$\frac{30^+}{30}$

12.805 Homework 1

Bryan Kaiser

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1 Overdetermined Least Squares

Three equations with two unknowns and three noise components are given

$$x_1 - x_2 + \eta_1 = 1,$$
$$x_1 + x_2 + \eta_2 = 2,$$
$$x_1 + 2x_2 + \eta_3 = 2,$$

which can be written in matrix form as

$$\mathbf{Ex} + \boldsymbol{\eta} = \mathbf{y}, \quad \Rightarrow \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}. \tag{1}$$

The system is overdetermined, therefore $\mathbf{E}$ is a tall matrix and it cannot be inverted to solve directly for $\mathbf{x}$. Therefore solutions for $\mathbf{x}$ can be obtained by rearranging Equation (1) and multiplying it by its transpose

$$\boldsymbol{\eta}^\dagger \boldsymbol{\eta} = (\mathbf{Ex} - \mathbf{y})^\dagger (\mathbf{Ex} - \mathbf{y}),$$

where $\dagger$ denotes the matrix transpose. Subsequently finding the minimum squared noise $\boldsymbol{\eta}^\dagger \boldsymbol{\eta}$ with respect to the vector $\mathbf{x}$

$$\frac{d}{d\mathbf{x}} \boldsymbol{\eta}^\dagger \boldsymbol{\eta} = 2\mathbf{E}^\dagger \mathbf{Ex} - 2\mathbf{E}^\dagger \mathbf{y} = 0, \quad \Rightarrow \quad \mathbf{x} = (\mathbf{E}^\dagger \mathbf{E})^{-1} \mathbf{E}^\dagger \mathbf{y}. \tag{2}$$

✓ Nice.

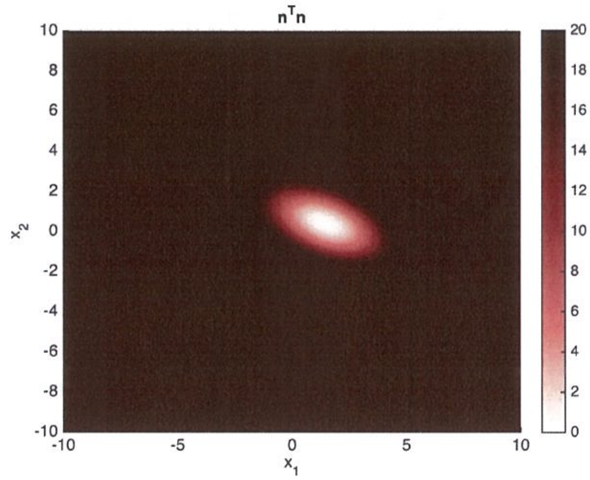
Substituting the given constraints into Equation (2) and solving for $\mathbf{x}$ yields

$$\mathbf{E}^\dagger \mathbf{E} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix},$$

$$(\mathbf{E}^\dagger \mathbf{E})^{-1} = \begin{bmatrix} 0.4286 & -0.1429 \\ -0.1429 & 0.2143 \end{bmatrix}, \quad \mathbf{E}^\dagger \mathbf{y} = \begin{bmatrix} 5 \\ 5 \end{bmatrix},$$

$$\Rightarrow \quad \mathbf{x} = (\mathbf{E}^\dagger \mathbf{E})^{-1} \mathbf{E}^\dagger \mathbf{y} = \begin{bmatrix} 1.4286 \\ 0.3571 \end{bmatrix}.$$

✓ perfect

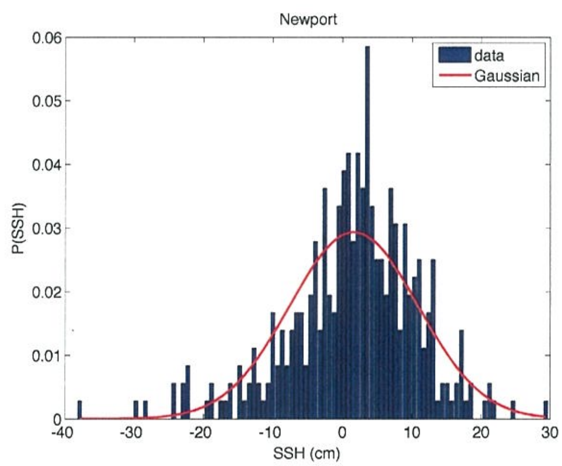


Interesting way to look at it. One global solution.

