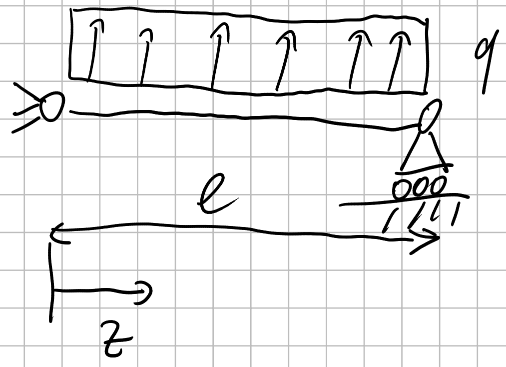
Exercise #2



EI Euler - Bernoulli

$$u \approx \tilde{c}_0 + \tilde{c}_1 z + \tilde{c}_2 z^2 + \tilde{c}_3 z^3 + \dots$$

is this approx OK for DBC? → NO

impose DBC :
$$\begin{cases} u(0) = 0 \\ u(l) = 0 \end{cases}$$

$$u(0) = \tilde{c}_0 = 0 \Rightarrow \tilde{c}_0 = 0$$

$$u(l) = \tilde{c}_1 l + \tilde{c}_2 l^2 + \tilde{c}_3 l^3 + \dots = 0$$

$$\tilde{c}_1 = - \tilde{c}_2 l - \tilde{c}_3 l^2 - \tilde{c}_4 l^4 - \dots$$

$$u(z) \simeq \left(-\tilde{c}_2 l - \tilde{c}_3 l^2 - \tilde{c}_4 l^3\right) z + \tilde{c}_2 z^2 + \tilde{c}_3 z^3 + \tilde{c}_4 z^4$$

re-arrange collecting $\tilde{c}_i$

and rename the unknowns: $c_1 = \tilde{c}_2$

$c_2 = \tilde{c}_3$

$$u(z) = c_1 \underbrace{\left(z^2 - l z\right)}_{\phi_1} + c_2 \underbrace{\left(z^3 - l^2 z\right)}_{\phi_2} + c_3 \underbrace{\left(z^4 - l^3 z\right)}_{\phi_3} + $$

one term approximation: $u = c_1\left(z^2 - l z\right)$ $\qquad \phi_1 = z^2 - l z$

$$\int_0^l \delta u'' EJ u'' \, dz = \int_0^l \delta u \, q \, dz$$

$u'' = 2 c_1 \qquad \delta u'' = 2 \delta c_1$

$$\delta c_1 \int_0^l 4 EI \, dz \, c_1 = \delta c_1 \int_0^l (z^2 - l z) q \, dz$$

$$\delta c_1 (4 EI l) c_1 = \delta c_1 \left(\frac{1}{3} l^3 - \frac{1}{2} l^3 \right) q = \delta c_1 \left(- \frac{1}{6} q l^3 \right)$$

$$\Rightarrow c_1 = \frac{-q l^2}{24 EI}$$

$\rightarrow$ two terms approximation.

$$u = c_1 (z^2 - l z) + c_2 (z^3 - l^2 z)$$

$$u'' = 2 c_1 + 6 z c_2 = \begin{bmatrix} 2 & 6z \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

$$\delta \{c\}^T \int_0^l \begin{bmatrix} 2 \\ 6z \end{bmatrix} EI \begin{bmatrix} 2 & 6z \end{bmatrix} dz \{c\} = \delta \{c\}^T \int_0^l \begin{Bmatrix} z^2 - l z \\ z^3 - l^2 z \end{Bmatrix} q \, dz$$

$$\delta\{c\}^T \int_0^l EI \begin{bmatrix} \boxed{4} & 12z \\ 12z & 36z^2 \end{bmatrix} dz \{c\} = \delta\{c\}^T \int_0^l \begin{Bmatrix} z^2 - lz \\ z^3 - l^2 z \end{Bmatrix} q\, dz$$

