Ritz method: convergence of the solution

Course of Spacecraft Structures

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Measures of the errors

- Results are presented with regard to four examples aimed at illustrating the convergence of the Ritz method for 1D problems
- Two norms are introduced to measure the error due to the Ritz approximation

Error measures

$$||e^f||_{L_2} = \frac{\sqrt{\int_0^l \left(f - \tilde{f}\right)^2 \mathrm{d}x}}{\int_0^l f \mathrm{d}x}$$

$$||e^f||_{\text{inf}} = \max |f - \tilde{f}|$$

f: exact solution

$\tilde{f}$: Ritz solution

- The L_2 norm provides a measure of the global error, i.e. over the entire domain. The normalization factor at the denominator is introduced to express the average percent difference between the exact and the approximate solution
- The inf norm provides a measure of the maximum error at local level (the norm is not normalized as the exact solution could be locally zero, leading to a singularity)

Measures of the errors

- The convergence in the L_2 and inf norm is defined as

Convergence

$$\lim_{N \to +\infty} ||e_N^f||_{L_2} = 0$$

$$\lim_{N \to +\infty} ||e_N^f||_{\infty} = 0$$

N: number of terms of the Ritz approximation

- Note that convergence in the inf norm implies convergence in the L_2 norm, but NOT viceversa. In other words, the convergence in the inf norm is a stronger requirement with respect to the convergence in the L_2 norm

Convergence

- For the Ritz method (at least for 1D problems) the following results hold:

  1. The convergence is guaranteed for the unknown function and its derivatives up to the highest order of differentiation entering the generalized deformation parameters. For the Euler-Bernoulli beam model, the convergence is thus guaranteed for the bending displacement up to the second derivative (bending moment). Note that convergence of the shear, which is associated with the third derivative, could be obtained but is not guaranteed

  2. The convergence is guaranteed in the inf norm for continuous fields. Thus displacements (and rotations) converge in the inf norm, as they are always continuous quantities. Bending moments converge in the inf norm when they are continuous, whilst the convergence can be only in the L_2 norm when a discontinuity exists

Convergence

- In addition:
  1. The total potential energy converges monotonically from beyond (or, equivalently, the strain energy converges monotonically from below):

     $\Pi_1 \geq \Pi_2 \geq \ldots \geq \Pi_N \geq \Pi_{exact}$ with Π_i total potential energy using i terms

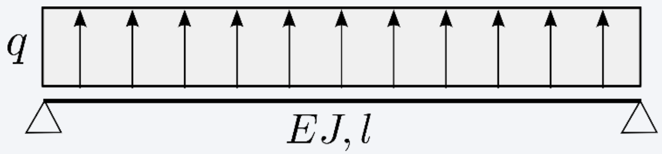
     $U_1 \leq U_2 \leq \ldots \leq U_N \leq U_{exact}$ with U_i strain energy using i terms

     In other words, <u>the stiffness of the structural model is globally relaxed as the number of degrees of freedom is increased.</u>

     This result does not imply that the local displacements obtained using M dofs are necessarily higher (in absolute value) with respect to the displacement obtained using N dofs, with M>N

  2. The rapidity of convergence reduces as the order of the derivatives is increased. It follows that bending moments converge slower with respect to the rotations which, in turn, converge slower than the displacements

Example 1: beam with distributed load



Essential conditions

$$\begin{cases} w(0) = 0 \\ w(l) = 0 \end{cases}$$

Ritz functions

$$\phi(x) = x^{i+1} - l^i x \qquad i = 1 \dots N$$

$$\phi(x) = \sin \frac{i\pi x}{l} \qquad i = 1 \dots N$$

Nondimensional parameters

$$\overline{w} = w\frac{EJ}{ql^4} \qquad \overline{M} = M\frac{1}{ql^2}$$

$$\overline{Q} = Q\frac{1}{ql} \qquad \overline{\Pi} = \Pi\frac{EJ}{q^2l^5}$$

Exact solution

$$w = \frac{ql^4}{24EJ}\left[\left(\frac{x}{l}\right)^4 - 2\left(\frac{x}{l}\right)^3 + \frac{x}{l}\right]$$

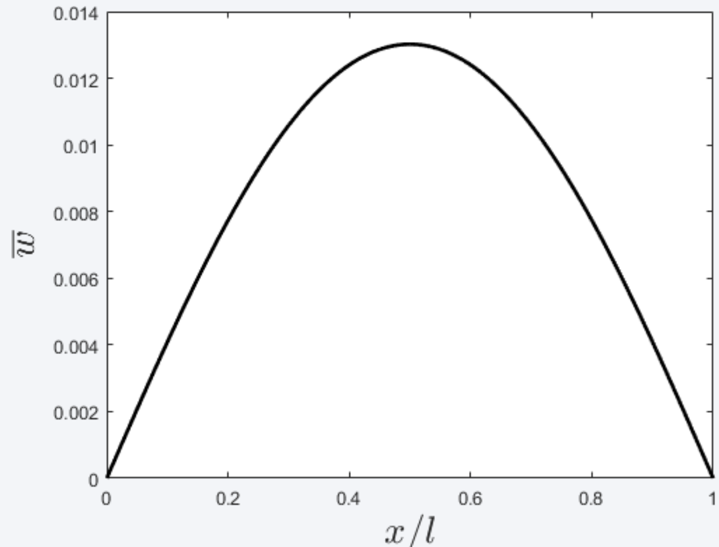
$$M = -EJw_{,xx} = qx\frac{l-x}{2}$$

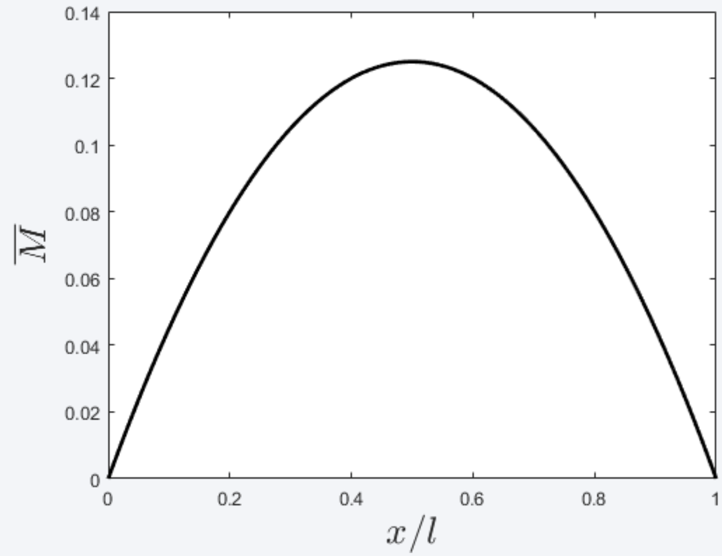
$$Q = -EJw_{,xxx} = q\left(\frac{l}{2} - x\right)$$

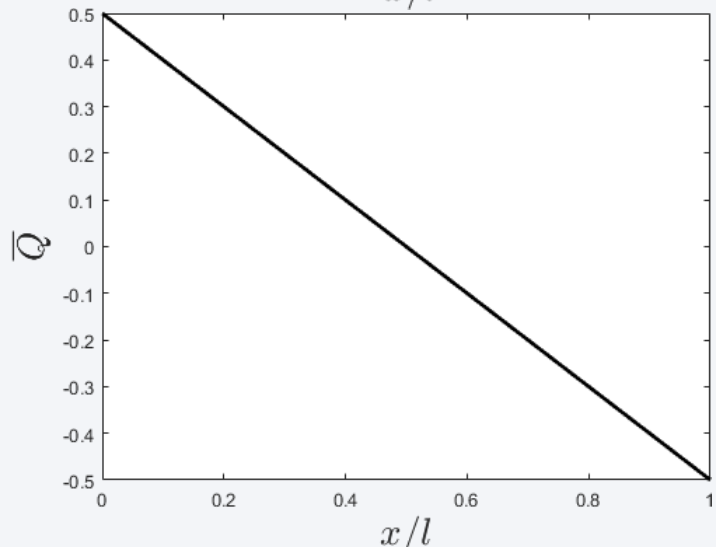
$$\Pi = -\frac{1}{240EJ}q^2l^5$$

Example 1: beam with distributed load

- Exact results







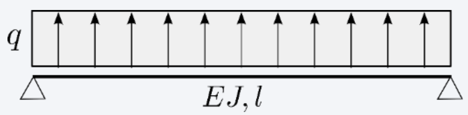
Note

- The bending moment (and, clearly, the displacement) is a continuous function: convergence is thus expected in the inf norm

Example 1: beam with distributed load

- Results using N trigonometric functions

	$\overline{w}(l/2)\cdot 10^2$	$\overline{M}(l/2)\cdot 10$	$\overline{\Pi}\cdot 10^3$
N = 1	1.3071	1.2901	-4.1606
N = 3	1.3017	1.2423	-4.1664
N = 5	1.3021	1.2526	-4.1666
N = 25	1.3021	1.2500	-4.1667
exact	1.3021	1.2500	-4.1667



