Floquet Theory Basics & Code Verification

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Consider the non-autonomous system,

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \mathbf{A}(t)\mathbf{x}(t) \tag{1}$$

where the operator $\mathbf{A}(t)$ is periodic:

$$\mathbf{A}(t+T) = \mathbf{A}(t) \tag{2}$$

Note that while $\mathbf{A}$ may be periodic, $\mathbf{x}$ need not be. If

$$\mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix} \tag{3}$$

then the general solution is a superposition of two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + c_2 \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \tag{4}$$

and c_1 and c_2 are arbitrary, time-independent constants. The fundamental solution matrix is a matrix in which the columns are the solutions

$$\mathbf{\Phi}(t) = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \tag{5}$$

The fundamental matrix Φ to the original system is *not* unique, as there are many ways to choose independent solutions and arbitrary constants. Therefore, we can write the general solution as:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{6}$$

Since the arbitrary constants $\mathbf{c}$ are independent of time, we can define them using (usually known) the initial conditions:

$$\mathbf{x}(0) = \mathbf{\Phi}(0)\mathbf{c} \tag{7}$$

Since the determinant of the initial fundamental solution matrix, $\det[\Phi_0]$, is the value at t_0 of the Wronskian of the independent solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, it is non-zero because the two solutions are linearly independent. Therefore a matrix inverse exists and the constants $\mathbf{c}$ can be obtained:

$$\mathbf{c} = \mathbf{\Phi}(0)^{-1}\mathbf{x}(0) \tag{8}$$

Equation 8 can be substituted into Equation 6 to obtain the forward propogator, which maps $\mathbf{x}_0$ onto $\mathbf{x}(t)$:

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{\Phi}(t)\mathbf{c} \\ &= \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}(0) \end{aligned} \tag{9}$$

Note that the fundamental solution matrix must also satisfy the equation:

$$\frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{dt}} = \mathbf{A}(t)\mathbf{\Phi}(t) \tag{10}$$

Since $\mathbf{\Phi}(t)$ is a fundamental matrix, then it can be shown that $\mathbf{Y}(t) = \mathbf{\Phi}(t)\mathbf{C}$ is also a fundamental matrix, so long as $\mathbf{C}$ is a non-singular constant matrix (note: for $\mathbf{C}$ to be non-singular, $\mathbf{x}(0) \neq \mathbf{0}$). We can then choose $\mathbf{Y}(t) = \mathbf{\Phi}(t+T)$ to obtain:

$$\mathbf{\Phi}(t+T) = \mathbf{\Phi}(t)\mathbf{C} \tag{11}$$

Now, *if* $\mathbf{\Phi}_0 = \mathbf{I}$ *then* $\mathbf{\Phi}(T)$ *is the prinicipal fundamental solution matrix*, denoted by *:

$$\begin{aligned}\mathbf{C} &= \mathbf{\Phi}^*(0)^{-1}\mathbf{\Phi}^*(T)\\ &= \mathbf{I}^{-1}\mathbf{\Phi}^*(T)\\ &= \mathbf{\Phi}^*(T)\end{aligned} \tag{12}$$

$\mathbf{\Phi}_0 = \mathbf{I}$ can be imposed without a loss of generality because of there is no unique choice for $\mathbf{\Phi}$ for any given system. Therefore the initial condition on $\mathbf{x}$ (Equation 7) is:

$$\mathbf{x}_0 = \mathbf{c} \tag{13}$$

and the forward propogator equation (Equation 9) is:

$$\mathbf{x}(t) = \mathbf{\Phi}^*(t)\mathbf{x}_0 \tag{14}$$

and therefore

$$\mathbf{x}(T) = \mathbf{\Phi}^*(T)\mathbf{x}_0 \tag{15}$$

Eigenanalysis of $\mathbf{\Phi}^*(T)$ selects the eigenvectors of $\mathbf{\Phi}^*(T)$ as the initial conditions for normal modes that are mapped from the initial conditions to the solutions $\mathbf{x}(T)$ by the eigenvalues of $\mathbf{\Phi}^*(T)$. The eigensystem:

$$\begin{aligned}\mathbf{x}(T)_k &= \mathbf{\Phi}^*(T)\mathbf{x}_{0,k}\\ &= \mathbf{\Phi}^*(T)\mathbf{v}_k\\ &= \mu_k\mathbf{I}\mathbf{v}_k\\ &= \mu_k\mathbf{x}_{0,k}\end{aligned} \tag{16}$$

where subscript k denotes each eigenvalue and eigenvector pair. This mapping is the core of Floquet analysis. μ_k are the Floquet multipliers and $\mathbf{x}_k$ are the Floquet modes.

