12.805 Homework 1

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1 Overdetermined Least Squares

Three equations with two unknowns and three noise components are given

$$\begin{aligned} x_1 - x_2 + \eta_1 &= 1, \\ x_1 + x_2 + \eta_2 &= 2, \\ x_1 + 2x_2 + \eta_3 &= 2, \end{aligned}$$

which can be written in matrix form as

$$\mathbf{E}\mathbf{x} + \boldsymbol{\eta} = \mathbf{y}, \quad \Rightarrow \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}. \tag{1}$$

The system is overdetermined, therefore $\mathbf{E}$ is a tall matrix and it cannot be inverted to solve directly for $\mathbf{x}$. Therefore solutions for $\mathbf{x}$ can be obtained by rearranging Equation (1) and multiplying it by its transpose

$$\boldsymbol{\eta}^\dagger \boldsymbol{\eta} = (\mathbf{E}\mathbf{x} - \mathbf{y})^\dagger (\mathbf{E}\mathbf{x} - \mathbf{y}),$$

where $\dagger$ denotes the matrix transpose. Subsequently finding the minimum squared noise $\boldsymbol{\eta}^\dagger \boldsymbol{\eta}$ with respect to the vector $\mathbf{x}$

$$\frac{d}{d\mathbf{x}} \boldsymbol{\eta}^\dagger \boldsymbol{\eta} = 2\mathbf{E}^\dagger \mathbf{E}\mathbf{x} - 2\mathbf{E}^\dagger \mathbf{y} = 0, \quad \Rightarrow \quad \mathbf{x} = (\mathbf{E}^\dagger \mathbf{E})^{-1} \mathbf{E}^\dagger \mathbf{y}. \tag{2}$$

Substituting the given constraints into Equation (2) and solving for $\mathbf{x}$ yields

$$\mathbf{E}^\dagger \mathbf{E} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix},$$

$$(\mathbf{E}^\dagger \mathbf{E})^{-1} = \begin{bmatrix} 0.4286 & -0.1429 \\ -0.1429 & 0.2143 \end{bmatrix}, \quad \mathbf{E}^\dagger \mathbf{y} = \begin{bmatrix} 5 \\ 5 \end{bmatrix},$$

$$\Rightarrow \quad \mathbf{x} = (\mathbf{E}^\dagger \mathbf{E})^{-1} \mathbf{E}^\dagger \mathbf{y} = \begin{bmatrix} 1.4286 \\ 0.3571 \end{bmatrix}.$$

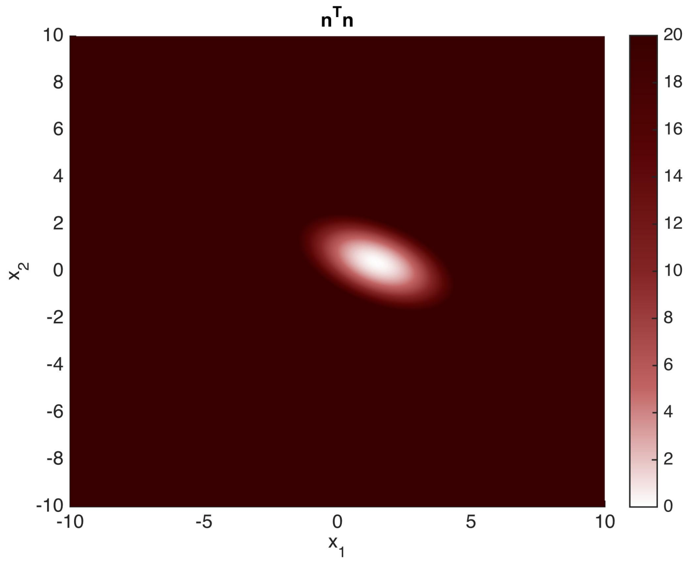

Figure 1: Squared noise $\boldsymbol{\eta}^{\dagger}\boldsymbol{\eta}$ solutions as a function of $\mathbf{x}$ for the given constraints. The minimum squared noise $\boldsymbol{\eta}^{\dagger}\boldsymbol{\eta} = 0.0714$ occurs at $x_1 = 1.4286, x_2 = 0.3571$.

2 Ocean Data

Solutions to problems 2.1 - 2.4 are shown below in Figures 2-4.