Figure 1: Squared noise $\boldsymbol{\eta}^\dagger\boldsymbol{\eta}$ solutions as a function of $\mathbf{x}$ for the given constraints. The minimum squared noise $\boldsymbol{\eta}^\dagger\boldsymbol{\eta} = 0.0714$ occurs at $x_1 = 1.4286, x_2 = 0.3571$.

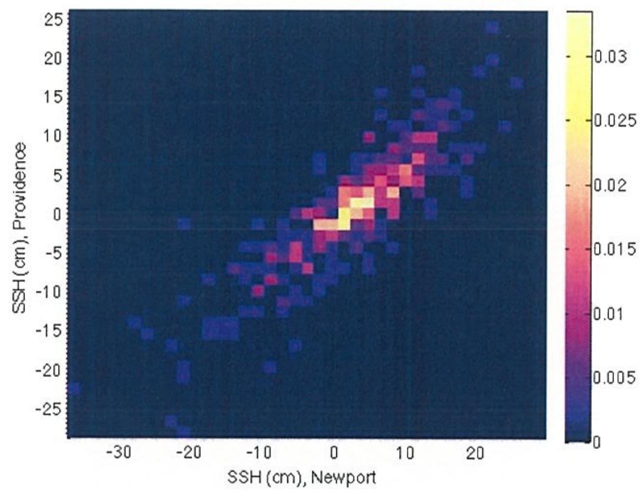
2 Ocean Data

Solutions to problems 2.1 - 2.4 are shown below in Figures 2-4.



← maybe look closer w/ less bins?

Figure 2: The probability density function of filtered sea level height near Newport, RI in 1995. The data is slightly skewed negatively (a skewness factor of -0.65) and slightly more peaked than a Gaussian distribution with an excess kurtosis of 1.46. The mean filtered sea level height is 1.60 cm and the standard deviation is 9.24 cm. ✓



← units $[cm^{-1}]$

very nice.

Figure 3: The joint probability density function of filtered sea level height near Newport, RI and Providence, RI in 1995. Note the positive trend indicating the two time series are linearly correlated. The bin values are dimensionless and sum to unity. $40 \times 40 = 1600$ bins were used. The correlation coefficient of the two time series is $r = 0.86$. ✓

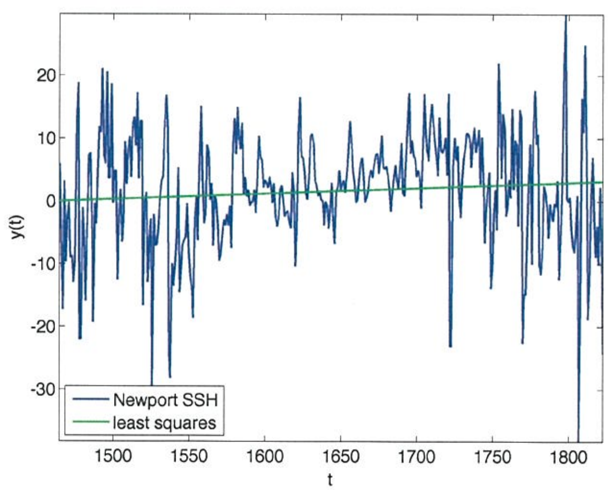


Figure 4: The filtered sea level height time series near Newport, RI in 1995 and the linear trend from overdetermined least squares for the time series. $a = -12.9835$ and $b = 0.0089$ for $\mathbf{y} \approx \mathbf{Ex} = a + b\mathbf{t}$. ✓ ✓

