Course of Aerospace Structures

Written test, February 13$^{\text{th}}$, 2024

Name ______________________________

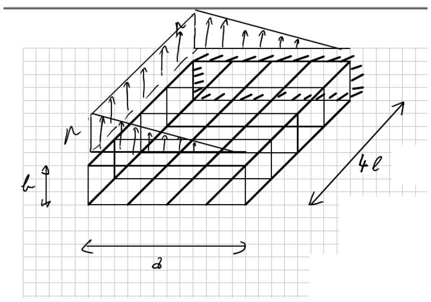
Surname ______________________________

Person code: | | | | | | | | |

Exercise 1

The semi-monocoque wing box model represented in the figure has five uniformly spaced ribs (including the rib located at the clamp) and ten stringers (five uniformly spaced on the upper side, five uniformly spaced on the lower side). Each stringer has a concentrated area A. The panels thickness is equal to t. The wing box is loaded by a pressure load that is constant along the wing span and has a triangular distribution along the chord, with the maximum value equal to p. Compute the upper right stringer jump of axial stress $\Delta\sigma_{zz}$ across the mid-span rib.

(Unit for result: MPa)



Data

$4l = 3$ m

$a = 1$ m

$b = 10$ cm

$t = 1$ mm

$A = 10$ cm^2

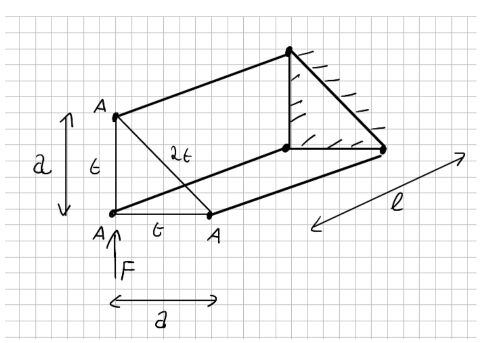
$p = 1000$ Pa

Answer ______________________________

Exercise 2

The semi-monocoque beam model sketched in the figure has a triangular cross-section with three concentrated areas A, two panels of thickness t and one panel of thickness $2t$. The beam has length l, is clamped at one extremity and loaded by the force F, aligned with the vertical panel, at the free extremity. Compute the overall torsional rotation of the loaded extremity section.

(Unit for result: rad)



Data

$a = 0.5$ m

$A = 4$ cm^2

$t = 1$ mm

$l = 4$ m

$E = 70000$ MPa

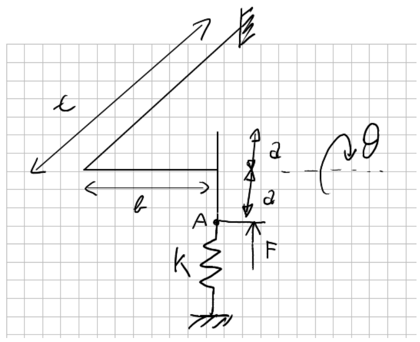
$\nu = 0.3$

$F = 10000$ N

Answer ______________________________

Exercise 3
Consider the thin beam model sketched in the figure, and loaded by the force F applied at point A. All the beams have the same cross-section. Compute the rotation θ of the point where the beam of length b is joined to the two beams of length a. Neglect shear deformability.

(Unit for result: rad)

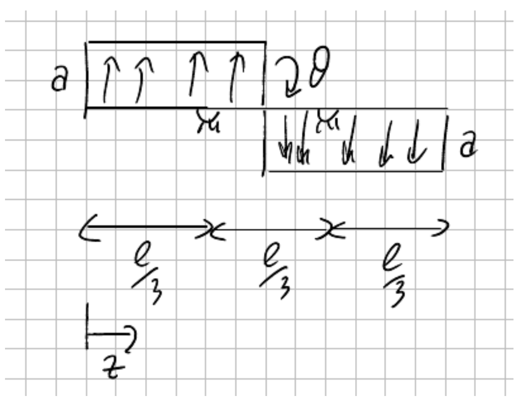


Data
$a = 1$ m
$b = 2$ m
$c = 3$ m
$EI_{xx} = EI_{yy} = 12 \times 10^{14}$ N mm^2
$EA = 6 \times 10^{10}$ N
$GJ = 7 \times 10^9$ N mm^2
$K = 1$ N/mm
$F = 10000$ N

Answer

Exercise 4
Consider the simply supported thin beam model sketched in the figure, with overall length l, and loaded by the piece-wise constant force per unit of length a. Compute the rotation $\theta(l/2)$ in the middle of the beam. Neglect shear deformability.

(Unit for result: rad)

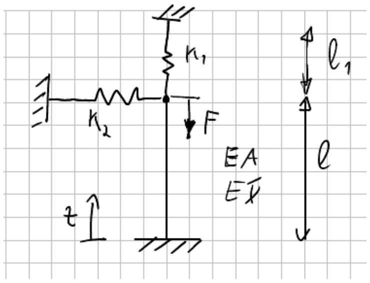


Data
$a = 1000$ N/mm
$l = 2000$ mm
$EI = 12 \times 10^{10}$ N mm^2

Answer

Exercise 5
The clamped thin beam sketched in the figure, of length l, is loaded by the compressive force F at its extremity. Two springs, with stiffness K_1 and K_2 are connected to the beam extremity. Compute the approximated value of critical buckling load by resorting to a suitable polynomial approximation of the transverse displacement truncated to the first non-null term.

(Unit for result: N)

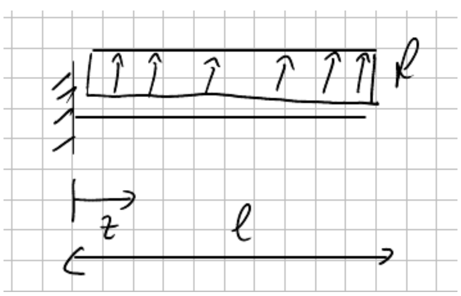


Data
$l = 2000$ mm
$l_1 = 1000$ mm
$EA = 6 \times 10^{10}$ N
$EI = 12 \times 10^{10}$ N mm^2
$K_1 = 1 \times 10^7$ N/mm
$K_2 = 1$ N/mm

Answer ________________________________

Exercise 6
The clamped beam sketched in the figure has a varying bending stiffness $EI(z) = a + bz$, and is loaded by the constant force per unit of length f. By resorting to a polynomial approximation of the transverse displacement, truncated to the first non-null term, estimate the vertical displacement $v(l)$ at the free extremity of the beam.

