Technical University of Denmark

DTU Electrical Engineering: Automation and Control

Part A

Course

31320 Robust and Fault-tolerant Control

Group 14

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1 Question 1

1.1 Overview

The investigated system has

- 15 constraints $\mathcal{C} = [\ c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6,\ d_7,\ d_8,\ d_9,\ d_10,\ d_11,\ d_12,\ m_13,\ m_14,\ m_15\ ]$,
- 4 known variables $\mathcal{K} = [\ u,\ y_1,\ y_2,\ y_3\ ]$ and
- 13 unknown variables $\mathcal{X} = [\ \theta_1,\ \dot{\theta}_1,\ \omega_1,\ \dot{\omega_1},\ \theta_2,\ \dot{\theta}_2,\ \omega_2,\ \dot{\omega_2},\ \theta_3,\ \dot{\theta}_3,\ \omega_3,\ \dot{\omega_3},\ d\ ]$.

The constraints of the system are as follows:

$$
\begin{array}{l|l}
c_1 & 0 = \dot{\theta}_1 - w_1 \\
c_2 & 0 = d - u + J_1\,\dot{\omega_1} + b_1\,\omega_1 + k_1\ (\theta_1 - \theta_2) \\
c_3 & 0 = \dot{\theta}_2 - \omega_2 \\
c_4 & 0 = J_2\,\dot{\omega_1} + b_2\,\omega_2 - k_1\ (\theta_1 - \theta_2) + k_2\ (\theta_2 - \theta_3) \\
c_5 & 0 = \dot{\theta}_3 - \omega_3 \\
c_6 & 0 = J_3\,\dot{\omega_3} + b_3\,w_3 - k_2\ (\theta_2 - \theta_3) \\
d_7 & 0 = \frac{d\theta_1}{dt} - \dot{\theta}_1 \\
d_8 & 0 = \frac{d\omega_1}{dt} - \dot{\omega_1} \\
d_9 & 0 = \frac{d\theta_2}{dt} - \dot{\theta}_2 \\
d_{10} & 0 = \frac{d\omega_2}{dt} - \dot{\omega_2} \\
d_{11} & 0 = \frac{d\theta_3}{dt} - \dot{\theta}_3 \\
d_{12} & 0 = \frac{d\omega_3}{dt} - \dot{\omega_3} \\
m_{13} & 0 = y_1 - \theta_1 \\
m_{14} & 0 = y_2 - \theta_2 \\
m_{15} & 0 = y_3 - \theta_3
\end{array}
$$

The analysis obtained 1 matchings that yield in total 2 parity equations.

1.2 Canonical Decomposition

The system consists of

- the over-determined subsystem $\mathcal{S}^+$ with $\mathcal{C}^+ = [\ c_3,\ c_4,\ c_5,\ c_6,\ d_9,\ d_10,\ d_11,\ d_12,\ m_13,\ m_14,\ m_15\ ]$ and $\mathcal{X}^+ = [\ \theta_1,\ \theta_2,\ \dot{\theta}_2,\ \omega_2,\ \dot{\omega_2},\ \theta_3,\ \dot{\theta}_3,\ \omega_3,\ \dot{\omega_3}\ ]$,
- the just-determined subsystem $\mathcal{S}^0$ with $\mathcal{C}^0 = [\ c_1,\ c_2,\ d_7,\ d_8\ ]$ and $\mathcal{X}^+ = [\ \dot{\theta}_1,\ \omega_1,\ \dot{\omega_1},\ d\ ]$ and
- the under-determined subsystem $\mathcal{S}^-$ with $\mathcal{C}^- = [\ \ ]$ and $\mathcal{X}^+ = [\ \ ]$.

1.3 Incidence Matrix

Table 2 presents the incidence matrix of the investigated system.

1.4 Matchings

Table 3 lists the obtained matchings. The fields either contain the matched unknown variables, zeros to indicate an unmatched constraints or nothing if constraints are not used in a matching.

#	$\mathcal{K}$				$\mathcal{X}$												
	u	y_1	y_2	y_3	θ_1	$\dot{\theta}_1$	ω_1	$\dot{\omega_1}$	θ_2	$\dot{\theta}_2$	ω_2	$\dot{\omega_2}$	θ_3	$\dot{\theta}_3$	ω_3	$\dot{\omega_3}$	d
c_1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
c_2	1	0	0	0	1	0	1	1	1	0	0	0	0	0	0	0	1
c_3	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
c_4	0	0	0	0	1	0	0	0	1	0	1	1	1	0	0	0	0
c_5	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
c_6	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	1	0
m_{13}	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
m_{14}	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0
m_{15}	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
d_7	0	0	0	0	X	1	0	0	0	0	0	0	0	0	0	0	0
d_8	0	0	0	0	0	0	X	1	0	0	0	0	0	0	0	0	0
d_9	0	0	0	0	0	0	0	0	X	1	0	0	0	0	0	0	0
d_{10}	0	0	0	0	0	0	0	0	0	0	X	1	0	0	0	0	0
d_{11}	0	0	0	0	0	0	0	0	0	0	0	0	X	1	0	0	0
d_{12}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	X	1	0

Table 2: Incidence matrix of the investigated system.

	c_1	c_2	c_3	c_4	c_5	c_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	m_{13}	m_{14}	m_{15}
1	ω_1	d	ω_2	$\dot{\omega_2}$	ω_3	$\dot{\omega_3}$	$\dot{\theta}_1$	$\dot{\omega_1}$	$\dot{\theta}_2$	0	$\dot{\theta}_3$	0	θ_1	θ_2	θ_3

Table 3: Matchings of the investigated system.