symmetric

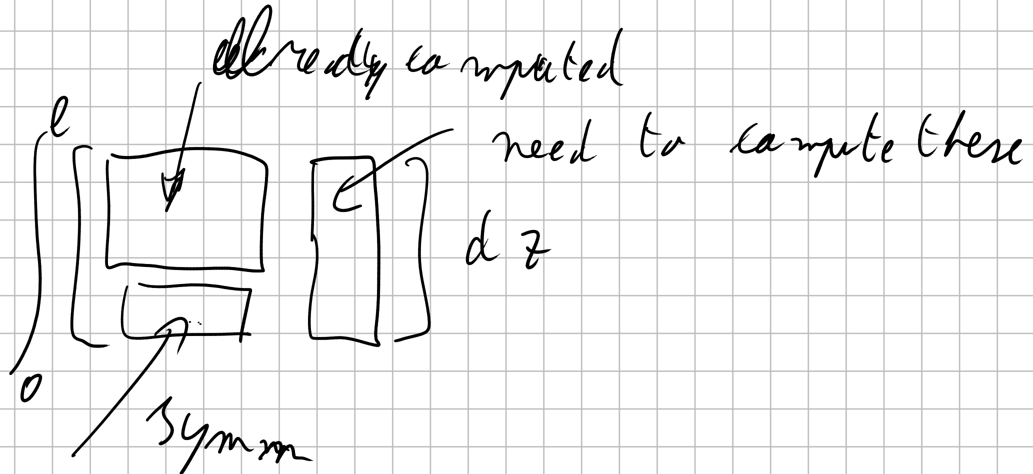
$$EI \begin{bmatrix} 4l & 6l^2 \\ 6l^2 & 12l^3 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} -q\frac{l^3}{6} \\ -q\frac{l^4}{4} \end{Bmatrix}$$

$$c_1 = \frac{-q l^2}{24 EI} \qquad c_2 = \emptyset$$

$\rightarrow$ three terms $\quad u = c_1 \underbrace{(z^2 - lz)}_{\phi_1} + c_2 \underbrace{(z^3 - l^2 z)}_{\phi_2} + c_3 \underbrace{(z^4 - l^3 z)}_{\phi_3}$

$$u'' = \begin{bmatrix} \phi_1'' & \phi_2'' & \phi_3'' \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

$$\int_0^l \begin{bmatrix} \phi_1'' \\ \phi_2'' \\ \phi_3'' \end{bmatrix} EI \begin{bmatrix} \phi_1'' & \phi_2'' & \phi_3'' \end{bmatrix} dz$$



$$k_{13} = EI \int_0^l \phi_1'' \, \phi_3'' \, dz = 8 \, EI \, l^3$$

$$k_{23} = EI \int_0^l \phi_2'' \, \phi_3'' \, dz = 18 \, EI \, l^4$$

$$k_{33} = EI \int_0^l \phi_3'' \, \phi_3'' \, dz = \frac{144}{5} \, EI \, l^5$$

$$f_3 = q \int_0^l \phi_3 \, dz = -\frac{3}{10} \, q \, l^5$$

$$\left[\quad K \quad\right]_{3\times 3} \left\{ t \right\}_{3\times 1} = \left\{ f \right\}_{3\times 1}$$

$$t_1 = \emptyset \qquad t_2 = \frac{-q\,l}{12\,EI} \qquad t_3 = \frac{q}{24\,EI}$$

$$u = \frac{q\,l^4}{2\,E\,\cancel{EI}} \left(\frac{z}{l} - 2\,\frac{z^3}{l^3} + \frac{z^4}{l^4} \right)$$

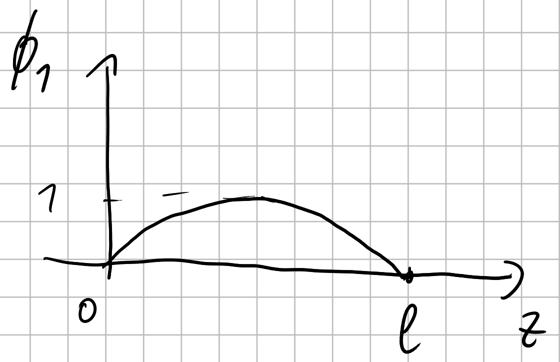
adding additional terms t_4 , t_5 . . . .

