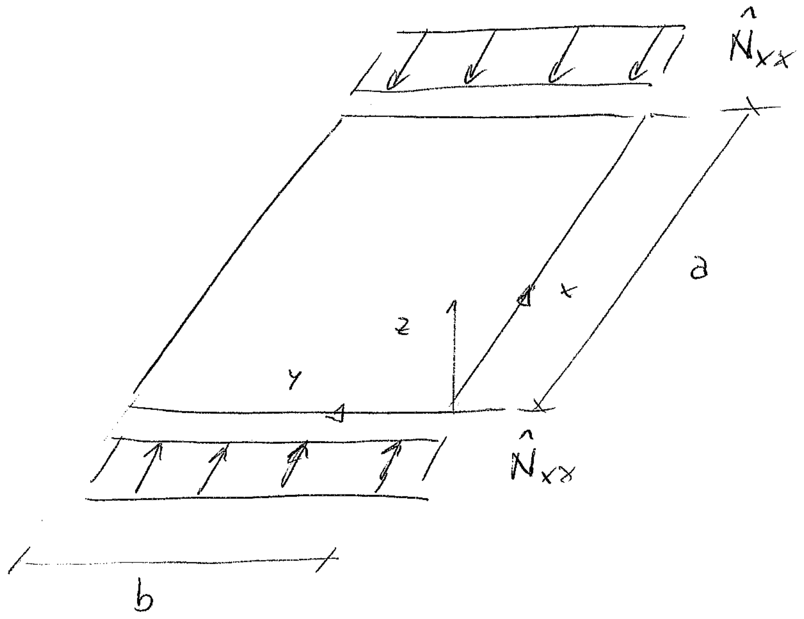
BUCKLING OF THIN PLATES

Consider a thin plate subjected to axial compression (the analysis can be extended to the case of multiaxial loading conditions). Goal of this section is to derive the equilibrium and buckling equations by means of the energy approach used for the Euler column.



The strain tensor components in the Donnell von Karman formulation are:

$$\epsilon_{ik} = \frac{1}{2}\left(u_{i/k} + u_{k/i} + w_{/i}\, w_{/k}\right) - w\, b_{ik} - z\, w_{/ik}$$

(for flat plates $b_{ik} = 0$)

$$\epsilon_{ik} = \varepsilon_{ik} + z\, k_{ik}$$

with $$\varepsilon_{ik} = \frac{1}{2}\left(u_{i/k} + u_{k/i} + w_{/i}\, w_{/k}\right) - w\, b_{ik}$$

$$k_{ik} = -\, w_{/ik}$$

$$\Pi = U + V$$

with $U = \frac{1}{2}\int_V \epsilon_{ik}\,\sigma_{ik}\, dV$

and $V = \hat{N}_x b\left(u(a) - u(0)\right)$

Recalling that $\epsilon_{ik} = \varepsilon_{ik} + z\, k_{ik}$, the strain energy reads:

$$U = \frac{1}{2}\int_V \left(\varepsilon_{ik}\,\sigma_{ik} + k_{ik}\, z\,\sigma_{ik}\right) dV$$

$$= \frac{1}{2}\int_A \left(\varepsilon_{ik} N_{ik} + k_{ik} M_{ik}\right) dA$$

with $N_{ik} = \int_t \sigma_{ik}\, dz$; $M_{ik} = \int_t z\,\sigma_{ik}\, dz$

Thus the total potential energy is:

$$\boxed{\Pi = \frac{1}{2}\int_A \left(\varepsilon_{ik} N_{ik} + k_{ik} M_{ik}\right) dA + \hat{N}_x b\left(u(a) - u(0)\right)}$$

Introduce now a perturbation:

$u \rightarrow u + \delta u$

$v \rightarrow v + \delta v$

$w \rightarrow w + \delta w$

$$\bullet \quad \gamma_{ik} + \delta\gamma_{ik} = \frac{1}{2}\left(u_{i/k} + u_{k/i} + w_{/i} w_{/k}\right) - w b_{ik}$$

$$+ \frac{1}{2}\left(\delta u_{i/k} + \delta u_{k/i} + \delta w_{/i} w_{/k} + \delta w_{/k} w_{/i}\right) - \delta w b_{ik}$$

$$+ \frac{1}{2} \delta w_{/i} \delta w_{/k}$$

$$\boxed{\delta\gamma_{ik} = \delta\gamma_{ik}^{L} + \delta\gamma_{ik}^{NL}}$$

with $\delta\gamma_{ik}^{L} = \frac{1}{2}\left(\delta u_{i/k} + \delta u_{k/i} + \delta w_{/i} w_{/k} + \delta w_{/k} w_{/i}\right) - \delta w b_{ik}$

$$\delta\gamma_{ik}^{NL} = \frac{1}{2} \delta w_{/i} \delta w_{/k}$$

$$\bullet \quad N_{ik} + \delta N_{ik} = A_{ikrs}\left(\gamma_{rs} + \delta\gamma_{rs}\right)$$

$$= A_{ikrs}\left(\gamma_{rs} + \delta\gamma_{rs}^{L} + \delta\gamma_{rs}^{NL}\right) \quad *$$

$$\boxed{\delta N_{ik} = \delta N_{ik}^{L} + \delta N_{ik}^{NL}}$$

with $\delta N_{ik}^{L} = A_{ikrs} \delta\gamma_{rs}^{L}$

$$\delta N_{ik}^{NL} = A_{ikrs} \delta\gamma_{rs}^{NL}$$

$$\bullet \quad \kappa_{ik} + \delta\kappa_{ik} = -w_{/ik} - \delta w_{/ik}$$

$$\boxed{\delta\kappa_{ik} = -\delta w_{/ik}}$$

$$\bullet \quad M_{ik} + \delta M_{ik} = D_{ikrs}\left(\kappa_{rs} + \delta\kappa_{rs}\right)$$

$$\boxed{\delta M_{ik} = D_{ikrs}\,\delta\kappa_{rs} = -D_{ikrs}\,\delta w_{/rs}} \quad *$$

