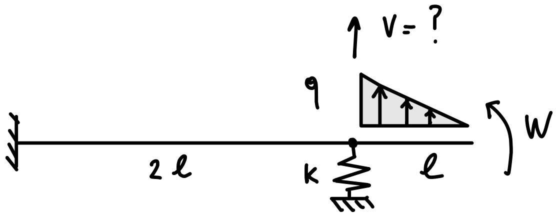
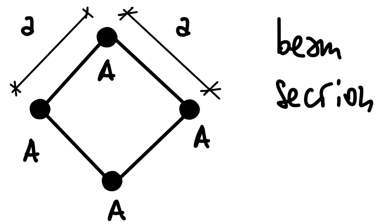
Exercise





Consider the structure in the figure. It is loaded with a distributed load, which is introduced continuously.
The section is square with dimension a; all the panels have thickness t, while the lumped area of the stringers, including the contribution of the panels, is A. The Young and shear moduli of the material are E and G, respectively.
Determine the vertical displacement v in correspondence of the spring by taking shear deformability effects into account.
For this purpose, refer to a semi-monocoque model of the section.
(Unit for result: mm)

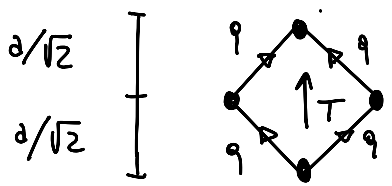
Data (solution for B = 0)
l = 200.; Units: mm
a = 40.; Units: mm
t = 0.2; Units: mm
A = 400.; Units: mm^2
G = 28000.; Units: MPa
E = 72000.; Units: MPa
k = 2000. * (1 + B / 10); Units: N / mm
q = 100.; Units: N / mm
W = 4.0 * 1e6; Units: N mm

Solution

The beam section properties are:

- $EJ = E\, 2A \left(\frac{a}{\sqrt{2}}\right)^2 = EA\, a^2$

- GA^*: due to the symmetry of the section, the shear flows due to the shear force T (applied at the shear center) are:



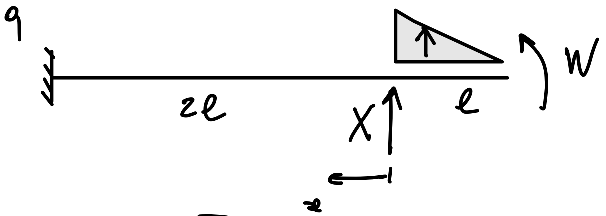
$$q \frac{a}{\sqrt{2}} \cdot 4 = T \quad , \text{ so:}$$

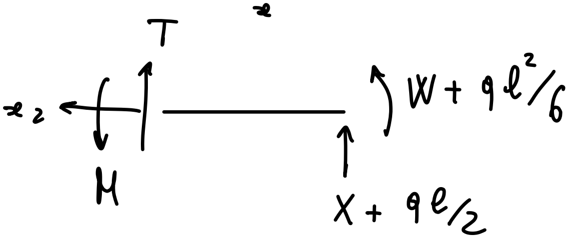
$$q = \frac{T}{4a}\sqrt{2}$$

From which the shear stiffness is:

$$GA^* = G \frac{T^2}{\sum_i q_i^2 l_i / t_i} = G \frac{T^2}{4 q^2 a / t} = G\, 2at$$

<u>Real system</u>

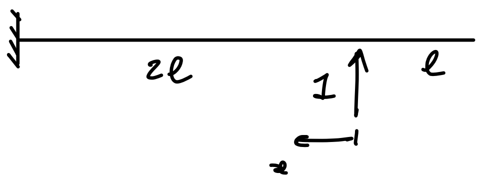




$$T = -\left(X + ql/2\right)$$

$$M = -W - ql^2/6 - \left(X + ql/2\right)x$$

<u>Dummy system</u>



$$\delta T = -1$$
$$\delta M = -x$$

By application of the PCVW:

$$\int_0^{2\ell}\left(\delta M\,\frac{M}{EJ}+\delta T\,\frac{T}{GA^*}\right)dx+\frac{X}{k}=0$$