(Unit for result: mm)



Data
$l = 4000$ mm
$f = 10$ N/mm
$a = 6 \times 10^{12}$ N mm^2
$b = 2.5 \times 10^9$ N mm

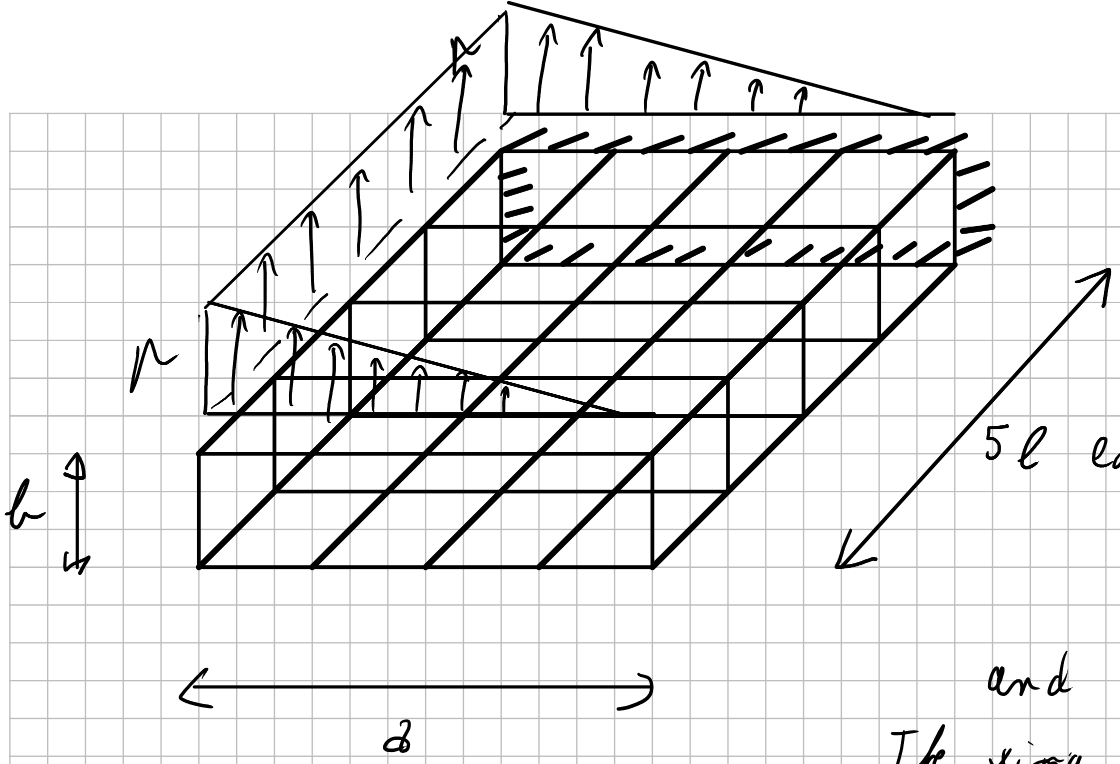
Answer ________________________________

True/False Questions
(Put a T (true) or F (false) at the end of the sentence)

1. "Differential bending" is a phenomenon that appears when the compressed panels of a wing are buckled, so that their bending stiffness contribution is different from that of the panels that remain stable
2. You can neglect shear deformability for beams if the number $EI/(l^2EA)$ is large, where EI is the bending stiffness, l is the beam length, A is the cross-section area and E is the material elastic modulus.
3. You need to use Hermitian shape functions for the displacement field when building a Timoshenko beam finite element

Multiple Choice questions
(Circle the correct answer)

1. $\sigma_{zz} = E\varepsilon_{zz}$ is:
   (a) wrong
   (b) correct for plates
   (c) correct for a state of axial stress
   (d) correct for a state of plane stress
   (e) correct for a state of plane strain
   (f) none of the above
2. The shear flux in a thin wing panel is equal to:
   (a) the average shear stress divided by the panel thickness
   (b) the average shear stress multiplied by the panel thickness
   (c) the average shear stress
   (d) the derivative, with respect to the chord direction, of the axial stress in the panel
   (e) none of the above
3. The shear stress transmitted by a glued connection is:
   (a) higher at the extremities
   (b) lower at the extremities
   (c) constant
   (d) described by a sin function
   (e) described by a cos function
   (f) described by a cubic polynomial function
   (g) none of the above



The semi-monocoque wing box model in the figure has six equispaced ribs, ten stringers each with a concentrated area A (five equispaced on the upper side, five equispaced on the lower side) and panels with thickness t.

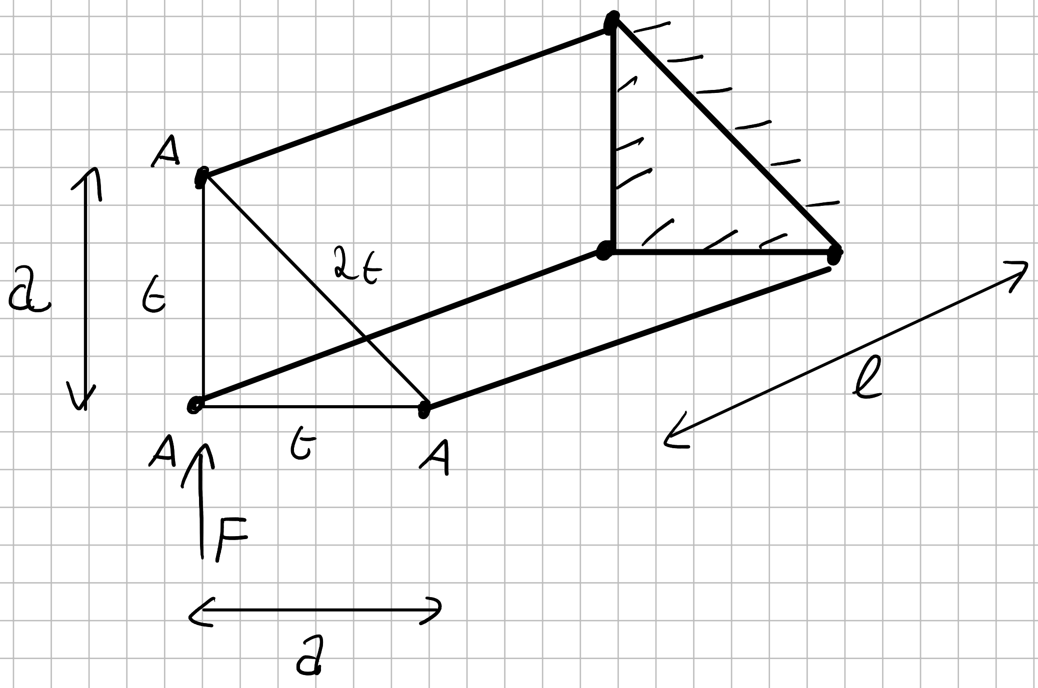
The wing box is loaded by a pressure load constant along the wing span and with a triangular distribution in chord.

