

## INSTABILITY CRITERIA

In the context of an energy-based approach, the stability of the elastic equilibrium can be studied referring to two different criteria:

1. Neutral equilibrium method
2. Adjacent equilibrium method (Treffitz criterion)

In many practical cases the two methods lead to identical equations (this is why they are sometimes confused); however, they are different criteria associated with different ideas.

### Neutral equilibrium method

The total potential energy is  $\Pi = U + V$

Consider the case of a N-dof problem, whose coordinates are given by  $\underline{q}$ .

The variation from  $\underline{q}$  to  $\underline{q} + \delta \underline{q}$  determines a variation of the total potential energy in the form:

$$\Delta \Pi = \Pi(\underline{q} + \delta \underline{q}) - \Pi(\underline{q})$$

Expand now the variation in Taylor series:

$$\Delta \Pi = \delta \Pi + \frac{1}{2!} \delta^2 \Pi + \frac{1}{3!} \delta^3 \Pi + \dots$$

Observing that, due to the equilibrium,  $\delta \Pi = 0$ , the variation  $\Delta \Pi$  reads:

$$\Delta \Pi = \frac{1}{2!} \delta^2 \Pi + \frac{1}{3!} \delta^3 \Pi + \dots$$

The sign of  $\Delta \Pi$  is thus governed by  $\delta^2 \Pi$ . As far as the system is in a stable equilibrium position  $\delta^2 \Pi > 0$

The transition from stability to instability happens when  $\delta^2 \Pi = 0$

More specifically the stability condition can be formulated as follows:

$$\boxed{\delta^2 \Pi|_{\text{@equil.}} = 0}$$

Recall the well-known sketches:



stable equil.

$\delta^2 \Pi > 0$  + virtual variation



unstable equil.

$\delta^2 \Pi < 0$  for at least one virtual variation



neutral equil

$\delta^2 \Pi = 0$  for + virtual variation

- Adjacent equilibrium method

Consider a reference condition (denoted by R = reference)

In this configuration the structure is characterized by a total potential energy  $\Pi_R$ .

After perturbing this configuration, the total potential energy will be:

$$\Pi_R + \Delta \Pi_R = \Pi_R + \delta \Pi_R + \frac{1}{2!} \delta^2 \Pi_R + \dots$$

$$= \Pi_R + \frac{1}{2!} \delta^2 \Pi_R + \dots \quad (\delta \Pi_R = 0 \text{ for the equilibrium})$$

Consider now another equilibrium condition, i.e. another configuration satisfying the equilibrium requirements, close to the reference condition R. Note: close but distinct.

Denote this configuration with A = adjacent (= close to the reference configuration)

The total potential energy of the adjacent configuration is then:

$$\boxed{\Pi_A = \Pi_R + \Delta \Pi_R}$$

( if you prefer, this equation defines what the adjacent equilibrium configuration is: a configuration of equilibrium whose total potential energy is equal to  $\Pi_R$  plus a small variation )

The adjacent configuration is an equilibrium configuration, so:

$$\delta \Pi_A = 0 \Rightarrow \delta \Pi_R + \delta \Delta \Pi_R = 0$$

but  $\delta \Pi_R = 0$  (the ref. configuration is in equilibrium)

and  $\Delta \Pi_R = \cancel{\delta \Pi_R} + \frac{1}{2!} \delta^2 \Pi_R + \dots$

$$\Rightarrow \delta \Delta \Pi_R = \frac{1}{2!} \delta (\delta^2 \Pi_R)$$

and so:

$$\delta \Pi_A = \frac{1}{2!} \delta (\delta^2 \Pi_R) = 0 \Rightarrow \boxed{\delta (\delta^2 \Pi_R) = 0}$$

Trefftz criterion

## Remark 5

1. In many cases, the two criteria lead to the same equations. However the criteria are well different each other from a conceptual point of view.

a. Neutral equilibrium method: check the stability of the equilibrium

b. Adjacent equilibrium method: for a given equilibrium configuration check if another, distinct configuration exists in the neighborhood of the reference one.

2. Often the expression of  $\Pi_R$  is quadratic, thus

$$\Delta \Pi_R = \frac{1}{2} \delta^2 \Pi_R$$

Assuming an initial condition with  $\Pi_R = 0$ , it follows that

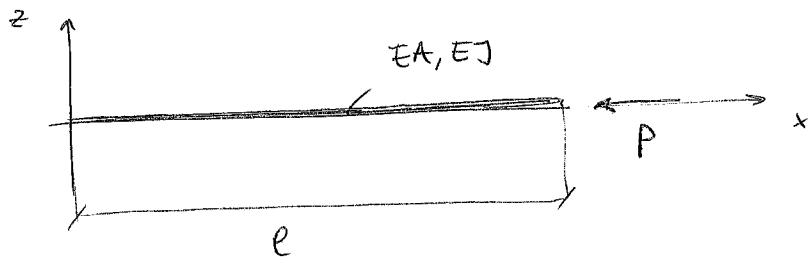
$$\Delta \Pi_R = \Pi_R \quad \text{and so} \quad \Pi_R = \frac{1}{2} \delta^2 \Pi_R$$

The Trefftz criterion, in these circumstances, becomes:

$$\delta(\delta^2 \Pi_R) = 0 \Rightarrow \boxed{\delta \Pi_R = 0}$$

It follows that the instability condition is sometimes denoted by  $\delta \Pi_R = 0$ . However, it is of paramount importance to understand that this is an "undercover" application of the Trefftz criterion.