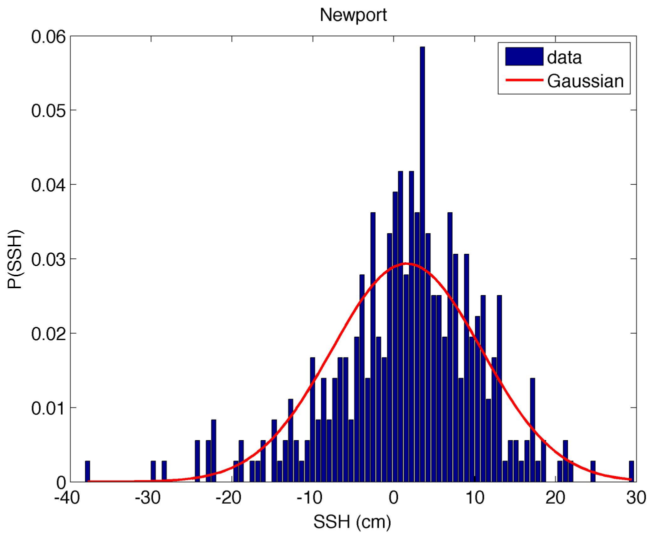

Figure 2: The probability density function of filtered sea level height near Newport, RI in 1995. The data is slightly skewed negatively (a skewness factor of -0.65) and slightly more peaked than a Gaussian distribution with an excess kurtosis of 1.46. The mean filtered sea level height is 1.60 cm and the standard deviation is 9.24 cm.

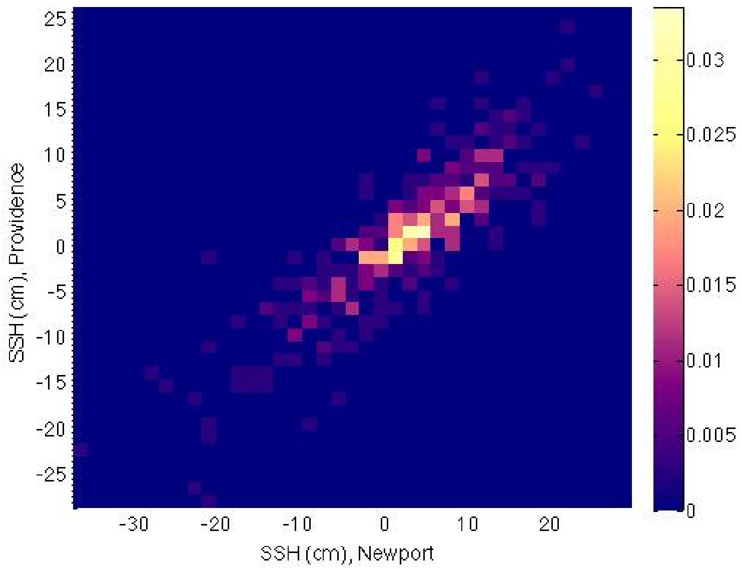

Figure 3: The joint probability density function of filtered sea level height near Newport, RI and Providence, RI in 1995. Note the positive trend indicating the two time series are linearly correlated. The bin values are dimensionless and sum to unity. $40 \times 40 = 1600$ bins were used. The correlation coefficient of the two time series is $r = 0.86$.

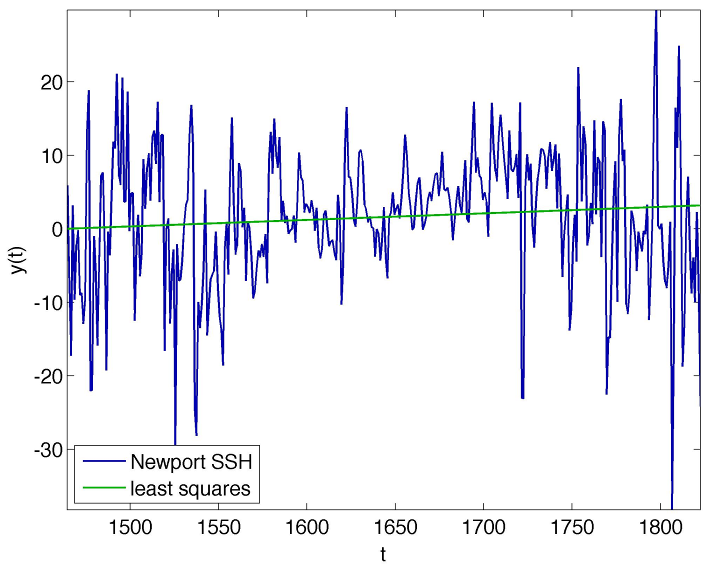

Figure 4: The filtered sea level height time series near Newport, RI in 1995 and the linear trend from overdetermined least squares for the time series. $a = -12.9835$ and $b = 0.0089$ for $\mathbf{y} \approx \mathbf{E}\mathbf{x} = a + b\mathbf{t}$.