Remarks

- The mid-displacement reaches convergence quickly: 5 trial functions are sufficient for reaching the exact solution
- The solution obtained with 1 term is a good approximation of the exact solution (the percent different with respect to the convergence solution is small). Note that the solution with N=1 is associated with a higher displacement with respect to the solutions obtained with N>1 (this result is local and does not inficiate the observation regarding the higher stiffness of models with less dofs)

Example 1: beam with distributed load

- Results using N trigonometric functions

	$\overline{w}(l/2) \cdot 10^2$	$\overline{M}(l/2) \cdot 10$	$\overline{\Pi} \cdot 10^3$
N = 1	1.3071	1.2901	-4.1606
N = 3	1.3017	1.2423	-4.1664
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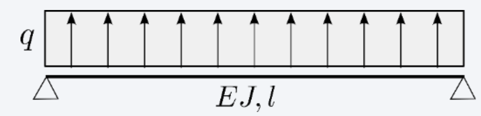
exact	1.3021	1.2500	-4.1667



Remarks

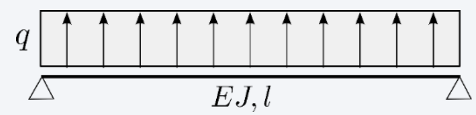
- The convergence of the bending moment, which is obtained by taking the second derivative of the displacement, is slower with respect to the transverse displacement. The convergence, at local level, is not monothonical
- The convergence of the total potential energy is quite fast. Few degrees of freedom allows to capture the exact value with a good degree of approximation
- The converegence of the total potential energy is monothonical (in this sense, the numerical model becomes more and more compliant as N increases)

Example 1: beam with distributed load

- Errors using N trigonometric functions

	$\|\|e^{\overline{w}}\|\|_{\infty} 10^5$	$\|\|e^{\overline{M}}\|\|_{\infty} 10^3$	$\|\|e^{\overline{Q}}\|\|_{\infty} 10^2$
N = 1	5.5734	5.3193	9.4715
N = 3	0.4938	1.4810	4.9684
N = 5	0.1063	0.6740	3.3472
N = 25	0.0003	0.0366	0.7790

	$\|\|e^{\overline{w}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{M}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{Q}}\|\|_{L_2} 10^2$
N = 1	0.4128	3.8013	12.0275
N = 3	0.0326	0.8673	4.7946
N = 5	0.0062	0.3364	2.6868
N = 25	0.0000	0.0091	0.3058

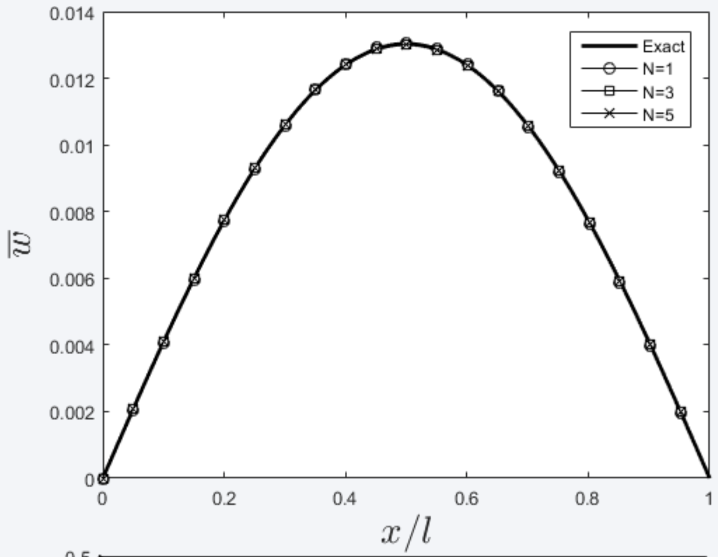


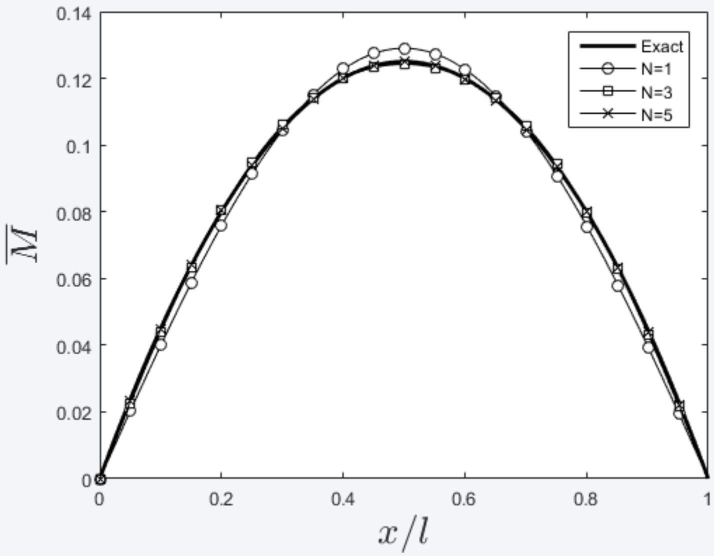
Remarks

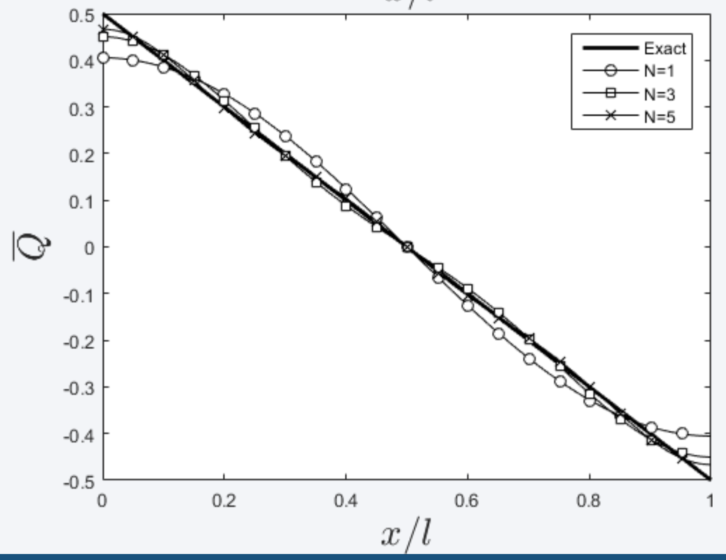
- The displacement and the bending moments are continuous functions and, as expected, the solution converges in the inf and L_2 norms
- Although not guaranteed, the convergence is achieved, in this specific case, also for the shear in both the norms
- The errors in the L_2 norm clearly illustrate the different rapidity of convergence of displacement, moment and shear. The errors obtained using N=5 are 0.01%, 0.34% and 2.69% for these three quantities, respectively

Example 1: beam with distributed load

- Plot of results using N=1, 3 and 5 terms (trigonometric functions)







Example 1: beam with distributed load

- Results using polynomial functions

	$\overline{w}(l/2) \cdot 10^2$	$\overline{M}(l/2) \cdot 10$	$\overline{\Pi} \cdot 10^3$
N = 1	1.0417	0.8333	-3.4722
N = 3	1.3021	1.2500	-4.1667
N = 5	1.3021	1.2500	-4.1667
exact	1.3021	1.2500	-4.1667

	$\|\|e^{\overline{w}}\|\|_\infty 10^2$	$\|\|e^{\overline{M}}\|\|_\infty 10^2$	$\|\|e^{\overline{Q}}\|\|_\infty 10^2$
N = 1	260.4167	83.3333	50.0000
N = 3	0.0000	0.0000	0.0000
N = 5	0.0000	0.0000	0.0000

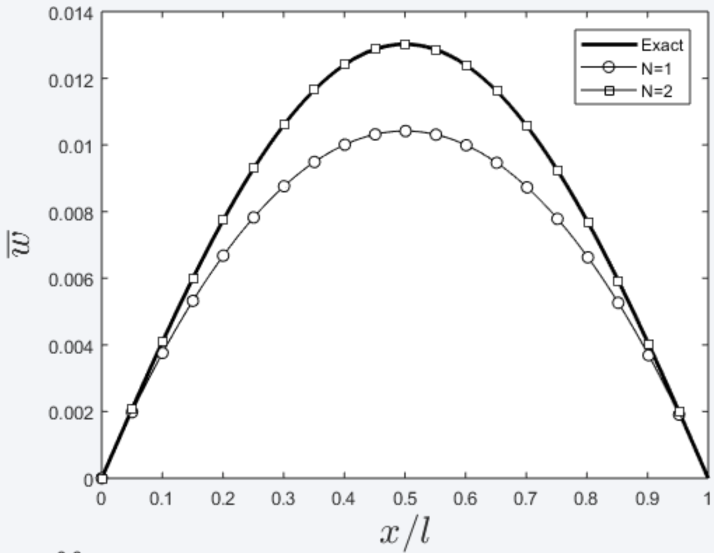
	$\|\|e^{\overline{w}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{M}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{Q}}\|\|_{L_2} 10^2$
N = 1	17.9605	40.8250	100.0000
N = 3	0.0000	0.0000	0.0000
N = 5	0.0000	0.0000	0.0000