# BUCKLING OF AN AXIALLY LOADED BEAM



## Strain-displacement relations

Recall the nonlinear expression of the Green-Lagrange strain tensor:

$$\epsilon_{ik} = \frac{1}{2} (u_{ik} + u_{ki} + u_i \cdot u_k)$$

The displacement components associated with the beam reported in the sketch are  $u$  and  $w$  ( $u_x$  and  $u_z$ ). The strain component  $\epsilon_{xx}$  is then:

$$\epsilon_{xx} = \frac{1}{2} (u_{xx} + u_{xx} + u_{xx}^2 + w_{xx}^2) = u_{xx} + \frac{1}{2} u_{xx}^2 + \frac{1}{2} w_{xx}^2$$

Assume now:

1. Infinitesimal in-plane displacements
2. Small but not infinitesimal out-of-plane displacements

It follows that:

$$\boxed{\epsilon_{xx} = u_{xx} + \frac{1}{2} w_{xx}^2}$$

Recall now the Euler-Bernoulli kinematic assumptions:

$$u = u_0 - z w_{0xx}$$

$$w = w_0$$

The strain component  $\epsilon_{xx}$ , expressed as function of the generalized displacement components of the Euler-Bernoulli beam model, is:

$$\epsilon_{xx} = u_{0xx} - z w_{0xxx} + \frac{1}{2} w_{0xx}^2$$

$$= u_{0/x} + \frac{1}{2} \underbrace{\omega_{0/x}^2}_{\text{membrane part}} - z \underbrace{\omega_{0/xx}}_{\text{bending part}}$$

$$\epsilon_{xx} = \epsilon_0 + z k$$

$$\text{with } \epsilon_0 = u_{0/x} + \frac{1}{2} \omega_{0/x}^2$$

$$k = -\omega_{0/xx}$$

Note: hereinafter the subscript "0" will be omitted. It will be implicit in the notation that  $u$  and  $w$  are displacements referred to the reference line.

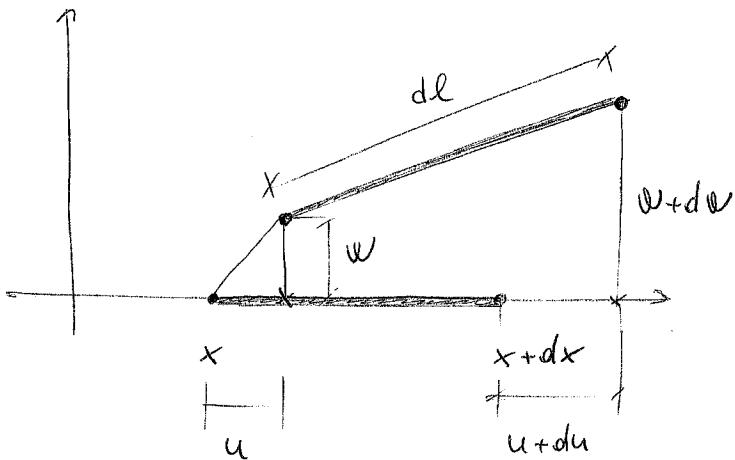
So:

$$\left| \begin{array}{l} \epsilon_{xx} = \epsilon_0 + z k \\ \epsilon_0 = u_{/x} + \frac{1}{2} \omega_{/x}^2 \\ k = -\omega_{/xx} \end{array} \right|$$

### Geometrical interpretation

It can be useful to adopt a geometrical interpretation for the results obtained for  $\epsilon_{xx}$ .

