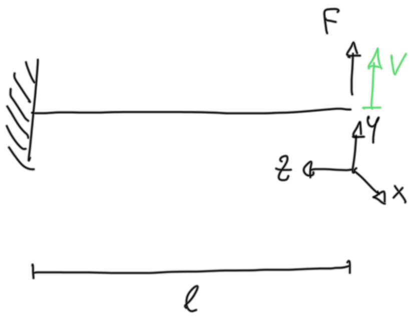
LABS

Displacement of Beams System I

1)



DATA

$l = 1000\ mm$

$F = 4000\ N$

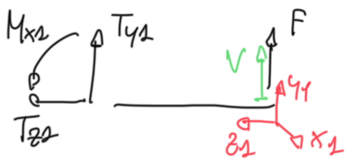
$E = 200\ GPa$

$J_{xx} = 500\,000\ mm^4$

Let's find v

- Internal Actions

REAL



$$\begin{cases} T_{z1} = \emptyset \\ T_{y1} = -F \\ M_{x1} = -F \cdot z_1 \end{cases}$$

The elastic problem is solved

But if I want to find v, I have to use PCVW.

VIRTUAL

$$\begin{cases} T'_{z1} = \emptyset \\ T'_{y1} = -1 \\ M'_{x1} = -z_1 \end{cases}$$

- PCVW

$$\delta W_e = 1 \cdot v$$

$$\delta W_i = \frac{1}{EJ_{xx}} \int_0^l M'_{x1} \cdot M_{x1}\, dz_1 = \frac{1}{EJ_{xx}} \int_0^l (-z_1) \cdot (-F \cdot z_1)\, dz_1 =$$

$$= \frac{1}{EJ_{xx}} \int_0^l F \cdot z_1^2\, dz_1 = \frac{1}{EJ_{xx}} \left[\frac{1}{3} F z_1^3 \right]_0^l = \frac{Fl^3}{3EJ_{xx}}$$

Fl^3

$dW_e = dW_i \qquad V = \frac{\ldots}{3EJ_{xx}} = 13.33\ mm$

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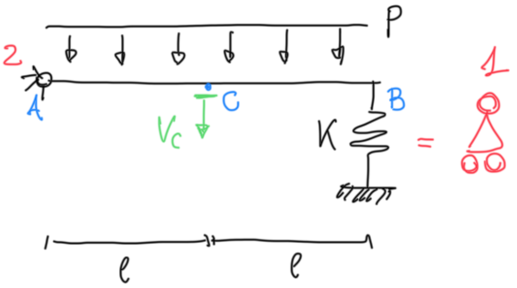
2) (Exam 06/09/2021)



DATA

$l = 1200\ mm$

$p = 12\ N/mm$

$EJ = 10^{12}\ N mm^2 \qquad \frac{N}{\cancel{mm^2}} \cdot mm^{\cancel{4}\,2} = N \cdot mm^2$

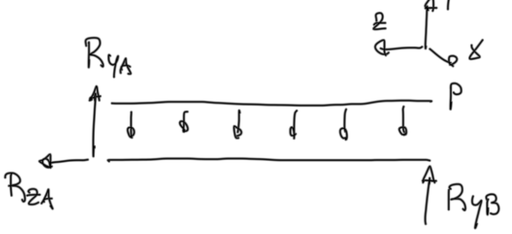
$K = 750\ N/mm$

Let's find V_c

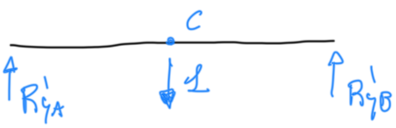
The system is isostatic.

- Reaction Forces

REAL



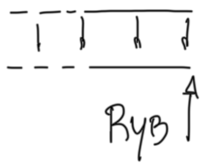
VIRTUAL



$$\begin{cases} R_{zA} = \emptyset \\ R_{yA} + R_{yB} - p \cdot 2l = \emptyset \ * \\ A)\ R_{yB} \cdot 2l = 2pl^2 \end{cases} \qquad R'_{yA} = R'_{yB} = \frac{1}{2}$$

$$* \begin{cases} R_{yA} = pl \\ R_{yB} = pl \end{cases} \longrightarrow F = K \cdot V_B$$

