1 A toy problem

Let's solve:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} + \sin(mz) \tag{1}$$

for $0 < z < \pi$ and $0 < t < T$ and Dirchlet boundary conditions:

$$u(0,t) = 0 \tag{2}$$

$$u(\pi,t) = 0 \tag{3}$$

and the initial condition:

$$u(z,0) = z(\pi - z) \tag{4}$$

1.1 Analytical solution

Start by solving:

$$\frac{\partial^2 u_s}{\partial z^2} = -\frac{1}{\nu}\sin(mz) \tag{5}$$

The solution is

$$u_s = \frac{1}{\nu m^2}\sin(mz) \tag{6}$$

Now let:

$$u(z,t) = u_s + v(z,t) \tag{7}$$

where

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} \tag{8}$$

$$v(0,t) = 0 \tag{9}$$

$$v(\pi,t) = 0 \tag{10}$$

$$v(z,0) = z(\pi - z) - u_s \tag{11}$$

Need a function that has $z(\pi - z)$ as it's time derivative and also it's second spatial derivative. The solution is a Fourier series for an odd parabola ($L = \pi, H = \pi^2/4$)

$$v(z,t) = -\frac{1}{\nu m^2}\sin(mz)\mathrm{e}^{-\nu m^2 t} + \sum_{n=1}^{\infty}\frac{4}{n^3\pi}(1 - (-1)^n)\sin(nz)\mathrm{e}^{-\nu n^2 t} \tag{12}$$

The full solution is then:

$$u(z,t) = \frac{1}{\nu m^2}\sin(mz)(1 - \mathrm{e}^{-\nu m^2 t}) + \sum_{n=1}^{\infty}\frac{4}{n^3\pi}(1 - (-1)^n)\sin(nz)\mathrm{e}^{-\nu n^2 t} \tag{13}$$

Verification:

$$\frac{\partial u}{\partial t} = \sin(mz)\mathrm{e}^{-\nu m^2 t} - \nu\sum_{n=1}^{\infty}\frac{4}{n\pi}(1 - (-1)^n)\sin(nz)\mathrm{e}^{-\nu n^2 t} \tag{14}$$

$$\nu\frac{\partial^2 u}{\partial z^2} = \sin(mz)(\mathrm{e}^{-\nu m^2 t} - 1) - \nu\sum_{n=1}^{\infty}\frac{4}{n\pi}(1 - (-1)^n)\sin(nz)\mathrm{e}^{-\nu n^2 t} \tag{15}$$

And so we see that Equation 1 is satisfied. The two boundary conditions and the initial condition are satisfied by Equation 13.

1.2 The Galerkin solution

Now we want to obtain a solution to Equation using "trial" functions of the form:

$$u(z,t) = \sum_{n=1}^{N} A_n(t)\sin(nz) \tag{16}$$

The error of approximation of the solution is:

$$\begin{aligned}\epsilon(z,t) &= \frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial z^2} - \sin(mz) &(17)\\ &= \sum_{n=1}^{N}\Big(\frac{\partial A_n}{\partial t} + n^2\nu A_n\Big)\sin(nz) - \sin(mz) &(18)\end{aligned}$$

To minimize ϵ wer require that its projection onto the functional subspace of N orthogonal functions ("test" functions $\phi_p(z)$) is zero. Basically, the inner product of the error and each test function is zero:

$$\epsilon \cdot \phi_p = \int_0^\pi \epsilon\phi_p \mathrm{dz} = 0 \qquad m = [1, ..., N] \tag{19}$$

For the Galerkin method, the test functions are chose to be the same as the trial functions, so

$$\phi_p = \sin(pz) \tag{20}$$

Note the orthogonality property:

$$\int_0^\pi \sin(nz)\sin(pz)\mathrm{dz} = \begin{matrix} 0 & \text{if} & p \neq n \\ \pi/2 & \text{if} & p = n \end{matrix} \tag{21}$$

Now evaluate:

$$\begin{aligned}\int_0^\pi \epsilon\phi_p \mathrm{dz} &= \int_0^\pi \Big(\sum_{n=1}^{N}\Big(\frac{\partial A_n}{\partial t} + n^2\nu A_n\Big)\sin(nz) - \sin(mz)\Big)\sin(pz)\mathrm{dz} &(22)\\ &= \int_0^\pi \Big(\sum_{n=1}^{N}\Big(\frac{\partial A_n}{\partial t} + n^2\nu A_n\Big)\sin(nz)\Big)\sin(pz)\mathrm{dz} - \int_0^\pi \sin(mz)\sin(pz)\mathrm{dz} &(23)\\ &= 0 &(24)\end{aligned}$$

An expression for the time derivative of the coefficients that satisfies the orthogonality condition is:

$$\frac{\partial A_n}{\partial t} = -n^2\nu A_n + \delta_{mn} \tag{25}$$

So now we can solve Equation 25 numerically to obtain A_n and subsequently u. To obtain the initial conditions we apply the orthogonality to the initial condition (Equation 4):

$$\begin{aligned}\int_0^\pi u(z,0)\sin(nz)\mathrm{dz} &= \int_0^\pi z(\pi - z)\sin(nz)\mathrm{dz} &(26)\\ A_n(0)\int_0^\pi \sin^2(nz)\mathrm{dz} &= \int_0^\pi z(\pi - z)\sin(nz)\mathrm{dz} &(27)\\ A_n(0)\frac{\pi}{2} &= \frac{2 - 2\cos(n\pi)}{n^3} &(28)\\ A_n(0) &= \frac{4(1 - \cos(n\pi))}{\pi n^3} &(29)\end{aligned}$$

Now we can advance the coefficients $A_n(t)$ forward in time.