* Assuming no coupling between in-plane and out-of-plane behaviour

$$\Pi + \Delta\Pi = \frac{1}{2}\int_A \left[(\varepsilon_{ik} + \delta\varepsilon_{ik})(N_{ik} + \delta N_{ik}) + (\kappa_{ik} + \delta\kappa_{ik})(M_{ik} + \delta M_{ik}) \right] dA$$

$$+ \hat{N}_x b \left[u(a) + \delta u(a) - u(0) - \delta u(0) \right]$$

$$\Delta\Pi = \frac{1}{2}\int_A \left(\varepsilon_{ik}\, \delta N_{ik} + \delta\varepsilon_{ik}\, N_{ik} + \delta\varepsilon_{ik}\, \delta N_{ik} + \kappa_{ik}\, \delta M_{ik} + \right.$$

$$\left. + \delta\kappa_{ik}\, M_{ik} + \delta\kappa_{ik}\, \delta M_{ik} \right) dA + \hat{N}_x \left[\delta u(a) - \delta u(0) \right]$$

$$\Delta\Pi = \frac{1}{2}\int_A \left[\varepsilon_{ik}\left(\delta N_{ik}^{L} + \delta N_{ik}^{NL}\right) + \left(\delta\varepsilon_{ik}^{L} + \delta\varepsilon_{ik}^{NL}\right) N_{ik} + \right.$$

$$\left(\delta\varepsilon_{ik}^{L} + \delta\varepsilon_{ik}^{NL}\right)\left(\delta N_{ik}^{L} + \delta N_{ik}^{NL}\right) + \kappa_{ik}\, \delta M_{ik} +$$

$$\left. + \delta\kappa_{ik}\, M_{ik} + \delta\kappa_{ik}\, \delta M_{ik} \right] dA + \hat{N}_x \left[\delta u(a) - \delta u(0) \right]$$

$$= \frac{1}{2}\int_A \left(\delta N_{ik}^{L}\, \varepsilon_{ik} + \delta N_{ik}^{NL}\, \varepsilon_{ik} + \delta\varepsilon_{ik}^{L}\, N_{ik} + \delta\varepsilon_{ik}^{NL}\, N_{ik} + \right.$$

$$+ \delta\varepsilon_{ik}^{L}\, \delta N_{ik}^{L} + \boxed{\cancel{\delta\varepsilon_{ik}^{L}\, \delta N_{ik}^{NL}}} + \boxed{\cancel{\delta\varepsilon_{ik}^{NL}\, \delta N_{ik}^{L}}} + \boxed{\cancel{\delta\varepsilon_{ik}^{NL}\, \delta N_{ik}^{NL}}}$$

$$\left. + \kappa_{ik}\, \delta M_{ik} + \delta\kappa_{ik}\, M_{ik} + \delta\kappa_{ik}\, \delta M_{ik} \right) dA +$$

$$+ \hat{N}_x \left[\delta u(a) - \delta u(0) \right]$$

$\boxed{}$: third and fourth order contributions

$$\Delta\Pi = \frac{1}{2}\int_A \left(\delta N_{ik}^L \varepsilon_{ik} + \delta\varepsilon_{ik}^L N_{ik} + \delta M_{ik} k_{ik} + \delta k_{ik} M_{ik} \right) dA +$$

$$+ \hat{N}_x b \left[\delta u(a) - \delta u(0) \right]$$

$$+ \frac{1}{2}\int_A \left(\delta N_{ik}^{NL} \varepsilon_{ik} + \delta\varepsilon_{ik}^{NL} N_{ik} + \delta\varepsilon_{ik}^L \delta N_{ik}^L + \delta k_{ik} \delta M_{ik} \right) dA$$

which is in the form $\Delta\Pi = \delta\Pi + \frac{1}{2!}\delta^2\Pi$

<u>Equilibrium equations</u>

$\delta\Pi = 0$

Note that: $\delta N_{ik}^L \varepsilon_{ik} = A_{ikrs}\, \delta\varepsilon_{rs}^L \varepsilon_{ik}$

$= A_{ikrs}\, \delta\varepsilon_{ik}^L \varepsilon_{rs}$ (symmetry of A_{ikrs})

$= \delta\varepsilon_{ik}^L N_{ik}$

and: $\delta M_{ik} k_{ik} = D_{ikrs}\, \delta k_{rs} k_{ik}$

$= D_{ikrs}\, \delta k_{ik} k_{rs}$ (symmetry of D_{ikrs})

$= \delta k_{ik} M_{ik}$

$$\delta\Pi = \int_A \left(\delta\varepsilon_{ik}^L N_{ik} + \delta k_{ik} M_{ik} \right) dA + \hat{N}_x b \left[\delta u(a) - \delta u(0) \right]$$

but: $\delta\varepsilon_{ik}^L = \frac{1}{2}\left(\delta u_{i/k} + \delta u_{k/i} + \delta w_{/i} w_{/k} + \delta w_{/k} w_{/i} \right) - \delta w\, b_{ik}$

$\delta k_{ik} = - \delta w_{/ik}$

$$\delta\Pi = \int_A \Big\{ \Big[\frac{1}{2}\left(\delta u_{i/k} + \delta u_{k/i} + \delta w_{/i} w_{/k} + \delta w_{/k} w_{/i} \right) - \delta w\, b_{ik} \Big] N_{ik} +$$

$$- \delta w_{/ik} M_{ik} \Big\} dA + \hat{N}_x b \left[\delta u(a) - \delta u(0) \right]$$

and observing that $N_{ik} = N_{ki}$

$$\delta\Pi = \int_A \Big[\left(\delta u_{i/k} + \delta w_{/k} w_{/i} - \delta w\, b_{ik} \right) N_{ik} - \delta w_{/ik} M_{ik} \Big] dA +$$

$$+ \hat{N}_x b \left[\delta u(a) - \delta u(0) \right]$$

integrating by parts:

$$= - \int_A \Big[\delta u_i\, N_{ik/k} + \delta w \left(N_{ik} w_{/i} \right)_{/k} + \delta w\, b_{ik} N_{ik} + \delta w\, M_{ik/ik} \Big] dA$$

$$+ \text{boundary terms}$$

The equilibrium equations are then.