solution t_4 , t_5 . . . . $= \emptyset$

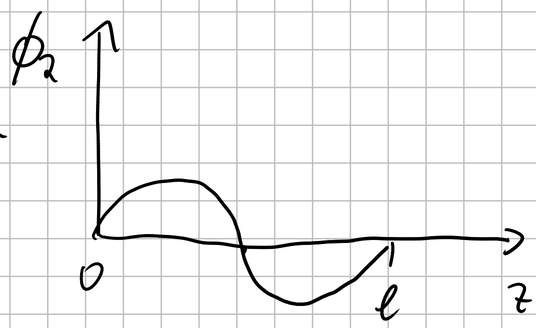
different approximation:

$$u(z) \approx \sum_{i=1}^{n} t_i \underbrace{\sin\left(\frac{i \pi z}{l}\right)}_{\phi_i}$$

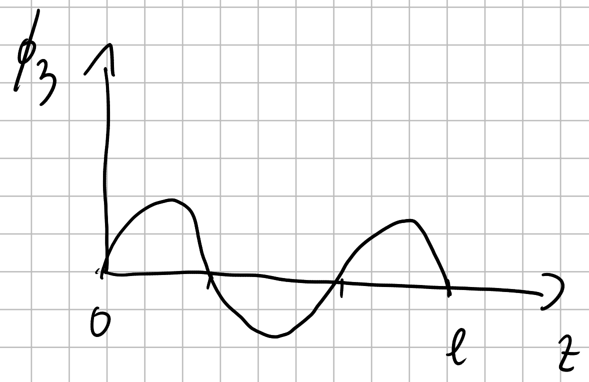
$i=1$



$i=2$



$i=3$



$$\int_0^l \delta u'' \, EI \, u'' \, dt = \int_0^l \delta u \, q \, dz$$

$$u = \sum_{i=1}^{n} t_i \sin\left(\frac{i\pi z}{l}\right) \qquad \delta u = \sum \delta t \left(\qquad \right)$$

$$u' = \sum_{i=1}^{n} t_1 \frac{i\pi}{l} \cos\left(\frac{i\pi z}{l}\right)$$

$$u'' = \sum_{i=1}^{n} -t_i \left(\frac{i\pi}{l}\right)^2 \sin\left(\frac{i\pi z}{l}\right) \qquad \delta u'' = \sum \delta t_1 \left(\qquad \right)$$

$$\int_0^l \sum_{i,j=1}^{n} \delta t_i \underbrace{\left(\frac{i\pi}{l}\right)^2 \sin\left(\frac{i\pi z}{l}\right)}_{-\phi_i} EI \underbrace{\left(\frac{j\pi}{l}\right)^2 \sin\left(\frac{j\pi z}{l}\right)}_{-\phi_j} t_j \, dz$$

$$= \int_0^l \sum_{i=1}^{n} \delta t_i \sin\left(\frac{i\pi z}{l}\right) q \, dz$$

$$
\begin{bmatrix} \phi_1'' \\ \phi_2'' \\ \phi_3'' \\ \\ \phi_n'' \end{bmatrix} EI \begin{bmatrix} \phi_1'' & \phi_2'' & \phi_3'' \end{bmatrix}
$$

$$
\begin{bmatrix} \phi_1''\phi_1'' & \phi_1''\phi_2'' & & \phi_1''\phi_n'' \\ \phi_2''\phi_1'' & & & \\ \phi_3''\phi_1'' & \cdots & & \phi_3''\phi_n'' \\ & & & \end{bmatrix}
$$

$$
K_{ij} = \phi_i'' \phi_j''
$$

$$\int_0^l \sin\left(\frac{i\pi z}{l}\right)\sin\left(\frac{j\pi z}{l}\right) dz = \begin{cases} \emptyset & \text{for } i \neq j \\ \int_0^l \sin^2\left(\frac{i\pi z}{l}\right) dz = \frac{l}{2} & i = j \end{cases}$$

$$K_{ij} \begin{cases} \emptyset & i \neq j \\ \neq \emptyset & i = j \end{cases}$$