Integration of Equation 10 for the fundamental solution matrix over one period

$$\int_0^T \frac{1}{\mathbf{\Phi}^*}\mathrm{d}\mathbf{\Phi}^* = \log(\mathbf{\Phi}^*(t))\Big|_0^T = \log(\mathbf{\Phi}^*(T)) = \int_0^T \mathbf{A}(t)\,\mathrm{dt} \tag{17}$$

yields

$$\mathbf{\Phi}^*(T) = \exp\Big[\int_0^T \mathbf{A}(t)\,\mathrm{dt}\Big] \tag{18}$$

which is equivalent in the limit of infinitesimal time steps to the ordered product of infinitesimal propogators:

$$\mathbf{\Phi}^*(T) = \lim_{\Delta t\to 0}\prod_{j=1}^{N} \mathrm{e}^{\mathbf{A}(t_j)\Delta t} \tag{19}$$

where $t_0 + (j-1)\Delta t < t_j < t_0 + j\Delta t$ and $T = t_0 + N\Delta t$. Alternatively, Equation 10 for the fundamental solution matrix can be solved using a conventional time stepping scheme, such as a Runge-Kutta method.

An example 2×2 system

Let the system of equations be:

$$\frac{\partial u}{\partial t} = -\alpha u - v\mathrm{e}^{i\omega t}, \qquad \frac{\partial v}{\partial t} = -\beta v \tag{20}$$

therefore

$$\mathbf{A}(t) = \begin{bmatrix} -\alpha & -\mathrm{e}^{i\omega t} \\ 0 & -\beta \end{bmatrix} \tag{21}$$

and the solutions are:

$$\mathbf{x}(t) = \begin{bmatrix} u_0\mathrm{e}^{-\alpha t} + \frac{v_0}{\beta - i\omega - \alpha}\mathrm{e}^{(i\omega-\beta)t} \\ v_0\mathrm{e}^{-\beta t} \end{bmatrix} \tag{22}$$

for the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} u_0 + \frac{v_0}{\beta - i\omega - \alpha} \\ v_0 \end{bmatrix} \tag{23}$$

The periodic term in $\mathbf{A}$ integrates to zero, therefore the principal fundamental solution matrix (denoted by *) is

$$\mathbf{\Phi}^*(T) = \begin{bmatrix} \mathrm{e}^{-\alpha T} & 0 \\ 0 & \mathrm{e}^{-\beta T} \end{bmatrix} \tag{24}$$

which has the eigenvalues (Floquet multipliers):

$$\mu_1 = \mathrm{e}^{-\alpha T} \qquad \mu_2 = \mathrm{e}^{-\beta T} \tag{25}$$

and the eigenvectors (initial conditions for the Floquet modes):

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{26}$$

The two independent solutions of $\mathbf{x}(T)$ can be use to form another fundamental solution matrix:

$$\mathbf{x}(T) = \mathbf{\Phi}(T)\mathbf{c} = \begin{bmatrix} \mathrm{e}^{-\alpha T} & \frac{\mathrm{e}^{-\beta T}}{\beta - i\omega - \alpha} \\ 0 & \mathrm{e}^{-\beta T} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \tag{27}$$

Note that this fundamental solution matrix is not the prinicipal fundamental solution matrix because it could not arise from the initial condition $\Phi_0 = \mathbf{I}$. Both sets of initial conditions from the eigenvectors of the prinicipal fundamental solution matrix (Equations 26) can be used to map the the solutions from time $t = 0$ to time $t = T$, as in Equation 15:

$$\mathbf{x}(T)_k = \mathbf{\Phi}^*(T)\mathbf{v}_k \tag{28}$$

Floquet mode $k = 1$

$$\mathbf{x}(T)_1 = \mathbf{\Phi}^*(T)\mathbf{v}_1 = \begin{bmatrix} \mathrm{e}^{-\alpha T} & 0 \\ 0 & \mathrm{e}^{-\beta T} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mu_1\mathbf{v}_1 \tag{29}$$