1.5 Parity Equations

Analytical:

$$0 = -\ddot{y_2} - \frac{k_1\ (y_2 - y_1) + k_2\ (y_2 - y_3) + b_2\,\dot{y_2}}{J_2} \tag{1}$$

$$0 = -\ddot{y_3} - \frac{k_2\ (y_3 - y_2) + b_3\,\dot{y_3}}{J_3} \tag{2}$$

Symbolic:

$$d_{10}(c_3(d_9(m_{14}(y_2))), c_4(m_{13}(y_1), m_{14}(y_2), c_3(d_9(m_{14}(y_2))), m_{15}(y_3))) \tag{3}$$

$$d_{12}(c_5(d_{11}(m_{15}(y_3))), c_6(m_{14}(y_2), m_{15}(y_3), c_5(d_{11}(m_{15}(y_3))))) \tag{4}$$

1.6 Detectability and isolability analysis

Table 4 lists the detectability and isolability properties of the parity equations separately and over all combined. Detectable (d), isolable (i) and non-failable constraints (n) are marked accordingly.

	c_1	c_2	c_3	c_4	c_5	c_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	m_{13}	m_{14}	m_{15}
1	d	d	d	d	d	d	n	n	n	n	n	n	i	i	i

Table 4: Detectability and isolability of the investigated system.

2 Question 2

2.1 Design of a proper residual generators

From the parity equations in analytic form, it is possible to obtain the residuals:

$$r_1 = -J_2\ddot{y}_2 - k_2(y_2 - y_3) + k_1(y_1 - y_2) + b_2\dot{y}_2 \tag{5}$$

$$r_2 = -J_3\ddot{y}_3 + k_2(y_2 - y_3) - b_3\dot{y}_3 \tag{6}$$

These residuals must ensure the properties of a good residual generator. Robustness to avoid false alarms or missed detections. Sensitivity to faults to be detected. Structured to fault isolation, residuals react different to a fault.

Stability and causality must be ensured too. Differentiating terms would introduce instability in the case of noise. For this reason, a second order low pass filter is needed in both residuals to make them implementable.

$$Q(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2} \tag{7}$$

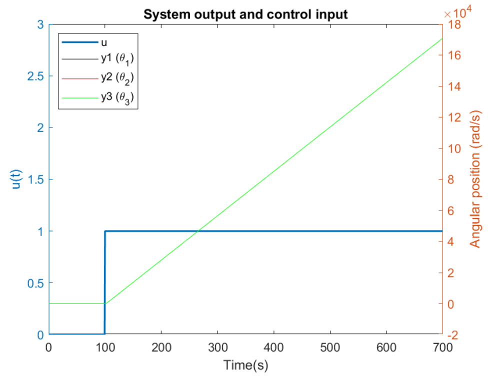
To eliminate the noise, $\omega = 1$ and $\zeta = 0.96$.

Then, the resulting stable and casual residual generators for the system are:

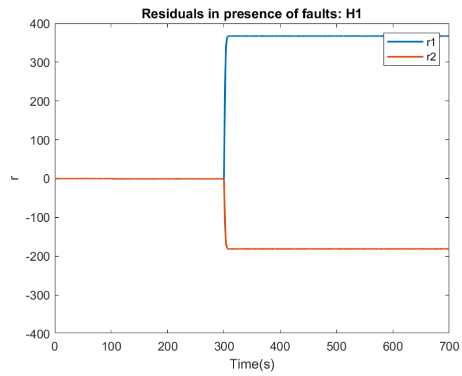
$$r_1(s) = \frac{-9y_2s^2 + y_2s + 13500y_1 - 26500y_2 + 13000y_3}{200(25s^2 + 48s + 25)} \tag{8}$$

$$r_2(s) = \frac{36y_3s^2 + 3y_3s - 52000y_2 + 52000y_3}{800(25s^2 + 48s + 25)} \tag{9}$$

2.2 Implementation of the residual generators



(a) Input and outputs



(b) Residuals

Figure 1: Response of the system with presence of the fault on 2nd sensor. The fault occurs at $t = 300s$

Simulation and test results show that residual generators can detect faults, are stable, casual, react differently to the same fault and are insensitive to input and disturbances. The figure 1 confirms that the residuals are sensitive to faults on simulation, while the figure 2 shows a detection of fault on the real system.

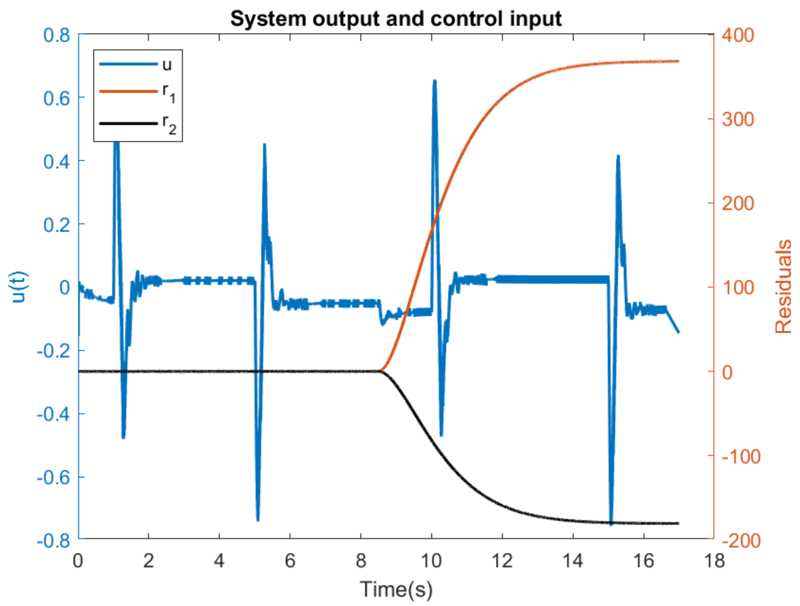


Figure 2: Residuals and control input of the ECP M502a torsional system. The fault occurs at $t = 8.5s$

Comparing the results on the Simulation, Figure 1 and experimental test, Figure 2, it can be concluded that residuals detect faults and also aren't sensitive to input changes.Residuals behaviour have the same performance in both cases. In the experimental case, the signals are also noisier.