Consider an infinitesimal beam element



After the deformation process, the length of the infinitesimal beam element is:

$$\begin{aligned} d\ell^2 &= (dx + du)^2 + d\psi^2 = dx^2 + du^2 + 2dxdu + d\psi^2 \\ &= dx^2 \left( 1 + u_{1x}^2 + 2u_{1x} + \psi_{1x}^2 \right) \end{aligned}$$

$$d\ell = dx \sqrt{1 + u_{1x}^2 + 2u_{1x} + \psi_{1x}^2}$$

Expand  $d\ell$  in Taylor series

$$\begin{aligned} d\ell &= dx + dx \frac{1}{2} (u_{1x}^2 + 2u_{1x} + \psi_{1x}^2) \\ &= dx \left( 1 + \frac{1}{2} (u_{1x}^2 + 2u_{1x} + \psi_{1x}^2) \right) \end{aligned}$$

The deformation is then:

$$\begin{aligned} \epsilon_{xx} &= \frac{d\ell - dx}{dx} = \frac{1}{2} u_{1x}^2 + u_{1x} + \frac{1}{2} \psi_{1x}^2 \\ &= u_{1x} + \frac{1}{2} \psi_{1x}^2 + \frac{1}{2} u_{1x}^2 \end{aligned}$$

## Energy approach

In contrast to what seen for the case of 1-dof and 2-dof problems, we consider now a beam, which is a continuum. It follows that we now deal with functionals instead of stiffness matrices (at least until the problem is not discretized).

Goal: identify the ODE and the relevant BCs describing the buckling behaviour of the beam

$$\Pi = U + V$$

$$U = \frac{1}{2} \int_0^l (N \epsilon_0 + M k) dx$$

$$\text{with } N = \int_A \sigma dA ; \quad M = \int_A \sigma z dA$$

$$V = -W_{ext} = P u(l)$$

So:

$$\left| \Pi = \frac{1}{2} \int_0^l (N \epsilon_0 + M k) dx + P u(l) \right|$$

Introduce now a perturbation to the reference configuration:

$$u \rightarrow u + \delta u$$

$$w \rightarrow w + \delta w$$

Note: we are dealing with a displacement-based approach, so the perturbations are introduced at the displacement components.

Despite  $\Pi$  is written (for brevity) in terms of  $N, M, \epsilon_0$  and  $k$  recall that the unknowns are  $u$  and  $w$

A perturbation of  $u$  and  $w$  determines a perturbation of  $N$ ,  $M$ ,  $\varepsilon_0$  and  $k$ . More specifically:

- $\varepsilon_0 = u_{xx} + \frac{1}{2} w_{xx}^2$

$$\varepsilon_0 + \delta\varepsilon_0 = u_{xx} + \delta u_{xx} + \frac{1}{2} w_{xx}^2 + \frac{1}{2} \delta w_{xx}^2 + w_{xx} \delta w_{xx}$$

$$\delta\varepsilon_0 = \delta u_{xx} + w_{xx} \delta w_{xx} + \frac{1}{2} \delta w_{xx}^2$$

$\delta\varepsilon_0 = \delta\varepsilon_0^L + \delta\varepsilon_0^{NL}$
--

with

$\delta\varepsilon_0^L = \delta u_{xx} + w_{xx} \delta w_{xx}$
--

$$\delta\varepsilon_0^{NL} = \frac{1}{2} \delta w_{xx}^2$$

- $k = -w_{xx}$

$$k + \delta k = -w_{xx} - \delta w_{xx}$$

$\delta k = -\delta w_{xx}$
-----------------------------

- $N = EA \varepsilon_0$

$$N + \delta N = EA (\varepsilon_0 + \delta\varepsilon_0)$$

$$= EA (\varepsilon_0 + \delta\varepsilon_0^L + \delta\varepsilon_0^{NL})$$

$\delta N = \delta N^L + \delta N^{NL}$
---

with:

$\delta N^L = EA \delta\varepsilon_0^L$
---

$$\delta N^{NL} = EA \delta\varepsilon_0^{NL}$$

- $M = EJ k$

$$M + \delta M = EJ (k + \delta k)$$

$\delta M = EJ \delta k$
--------------------------

$$\Pi = \frac{1}{2} \int_0^L (N\epsilon_0 + M\kappa) dx + P u(L)$$

After perturbing the system:

$$\Pi + \Delta\Pi = \frac{1}{2} \int_0^L \left[ (N + \delta N^L + \delta N^{NL}) (\epsilon_0 + \delta \epsilon_0^L + \delta \epsilon_0^{NL}) + (M + \delta M) (\kappa + \delta \kappa) \right] dx + P \delta u(L)$$

$$\begin{aligned} \Delta\Pi = & \frac{1}{2} \int_0^L \left( N \delta \epsilon_0^L + N \delta \epsilon_0^{NL} + \delta N^L \epsilon_0 + \delta N^L \delta \epsilon_0^L + \boxed{\delta N^L \delta \epsilon_0^{NL}} + \right. \\ & + \delta N^{NL} \epsilon_0 + \boxed{\delta N^{NL} \delta \epsilon_0^L} + \boxed{\delta N^{NL} \delta \epsilon_0^{NL}} + M \delta \kappa + \delta M \kappa + \delta M \delta \kappa \Big) dx + \\ & + P \delta u(L) \end{aligned}$$

