Kinematic formulation for beams

The beam solutions derived within the framework of the DSV beam theory provide a fundamental background in the structural analysis.
It is useful to remind that the solution for σ_{zz} was found as an exact solution of the elastic problem, whilst shear stresses were obtained by introducing different kinds of approximations. Bredt's solution was derived by assuming constant stresses through the thickness (thus violating the compatibility along the thickness direction), and similarly was done for shear stresses in the context of the semi-monocoque approximation.
It is then clear that structural theories aim at identifying global or average quantities over a given region of the structure, which are then used for simplifying the solution of the problem. The shear stresses were indeed assumed constant along the thickness, and the shear flows (which are an average quantity: $q = \int_t \sigma_{sz} \, dt$) used as unknown of the problem for seeking a solution.
These global/average quantities are also denoted as

generalized variables.

Despite the usefulness of the DSV solutions, it is intuitive to formulate the beam problem from a different perspective, by defining generalized variables starting from hypothesis that regard the kinematics of the beam. This means that the starting point of a kinematic formulation is the description of the displacement field by means of generalized variables. The choice of the kinematic field is associated with a degree of approximation, exactly as the assumption of constant thicknesswise stress was an approximation for DSV solutions.

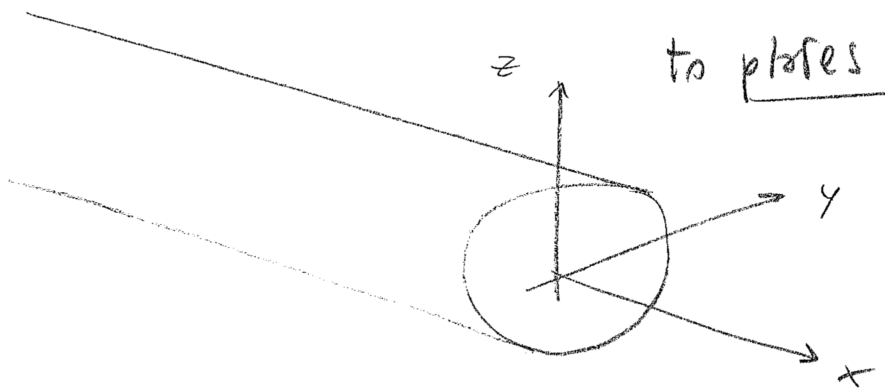
The reasons for introducing the kinematic approach are related to

1. facility of implementation in computer codes: kinematic formulations are "naturally" suitable for a computer implementation
2. Ease of extension of the beam model to the case of nonlinear problems (both materially and geometrically)
3. possibility of extending the same kind of

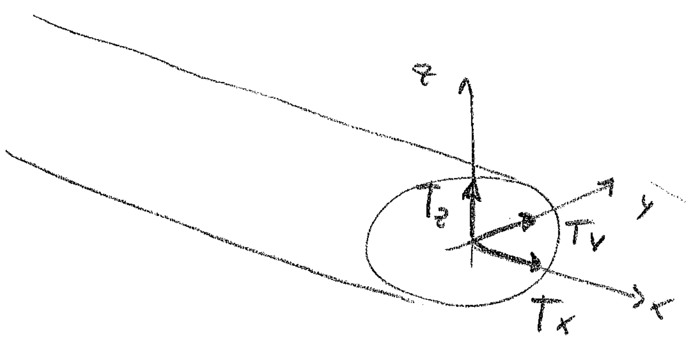
approach to <u>plates</u> and <u>shells</u> (in these latest cases the kinematic strategy is even more relevant as ΔSV-like solutions for 2D structures are very complex to develop)

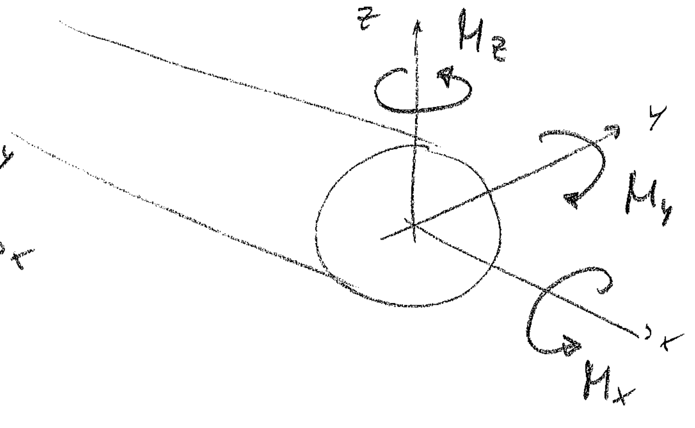
A preliminary remark on the conventions

For consistency with the vast majority of the literature dealing with kinematic formulations, the sign convention used hereinafter considers <u>x as the beam axis</u> (in contrast to z, as done in the context of the ASV formulation), while <u>yz</u> describes the <u>plane of the beam section</u>. This convention is also useful for extending in the following the approach to <u>plates and shells</u>, according to the classical conventions used.



The internal actions are defined, in consistency with the convention used in the context of ASV approach, as:





where

$$M_x = \int_A \left(\sigma_{xz}\, y - \sigma_{xy}\, x \right) dA \qquad \text{(twisting moment)}$$

$$M_y = \int_A \sigma_{xx}\, z \, dA \qquad \text{(bending moment)}$$

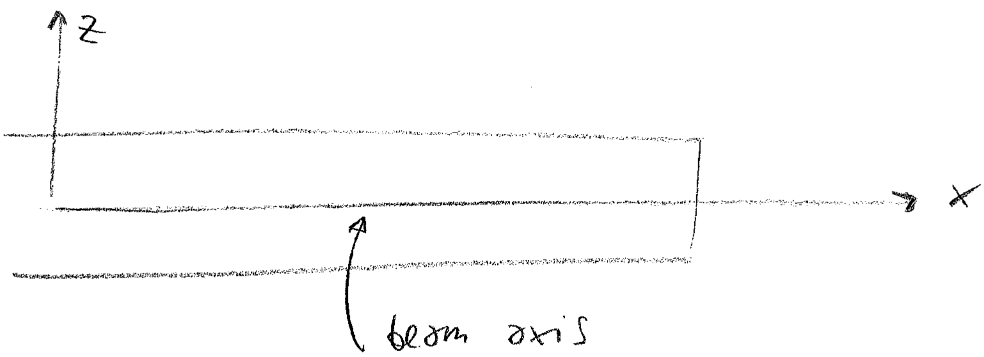
$$M_z = \int_A - \sigma_{xx}\, z \, dA \qquad \text{(bending moment)}$$

According to this convention the DSV solution for the axial stress reads:

$$\sigma_{xx} = \frac{T_x}{A} + \frac{M_y}{J_{yy}} z - \frac{M_z}{J_{zz}} y$$

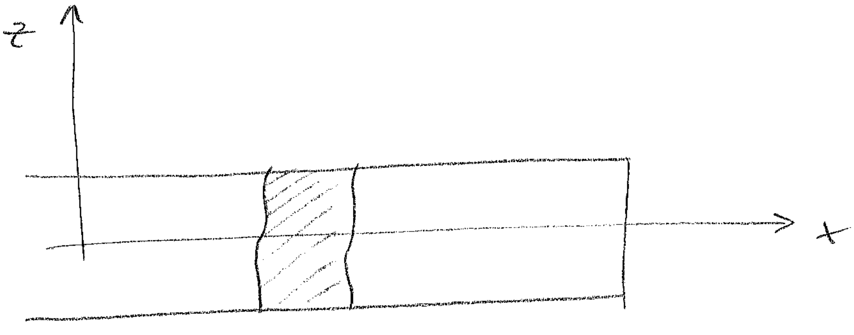
- Timoshenko beam model

Consider a planar beam (for simplicity and with no loss of generality; everything can be easily extended to the case of 3D beams).



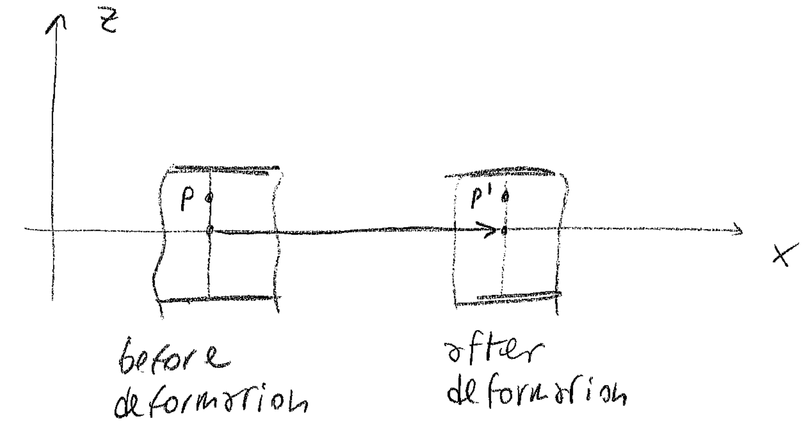
The kinematic model proposed by Timoshenko is formulated by assuming that the displacement field is such that the sections normal to the beam axis are free to rotate during the deformation process, but are constrained to remain straight.