3 Question 3

3.1 Standard form of the system

The linear state space system model:

$$\dot{x}(t) = Ax(t) + Bu(t) + E_x d(t) \tag{10}$$

$$y(t) = Cx(t) + Du(t) + E_y d(t) \tag{11}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{J_1} & -\frac{b_1}{J\_1} & \frac{k_1}{J_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{J_2} & 0 & -\frac{k_1+k_2}{J_2} & -\frac{b_2}{J_2} & \frac{k_2}{J_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_2}{J_3} & 0 & -\frac{k_2}{J_3} & -\frac{b_3}{J_3} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \omega_1 \\ \theta_2 \\ \omega_2 \\ \theta_3 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -\frac{1}{J_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} d(t) \tag{12}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \omega_1 \\ \theta_2 \\ \omega_2 \\ \theta_3 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} d(t) \tag{13}$$

4 Question 4

4.1 Calculation of the transfer function matrices Hyu(s) and Hyd(s)

From the linear state space system model:

$$H_{yu} = C(sI - A)^{-1}B + D \tag{14}$$

$$H_{yd} = C(sI - A)^{-1}E_x + E_y \tag{15}$$

4.2 Residual generators design using left nullspace

Deterministic LTI systems in frequency domain with zero initial conditions, the output of the system:

$$\dot{x}(t) = Ax(t) + Bu(t) + E_x d(t) + F_x f(t) \tag{16}$$

$$y(t) = Cx(t) + Du(t) + E_y d(t) + F_y f(t) \tag{17}$$

In the frequency (Laplace) domain is given by:

$$y(s) = H_{yu}(s)u(s) + H_{yd}(s)d(s) + H_{yf}(s)f(s) \tag{18}$$

As the general form of a residual in the frequency domain is:

$$r(s) = V_{ry}(s)y(s) + V_{ru}(s)u(s) \tag{19}$$

As the behaviour of the residual should only depend on faults, changes in control input and disturbances shouldn't be reflected on them:

$$\begin{bmatrix} V_{ry}(s) & V_{ru}(s) \end{bmatrix} \begin{bmatrix} H_{yu}(s) & H_{yd}(s) \\ I & 0 \end{bmatrix} \begin{bmatrix} u(s) \\ d(s) \end{bmatrix} = 0 \tag{20}$$

Where:

$$F(s) = \begin{bmatrix} V_{ry}(s) & V_{ru}(s) \end{bmatrix} \tag{21}$$

$$H(s) = \begin{bmatrix} H_{yu}(s) & H_{yd}(s) \\ I & 0 \end{bmatrix} \tag{22}$$

Therefore, F(s) has to be a basis of the left nullspace of H(s) to guarantee dependency only on faults. Also have a number of rows k (residuals) with greater number of measurements than disturbances. Matrix F(s) is obtained with the command "null" in Matlab.

Since there can be measurement noise in the residuals, filtering should be considered. Filters should be designed to avoid pure integration and differentiation, as well as making the residual proper, to guarantee causal and stable residuals.

To eliminate the noise a low pass filter is applied to each residual. Q(s) is a diagonal matrix where the i^{th} element of the diagonal is a filter that is applied to the i^{th} residual.

$$Q(s) = \begin{bmatrix} \frac{\omega^2}{s^2+2\zeta\omega s+\omega^2} & 0 \\ 0 & \frac{\omega^2}{s^2+2\zeta\omega s+\omega^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2+1.92s+1} & 0 \\ 0 & \frac{1}{s^2+1.92s+1} \end{bmatrix} \tag{23}$$

Where the cut off frequency is chosen $\omega_0 = 1rad$ and damping ratio is $\zeta = 0.96$.

The resulting residual generator for the system based on nullspace design in frequency domain:

$$r(s) = Q(s)F(s) \begin{bmatrix} y(s) \\ u(s) \end{bmatrix} \tag{24}$$

where:

Residual 1

$r_{y1}(s) = \frac{-2.295e24s^2-1.913e23s-3.315e27}{1.53e21s^4+2.975e20s^3+6.715e24s^2+6.21e23s+3.315e27}$

$r_{y2}(s) = \frac{1}{s^2+1.92s+1}$

$r_{y3}(s) = 0$
$r_u(s) = 0$

Residual 2
$r_{y1}(s) = \frac{-3.315e27s^2}{1.53e21s^4+2.975e20s^3+6.715e24s^2+6.21e23s+3.315e27}$
$r_{y2}(s) = 0$
$r_{y3}(s) = \frac{1}{s^2+1.92s+1}$
$r_u(s) = 0$

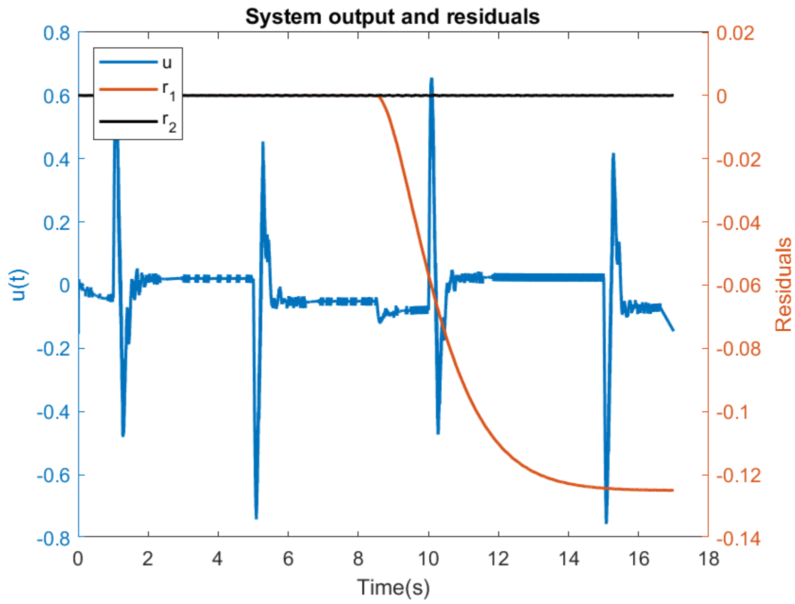


Figure 3: Residuals and control input of the ECP M502a torsional system. The fault occurs at $t = 8.5s$

On figure 3 the residuals and control of the ECP M502a torsional system can be seen. In the absence of faults, residuals remain around zero value with certain noise. As expected, the control input is not causing the residuals to change their values. When the fault on the incremental position encoder of the second disc occurs, the first residual changes its value while the second one does not. It is expected as $r_2(s)$ does not depend of the measurement y_2.

