Floquet Theory Basics

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January 31, 2019

Consider the non-autonomous system,

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \mathbf{A}(t)\mathbf{x}(t) \tag{1}$$

where the operator $\mathbf{A}(t)$ is periodic:

$$\mathbf{A}(t+T) = \mathbf{A}(t) \tag{2}$$

Note that while $\mathbf{A}$ may be periodic, $\mathbf{x}$ need not be. If

$$\mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix} \tag{3}$$

then the general solution is a superposition of two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + c_2 \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \tag{4}$$

and c_1 and c_2 are arbitrary, time-independent constants. The fundamental solution matrix is a matrix in which the columns are the solutions

$$\mathbf{\Phi}(t) = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \tag{5}$$

The fundamental matrix Φ to the original system is *not* unique, as there are many ways to choose independent solutions and arbitrary constants. Therefore, we can write the general solution as:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{6}$$

Since the arbitrary constants $\mathbf{c}$ are independent of time, we can define them using (usually known) the initial conditions:

$$\mathbf{x}(0) = \mathbf{\Phi}(0)\mathbf{c} \tag{7}$$

Since the determinant of the initial fundamental solution matrix, $\det[\Phi_0]$, is the value at t_0 of the Wronskian of the independent solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, it is non-zero because the two solutions are linearly independent. Therefore a matrix inverse exists and the constants $\mathbf{c}$ can be obtained:

$$\mathbf{c} = \mathbf{\Phi}(0)^{-1}\mathbf{x}(0) \tag{8}$$

Equation 8 can be substituted into Equation 6 to obtain the forward propogator, which maps $\mathbf{x}_0$ onto $\mathbf{x}(t)$:

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{\Phi}(t)\mathbf{c} \\ &= \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}(0) \end{aligned} \tag{9}$$

Note that the fundamental solution matrix must also satisfy the equation:

$$\frac{\mathrm{d}\mathbf{\Phi}}{\mathrm{dt}} = \mathbf{A}(t)\mathbf{\Phi}(t) \tag{10}$$

Since $\mathbf{\Phi}(t)$ is a fundamental matrix, then it can be shown that $\mathbf{Y}(t) = \mathbf{\Phi}(t)\mathbf{C}$ is also a fundamental matrix, so long as $\mathbf{C}$ is a non-singular constant matrix. We can then choose $\mathbf{Y}(t) = \mathbf{\Phi}(t+T)$ to obtain:

$$\mathbf{\Phi}(t+T) = \mathbf{\Phi}(t)\mathbf{C} \tag{11}$$

Now, *if* $\mathbf{\Phi}_0 = \mathbf{I}$ *then* $\mathbf{\Phi}(T)$ *is the prinicipal fundamental solution matrix*, denoted by *:

$$\begin{aligned}\mathbf{C} &= \mathbf{\Phi}^*(0)^{-1}\mathbf{\Phi}^*(T) \\ &= \mathbf{I}^{-1}\mathbf{\Phi}^*(T) \\ &= \mathbf{\Phi}^*(T)\end{aligned} \tag{12}$$

$\mathbf{\Phi}_0 = \mathbf{I}$ can be imposed without a loss of generality because of there is no unique choice for $\mathbf{\Phi}$ for any given system. Therefore the initial condition on $\mathbf{x}$ (Equation 7) is:

$$\mathbf{x}_0 = \mathbf{c} \tag{13}$$

and the forward propogator equation (Equation 9) is:

$$\mathbf{x}(t) = \mathbf{\Phi}^*(t)\mathbf{x}_0 \tag{14}$$

and therefore

$$\mathbf{x}(T) = \mathbf{\Phi}^*(T)\mathbf{x}_0 \tag{15}$$

Eigenanalysis of $\mathbf{\Phi}^*(T)$ selects the eigenvectors of $\mathbf{\Phi}^*(T)$ as the initial conditions for normal modes that are mapped from the initial conditions to the solutions $\mathbf{x}(T)$ by the eigenvalues of $\mathbf{\Phi}^*(T)$. The eigensystem:

$$\begin{aligned}\mathbf{x}(T)_k &= \mathbf{\Phi}^*(T)\mathbf{x}_{0,k} \\ &= \mathbf{\Phi}^*(T)\mathbf{v}_k \\ &= \mu_k\mathbf{I}\mathbf{v}_k \\ &= \mu_k\mathbf{x}_{0,k}\end{aligned} \tag{16}$$

where subscript k denotes each eigenvalue and eigenvector pair. This mapping is the core of Floquet analysis. μ_k are the Floquet multipliers and $\mathbf{x}_k$ are the Floquet modes.

Integration of the principal fundmental matrix $\Phi(t)$ (for $\Phi_0 = \mathbf{I}$) over one period

$$\int_0^T \mathrm{d}\mathbf{\Phi}^* = \mathbf{\Phi}^*(T) - \mathbf{\Phi}^*(0) = \mathbf{\Phi}^*(T) - \mathbf{I} = \int_0^T \mathbf{A}(t)\cdot\mathbf{\Phi}^*(0)\,\mathrm{dt} = \int_0^T \mathbf{A}(t)\cdot\mathbf{I}\,\mathrm{dt} \tag{17}$$

therefore

$$\mathbf{\Phi}^*(T) = \mathbf{I} + \int_0^T \mathbf{A}(t)\ \mathrm{dt} \tag{18}$$

An example 2×2 system

Let the system of equations be:

$$\frac{\partial u}{\partial t} = -\alpha u - v\mathrm{e}^{i\omega t}, \qquad \frac{\partial v}{\partial t} = -\beta v \tag{19}$$

therefore

$$\mathbf{A}(t) = \begin{bmatrix} -\alpha & -\mathrm{e}^{i\omega t} \\ 0 & -\beta \end{bmatrix} \tag{20}$$

and the solutions are:

$$\mathbf{x}(t) = \begin{bmatrix} u_0\mathrm{e}^{-\alpha t} + \frac{v_0}{\beta - i\omega - \alpha}\mathrm{e}^{(i\omega-\beta)t} \\ v_0\mathrm{e}^{-\beta t} \end{bmatrix} \tag{21}$$

for the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} u_0 + \frac{v_0}{\beta - i\omega - \alpha} \\ v_0 \end{bmatrix} \tag{22}$$

The periodic term in $\mathbf{A}$ integrates to zero, therefore the principal fundamental solution matrix (denoted by *) is

$$\mathbf{\Phi}^*(T) = \begin{bmatrix} \mathrm{e}^{-\alpha T} & 0 \\ 0 & \mathrm{e}^{-\beta T} \end{bmatrix} \tag{23}$$

which has the eigenvalues (Floquet multipliers):

$$\mu_1 = \mathrm{e}^{-\alpha T} \qquad \mu_2 = \mathrm{e}^{-\beta T} \tag{24}$$

and the eigenvectors (initial conditions for the Floquet modes):

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{25}$$

The two independent solutions of $\mathbf{x}(T)$ can be use to form another fundamental solution matrix:

$$\mathbf{x}(T) = \mathbf{\Phi}(T)\mathbf{c} = \begin{bmatrix} \mathrm{e}^{-\alpha T} & \frac{\mathrm{e}^{-\beta T}}{\beta - i\omega - \alpha} \\ 0 & \mathrm{e}^{-\beta T} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \tag{26}$$