$$\begin{cases} N_{ik/k} = 0 \\ M_{ik/ik} + \left(N_{ik} w_{/i} \right)_{/k} + b_{ik} N_{ik} = 0 \end{cases}$$

$$\begin{cases} N_{ik/k} = 0 \\ M_{ik/ik} + \cancel{N_{ik/k}} w_{/i} + N_{ik} w_{/ik} + b_{ik} N_{ik} = 0 \end{cases}$$

$$\boxed{\begin{cases} N_{ik/k} = 0 \\ M_{ik/ik} + N_{ik} w_{/ik} + b_{ik} N_{ik} = 0 \end{cases}}$$

or, in their extended form:

$$
\boxed{\begin{cases}
N_{xx/x} + N_{xy/y} = 0 \\
N_{xy/x} + N_{yy/y} = 0 \\
M_{xx/xx} + 2M_{xy/xy} + M_{yy/yy} + N_{xx}\,w_{/xx} + 2N_{xy}\,w_{/xy} + N_{yy}\,w_{/yy} + \\
\quad b_{xx}N_{xx} + b_{yy}N_{yy} = 0
\end{cases}}
$$

Remarks

1. The equilibrium equations are, in general, non linear equations. Often the pre-buckling analysis is performed by considering their linearized counterpart but, in principle, the equilibrium condition can be found by solving the non linear problem

2. In the non linear case, the in-plane and out-of-plane response are coupled.
   In the linearized case, the coupling between in-plane and out-of-plane behaviour is due to the curvature.
   For flat panels ($b_{ik} = 0$) the two responses are uncoupled.

Buckling equations

Consider the Trefftz criterion $\delta\left(\delta^2\Pi\big|_{@\,equil}\right)=0$

$$\frac{1}{2}\delta^2\Pi=\frac{1}{2}\int_A\left(\delta N_{ik}^{NL}\,\varepsilon_{ik}+\delta\varepsilon_{ik}^{NL}\,N_{ik}+\delta\varepsilon_{ik}^{L}\delta N_{ik}^{L}+\delta\kappa_{ik}\delta M_{ik}\right)dA$$

Note that:
$$\delta N_{ik}^{NL}\,\varepsilon_{ik}=A_{ikrs}\,\delta\varepsilon_{rs}^{NL}\,\varepsilon_{ik}$$
$$=A_{ikrs}\,\delta\varepsilon_{ik}^{NL}\,\varepsilon_{rs}\quad(\text{symmetry of } A_{ikrs})$$
$$=\delta\varepsilon_{ik}^{NL}\,N_{ik}$$

$$\frac{1}{2}\delta^2\Pi=\frac{1}{2}\int_A\left(2\,\delta\varepsilon_{ik}^{NL}\,N_{ik}+\delta\varepsilon_{ik}^{L}\delta N_{ik}^{L}+\delta\kappa_{ik}\,\delta M_{ik}\right)dA$$

$$\delta\varepsilon_{ik}^{L}=\frac{1}{2}\left(\delta u_{i/k}+\delta u_{k/i}+w_{/i}\,\delta w_{/k}+w_{/k}\,\delta w_{/i}\right)-\delta w\,b_{ik}$$

$$\delta\varepsilon_{ik}^{NL}=\frac{1}{2}\delta w_{/i}\,\delta w_{/k}$$

$$\frac{1}{2}\delta^2\Pi\Big|_{@\,equil.}=\frac{1}{2}\int_A\Big[\delta w_{/i}\,\delta w_{/k}\,\bar{N}_{ik}+\left(\delta u_{i/k}+\bar{w}_{/i}\,\delta w_{/k}-\delta w\,b_{ik}\right)\delta N_{ik}^{L}$$
$$-\delta w_{/ik}\,\delta M_{ik}\Big]\,dA$$

To take the first variation of $\delta^2\Pi$, operate the following replacement:

$\delta w\to w$

$\delta u\to u$ — However w,u,N,M should be still

$\delta N\to N$ — intended as variations!

$\delta M\to M$

$$\frac{1}{2}\delta^2\Pi\Big|_{@\text{ equil.}} = \frac{1}{2}\int_A \Big[w_{/i} w_{/k} \bar{N}_{ik} + \left(u_{i/k} + \bar{w}_{/i} w_{/k} - w b_{ik} \right) N_{ik}^L + $$
$$- w_{/ik} M_{ik} \Big] dA$$

Setting now $\delta\left(\delta^2\Pi\big|_{@\text{ equil.}}\right) = 0$ leads to:

$$\delta\left(\delta^2\Pi\big|_{@\text{ equil.}}\right) = \int_A \Big[\delta w_{/k} w_{/i} \bar{N}_{ik} + \left(\delta u_{i/k} + \delta w_{/k} \bar{w}_{/i} - \delta w b_{ik} \right) N_{ik}^L +$$
$$- \delta w_{/ik} M_{ik} \Big] dA = 0$$

Integrating by parts:

$$= -\int_A \Big[\delta w \left(w_{/i} \bar{N}_{ik} \right)_{/k} + \delta u_i N_{ik/k}^L + \delta w \left(N_{ik}^L \bar{w}_{/i} \right)_{/k}$$
$$+ \delta w \, b_{ik} N_{ik}^L + \delta w \, M_{ik/ik} \Big] dA$$

+ boundary terms

Thus, the buckling equations are:

$$\begin{cases} N_{ik/k}^L = 0 \\ M_{ik/ik} + \left(\bar{N}_{ik} w_{/i} + N_{ik}^L \bar{w}_{/i} \right)_{/k} + b_{ik} N_{ik}^L = 0 \end{cases}$$

$$\begin{cases} N_{ik/k}^L = 0 \\ M_{ik/ik} + \cancel{\bar{N}_{ik/k} w_{/i}} + \bar{N}_{ik} w_{/ik} + \cancel{N_{ik/k}^L \bar{w}_{/i}} + N_{ik}^L \bar{w}_{/ik} + b_{ik} N_{ik}^L = 0 \end{cases}$$

$$\boxed{\begin{cases} N_{ik/k}^L = 0 \\ M_{ik/ik} + \bar{N}_{ik} w_{/ik} + N_{ik}^L \bar{w}_{/ik} + b_{ik} N_{ik}^L = 0 \end{cases}}$$

Consider the case of

1. flat panel $\to b_{ik} = 0$
2. no pre-buckling out-of-plane displacements ($\bar{w} = 0$)

The equations become:

$$\boxed{\begin{cases} N_{ik/k}^{L} = 0 & \to \text{2 in-plane buckling equations} \\ M_{ik/ik} + \bar{N}_{ik} w_{/ik} = 0 & \to \text{1 out-of-plane buckling equation} \end{cases}}$$

Remarks

1. For isotropic, homogeneous plates or composite plates with symmetric lay-ups, the pre-buckling solution is characterized by $\bar{w} = 0$. In the presence of coupled in-plane/out-of-plane response (e.g. non-symmetric laminates), $\bar{w} \neq 0$. In this case, assuming $\bar{w} = 0$ is an assumption which is worth checking.