5 Question 5

5.1 Hrf(s) transfer function

The LTI state space model of the system that includes possible faults on the actuator and the three sensors is:

$$\dot{x}(t) = Ax(t) + Bu(t) + E_x d(t) + F_x f(t) \tag{25}$$

$$y(t) = Cx(t) + Du(t) + E_y d(t) + F_y f(t) \tag{26}$$

Where:

$$f = \begin{bmatrix} f_u \\ f_{y1} \\ f_{y2} \\ f_{y3} \end{bmatrix} F_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F_y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

$$r(s) = Q(s)F(s) \begin{bmatrix} H_{yf}(s) \\ 0 \end{bmatrix} f(s) \tag{28}$$

Therefore:

$$H_{rf}(s) = Q(s)F(s)H_{yf}(s) \tag{29}$$

5.2 Strong and weak detectability of the faults

From the residuals behaviour, it can be deduced that faults in any of the incremental position encoders, are strongly detectable. That can be concluded due to the fact that once the fault occurs, the print on the residual stays over the time.On the contrary, faults on the actuators can't be detected due the fact that residuals are designed to have only dependency on sensor faults. The result can be checked if the transfer functions of the system hold the conditions of weak and strong detectability:

For a step change in a fault fi(t) from 0 to fc, the i^{th} fault is weakly detectable if and only if

$$rank \begin{bmatrix} H_{yd}(s) & H^i_{yf}(s) \end{bmatrix} > rank \begin{bmatrix} H_{yd}(s) \end{bmatrix} \tag{30}$$

Fault 1

$$rank \begin{bmatrix} H_{yd}(s) & H^i_{yf}(s) \end{bmatrix} = 1 < rank \begin{bmatrix} H_{yd}(s) \end{bmatrix} = 2 \tag{31}$$

Fault 2, 3, 4

$$rank \begin{bmatrix} H_{yd}(s) & H^i_{yf}(s) \end{bmatrix} = 2 > rank \begin{bmatrix} H_{yd}(s) \end{bmatrix} = 2 \tag{32}$$

The i^{th} fault is strongly detectable if and only if

$$\lim_{s\to\ 0} sr(s) = sQ(s)F(s) \begin{bmatrix} H^i_{yf}(s) \\ 0 \end{bmatrix} \frac{f_c}{s} \neq 0 \tag{33}$$

For the left null space residual design for the system, in steady state the result is:

$$\lim_{s\to\ 0} sr(s) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{34}$$

That result confirms the experimental response obtained in the residuals. Faults on incremental position encoders are strongly detectable. They have non zero value in steady state, whereas faults in the actuator have zero value in steady state, therefore not strongly detectable. Also, fault in the actuator is neither weak detectable, therefore not detectable at all on the residuals.

5.3 Presence of Coulomb friction function $T_C(\omega_1)$

In case the disturbance is known, it is included as a state. The system is augmented:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{J_1} & -\frac{b_1}{J_1} & \frac{k_1}{J_1} & 0 & 0 & 0 & -\frac{1}{J_1} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{k_1}{J_2} & 0 & -\frac{k_1+k_2}{J_2} & -\frac{b_2}{J_2} & \frac{k_2}{J_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{k_2}{J_3} & 0 & -\frac{k_2}{J_3} & -\frac{b_3}{J_3} & 0 \\ 0 & Tc & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \omega_1 \\ \theta_2 \\ \omega_2 \\ \theta_3 \\ \omega_3 \\ d \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \tag{35}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \omega_1 \\ \theta_2 \\ \omega_2 \\ \theta_3 \\ \omega_3 \\ d \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u(t) \tag{36}$$

Doing the same calculations like in problems 5.1 and 5.2, we get that the sensor faults are strongly detectable. However, the actuator fault is also strongly detectable in this case:

$$rank \begin{bmatrix} H_{yd}(s) & H^i_{yf}(s) \end{bmatrix} = 1 > rank \begin{bmatrix} H_{yd}(s) \end{bmatrix} = 0 \tag{37}$$

$$\lim_{s \to 0} sr(s) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & -Tc & 0 & 0 \end{bmatrix} \tag{38}$$

Therefore, in case disturbance is know, all the faults are strongly detectable.

6 Question 6

6.1 Discretization of the residual generator with T_s=0.004 s

The Malab command "c2d" with the discrete option "tustin" and sample time Ts is used to obtain the following transfer function.

$$r(z) = \frac{N_{y1}(z)}{D(z)} y_1(z) + \frac{N_{y2}(z)}{D(z)} y_2(z) + \frac{N_{y3}(z)}{D(z)} y_3(z) \tag{39}$$

where:

$$N_{y1}(z) = 0.03693z^2 + 0.07387z + 0.03693 \tag{40}$$

$$N_{y2}(z) = -6.13z^2 + 11.98z - 6.128 \tag{41}$$

$$N_{y3}(z) = 0.03357z^2 + 0.06715z + 0.03357 \tag{42}$$

$$D(z) = z^2 - 1.652z + 0.7047 \tag{43}$$

6.2 Minimal detectable fault on the 2nd sensor

The CUSUM conditions for the fault detection time (τ_D) and time between false alarms (τ_F) are:

$$\tau_D \leq 500 \tag{44}$$

$$\tau_F \geq \frac{3 \cdot 30 \cdot 24 \cdot 3600}{T_s} \tag{45}$$

Sensitivity of faults, must be guaranteed in residual generators. However, a decision system is needed to evaluate the residual and determine the existence of the fault. Sensors present noise, they introduce a random part in the residual. Therefore, a statistical change detection is needed to determine if the residual is considered to have zero or non-zero value and so, the presence of the fault.

The method required in this problem is CUSUM test, that aims to detect a known change of mean and/or variance when the fault occurs. In this system, sensors present the same uncorrelated noise that doesn't vary over the time.

The aim is to detect a change in mean from μ_0 to μ_1 with variance σ. The cumulative sum used as decision function is:

$$g(k) = g(k-1) + \frac{\mu_1 - \mu_0}{\sigma^2}\left[z(k) - \tfrac{\mu_1 - \mu_0}{2}\right] \tag{46}$$

where:

g(k): is the cumulative sum
z(k): is the residual in discrete time

CUSUM test can be used to detect faults in different ways. The recursive test is the one implemented to solve the problem. It assumes no fault present until a threshold "h" is reached. Let g(k)=max(g(k),0).

In this design case, time to detect, time between false alarms and residual variance are known parameters, whereas the fault magnitude to detect as well as the threshold are unknown. They have to be calculated to implement the recursive test.