Compute the jump $\Delta\sigma_{zz}$ of axial stress in the stringer indicated by the arrow, across the middle rib.

$6l = 3\ m$ $\quad a = 1\ m$ $\quad b = 10\ cm$

$t = 1\ mm$ $\quad A = 10\ cm^2$ $\quad p = 1000\ Pa$

$\Delta\sigma_{zz} = 0\ MPa$



The semi-monocoque beam model sketched in the figure has a triangular cross-section with three concentrated areas A, two panels of thickness t and one panel of thickness $2t$.
The beam has length l, is clamped at one extremity and loaded by the force F, aligned with the vertical panel, at the other extremity.

Compute the overall torsional rotation of the loaded section.

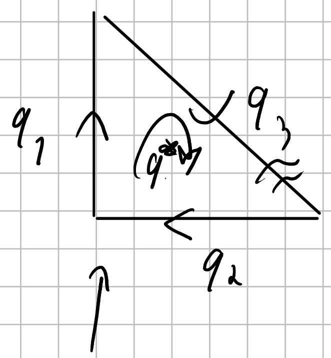
$a = 0{,}5 \ m$ $\quad A = 4 \ cm^2$ $\quad t = 1 \ mm$

$E = 70000 \ MPa$ $\quad \nu = 0{,}3$ $\quad F = 10000 \ N$

$l = 4 \ m$

$$G = \frac{E}{2(1+\nu)} = 2{,}6923\,E4 \text{ MPa}$$

$a = 500$ mm $\qquad A = 4\,E2$ mm^2 $\qquad l = 4000$ mm

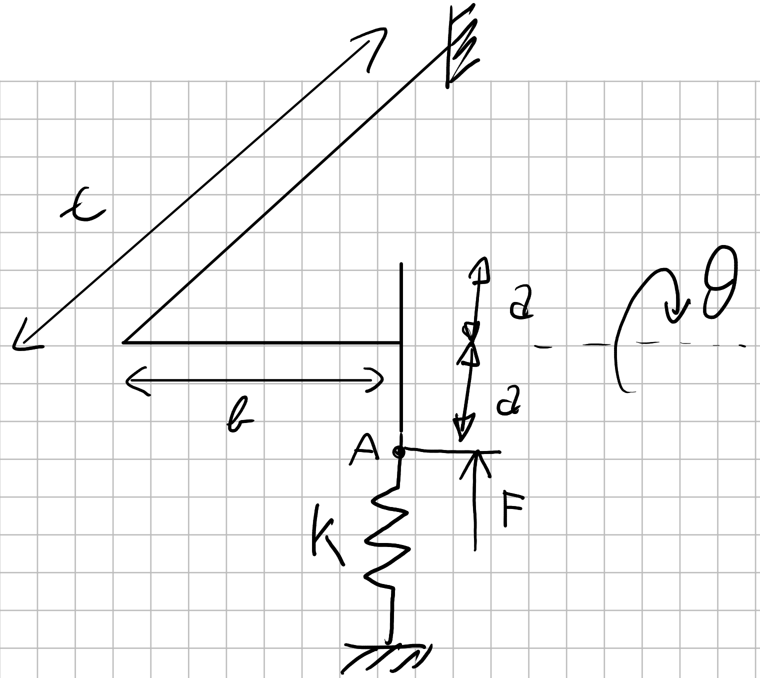


$q_1' = \frac{F}{a}$ $\qquad q_2' = \emptyset$ $\qquad q^* = \emptyset$

$q_1 = \frac{F}{a}$ $\qquad q_2 = \emptyset$ $\qquad q_3 = \emptyset$

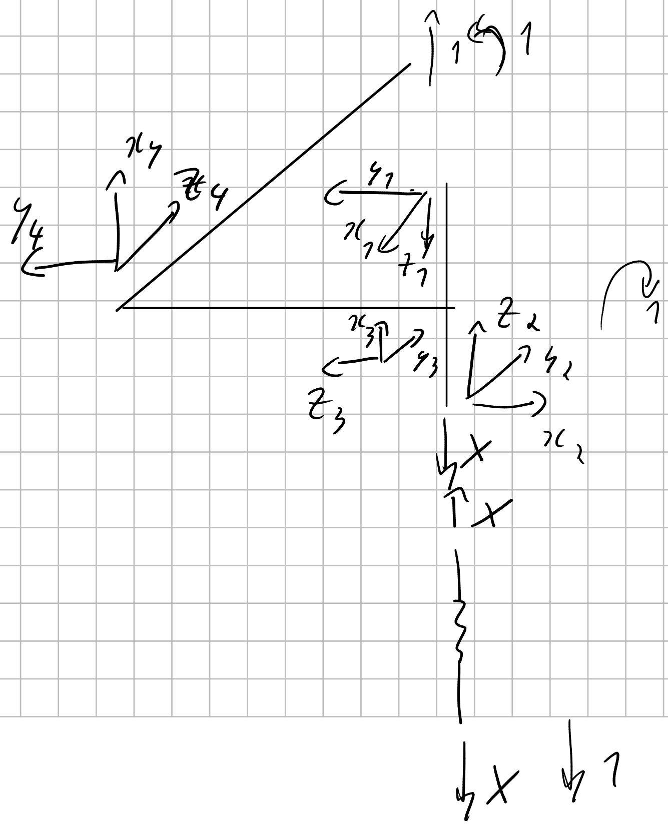
$$\theta' = \frac{1}{a^2 G}\,\frac{q_1 a}{t} = 1{,}4857\,E-6 \text{ rad mm}^{-1}$$

$$\theta = \theta' \cdot l = 5{,}9429\,E-3 \text{ rad}$$



$a = 1\ m$
$b = 2\ m$
$c = 3\ m$
$F = 10000\ N$
$k = 1\ N/mm$

$EJ = 12E14\ N\,mm^2$
$EA = 6E10\ N$
$GJ = 7E9\ N\,mm^2$



$F_{z_2} = X - F$
$M_{y_3} = (F - X)\, z_3$
$M_{y_4} = (X - F)\, z_4$
$M_{z_4} = (F - X)\, b$

