Lagrange polynomials: the first derivative

$k+1$ grid points:

$$\frac{\partial \phi}{\partial z} = \sum_{j=0}^{k} \phi_j l_j^{(1)}(z)$$

where

$$l_j^{(1)}(z) = \sum_{i=0, i\neq j}^{k} \Big[\frac{1}{z_j - z_i} \prod_{m=0, m\neq(i,j)}^{k} \frac{z - z_m}{z_j - z_m} \Big]$$

Choose 4 grid points ($k=3$):

$$z_0, z_1, z_2, z_3$$

Then:

$$\prod_{m=0, m\neq(i,j)}^{k} \frac{z - z_m}{z_j - z_m} = \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_2}{z_j - z_2} \cdot \frac{z - z_3}{z_j - z_3}$$

$$\begin{aligned} l_j^{(1)}(z) = & \frac{1}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_2}{z_j - z_2} \cdot \frac{z - z_3}{z_j - z_3} \\ & + \frac{1}{z_j - z_1} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_2}{z_j - z_2} \cdot \frac{z - z_3}{z_j - z_3} \\ & + \frac{1}{z_j - z_2} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_3}{z_j - z_3} \\ & + \frac{1}{z_j - z_3} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_2}{z_j - z_2} \end{aligned}$$

Finally, the derivative of ϕ is given by evaluating:

$$\begin{aligned} \sum_{j=0}^{k} \phi_j l_j^{(1)}(z) = & \phi_0 \Big(\frac{1}{z_0 - z_1} \cdot \frac{z - z_2}{z_0 - z_2} \cdot \frac{z - z_3}{z_0 - z_3} + \frac{1}{z_0 - z_2} \cdot \frac{z - z_1}{z_0 - z_1} \cdot \frac{z - z_3}{z_0 - z_3} + \frac{1}{z_0 - z_3} \cdot \frac{z - z_1}{z_0 - z_1} \cdot \frac{z - z_2}{z_0 - z_2} \Big) \\ & + \phi_1 \Big(\frac{1}{z_1 - z_0} \cdot \frac{z - z_2}{z_1 - z_2} \cdot \frac{z - z_3}{z_1 - z_3} + \frac{1}{z_1 - z_2} \cdot \frac{z - z_0}{z_1 - z_0} \cdot \frac{z - z_3}{z_1 - z_3} + \frac{1}{z_1 - z_3} \cdot \frac{z - z_0}{z_1 - z_0} \cdot \frac{z - z_2}{z_1 - z_2} \Big) \\ & + \phi_2 \Big(\frac{1}{z_2 - z_0} \cdot \frac{z - z_1}{z_2 - z_1} \cdot \frac{z - z_3}{z_2 - z_3} + \frac{1}{z_2 - z_1} \cdot \frac{z - z_0}{z_2 - z_0} \cdot \frac{z - z_3}{z_2 - z_3} + \frac{1}{z_2 - z_3} \cdot \frac{z - z_0}{z_2 - z_0} \cdot \frac{z - z_1}{z_2 - z_1} \Big) \\ & + \phi_3 \Big(\frac{1}{z_3 - z_0} \cdot \frac{z - z_1}{z_3 - z_1} \cdot \frac{z - z_2}{z_3 - z_2} + \frac{1}{z_3 - z_1} \cdot \frac{z - z_0}{z_3 - z_0} \cdot \frac{z - z_2}{z_3 - z_2} \cdot + \frac{1}{z_3 - z_2} \cdot \frac{z - z_0}{z_3 - z_0} \cdot \frac{z - z_1}{z_3 - z_1} \Big) \end{aligned}$$

Lagrange polynomials: the second derivative

$k+1$ grid points:

$$\frac{\partial^2 \phi}{\partial z^2} = \sum_{j=0}^{k} \phi_j l_j^{(2)}(z)$$

$$l_j^{(2)}(z) = \sum_{i=0, i\neq j}^{k} \frac{1}{z_j - z_i} \left[\sum_{m=0, m\neq (i,j)}^{k} \left[\frac{1}{z_j - z_m} \prod_{l=0, l\neq (i,j,m)}^{k} \frac{z - z_l}{z_j - z_l} \right] \right]$$

Choosing 4 data points $k = 3$:

$$\prod_{l=0, l\neq (i,j,m)}^{3} \frac{z - z_l}{z_j - z_l} = \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_2}{z_j - z_2} \cdot \frac{z - z_3}{z_j - z_3}$$

$$\begin{aligned} \sum_{m=0, m\neq (i,j)}^{3} \left[\frac{1}{z_j - z_m} \prod_{l=0, l\neq (i,j,m)}^{3} \frac{z - z_l}{z_j - z_l} \right] &= \frac{1}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_2}{z_j - z_2} \cdot \frac{z - z_3}{z_j - z_3} \\ &+ \frac{1}{z_j - z_1} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_2}{z_j - z_2} \cdot \frac{z - z_3}{z_j - z_3} \\ &+ \frac{1}{z_j - z_2} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_3}{z_j - z_3} \\ &+ \frac{1}{z_j - z_3} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_2}{z_j - z_2} \end{aligned}$$