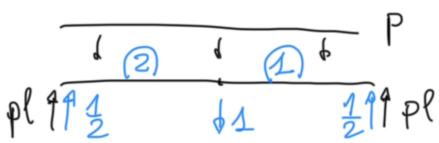
spring's internal force



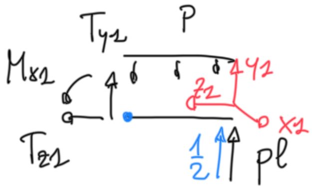
interface equilibrium

$$\boxed{F = -R_{yB}}$$

- Internal Actions



1



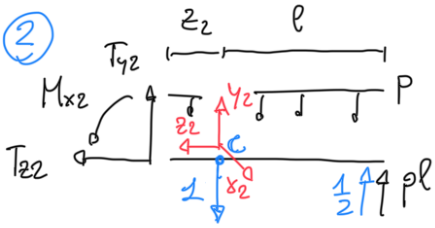
REAL

$$\begin{cases} T_{z1} = \emptyset \\ T_{y1} = p \cdot z_1 - pl \\ M_{x1} = \frac{1}{2} p z_1^2 - pl \cdot z_1 \\ \quad = p\left(\frac{1}{2} z_1^2 - l z_1\right) \end{cases}$$

VIRTUAL

$$\begin{cases} T'_{z1} = \emptyset \\ T'_{y1} = -\frac{1}{2} \\ M'_{x1} = -\frac{1}{2} z_1 \end{cases}$$

2



REAL

$$\begin{cases} T_{z2} = \emptyset \\ T_{y2} = p(l+z_2) - pl \\ M_{x2} = \frac{1}{2} p (l+z_2)^2 - pl(l+z_2) = \\ \quad = p\left(\frac{1}{2}l^2 + \frac{1}{2}z_2^2 + \cancel{l z_2} - l^2 - \cancel{l z_2}\right) = \\ \quad = \frac{p}{2}\left(z_2^2 - l^2\right) \end{cases}$$

VIRTUAL

$$\begin{cases} T'_{z2} = \emptyset \\ T'_{y2} = 1 - \frac{1}{2} = \frac{1}{2} \\ M'_{x2} = z_2 - \frac{1}{2}(l + z_2) = \\ \quad = \frac{1}{2} z_2 - \frac{1}{2} l \end{cases}$$

- PCVW

the virtual work of R'_{yB} is negative because its sign is opposite to the one of F, thus of V_B

$$\delta W_e = 1 \cdot V_C - R'_{yB} \cdot V_B = V_C - \frac{1}{2} \cdot \frac{F}{K} = V_C - \frac{pl}{2K}$$

$$\delta W_i = \frac{1}{EJ} \int_0^l M'_{x1} \cdot M_{x1} \, dz_1 + \frac{1}{EJ} \int_0^l M'_{x2} \cdot M_{x2} \, dz_2 =$$

$$= \frac{p}{EJ}\left(\int_0^l \left(-\frac{1}{2} z_1\right)\left(\frac{1}{2} z_1^2 - l z_1\right) dz_1 + \int_0^l \frac{1}{4}\left(z_2 - l\right)\left(z_2^2 - l^2\right) dz_2 \right) =$$

$$= \frac{p}{EJ}\left(\int_0^l \left(-\frac{z_1^3}{4} + \frac{l z_1^2}{2}\right) dz_1 + \frac{1}{4}\int_0^l \left(z_2^3 - l^2 z_2 - l z_2^2 + l^3\right) dz_2 \right) =$$

$$= \frac{p}{EJ}\left(\left[-\frac{z_1^4}{16} + \frac{l z_1^3}{6} \right]_0^l + \left[\frac{z_2^4}{16} - \frac{l^2 z_2^2}{8} - \frac{l z_2^3}{12} + \frac{l^3}{4} z_2 \right]_0^l \right) =$$

$$= \frac{p}{EJ}\left(\left[-\cancel{\frac{l^4}{16}} + \frac{l^4}{6} + \cancel{\frac{l^4}{16}} - \frac{l^4}{8} - \frac{l^4}{12} + \frac{l^4}{4} \right) = \frac{5}{24} \frac{pl^4}{EJ}$$

$$\delta W_i = \delta W_e \qquad \frac{5}{24}\frac{pl^4}{EJ} = V_C - \frac{pl}{2K} \qquad V_C = \frac{5}{24}\frac{pl^4}{EJ} + \frac{pl}{2K} = 14.78 \text{ mm}$$

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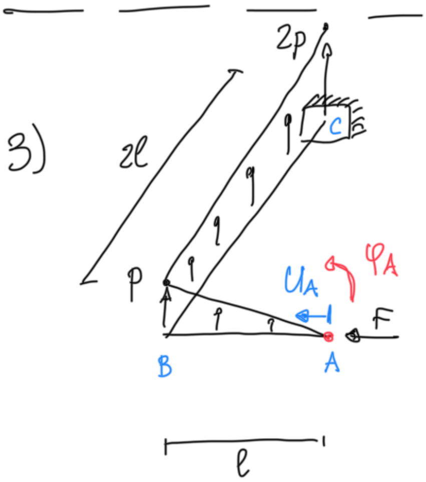
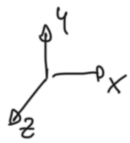
3)