The geometrical interpretation is illustrated here below

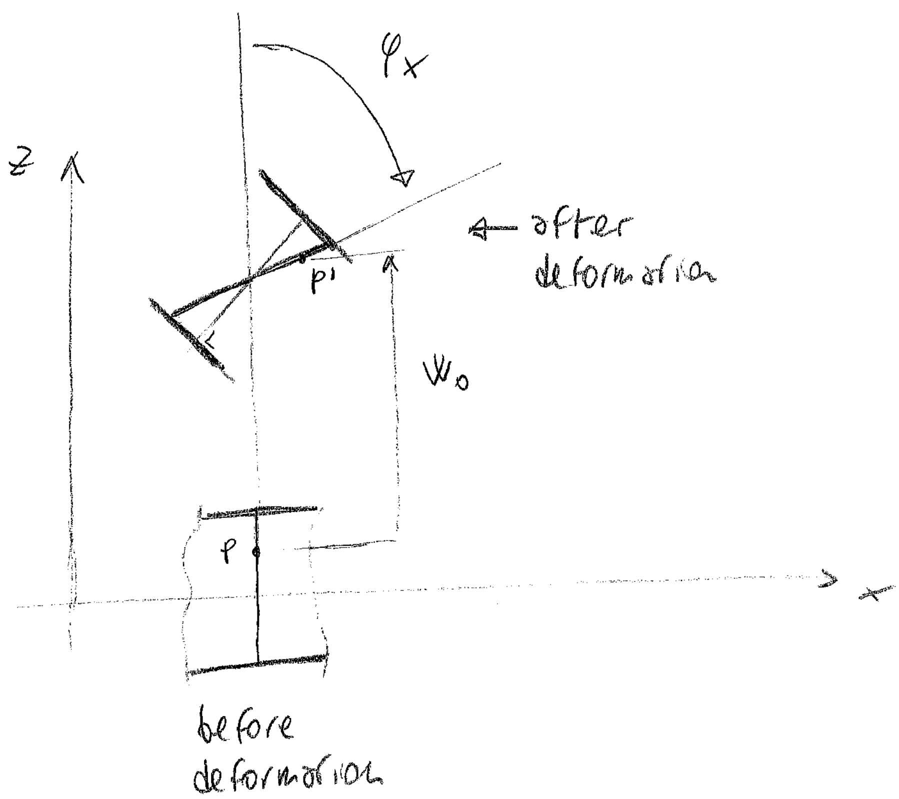


Consider a portion of beam

The displacement of the generic portion of beam can be decomposed into an axial contribution and a bending contribution.



axial displacement



bending displacement

It is then assumed that the kinematic field can be represented as:

$$\boxed{\begin{aligned} u(x,z) &= u_0(x) + z\,\varphi_x(x) \\ w(x,z) &= w_0(x) \end{aligned}}$$

u_0 : axial displacement of beam axis

w_0 : bending displ. of beam axis

φ_x : rotation of the section

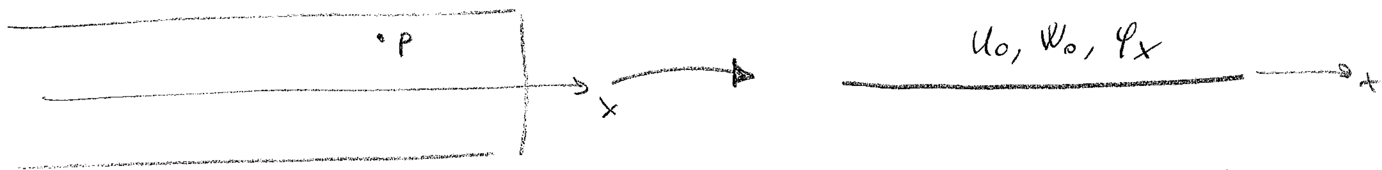
This is the key-point of kinematic models: starting from the knowledge of few generalized variables - in this case $u_0(x)$, $w_0(x)$, $\varphi_x(x)$ -

it is possible to recover the full three-dimensional displacement field (in this case to the full 2D displacement field as the problem is planar).

It is important to note that the generalized parameters u_0, w_0 and φ_x are functions of the axis coordinate x only. They do not depend on z.

In other words, the displacement of a generic point is readily available as the generalized variables are known quantities.

This interpretation of the kinematic model leads to a clear identification of the beam as a line which condenses the overall section behaviour by means of a kinematic description.



The displacement of any point can be obtained once u_0, w_0 and φ_x are available.

The set of deformations which are associated with the kinematic model due to Timoshenko are:

$$\boxed{\begin{aligned} \varepsilon_{xx}(x,z) &= u(x,z)_{/x} = u_0(x)_{/x} + z\varphi_x(x)_{/x} \\ \gamma_{xz}(x,z) &= w(x,z)_{/x} + u(x,z)_{/z} = w_0(x)_{/x} + \varphi_x(x) \\ \varepsilon_{zz}(x,z) &= w(x,z)_{/z} = 0 \end{aligned}}$$

or, by separating the dependence on x and z:

$$\boxed{\begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} u_{0/x} \\ w_{0/x} + \varphi_x \end{Bmatrix} + z \begin{Bmatrix} \varphi_{x/x} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{0xx} \\ t_z \end{Bmatrix} + z \begin{Bmatrix} k_x \\ 0 \end{Bmatrix}}$$

The terms ε_{0xx}, t_x and k_x are defined as

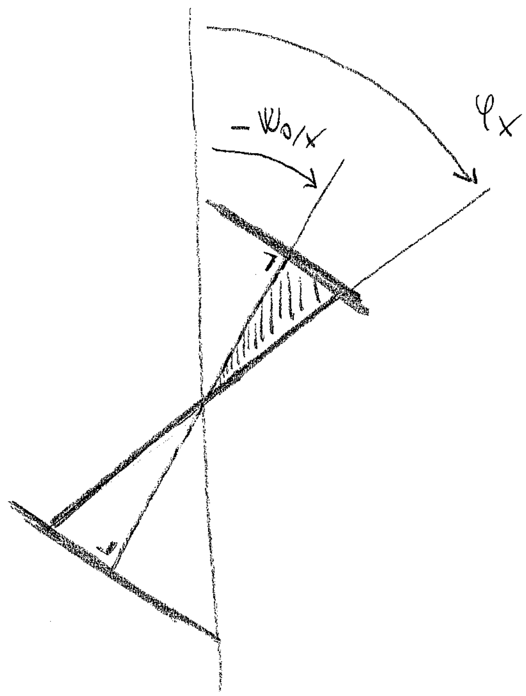
$$\boxed{\begin{aligned} \varepsilon_{0xx} &= u_{0/x} \\ t_z &= w_{0/x} + \varphi_x \\ k_x &= \varphi_{x/x} \end{aligned}}$$

generalized deformations

and represent the generalized deformations of the Timoshenko beam model. Note that these terms depend on the x-position only, thus they are referred to the beam axis. They are denoted as "generalized" because they do not represent the actual deformation in a point of the structure. (e.g. $\varepsilon_{xx} = u_0 + z\varphi_{x/x}$), but are the quantities that allow the evaluation of the deformation in any point of the beam

It can be observed that the axial deformation ε_{xx} given by a membrane contribution, $u_{0/x}$, which is independent on z, and a bending contribution, $z\varphi_{x/x}$, which is linear with z.

The shearing deformation γ_{xz} is different from zero, meaning that Timoshenko beam model accounts for shear deformability effects. The shearing deformation, which is due to the rotation of the section, can be clearly seen from the sketch here reported.