And by integrating:

$$\frac{\ell^2}{3EJ}\left(5q\ell^2+8X\ell+6W\right)+\frac{\ell}{GA^*}\left(q\ell+2X\right)+X/k=0$$

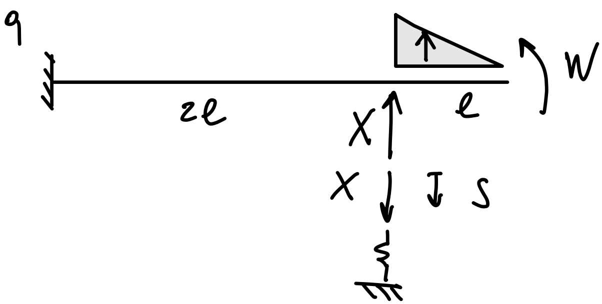
And so:

$$\left(\frac{\ell^3}{3EJ}8+\frac{\ell}{GA^*}2+1/k\right)X=-\frac{\ell^2}{3EJ}\left(5q\ell^2+6W\right)-\frac{\ell}{GA^*}q\ell$$

and so:

$$X=-12615\text{ N}$$

And so, the displacement is

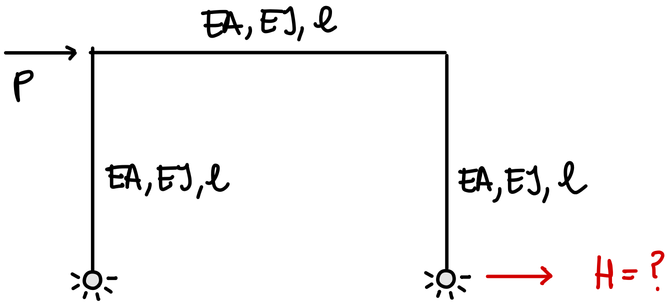


$$s=X/k=-5.84\text{ mm}$$

(s positive in the downward direction)

So: $v=5.84$ mm

Exercise



The structure in the figure is composed of three beams with same length L, bending stiffness EJ and axial stiffness EA. Shear deformability is negligible. Determine the reaction force H.
(Unit for result: N)

Data (solution for D = 0)
l = 1000. * (1 + D / 10); Units: mm
EJ = 1.0 * 1e12; Units: N mm^2
EA = 1.0 * 1e6; Units: N
P = 500.; Units: N