$$\begin{aligned} l_j^{(2)}(z) = &\frac{1}{z_j - z_0} \left[\frac{1}{z_j - z_1} \cdot \frac{z - z_2}{z_j - z_2} \cdot \frac{z - z_3}{z_j - z_3} + \frac{1}{z_j - z_2} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_3}{z_j - z_3} + \frac{1}{z_j - z_3} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_2}{z_j - z_2} \right] \\ &+ \frac{1}{z_j - z_1} \left[\frac{1}{z_j - z_0} \cdot \frac{z - z_2}{z_j - z_2} \cdot \frac{z - z_3}{z_j - z_3} + \frac{1}{z_j - z_2} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_3}{z_j - z_3} + \frac{1}{z_j - z_3} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_2}{z_j - z_2} \right] \\ &+ \frac{1}{z_j - z_2} \left[\frac{1}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_3}{z_j - z_3} + \frac{1}{z_j - z_1} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_3}{z_j - z_3} + \frac{1}{z_j - z_3} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \right] \\ &+ \frac{1}{z_j - z_3} \left[\frac{1}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \cdot \frac{z - z_2}{z_j - z_2} + \frac{1}{z_j - z_1} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_2}{z_j - z_2} + \frac{1}{z_j - z_2} \cdot \frac{z - z_0}{z_j - z_0} \cdot \frac{z - z_1}{z_j - z_1} \right] \end{aligned}$$

At each j grid point the Lagrange polynomial weights for the second derivative are:

$$\begin{aligned} l_0^{(2)}(z) = &\frac{1}{z_0 - z_1} \left[\frac{1}{z_0 - z_2} \cdot \frac{z - z_3}{z_0 - z_3} + \frac{1}{z_0 - z_3} \cdot \frac{z - z_2}{z_0 - z_2} \right] \\ &+ \frac{1}{z_0 - z_2} \left[\frac{1}{z_0 - z_1} \cdot \frac{z - z_3}{z_0 - z_3} + \frac{1}{z_0 - z_3} \cdot \frac{z - z_1}{z_0 - z_1} \right] \\ &+ \frac{1}{z_0 - z_3} \left[\frac{1}{z_0 - z_1} \cdot \frac{z - z_2}{z_0 - z_2} + \frac{1}{z_0 - z_2} \cdot \frac{z - z_1}{z_0 - z_1} \right] \end{aligned}$$

$$\begin{aligned} l_1^{(2)}(z) = &\frac{1}{z_1 - z_0} \left[\frac{1}{z_1 - z_2} \cdot \frac{z - z_3}{z_1 - z_3} + \frac{1}{z_1 - z_3} \cdot \frac{z - z_2}{z_1 - z_2} \right] \\ &+ \frac{1}{z_1 - z_2} \left[\frac{1}{z_1 - z_0} \cdot \frac{z - z_3}{z_1 - z_3} + \frac{1}{z_1 - z_3} \cdot \frac{z - z_0}{z_1 - z_0} \right] \\ &+ \frac{1}{z_1 - z_3} \left[\frac{1}{z_1 - z_0} \cdot \frac{z - z_2}{z_1 - z_2} + \frac{1}{z_1 - z_2} \cdot \frac{z - z_0}{z_1 - z_0} \right] \end{aligned}$$

$$
\begin{aligned}
l_2^{(2)}(z) = &\frac{1}{z_2 - z_0}\left[\frac{1}{z_2 - z_1} \cdot \frac{z - z_3}{z_2 - z_3} + \frac{1}{z_2 - z_3} \cdot \frac{z - z_1}{z_2 - z_1}\right] \\
&+ \frac{1}{z_2 - z_1}\left[\frac{1}{z_2 - z_0} \cdot \frac{z - z_3}{z_2 - z_3} + \frac{1}{z_2 - z_3} \cdot \frac{z - z_0}{z_2 - z_0}\right] \\
&+ \frac{1}{z_2 - z_3}\left[\frac{1}{z_2 - z_0} \cdot \frac{z - z_1}{z_2 - z_1} + \frac{1}{z_2 - z_1} \cdot \frac{z - z_0}{z_2 - z_0}\right]
\end{aligned}
$$

$$
\begin{aligned}
l_3^{(2)}(z) = &\frac{1}{z_3 - z_0}\left[\frac{1}{z_3 - z_1} \cdot \frac{z - z_2}{z_3 - z_2} + \frac{1}{z_3 - z_2} \cdot \frac{z - z_1}{z_3 - z_1}\right] \\
&+ \frac{1}{z_3 - z_1}\left[\frac{1}{z_3 - z_0} \cdot \frac{z - z_2}{z_3 - z_2} + \frac{1}{z_3 - z_2} \cdot \frac{z - z_0}{z_3 - z_0}\right] \\
&+ \frac{1}{z_3 - z_2}\left[\frac{1}{z_3 - z_0} \cdot \frac{z - z_1}{z_3 - z_1} + \frac{1}{z_3 - z_1} \cdot \frac{z - z_0}{z_3 - z_0}\right]
\end{aligned}
$$