2. If $\bar{w} = 0$ (or can be reasonably assumed $= 0$), then the last buckling equation can be solved independently from the previous ones.

3. Although boundary conditions where not considered, it is clear from the expression of $\Delta\Pi$ that the boundary conditions are
   a. homogeneous for the buckling case
   b. non-homogeneous for the equilibrium case

4. In the present approach, the formulation is aimed at deriving the buckling equations (so the problem is formulated in a strong form manner).
However, it is worth observing that even the buckling problem can be formulated by means of a direct method (e.g. the method of Ritz), so in a weak form manner.
In this case, the Ritz method could be applied to the functional

$$\frac{1}{2}\delta^2\Pi\Big|_{@\,equil.} = \frac{1}{2}\int_A \Big[w_{/i}\, w_{/k}\, \bar{N}_{ik} + (u_{i/k} + \bar{w}_{/i}\, w_{/k} - w b_{ik}) \cdot N^L_{ik} - w_{/ik} M_{ik} \Big] dA$$

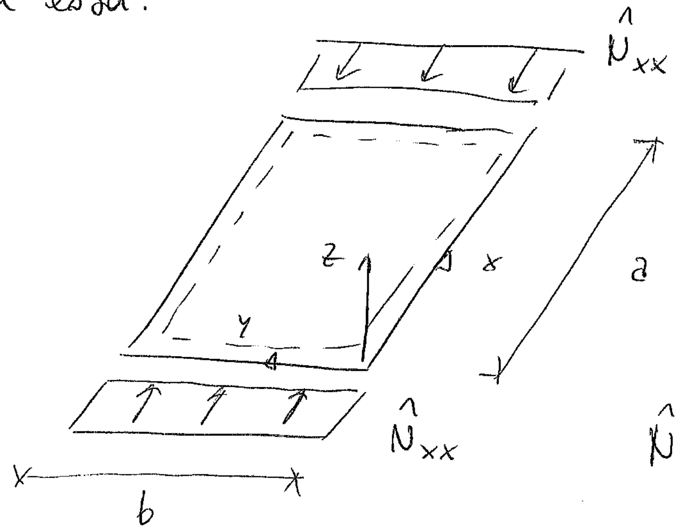
For the case of flat plates with no coupling between in-plane and out-of-plane response, the functional simplifies to:

$$\boxed{\frac{1}{2}\delta^2\Pi\Big|_{@\,equil.} = \frac{1}{2}\int_A \Big[w_{/i}\, w_{/k}\, \bar{N}_{ik} - w_{/ik} M_{ik} \Big] dA}$$

($b_{ik} = 0$ for flat plates;
$\bar{w} = 0$
u_i are, in general, different from zero but uncoupled from the bending displacement w)

• Solution for axially compressed simply-supported plate

The equilibrium and buckling equations are now solved for the special case of simply-supported panel, subjected to axial load.



$\hat{N}_{xx} > 0$ in compression

1. Equilibrium

It is straightforward to verify that the solution of the pre-buckling equations can be found as:

$N_{xx} = -\hat{N}_{xx}$

$N_{xy} = N_{yy} = 0$ → equilibrium equations, as well as

$w = 0$ the boundary conditions, are satisfied

2. Buckling

The out of plane buckling equation is

$M_{ik/ik} + \bar{N}_{ik}\, w_{/ik} = 0$

Expanding the components, it is:

$$M_{xx/xx} + 2M_{xy/xy} + M_{yy/yy} + \bar{N}_{xx}\, w_{/in} = 0$$

but $\bar{N}_{xx} = -\hat{N}_{xx}$, so:

$$M_{xx/xx} + 2M_{xy/xy} + M_{yy/yy} - \hat{N}_{xx}\, w_{/xx} = 0$$

Introduce now the constitutive equation (for an isotropic plate)

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} -w_{/xx} \\ -w_{/yy} \\ -2w_{/xy} \end{Bmatrix}$$

So the buckling equation is:

$$-D\left[w_{/xxxx} + \nu w_{/xxyy} + w_{/yyyy} + \nu w_{/xxyy} + 2(1-\nu)\, w_{/xxyy}\right] - \hat{N}_{xx} w_{/xx} = 0$$

$$\boxed{D\left(w_{/xxxx} + 2w_{/xxyy} + w_{/yyyy}\right) + \hat{N}_{xx}\, w_{/xx} = 0}$$

or

$$\boxed{D\,\Delta\Delta w + \hat{N}_{xx}\, w_{/xx} = 0}$$

The boundary conditions for the simply-supported plate are:

$$\boxed{\begin{array}{lll} w = 0 & \text{on} & \partial\Omega \\ w_{/xx} = 0 & \text{on} & x = 0, a \\ w_{/yy} = 0 & \text{on} & y = 0, b \end{array}}$$

The solution can be found by expressing w as:

$$w = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

This solution (trial, so far) satisfies both the essential ($w=0$ on $\partial\Omega$) and the natural conditions ($w_{,xx}=0$ and $w_{,yy}=0$ on the transverse and longitudinal sides)

The buckling equation becomes:

$$\sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} q_{mn} \left\{ D \left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4 \right] - \hat{N}_{xx} \left(\frac{m\pi}{a}\right)^2 \right\} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = 0$$

The solution $q_{mn}=0 \quad \forall m,n$ is the trivial solution.
The buckled configuration is found by imposing:

$$\hat{N}_{xx} \left(\frac{m\pi}{a}\right)^2 = D \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]^2$$

$$\hat{N}_{xx} = D \left(\frac{a}{m\pi}\right)^2 \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]^2$$

$$= D \left[\frac{m\pi}{a} + \frac{n^2\pi}{b^2} \frac{a}{m} \right]^2 \qquad \text{Multiply with } \frac{b^2}{b^2}$$