• <u>Solution</u>

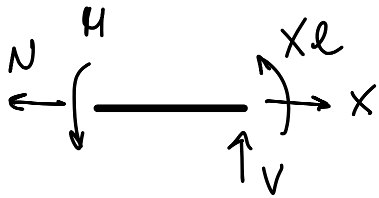
<u>Real system</u>

• Beam 1

$$N = -V$$

$$M = -X x_1$$

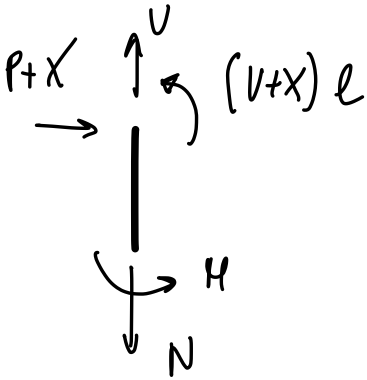
• Beam 2



$$N = X$$

$$M = -Xl - V x_2$$

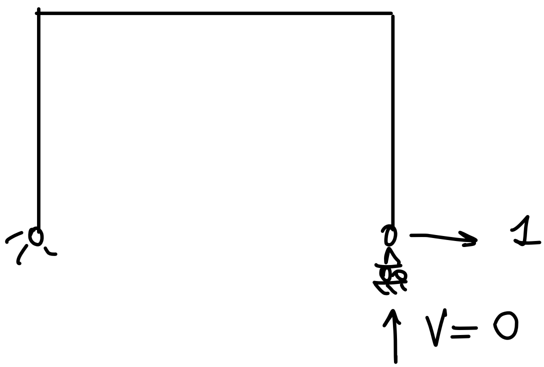
- Beam 3



$$N = V$$

$$M = (P+X)x_3 - (V+X)\ell$$

- Dummy system



Beam 1 : $\delta N = 0$

$\delta M = -x_1$

Beam 2 : $\delta N = 1$

$\delta M = -\ell$

Beam 3 : $\delta N = 0$

$\delta M = -\ell + x_3$

By application of the PCVW:

$$\int_0^{\ell} \delta N \, N/_{EA} \, dx_2 + \sum_{i=1}^{3} \int_0^{\ell} \delta M \, M/_{EJ} \, dx_i = 0$$

which leads to:

$$\frac{X\ell}{EA} + \frac{5}{3EJ}\ell^3 X + \frac{5}{6EJ}\ell^3 P = 0$$

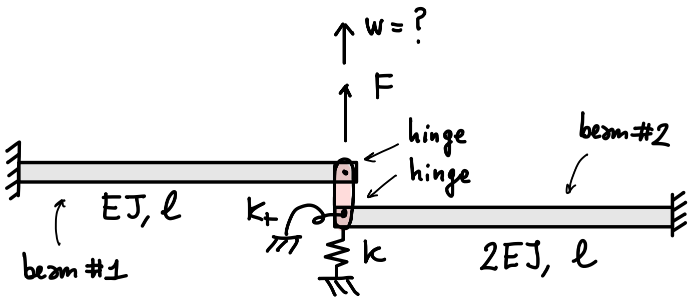
From which

$$X\left(\frac{\ell}{EA} + \frac{5\ell^3}{3EJ}\right) = -\frac{5}{6EJ}\ell^3 P$$

and then

$$X = -5\,\frac{EA\ell^2}{6EJ + 10EA\ell^2}\,P = -156.25 \text{ N}$$

Exercise



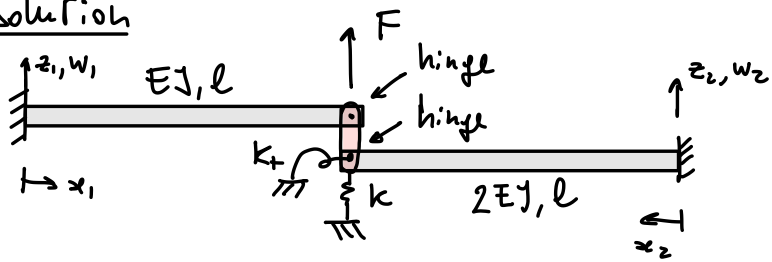
Two beams are hinged through an inextensible joint (see figure). Determine the vertical displacement w in correspondence of the applied force F using the Euler-Bernoulli beam model in conjunction with the Ritz method. For this purpose, adopt a polynomial expansion of the displacement field where:

+ the displacement is expanded up to the third order for the first beam
+ the displacement is expanded up to the second order for the second beam

(Unit for result: mm)

Data (solution for E = 0)
l = 500.; Units: mm
EJ = 1.0 * 1e10; Units: N mm^2
k = 80. * (1 + E / 10); Units: N / mm
kt = 2.0 * 1e7; Units: N mm
F = 2500.; Units: N

Solution



The displacement field is expanded as:

$$w_1 = c_1 \left(\frac{x_1}{\ell}\right)^2 + c_2 \left(\frac{x_1}{\ell}\right)^3 ; \quad w_2 = c_3 \left(\frac{x_2}{\ell}\right)^2$$

