The Continuous Wavelet Transform

Bryan Kaiser

March 9, 2015

1 Notes for Implementation

1. Find the Fourier transform $\hat{x}(\omega)$ of the time series $x(t)$,

$$x_n = x(t_n), \quad t_n = n \cdot \Delta t, \quad n = [0, 1, ..., N-1]$$

$$\hat{x}_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-i \frac{2\pi k}{N \Delta t} n \Delta t} = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-i \frac{2\pi k}{N} n} = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-i \omega_k n}$$

$$\hat{x}_k = \hat{x}(\omega_k) \quad \omega_k = \frac{2\pi k}{N \Delta t}, \quad k = [0, 1, ..., N-1]$$

Here $N-1$ is the number of harmonics of the fundamental frequency $\omega_0 = 2\pi/N$, where $k = 0$ represents a non-oscillatory, background signal.

2. Select a mother wavelet. The Fourier transform of the Morlet wavelet,

$$\Psi_0(t_n) = \pi^{-1/4} e^{i\omega_0 t_n - t_n^2/2},$$

is:

$$\hat{\Psi}_0(s\omega_k) = \pi^{-1/4} e^{i\omega_0 (s\omega_k) - (s\omega_k)^2/2}.$$

Where $\omega_0 = 6$ and the e-folding time is $\tau_s = \sqrt{2}s$. The cone of influence is constructed using the e-folding time to truncate the ends of the time series that the wavelet transform produces for each scale s.

3. Select the set of scales to use the wavelet transform. For an orthogonal wavelet, one is limited to a discrete set of scales. For nonorthogonal wavelet analysis, one can use an arbitrary set of scales to build up a more complete picture. Discrete wavelet transforms require orthogonal basis functions, while continuous wavelet transforms do not.

$$s_j = s_0 2^{j \cdot \Delta j}, \quad s_0 = 2\Delta t, \quad j = [0, 1, ..., J]$$

$$J = \frac{1}{\Delta j} \log_2 \left(\frac{N \Delta t}{s_0} \right), \quad \Delta j = 0.125 \quad (Morlet)$$

Δj is a factor for scale averaging. s_0 is the smallest resolvable scale. J determines the largest resolvable scale.

4. The normalized Fourier transform of the mother wavelet is:

$$\hat{\Psi}(s\omega_k) = \left(\frac{2\pi s}{\Delta t} \right)^{1/2} \hat{\Psi}_0(s\omega_k)$$

The continous wavelet transform is:

$$W_n(s) = \sum_{k=0}^{N-1} \hat{x}_k \hat{\Psi}^*(s\omega_k) e^{i\omega_k n \Delta t} = \sum_{k=0}^{N-1} \hat{x}_k \left[\left(\frac{2\pi s}{\Delta t} \right)^{1/2} \hat{\Psi}_0(s\omega_k) \right]^* e^{i\omega_k n \Delta t}$$

$$= \sum_{k=0}^{N-1} \hat{x}_k \left[\left(\frac{2\pi s}{\Delta t} \right)^{1/2} \pi^{-1/4} e^{i\omega_0 (s\omega_k) - (s\omega_k)^2/2} \right]^* e^{i\omega_k n \Delta t}$$

Where $*$ represents the complex conjugate, n is the localized time index (which is translated across the time series), k represents the harmonics of the fundamental frequency, and s is the wavelet scale.

$$\omega_k = \begin{cases} \frac{2\pi k}{N\Delta t} & \text{if } k \leq N/2; \\ -\frac{2\pi k}{N\Delta t} & \text{if } k > N/2. \end{cases}$$

5. Energy is conserved under the wavelet transform. The variance may be calculated:

$$\sigma^2 = \frac{\Delta j \Delta t}{C_\delta N} \sum_{n=0}^{N-1} \sum_{j=0}^{J} \frac{|W_n(s_j)|^2}{s_j} \qquad C_\delta = 0.776 \ \ (Morlet)$$