1

$F'_{z_2} = 0$
$M'_{y_3} = 0$
$M'_{y_4} = -1$
$M'_{z_4} = 0$

1

$F''_{z_2} = 1$
$M''_{y_3} = -z_3$
$M''_{y_4} = z_4$
$M_{z_4} = -b$

Compute X

$$\int_0^a \frac{(X-F)}{EA}\,dz_2 + \int_0^b \frac{-(F-X)z_3^2}{EI} + \int_0^c \frac{(X-F)z_4^2}{EI} - \frac{(F-X)b^2}{GJ}\,dz_4 + \frac{X}{K} = 0$$

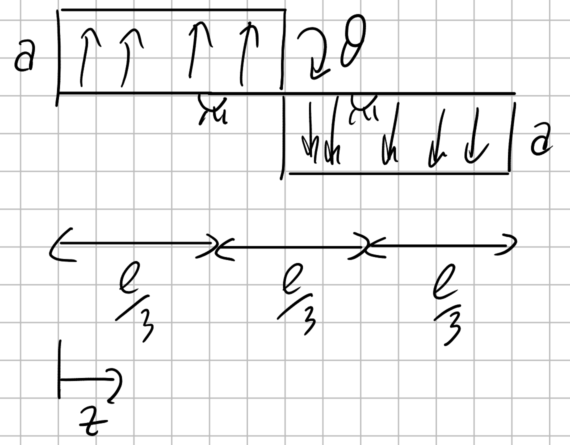
$$X\frac{a}{EA} - \frac{Fa}{EA} - \frac{1}{3}\frac{Fb^3}{EI} + X\frac{1}{3}\frac{b^3}{EI} + X\frac{1}{3}\frac{c^3}{EI} + \frac{X}{K} - \frac{1}{3}\frac{Fc^3}{EI} - \frac{Fb^2c}{GJ} + X\frac{b^2c}{GJ}$$

$$X\left(\frac{a}{EA} + \frac{1}{3}\frac{b^3}{EI} + \frac{1}{3}\frac{c^3}{EI} + \frac{b^2c}{GJ} + \frac{1}{K}\right) = \frac{Fa}{EA} + \frac{1}{3}\frac{Fb^3}{EI} + \frac{1}{3}\frac{Fc^3}{EI} + \frac{Fb^2c}{GJ}$$

$$X = 6317{,}1\ N$$

Compute θ

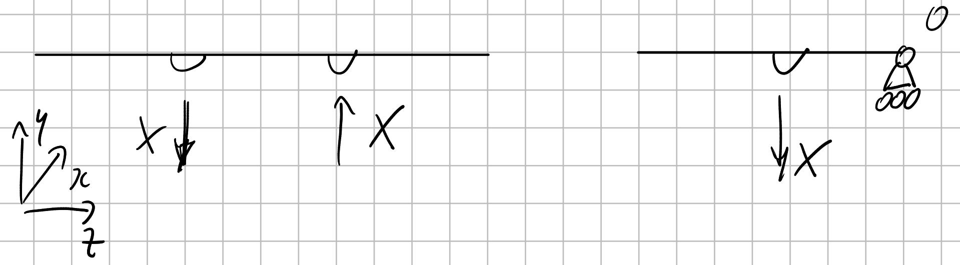
$$\int_0^c -\frac{(X-F)z_4}{EI} = \theta = -\frac{1}{2}\frac{Xc^2}{EI} + \frac{1}{2}\frac{Fc^2}{EI} = 0{,}00001381\ \text{rad}$$



$q = 1000$ N/mm

$l = 2000$ mm

$EJ = 12E10$ N mm^2

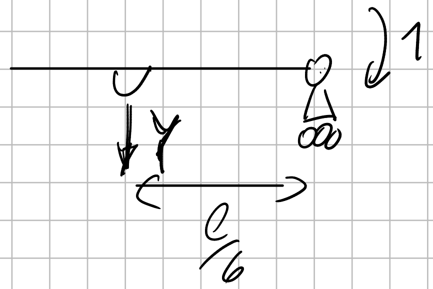


$$M_x\left(z=\frac{l}{2}\right) = q\frac{l^2}{8} - X\frac{l}{6} = 0$$

$$X = \frac{6}{8} q l = \frac{3}{4} q l$$

$$M_x = -q\frac{z^2}{2} \qquad \forall z \in \left[0, \frac{l}{3}\right]$$

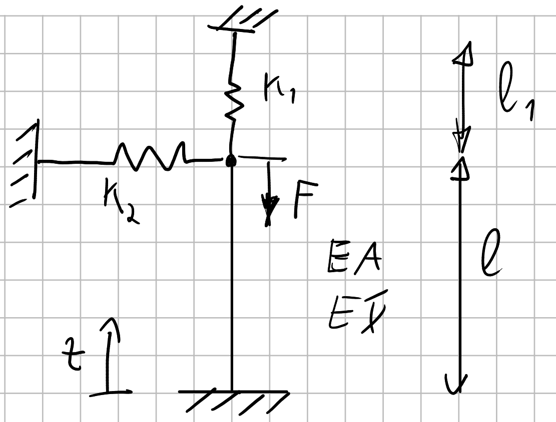
$$-\frac{q z^2}{2} + \frac{3}{4} q l\left(z - \frac{l}{3}\right) \qquad \forall z \in \left[\frac{l}{3}, \frac{l}{2}\right]$$



$$Y = \frac{6}{\ell}$$

$$M'_x = Y\left(z - \frac{\ell}{3}\right)$$

$$\int_{\ell/3}^{\ell/2} \frac{M_x \, M'_x}{E\cancel{I}}\, dz = \vartheta = \frac{7 a \ell^3}{5184} \cdot \frac{1}{E\cancel{I}} = -0{,}090021 \text{ rad}$$