The two expansions do not include the constant and the linear terms to guarantee the fulfillment of the essential conditions:

$$\begin{cases} w_1(0) = 0 \\ w_2(0) = 0 \end{cases}$$

The additional essential condition introduced by the joint is:

$$w_1(\ell) = w_2(\ell) = 0$$

So:

$$c_1 + c_2 = c_3$$

and then:

$$w_1 = c_1 \left(\frac{x_1}{\ell}\right)^2 + c_2 \left(\frac{x_1}{\ell}\right)^3 ;$$

$$w_2 = (c_1 + c_2) \left(\frac{x_2}{\ell}\right)^2$$

The PVW reads:

$$\int_0^\ell \delta w_1'' \, EJ \, w_1'' \, dx_1 + \int_0^\ell \delta w_2'' \, 2EJ \, w_2'' \, dx_2 +$$

$$+ \delta w_2(\ell) \, k w_2(\ell) + \delta w_2'(\ell) \, k_t w_2'(\ell) = \delta w_2(\ell) F$$

where the derivatives are:

$$w_1' = c_1 \, 2x_1/\ell^2 + c_2 \, 3x_1^2/\ell^3 \; ; \quad w_2' = (c_1 + c_2) \, 2x_2/\ell^2$$

$$w_1'' = c_1 \, 2/\ell^2 + c_2 \, 6x_1/\ell^3 \; ; \quad w_2'' = (c_1 + c_2) \, 2/\ell^2$$

And so:

$$\delta c_1 \left(k + 12EJ/\ell^3 + 4k_t/\ell^2 \right) c_1 +$$
$$+ \delta c_1 \left(k + 14EJ/\ell^3 + 4k_t/\ell^2 \right) c_2 +$$
$$+ \delta c_2 \left(k + 14EJ/\ell^3 + 4k_t/\ell^2 \right) c_1 +$$
$$+ \delta c_2 \left(k + 20EJ/\ell^3 + 4k_t/\ell^2 \right) c_2 = \delta c_1 F + \delta c_2 F$$

Defining $\hat{k} = k\ell^3/EJ$; $\hat{k}_t = k_t \ell/EJ$, the linear system to be solved is

$$\begin{bmatrix} \hat{k} + 12 + 4\hat{k}_t & \hat{k} + 14 + 4\hat{k}_t \\ \hat{k} + 14 + 4\hat{k}_t & \hat{k} + 20 + 4\hat{k}_t \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \frac{F\ell^3}{EJ}$$

And so:

$$\begin{bmatrix} 17.00 & 19.00 \\ 19.00 & 25.00 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 31.25 \\ 31.25 \end{Bmatrix}$$

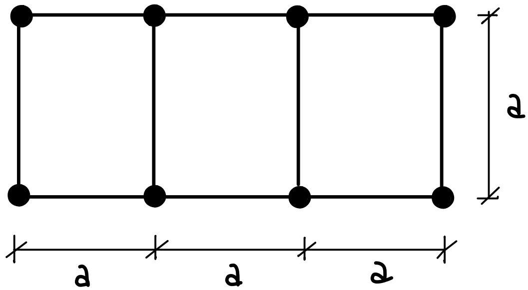
From which:

$$c_1 = 2.93 \text{ mm}$$
$$c_2 = -0.98 \text{ mm}$$

And the displacement at $x_2 = \ell$ is:

$$w_2 = (c_1 + c_2)\left(\ell/\ell\right)^2 = c_1 + c_2 = 1.95 \text{ mm}$$

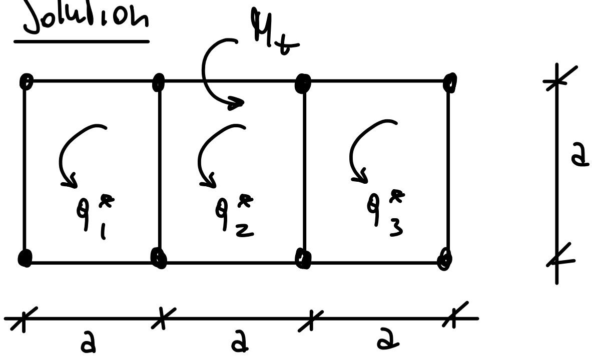
Exercise



Determine the torsional stiffness GJ of the three-cell section in the figure. All the panels have thickness t, while the lumped area of the stringers, including the contribution of the panels, is A.
The shear modulus of the material is G.
To solve the exercise, use a semi-monocoque approximation of the section.
Report the result as GJ / GJref, where GJref is reference value available in the data.
(Unit for result: adim)