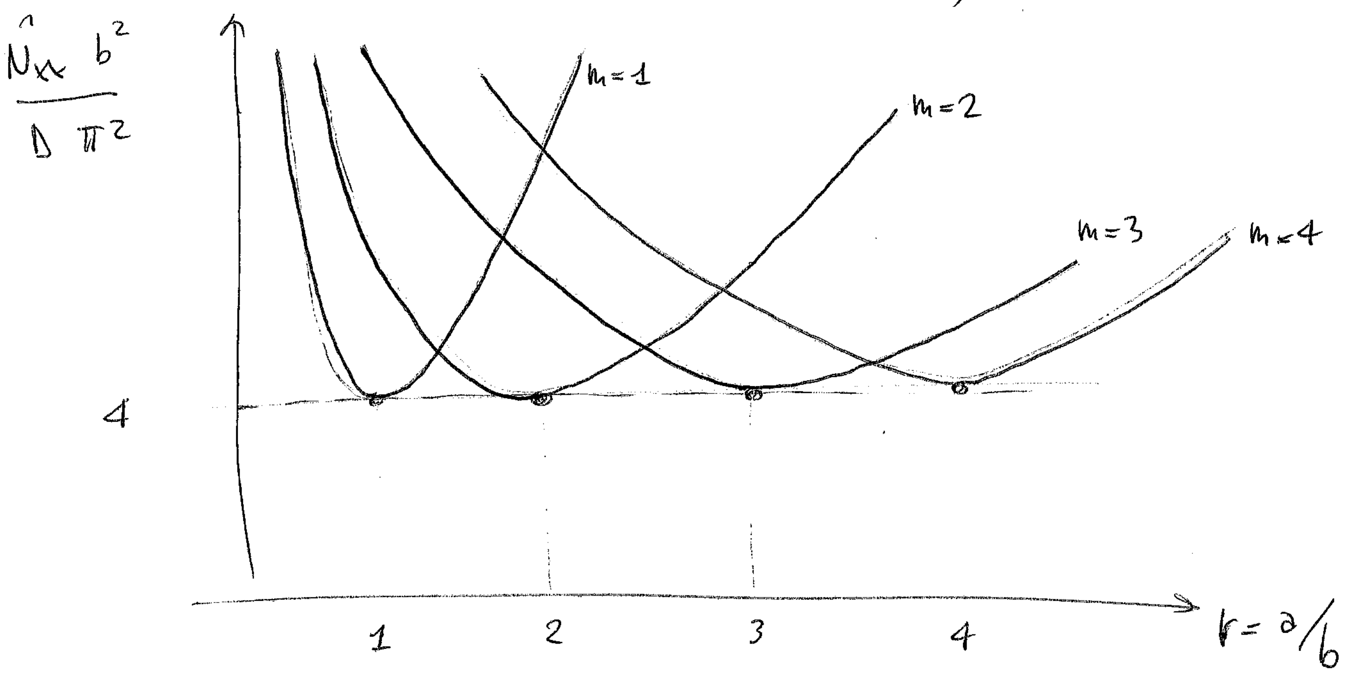
$$= \frac{D\pi^2}{b^2} \left[m\frac{b}{a} + n^2 \frac{a}{b} \frac{1}{m} \right]^2 \qquad \text{define } r = a/b$$

$$\hat{N}_{xx} = \frac{D\pi^2}{b^2}\left(\frac{m}{r} + n^2\frac{r}{m}\right)^2$$

A minimum is reached for $n = 1$, so:

$$\hat{N}_{xx} = D\frac{\pi^2}{b^2}\left(\frac{m}{r} + \frac{r}{m}\right)^2$$

The number of m that makes $\hat{N}_{xx}$ minimum has to be found. To this aim a graphical representation of the nondimensional parameter $\left(\frac{m}{r} + \frac{r}{m}\right)^2$ is presented



Remarks

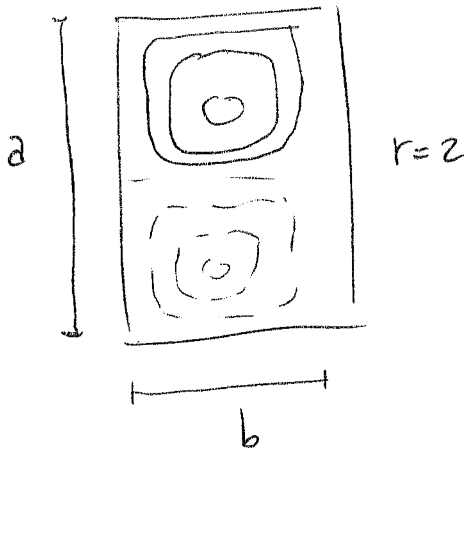
1. The minimum value assumed by the nondimensional parameter

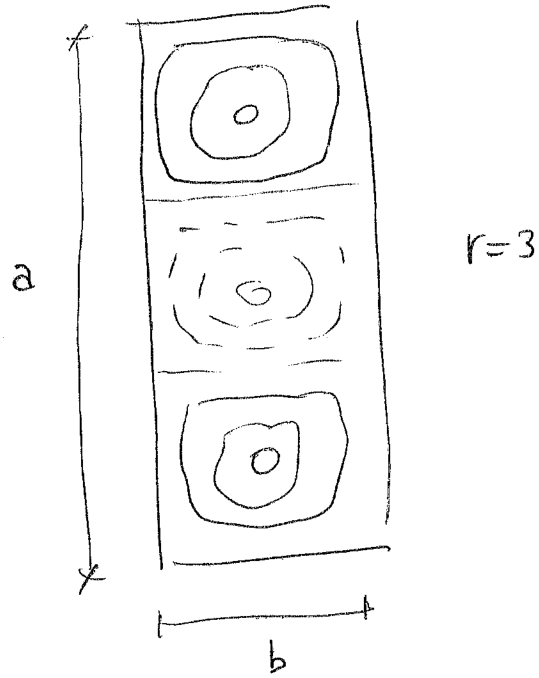
$$\frac{\hat{N}_{xx} b^2}{\Delta \pi^2} = \left(\frac{m}{r} + \frac{r}{m}\right)^2$$ is 4. This happens for

integers value of r. Whenever r is non-integer, the value of $\left(\frac{m}{r} + \frac{r}{m}\right)^2$ is higher than 4.