3D

DATA

$l = 750$ mm

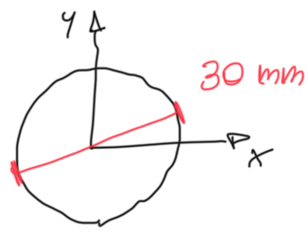
$p = 0.08$ N/mm

$E = 200$ GPa

$G = 77$ GPa

$F = 1000$ N

Let's find u_A and φ_A



- Section Properties

$$A = \pi r^2 = 706.86 \text{ mm}^2$$

$$J_{xx} = J_{yy} = \frac{\pi r^4}{4} = 39761 \text{ mm}^4$$

Polar Inertia Moment

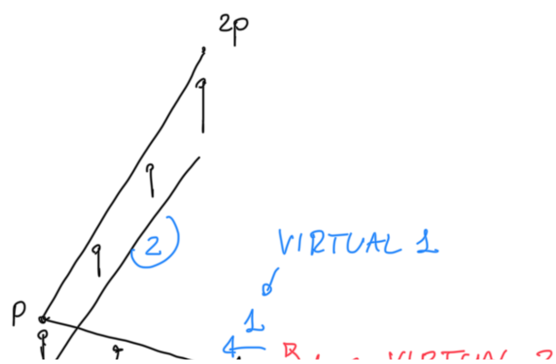
$$J_p = J_{xx} + J_{yy} = 79522 \text{ mm}^4$$

if you have an axisymmetric section $\rightarrow$ Torsional Stiffness GJ_p

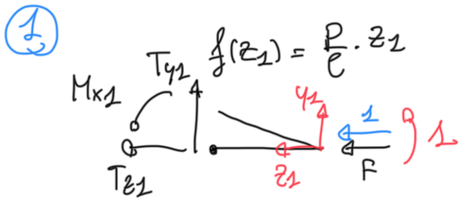
for the other types of section J_p must be corrected to consider the real distriubtion of shear and warping

$$J_p \rightarrow J_p^* \quad \longrightarrow \quad \text{Torsional Stiffness } GJ_p^*$$

- Internal Actions



$)1 \leftarrow$ VIRTUAL 2



REAL

$T_{z1} = -F$

$$T_{y1} = -\frac{1}{2}\underbrace{\frac{p}{\ell}z_1}_{\text{heigth}}\cdot\underbrace{z_1}_{\text{base}} = -\frac{p z_2^2}{2\ell}$$

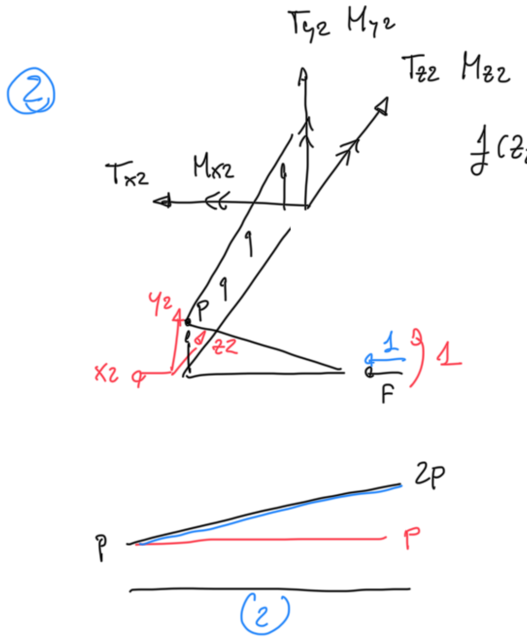
$$M_{x1} = -\frac{1}{2}\frac{p z_1^2}{\ell}\cdot\frac{1}{3}z_1 = -\frac{1}{6}\frac{p}{\ell}z_1^3$$

VIRTUAL 1

$T'_{z1} = -1$

VIRTUAL 2

$M''_{x1} = -1$



$$f(z_2) = \frac{p}{2\ell}\cdot z_2 + p$$

REAL

$$\begin{cases} T_{z2} = \emptyset \\ T_{y2} = -\frac{1}{2}p\ell - p z_2 - \frac{1}{4}\frac{p}{\ell}\cdot z_2^2 \\ T_{x2} = -F \\ M_{z2} = \frac{p\ell}{2}\cdot\frac{1}{3}\ell = \frac{1}{6}p\ell^2 \\ M_{y2} = F\cdot z_2 \\ M_{x2} = \underbrace{-\frac{1}{2}p\ell\cdot z_2}_{1} - p\frac{z_2^2}{2} - \underbrace{\frac{p}{2\ell}z_2\cdot\frac{1}{3}z_2}_{2} \end{cases}$$