$l = 2000$ mm

$l_1 = 1000$ mm

$EI = 12E10$ N mm^2

$EA = 6E10$ N

$k_1 = 1E7$ N/mm

$k_2 = 1$ N/mm

$$u = \frac{F}{k_1 + \frac{EA}{l}}$$

$$P_{EA} = \frac{EA/l}{k_1 + \frac{EA}{l}} F = A \cdot F \quad \text{with } A = \frac{EA/l}{k_1 + \frac{EA}{l}}$$

$$F_{k_1} = \frac{k_1}{k_1 + \frac{EA}{l}} = B \cdot F \quad \text{with } B = \frac{k_1}{k_1 + \frac{EA}{l}}$$

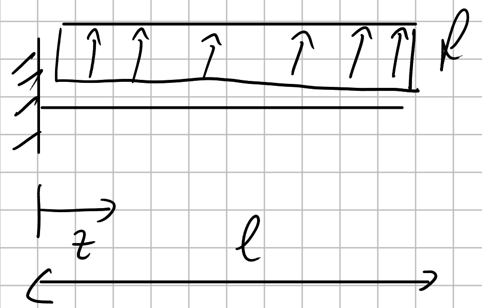
$$w = a z^2 \qquad w'' = 2a \qquad w(l) = a l^2 \qquad w' = 2az$$

$$\int_0^l \delta w'' EI w'' - \delta w' P_{EA} w' \, dz + \delta w(l) k_2 w(l) + \delta w(l) \frac{F_{k_1}}{l_1} w(l) = 0$$

$$\delta a \left(4EIl - \frac{4}{3} l^3 A F + k_2 l^4 + \frac{BF}{l_1} l^4 \right) a = 0$$

$$F\left(-\frac{4}{3} l^3 A + B \frac{l^4}{l_1} \right) = -4EIl - k_2 l^4$$

$$F = \frac{-4E\cancel{I}\ell - K_2 \ell^4}{-\frac{4}{3}\ell^3 A + B\frac{\ell^4}{\ell_1}} = 2.44E5 \qquad N$$



$p = 10 \quad N/mm$ $\qquad a = 6E12 \; Nmm^2$

$EI = a + bz$ $\qquad b = 2{,}5E9 \; Nmm$

$l = 4000 \quad mm$

1-term polynomial approximation.

$v(l) = ?$

$v = c z^2$

$v'' = 2c \qquad \delta v'' = 2\delta c \qquad \delta v = \delta c \, z^2$

$$\int_0^l \delta c \, (4a + 4bz)\, c \, dz = \int_0^l \delta c \, z^2 p \, dz$$

$$(4al + 2bl^2)\, c = \frac{1}{3} l^3 p \qquad c = \left(\frac{1}{3} l^3 p\right) / \left(4al + 2bl^2\right)$$

