourier approximation method, it was found that it is sometimes necessary to solve a sequence of lower waves, extrapolating forward in height steps until the desired height is reached. The reason is that for very long waves these methods can occasionally converge to the wrong solution, that of a wave 1/3 of the length, which is obvious from the Fourier coefficients which result, as only every third is non-zero. This problem can be avoided by using the sequence of height steps. For waves up to about half the highest $H \approx H_m/2$ it is not necessary to do this, but thereafter it is better to take 2 or more height steps. For waves very close to H_m for a given length it might be necessary to take as many as 10. The evidence as to whether enough have been taken is provided by the spectrum, as noted above.

5. Output files

The program produces output to the screen showing how the process of convergence is working. It is best if run from an MS-DOS prompt, by going to the correct directory and typing ”Fourier”, as in this case the output remains on the screen. Two files are produced (unless the ”Nil” option has been chosen):

5.1 SOLUTION.TXT

This initially contains the global parameters of the solution, in accordance with the reference numbering of Table 1 of Fenton (1990), but where all variables are non-dimensionalised with respect to g and d. All variables are presented in machine-readable format using 14 significant figures, should another program be written to use these results without having to run FOURIER repeatedly. The results are:

Number	Quantity	
1	λ/d	Wave length
2	H/d	Wave height
3	$\tau\sqrt{g/d}$	Period
4	$c/\sqrt{gd}$	Wave speed
5	$\bar{u}_1/\sqrt{gd}$	Eulerian current
6	$\bar{u}_2/\sqrt{gd}$	Stokes current
7	$\bar{U}/\sqrt{gd}$	Mean fluid speed in frame of wave
8	$Q/\sqrt{gd^3}$	Discharge
9	R/gd	Bernoulli constant

After this the value of N and then the spectra of the velocity potential coefficients B_j and the surface elevation coefficients E_j are given, for $j = 1, \ldots, N$, the two corresponding coefficients on each row, once again to 14 figures and readable by a subsequent computer program. These spectra should be checked, as suggested above, to ensure that the coefficients become small enough that the solution has converged satisfactorily.

5.2 SUMMARY.TXT

An overall summary of results, not necessarily in machine-readable form.

6. Post-processing to obtain quantities for practical use – POST.EXE

Another computer program, POST.EXE reads the data and prints more out for practical use.

6.1 Data files

6.1.1 SOLUTION.TXT

This has been described above.

6.1.2 CONTROL.DAT

A two-line file which tells how many vertical velocity profiles to be printed, for example:

An identifying line
16 Number of velocity profiles to print out

6.2 Results files

The program produces three text files:

6.2.1 RESULTS.PRN

This is a summary of results, including an energy check for a number of points on the surface.

6.2.2 SURFACE.PRN

The co-ordinates of points on the surface.

6.2.3 FIELD.PRN

Prints out a number of velocity profiles, as determined by CONTROL.DAT.

7. References

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Appendix A. Theory

Here we present an outline of the theory. If the fluid is incompressible, in two dimensions a stream function $\psi(X, Y)$ exists such that the velocity components are given by

$$U = \partial\psi/\partial Y, \quad \text{and} \quad V = -\partial\psi/\partial X.$$

If motion is irrotational, then $\nabla \times \mathbf{u} = \mathbf{0}$ and it follows that ψ satisfies Laplace's equation throughout the fluid:

$$\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial Y^2} = 0. \tag{A-1}$$

The kinematic boundary conditions to be satisfied are

$$\psi(X, 0) = 0 \quad \text{on the bottom, and} \tag{A-2}$$

$$\psi(X, \eta(X)) = -Q \quad \text{on the free surface,} \tag{A-3}$$

where $Y = \eta(X)$ on the free surface and Q is a positive constant denoting the volume rate of flow per unit length normal to the flow underneath the stationary wave in the (X, Y) co-ordinates. In these co-ordinates the apparent flow is in the negative X direction. The dynamic boundary condition to be satisfied is that pressure is zero on the surface so that Bernoulli's equation becomes

$$\frac{1}{2}\left(\left(\frac{\partial\psi}{\partial X}\right)^2+\left(\frac{\partial\psi}{\partial Y}\right)^2\right)+g\eta=R \quad \text{on the free surface,} \tag{A-4}$$

where R is a constant.