$\Rightarrow$ diagonal stiffness matrix

$$\begin{bmatrix} \ddots & & \\ & \text{diag} & \\ & & \ddots \end{bmatrix} \{ x \} = \{ f \}$$

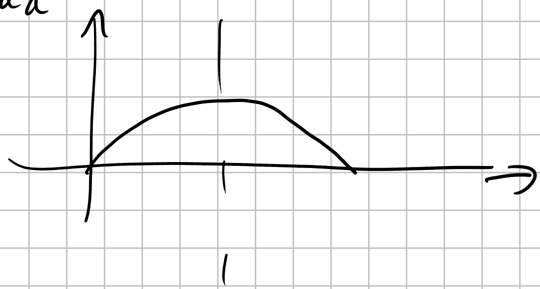
$$\sum_{i=1}^{n} \delta t_i \, EI\left(\frac{i\pi}{l}\right)^4 \cdot \frac{l}{2} \, t_i = \int_0^l \delta u \, q \, dz$$

$$= \sum_{i=1}^{n} \delta t_i \int_0^l \sin\left(\frac{i\pi z}{l}\right) q \, dz$$

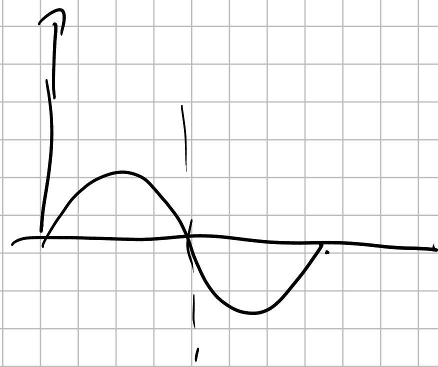
$$\int_0^l \sin\left(\frac{i\pi z}{l}\right) dz = -\cos\left(\frac{i\pi z}{l}\right) \cdot \frac{l}{i\pi}\Bigg|_0^l = -\cos(i\pi) + \cos(0)$$

$$= \begin{cases} \emptyset & i \text{ even} \\ 2 & i \text{ odd} \end{cases}$$

i odd



i even

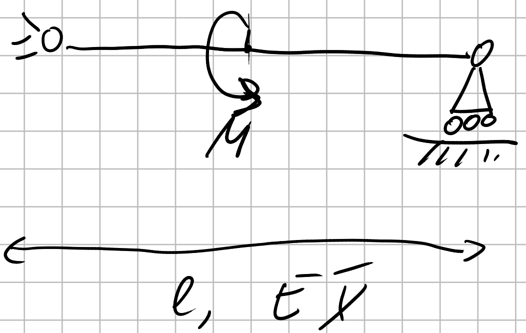


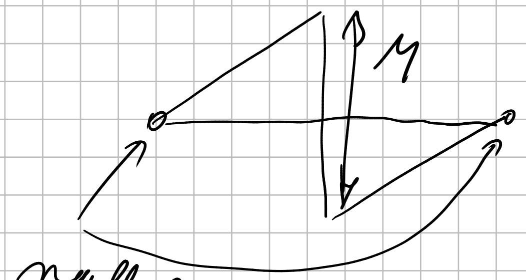
i even $\qquad EI\left(\frac{i\pi}{l}\right)^4 \frac{l}{2}\, c_i = 0 \qquad \Rightarrow \quad c_i = 0$

i odd $\qquad EI\left(\frac{i\pi}{l}\right)^4 \frac{l}{2}\, c_i = \frac{2l}{i\pi}\, q$

$$\Rightarrow c_i = \frac{4}{i\pi}\left(\frac{l}{i\pi}\right)^4 \cdot \frac{1}{EI} \cdot q$$