Asymptotically, i.e. for $r \to +\infty$, the value is equal to 4.

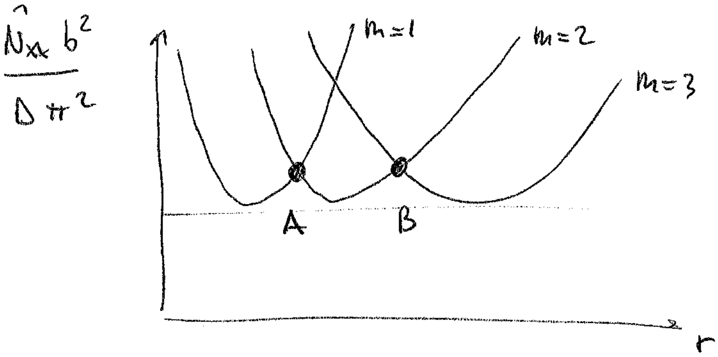
2. The buckling mode is always characterized by $n=1$.
On the contrary, the value of m depends on r.
For integer values of r, $m=r$. This means that the buckling pattern is characterized by square buckles





If r is non integer, the buckles will be not exactly square, but mostly square.

3. For some specific values of r $\left(r = \sqrt{m(m+1)}\right)$, two coincident buckling modes exist



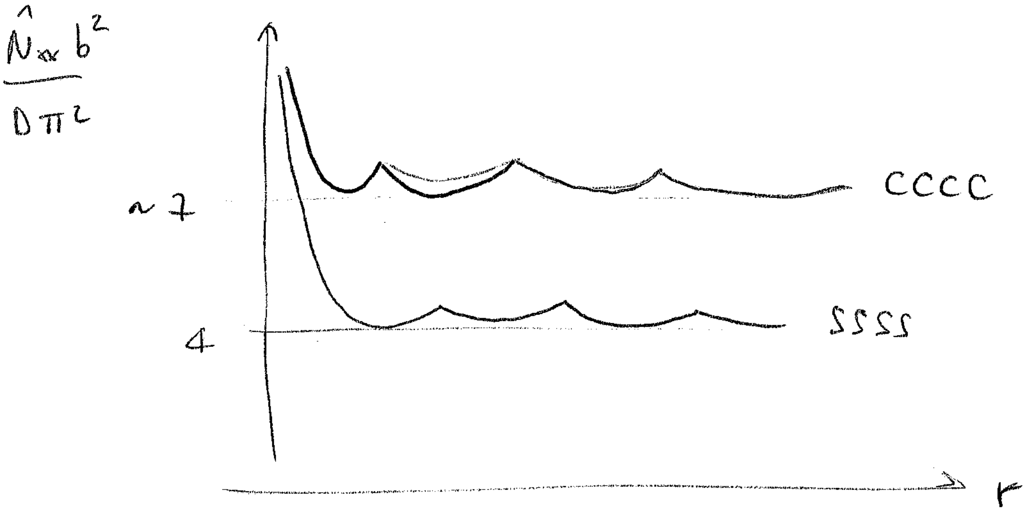
In A: $m = 1$ and $m = 2$

in B: $m = 2$ and $m = 3$

This means that two buckled configurations are associated with the same buckling load.

4. Different boundary conditions or loading conditions can be handled by means of approximate techniques (Ritz, Galerkin, Levy, ...).

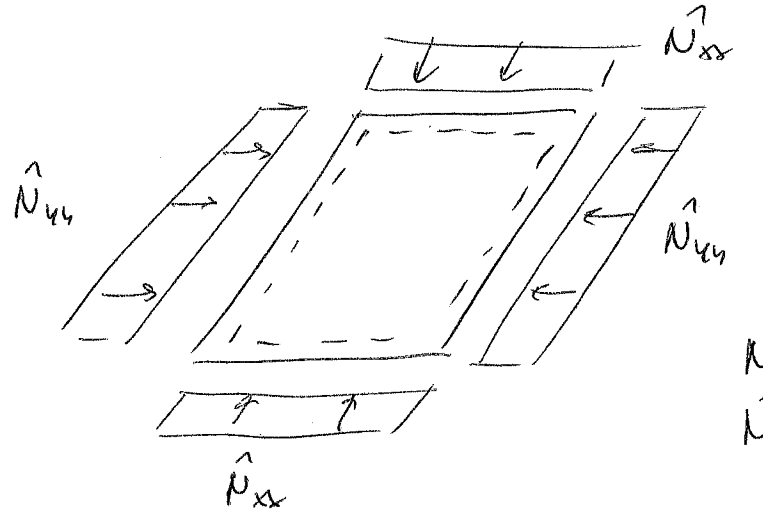
In general the results can still be represented as function of a nondimensional parameter in the form $\frac{\hat{N}_{xx} b^2}{D\pi^2}$ (for the compressive case)



Multiaxial loading conditions - biaxial compression

In many cases the loading condition is characterized by the presence of transverse and longitudinal compression / traction, shear, ...

For simplicity, consider now the case of biaxial compression



$\hat{N}_{xx} > 0$
$\hat{N}_{yy} > 0$ in compression

Consider again simply-supported ~~loading~~ conditions.

1. Equilibrium

$$N_{xx} = -\hat{N}_{xx}$$

$$N_{yy} = -\hat{N}_{yy}$$

$$N_{xy} = 0$$

$$w = 0$$

2. Buckling

$$D\left(w_{/xxxx} + 2w_{/xxyy} + w_{/yyyy}\right) + \hat{N}_{xx} w_{/xx} + \hat{N}_{yy} w_{/yy} = 0$$

+

Boundary conditions (same as previous)

The solution is still found by taking

$$w = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

After substituting w into the buckling equation and seeking for a non-trivial solution, the following equation is found:

$$\hat{N}_{xx}\left(\frac{m\pi}{a}\right)^2 + \hat{N}_{yy}\left(\frac{n\pi}{b}\right)^2 = D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2$$