$$v(l) = c\, l^2 = 19{,}394 \quad mm$$

EXAM 13/02/2024

Ex 1

$6\ell = 3000\ mm$

$a = 1000\ mm$

$b = 100\ mm$

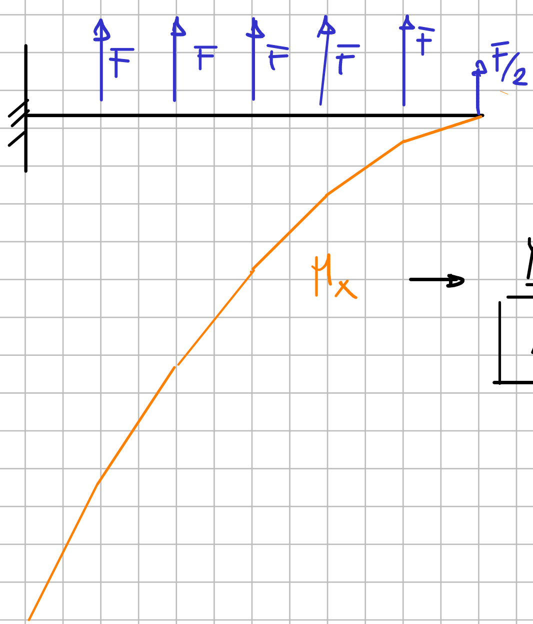
$t = 1\ mm$

$A = 1000\ mm^2$

$p = 1 \cdot 10^{-3}\ MPa$

SOL

MOM DISTRIBUTION

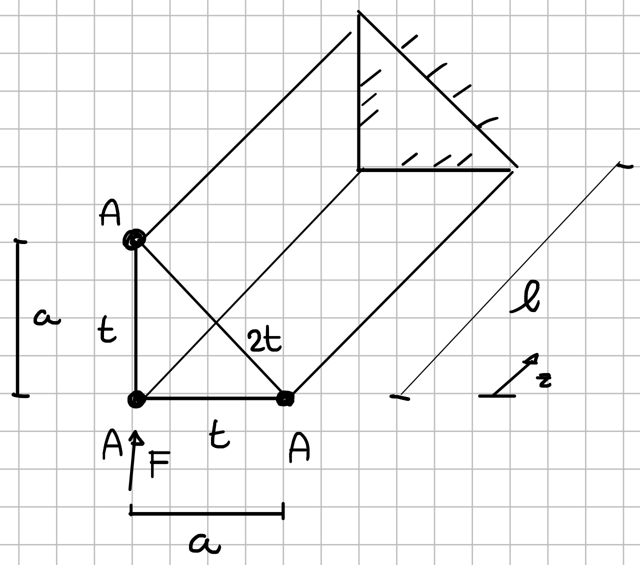


WHERE $F = \frac{1}{2} p \cdot a \cdot a$

$\rightarrow$ $\Delta M_x = 0$

$\boxed{\Delta \sigma_{xx} = 0}$

Ex 2



DATA

$a = 500\ mm$

$A = 400\ mm^2$

$t = 1\ mm$

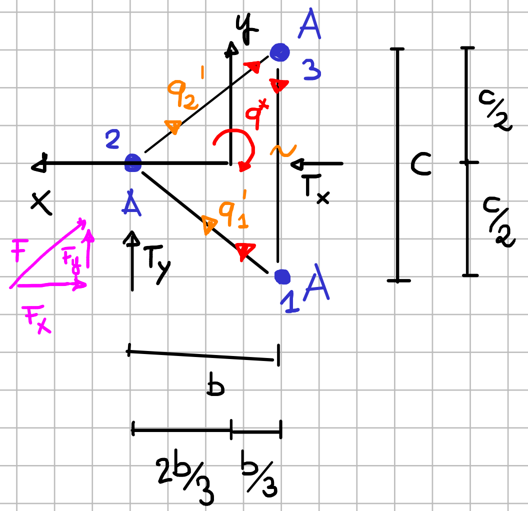
$\ell = 4000\ mm$

$E = 70000\ MPa$

$\nu = 0.3$

$F = 10000\ N$

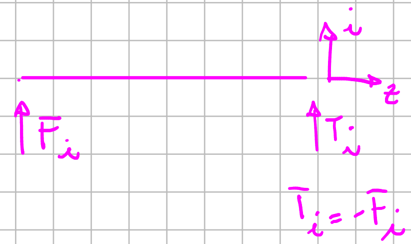
SOL



$$T_y = -\frac{\sqrt{2}}{2} F = -T_x$$

$$b = \frac{\sqrt{2}}{2} a$$

$$c = 2b$$



$$J_{xx} = 2Ab^2$$

$$J_{yy} = 2A\frac{b^2}{9} + A\frac{4b^2}{9} = \frac{2}{3}Ab^2$$

$$S_{x_1} = -S_{x_3} = -A\frac{c}{2} = -Ab \qquad S_{x_2} = 0$$

$$S_{y_1} = S_{y_3} = -A\frac{b}{3} \qquad S_{y_2} = +2Ab/3$$

$$\rightsquigarrow q_1' = -T_y\frac{S_{x_1}}{J_{xx}} - T_x\frac{S_{y_1}}{J_{yy}}$$

$$q_2' = T_y\frac{S_{x_3}}{J_{xx}} + T_x\frac{S_{x_3}}{J_{yy}}$$

MOM. EQ. $0 = 2\Omega_{CELL}\, q^*$ WHERE $\Omega_{CELL} = \frac{1}{2} c \cdot b$

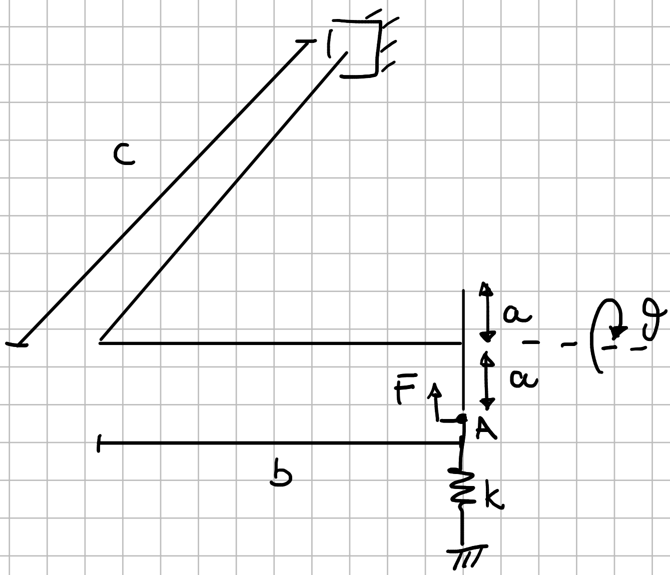
ROT. $\vartheta' = \frac{1}{2\Omega_{CELL} G} \cdot \left(\frac{q^* \cdot c}{2t} + \frac{q_1' \cdot a}{t} + \frac{q_2' \cdot a}{t} \right)$ WITH $G = \frac{E}{2(1+\nu)}$

$$\vartheta = \underline{-\left(\vartheta' \cdot \ell\right)} = +0.059 \text{ rad}$$

$$\hookrightarrow \vartheta = \int_{\ell}^{0} \vartheta' \, dz + \cancel{\vartheta(\ell)}$$

$$= -\vartheta' \cdot \ell$$

Es 3



DATA

$a = 1000\ mm$

$b = 2000\ mm$

$c = 3000\ mm$

$EJ_{xx} = EJ_{yy} = 12 \cdot 10^{14}\ Nmm^2$

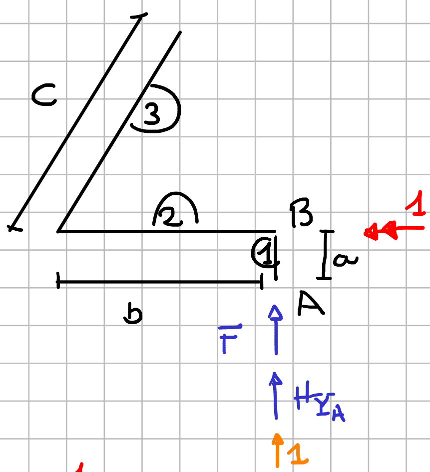
$EA = 6 \cdot 10^{10}\ N$

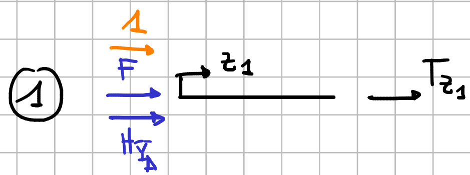
$GJ = 7 \cdot 10^{9}\ Nmm^2$

$k = 1\ N/mm$

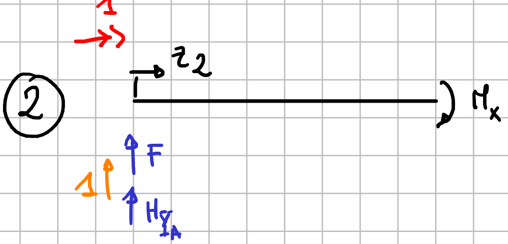
$F = 10000\ N$

SOL



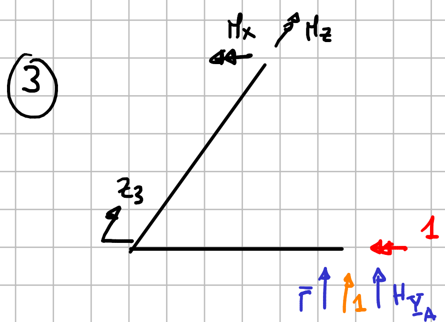


$$\begin{cases} T_{z_1} = -F - H_{Y_A} \\ T'_{z_1} = -1 \\ T''_{z_1} = 0 \end{cases}$$