The time it takes the cumulative sum to reach "h", is the Average Run Length, ARL.
Let L denote ARL and let

$$arg = \left[2\tfrac{\mu_s h}{\sigma_s^2} + 2.2232\tfrac{\mu_s}{\sigma_s}\right] \tag{47}$$

Then

$$L(\mu_s, \sigma_s, h) = \frac{\sigma_s^2}{2\mu_s^2}(exp(-arg) + arg - 1) \tag{48}$$

For no fault hypothesis, mean value is zero, $\mu_0 = 0$. Variance remains unchanged over the time in the residual, σ obtained with the Matlab function "var". Solving the ARL equation with Matlab "vpasolve" for $\mu_0 = 0$, time to detect and time between false alarms conditions:

$$500 > \tau_D = L(\mu_s, \sigma_s, h) = L(\frac{(\mu_1 - \mu_0)^2}{2\sigma^2}, \frac{(\mu_1 - \mu_0)^2}{\sigma^2}, h) \tag{49}$$

$$1.944e09 < \tau_F = L(\mu_s, \sigma_s, h) = L(-\frac{(\mu_1 - \mu_0)^2}{2\sigma^2}, \frac{(\mu_1 - \mu_0)^2}{\sigma^2}, h) \tag{50}$$

Then we get:

$\mu_1 = -0.0172$
$h = 17.72$

When the fault occurs on the 2nd sensor (f_{y_2}) in the system, given residual's mean value will change for -5.3f_{y_2}. Therefore, the minimal magnitude of the fault that can be detected is -5.3 times smaller than the obtained mean value μ_1. That gives that the minimal magnitude of the fault is: $f_2 = 0.0033$. The optimally of this solution is proved in the τ_D and τ_F ARL Matlab function solution for a given $\mu_1 = -0.0172$:

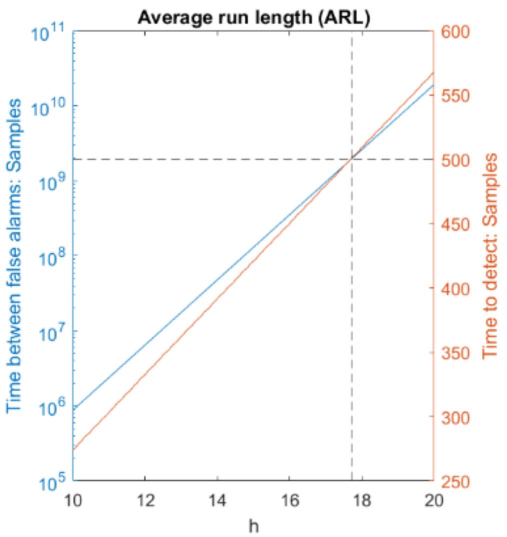


Figure 4: Graphical solution for the given τ_F and τ_D: $h = 17.72$ and $\mu_1 = -0.0172$

The CUSUM test is based on Neyman–Pearson's approach to detection. It assumes that no prior knowledge is available to know if the system is on the condition of no fault θ_0 or fault θ_1. In order to know if the available data corresponds to any of these conditions, the likelihood ratio for an observation r(i) is used:

$$s(r(i)) = ln\left[\frac{p(r(i)|\theta_1)}{p(r(i)|\theta_0}\right] \tag{51}$$

The result of the likelihood ratio is negative in case of θ_0 distribution and positive in case of θ_1 distribution. However, as the CUSUM recursive test takes the maximum of the cumulative sum or zero, the function remains at zero value until a fault is detected and then, growing over the time. This behaviour can be seen on the implementation of the CUSUM recursive test with all the known parameters on the ECP M502a torsional system:

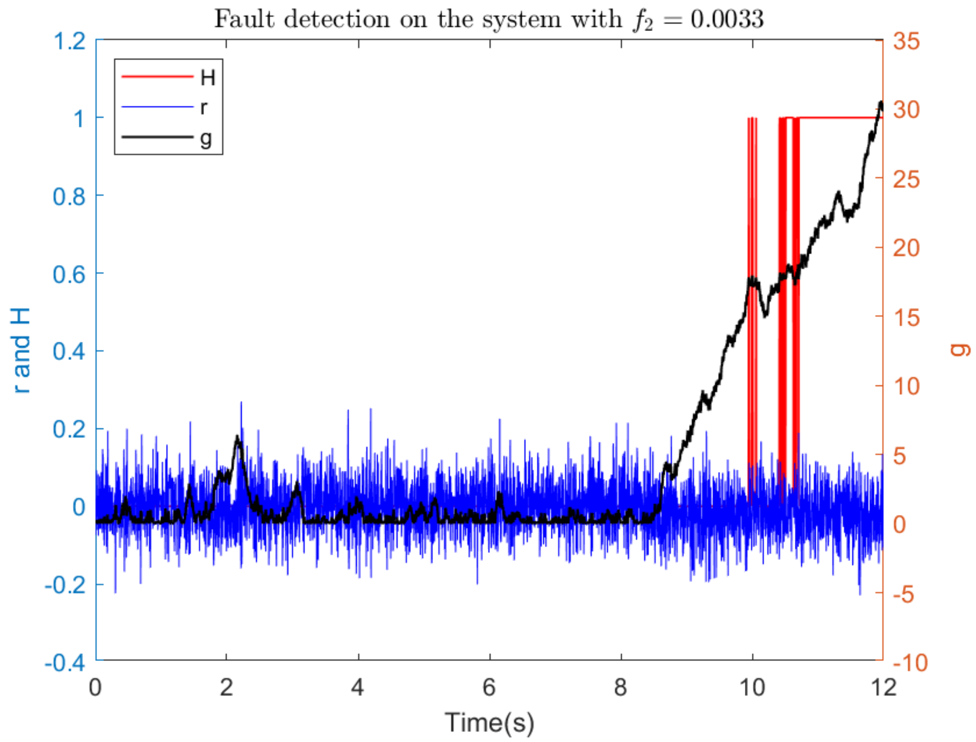


Figure 5: CUSUM test. The fault occurs on $t = 8.5s$ and is detected at $t = 10.7s$

It can be seen that the value of g is around 0 for most of the time before the fault's occurence. If it is not 0, it is still lower than the threshold that is equal to $h = 17.72$ in this problem. After the fault occurs, the value of g starts to grow. When it becomes larger than h, the fault decision function H changes to 1 and the fault is detected. The fault is detected 2.2 seconds after occurring, instead of desired 2 seconds proving that practice and theory are not always aligned.