Note that this fundamental solution matrix is not the prinicipal fundamental solution matrix because it could not arise from the initial condition $\Phi_0 = \mathbf{I}$. Both sets of initial conditions from the eigenvectors of the prinicipal fundamental solution matrix (Equations 25) can be used to map the the solutions from time $t = 0$ to time $t = T$, as in Equation 15:

$$\mathbf{x}(T)_k = \mathbf{\Phi}^*(T)\mathbf{v}_k \tag{27}$$

Floquet mode $k = 1$

$$\mathbf{x}(T)_1 = \mathbf{\Phi}^*(T)\mathbf{v}_1 = \begin{bmatrix} \mathrm{e}^{-\alpha T} & 0 \\ 0 & \mathrm{e}^{-\beta T} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mu_1\mathbf{v}_1 \tag{28}$$

corresponds to selecting:

$$u_0 = 0, \qquad v_0 = 1 \tag{29}$$

in Equation 21.Floquet mode $k = 2$

$$\mathbf{x}(T)_2 = \mathbf{\Phi}^*(T)\mathbf{v}_2 = \begin{bmatrix} \mathrm{e}^{-\alpha T} & 0 \\ 0 & \mathrm{e}^{-\beta T} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mu_2\mathbf{v}_2 \tag{30}$$

corresponds to selecting

$$u_0 = -\Big(\frac{\mathrm{e}^{-\beta T}}{\beta - i\omega - \alpha} + \mathrm{e}^{-\alpha T}\Big), \qquad v_0 = 1 \tag{31}$$

in Equation 21.The Floquet exponents are defined as

$$\mu_k = \mathrm{e}^{\gamma_k T}$$

which indicate Floquet mode frequencies exist for imaginary γ_k and that if the real part of γ_k is greater than unity growth will occur.

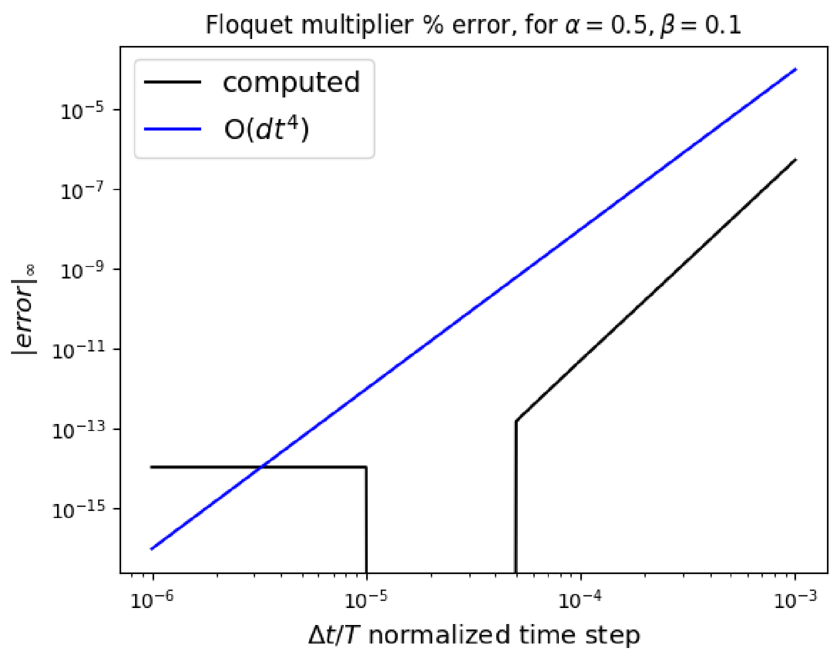

Figure 1: L_∞ norm of the local (i.e. one time step) Floquet multiplier percent error ($\max[|\lambda_{k,\text{computed}} - \lambda_{k,\text{analytical}}|/|-\lambda_{k,\text{analytical}}|]$) for the system of equations in Equation 19 as a function of time step divided by period. The principal fundamental solution matrix $\mathbf{\Phi}^*(t)$ was advanced in time explicitly using a 4th order Runge Kutta method (for which the local truncation error is $\mathcal{O}(\Delta t^4)$) one time step Δt and then the percent error was computed between the eigenvalues (i.e. Floquet multipliers) of the computed matrix $\mathbf{\Phi}^*(t)$ and the exact analytical solutions (Equation 24). Machine precision is acheived at time step size corresponding to about $3 \cdot 10^4$ time steps per period.