$\boxed{\quad}$  : cubic/quartic contributions

Rearranging the terms:

$$\begin{aligned} \Delta\Pi = & \frac{1}{2} \int_0^L (N \delta \epsilon_0^L + \delta N^L \epsilon_0 + M \delta \kappa + \delta M \kappa) dx + P \delta u(L) + \\ & + \frac{1}{2} \int_0^L (N \delta \epsilon_0^{NL} + \delta N^L \delta \epsilon_0^L + \delta N^{NL} \epsilon_0 + \delta M \delta \kappa) dx \end{aligned}$$

Consider now the Taylor expansion of  $\Delta\Pi$ :

$$\begin{aligned} \Delta\Pi &= \delta\Pi + \frac{1}{2!} \delta^2\Pi + \frac{1}{3!} \delta^3\Pi + \frac{1}{4!} \delta^4\Pi \\ &\approx \delta\Pi + \frac{1}{2!} \delta^2\Pi \end{aligned}$$

$$\delta\Pi = \frac{1}{2} \int_0^L (N \delta \epsilon_0^L + \delta N^L \epsilon_0 + M \delta \kappa + \delta M \kappa) dx + P \delta u(L)$$

$$\frac{1}{2!} \delta^2\Pi = \frac{1}{2} \int_0^L (N \delta \epsilon_0^{NL} + \delta N^L \delta \epsilon_0^L + \delta N^{NL} \epsilon_0 + \delta M \delta \kappa) dx$$

From  $\delta\bar{\Pi}$  we obtain the equilibrium equations

From  $\delta^2\bar{\Pi}$  we can assess the stability by means of the Trefftz criterion (we obtain the buckling equations)

### • Equilibrium

$$\delta\bar{\Pi} = \frac{1}{2} \int_0^l (N \delta\epsilon_o^L + \delta N^L \epsilon_o + M \delta k + \delta M k) dx + P \delta u(l)$$

$$= \frac{1}{2} \int_0^l (EA \delta\epsilon_o^L + EA \delta\epsilon_o^L \epsilon_o + EI_k \delta k + EI_k k k) dx + P \delta u(l)$$

$$= \frac{1}{2} \int_0^l (2N \delta\epsilon_o^L + 2M \delta k) dx + P \delta u(l)$$

$$= \int_0^l (N \delta\epsilon_o^L + M \delta k) dx + P \delta u(l)$$

Recall now that:  $\delta\epsilon_o^L = \delta u_{xx} + \omega_{xx} \delta\omega_{xx}$

$$\delta k = -\delta\omega_{xx}$$

$$\delta\bar{\Pi} = \int_0^l [(\delta u_{xx} + \omega_{xx} \delta\omega_{xx}) N - \delta\omega_{xx} M] dx + P \delta u(l)$$

and integrating by parts:

$$= \int_0^l [-\delta u N_{xx} - \delta\omega (N\omega_{xx})_{xx} - \delta\omega M_{xx}] dx + P \delta u(l)$$
$$+ \left. \delta u N + \delta\omega \omega_{xx} N - \delta\omega_{xx} M + \delta\omega M_{xx} \right|_0^l$$

The equilibrium conditions are obtained by letting  $\delta\bar{\Pi}=0$ :

$$\begin{cases} N_{xx} = 0 \\ M_{xx} + (N\omega_{xx})_{xx} = 0 \end{cases} \rightsquigarrow M_{xx} + \cancel{\frac{N_{xx}\omega_{xx}}{0} + N\omega_{xx}} = 0$$

$$N_{xx} = 0$$

$$M_{xx} + NW_{xx} = 0$$

$$N + P = 0$$

$$N = 0$$

$$M_{xx} + NW_{xx} = 0 \quad \text{or}$$

$$M_{\cancel{xx}} = 0$$

or

$$\delta u(l) = 0$$

or

$$\delta u(0) = 0$$

$$\delta W = 0 \quad \text{in } x=0, l$$

or

$$\delta W_{xx} = 0 \quad \text{in } x=0, l$$

→ Equilibrium  
equations

→ Boundary  
conditions

↑  
natural

↑  
essential

### Remarks

1. The equilibrium equations here obtained are NONLINEAR (where nonlinearity is referred to the displacement unknowns!)

$$N = EA \epsilon_s$$

$$= EA \left( u_{xx} + \frac{1}{2} W_{xx}^2 \right)$$

So, the first equation is :  $N_{xx} = \underbrace{\left[ EA \left( u_{xx} + \frac{1}{2} W_{xx}^2 \right) \right]_{xx}}_{\text{nonlinear term}} = 0$

Similarly the second equation contains nonlinear terms due to the term  $NW_{xx}$ .

