Weakly Nonlinear Oscillators

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1 Van der Pol

$$\frac{\partial^2 T}{\partial t^2} + T + \varepsilon \mathcal{F}\Big(T, \frac{\partial T}{\partial t}\Big) = 0$$

where $0 \leq \varepsilon << 1$ and $\mathcal{F}()$ is an arbitrary smooth function. This represents small perturbations from

$$\frac{\partial^2 T}{\partial t^2} + T = 0$$

The van der Pol equation

$$\frac{\partial^2 T}{\partial t^2} + T + \varepsilon(T^2 - 1)\frac{\partial T}{\partial t} = 0 \qquad \mathcal{F}\Big(T, \frac{\partial T}{\partial t}\Big) = (T^2 - 1)\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial t^2} - \delta\frac{\partial T}{\partial t} + \alpha T + \delta T^2 \frac{\partial T}{\partial t} = 0 \qquad \alpha = 1 \quad \delta = \varepsilon$$

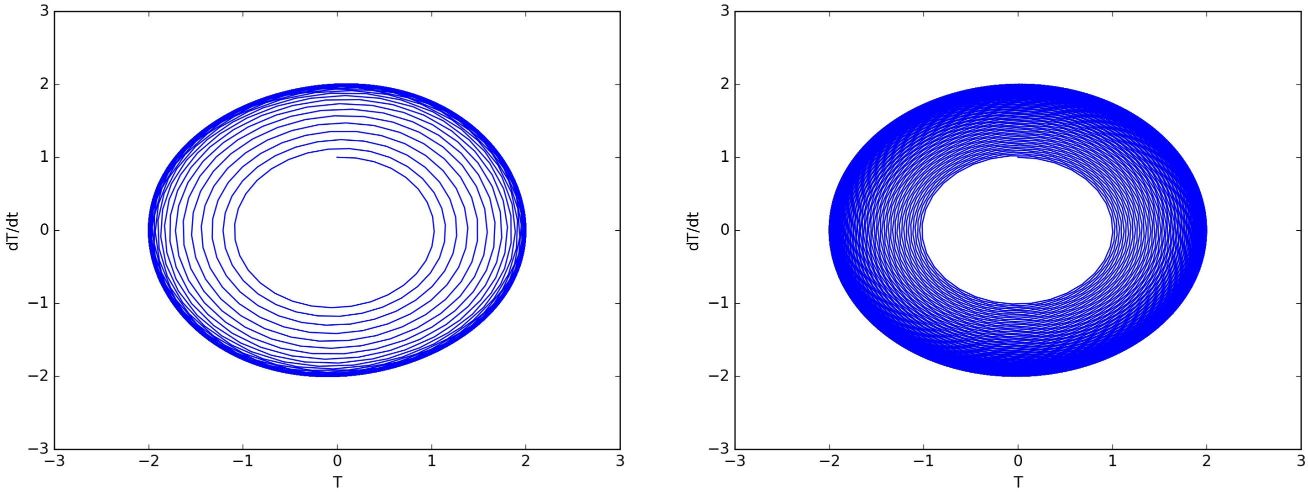

Figure 1: Van der Pol equation phase trajectories for $\alpha = 1$, $\delta = 0.05$ $(left)$, and $\delta = 0.01$, $(right)$ for the same total time interval.

2 Duffing

The undamped, undriven Duffing equation

$$\frac{\partial^2 T}{\partial t^2} + \alpha T + \beta T^3 = 0 \qquad \mathcal{F}\Big(T, \frac{\partial T}{\partial t}\Big) = T^3 \quad \alpha = 0$$

A SST duffing oscillator for T is (take one derivative of the heat equation?)

$$\frac{\partial^2 T}{\partial t^2} + \delta\frac{\partial T}{\partial t} + \alpha T + \beta T^3 = \gamma\cos(\omega t) \qquad \delta = 0 \quad \alpha = 1 \quad 0 \leq \beta << 1$$

does not exactly obey Hooke's law.