Data (solution for B = 0)
a = 100. * (1 + B / 10); Units: mm
t = 1.5; Units: mm
A = 400.; Units: mm^2
G = 27000.; Units: MPa
GJref = 1.0 * 1e10; Units: N mm^2

Solution



- Moment equivalence

$$2\Omega\left(q_1^* + q_2^* + q_3^*\right) = M_t \quad \text{with } \Omega = a^2$$

- Compatibility

$$2G\Omega t\theta_1' = 4a\,q_1^* - a\,q_2^*$$

$$2G\Omega t\theta_2' = 4a\,q_2^* - a\,q_1^* - a\,q_3^*$$

$$2G\Omega t\theta_3' = 4a\,q_3^* - a\,q_2^*$$

From which:

$$\theta_1' = \theta_2' \quad \rightarrow \quad 5q_1^* - 5q_2^* + q_3^* = 0$$

$$\theta_1' = \theta_3' \quad \rightarrow \quad q_1^* = q_3^*$$

And so:

$$\begin{cases} 6q_1^* - 5q_2^* = 0 \\ 2q_1^* + q_2^* = M_t/2\Omega \end{cases}$$

From which:

$$q_1^* = \frac{5}{16} M_t/2\Omega$$

$$q_2^* = \frac{3}{8} M_t/2\Omega$$

And recalling that: $M_t = GJ\theta' \rightarrow GJ = M_t/\theta'$

with: $\theta' = \theta_1'$, so:

$$2G\Omega t\theta_1' = 4a q_1^* - a q_2^* \text{ , so:}$$

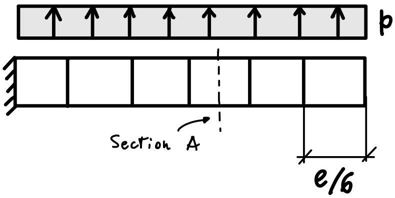
$$\theta_1' = \frac{1}{2G\Omega t} \frac{M_t a}{2\Omega} \left(\frac{5}{4} - \frac{3}{8} \right)$$

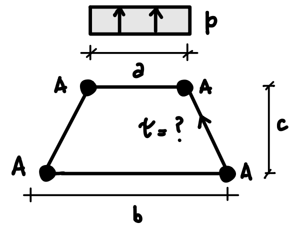
$$= \frac{7}{32} \frac{M_t}{G a^3 t} \text{ , so:}$$

$$GJ = M_t/\theta_1' = G \frac{32}{7} a^3 t = 1.85 \cdot 10^{11} \text{ N mm}^2$$

$$GJ/GJ_{ref} = 18.51$$

Exercise

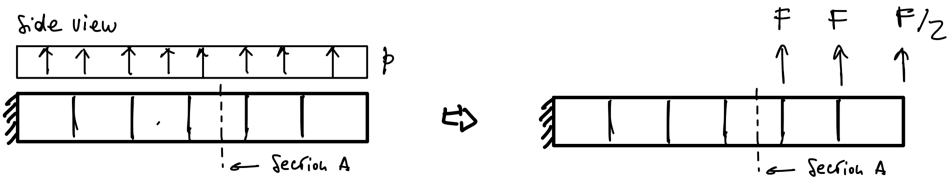




The thin-walled beam in the figure is loaded with a uniform pressure p. The section is modeled with a semi-monocoque approximation, where the lumped stringers (including the contribution of the panels) have area A, and the thickness of the panels is t. Assuming that the load is introduced in correspondence of the ribs, determine the shear stress in the panel indicated in the figure at the section A.
(Unit for result: MPa)