Generalized stresses

The identification of three global quantities – u_0, w_0, φ_x – for describing the displacement field led to the evaluation of generalized strain measures by application of the infinitesimal displacement strain tensor.
It is now possible to identify generalized stress measures descending from the kinematic field initially assumed by applying the PVW:

$$\delta W_i = \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} \, dV$$

$$= \int_V \left(\delta\varepsilon_{xx}\sigma_{xx} + \cancel{\delta\varepsilon_{zz}\sigma_{zz}} + \delta\gamma_{xz}\sigma_{xz} \right) dV$$

but:

$$\delta\varepsilon_{xx} = \delta u_{0/x} + z\,\delta\varphi_{x/x}$$

$$\delta\varepsilon_{zz} = 0$$

$$\delta\gamma_{xz} = \delta w_{0/x} + \delta\varphi_x \qquad \text{and so:}$$

$$= \int_V \left(\delta u_{0/x}\,\sigma_{xx} + \delta\varphi_{x/x} \cdot \sigma_{xx} z + \delta\left(w_{0/x} + \varphi_x\right)\sigma_{xz} \right) dV$$

$$= \int_\ell \delta u_{0/x} \int_A \sigma_{xx}\, dA\, dx + \int_\ell \delta\varphi_{x/x} \int_A \sigma_{xx} z \, dA\, dx$$

$$+ \int_\ell \delta\left(w_{0/x} + \varphi_x\right) \int_A \sigma_{xz}\, dA\, dx$$

(the variables u_0, w_0, φ_x do not depend on A)

and defining:

$$N_x = \int_A \sigma_{xx} \, dA$$

$$M_y = \int_A z \sigma_{xx} \, dA$$ (in the next it will be simply denoted a M

$$Q_z = \int_A \sigma_{xz} \, dA$$

it is obtained that:

$$\boxed{\delta W_i = \int_0^\ell \left(\delta u_{0/x} N_x + \delta \varphi_{x/x} M + \delta (w_{0/x} + \varphi_x) Q_z \right) dx}$$

It is then obtained that <u>the generalized stress measures</u> which are in <u>energy-conjugacy relation with the generalized deformation measures</u> <u>are the internal actions</u>.

Generalized forces

The external generalized forces can be obtained, as done for the generalized stress measures, by expressing the external virtual work.

Consider the presence of the volume forces F_x and F_z

$$\delta W_e = \int_V (\delta u\, F_x + \delta w\, F_z)\, dV$$

$$= \int_V (\delta u_0 F_x + \delta \varphi_x z F_x + \delta w_0 F_z)\, dV$$

$$= \int_\ell \delta u_0 \int_A F_x\, dA\, dx + \int_\ell \delta \varphi_x \int_A z F_x\, dA\, dx +$$

$$\int_\ell \delta w_0 \int_A F_z\, dA\, dx$$

$$\boxed{\delta W_e = \int_\ell (\delta u_0\, \hat{n}_x + \delta \varphi_x\, \hat{m} + \delta w_0\, \hat{n}_z)\, dz}$$

where :

$\hat{n}_x = \int_A F_x\, dA$ $\longrightarrow$ axial force per unit length

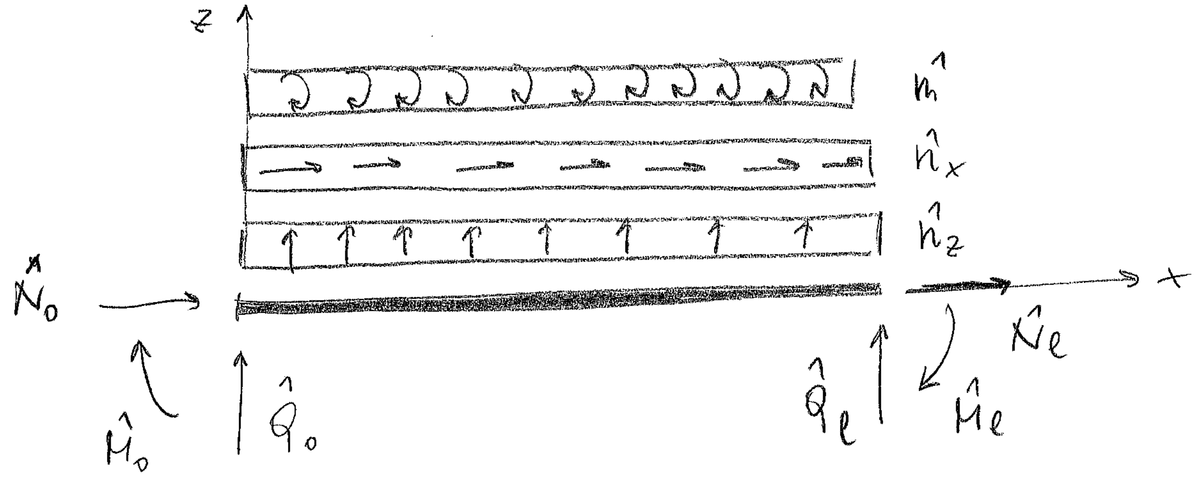
$\hat{m} = \int_A z F_x\, dA$ $\longrightarrow$ bending moment per unit length

$\hat{n}_z = \int_A F_z\, dA$ $\longrightarrow$ shear force per unit length

Equilibrium conditions

It is now possible to derive the equilibrium conditions for the Timoshenko beam.

In addition to the external loads consistent with the kinematic assumptions, any possible concentrated load is now introduced at the ends of the beam.



While compatibility requirements are intrinsically satisfied by the model, the equilibrium conditions can be imposed referring to the PVW.

$$\int_0^\ell \left[\delta u_{0/x} N_x + \delta \varphi_{x/x} M + \delta(\varphi_x + w_{0/x}) Q_z \right] dx =$$

$$= \int_0^\ell \left(\delta u_0 \hat{n}_x + \delta w_0 \hat{n}_z + \delta \varphi_x \hat{m}_x \right) dx + \quad \rightarrow \text{distributed loads}$$

$$\left.\begin{aligned} &+ \delta u_0(0) \hat{N}_0 + \delta w_0(0) \hat{Q}_0 + \delta \varphi_x(0) \hat{M}_0 + \\ &+ \delta u_0(\ell) \hat{N}_\ell + \delta w_0(\ell) \hat{Q}_\ell + \delta \varphi_x(\ell) \hat{M}_\ell \end{aligned}\right] \rightarrow \text{concentrated loads}$$

The internal virtual work can now be integrated by parts:

$$\delta W_i = -\int_0^\ell \left(\delta u_0 N_{x/x} + \delta\varphi_x M_{/x} + \delta w_0 Q_{z/x} - \delta\varphi_x Q_z\right)dx$$

$$+ \delta u_0 N_x + \delta\varphi_x M_x + \delta w_0 Q_z \Big|_0^\ell$$

The PVW is then:

$$\int_0^\ell \left[\delta u_0 \left(N_{x/x} + \hat{n}_x\right) + \delta\varphi_x \left(M_{/x} - Q_z + \hat{m}\right) + \delta w_0 \left(Q_{z/x} + \hat{n}_z\right)\right]dx +$$

$$- \delta u_0(\ell)\left[N_x(\ell) - \hat{N}_\ell\right] + \delta u_0(0)\left[N_x(0) + \hat{N}_0\right]$$

$$- \delta\varphi_x(\ell)\left[M_x(\ell) - \hat{M}_\ell\right] + \delta\varphi_x(0)\left[M_x(0) + \hat{M}_0\right]$$

$$- \delta w_0(\ell)\left[Q_z(\ell) - \hat{Q}_\ell\right] + \delta w_0(0)\left[Q_z(0) + \hat{Q}_0\right] = 0$$

and due to the arbitrarity of the virtual displacements, the equilibrium conditions expressed by the PVW are:

$$\begin{cases} N_{x/x} + \hat{n}_x = 0 \\ M_{/x} - Q_z + \hat{m} = 0 \\ Q_{z/x} + \hat{n}_z = 0 \end{cases} \quad \text{in } x \in [0\ \ell]$$

equilibrium equations

$$N_x(0) = -\hat{N}_0 \quad \text{or} \quad \delta u_0(0) = 0$$

$$N_x(\ell) = +\hat{N}_\ell \quad \text{or} \quad \delta u_0(\ell) = 0$$

$$M(0) = -\hat{M}_0 \quad \text{or} \quad \delta\varphi_x(0) = 0$$

$$M(\ell) = +\hat{M}_\ell \quad \text{or} \quad \delta\varphi_x(\ell) = 0$$

$$Q_z(0) = -\hat{Q}_0 \quad \text{or} \quad \delta w_0(0) = 0$$

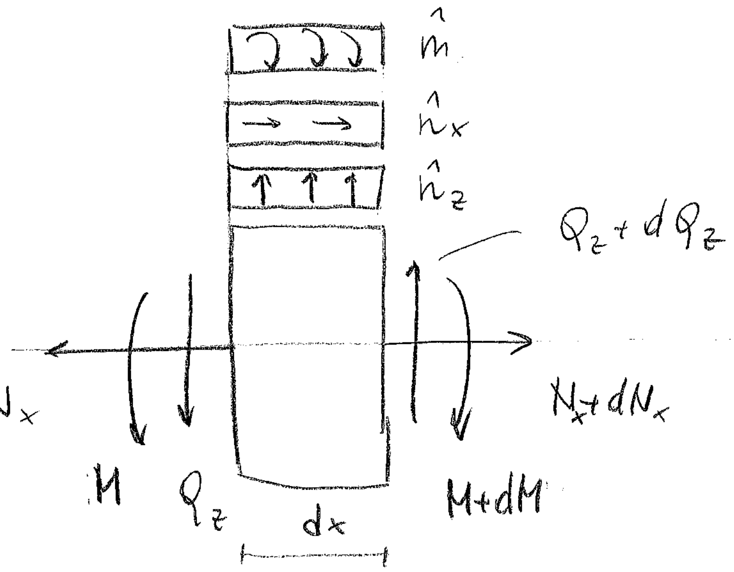
$$Q_z(\ell) = +\hat{Q}_\ell \quad \text{or} \quad \delta w_0(\ell) = 0$$