corresponds to selecting:

$$u_0 = 0, \qquad v_0 = 1 \tag{30}$$

in Equation 22.Floquet mode $k = 2$

$$\mathbf{x}(T)_2 = \mathbf{\Phi}^*(T)\mathbf{v}_2 = \begin{bmatrix} \mathrm{e}^{-\alpha T} & 0 \\ 0 & \mathrm{e}^{-\beta T} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mu_2\mathbf{v}_2 \tag{31}$$

corresponds to selecting

$$u_0 = -\Big(\frac{\mathrm{e}^{-\beta T}}{\beta - i\omega - \alpha} + \mathrm{e}^{-\alpha T}\Big), \qquad v_0 = 1 \tag{32}$$

in Equation 22.The Floquet exponents are defined as

$$\mu_k = \mathrm{e}^{\gamma_k T}$$

which indicate Floquet mode frequencies exist for imaginary γ_k and that if the real part of γ_k is greater than unity growth will occur.

Mathieu equation example

Hill's equation are the form:

$$\frac{\partial^2 y}{\partial t^2} + f(t)y = 0,$$

which can be expressed in matrix form as:

$$\begin{bmatrix} y_t \\ y_{tt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -f(t) & 0 \end{bmatrix} \begin{bmatrix} y \\ y_t \end{bmatrix}$$

where y_t and y_{tt} are the first and second temporal derivatives of y. Mathieu's equation is a special case of a Hill equation in which the function $f(t)$ takes the form:

$$f(t) = \delta + \varepsilon \cos(t)$$

Hill equations possess the property:

$$\det[\Phi^*(T)] = 1$$

Which means that the eigenvalues (Floquet multipliers) of $\Phi^*(T)$ can be obtained through the relationship:

$$\mu_{1,2} = \frac{\mathrm{tr}[\Phi^*(T)] \pm \sqrt{\mathrm{tr}[(\Phi^*(T))^2] - 4}}{2} \tag{33}$$

So for Hill equations the stability can be determined either by the Floquet multipliers or the trace of the monodromy matrix ($\mathrm{tr}[(\Phi^*(T))^2]$). The stability of the system in terms of Floquet multipliers is:

(a) If $|\mu| < 1$ (where $|\mu|$ is the complex modulus of μ) then the real part of the corresponding Floquet exponent is negative ($\mathrm{Real}[\gamma] < 0$) and the Floquet mode is damped. If all of the multipliers μ_k satisfy this property then the system is **stable** and decays as $t \to \infty$.

(b) If $|\mu| = 1$ then the corresponding Floquet exponent is zero ($\mathrm{Real}[\gamma] = 0$) and the mode is periodic, although not necessarily oscillating with the base period. If $\mu \pm 1 + 0i$ the mode is periodic with exactly the base period. If all modes in the system satisfy this property then the system is **purely periodic** for all time.

(c) If $|\mu| > 1$ then the real part of the corresponding Floquet exponent is positive ($\mathrm{Real}[\gamma] > 0$) and the mode will grow in amplitude as $t \to \infty$. If any mode satisfies this property, then the system is **unstable** and grows as $t \to \infty$.

Equivalently, the stability conditions in terms of the trace of the monodromy matrix for Hill equations are:

(a) For $|\mathrm{tr}[\Phi^*(T)]| < 2$ Equation 33 has a pair of complex conjugate roots. Since the product of the roots is unity, they both must lie in the unit circle. Therefore the system is **stable**.

(b) For $|\mathrm{tr}[\Phi^*(T)]| = 2$ the roots of Equation 33 are $\mu = 1, 1$ and $\mu = -1, -1$. For $\mu = 1, 1$ the system is **purely periodic** with period T. Plug $\mu = -1, -1$ in Equation 33 and the system is **purely periodic** with period $2T$.

(c) For $|\mathrm{tr}[\Phi^*(T)]| > 2$ Equation 33 has real roots. The product of the roots is unity, but one of the complex moduli of the Floquet multipliers is less than unity and the other must be greater than unity, so this case the system is **unstable**.

Here we verify the estimate of system stability obtained from the Floquet multipliers with the stability obtained from the trace of the monodromy matrix for the Mathieu equation.

References