Exampl #3





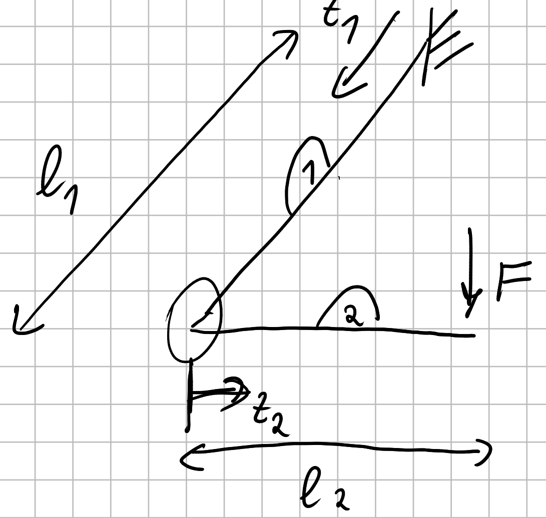
null bending moment at hinges

$$u \approx \sum_{i=1}^{n} c_i \sin\left(\frac{i\pi z}{l}\right)$$

$$\delta \mathcal{L}_i = \int_0^l \delta u'' EI\, u''\, dz = \delta \mathcal{L}_e = \delta u'\left(\frac{l}{2}\right) \cdot M$$

$$\delta u'\left(\frac{l}{2}\right) = \sum_{i=1}^{n} \delta c_i \left(\frac{i\pi}{l}\right) \cos\left(\frac{i\pi}{2}\right) \begin{cases} = \emptyset & i \text{ is odd} \\ \pm 1 & i \text{ is even} \end{cases}$$

$$\Rightarrow \quad \begin{aligned} c_i &= \emptyset \quad i \text{ odd} \\ c_i &\neq \emptyset \quad i \text{ even} \end{aligned}$$

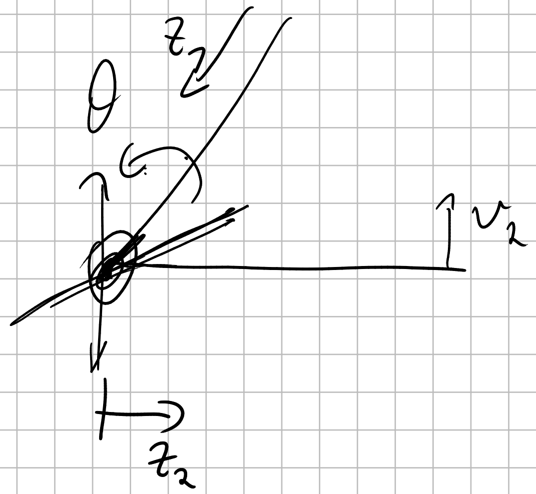


$$v_1 = \sum c_i f_i(z_1) \Rightarrow v_1'' \quad \delta v_1'' \rightarrow \delta \mathcal{W}_i \text{ (bending moment in beam \#1)}$$

$$\theta_1 = \sum a_i g_i(z_1) \Rightarrow \theta_1' \quad \delta \theta_1' \rightarrow \delta \mathcal{W}_i \text{ (+ torsional moment in beam \#1)}$$

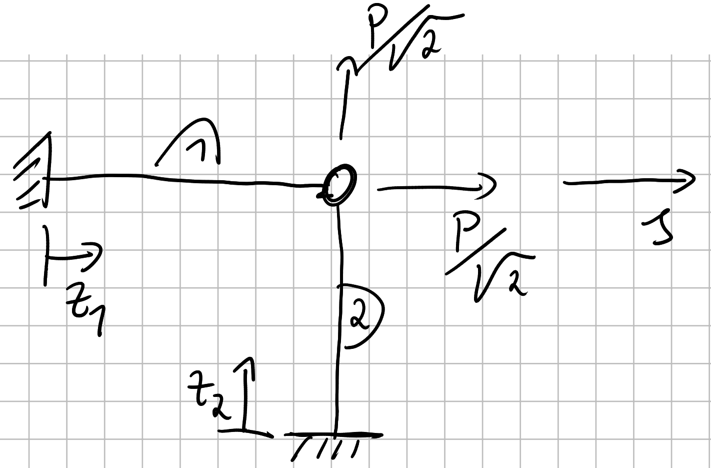
$$v_2 = \sum d_i e_i(z_2) \Rightarrow v_2'' \quad \delta v_2'' \rightarrow \delta \mathcal{W}_i \text{ (+ bending beam \#2)}$$