2. To identify the equilibrium condition (pre-buckling configuration) the solution of the two equilibrium equations is needed. Again, notice that the two equations are nonlinear (see remark 1) and coupled ( $u$  and  $W$  appear in both the equations)

However, for the case of axially loaded beam, the solution of the equilibrium conditions can be found as follows:

$$N_x = 0 \quad \text{so} \quad N = \text{const} \quad \text{but} \quad N + P = 0 \quad \text{in } x = l$$

$$\text{so} \quad \boxed{N = -P}$$

This means that the axial force is equal to applied load  $P$  (the minus sign is due to the fact that the load is compressive)

For solving the buckling equations (see later) the knowledge of  $N$  is enough. Once we are in the position of obtaining the value of  $N$ , we can try to solve the buckling equations.

For the simple case here considered  $N = -P$ , thus  $N$  is constant. To a more general extent, it should be clear that  $N$  is the axial force associated with the solution of the pre-buckling equilibrium equations.

## Stability - buckling equations

The buckling equations are now derived using the Trefftz criterion:

$$\delta \left( \delta^2 \Pi \Big|_{\text{@ equil.}} \right) = 0$$

Recall:

$$\begin{aligned} \frac{1}{2} \delta^2 \Pi &= \frac{1}{2} \int_0^L \left( N \delta \epsilon_o^{NL} + \delta N^L \delta \epsilon_o^L + \underbrace{\delta N \frac{NL}{E_o} \epsilon_o}_{\downarrow} + \delta M \delta k \right) dx \\ &= \frac{1}{2} \int_0^L \left( N \delta \epsilon_o^{NL} + \delta N^L \delta \epsilon_o^L + \underbrace{\delta A \frac{NL}{E_o} \epsilon_o}_{\downarrow} + \delta M \delta k \right) dx \\ &= \frac{1}{2} \int_0^L \left( N \delta \epsilon_o^{NL} + \delta N^L \delta \epsilon_o^L + \underbrace{\delta N \frac{NL}{E_o}}_{\downarrow} + \delta M \delta k \right) dx \\ &= \frac{1}{2} \int_0^L \left( 2N \delta \epsilon_o^{NL} + \delta N^L \delta \epsilon_o^L + \delta M \delta k \right) dx \end{aligned}$$

$$\delta \epsilon_o^{NL} = \frac{1}{2} \delta \omega_{1x}^2 \quad \text{and} \quad \delta \epsilon_o^L = \delta u_{1x} + \omega_{1x} \delta \omega_{1x}$$

$$\delta k = - \delta \omega_{1xx}$$

$$\frac{1}{2} \delta^2 \Pi = \frac{1}{2} \int_0^L \left[ N \delta \omega_{1x}^2 + \delta N^L \left( \delta u_{1x} + \omega_{1x} \delta \omega_{1x} \right) - \delta \omega_{1xx} \delta M \right] dx$$

$$\delta^2 \Pi \Big|_{\text{@ equil.}} = \int_0^L \left[ \bar{N} \delta \omega_{1x}^2 + \delta N^L \left( \delta u_{1x} + \bar{\omega}_{1x} \delta \omega_{1x} \right) - \delta \omega_{1xx} \delta M \right] dx$$

$\bar{N}, \bar{\omega}$ : axial force and out-of-plane displacement  
as obtained from the solution of the  
equilibrium (pre-buckling equations)

We must now impose the variation of  $\delta^2 \Pi|_{\text{@ equil.}}$ :

$$\delta (\delta^2 \Pi|_{\text{@ equil.}}) = 0$$

To avoid confusion with the  $\delta$  symbol let's operate the following change of nomenclature:

$$\delta w \rightarrow w$$

$$\delta u \rightarrow u$$

$$\delta N^L \rightarrow N^L$$

$$\delta M \rightarrow M$$

(We should not forget, however, that  $w, u, N^L, M$  should be intended as variations with respect to the equilibrium configuration)

$$\delta^2 \Pi|_{\text{@ equil.}} = \int_0^l [\bar{N} w_{xx}^2 + N^L (u_{xx} + \bar{w}_{xx} w_{xx}) - w_{xxx} M] dx$$

$$\begin{aligned} \delta (\delta^2 \Pi|_{\text{@ equil.}}) &= \int_0^l [2(\bar{N} w_{xx} \delta w_{xx} + N^L \delta u_{xx} + N^L \bar{w}_{xx} \delta w_{xx} - M \delta w_{xxx}) dx \\ &= 2 \int_0^l [\delta w_{xx} (\bar{N} w_{xx} + N^L \bar{w}_{xx}) + \delta u_{xx} N^L - \delta w_{xxx} M] dx \end{aligned}$$

Integrating by parts:

$$\begin{aligned} &= -2 \int_0^l [\delta w_{xx} (\bar{N} w_{xx} + N^L \bar{w}_{xx})_{xx} + \delta u_{xx} N^L + \delta w_{xxx} M] dx \\ &\quad + 2 \left. \delta w (\bar{N} w_{xx} + N^L \bar{w}_{xx}) + \delta u N^L - \delta w_{xx} M + \delta w M_{xx} \right|_0^l \end{aligned}$$