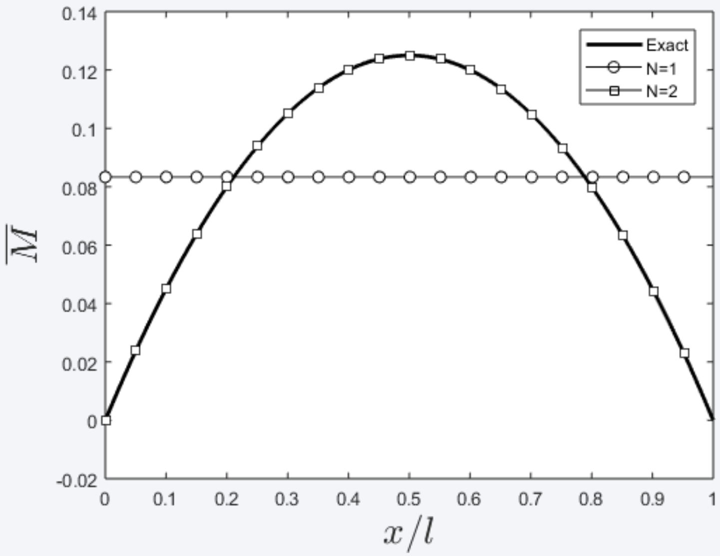
Remarks

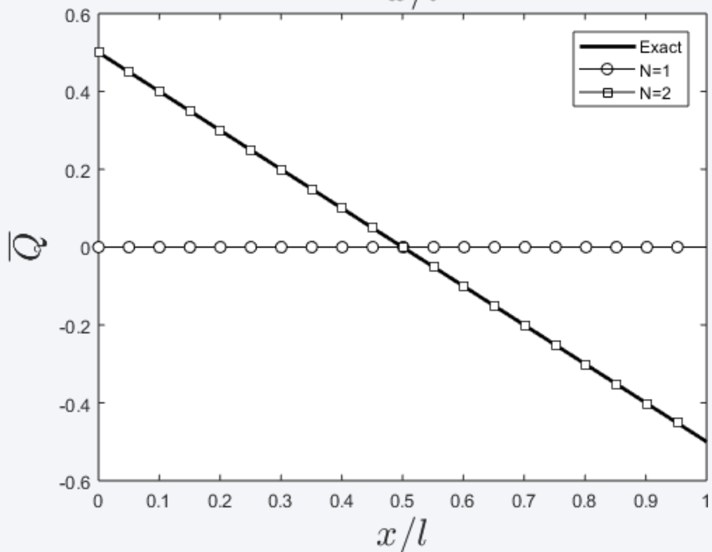
- In this case, the space spanned by the Ritz basis contains also the exact solution. Indeed the Ritz approximation converges to the exact solution with 3 dofs (when the quartic term is accounted for)
- Note that the 1 dof solution is less accurate with respect to the 1 dof solution obtained using the trigonometric expansion

Example 1: beam with distributed load

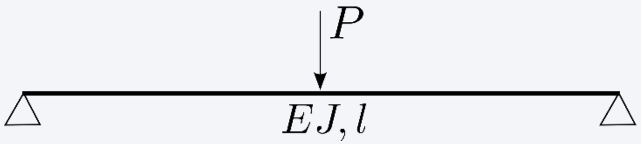
- Plot of results using N=1, 3 and 5 terms (polynomial functions)







Example 2: beam with concentrated load



Essential conditions

$$\begin{cases} w(0) = 0 \\ w(l) = 0 \end{cases}$$

Ritz functions

$$\phi(x) = x^{i+1} - l^i x \qquad i = 1 \dots N$$

$$\phi(x) = \sin\frac{i\pi x}{l} \qquad i = 1 \dots N$$

Nondimensional parameters

$$\overline{w} = w\frac{EJ}{Pl^3} \qquad \overline{M} = M\frac{1}{Pl}$$

$$\overline{Q} = Q\frac{1}{P} \qquad \overline{\Pi} = \Pi\frac{EJ}{P^2l^3}$$

Exact solution

Solution for $0 \leq x \leq l/2$ (anti-symmetric)

$$w = \frac{Pl^3}{48EJ}\left[4\left(\frac{x}{l}\right)^3 - 3\frac{x}{l}\right]$$

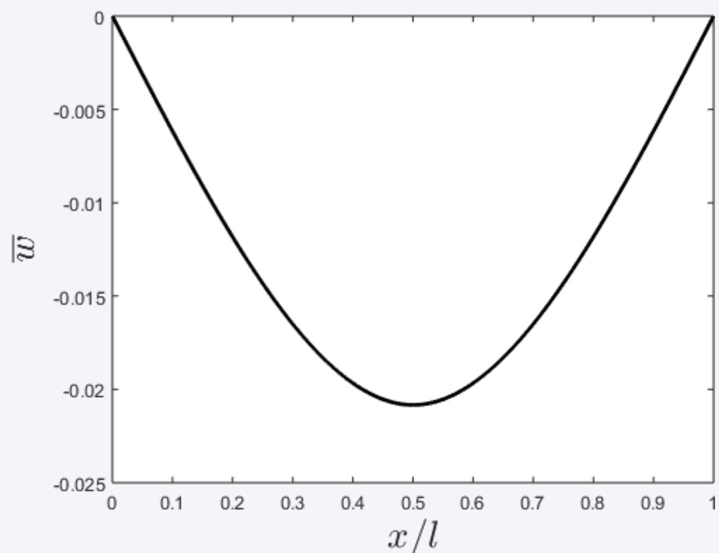
$$M = -\frac{1}{2}Px$$

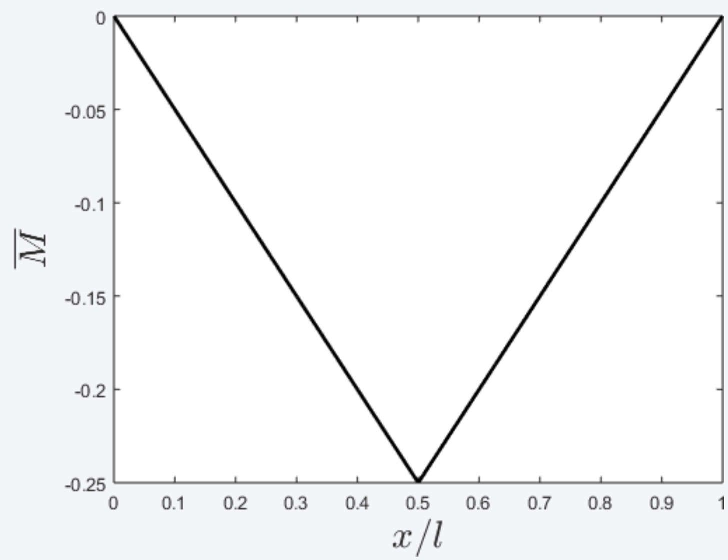
$$Q = -\frac{P}{2}$$

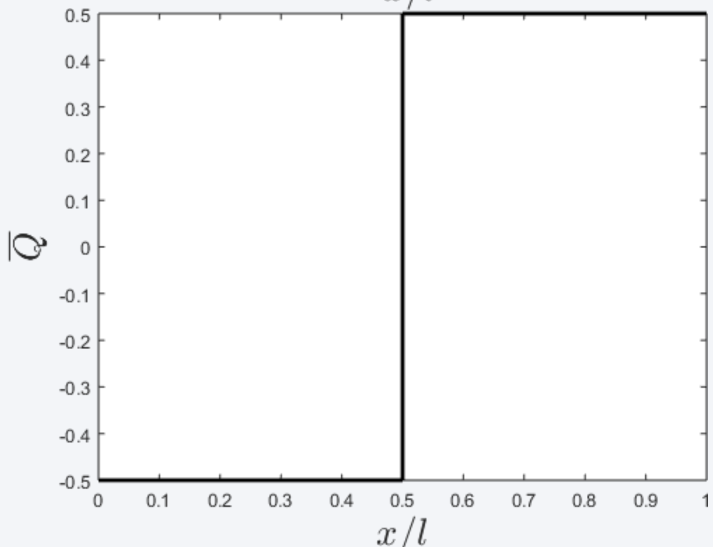
$$\Pi = -\frac{P^2l^3}{96EJ};$$

Example 2: beam with concentrated load

- Exact results







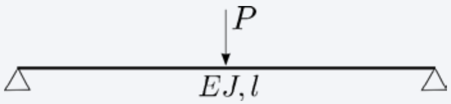
Note

- The shear is discontinuous (and the bending moment is characterized by a discontinuity of the first derivative). The exact solution cannot be achieved using global functions