$M_x = (-F - H_{Y_A})\, z_2$

$M'_x = (-1) \cdot z_2$



$M_{z_3} = (F + H_{Y_A})\, b$

$M'_{z_3} = (1)\, b$

$M_{x_3} = (-F - H_{Y_A}) \cdot z_3$

$M'_{x_3} = (-1)\, z_3$

$M''_{x_3} = -1$

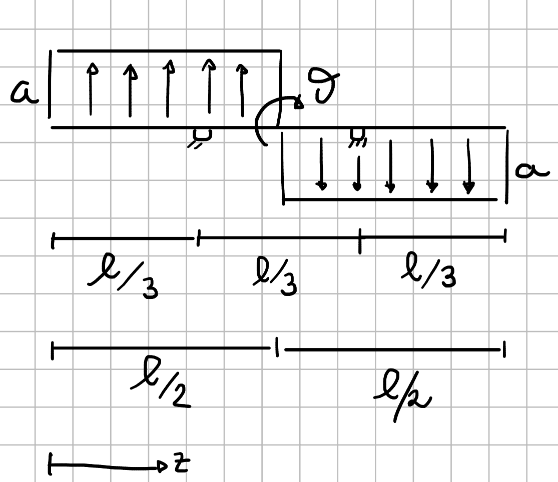
PCVW

$$\delta W_e' = \begin{bmatrix} -1 \cdot \frac{M_{YA}}{K} \\ 1 \cdot \vartheta \end{bmatrix}$$

$$\delta W_i' = \begin{bmatrix} \int_0^a T_{z_1}' \cdot \frac{T_{z_1}}{EA} dz_1 + \int_0^b M_{x_2}' \cdot \frac{M_{x_2}}{EJ} dz_2 + \int_0^c M_{z_3}' \cdot \frac{M_{z_3}}{GJ} dz_3 + \int_0^c M_{x_3}' \cdot \frac{M_{x_3}}{EJ} dz_3 \\ \int_0^c M_{x_3}'' \cdot \frac{M_{x_3}}{EJ} dz_3 \end{bmatrix}$$

SOLVE $\delta W_e' = \delta W_i'$ FOR $\boxed{\vartheta = 1.3816 \cdot 10^{-5} \text{ rad}}$

Ex 4

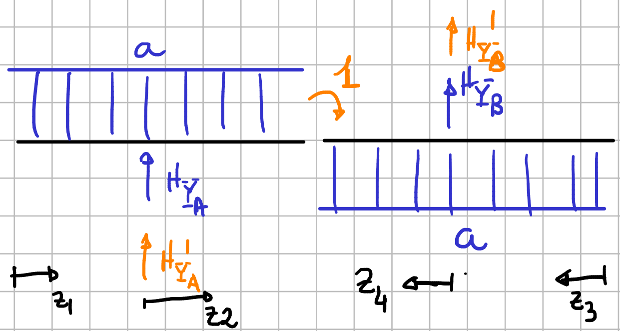


$a = 1000\ N/mm$

$l = 2000\ mm$

$EJ = 12 \cdot 10^{10}\ N\,mm^2$

SOL



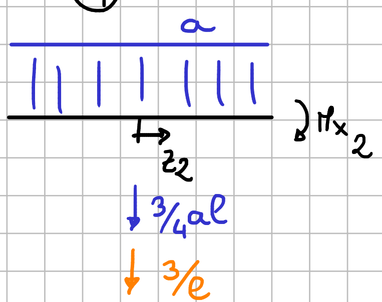
GLOBAL EQUILIBRIUM:

$$\begin{cases} H_{Y_A} = -H_{Y_B} \rightarrow H_{Y_A} = -\frac{3}{4}al \\ 2H_{Y_B} = \dfrac{2a \cdot \frac{l}{2} \cdot \frac{l}{4}}{l/6} = \dfrac{3}{4}al \end{cases}$$

$$\begin{cases} H'_{Y_A} = -H'_{Y_B} \\ 1 = 2H'_{Y_B} \cdot \dfrac{l}{6} \end{cases} \Rightarrow \quad H'_{Y_A} = -3/l \qquad H'_{Y_B} = \frac{3}{l}$$

① − ③ DUMMY UNLOADED

② = − ④



$$M_{x_2} = -M_{x_4} = -\left(a \cdot \left(\frac{l}{3} + z_2\right)\right) \cdot \left(\frac{l}{3} + z_2\right) \cdot \frac{1}{2} + \frac{3}{4}al \cdot z_2$$

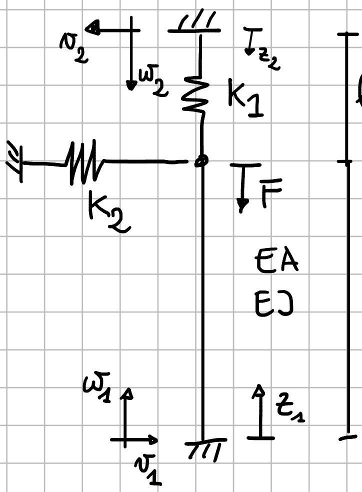
$$M'_{x_2} = -M'_{x_4} = \frac{3}{l} \cdot z_2$$

PCVW

$$\vartheta = 2\int_0^{l/6} M'_{x_2} \cdot \frac{M_{x_2}}{EJ}\, dz_2 = 0.09\ rad$$