The basis of the method is to write the analytical solution for ψ in separated variables form

$$\psi(X,Y)=-\bar{U}\,Y+\sqrt{\frac{g}{k^3}}\sum_{j=1}^{N}B_j\frac{\sinh jkY}{\cosh jkd}\cos jkX, \tag{A-5}$$

where $\bar{U}$ is the mean fluid speed on any horizontal line underneath the stationary waves, the minus sign showing that in this frame the apparent dominant flow is in the negative x direction. The $B_{1,}\ldots,B_N$ are dimensionless constants for a particular wave, and N is a finite integer. The truncation of the series for finite N is the only mathematical or numerical approximation in this formulation. The quantity k is the wavenumber $k=2\pi/\lambda$ where λ is the wavelength, which may or may not be known initially, and d is the mean depth as shown on Figure 3-1. Each term of this expression satisfies the field equation (A-1) and the bottom boundary condition (A-2) identically. It might be thought that the use of the denominator $\cosh jkd$ is redundant, but it serves the useful function that for large j the B_j do not have to decay exponentially, thereby making solution rather more robust. Possibly more importantly, it also allows for the treatment of deep water, such that if we introduce a vertical co-ordinate Y_* with origin at the mean water level such that $Y=d+Y_*$, then in the limit as $kd\to\infty$,

$$\frac{\sinh jkY}{\cosh jkd}\sim e^{jkY_*},$$

which can be used as the basis for variation in the vertical.

If one were proceeding to an analytical solution, the coefficients B_j would be found by using a perturbation expansion in wave height. Here they are found numerically by satisfying the two nonlinear equations (A-3) and (A-4) from the surface boundary conditions, which become, after dividing through to make them dimensionless:

$$-\bar{U}\sqrt{k/g}\,k\eta(X)+\sum_{j=1}^{N}B_j\frac{\sinh jk\eta(X)}{\cosh jkd}\cos jkX+Q\sqrt{\frac{k^3}{g}}=0, \quad \text{and} \tag{A-6}$$

$$\frac{1}{2}\left(-\bar{U}\sqrt{k/g}+\sum_{j=1}^{N}jB_j\frac{\cosh jk\eta(X)}{\cosh jkd}\cos jkX\right)^2+\frac{1}{2}\left(\sum_{j=1}^{N}jB_j\frac{\sinh jk\eta(X)}{\cosh jkd}\sin jkX\right)^2+k\eta(X)-Rk/g=0, \tag{A-7}$$

both to be satisfied for all x. To solve the problem numerically these two equations are to be satisfied at a sufficient number of discrete points so that we have enough equations for solution. If we evaluate the equations at $N+1$ discrete points over one half wave from the crest to the trough for $m=0,1,\ldots,N$, such that $x_m=m\lambda/2N$ and $kx_m=m\pi/N$, and where $\eta_m=\eta(x_m)$, then (A-6) and (A-7) provide $2N+2$ nonlinear equations in the $2N+5$ dimensionless variables: $k\eta_m$ for $m=0,1,\ldots,N$; B_j for $j=1,2,\ldots,N$; $\bar{U}\sqrt{k/g}$; kd; $Q\sqrt{k^3/g}$; and Rk/g. Clearly, three extra equations are necessary for solution. One is the expression for the dimensionless mean depth kd in terms of the dimensionless depths $k\eta_m$ evaluated using the trapezoidal rule:

$$\frac{1}{N}\left(\frac{1}{2}\left(k\eta_0+k\eta_N\right)+\sum_{m=1}^{N-1}k\eta_m\right)-kd=0. \tag{A-8}$$

For quantities which are periodic such as here, the trapezoidal rule is very much more accurate than usually believed. It can be shown that the error is of the order of the last (Nth) coefficient of the Fourier series of the function being integrated. As that is essentially the approximation used throughout this work (where it is assumed that the series can be truncated at a finite value of N) this is in keeping with the overall accuracy.

The remaining two equations necessary could be provided by specifying numerical values of any two of the parameters introduced. However in practice it is often the physical dimensions of wavelength λ, mean water depth d and wave height H which are known, giving a numerical value for the dimensionless wave height kH for which an equation can be provided connecting the crest and trough heights $k\eta_0$ and $k\eta_N$ respectively: $H=\eta_0-\eta_N$,which we write in terms of our dimensionless variables as

$$k\eta_0 - k\eta_N - kd\,\frac{H}{d} = 0, \tag{A-9}$$

because in some problems we know the wave period rather than the wavelength and we do not know kd initially. If we do know the wavelength, we have a trivial equation for kd:

$$kd - 2\pi\frac{d}{\lambda} = 0. \tag{A-10}$$

There are now $2N+5$ equations in the $2N+5$ dimensionless variables, and the system can be solved. Some formulations of the problem (*e.g.* Dean, 1965) allow more surface collocation points and the equations are solved in a least-squares sense. In general this would be thought to be desirable, but in practice seems not to make much difference, and the system of equations appears quite robust.