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% 12.805 homework 1, problem 2
% Bryan Kaiser
% 2/22/16

close all; clear all; clc;
addpath('./functions'); run('./functions/mcolormaps.m');

%==============================================================================
% problem 1: overdetermined least squares

% solution
E=[1 -1; 1 1; 1 2];
y=[1; 2; 2];
x = (inv(E'*E))*(E'*y)

% minimized squared noise norm
n=[1-x(1)+x(2); 2-x(1)-x(2); 2-x(1)-2*x(2)];
err=n'*n

% plot of minimized squared noise norm for a spread of x1,x2
N = 400;
x1 = linspace(-10,10,N);
x2 = linspace(-10,10,N);
nTn = nan(N,N);
for i = 1:N
    for j = 1:N
        n=[1-x1(i)+x2(j); 2-x1(i)-x2(j); 2-x1(i)-2*x2(j)];
        nTn(i,j)=n'*n;
        clear n
    end
end
[X1,X2]=meshgrid(x1,x2);
surf(X1,X2,nTn'); view(2); %shading flat
colorbar; colormap(reds/255); %colormap(flipud(colormap));
shading flat; grid off; axis tight
xlabel('x_1','FontSize',14);
ylabel('x_2','FontSize',14);
title('n^Tn')
caxis([0,20]);

% find x1,x2 corresponding to the minimized squared noise norm
loc_min_nTn = min(min(nTn));
loc_min_nTn = find(nTn'==loc_min_nTn);
x1_min=X1(loc_min_nTn)
x2_min=X2(loc_min_nTn)


%==============================================================================
% problem 2: sea level data

% load data
data = importdata('hwa4ar.mat');
time = data(:,1); % days
ssh_P = data(:,2); % cm
ssh_N = data(:,3); % cm

% 1) Compute a histogram of Newport Sea level. Does it look Gaussian?
% 2) Compute the mean and standard deviation of Newport sea level. Plot the
% Gaussian given by these statistics and compare to the PDF derived from
% the histogram.

[ muN sigN minN maxN varN skN flN ] = pdf_1D(ssh_N,2,100,50)
xlabel('SSH (cm)'); ylabel('P(SSH)'); title('Newport');
legend('data','Gaussian')

[ muP sigP minP maxP varP skP flP ] = pdf_1D(ssh_P,2,100,50)
xlabel('SSH (cm)'); ylabel('P(SSH)'); title('Providence');
```

later "mldivide" will work best.

```
legend('data','Gaussian')


% 3) Compute the correlation of the two sealevel (Portland, Newport) time
% series. Write your own MATLAB code (and you may compare to a built-in
% MATLAB function). (Extra credit: plot the joint PDF of these two
% timeseries).

N = length(ssh_N);
R = (sum((ssh_N-muN).*(ssh_P-muP)))./(N-1); % cm^2, covariance at zero lag
r = R./(sigN*sigP) % unitless

pdf = pdf_2D(ssh_N,ssh_P,1,40,1000);
xlabel('SSH (cm), Newport'); ylabel('SSH (cm), Providence'); zlabel('P')
view(0,90); colorbar; colormap(dawn/255);  colormap(flipud(colormap)); shading flat
axis tight
% figure; scatter(ssh_N,ssh_P);
% xlabel('SSH (cm), Newport'); ylabel('SSH (cm), Providence');


% 4) Using overdetermined least squares, find the linear trend that best
% fits the Newport sealevel data by minimizing the squared 2-norm of the
% misfit. Plot the data and the estimated trend.

y = ssh_N; % observations
E = [ones(length(time),1) time];
x = ((E'*E)\E')*y; % least squares solutions for n'n minimum
a = x(1)
b = x(2)
yLS = a+time.*b; % least squares solution

figure;
p1=plot(time,y,'Color',[0 0 0.7],'LineWidth',1.2); hold on
p2=plot(time,yLS,'Color',[0 0.7 0],'LineWidth',1.2);
xlabel('t'); ylabel('y(t)'); legend([p1 p2],'Newport SSH','least squares',3)
axis tight
```