7 Question 7

7.1 GLR detector design

GLR is another type of decision system to evaluate the existence of the fault in the residual. Its main difference with CUSUM method is that it's used for unknown magnitude of change. GLR is also based on the Neyman–Pearson's approach to detection:

$$S_M(k) = \sum_{i=k-M+1}^{k} \frac{p_{\theta 1}(z(i))}{p_{\theta 0}(z(i))} = \frac{1}{2\sqrt{M}\sigma} \sum_{i=k-M+1}^{k} [z(i) - \mu_0] \tag{52}$$

Assuming the residual a Gaussian sequence:

$$H_0 : p(S_M(k)) = \mathcal{N}(0,\, 1) \tag{53}$$

$$H_1 : p(S_M(k)) = \mathcal{N}(\frac{\sqrt{M}(\mu_1 - \mu_0)}{\sigma},\, 1) \tag{54}$$

Different from CUSUM algorithm, GLR decision function is not calculated based on μ_1 value, that is unknown. It's based on the likelihood ratio, but maximised for μ_1, that is:

$$g(k) = \frac{1}{2M\sigma^2} \sum_{i=k-M+1}^{k} [z(i) - \mu_0]^2 \tag{55}$$

where:

z(k): residual generator
M: window length

Since $S_M(k)$ is Gaussian and its relation with the decision function has the form of $S_M(k) = \sqrt{2g(k)}$, the decision function has a chi-square distribution with 1 degree of freedom:

H_0: chi-square distribution

$$p(2g_M(k)) = \mathcal{X}^2(1)$$

H_1: non-central chi-square distribution

$$p(2g_M(k)) = \mathcal{X}^2(1,\, \lambda)$$

where the centrality parameter is: $\lambda = \frac{M(\mu_1 - \mu_0)^2}{\sigma^2}$

The design method of the detector, consists in finding a threshold value "h" to detect faults and a time analysis window "M". Both parameters are calculated to meet a given probability of P_F, probability of false alarm and P_D, probability to detect the change μ_0 to μ_1:

$$P_F = p(2g > 2h \mid H_0) = \int_h^{\infty} p(2h \mid H_0) dg \tag{56}$$

$$P_D = p(2g \geq 2h \mid H_1) = \int_h^{\infty} p(2h \mid H_1) dg \tag{57}$$

The value of μ_0 is taken to be 0 and the values of σ and μ_1 are estimated to be 0.07 and 0.1325 respectively. Solving with Matlab command "chi2cdf" given $P_F = 0.01$, is obtained the threshold $h = 3.3174$. For $P_M = 0.01$ and $P_M = 1 - P_D$, P_D is calculated and used with the Matlab command

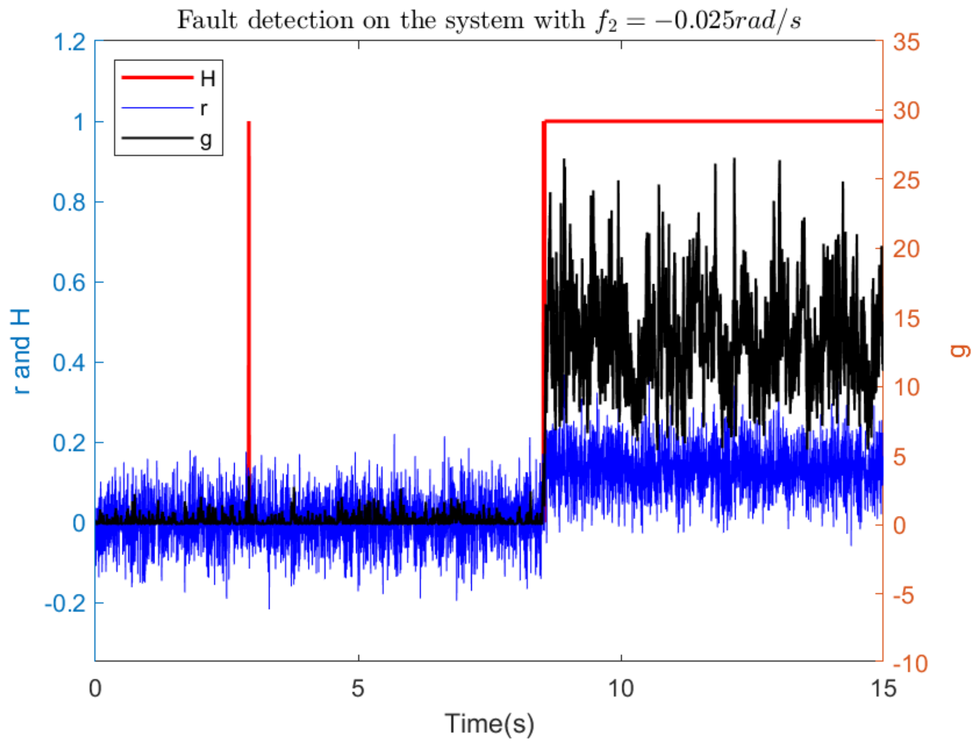


Figure 6: CUSUM test. The fault occurs on $t = 8.5s$

"ncx2cdf", to obtain $\lambda = 24.0313$. Then, $M = 7$ is calculated from the previously mentioned centrality parameter λ. Where M window size is chosen to be big enough to avoid false alarms but also bounded to avoid excessive calculations.

The GLR detector is then implemented on the ECP M502a torsional system. Input to the system is the same like in problem 6 and the GLR detector's response is shown in figure 6. It can be seen that the GLR detector sounded a false alarm before the fault has occurred. The reason for that is a relatively small value for the window size ($M = 7$). As the measurements are quite noisy, a higher value of M would take into account more values of the residual and would be more likely to not trigger false alarms.