So:

$$\boxed{N^L_{xx} = 0}$$

$$M_{xx} + (\bar{N} w_{xx} + N^L \bar{w}_{xx})_{xx} = 0 \rightarrow M_{xx} + \boxed{\bar{N}_{xx} w_{xx} + \bar{N} w_{xxx} +}$$

$$\begin{aligned} &+ \boxed{N^L_{xx} w_{xx} + N^L \bar{w}_{xxx}} = 0 \\ &\quad \boxed{0} \quad \rightarrow \text{buckling equation } N^L_{xx} = 0 \end{aligned}$$

$$N_x^L = 0$$

### Buckling equations

$$M_{xx} + \bar{N}W_{xx} + N^L \bar{W}_{xx} = 0$$

$$N^L = 0$$

or  $\delta u = 0$  in  $x=0, l$

$$M_x + \bar{N}W_x + N^L \bar{W}_x = 0$$

or  $\delta w = 0$  in  $x=0, l$

$$M = 0$$

or  $\delta M_x = 0$  in  $x=0, l$

BCs

↑  
Natural

↑  
Essential

In the context of a displacement-based approach, the equations should be intended as function of the displacement component unknowns.

To this aim, recall that  $N^L = EA \epsilon_s^L = EA(u_x + \bar{W}_x W_x)$ , so:

$$\left\{ \begin{array}{l} [EA(u_x + \bar{W}_x W_x)]_x = 0 \\ -(EIW_{xx})_{xx} + \bar{N}W_{xx} + EA(u_x + \bar{W}_x W_x) \bar{W}_{xx} = 0 \\ + \\ \text{BCs} \end{array} \right.$$

If the pre-buckling condition is characterized by  $\bar{W}=0$  the equations simplify to:

$$EAu_{xx} = 0$$

$$-EIW_{xxx} + \bar{N}W_{xx} = 0 \quad \sim EIW_{xxxx} + P W_{xx} = 0$$

$$EAu_x = 0$$

or  $\delta u = 0$  in  $x=0, l$

$$-EIW_{xxx} + \bar{N}W_x = 0$$

or  $\delta w = 0$  in  $x=0, l$

$$EIW_{xx} = 0$$

or  $\delta W_x = 0$  in  $x=0, l$

## Remarks

1. The buckling equations depend on the solution of the pre-buckling equilibrium equations (they depend on  $\bar{w}$  and  $\bar{N}$ )

To solve the buckling problem it is necessary to solve the equilibrium problem first!

2. For an axially loaded beam, the pre-buckling equations can be solved easily in terms of axial force:

$$\bar{N} = -P$$

However the evaluation of  $\bar{w}$  requires the solution of the pre-buckling equations in terms of displacements.

3. If  $\bar{w} = 0$  the only result from the pre-buckling analysis is  $\bar{N} = -P$ . Nothing else is needed to solve the buckling problem.

4. It is fundamental to recall that  $u$  and  $w$  are variations with respect to the equilibrium condition the buckled configuration is then the superposition of the equilibrium configuration and its variation as determined from the buckling equations

5. As a follow up to the previous remark, it is worth noting that the boundary conditions are always homogeneous

Consider the BC associated with the axial force.

Equil. equations (BC)	Buckling equations (BC)
$EAu_{xx} = -P \quad \text{in } x=l$ $+ \frac{1}{2} w_x^2$	$EAu_{xx} = 0 \quad \text{in } x=l$

The buckling equations refer to a variation with respect to a configuration that already satisfies the equilibrium requirements (both in terms of equations and BCs). It is then clear that the non-homogeneous conditions are already satisfied by the pre-buckling configuration.

6. There is a substantial difference between the equilibrium and the buckling equations:

- equilibrium equations: they are a classical boundary value problem. For a given load it is possible to find the deformed shape
- buckling equations: they are an eigenvalue problem. The solution  $u=w=0$  is a solution of the equations (trivial). The idea is to identify the value of  $P$  such that a solution, different from the trivial one, exists.

7. The bending part of the buckling equations is uncoupled from the axial one. It is then possible to solve the buckling problem by considering the bending buckling equation only

8. The buckling equations are not equilibrium equations!  
They provide information regarding the stability of the  
equilibrium, not regarding the equilibrium itself.

## Solution of the buckling equations

Due to the uncoupling between in-plane and out-of-plane behaviour, the second equation can be solved independently from the first one.

$$+ EI w_{xxxx} + P w_{xx} = 0$$

The characteristic polynomial is:

$$\lambda^2 (EI\lambda^2 + P) = 0 \quad , \text{ so:}$$

$$\lambda_{1,2} = 0 \quad \text{and} \quad \lambda_{3,4} = \pm i \sqrt{\frac{P}{EI}}$$

The solution is in the form

$$w = Ax + B + C \sin x \sqrt{\frac{P}{EI}} + D \cos x \sqrt{\frac{P}{EI}}$$

where  $A, B, C$  and  $D$  have to be determined by imposing the boundary conditions.