essential boundary conditions

natural boundary conditions

Remarks

- The equilibrium equations could have been obtained by considering an infinitesimal slice of beam



a. $N_x + dN_x + \hat{n}_x dx - N_x = 0 \quad \Rightarrow \quad \boxed{N_{x/x} + \hat{n}_x = 0}$

b. $Q_z + dQ_z + \hat{n}_z dx - Q_z = 0 \quad \Rightarrow \quad \boxed{Q_{z/x} + \hat{n}_z = 0}$

c. $M + dM + \hat{m}_x dx - Q_z dx - M = 0$

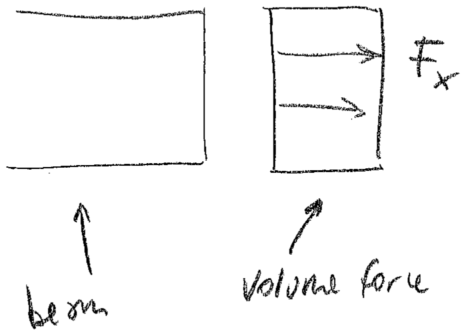
$\Rightarrow \quad \boxed{M_{/x} - Q_z + \hat{m} = 0}$

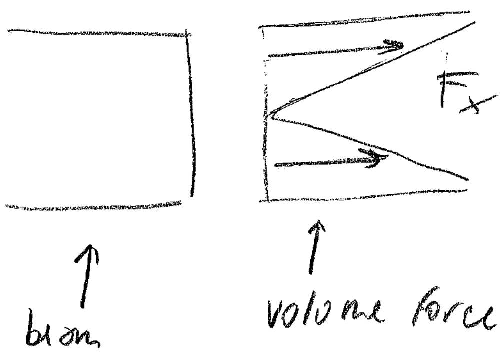
What is worth highlighting is that <u>the equilibrium equations are automatically obtained by applying the PVW</u>.

- The equilibrium conditions obtained define the equilibrium requirements at <u>global level</u>, which is exactly what is intrinsic in the definition of a kinematic model where the problem is formulated by identifying global quantities, or <u>generalized variables</u>.

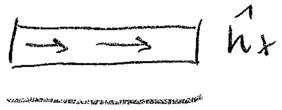
In other words, the equilibrium equations here obtained do not imply the <u>local</u> satisfaction of the 3D equilibrium equations. The 3D equilibrium is satisfied in a <u>global</u> sense, in consistency with the idea of a structural theory based on generalized variables.

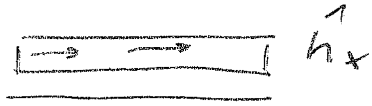
This idea can be understood by considering an example where two different volume forces are considered





As observed, the volume force is "seen" by the model as $\hat{n}_x = \int_A F_x \, dA$, so the two cases would lead to the same value of $\hat{n}_x$





This also means that the two different distributions of F_x will lead to identical equations, and so identical solutions. How F_x is distributed is not relevant. What is relevant is the average / global effect of F_x intended as the integral over the beam section.

- It is finally noted that the equations derived so far are independent on the constitutive law, which has not been introduced yet.

Constitutive law - elastic beam

The beam constitutive law is now introduced in order to express the equilibrium conditions in terms of displacements (recall that kinematic formulations are adopted in the context of displacement-based approaches).

The constitutive law establishes a relation between generalized stresses and generalized deformations.

It is then a relation in the form

$$\underbrace{\begin{Bmatrix} N_x \\ M \\ Q_z \end{Bmatrix}}_{\text{generalized stresses}} = \underbrace{[\,?\,]}_{\substack{\text{beam section}\\ \text{constitutive law}}} \underbrace{\begin{Bmatrix} \varepsilon_{0xx} \\ k_x \\ t_x \end{Bmatrix}}_{\text{generalized deformations}}$$

The choice of the constitutive law is a crucial aspect in the development of the model. The more natural way for identifying this relation consists in assuming an axial state of stress, meaning that the relation between stress and strain components is:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} = \tau \end{Bmatrix} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{Bmatrix}$$

Recalling how the definition of generalized stresses:

$$N_x = \int_A \sigma_{xx}\, dA = \int_A E\, \varepsilon_{xx}\, dA$$

$$= \int_A E \left(u_{0/x} + z\, \varphi_{x/x} \right) dA$$

$$= \int_A E\, u_{0/x}\, dA + \int_A E\, z\, \varphi_{x/x}\, dA$$

principal axes

$$= EA\, u_{0/x} + \cancel{E \int_A z\, dA\, \varphi_{x/x}}$$

$$= EA\, u_{0/x} = EA\, \varepsilon_{0xx}$$

$$M = \int_A \sigma_{xx}\, z\, dA = \int_A E\, z\, \varepsilon_{xx}\, dA$$

$$= \int_A E\, z\, u_{0/x}\, dA + \int_A E\, z^2\, \varphi_{x/x}\, dA$$

$$= E \int_A z^2\, dA\, \varphi_{x/x}$$

$$= EJ\, \varphi_{x/x} = EJ\, k_x$$

$$Q_z = \int_A \sigma_{xz}\, dA = \int_A G\, \gamma_{xz}\, dA$$

$$= \int_A G \left(\varphi_x + w_{0/x} \right) dA$$

$$= G \int_A dA \, (\varphi_x + w_{0/x})$$

$$= GA \, (\varphi_x + w_{0/x}) = GA \, t_z$$

The beam constitutive law is then obtained as:

$$\begin{Bmatrix} N_x \\ M \\ Q_z \end{Bmatrix} = \begin{bmatrix} EA & & \\ & EJ & \\ & & GA \end{bmatrix} \begin{Bmatrix} \varepsilon_{0xx} \\ k_x \\ t_z \end{Bmatrix}$$

Accordingly, the strain energy can be written as:

$$U = \frac{1}{2} \int_0^{\ell} \left(N_x \varepsilon_{0xx} + M k_x + Q_z t_z \right) dx$$

$$= \frac{1}{2} \int_0^{\ell} \left(N_x \frac{N_x}{EA} + M \frac{M}{EJ} + Q_z \frac{Q_z}{GA} \right) dx$$

This expression can now be compared with the expression obtained in the context of the DSV beam formulation:

$$U^{DSV} = \frac{1}{2} \int_{\ell} \left(N_x \frac{N_x}{EA} + M \frac{M}{EJ} + Q_z \frac{Q_z}{GA^*} \right) dx$$

It can be noted that the energy contribution due to shear is different. In particular, it is:

$$u_{shear} = Q_z \frac{Q_z}{GA} \qquad U^{DSV}_{shear} = Q_z \frac{Q_z}{GA^*}$$

The Timoshenko beam model tends, indeed, to overestimate the shear stiffness (recall that $GA > GA^*$). While the term GA descends naturally from the kinematic assumptions, it is generally corrected as GA^* in order to provide additional shear flexibility.

Concluding, the beam constitutive law adopted for the Timoshenko beam is:

$$\boxed{\begin{Bmatrix} N_x \\ M \\ Q_z \end{Bmatrix} = \begin{bmatrix} EA & & \\ & EJ & \\ & & GA^* \end{bmatrix} \begin{Bmatrix} \varepsilon_{0xx} \\ k_x \\ t_z \end{Bmatrix}}$$

- Remarks (important)

- The correction of the stiffness GA as GA^* can be interpreted as link between the kinematic formulation and the DSV results.

Indeed the kinematic approach, which represents another strategy for dealing with beam problems, has not to be interpreted as an approach completely independent from DSV results.

The formulation is based on initial kinematic assumptions (the displacement field) leading to an expression of the strain energy that is consistent with the model itself, but not necessarily correct (indeed the kinematic field is an approximation, not the exact solution!).

The idea of fixing, ex-post, the constitutive law in order to recover an improved expression of the strain energy represents the link with DSV. In other words, the correction from GA to GA^* can be done only because we know from DSV theory that the shearing stiffness should be GA^* and not GA.