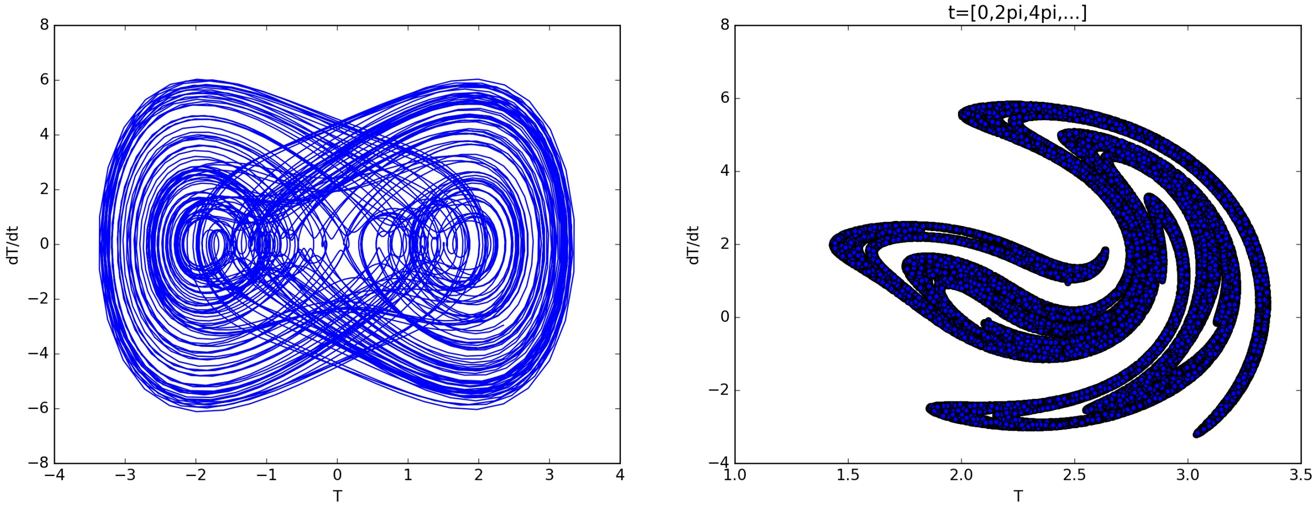

Figure 2: Duffing oscillator phase trajectory ($left$) and Poincare section ($right$) for $\alpha = 0$, $\beta = 1$, $\delta = 0.05$, $\gamma = 7.5$, and $\omega = 1$.

3 Initial conditions

At time $t = 0$ let

$$T(0) = 0 \qquad \frac{\partial T}{\partial t} = 0$$

Find the paper that Marianna had on T3 for ENSO

4 RK4 Numerical Solution

Let

$$\mathcal{F} = T \qquad \mathcal{G} = \frac{\partial T}{\partial t}$$

where

$$\dot{\mathcal{F}} = \frac{\partial \mathcal{F}}{\partial t} = \frac{\partial T}{\partial t} \qquad \dot{\mathcal{G}} = \frac{\partial \mathcal{G}}{\partial t} = \frac{\partial^2 T}{\partial t^2}$$

then we can rewrite Equation BLANK

$$\dot{\mathcal{F}}_n = \dot{\mathcal{F}}(\mathcal{G}_n) = \mathcal{G}_n$$

$$\dot{\mathcal{G}}_n = \dot{\mathcal{G}}(\mathcal{G}_n, \mathcal{F}_n, t_n) = -\delta \mathcal{G}_n - \alpha \mathcal{F}_n - \beta \mathcal{F}_n^3 + \gamma \cos(\omega t_n)$$

RK4:

$$\mathcal{G}_{n+1} = \mathcal{G}_n + \frac{\Delta t}{6}(k_{1g} + 2k_{2g} + 2k_{3g} + k_{4g})$$

$$\mathcal{F}_{n+1} = \mathcal{F}_n + \frac{\Delta t}{6}(k_{1f} + 2k_{2f} + 2k_{3f} + k_{4f})$$

$$t^{n+1} = t^n + \Delta t$$

start with initial values

$$\mathcal{F}_n = T_n \qquad \mathcal{G}_n = \frac{\partial T}{\partial t}\Big|_n$$

$$\mathcal{F}(0) = \mathcal{F}_0 = 0 \qquad \mathcal{G}(0) = \mathcal{G}_0 = 0$$

and use coefficients

$$k_{1f} = \dot{\mathcal{F}}(\mathcal{G}_n) = \mathcal{G}_n$$

$$k_{2f} = \dot{\mathcal{F}}\Big(\mathcal{G}_n + \frac{\Delta t}{2} k_{1f}\Big) = \mathcal{G}_n + \frac{\Delta t}{2} k_{1f}$$

$$k_{3f} = \dot{\mathcal{F}}\Big(\mathcal{G}_n + \frac{\Delta t}{2} k_{2f}\Big) = \mathcal{G}_n + \frac{\Delta t}{2} k_{2f}$$

$$k_{4f} = \dot{\mathcal{F}}(\mathcal{G}_n + \Delta t k_{3f}) = \mathcal{G}_n + \Delta t k_{3f}$$

$$
\begin{aligned}
k_{1g} &= \dot{\mathcal{G}}(\mathcal{G}_n, \mathcal{F}_n, t_n) = -\delta\mathcal{G}_n - \alpha\mathcal{F}_n - \beta\mathcal{F}_n^3 + \gamma\cos(\omega t_n) \\
k_{2g} &= \dot{\mathcal{G}}\Big(\mathcal{G}_n + \frac{\Delta t}{2}k_{1g}, \mathcal{F}_n + \frac{\Delta t}{2}k_{1g}, t_n + \frac{\Delta t}{2}\Big) \\
k_{3g} &= \dot{\mathcal{G}}\Big(\mathcal{G}_n + \frac{\Delta t}{2}k_{2g}, \mathcal{F}_n + \frac{\Delta t}{2}k_{2g}, t_n + \frac{\Delta t}{2}\Big) \\
k_{4g} &= \dot{\mathcal{G}}\Big(\mathcal{G}_n + \Delta t k_{3g}, \mathcal{F}_n + \Delta t k_{3g}, t_n + \Delta t\Big)
\end{aligned}
$$