Data (solution for D = 0)
l = 2000. * (1 + D / 10); Units: mm
a = 70.; Units: mm
b = 100.; Units: mm
c = 30.; Units: mm
t = 1.2; Units: mm
A = 400.; Units: mm^2
p = 1.0 * 1e-2; Units: N / mm

Solution

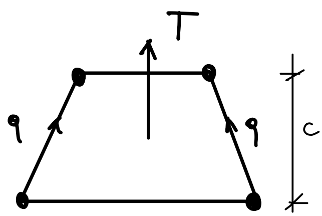


$$F = p\,a\,\ell/6 = 233.33\ \text{N}$$

The shear force at the section A is then:

$$T = \frac{5}{2} F = 583.33\ \text{N}$$

Due to the symmetry of the section, the torsional moment with respect to the shear center is null.



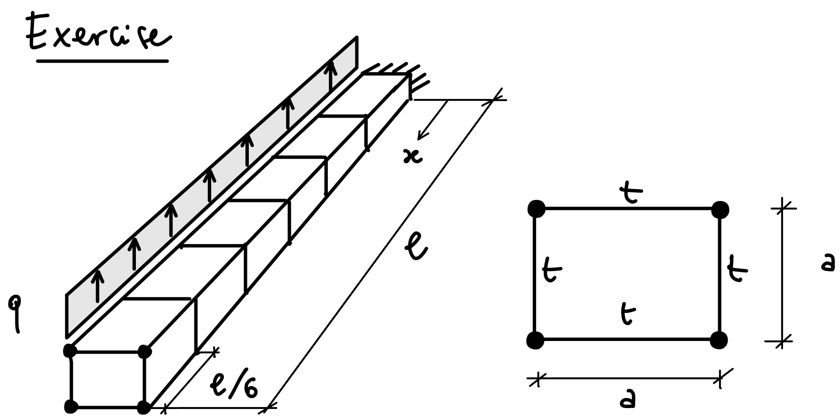
The shear flows are readily available by exploiting the symmetry of the section:

$$q = \frac{T}{2c} = 9.72\ \text{N/mm}$$

And the corresponding shear stresses are:

$$\tau = q/t = 8.10\ \text{MPa}$$

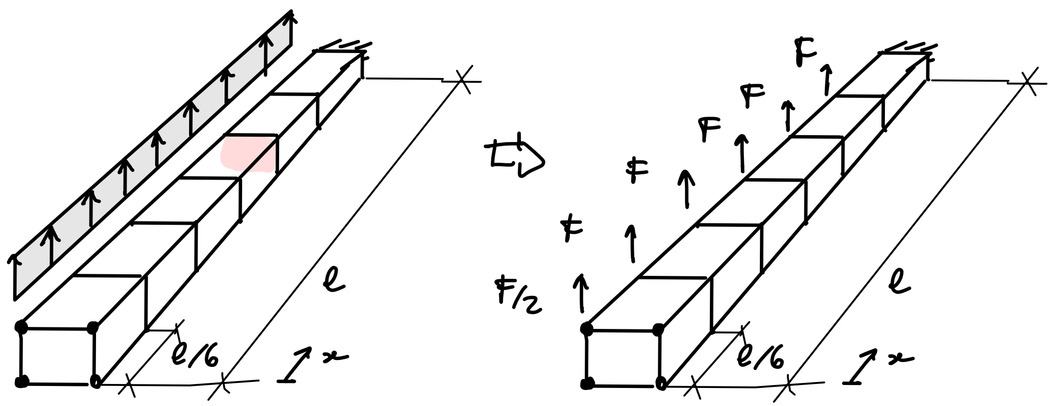
Exercise



The thin-walled beam in the figure is loaded with a force per unit length q. The lumped stringers, including the contribution of the panels, have area A, and the thickness of the panels is t. The shear modulus of the material is G. Assume that the load is introduced in correspondence of the ribs. By using a semi-monocoque approximation, determine the rotation of the section at x = l / 3. Report the absolute value of the rotation angle expressed in degrees. (Unit for result: deg - absolute value)