- It is important to avoid any possible misunderstanding with regard to the displacement field initially introduced ($u = u_0 + z\varphi_x$; $w = w_0$).

  This displacement field is an <u>approximation</u> of the exact solution. It is an intuitive way of describing how the displacement field could be reasonably be, starting from the knowledge of few generalized displacement components (u_0, φ_x, w_0) referred to the beam axis.

  The displacement field is <u>NOT</u> an initial guess solution that is finally verified to be also the exact solution (as it happened, for instance, for the torsion problem in the context of DSV theory)

Equations in terms of displacement components

By introducing the constitutive equation for the beam section, it is possible to express the equilibrium conditions in terms of displacement components:

$$N_x = EA\, u_{0/x}$$

$$M = EJ\, \varphi_{x/x}$$

$$Q_z = GA^* (\varphi_x + w_{0/x})$$

and so:

$$\begin{cases} (EA\, u_{0/x})_{/x} + \hat{n}_x = 0 \\ (EJ\, \varphi_{x/x})_{/x} - GA^* (\varphi_x + w_{0/x}) + \hat{m} = 0 \\ \left[GA^* (\varphi_x + w_{0/x}) \right]_{/x} + \hat{n}_z = 0 \\ + \\ \text{Boundary conditions} \end{cases}$$

If the section stiffnesses are constant along the beam axis, the equation simplify as reported next.

$EA\, u_{0/xx} + \hat{n}_x = 0$	axial equilibrium
$EJ\, \varphi_{x/xx} - GA^*(\varphi_x + w_{0/x}) + \hat{m} = 0$	bending equilibrium
$GA^*(\varphi_{x/x} + w_{0/xx}) + \hat{n}_z = 0$	shear equilibrium

$$EA\, u_{/x} = \mp \hat{N} \quad \text{or} \quad \delta u_0 = 0 \quad \text{in } x = 0, \ell$$

$$EJ\, \varphi_{x/x} = \mp \hat{M} \quad \text{or} \quad \delta \varphi_x = 0 \quad \text{in } x = 0, \ell$$

$$GA^*(\varphi_x + w_{0/x}) = \mp \hat{Q} \quad \text{or} \quad \delta w_0 = 0 \quad \text{in } x = 0, \ell$$

natural BCs   essential BCs

The elastic problem is then formulated in terms of 3 ordinary differential equations and <u>3 unknown</u> functions representing the generalized displacement components.

After the solution has been obtained it is possible, if needed, to recover the full displacement field by using the initial expression of the beam kinematics.

It can be observed that the first equation, representing the axial equilibrium, is <u>uncoupled</u> from the other two equations.

Evaluation of the stresses (extra)

The solution of the structural problem, according to the formulation proposed by Timoshenko, leads to the evaluation of the generalized displacement components u_0, w_0 and φ_x.

Once the generalized displacement components are available it is possible to determine:

1. – the displacement field (by making use of the kinematic model)
2. – the strain field (by deriving the components of the displacement field)
3. – the stress field (by using the constitutive relation)

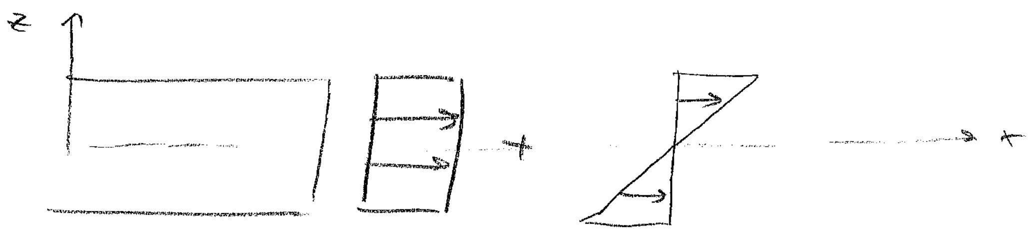
So:

1. $$u = u_0 + z\,\varphi_x$$
   $$w = w_0$$

2. $$\varepsilon_{xx} = u_{0/x} + z\,\varphi_{x/x}$$
   $$\gamma_{xz} = w_{0/z} + \varphi_x$$

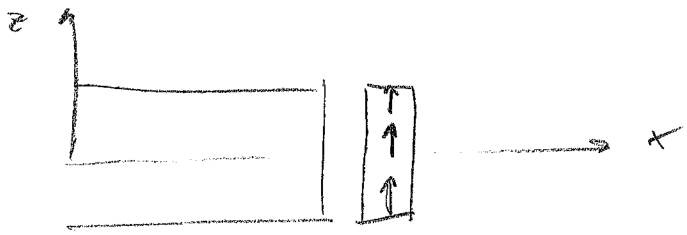
3. $$\boxed{\begin{aligned} \sigma_{xx} &= E\,(u_{0/x} + z\,\varphi_{x/x}) \\ \sigma_{xz} &= G\,(w_{0/z} + \varphi_x) \end{aligned}}$$

It can be noted that the axial stress σ_{xx} is characterized by a constant part (the axial contribution) and a linear part (due to bending)



Axial stress σ_{xx}

The shear stress σ_{xz} is constant throughout the thickness direction (does not depend on z)



Transverse shear stress σ_{xz}

It follows that the prediction of the transverse shear stress σ_{xz} is rather poor. From beam theory it is well known (see Jourawski solution) that the transverse shear stresses are expected to be quadratic with z. The prediction of the Timoshenko model, obtained by making use of the constitutive law, is then a raw approximation.

These results are not surprising as it was previously observed that

- the strain energy due to bending and axial behaviour obtained with Timoshenko

model is equal to the strain energy obtained in the context of DSV theory.
Similary the axial and bending stresses exhibit the expected behaviour

- The strain energy due to the transverse shear, as obtained by Timoshenko model, required a correction to recover the DSV results.

$$\delta(w_{0/x}+\varphi_x)\, GA\, (w_{0/x}+\varphi_x)$$

correction

$$\delta(w_{0/x}+\varphi_x)\, GA^*\, (w_{0/x}+\varphi_x)$$

This correction (i.e. the introduction of the shear factor) is indeed necessary to correct the inability of the model to capture the actual behaviour of the transverse shear stresses.
Note that the correction into GA^* operates at energetic level so:

- the effect of transverse shear flexibility is properly accounted for at global level

- the local evaluation of the transverse shear stresses remains poor. Alternative procedures are usually implemented for obtaining more accurate predictions

A typical approach (also used in the context of plate models) consists in evaluating the shear stress by integration of the 3D equilibrium equations:

$$\sigma_{xx/x} + \sigma_{xz/z} = 0$$

From which:

$$\sigma_{xz/z} = - \sigma_{xx/x}$$

for a given position $x = \bar{x}$ along the beam axis, the thicknesswise behaviour reads:

$$\sigma_{xz}(\bar{x}, z) = - \int_{-t/2}^{z} \sigma_{xx/x}(\bar{x}, \xi)\, d\xi$$

and recalling that

$$\sigma_{xx} = E\left(u_{0/x} + z\varphi_{x/x}\right) \qquad \text{So:}$$

$$\sigma_{xx/x} = E\left(u_{0/xx} + z\varphi_{x/xx}\right)$$

From which

$$\sigma_{xz}(\bar{x}, z) = - \int_{-t/2}^{z} E\left(u_{0/xx} + \xi\varphi_{x/xx}\right) d\xi$$

$$= - E\left[u_{0/xx}\left(z + \frac{t}{2}\right) + \varphi_{x/xx}\left(\frac{z^2}{2} - \frac{t^2}{8}\right)\right]$$

which is characterized by a quadratic behaviour as expected.

- Boundary conditions

The definition of the boundary conditions depends on the set of constraints applied at the two ends. Before illustrating a few examples, it is important to remark that

1. the set of boundary conditions here obtained has been automatically derived by applying the PVW. This illustrates the utility of the PVW: starting from one single (and simple) scalar equations it is possible to obtain the equilibrium equations together with the relevant boundary conditions

2. As far as the Timoshenko kinematic assumptions entered the expression of the PVW, it follows that the boundary conditions depend on the kinematic model. Different kinematic assumptions (see later Euler-Bernoulli) would lead to different boundary conditions. In this sense the boundary conditions do not exist in an absolute sense, regardless of the beam theory. Rather they are the conditions that can be imposed in consistency with

the Kinematic theory

3. A distinction of paramount importance has been introduced between natural and essential conditions.

Essential conditions are of geometric type and regard the generalized displacement components. As far as they do not involve the derivatives of the functions, they are sometimes referred to as Dirichlet boundary conditions.

Natural conditions regard the equilibrium at the boundaries with the external loads. The derivatives of the unknown functions are involved in their expression and, for this reason, they are sometimes denoted as Neumann-type boundary conditions.