2 Test Function (MATLAB/PYTHON scripts)

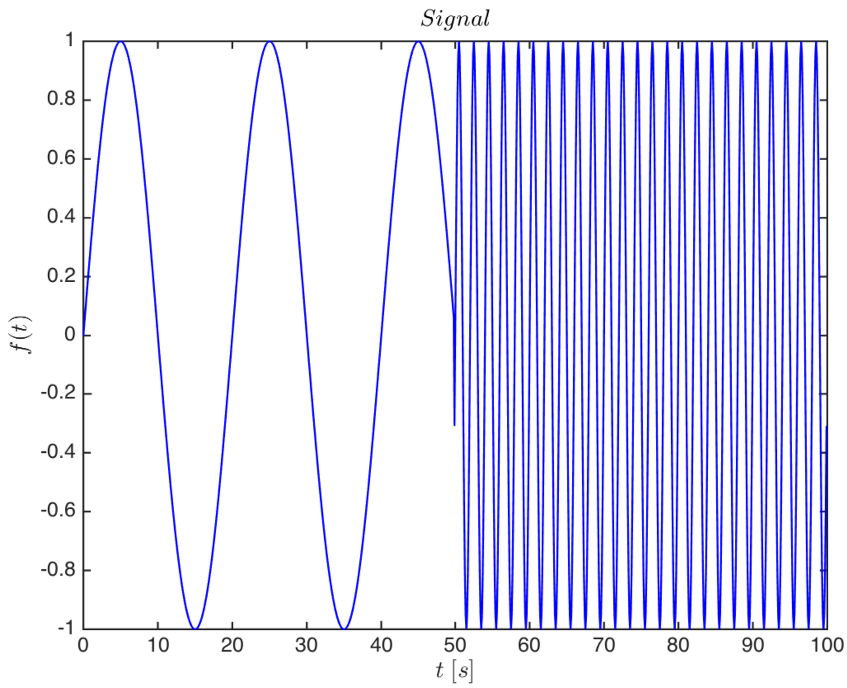

Figure 1: Time-varying signal.

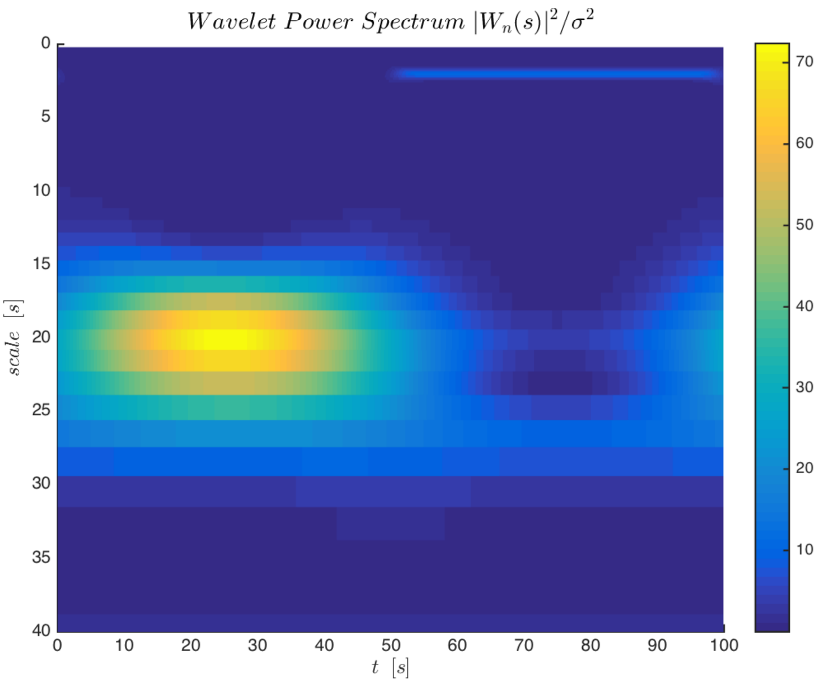

Figure 2: Normalized Wavelet Power Spectrum Confidence Levels.