VIRTUAL 1

$$\begin{cases} T'_{x2} = -1 \\ M'_{y2} = z_2 \end{cases}$$

VIRTUAL 2

$M''_{z2} = 1$

- PCVW

VIRTUAL 1

$\delta W_e = 1\cdot u_A$

$$\delta W_i = \int_0^{\ell} T'_{z1}\cdot\frac{T_{z1}}{EA}dz_1 + \int_0^{2\ell} M'_{y2}\cdot\frac{M_{y2}}{EJ}dz_2 =$$

$$= \int_0^{\ell}(-1)\left(-\frac{F}{EA}\right)dz_1 + \int^{\ell} z_2\cdot\frac{F z_2}{EJ}dz_2 =$$

$$= \left[\frac{F}{EA} z_2\right]_0^{\ell} + \left[\frac{1}{3}\frac{F z_2^3}{EJ}\right]_0^{2\ell} = \frac{F\ell}{EA} + \frac{8}{3}\frac{F\ell^3}{EJ}$$

$$\delta W_i = \delta W_e \quad \rightarrow \quad u_A = 141.48 \text{ mm}$$

VIRTUAL 2

$$\delta W_e = 1 \cdot \varphi_A$$

$$\delta W_i = \int_0^{\ell} M_{x1}'' \cdot \frac{M_{x1}}{EJ} dz_1 + \int_0^{2\ell} M_{z2}'' \cdot \frac{M_{z2}}{GJ_p} dz_2$$

$$\delta W_i = \delta W_e$$

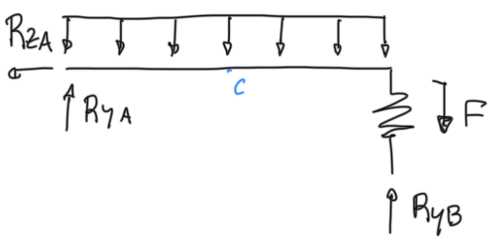
$$\varphi_A = \int_0^{\ell} (-1)\cdot\left(-\frac{1}{6} p z_1^3 \cdot \frac{1}{\ell}\right)\frac{1}{EJ} \cdot dz_1 + \int_0^{2\ell} 1 \cdot \frac{1}{6} p\ell \cdot \frac{1}{GJ_p} dz_2 =$$

$$= \frac{1}{24}\frac{p\ell^3}{EJ} + \frac{1}{3}\frac{p\ell^3}{GJ_p} = 2.014 \cdot 10^{-3} \text{ rad}$$

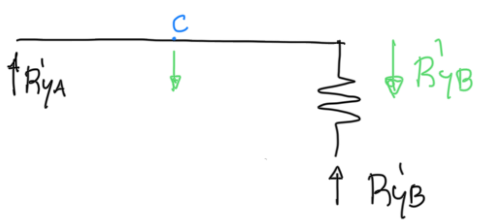
Alternative solutions to exercise 2

I) Let's consider the spring among the deformable structures

REAL



VIRTUAL



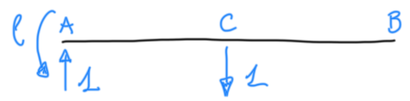
Now the spring work is included in the INTERNAL virtual work

$$\delta W_e = 1 \cdot v_C$$

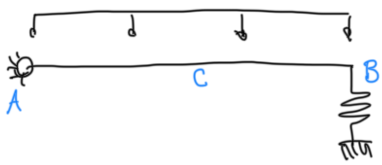
$$\delta W_i = \int_0^{\ell} M'_{x1} \frac{M_{x1}}{EJ} dz_1 + \int_0^{\ell} M'_{x2} \frac{M_{x2}}{EJ} dz_2 + R'_{yB} \cdot \frac{R_{yB}}{K}$$

II) Let's consider an alternative virtual system

VIRTUAL



REAL



In this way, the right part of the structure is unloaded in the virtual system → NO virtual work by the forces in the spring

$$\delta W_i = 1 \cdot u_C + 1 \cdot u_A + \ell \cdot \varphi_A = $$
$$= 1 \cdot u_C + 1 \cdot 0 + \ell \cdot \varphi_A$$

φ_A is the REAL rotation of point A and it is unknown.
We shoud create a second virtual system to compute it.

→ Not so convenient