8 Question 8

8.1 Discretization of the system

Discretizing the LTI state space model in open loop defined in Question 3 with Matlab command "c2d", sample time Ts and assuming no disturbances:

$$x(k+1) = Fx(k) + Gu(k) \tag{58}$$

$$y(k) = Cx(k) + Du(k) \tag{59}$$

$$F = \begin{bmatrix} 0.9914 & 0.0039 & 0.0085 & 0 & 0 & 0 \\ -4.2804 & 0.9867 & 4.2638 & 0.0085 & 0.0165 & 0 \\ 0.0119 & 0 & 0.9765 & 0.0039 & 0.0114 & 0 \\ 5.9345 & 0.0119 & -11.6436 & 0.9761 & 5.7091 & 0.0114 \\ 0 & 0 & 0.0114 & 0 & 0.9884 & 0.0039 \\ 0.0230 & 0 & 5.7094 & 0.0114 & -5.7324 & 0.9881 \end{bmatrix} G = \begin{bmatrix} 0.0031 \\ 1.5917 \\ 0 \\ 0.0063 \\ 0 \\ 0 \end{bmatrix} \tag{60}$$

8.2 Full state-feedback DLQR

The closed loop system has the following form:

$$x(k+1) = (F - GK_c)x(k) + GK_cC_{ref}\theta_{ref} \tag{61}$$

$$y(k) = Cx(k) \tag{62}$$

With the scaling matrix C_{ref} for the scalar reference signal:

$$C_{ref} = (C_3(I - F + GK_c)^{-1}GK_c)^{+} \tag{63}$$

For the hypothesis of no fault θ_0, the Linear Quadratic Control, LQR, provides a linear quadratic solution for the state feedback control:

$$Kc = R^{-1}B^TP \tag{64}$$

Where P is obtained as the solution of the Ricatti equation and it's constant. In general case, P and K vary over the time. However, the fault tolerant control problem assumes (A,B) matrices controllable, therefore the response in close loop stabilizes. In a time infinite horizon, P and K are constant most of the time. Therefore they can be considered as such and calculated with Matlab command "dlqr" and the relative weighing matrices Qc and Rc.

$$Kc = \begin{bmatrix} 0.5962 & 0.0541 & 0.0671 & 0.0219 & 0.1080 & 0.0237 \end{bmatrix} \tag{65}$$

According to the relative value assigned to Qc and Rc, the control action is more penalized than the deviations from steady state in the transition from the original value to the reference.
The control action has to ensure that top disc incremental encoder, $y_3 = \theta_3$ tracks a reference, as can be seen on figure 7:

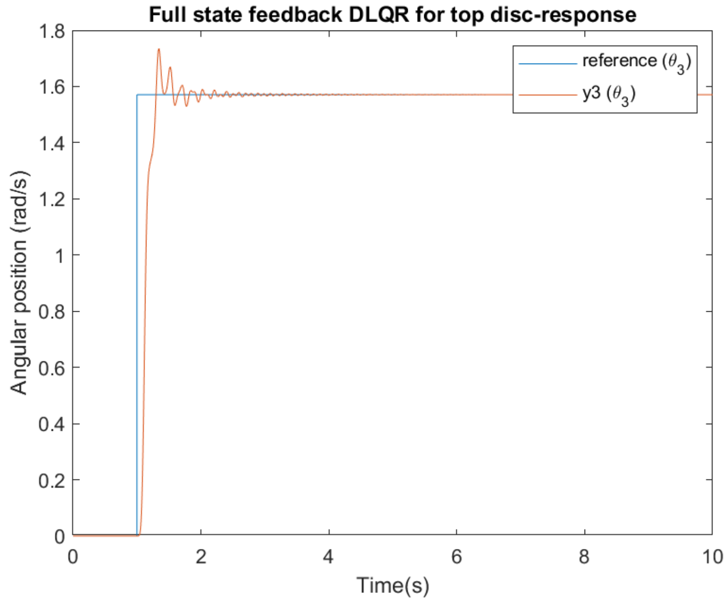


Figure 7: Response of the output y_3 when the reference $theta_3$ is given.

9 Question 9

9.1 Virtual sensor design

In the absence of faults, the system works with the controller designed in Question 8 for the nominal plant. The set of residuals used to sense the faults are the ones implemented in Question 6.
The effect of the additive fault $y_2(t) = y_2(t) + f_m(t)$ at $t_f = 15s$ on the residual:

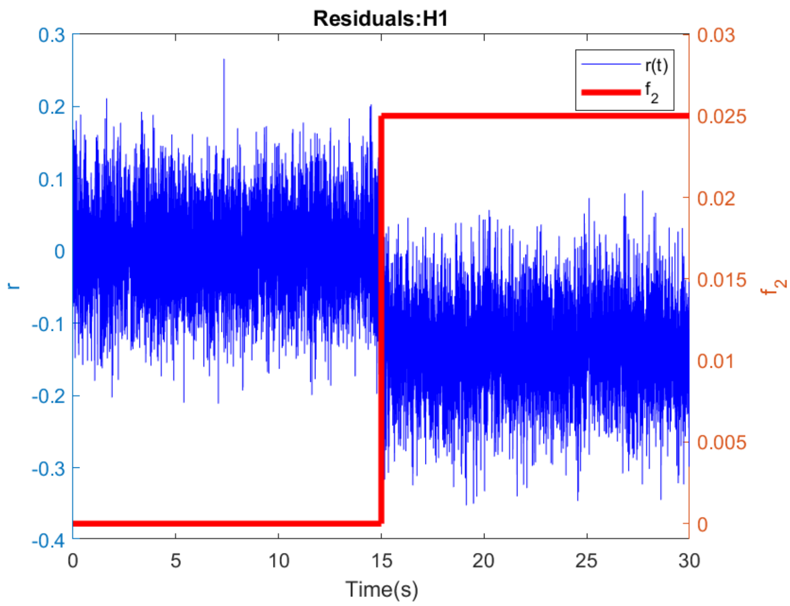


Figure 8: Response of the residual after the fault is introduced in the system

Once the fault is detected with CUSUM algorithm, the faulty sensor y_2 is discarded:

$$C_f = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{66}$$

Then, the decision module selects between y or y_c as input to the state feedback controller. This decision is based on the H_{CUSUM} binary value. This parameter is true just when the cumulative sum in the recursive test has a greater value than the threshold "h".If that is the case, a new controller is required to meet the same or similar performance.