Example 2: beam with concentrated load

- Results using N trigonometric functions

	$\overline{w}(l/2)\cdot 10^2$	$\overline{M}(l/2)\cdot 10$	$\overline{\Pi}\cdot 10^3$
N = 1	-2.0532	-2.0264	-10.2660
N = 3	-2.0785	-2.2516	-10.3927
N = 5	-2.0818	-2.3326	-10.4091
N = 25	-2.0833	-2.4610	-10.4166
exact	-2.0833	-2.5000	-10.4167



Remarks

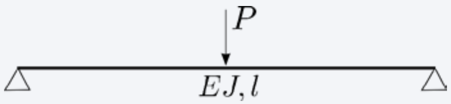
- As observed in the previous example, the convergence in terms of displacements is quite fast, and so is the convergence of the total potential energy
- An accurate estimate of the bending moment is possible only if a relatively large number of dofs is used. The convergence of the second derivatives is slower

Example 2: beam with concentrated load

- Errors using N trigonometric functions

	$\|\|e^{\overline{w}}\|\|_{\infty} 10^5$	$\|\|e^{\overline{M}}\|\|_{\infty} 10^3$	$\|\|e^{\overline{Q}}\|\|_{\infty} 10^2$
N = 1	30.1369	47.3576	50.0000
N = 3	4.7888	24.8418	50.0000
N = 5	1.5037	16.7361	50.0000
N = 25	0.0194	3.8951	50.0000

	$\|\|e^{\overline{w}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{M}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{Q}}\|\|_{L_2} 10^2$
N = 1	1.2456	12.0275	43.5238
N = 3	0.1662	4.7946	31.5230
N = 5	0.0451	2.6868	25.8744
N = 25	0.0003	0.3058	12.4890

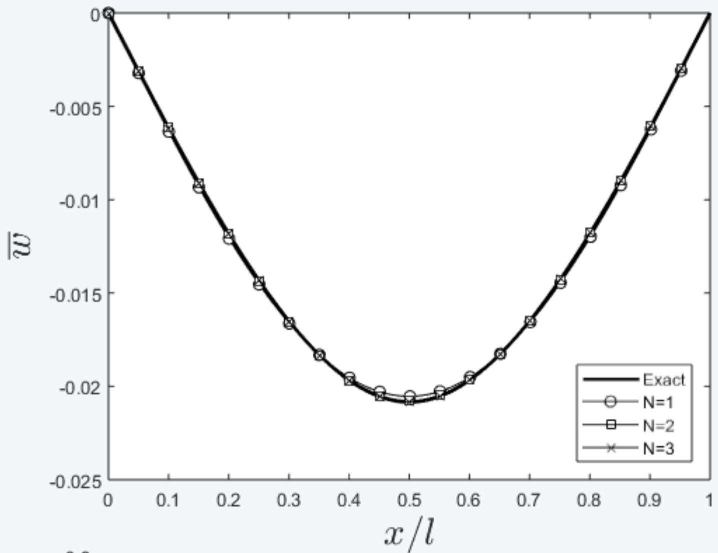


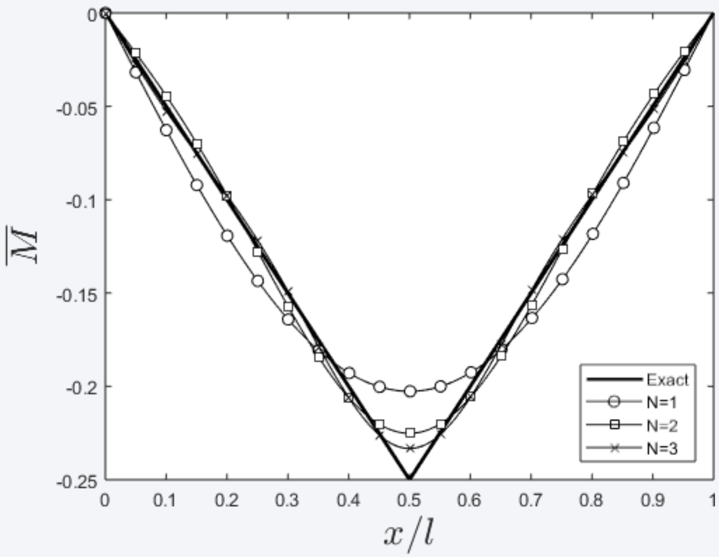
Remarks

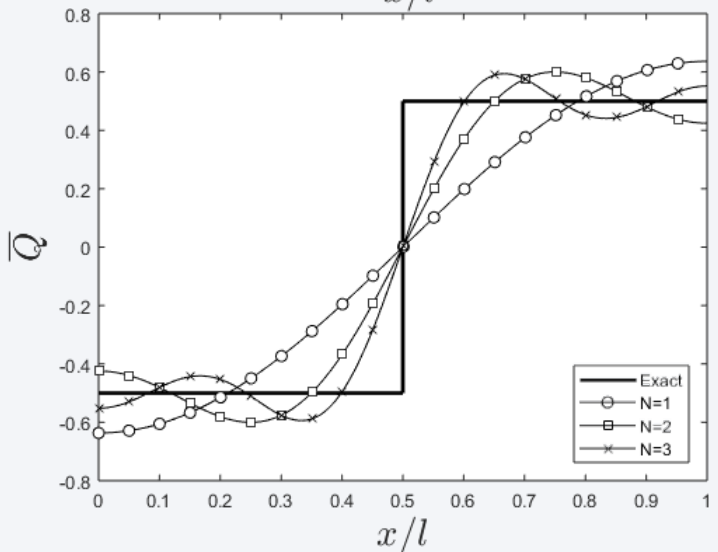
- The displacements and the bending moment converge in the inf norm (they are continuous functions, so convergence is guaranteed in the inf norm)
- On the contrary, the shear force does not converge in the inf norm. Convergence is observed only at global level (although for the shear it is not guaranteed a priori), but it is very slow: when N=25 the error is 12.5%!

Example 2: beam with concentrated load

- Plot of results using N=1, 3 and 5 terms (trigonometric functions)

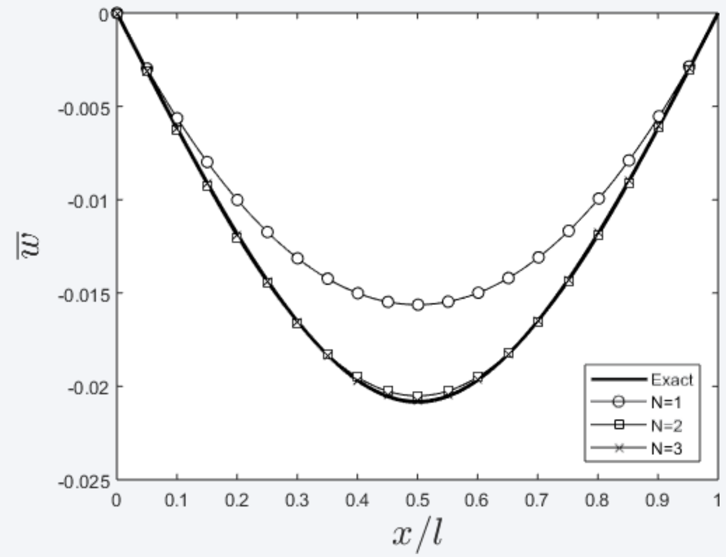


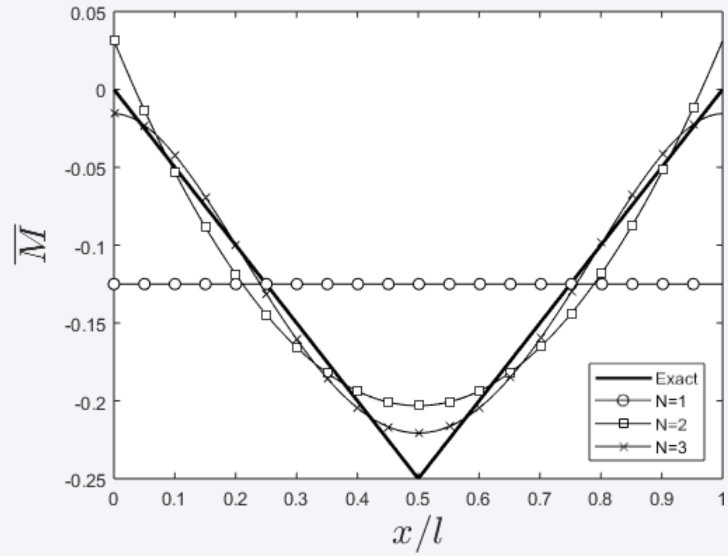


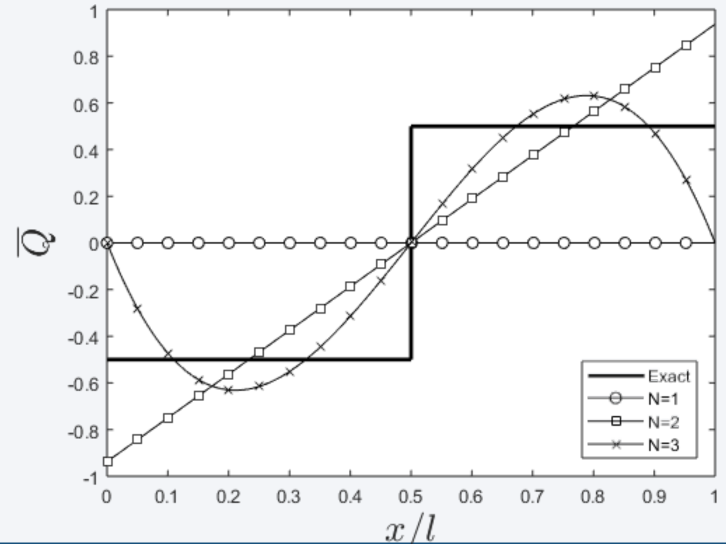


Example 2: beam with concentrated load

- Plot of results using N=1, 3 and 5 terms (polynomial functions)