Data (solution for E = 0)
l = 2500.; Units: mm
a = 150. * (1 + E / 10); Units: mm
A = 500.; Units: mm^2
t = 1.2; Units: mm
G = 27000.; Units: MPa
q = 100.; Units: N / mm

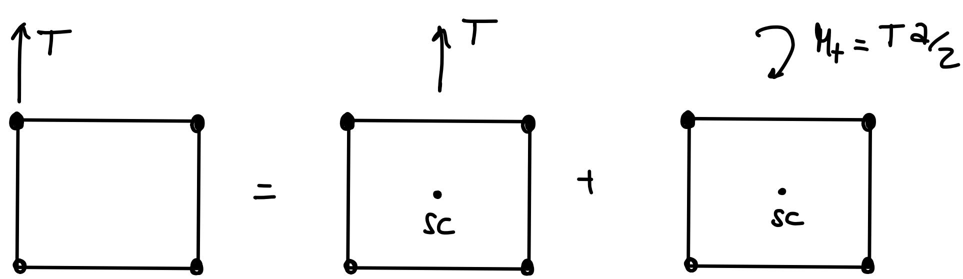
Solution



this is a 1-cell section, so the Bredt's formula can be applied.

$$J = \frac{4\Omega^2}{\oint \frac{1}{t} d\Gamma} = \frac{4a^4 t}{4a} = a^3 t$$

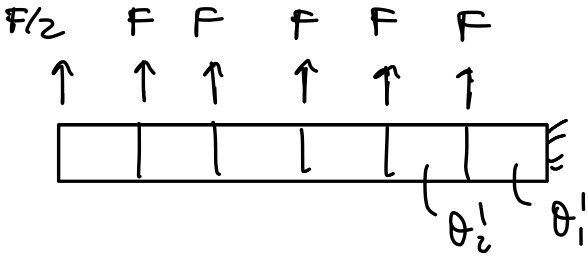
the torsion θ' is then evaluated by referring to the section's shear center, which is readily available due to the section's symmetry.



So:

$$\theta' = M_t / GJ \quad \text{with} \quad M_t = Ta/2$$

$$\theta' = \frac{Ta}{2} \frac{1}{Ga^3 t} = \frac{T}{2Ga^2 t}$$



$F = ql/6$

For the first bay: $T_1 = (5 + 1/2)F$

For the second bay: $T_2 = (4 + 1/2)F$

So:

$$\theta_1' = \frac{T_1}{2Ga^2 t}$$

$$\theta_2' = \frac{T_2}{2Ga^2 t}$$

The rotation is then:

$$\theta = \theta_1' \, l/6 + \theta_2' \, l/6 = 6.82 \text{ deg}$$

- The Principle of Virtual Work:
  - is used to impose the equilibrium
  - ~~is used to impose the equilibrium and compatibility~~
  - ~~is used to impose the compatibility~~
- The shear force is an Euler-Bernoulli beam:
  - ~~is null because the shear deformation is negligible~~
  - is different from zero and can be computed from the derivative of the bending moment
  - ~~is infinite so that the shear deformation is null~~
  - ~~cannot be computed~~
- In a thin-walled beam, a rib contributes to:
  - preserve the shape of the section
  - ~~reduce the shear flows in the panels~~
  - ~~reduce the force carried by the stringers~~
- The Finite Element method requires the natural boundary conditions to be identically fulfilled
  - ~~True~~
  - False
- The trial functions used in the Ritz approximation must be part of a complete set of functions
  - True
  - ~~False~~
- The position of the shear center depends on the loading conditions
  - ~~True~~
  - False