To hide the fault to the nominal controller and in order to avoid its recalculation, an additional block is needed between the faulty plant and the nominal controller. The addition of the reconfiguration block, allows the use of the same controller even in the case of fault and still comply with the system objective.

From the possible static reconfiguration matching, it is checked the condition for model matching:

$$rank(C_f) = rank(\begin{bmatrix} C \\ C_f \end{bmatrix}) \tag{67}$$

where

$$rank(C_f) = 2; rank(\begin{bmatrix} C \\ C_f \end{bmatrix}) = 3 \tag{68}$$

Therefore, close loop dynamics can't be perfectly matched just with the static part of the reconfiguration block. The dynamic part of the Virtual Sensor, is needed. As there are no disturbances, the Integral Virtual Sensor is not considered to implement.

The Virtual Sensor reconstructs the observable state space. Observability of (F, C_f), S_o matrix full rank has to be checked:

$$rank(S_o) = rank(F) = 6$$

The observer gain was obtained using Matlab's function "place". The desired eigenvalues for the observer were taken to have 10 times larger real parts than the corresponding eigenvalues of the closed loop system.

$$F_V = F - L_{Vd}C_f \tag{69}$$
$$G_V = G \tag{70}$$
$$P_{Vd} = CC_f^+ \tag{71}$$
$$C_{Vd} = C - P_{Vd}C_f \tag{72}$$
$$\tag{73}$$

As $P \neq 0$, that means that all measurements are used.

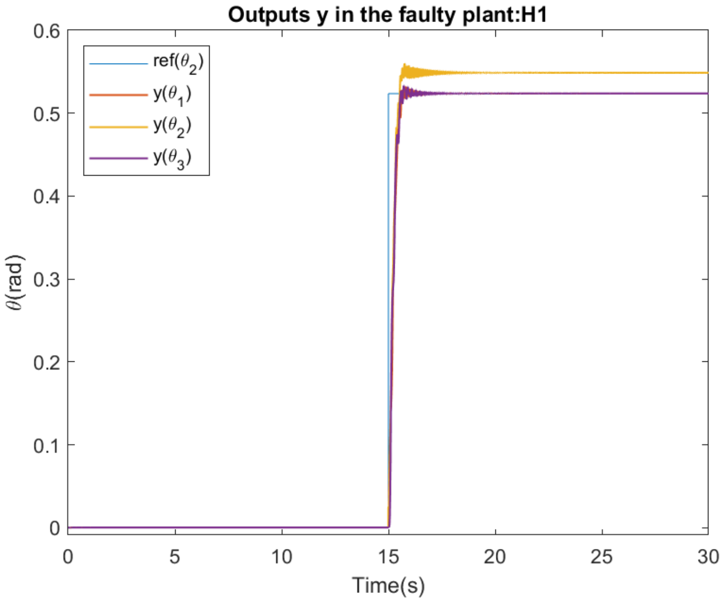


Figure 9: The outputs of the system measured by the sensors. The fault occurrs on $t = 15s$

Behaviour of the system when the fault occurs is shown on figure 9. It can be seen the outputs y_1 and y_3 follow the reference $\theta_{ref} = \frac{\pi}{6}$. On the other hand, the output y_2 is above the reference value due to a fault on the sensor. When the virtual sensor is introduced into the control system, the offset due to the sensor fault that the output y_2 had is no longer present in the system. This can be seen on figure 10.

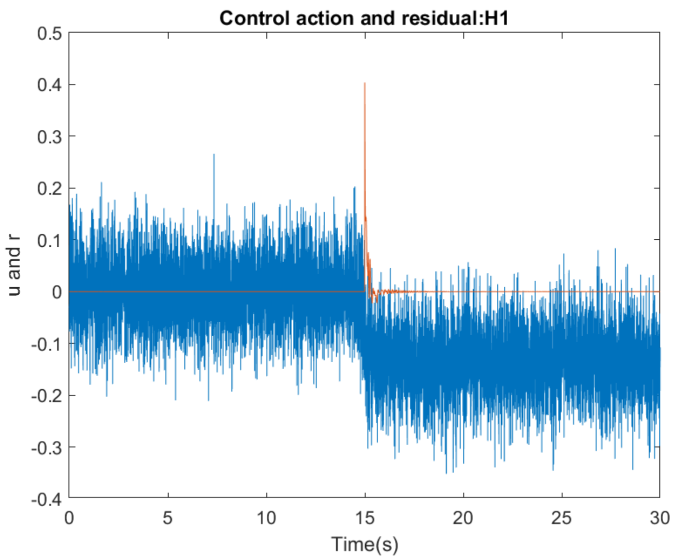


Figure 11: Residual and control input of the system. The fault occurs on $t = 15s$

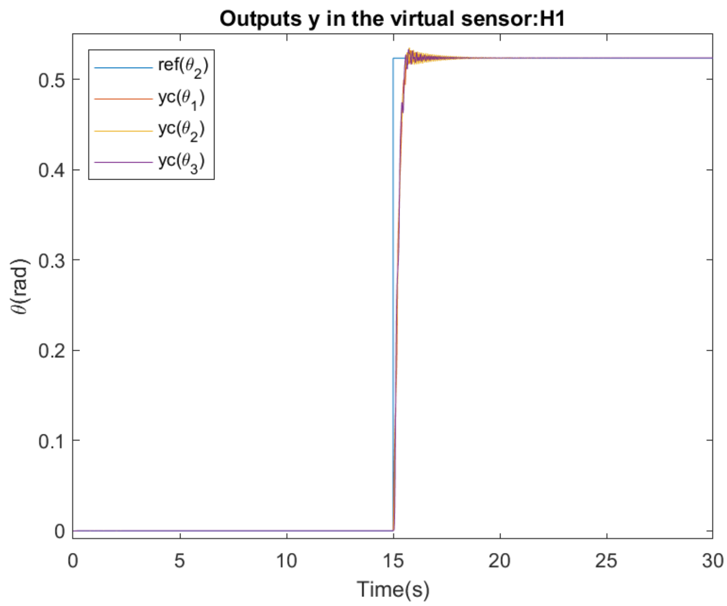


Figure 10: The outputs of the system estimated by the virtual sensor. The fault occurs on $t = 15s$