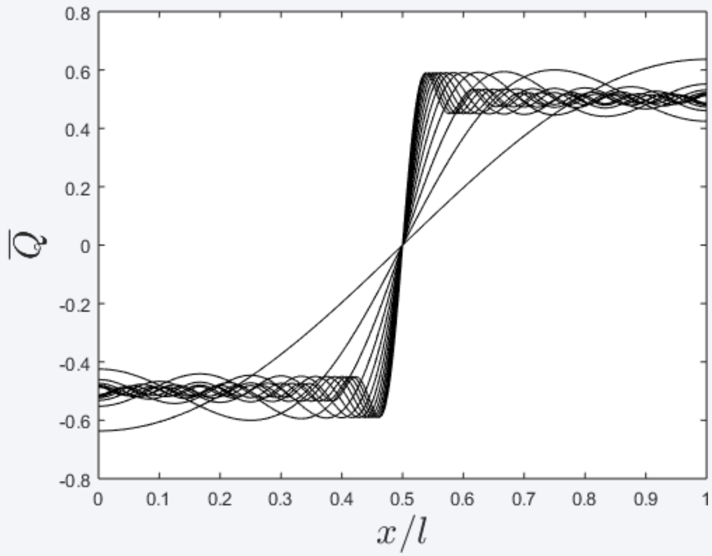




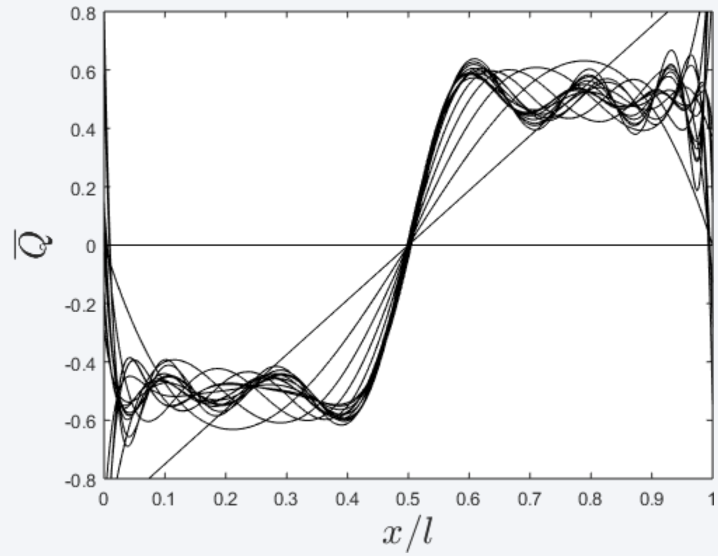


Example 2: beam with concentrated load

- Comparison of internal shear approximation using N=1,...,25 terms



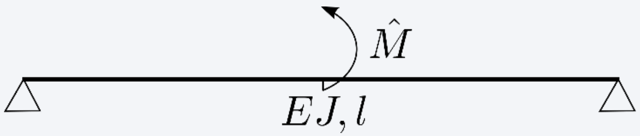
Trigonometric



Polynomials

- As N is increased, the description of the shear force is better approximated in a global sense only. For polynomials, a Gibbs-like effect can be noted, with strong oscillations and local errors close to the boundaries

Example 3: beam with concentrated moment



Essential conditions

$$\begin{cases} w(0) = 0 \\ w(l) = 0 \end{cases}$$

Ritz functions

$$\phi(x) = x^{i+1} - l^i x \qquad i = 1 \dots N$$

$$\phi(x) = \sin\frac{i\pi x}{l} \qquad i = 1 \dots N$$

Nondimensional parameters

$$\overline{w} = w\frac{EJ}{\hat{M}l^2} \qquad \overline{M} = M\frac{1}{\hat{M}}$$

$$\overline{Q} = Q\frac{l}{\hat{M}} \qquad \overline{\Pi} = \Pi\frac{EJ}{\hat{M}^2 l}$$

Exact solution

Solution for $0 \leq x \leq l/2$ (anti-symmetric)

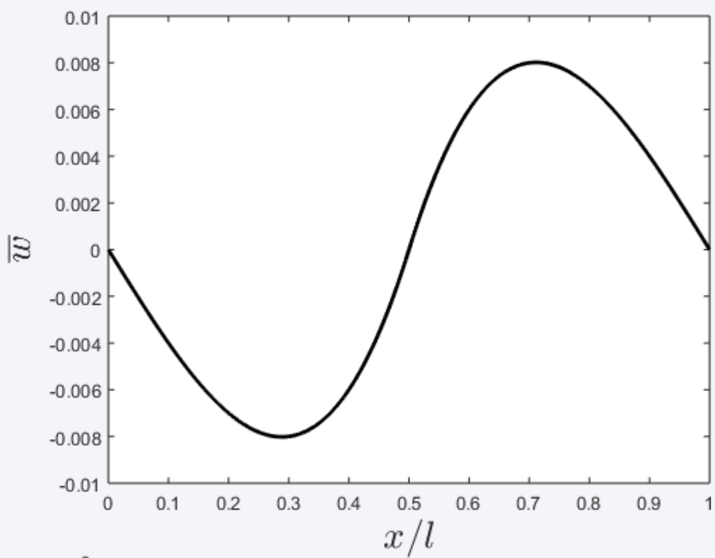
$$w = \frac{\hat{M}l^2}{EJ}\left[\frac{1}{6}\left(\frac{x}{l}\right)^3 - \frac{1}{24}\left(\frac{x}{l}\right)\right]$$

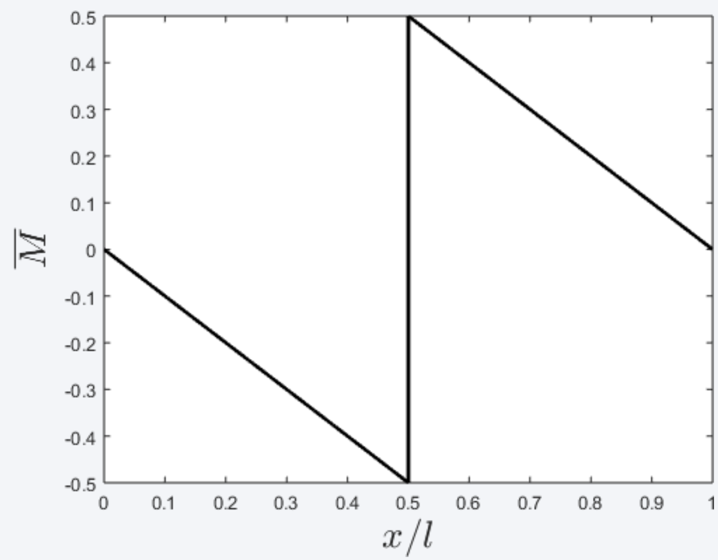
$$M = -\hat{M}\frac{x}{l}$$

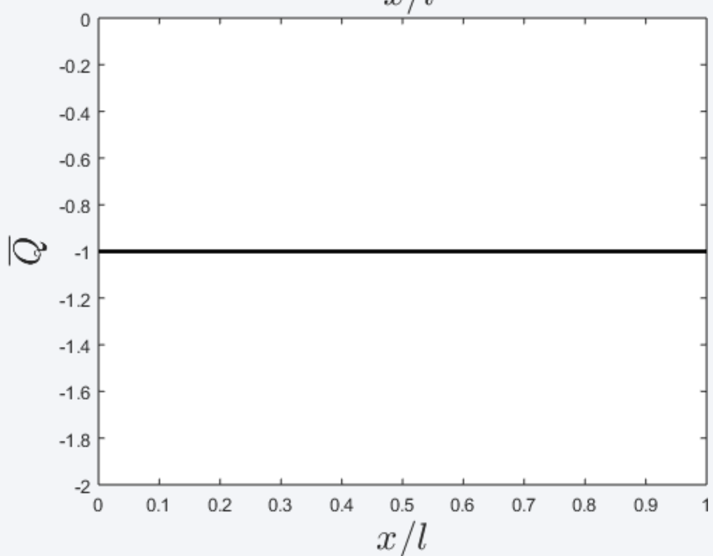
$$Q = -\frac{\hat{M}}{l}$$

$$\Pi = -\frac{\hat{M}^2 l}{24EJ}$$

Example 3: beam with concentrated moment







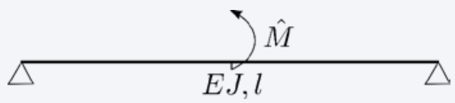
Note

- The bending moment is now discontinuous: it is then expected that convergence can be achieved only in the L_2 norm

