Adaptive Runge-Kutta methods

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Advance two exact solutions from t to $t + \Delta t$. Advance one by taking one time step of magnitude Δt (called $\mathbf{x}_1$) and the other by taking two time steps of Δt (called $\mathbf{x}_2$), such that:

$$\begin{aligned}\mathbf{x}(t+\Delta t) &= \mathbf{x}_1 + \mathbf{c}(\Delta t)^{p+1} + \mathcal{O}(\Delta t^{p+2}) \\ \mathbf{x}(t+\Delta t) &= \mathbf{x}_2 + \mathbf{c}2(\Delta t/2)^{p+1} + \mathcal{O}(\Delta t^{p+2})\end{aligned}$$

where the value $\mathbf{c}$ remains constant over the step and $\mathcal{O}(\Delta t^{p+2})$ represents the sixth and higher order terms. The difference between the two equations is an indicator of the truncation error:

$$\begin{aligned}0 &= \mathbf{x}_1 + \mathbf{c}(\Delta t)^{p+1} - \mathbf{x}_2 - \mathbf{c}2(\Delta t/2)^{p+1} \\ 0 &= \mathbf{x}_1 - \mathbf{x}_2 + \mathbf{c}\Delta t^{p+1}(1 - 2^{-p}) \\ \mathbf{c} &= \frac{\mathbf{x}_1 - \mathbf{x}_2}{\Delta t^{p+1}(2^{-p} - 1)}\end{aligned}$$

Substitute $\mathbf{c}$ into the second estimate to get:

$$\begin{aligned}\mathbf{x}(t+\Delta t) &= \mathbf{x}_1 + \frac{\mathbf{x}_1 - \mathbf{x}_2}{\Delta t^{p+1}(2^{-p} - 1)}\Delta t^{p+1}(1/2)^p + \mathcal{O}(\Delta t^{p+2}) \\ &= \mathbf{x}_2 + \frac{\mathbf{x}_1 - \mathbf{x}_2}{2^p - 1} + \mathcal{O}(\Delta t^{p+2})\end{aligned}$$

So if we choose a 4th order method, $p = 4$, and to 6th order the truncation error is given

$$\varepsilon = \frac{|\mathbf{x}_1 - \mathbf{x}_2|}{2^p - 1} = \frac{|\mathbf{x}_1 - \mathbf{x}_2|}{15}$$

Therefore if we want the truncation error of a given step to be below some value, ε_0:

$$\begin{aligned}\mathbf{c}\Delta t_{\mathrm{new}}^{p+1} &= \frac{\mathbf{x}_1 - \mathbf{x}_2}{(2^{-p} - 1)}\frac{\Delta t_{\mathrm{new}}^{p+1}}{\Delta t^{p+1}} \\ &= \varepsilon\frac{\Delta t_{\mathrm{new}}^{p+1}}{\Delta t^{p+1}} \\ &\leq \varepsilon_0\end{aligned}$$