Given the uncoupling between axial and flexural response, even the boundary conditions are conviniently treated separately.

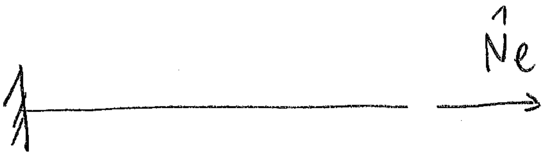
Axial behaviour

- 

Free - Free

B.Cs. $\quad EA\,u_{/x}(0) = -\hat{N}_o$

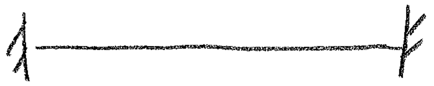
$\quad\quad EA\,u_{/x}(\ell) = +\hat{N}_\ell$

- 

Fixed - free

B.Cs: $\quad u(0) = 0$

$\quad\quad EA\,u_{/x}(\ell) = \hat{N}_e$

- 

Fixed - fixed

B.Cs $\quad u(0) = 0$

$\quad\quad u(\ell) = 0$

Bending behaviour

• 

Free - Free

BCs:

$$GA^*(\varphi_x(0) + w_{0/x}(0)) = -\hat{Q}_0$$

$$GA^*(\varphi_x(\ell) + w_{0/x}(0)) = \hat{Q}_\ell$$

$$EJ\,\varphi_{x/x}(0) = -\hat{M}_0$$

$$EJ\,\varphi_{x/x}(\ell) = \hat{M}_\ell$$

• 

Simply-supported / Simply-supported

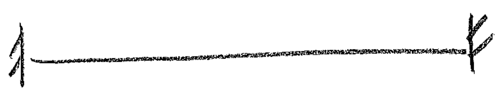
BCs:

$$w_0(0) = 0$$

$$EJ\,\varphi_{x/x}(0) = -\hat{M}_0$$

$$w_0(\ell) = 0$$

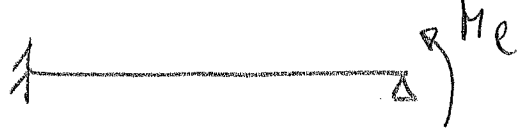
$$EJ\,\varphi_{x/x}(\ell) = \hat{M}_\ell$$

• 

Clamped - Clamped

BCs: $w_0(0) = 0$ $\quad$ $w_0(\ell) = 0$

$\varphi_0(0) = 0$ $\quad$ $\varphi_0(\ell) = 0$

• 

Clamped – simply – supported

BCs: $w_0(0) = 0$ $\quad$ $w_0(\ell) = 0$

$\varphi_0(0) = 0$ $\quad$ $EJ\,\varphi_{x/x}(\ell) = \hat{M}_e$

and similarly for all the other combinations. To summarize:

Free constraint $\rightarrow$ No essential conditions (all natural)

Simply-supported $\rightarrow$ $w_0 = 0$ essential condition (natural condition on moment)

Clamped $\rightarrow$ $w_0 = 0$; $\varphi_x = 0$ essential conditions (no natural conditions)

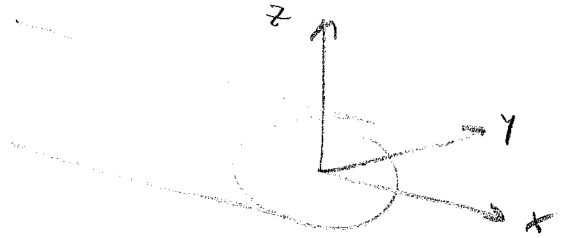
Final critical remarks

- It is important to observe that the theory was developed from a "displacement" point of view. The initial step regarded the definition of a "rule" for describing the displacement field, viz. the kinematic model.
  The strains were then expressed by applying the definition of strain tensor $\varepsilon_{ik} = \frac{1}{2}\left(u_{i/k} + u_{k/i} \right)$.
  $\left(\varepsilon_{xx} = u_{/x} \; ; \; \gamma_{xz} = w_{/x} + u_{/z} \; ; \; \varepsilon_{zz} = w_{/x} \right)$
  In this sense, the compatibility is intrinsically satisfied by the formulation. The PVW, which is applied to impose the equilibrium, is written by expressing the strains according to the expression of the strain tensor. Thus compatibility is automatically fulfilled
- According to the kinematic model, the strain component ε_{zz} is obtained as zero.
  This is inconsistent with the assumption of axial state of stress, as detailed here below, and for this reason represents a static inconsistency of model

Recall that the axial stress state is obtained as a "special" case of the generic 3D state of stress

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & 1/G & & \\ & & & & 1/G & \\ & & & & & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}$$ 3D law

Consider a beam with axis along x (as in the case here analyzed). According to DSV solution:



$$\sigma_{zz} = \sigma_{yy} = \sigma_{zy} = 0$$

It follows that:

$$\varepsilon_{xx} = 1/E \, \sigma_{xx}$$

$$\varepsilon_{yy} = -\nu/E \, \sigma_{xx}$$

$$\varepsilon_{zz} = -\nu/E \, \sigma_{xx}$$

$$\gamma_{xy} = 1/G \, \sigma_{xy}$$

$$\gamma_{xz} = 1/G \, \sigma_{xz}$$

$$\gamma_{yz} = 0$$

$$\Rightarrow \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} E & & \\ & G & \\ & & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix}$$

(3D beam)

or, as in the 2D planar case here considered,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} E & \\ & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{Bmatrix}$$ ← constitutive law used in Timoshenko model

(2D beam)

In the context of this constitutive law, as seen in the previous page, $\varepsilon_{zz} \neq 0$. In particular:

$$\boxed{\varepsilon_{zz} = -\frac{\nu}{E}\,\sigma_{xx}}$$

This is clearly in contrast with

$\boxed{\varepsilon_{zz} = 0}$ as predicted by Timoshenko model.

Which is the inconsistency between constitutive law and kinematic field initially introduced.

How is this inconsistency fixed?

The energy contribution associated with ε_{zz} is

$$U^{zz} = \frac{1}{2}\int_{e} \sigma_{zz}\,\varepsilon_{zz}\,dx$$ it follows that:

$$U^{zz} = \frac{1}{2}\int_{e} \sigma_{zz}\,\underset{\substack{\parallel \\ 0}}{\varepsilon_{zz}}\,dx = 0$$ (referring to the kinematic model)

$$U = \frac{1}{2}\int_{\ell} \underset{\overset{\|}{0}}{\sigma_{zz}}\, \varepsilon_{zz}\, dx = 0$$ (referring to the constitutive law)

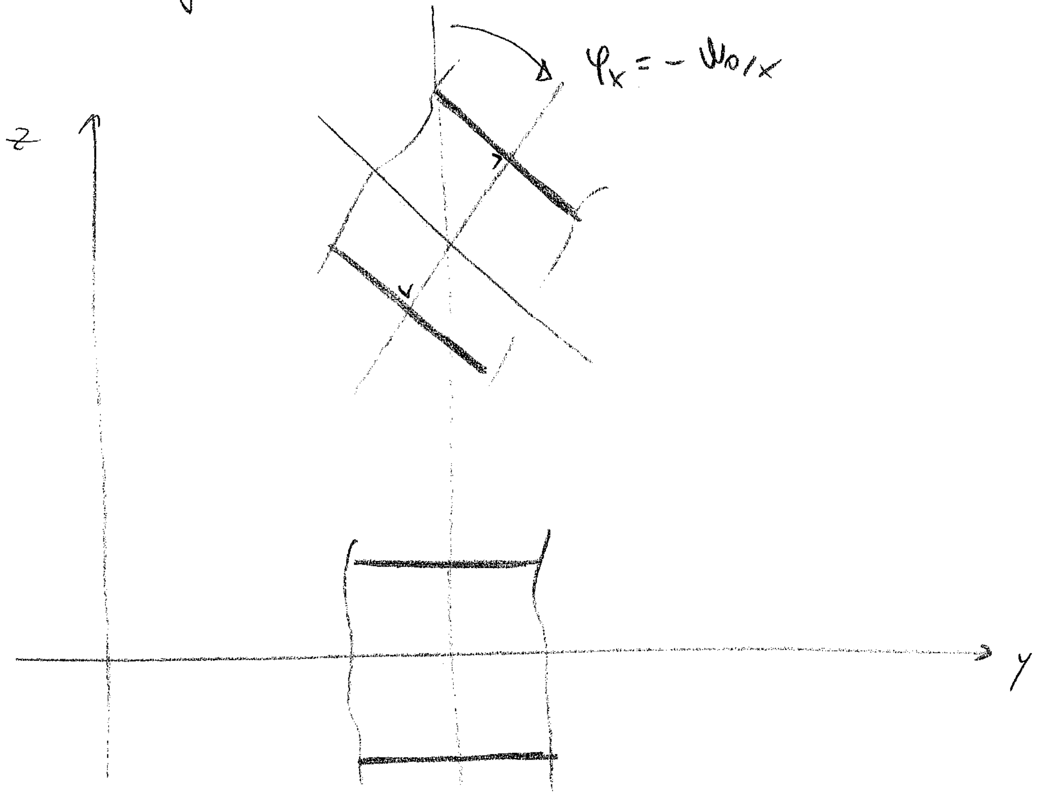
Concluding, the inconsistency of the model over the ε_{zz} does not alter the expected value of the strain energy U^{zz}, which is zero, in both cases, although for different reasons.