Example 3: beam with concentrated moment

- Results using N trigonometric functions

	$\overline{w}(l/4) \cdot 10^2$	$\overline{M}(l/4) \cdot 10$	$\overline{\Pi} \cdot 10^3$
N = 2	-0.8063	-3.1831	-25.3303
N = 4	-0.8063	-3.1831	-31.6629
N = 6	-0.7764	-2.1221	-34.4773
N = 26	-0.7814	-2.6131	-39.7912
exact	-0.7813	-2.5000	-41.6667



Remarks

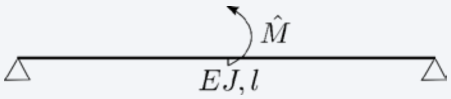
- The displacements and bending moments are now evaluated in l/4 as they are zero and discontinuous, respectively, in l/2
- The total potential energy converges monotonically, as guaranteed for any any Ritz approximation
- In this case, also the displacement converges monothonically (this is not always guaranteed – see example 1 –, but in the presence of concentrated loads it is).
- On the contrary, the bending moment is now converging with an oscillating behaviour

Example 3: beam with concentrated moment

- Errors using N trigonometric functions

	$\|\|e^{\overline{w}}\|\|_\infty 10^5$	$\|\|e^{\overline{M}}\|\|_\infty 10^3$	$\|\|e^{\overline{Q}}\|\|_\infty 10^2$
N = 1	127.6503	500.00	300.00
N = 3	48.1164	500.00	500.00
N = 5	24.8906	500.00	700.00
N = 25	1.6957	500.00	2700.00

	$\|\|e^{\overline{w}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{M}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{Q}}\|\|_{L_2} 10^2$
N = 1	13.0566	62.6161	173.2051
N = 3	4.1095	49.0001	223.6068
N = 5	1.8450	41.5400	264.5751
N = 25	0.0660	21.2286	519.6152

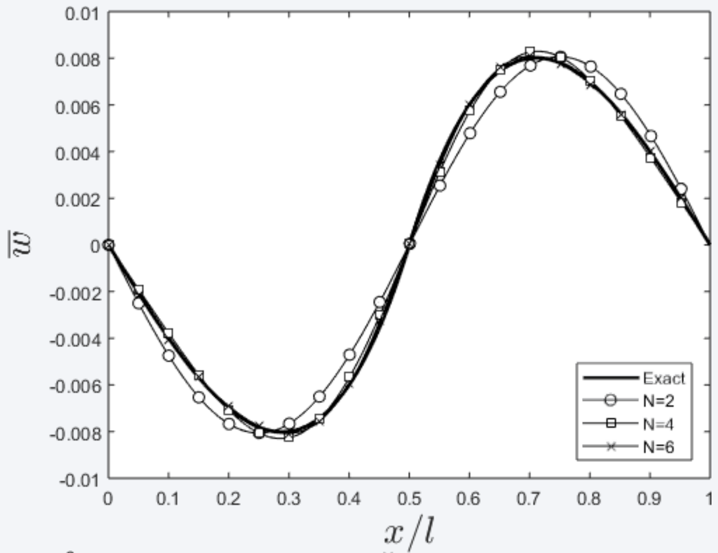


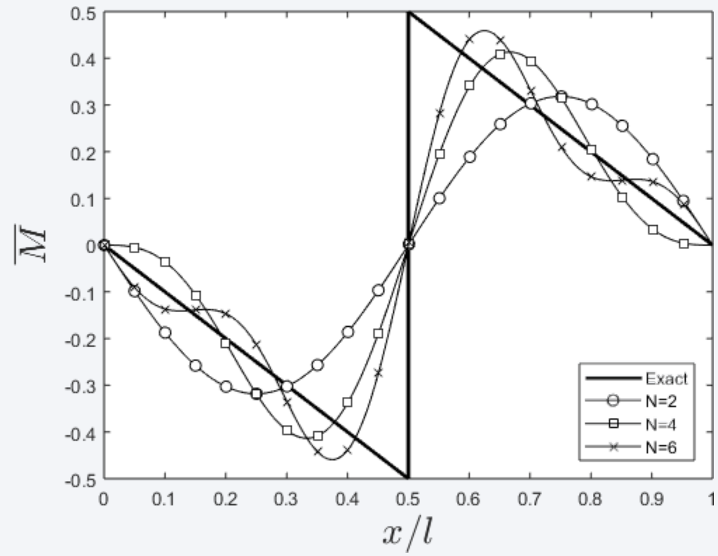
Remarks

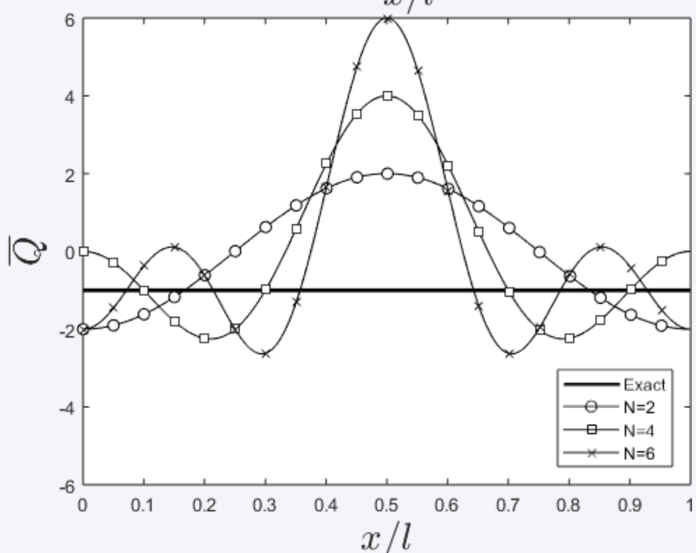
- As guaranteed by the Ritz method, the displacements converge in the inf norm (and, consequently, also in the L_2 norm)
- The bending moment does not converge locally (inf norm), but only globally. This is due to the presence of a discontinuity in x=l/2
- The shear, for which convergence is not guaranteed, does not converge, nor in the inf norm, nor in the L_2 norm

Example 3: beam with concentrated moment

- Plot of results using N=1, 3 and 5 terms (trigonometric functions)