CONTRIBUTE FROM ② & ④

Ex 5



$\ell = 2000$ mm $\qquad \ell_1 = 1000$ mm

$EA = 6 \cdot 10^{10}$ N $\qquad k_1 = 1 \cdot 10^{7}$ N/mm

$EJ = 12 \cdot 10^{10}$ N mm^2

$k_2 = 1$ N/mm

SOL TAKE AS KNOWN THE WHOLE DERIVATION IN 02|2023 RESULTS

TAKE LINEAR $w_i \rightarrow w_1(z_1) = a_{w_1} z_1$
$w_2(z_2) = a_{w_2} z_2$

$BC' \rightarrow a_{w_1} = -a_{w_2} \frac{\ell_1}{\ell} = a_w$

TAKE QUADRATIC $v_1 \rightarrow v_1(z_1) = a_1 z_1^2$

ITS A SPRING $\rightsquigarrow$ LINEAR $v_2 \rightarrow v_2(z_2) = a_2 z_2$

$BC \rightarrow a_1 = -a_2 \frac{\ell_1}{\ell^2} = a$

$$\delta W_i = \int_0^{\ell} \delta w_{1/z_1} \cdot N dz_1 + \int_0^{\ell_1} \delta w_{2/z_2} N_{k_1} dz_2 + \int_0^{\ell} \left(\delta v_{1/z_1} v_{1/z_1} N + \delta v_{1/z_1 z_1} EJ v_{1/z_1 z_1} \right) dz_1 + \int_0^{\ell_1} \delta v_{2/z_2} v_{2/z_2} N_{k_1} dz_2$$

$$\delta W_e = -\delta w_1(\ell) F - \delta v_1(\ell) \cdot k_2 v_1(\ell)$$

WHERE $N_{k_1} = +k_1 w_2(\ell_1) = -k_1 a_w \ell$ $\qquad N = EA w_{1/z_1} = EA a_w$

$\delta w_{1/z_1} = \delta a_w$ $\qquad \delta v_{1/z_1 z_1} = 2 \delta a$

$\delta w_{2/z_2} = +\delta a_{w_2} = -\delta a_w \cdot \frac{\ell}{\ell_1}$ $\qquad v_{1/z_1 z_1} = 2a$

$\delta v_{1/z_1} = 2 \delta a z_1$ $\qquad \delta v_{2/z_2} = \delta a_2 = -\delta a \frac{\ell^2}{\ell_1}$

$v_{1/z_1} = 2a z_1$ $\qquad v_{2/z_2} = a_2 = -a \frac{\ell^2}{\ell_1}$

$w_{1/z_1} = a_w$ $\qquad \delta w_1(\ell) = \delta a_w \cdot \ell$

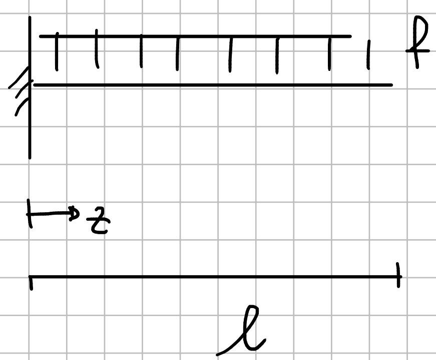
$\delta v_1(\ell) = \delta a \cdot \ell^2$ $\qquad v_1(\ell) = a\ell$

- $$\delta a_w \left(\int_0^{l} EA a_w \, dz_1 + \int_0^{l_1} \left(-\frac{l}{l_1} \right) \cdot \left(+k_1 \cdot \left(-a_w \cdot \frac{l}{l_1} \cdot l_1 \right) \right) dz_2 \right) = -\delta a_w \cdot F \cdot l \longrightarrow$$ FIND a_w (ALSO USING SPRING APPROX)

- $$\delta a \left(\int_0^{l} 2z_1 \, 2az_1 \, EA a_w + 2EJ \, 2a \; dz_1 + \int_0^{l_1} \left(-\frac{l^2}{l_1} \right) \cdot a \left(-\frac{l^2}{l_1} \right) \cdot \left(-k_1 a_w \cdot l \right) dz_2 = -\delta a \cdot l^2 k_2 a l^2$$

$\longrightarrow a = 0$ | $\boxed{F = 244000 \text{ N}}$

Ex 6



DATA

$\ell = 4000$ mm

$f = 10\ N/mm$

$a = 6 \cdot 10^{12}\ N\,mm^2$

$b = 2.5 \cdot 10^{9}\ N\,mm$

$EJ = a + bz$

SOL $\quad v = A z^2 \qquad v_{/zz} = 2A$

$$\delta W_i = \int_0^{\ell} \delta v_{/zz}\, EJ\, v_{/zz}\, dz$$

$$\delta W_e = \int_0^{\ell} \delta v\, f(z)\, dz \qquad \text{WITH } f(z) = f$$

PCVW

$$\cancel{\delta A} \int_0^{\ell} 2EJ\, 2A\, dz = \cancel{\delta A} \int_0^{\ell} z^2 \cdot f(z)\, dz$$

$$\int_0^{\ell} 4Aa\, dz + \int_0^{\ell} 4Abz\, dz - \int_0^{\ell} z^2 \cdot f\, dz = 0$$

$$4Aa\ell + \frac{4}{2} Ab\ell^2 = \frac{1}{3} f\ell^3$$

$$\rightarrow A = \left(\frac{1}{3} f\ell^3\right) \cdot \left(4a\ell + 2b\ell^2\right)^{-1}$$

$$v(\ell) = A \cdot \ell^2 = 19.394 \text{ mm}$$

True/False Questions

(Put a T (true) or F (false) at the end of the sentence)

1. "Differential bending" is a phenomenon that appears when the compressed panels of a wing are buckled, so that their bending stiffness contribution is different from that of the panels that remain stable

   - False

2. You can neglect shear deformability for beams if the number $EI/(l^2 EA)$is large, where EI is the bending stiffness, l is the beam length, A is the cross-section area and E is the material elastic modulus.

   - False

3. You need to use Hermitian shape functions for the displacement field when building a Timoshenko beam finite element

   - False

Multiple Choice questions

(Circle the correct answer)

1. $\sigma_{zz} = E\varepsilon_{zz}$ is:

   (a) wrong
   (b) correct for plates
   (c) correct for a state of axial stress
   (d) correct for a state of plane stress
   (e) correct for a state of plane strain
   (f) non of the above

2. The shear flux in a thin wing panel is equal to:

   (a) the average shear stress divided by the panel thickness
   (b) the average shear stress multiplied by the panel thickness
   (c) the average shear stress
   (d) the derivative, with respect to the chord direction, of the axial stress in the panel
   (e) none of the above

3. The shear stress transmitted by a glued connection is:

   (a) higher at the extremities
   (b) lower at the extremities
   (c) constant
   (d) described by a sin function
   (e) described by a cos function
   (f) described by a cubic polynomial function
   (g) none of the above