Euler - Bernoulli beam model

An additional level of approximation is introduced by the Euler-Bernoulli beam model (with respect to the Timoshenko one).

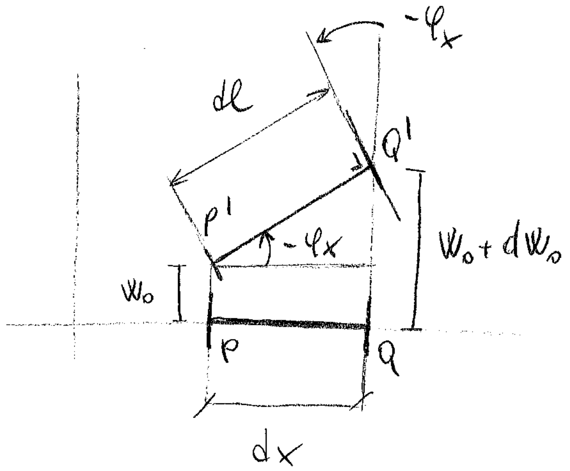
In particular, it is assumed that sections initially straight and normal to the beam axis remain straight and normal after the deformation happens.

The bending displacement is then approximated as:



In other words the rotation of the section is equal to the first derivative of the bending displacement (apart from the sign)

This can be clearly seen in the sketch here below where an infinitesimal portion of beam is depicted



The bending displacement of the generic point P is w_o. The point Q, infinitesimally close to P, undergoes a displacement which is $w_o + dw_o$.

It follows that

$$\begin{aligned} dl \cos(-\varphi_x) &= dx \\ dl \sin(-\varphi_x) &= dw_o \end{aligned} \Rightarrow \quad w_{o/x} = tg(-\varphi_x) \simeq -\varphi_x \quad (\text{if } \varphi_x \text{ is small})$$

The kinematic model of the Euler-Bernoulli beam model is then:

$$\boxed{\begin{aligned} u(x,z) &= u_o(x) - z\, w_{o/x}(x) \\ w(x,z) &= w_o(x) \end{aligned}}$$

where u_o and w_o are the generalized displacement components, function of the position along the beam axis.

The assumption of sections remaining perpendicular to the beam axis allows then to reduce the number of unknowns from 3 (in Timoshenko) to 2.

The strains are expressed by substituting the displacement field into the expression of the strain tensor, leading to:

$$\boxed{\begin{aligned} \varepsilon_{xx} &= u_{0/x} - z\, w_{0/xx} \\ \gamma_{xz} &= w_{/x} + u_{/z} = + w_{0/x} - w_{0/x} = 0 \\ (\varepsilon_{zz} &= w_{/z} = 0) \end{aligned}}$$

or, by separating the terms dependent on x and z:

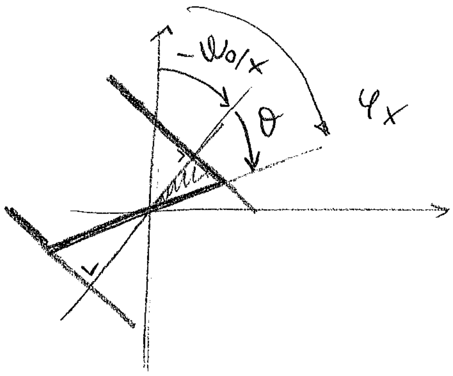
$$\boxed{\varepsilon_{xx} = \varepsilon_{0xx} + z\, k_x}$$

where

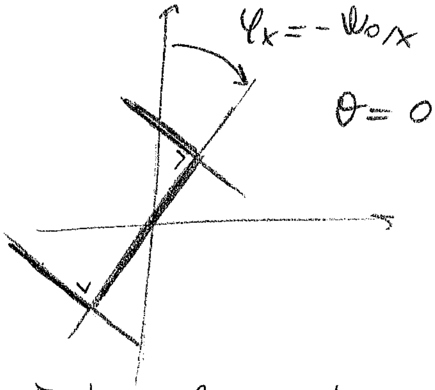
$$\boxed{\begin{aligned} \varepsilon_{0xx} &= u_{0/x} \\ k_x &= w_{0/xx} \end{aligned}}$$

generalized deformations

It is important to note that, in contrast to the Timoshenko beam, the shearing strain γ_{xz} is now zero. The Euler-Bernoulli beam is thus incapable of describing shearing deformations (and consequently to account for their contribution to the strain energy). This result is not surprising as far as it is implicit in the initial assumption of sections remaining normal.



Timoshenko



Euler-Bernoulli

The angle θ, which is indeed the shearing deformation, is identically null in Euler-Bernoulli.

- <u>Generalized stress measures</u>

As done for Timoshenko model, the generalized stress measures energetically conjugate with the generalized deformations obtained are derived by writing the internal virtual work

$$\delta W_i = \int_V \delta\varepsilon_{xx}\,\sigma_{xx}\,dV = \qquad \text{(the other } \varepsilon_{ik} \text{ are null!)}$$

$$= \int_V \left(\delta\varepsilon_{0xx} + z\,\delta K_x\right)\sigma_{xx}\,dV$$

$$= \int_\ell \delta\varepsilon_{0xx} \int_A \sigma_{xx}\,dA\,dx + \int_\ell \delta K_x \int_A z\,\sigma_{xx}\,dA\,dx$$

$$= \int_\ell \left(\delta\varepsilon_{0xx}\,N_x + \delta K_x\,M\right)dx$$

Hence the generalized stress measures are the axial force N_x and the bending moment M.

- Generalized forces

The distributed forces consistent with the Euler-Bernoulli beam model are derived by expressing the external virtual work by considering a volume force $\underline{F}$ with components F_x and F_z:

$$\delta W_e = \int_V (\delta u\, F_x + \delta w\, F_z)\, dV$$

$$= \int_V (\delta u_0\, F_x - \delta w_{0/x}\, z F_x + \delta w_0\, F_z)\, dV$$

$$= \int_e \left[\delta u_0 \int_A F_x\, dA + \delta w_0 \int_A F_z\, dA - \delta w_{0/x} \int_A F_x z\, dA\right] dx$$

$$= \int_e \left(\delta u_0\, \hat{n}_x + \delta w_0\, \hat{n}_z - \delta w_{0/x}\, \hat{m}_x\right) dx$$

The last contribution can be integrated by parts, leading to a contribution in the form:

$$-\int_e \delta w_{0/x}\, \hat{m}_x\, dx \longrightarrow \int_e \delta w_0\, \hat{m}_{x/x}\, dx$$

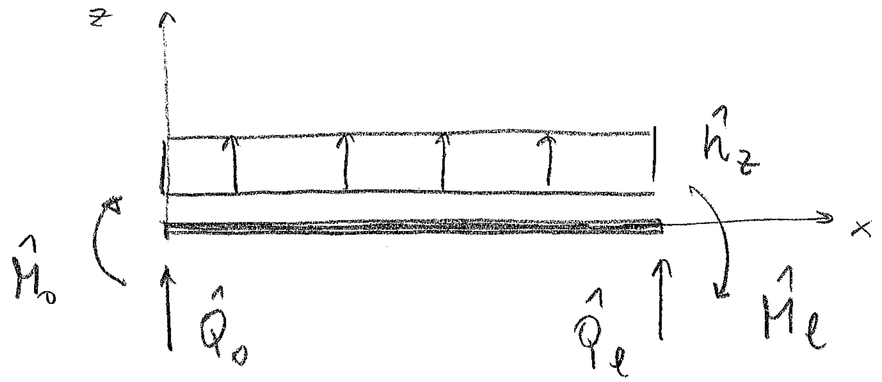
This means that the Euler-Bernoulli model is not influenced by $\hat{m}_x$, but only by $\hat{m}_{x/x}$. In other words, the model cannot deal with a set of distributed moments as they would promote shearing deformations which are outside the capabilities of the Euler-Bernoulli beam model.