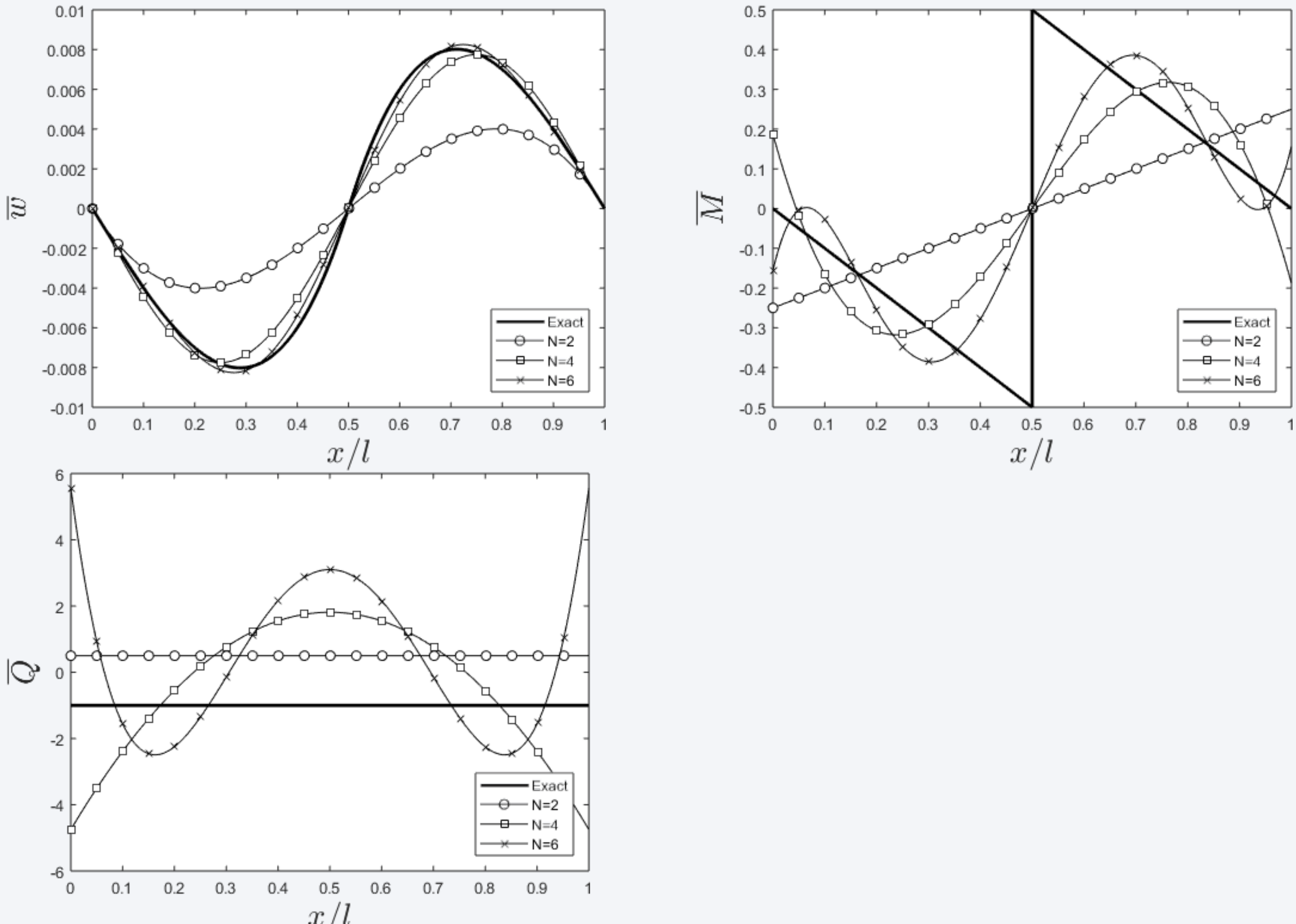






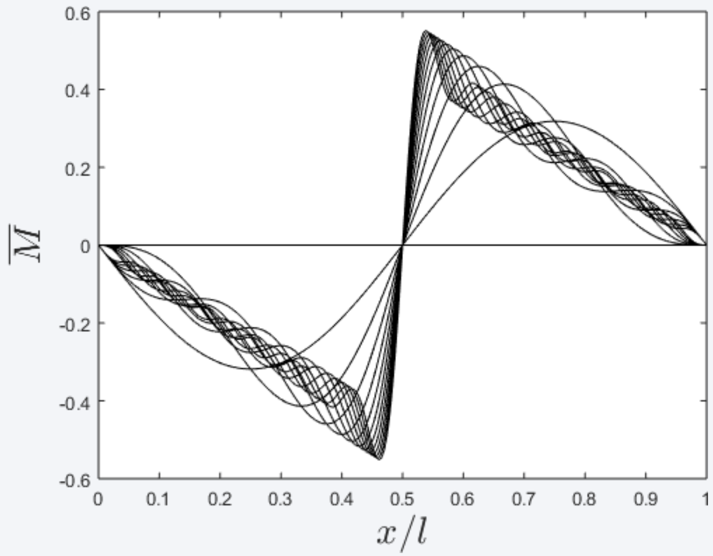
Example 3: beam with concentrated moment

- Plot of results using N=1, 3 and 5 terms (polynomial functions)

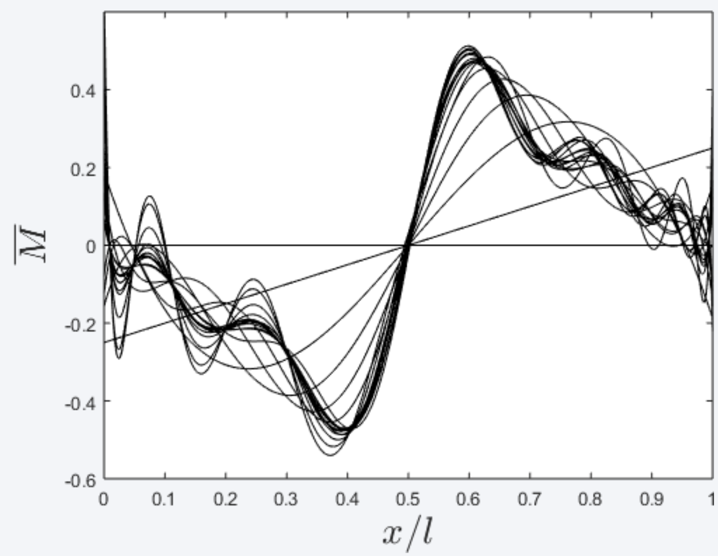


Example 3: beam with concentrated moment

- Comparison of bending moment using N=1,...,25 terms



Trigonometric

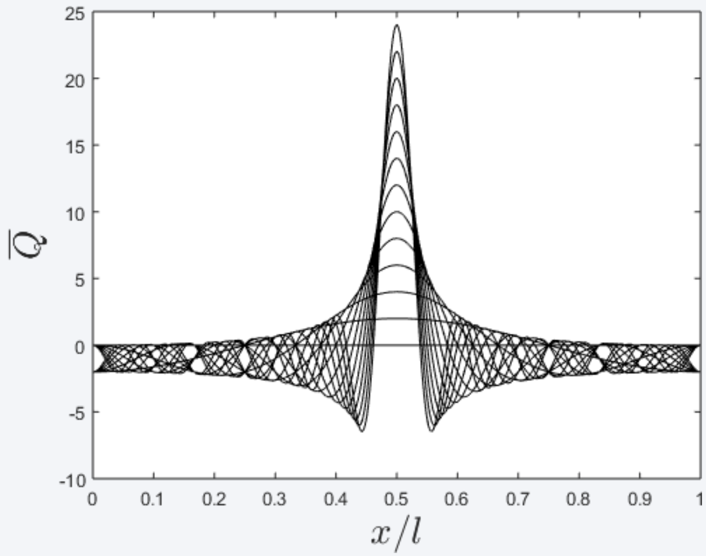


Polynomials

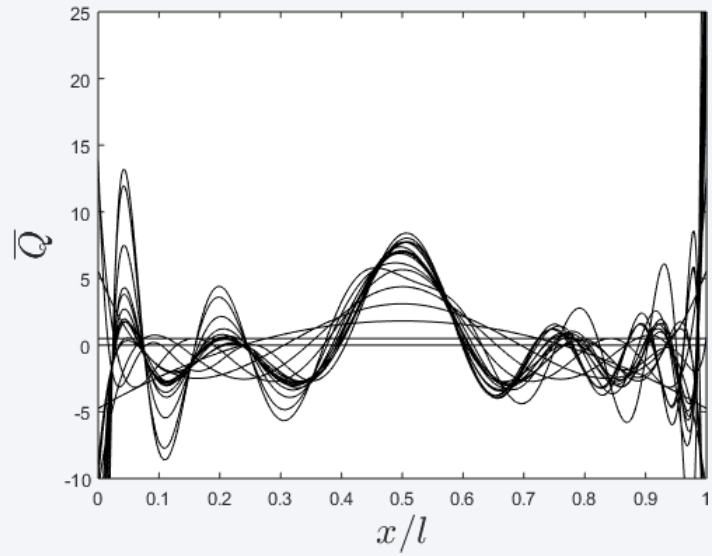
- Despite the presence of a discontinuity, the convergence of the bending moment is guaranteed at global level. As seen in the figure, the quality of the approximation improves for increasing number of functions

Example 3: beam with concentrated moment

- Comparison of internal shear using N=1,...,25 terms



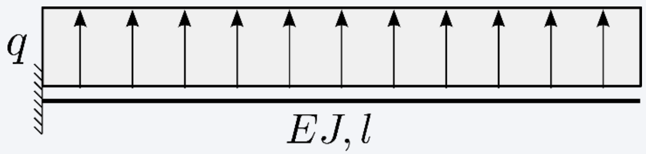
Trigonometric



Polynomials

- On the contrary, the shear does not converge, nor locally, nor globally. As seen in the two figures, the approximation of the behavior is poor, even when the number of functions is relatively high
- Note again the strong oscillations associated with the use of polynomials (despite the field to be represented does not display any discontinuity)

Example 4: cantilever beam with distributed load



Essential conditions

$$\begin{cases} w(0) = 0 \\ w_{,x}(0) = 0 \end{cases}$$

Ritz functions

$$\phi(x)_i = 1 - \cos\frac{i\pi x}{l} \qquad i = 1 \dots N$$

Nondimensional parameters

$$\overline{w} = w\frac{EJ}{ql^4} \qquad \overline{M} = M\frac{1}{ql^2}$$

$$\overline{Q} = Q\frac{1}{ql} \qquad \overline{\Pi} = \Pi\frac{EJ}{q^2l^5}$$

Exact solution

$$w = \frac{ql^4}{24EJ}\left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right)^3 + 6\left(\frac{x}{l}\right)^2\right]$$

$$M = -ql^2\left[\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{x}{l} + \frac{1}{2}\right]$$

$$Q = -ql\left(\frac{x}{l} - 1\right)$$

$$\Pi = -\frac{q^2l^5}{40EJ}$$

Example 4: cantilever beam with distributed load

- A few remarks on the shape functions $\phi(x)_i = 1 - \cos\frac{i\pi x}{l} \qquad i = 1 \ldots N$
  - They respect the essential conditions
  - They are C^{∞}, thus satisfy the requirements of the problem in terms of regularity
  - They are a complete set of functions (if completeness is intended as ability to represent any function in the norm L_2)