D.B.C. :
$$\begin{cases} v_1(z_1=\emptyset) = \emptyset \\ v_1'(z_1=\emptyset) = \emptyset \\ \theta_1(z_1=\emptyset) = \emptyset \end{cases}$$



$$\begin{cases} v_1(z_1=\ell_1) = v_2(z_2=\emptyset) \\ \theta_1(z_1=\ell_1) = v_2'(z_2=\emptyset) \end{cases}$$

Example #4



axial displacement 1 $\quad u_1 = c\, z_1$

transverse displacement 1 $\quad v_1 = d\, z_1^2$

axial displacement 2 $\quad u_2 = e\, z_2$

transverse displ 2 $\quad v_2 = f\, z_2^2$

already accounting for the clamp B.C.

need to account for the hinge:

$$u_1(\ell) = v_2(\ell) \qquad c\,\ell = f\,\ell^2 \Rightarrow c = f\,\ell$$

$$v_1(\ell) = u_2(\ell) \qquad d\,\ell^2 = e\,\ell \Rightarrow e = d\,\ell$$

$$u_1 = k \ell z_1 \qquad u_2 = d \ell z_2$$

$$v_1 = d z_1^2 \qquad v_2 = k z_2^2$$

unknowns? k, d

$$\varepsilon_1 = \frac{\partial u_1}{\partial z_1} = k \ell \qquad \varepsilon_2 = \frac{\partial u_2}{\partial z_2} = d \ell$$

$$v_1'' = 2d \qquad v_2'' = 2k$$

$$\delta\varepsilon_1 = \delta k \ell \qquad \delta\varepsilon_2 = \delta d \ell$$

$$\delta v_1'' = 2\delta d \qquad \delta v_2'' = 2\delta k$$

$$\int_0^{\ell_1} \left(\delta\varepsilon_1 \, EA_1 \, \varepsilon_1 + \delta v_1'' \, EI_1 \, v_1'' \right) dz_1 + \int_0^{\ell_2} \left(\delta\varepsilon_2 \, EA_2 \, \varepsilon_2 + \delta v_2'' \, EI_2 \, v_2'' \right) dz_2$$

$$= \delta u_1(\ell_1) \cdot \frac{P}{\sqrt{2}} + \delta u_2(\ell_2) \cdot \frac{P}{\sqrt{2}}$$

$$\int_0^{\ell} \left(\boxed{\delta k}\, EA_1\, \ell^2 k + \boxed{\delta d}\, 2\, EI_1\, 2 d \right) dz_1 + \int_0^{\ell} \left(\boxed{\delta d}\, \ell^2\, E E_2\, d + \boxed{\delta k}\, 2\, EI_2\, 2 k \right) dz_2$$

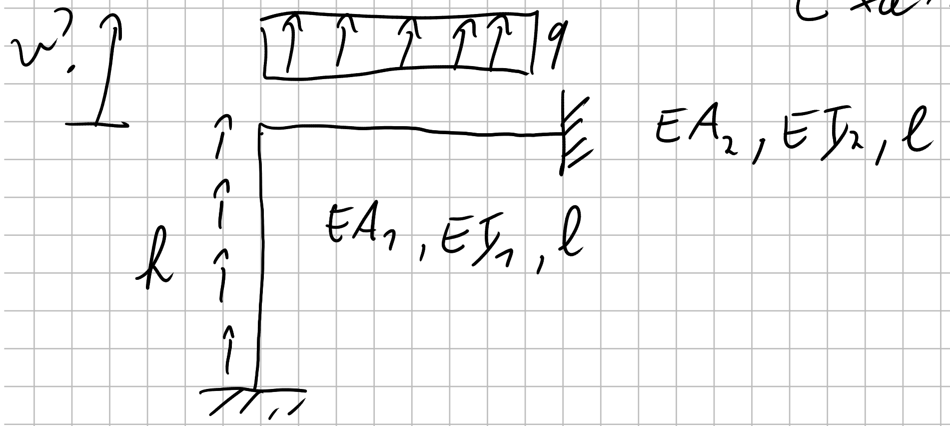
$$= \boxed{\delta k}\, \ell^2 \frac{P}{\sqrt{2}} + \boxed{\delta d}\, \ell^2 \frac{P}{\sqrt{2}}$$