Appendix B. Specification of wave period and current

In many problems it is not the wavelength λ which is known but the wave period τ as measured in a stationary frame. The two are connected by the simple relationship

$$c = \frac{\lambda}{\tau}, \tag{B-1}$$

where c is the wave speed, however it is not known *a priori*, and in fact depends on the current on which the waves are travelling. In the frame travelling with the waves at velocity c the mean horizontal fluid velocity at any level is $-\bar{U}$, hence in the stationary frame the time-mean horizontal fluid velocity at any point denoted by $\bar{u}_1$, the mean current which a stationary meter would measure, is given by

$$\bar{u}_1 = c - \bar{U}. \tag{B-2}$$

In the special case of no mean current at any point, $\bar{u}_1 = 0$ and $c = \bar{U}$, which is Stokes' first approximation to the wave speed, usually incorrectly referred to as his "first definition of wave speed", and is that relative to a frame in which the current is zero. Most wave theories have presented an expression for $\bar{U}$, obtained from its definition as a mean fluid speed. It has often been referred to, incorrectly, as "the wave speed".

A second type of mean fluid speed or current is the depth-integrated mean speed of the fluid under the waves in the frame in which motion is steady. If Q is the volume flow rate per unit span underneath the waves in the (X, Y) frame, the depth-averaged mean fluid velocity is $-Q/d$, where d is the mean depth. In the physical (x, y) frame, the depth-averaged mean fluid velocity, the "mass-transport velocity", is $\bar{u}_2$, given by

$$\bar{u}_2 = c - Q/d. \tag{B-3}$$

If there is no mass transport, such as in a closed wave tank, $\bar{u}_2 = 0$, and Stokes' second approximation to the wave speed is obtained: $c = Q/d$. In general, neither of Stokes' first or second approximations is the actual wave speed, and the waves can travel at any speed. Usually the overall physical problem will impose a certain value of current on the wave field, thus determining the wave speed. To apply the methods of this section, where wave period is known, to obtain a unique solution it is also necessary to specify the magnitude and nature of that current.

We now eliminate c between (B-1) and (B-2) or (B-3) so that they can be re-written in terms of the physical variables $\tau\sqrt{g/d}$ and $\bar{u}_1/\sqrt{gd}$ or $\bar{u}_2/\sqrt{gd}$ which have to be specified. Equations (B-2) and (B-3) respectively become

$$\sqrt{kd}\,\bar{U}\sqrt{k/g} + kd\,\frac{\bar{u}_1}{\sqrt{gd}} - \frac{2\pi}{\tau\sqrt{g/d}} \quad = \quad 0 \quad \text{and} \tag{B-4}$$

$$\frac{Q\sqrt{k^3/g}}{\sqrt{kd}} + kd\,\frac{\bar{u}_2}{\sqrt{gd}} - \frac{2\pi}{\tau\sqrt{g/d}} \quad = \quad 0. \tag{B-5}$$

There are $2N+5$ equations: two free surface equations (A-6) and A-7 at each of $N+1$ points, the mean depth condition A-8, the wave height condition (A-9), and either (A-10) if the relative wavelength is known or (B-4) or (B-5) if the wave period and current are known. The variables to be solved for are the $N+1$ values of surface elevation $k\eta_m$, the N coefficients B_j, $\bar{U}\sqrt{k/g}$, kd, $Q\sqrt{k^3/g}$, and Rk/g, the input data requires values of H/d

and either λ/d or values of dimensionless period $\tau\sqrt{g/d}$ and one of the known mean currents $\bar{u}_1/\sqrt{gd}$ or $\bar{u}_2/\sqrt{gd}$.