The pre-buckling loading condition can be parametrized by defining

$$\gamma = \frac{\hat{N}_{yy}}{\hat{N}_{xx}} \quad \text{so that:}$$

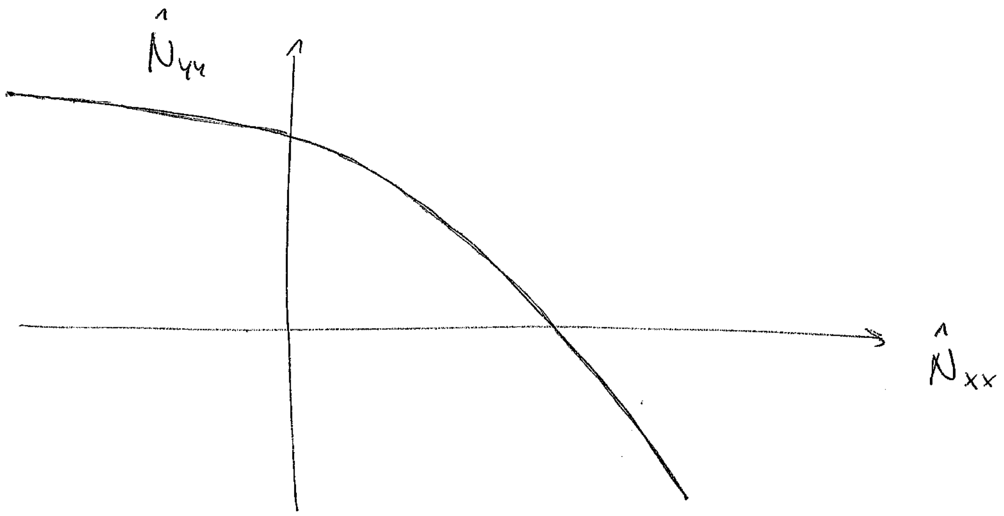
$$\hat{N}_{xx} = \frac{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2}{\left(\frac{m\pi}{a}\right)^2 + \gamma\left(\frac{n\pi}{b}\right)^2}$$

and, after rearranging the expression:

$$\hat{N}_{xx} = \frac{D\pi^2}{b^2} \frac{\left[\left(\frac{m}{r}\right)^2 + n^2\right]^2}{\left(\frac{m}{r}\right)^2 + n^2\gamma}$$

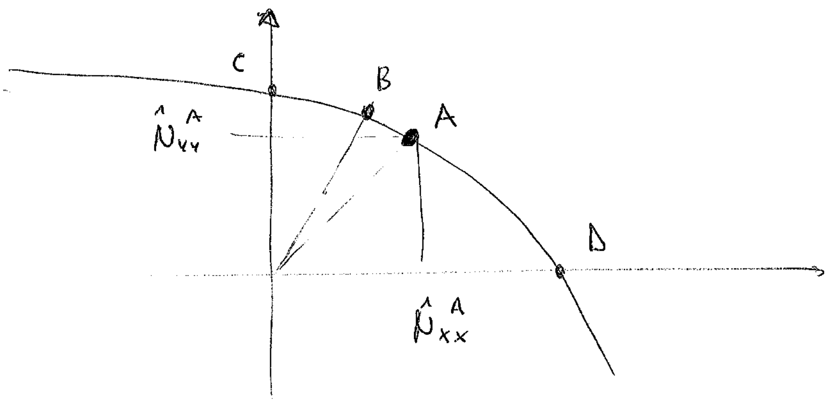
Notice that the expression is now function of γ
($\Rightarrow$ $\hat{N}_{xx}$ depends on the loading condition, i.e. the ratio between the transverse and the longitudinal load)

It is then possible to represent the buckling loads obtained for different values of γ in the form of interaction curves



Remarks

1. Each point of the curve is associated with a different value of γ, e.g.

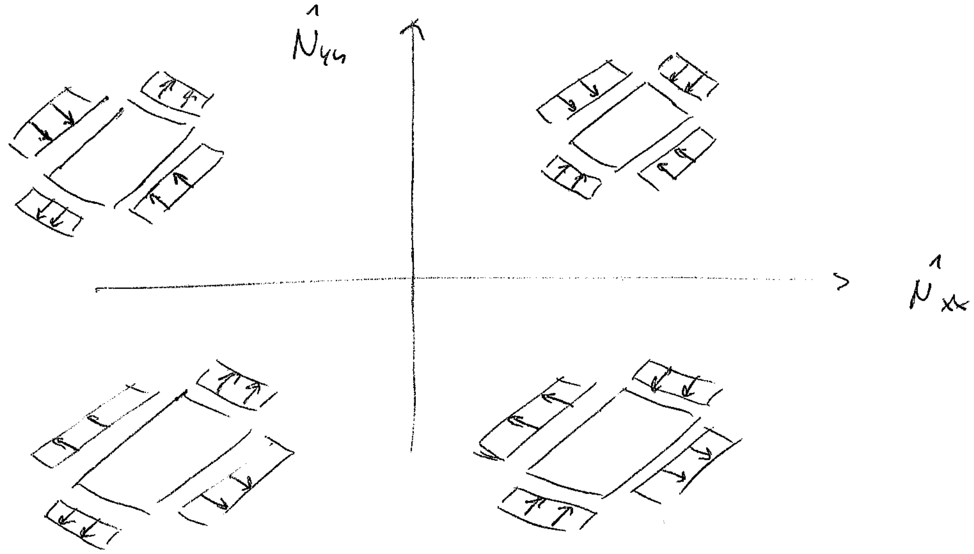


- Point A is obtained by considering $\gamma = 1$
- Point B " $\gamma = 2$

and so forth.