It can be useful to represent the solution as

$$w = Ax + B + C \sin\left(x \sqrt{\frac{P}{EI}} + \phi\right)$$

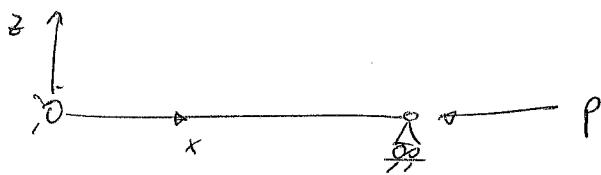
where the constants are now  $A, B, C$  and  $\phi$ .

This latest expression highlights the relation between the load  $P$  and the buckled configuration. In particular:

$$\sin \frac{\pi x}{\lambda} \quad \lambda: \text{half wave length}$$

$$\left(\frac{\pi}{\lambda}\right)^2 = \frac{P}{EI} \Rightarrow \boxed{P = \frac{EI\pi^2}{\lambda^2}} \quad \lambda \text{ depends upon the boundary conditions}$$

• Hinged-hinged



$$w = Ax + B + C \sin \beta x + D \cos \beta x \quad \text{with} \quad \beta = \sqrt{\frac{P}{EI}}$$

Boundary conditions:

$$\begin{cases} w(0) = 0 \\ w_{xx}(0) = 0 \\ w(l) = 0 \\ w_{xx}(l) = 0 \end{cases} \quad \begin{cases} B + D = 0 \\ D = 0 \\ Al + B + C \sin \beta l + D \cos \beta l = 0 \\ -\beta^2 C \sin \beta l - \beta^2 D \cos \beta l = 0 \end{cases}$$

$$\begin{cases} B = 0 \\ D = 0 \\ Al + C \sin \beta l = 0 \\ -\beta^2 C \sin \beta l = 0 \end{cases} \quad (\text{if } C=0 \Rightarrow A=0 \Rightarrow \text{Trivial solution})$$

$\sin \beta l = 0 \quad \text{if} \quad \beta l = n\pi \quad \text{so:}$

$$\sqrt{\frac{P}{EI}} l = n\pi \quad \Rightarrow \quad P = \frac{\pi^2 n^2}{l^2} EI$$

$$P_{cn} = \frac{\pi^2 EI}{l^2} \quad (n=1)$$

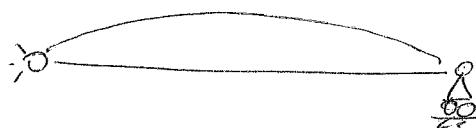
By comparison with  $P = \frac{\pi^2 EI}{l^2} \quad \boxed{l = \lambda}$

Evaluation of the buckling mode:

$$\left\{ \begin{array}{l} B = 0 \\ D = 0 \\ Al + C \sin \beta l = Al = 0 \Rightarrow A = 0 \\ -\beta^2 C \sin \beta l = 0 \end{array} \right.$$

$$w = C \sin \beta x \quad \text{but} \quad \beta l = n\pi \quad (n=1) \\ = \pi$$

$$w = C \sin \frac{\pi}{l} x \quad \text{buckling mode}$$



• Clamped - clamped



Boundary conditions:

$$\begin{cases} w(0) = 0 \\ w_x(0) = 0 \end{cases} \quad \begin{cases} B + D = 0 \\ A + \beta C = 0 \end{cases} \quad (1)$$

$$\begin{cases} w(l) = 0 \\ w_x(l) = 0 \end{cases} \quad \begin{cases} Al + B + C \sin \beta l + D \cos \beta l = 0 \\ A + \beta C \cos \beta l - \beta D \sin \beta l = 0 \end{cases} \quad (2)$$

$$(3)$$

$$(4)$$

$$B = -D$$

$A = -\beta C$ , so the (3) and (4) become:

$$\begin{cases} -\beta l C - D + C \sin \beta l + D \cos \beta l = 0 \\ -\beta C + \beta C \cos \beta l - \beta D \sin \beta l = 0 \end{cases}$$

$$\begin{cases} C (\sin \beta l - \beta l) + D (\cos \beta l - 1) = 0 \\ C (\cos \beta l - 1) - D \sin \beta l = 0 \end{cases}$$

As usual  $C = D = 0$  satisfies the equations, but it implies that  $A = B = 0 \Rightarrow$  trivial solution