Appendix C. Computational method

The system of nonlinear equations can be iteratively solved using Newton's method. If we write the system of equations as

$$F_i(\mathbf{x}) = 0, \quad \text{for } i = 1, \ldots, 2N+5**,$$

where F_i represents equation i and $\mathbf{x} = \{x_j, j = 1, \ldots, 2N+5**\}$, the vector of variables x_j (there should be no confusion with that same symbol as a space variable), then if we have an approximate solution $\mathbf{x}^{(n)}$ after n iterations, writing a multi-dimensional Taylor expansion for the left side of equation i obtained by varying each of the $x_j^{(n)}$ by some increment $\delta x_j^{(n)}$:

$$F_i\left(\mathbf{x}^{(n+1)}\right) \approx F_i\left(\mathbf{x}^{(n)}\right) + \sum_{j=1}^{2N+5}\left(\frac{\partial F_i}{\partial x_j}\right)^{(n)}\delta x_j^{(n)}.$$

If we choose the $\delta x_j^{(n)}$ such that the equations would be satisfied by this procedure such that $F_i\left(\mathbf{x}^{(n+1)}\right) = 0$, then we have the set of linear equations for the $\delta x_j^{(n)}$:

$$\sum_{j=1}^{2N+5}\left(\frac{\partial F_i}{\partial x_j}\right)^{(n)}\delta x_j^{(n)} = -F_i\left(\mathbf{x}^{(n)}\right) \quad \text{for } i = 1, \ldots, 2N+5**,$$

which is a set of equations linear in the unknowns $\delta x_j^{(n)}$ and can be solved by standard methods for systems of linear equations. Having solved for the increments, the updated values of all the variables are then computed for $x_j^{(n+1)} = x_j^{(n)} + \delta x_j^{(n)}$ for all the j. As the original system is nonlinear, this will in general not yet be the required solution and the procedure is repeated until it is.

It is possible to obtain the array of derivatives of every equation with respect to every variable, $\partial F_i/\partial x_j$ by performing the analytical differentiations, however as done in Fenton (1988) it is rather simpler to obtain them numerically. That is, if variable x_j is changed by an amount ε_j, then on numerical evaluation of equation i before and after the increment (after which it is reset to its initial value), we have the numerical derivative

$$\frac{\partial F_i}{\partial x_j} \approx \frac{F(x_1, \ldots, x_j + \varepsilon_j, \ldots, x_{2N+5}) - F(x_1, \ldots, x_j, \ldots, x_{2N+5})}{\varepsilon_j}.$$

The complete array is found by repeating this for each of the $2N+5**$ equations for each of the $2N+5**$ variables. Compared with the solution procedure, which is $O(N^3)$, this is not a problem, and gives a rather simpler program.

To begin this procedure it is necessary to have some initial estimate for each of the variables. It is relatively simple to use linear wave theory assuming no current. If the wave period is known, it is necessary to solve for the wavenumber. The solution from that simple theory is

$$\sigma^2 = gk\tanh kd, \tag{C-1}$$

where the angular frequency $\sigma = 2\pi/\tau$. This equation which is transcendental in kd could be solved using standard methods for solution of a single nonlinear equation, however Fenton and McKee (1990) have given an approximate explicit solution:

$$kd \approx \frac{\sigma^2 d}{g}\left(\coth\left(\sigma\sqrt{d/g}\right)^{3/2}\right)^{2/3}. \tag{C-2}$$

This expression is an exact solution of (C-1) in the limits of long and short waves, and between those limits its greatest error is 1.5%. Such accuracy is adequate for the present approximate purposes. Having solved for kd

linear theory can be applied:

$$\begin{aligned}
k\eta_m &= kd + \frac{1}{2}kH\cos\frac{m\pi}{N}, \quad \text{for } m = 1,\ldots,N,\\
\bar{U}\sqrt{k/g} &= \sqrt{\tanh kd},\\
B_1 &= \frac{1}{2}\frac{kH}{\sqrt{\tanh kd}}, \quad B_j = 0 \text{ for } j = 2,\ldots,N,\\
Q\sqrt{k^3/g} &= kd\sqrt{\tanh kd},\\
Rk/g &= kd + \frac{1}{2}\frac{\bar{U}^2 k}{g}.
\end{aligned}$$

The present Fourier approach breaks down in the limit of very long waves, when the spectrum of coefficients becomes broad-banded and many terms have to be taken, as the Fourier approximation has to approximate both the short rapidly-varying crest region and the long trough where very little changes. The numerical cnoidal theory described below could be used, but for many applications, say for wavelengths as long as 50 times the depth, the Fourier method provides good solutions. More of a problem is that it is difficult to get the method to converge to the solution desired for high waves which are moderately long. In many cases the tendency of the method is to converge to a solution with a wavelength 1/3 of that desired. The results should be monitored, and if that has happened the problem is simply remedied by solving for two lower waves and using the results to extrapolate upwards to provide better initial conditions for the solution at the desired height.