For this reason $\hat{m}_x$ will not be considered.

In addition, even the presence of $\hat{n}_x$ is not accounted for, as the axial behaviour of the Euler-Bernoulli beam model is equal to the Timoshenko one.

Equilibrium equations

Consider then a beam subjected to a distributed shearing load and end concentrated forces:



The equilibrium conditions are found by means of the PVW:

$$\int_\ell \left(\delta\varepsilon_{0_{xx}} N_x + \delta K_x M \right) dx = \int_0^\ell \left(\delta u_0 \hat{n}_x + \delta w_0 \hat{n}_z \right) dx$$

+ concentrated forces

not considered as they lead to the same equations already obtained in Timoshenko

$$\int_0^\ell \delta K_x M \, dx =$$

$$\int_0^\ell - \delta w_{0/xx} M \, dx = \quad \text{(integrate by parts two times)}$$

$$= \int_0^\ell \delta w_{0/x} M_{/x} \, dx - \delta w_{0/x} M \Big|_0^\ell$$

$$= - \int_0^\ell \delta w_0 M_{/xx} \, dx + \delta w_0 M_{/x} \Big|_0^\ell - \delta w_{0/x} M \Big|_0^\ell$$

So:

$$\delta W_i = - \int_0^\ell \delta w_0 M_{/xx} \, dx + \delta w_0(\ell) M_{/x}(\ell) - \delta w_0(0) M_{/x}(0)$$
$$- \delta w_{0/x}(\ell) M(\ell) + \delta w_{0/x}(0) M(0)$$

The external work is

$$\delta W_e = \int_0^\ell \delta w_0 \hat{n}_z \, dx + \delta \varphi(0) \hat{M}_0 + \delta \varphi(\ell) \hat{M}_\ell + \delta w_0 \hat{Q}_0 + \delta w_\ell \hat{Q}_\ell$$

the rotation φ is : $\boxed{\varphi = - w_{0/x}}$ so:

$$= \int_0^\ell \delta w_0 \hat{n}_z \, dx - \delta w_{0/x}(0) \hat{M}_0 - \delta w_{0/x}(\ell) \hat{M}_\ell +$$
$$+ \delta w_0 \hat{Q}_0 + \delta w_\ell \hat{Q}_\ell$$

Imposing how $\delta W_i = \delta W_e$, it is obtained the set of equilibrium equations and boundary conditions

$$
\boxed{\begin{array}{llll}
M_{/xx} + \hat{n}_z = 0 & \text{in } x \in [0 \; \ell] & & \leftarrow \text{equilibrium equation} \\
M(0) = -\hat{M}_0 & \text{or} & \delta w_{0/x}(0) = 0 & \\
M(\ell) = \hat{M}_\ell & \text{or} & \delta w_{0/x}(\ell) = 0 & \text{boundary} \\
M_{/x}(0) = -\hat{Q}_0 & \text{or} & \delta w_0(0) = 0 & \text{conditions} \\
M_{/x}(\ell) = \hat{Q}_\ell & \text{or} & \delta w_0(\ell) = 0 &
\end{array}}
$$

natural ↑ ↑ essential

<u>Constitutive law and equilibrium equations in terms of displacements</u>

Similarly to the case of the Timoshenko beam, it is assumed a state of axial stress, so:

$$\sigma_{xx} = E\,\varepsilon_{xx}$$

It follows that the beam section constitutive law is:

$$\left(N_x = \int_A \sigma_{xx}\, dA = EA\,\varepsilon_{0xx} \right) \leftarrow \text{not considered}$$

$$M = \int_A \sigma_{xx}\, z\, dA = EJ\, k_x = -EJ\, w_{0/xx}$$

$$Q_z = \int_A \sigma_{xz}\, dA = \int_A G\gamma_{xz}\, dA = 0 \qquad (\text{as } \gamma_{xz} = 0)$$

the equilibrium conditions, expressed in terms of displacements, are then:

$$(EJ\, w_{0/xx})_{/xx} - \hat{n}_z = 0 \qquad \text{in } x \in [0\ \ell]$$

$$-EJ\, w_{0/xx} = -\hat{M}_0 \qquad \text{or} \qquad \delta w_{0/x} = 0 \qquad \text{in } x = 0$$

$$-EJ\, w_{0/xx} = \hat{M}_\ell \qquad \text{or} \qquad \delta w_{0/x} = 0 \qquad \text{in } x = \ell$$

$$-(EJ\, w_{0/xx})_{/x} = -\hat{Q}_0 \qquad \text{or} \qquad \delta w_0 = 0 \qquad \text{in } x = 0$$

$$-(EJ\, w_{0/xx})_{/x} = \hat{Q}_\ell \qquad \text{or} \qquad \delta w_0 = 0 \qquad \text{in } x = \ell$$

if the beam properties are constant along x:

$$\boxed{EJ\, w_{0/xxxx} - \hat{n}_z = 0 \qquad \text{in } x \in [0\ \ell]}$$

equilibrium equation

$$EJ\, w_{0/xx} = \pm \hat{M} \qquad \text{or} \qquad \delta w_{0/x} = 0 \qquad \text{in } x = 0, \ell$$

$$EJ\, w_{0/xxx} = \pm \hat{Q} \qquad \text{or} \qquad \delta w_0 = 0 \qquad \text{in } x = 0, \ell$$

boundary conditions

natural

essential

Remarks

- The number of unknowns is reduced to 2 (u_0 and w_0. Again note that u_0 is not entering the equations here reported because the axial behaviour is equal to the one obtained for the Timoshenko beam, so: $EA\, u_{0,xx} + \hat{n}_x = 0$).

- The set of solving equations is given by two uncoupled ordinary differential equations:

$$\begin{cases} EA\, u_{0,xx} + \hat{n}_x = 0 & \text{(axial equilibrium)} \\ EJ\, w_{0,xxxx} - \hat{n}_z = 0 & \text{(bending equilibrium)} \end{cases}$$

The two equations express, respectively, the axial and the bending equilibrium.

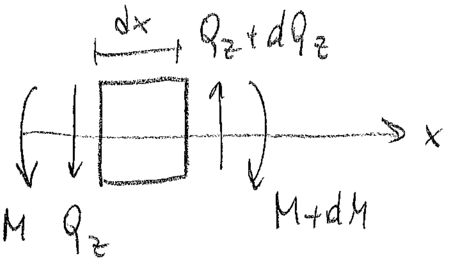
The equation expressing the equilibrium along z (the shear equilibrium) is not part of the solving equations. Similarly, the internal shear force q_z cannot be obtained from the constitutive law ($q_z = GA\, \gamma_{xz} = 0$).

However, it is important to clarify that the internal shear force q_z is, in general, different from zero.

If the shear force is needed, it can be determined ex-post by recalling that:

$$Q_z = M_{/x}$$
$$= -(EJ\, w_{0/xx})_{/x}$$

(The equation $Q_z = M_{/x}$ is available from the equilibrium to the moment of an infinitesimal portion of beam



)

With this regard the internal shear force Q_z can be seen as the reaction force imposing the condition of sections remaining normal.

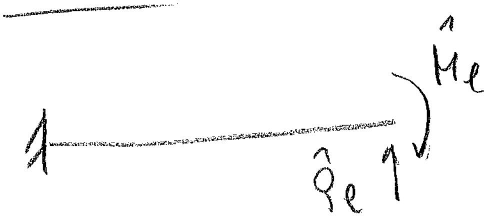
- The assumption of sections remaining normal to the beam axis was also obtained in the DSV solution for pure bending. In this sense the Euler Bernoulli displacement field can be interpreted as the application of the DSV solution for pure bending in the context of an approximate kinematic theory

- ## Boundary conditions

Regarding the bending behaviour, the boundary conditions can be

1. Free: no geometric constraints + natural conditions
2. Simply-supported: $w_0 = 0$ + natural condition on moment equilibrium.
3. Clamped: $w_0 = 0$; $\varphi_x = 0$

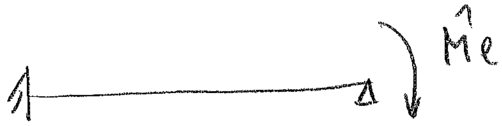
Examples

- 

Clamped-free

$w_0(0) = 0$ $\quad$ $EJ\, w_{0/xx}(\ell) = -\hat{M}_\ell$

$\varphi_x(0) = 0$ $\quad$ $EJ\, w_{0/xxx}(\ell) = -\hat{P}_\ell$

- 

Clamped-simply-supported

$w_0(0) = 0$ $\quad$ $w_0(\ell) = 0$

$\varphi_x(0) = 0$ $\quad$ $EJ\, w_{0/xx}(\ell) = -\hat{M}_\ell$