$$\delta k \left(EA_1 \ell^3 + 4 EI \ell \right) k = \delta k\, \ell^2 \frac{P}{\sqrt{2}}$$

$$\Rightarrow k = \left(\qquad \right)$$

$$s = u_1(z_1 = \ell) = k\, \ell^2$$

Exampl #5



determine the vertical displ v
Solve the problem using Ritz and consider the smallest number of dofs using a polynomial representation.

$l = 1300$ mm

$EA_1 = 3E6$ N

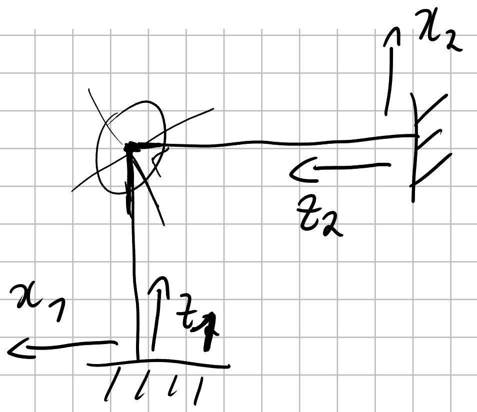
$EJ_1 = 4E12$ N mm^2

$EA_2 = 6E6$ N

$EJ_2 = 8E12$ N mm^2

$q = 100$ N mm^{-1}

$k = 70$ N mm^{-1}



$$w_1 = \boxed{a_0} + a_1 \left(z_1/\ell\right)$$
$$u_1 = \boxed{b_0} + \boxed{b_1}\left(\frac{z_1}{\ell}\right) + b_2 \left(\frac{z_1}{\ell}\right)^2$$

$\boxed{}$: null because of clamps

$$w_2 = \boxed{c_0} + c_1 \left(\frac{z_2}{\ell}\right)$$
$$u_2 = \boxed{d_0} + \boxed{d_1}\left(\frac{z_2}{\ell}\right) + d_2 \left(\frac{z_2}{\ell}\right)^2$$

4 unknowns: a_1, b_2, c_1, d_2

$$\left.\begin{array}{l} w_1(\ell) = u_2(\ell) \\ w_2(\ell) = u_1(\ell) \\ \boxed{u_1'(\ell) = -u_2'(\ell)} \end{array}\right]$$

3 additional eqs to be accounted for

$$d_2 = a_1$$
$$t_1 = -a_1 \qquad \Rightarrow \text{ only 1 independent}$$
$$b_2 = -a_1 \qquad \text{unknown : } a_1$$

$$\int_0^\ell \left(\delta w_1' EA_1 w_1' + \delta u_1'' EI_1 u_1''\right) dz_1 + \int_0^\ell \left(\delta w_2' EA_2 w_2' + \delta u_2'' EI_2 u_2''\right) dz_2$$
$$= \int_0^\ell \delta w_1 \, k \, dz_1 + \int_0^\ell \delta u_2 \, q \, dz_2$$

$$K a = f \qquad K = \frac{EA_1}{\ell} + \frac{EA_2}{\ell} + \frac{EI_1}{\ell^3} + \frac{4EI_2}{\ell^3}$$

$$f = \frac{k}{2}\ell + \frac{q}{3}\ell$$

$$\Rightarrow a_1 = 3{,}0876 \text{ mm} \qquad \Rightarrow w_1(\ell) = a_1 = 3{,}0876 \text{ mm}$$