Appendix D. Post-processing to obtain quantities for practical use

Once the solution has been obtained in these dimensionless variables oriented towards then quantities rather more useful for physical calculations can be evaluated. Another computer program, POST.EXE reads the data from SOLUTION.TXT. Often it is more convenient to process the results in terms of the water depth as being the fundamental length scale. Here we assume that all physical quantities are available, for example, having solved for all the dimensionless variables $k\eta_m$ the numerical value of k has been used to calculate all the η_m, and so on.

It can be shown from (A-5) and the Cauchy-Riemann equations

$$\frac{\partial\Phi}{\partial X} = \frac{\partial\psi}{\partial Y} \quad \text{and} \quad \frac{\partial\Phi}{\partial Y} = -\frac{\partial\psi}{\partial X},$$

where Φ is the velocity potential in the frame moving with the wave, that

$$\Phi(X,Y) = -\bar{U}\,X + \sqrt{\frac{g}{k^3}}\sum_{j=1}^{N} B_j \frac{\cosh jkY}{\cosh jkd}\sin jkX,$$

and considering the physical frame, the now unsteady velocity potential $\phi(x,y,t)$ is given by

$$\phi(x,y,t) = \left(c - \bar{U}\,\right)x + \sqrt{\frac{g}{k^3}}\sum_{j=1}^{N} B_j \frac{\cosh jky}{\cosh jkd}\sin jk\,(x - ct) + C(t), \tag{D-1}$$

where we have shown the additional function of time $C(t)$ for purposes of generality. The velocity components anywhere in the fluid are given by $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$:

$$u(x,y,t) = c - \bar{U} + \sqrt{\frac{g}{k}}\sum_{j=1}^{N} jB_j \frac{\cosh jky}{\cosh jkd}\cos jk\,(x - ct)\,, \tag{D-2}$$

$$v(x,y,t) = \sqrt{\frac{g}{k}}\sum_{j=1}^{N} jB_j \frac{\sinh jky}{\cosh jkd}\sin jk\,(x - ct)\,. \tag{D-3}$$

Acceleration components can be obtained simply from these expressions by differentiation, and from the Cauchy-

Riemann equations, and are given by.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -c \times \frac{\partial u}{\partial x}, \quad \text{where} \quad \frac{\partial u}{\partial x} = -\sqrt{gk} \sum_{j=1}^{N} j^2 B_j \frac{\cosh jky}{\cosh jkd} \sin jk\,(x-ct)\,, \\
\frac{\partial v}{\partial t} &= -c \times \frac{\partial v}{\partial x}, \quad \text{where} \quad \frac{\partial v}{\partial x} = \sqrt{gk} \sum_{j=1}^{N} j^2 B_j \frac{\sinh jky}{\cosh jkd} \cos jk\,(x-ct)\,, \\
\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}, \\
\frac{\partial v}{\partial y} &= -\frac{\partial u}{\partial x}.
\end{aligned}$$

The free surface elevation at an arbitrary point requires more effort, as we only have it at discrete points η_m. We take the cosine transform of the $N+1$ surface elevations:

$$E_j = \sum_{m=0}^{N}{}'' \, \eta_m \cos \frac{jm\pi}{N} \quad \text{for } j = 1, \ldots, N,$$

where $\sum''$ means that it is a trapezoidal-type summation, with factors of 1/2 multiplying the first and last contributions. This could be performed using fast Fourier methods, but as N is not large, simple evaluation of the series is reasonable. It can be shown that the interpolating cosine series for the surface elevation is

$$\eta(x,t) = 2 \sum_{j=0}^{N}{}'' \, E_j \cos jk\,(x-ct)\,, \tag{D-4}$$

which can be evaluated for any x and t.

The pressure at any point can be evaluated using Bernoulli's theorem, but most simply in the form from the steady flow, but using the velocities as computed from (D-2) and (D-3):

$$\frac{p(x,y,t)}{\rho} = R - gy - \frac{1}{2}\left((u(x,y,t) - c)^2 + v^2(x,y,t) \right).$$

In fact, it can be shown that the Bernoulli constants in the two frames are related by

$$R = C + \frac{1}{2}\,c^2.$$