The nontrivial solution is found by setting to zero the determinant of the matrix of coefficients (note: this is a nonlinear eigenvalue problem)

$$-(\sin \beta l - \beta l) \cdot \sin \beta l - (\cos \beta l - 1)^2 = 0$$

$$-\sin \beta l^2 + \beta l \sin \beta l - \cos \beta l^2 - 1 + 2 \cos \beta l =$$

$$-2 + \beta l \sin \beta l + 2 \cos \beta l = 0$$

$$\left| \frac{\beta l}{2} \sin \beta l + \cos \beta l = 1 \right|$$

The equation is satisfied if  $\boxed{\beta l = 2n\pi}$

$$(\sin 2n\pi = 0; \cos 2n\pi = 1)$$

So:

$$\sqrt{\frac{P}{EI}} l = 2n\pi \quad P = h^2 \frac{4(\pi^2 EI)}{l^2}$$

$$\left| P_{cn} = \frac{4\pi^2 EI}{c^2} \right| \quad (n=1)$$

$$= \frac{\pi^2 EI}{l^2} \Rightarrow \lambda^2 = \frac{l^2}{4} \Rightarrow \boxed{\lambda = l/2}$$

Evaluation of the buckling mode:

$$\text{if } \beta l = 2n\pi \Rightarrow \sin \beta l = 0$$

$$\cos \beta l = 1$$

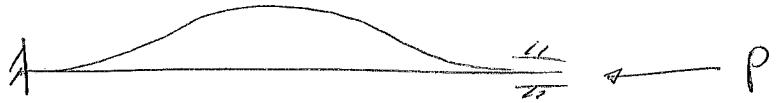
The boundary conditions are:

$$-C\beta l = 0 \Rightarrow C = 0 \Rightarrow A = 0; B = -D$$

The buckling mode is then:

$$w = B + D \cos \beta x = B(1 - \cos \beta x) \quad \text{with } \beta = \frac{2\pi}{l}$$

$$\Rightarrow \boxed{w = B(1 - \cos \frac{2\pi}{l} x)}$$



• Fixed-free



$$\begin{cases} w(0) = 0 \\ w_{xx}(0) = 0 \\ w_{xxx}(l) = 0 \\ EIw_{xxx}(l) + Pw_{xx}(l) = 0 \end{cases}$$

$$\begin{cases} B + D = 0 \\ A + \beta C = 0 \\ -\beta^2 C \sin \beta l - \beta^2 D \cos \beta l = 0 \\ (-\beta^3 C \cos \beta l + \beta^3 D \sin \beta l) EI + (A + \beta C \cos \beta l - \beta D \sin \beta l) P = 0 \end{cases} \quad \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix}$$

Express the constants as functions of  $C$ .

The (1)-(3) become:

$$\begin{array}{ll} B + D = 0 & B = C \operatorname{tg} \beta l \\ A = -\beta C & A = -\beta C \\ D = -C \operatorname{tg} \beta l & D = -C \operatorname{tg} \beta l \end{array}$$

The (4) is then:

$$(-\beta^3 C \cos \beta l - \beta^3 C \operatorname{tg} \beta l \sin \beta l) EI + (-\beta C + \beta C \cos \beta l + \beta C \operatorname{tg} \beta l \cdot \sin \beta l) P = 0$$

$$[\operatorname{tg} \beta l \sin \beta l (P\beta - \beta^3 EI) + \cos \beta l (P\beta - \beta^3 EI) - P\beta] C = 0$$

Multiply both sides with  $\cos \beta l$  and divide by  $\beta$ :

$$\left[ \sin \beta l^2 (P - \beta^2 EJ) + \cos \beta l^2 (P - \beta^2 EJ) - P \cos \beta l \right] C = 0$$

but  $\beta = \sqrt{\frac{P}{EJ}}$  so:

$$\boxed{P \cos \beta l \ C = 0} \quad C = 0 \Rightarrow \text{trivial solution}$$

$$\cos \beta l = 0 \quad \text{if} \quad \beta l = \frac{\pi}{2} + n\pi \quad n = 0, \dots, N$$

$$\frac{P}{EJ} l^2 = \left( \frac{\pi}{2} + n\pi \right)^2$$

$$\boxed{P_{nl} = \frac{EJ \pi^2}{4l^2}} \quad (n=0)$$

$$\boxed{\lambda = 2l}$$

Evaluation of the buckling mode

$$\beta l = \frac{\pi}{2} \Rightarrow \cos \beta l = 0 \quad \sin \beta l = 1$$

The boundary conditions are then:

$$\left. \begin{array}{l} B + D = 0 \\ A + \beta C = 0 \\ -\beta^2 C = 0 \\ \beta^3 D EJ + (A - \beta D) P = 0 \end{array} \right\} \quad \left. \begin{array}{l} B = -D \\ A = 0 \\ C = 0 \\ \beta^3 D EJ - \beta D P = 0 \end{array} \right\}$$

$D = 0 \Rightarrow \text{trivial solution}$

$D \neq 0$  and  $B = -D \Rightarrow$  buckling mode

$$w = B + D \cos \beta x$$

$$= -D + D \cos \beta x \quad \text{with} \quad \beta = \frac{\pi}{2l}$$

$$\boxed{w = D \left( -1 + \cos \frac{\pi}{2l} x \right)}$$