- Point C : pure transverse compression
- Point D : pure longitudinal compression

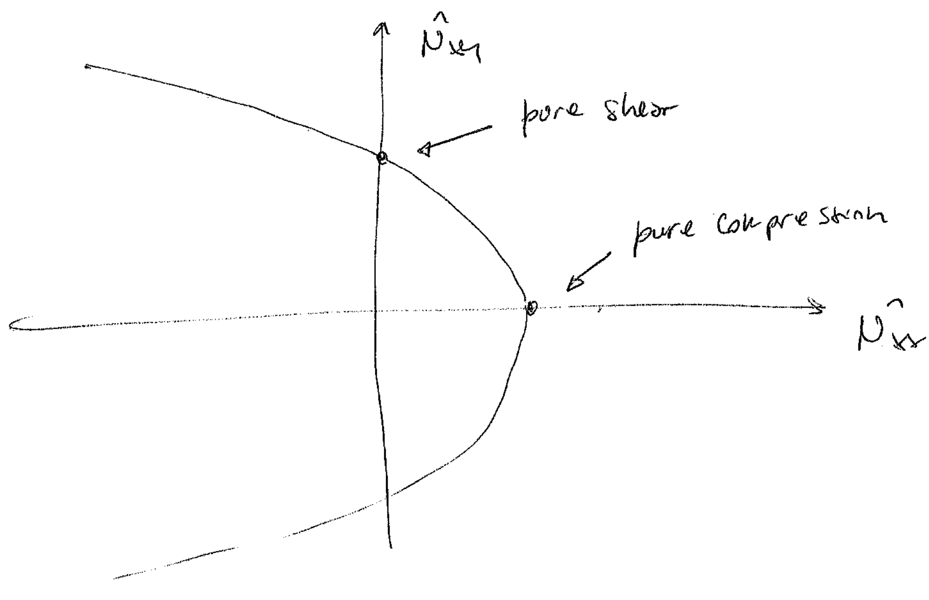
2. $\hat{N}_{xx}$ and $\hat{N}_{yy}$ were defined >0 in compression, so



a. No instability phenomena is associated with a biaxial tensile loading condition (pretty obvious!)

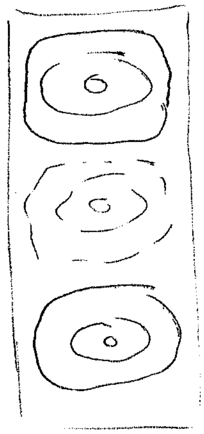
b. The presence of tensile loads has the effect of raising the buckling load

In a similar way, interaction curves can be derived for another common combined loading condition, i.e. compression and shear.
In this case the curve is in the form:

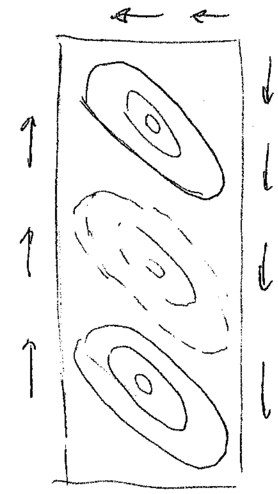


As seen, the buckled shape for pure compression is characterized by almost square halfwaves.

For pure shear, the buckled surface is characterized by skew halfwaves, e.g.



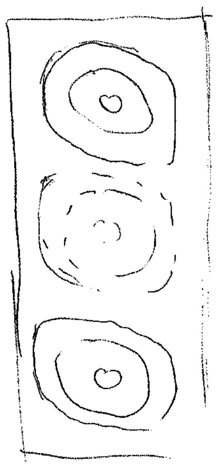
pure compression



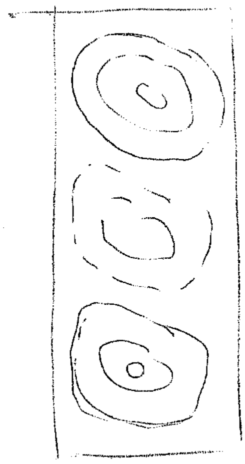
pure shear

skew angle ~ 35°

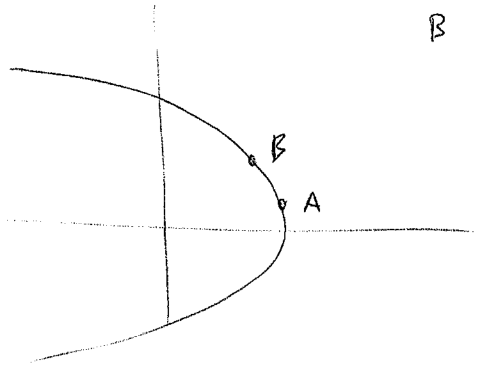
All the loading conditions with compression AND shear will be characterized by an increasing skew angle for an increasing amount of shear



A

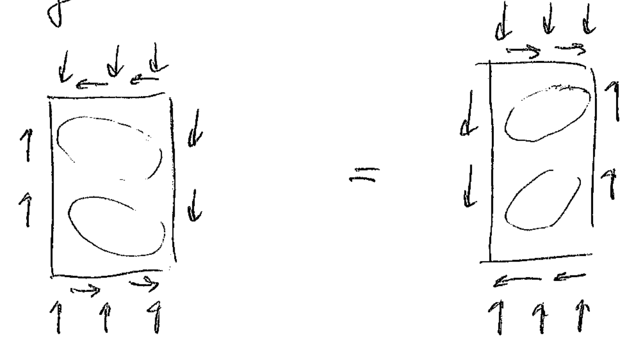


B

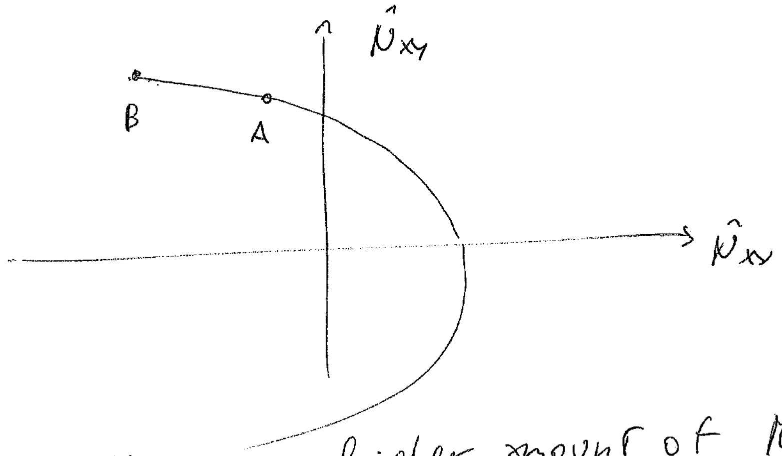


Remarks

1. The interaction curve is symmetric with respect to the x-axis $\Rightarrow$ the sign of the shear does not affect the buckling load



2. The presence of a tensile load is beneficial in terms of buckling behaviour.
Buckling phenomena can arise in the presence of tensile loads if shear is applied, e.g. points A and B



Note that a higher amount of the traction load (point B) determines a higher buckling load $\hat{N}_{xy}$.