- The problem is characterized by continuous displacement, moment and shear. Convergence is then expected in the inf norm for displacements and bending moments. Although not guaranteed, convergence could be reasonably expected also for the shear, given the regularity of the solution

Example 4: the “paradox lost”

	$\overline{w}(l/2) \cdot 10^2$	$\overline{M}(l/2) \cdot 10$	$\overline{\Pi} \cdot 10^3$
N = 1	2.0532	-0.0000	-10.2660
N = 10	2.3432	0.4248	-11.1082
N = 25	2.3437	0.4151	-11.1109
N = 100	2.3437	0.4166	-11.1111
exact	4.4271	-1.2500	-25.0000

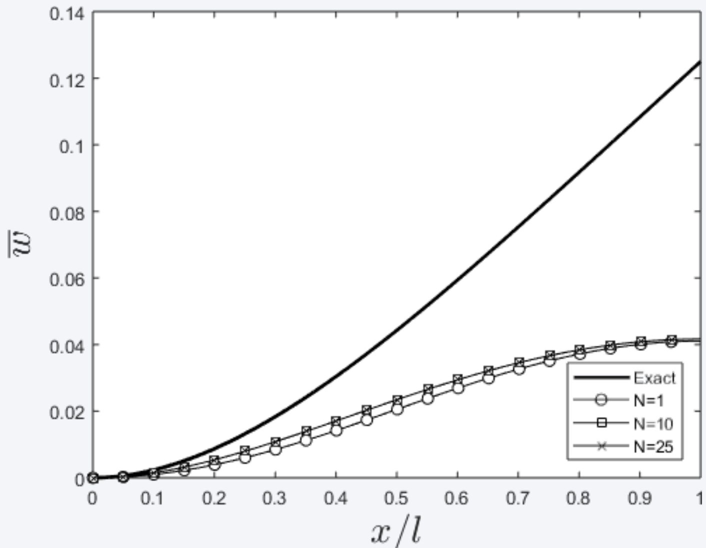
	$\|\|e^{\overline{w}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{M}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{Q}}\|\|_{L_2} 10^2$
N = 1	60.6619	76.7698	62.6158
N = 10	58.8418	74.5435	24.0551
N = 25	58.8353	74.5361	15.4476
N = 100	58.8349	74.5356	7.8422

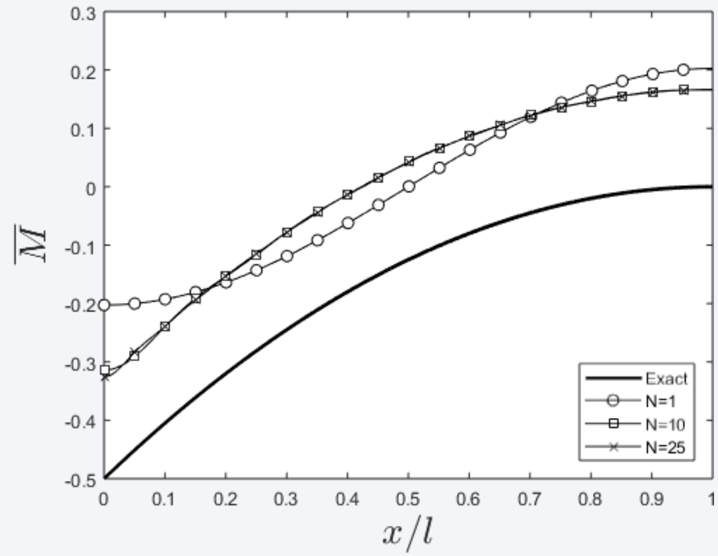
Remark

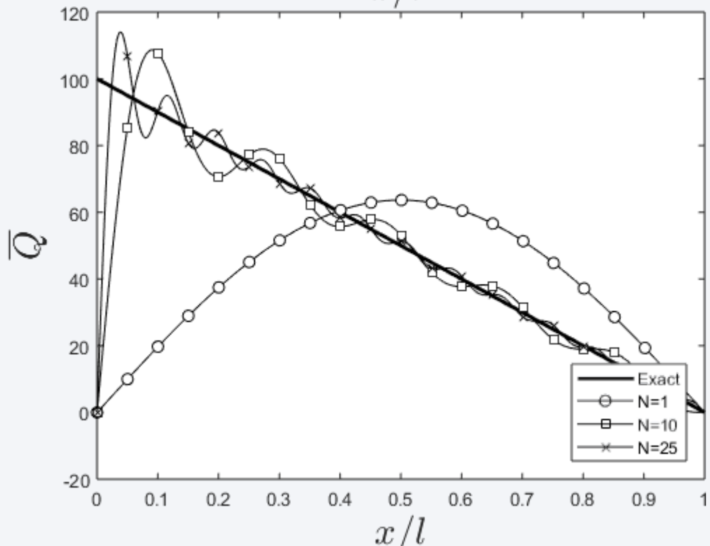
- The solution does not converge to the exact result (nor the displacements, nor the moments, nor the energies)!
- The L_2 error over the dispacements is well beyond 60% when 100 functions are used
- The Ritz method seems to fail. This is the «Paradox Lost» discussed by Storch and Strang*. The interested reader is strongly suggested to read the paper

*J. Storch, G. Strang, "Paradox Lost: Natural Boundary Conditions in the Ritz-Galerkin Method", International Journal for Numerical Methods in Engineering, Vol. 26, 1988, pp. 2255-2266.

Example 4: the “paradox lost”







Note

- The quality of the approximate solution is very poor, even for the displacement field!

Example 4: on the completeness of the trial functions

- Why does the Ritz method apparently fail?

  Because the choice of the trial function is wrong, and indeed they do not satisfy the requirement on the completeness.

- The question to be answered is then: what does completeness mean?

  Completeness is the ability of representing a generic function f with the desired level of accuracy. The "level of accuracy" is the distance between the function f and its expansion in trigonometric series according to a given norm.

  The common way of intending completeness refers to the L_2 distance between the functions

$$||e^f||_{L_2} = \sqrt{\int_0^l \left(f - \tilde{f}\right)^2 \mathrm{d}x}$$

f: generic function
$\tilde{f}$: approximation (trigonometric expansion)

With this regard, the trial functions here considered are a complete set, and the cosine expansion allows to represent any function f, by progressively reducing the L_2 distance with f

Example 4: on the completeness of the trial functions

- This is the key-point: completeness, in the context of the Ritz method, refers to a norm which is not the L_2 one. The correct norm to measure the distance between the approximated function and the exact one is the energy norm, which is descending from the variational formulation of the problem. For the problem at hand, the internal energy reads:

$$U = \frac{1}{2}\int_0^a EJw_{,xx}^2 \mathrm{d}x$$

- The energy norm is then associated with the second derivatives, which are the ones entering the expression of the internal energy U. So the correct norm is:

$$||e^f||_{energy} = \sqrt{\int_0^l \left(f_{,xx} - \tilde{f}_{,xx}\right)^2 \mathrm{d}x}$$

- The trial functions used for approximating f must then guarantee the ability of reaching arbitrary accuracy in the description of the second derivatives
- The question is then: can the trigonometric functions come arbitrarily close to any function according to the energy norm? The answer is no.

Example 4: on the completeness of the trial functions

- Indeed, the second derivative of the exact solution is:

$$f = w_{,xx} = \frac{ql^2}{EJ}\left[\frac{1}{2}\left(\frac{x}{l}\right)^2 - \frac{x}{l} + \frac{1}{2}\right]$$

- while the second derivatives of the trial functions are:

$$\tilde{f} = c_i \phi\left(x\right)_{i,xx} = c_i \left(\frac{\pi i}{l}\right)^2 \cos\frac{i\pi x}{l}$$

- As observed, the second derivatives of the approximate solution are missing the constant term (which is not null in the exact solution). It follows that the trial functions are not complete (the constant term is missing) and the distance in the energy norm cannot be arbitrarily reduced by increasing the number of terms.

Example 4: on the completeness of the trial functions

- A complete, in the energy norm sense, trigonometric expansion would be:

$$\tilde{f} = c_i \phi_{i,xx} = c_0 + c_1 \left(\frac{\pi x}{l}\right) \cos\frac{\pi x}{l} + c_2 \left(\frac{\pi x}{l}\right)^2 \cos\frac{2\pi x}{l} + \dots$$

- which is obtained by introducing a quadratic term in the set of trial functions:

$$\phi\left(x\right)_1 = x^2$$

$$\phi\left(x\right)_i = 1 - \cos\frac{(i-1)\pi x}{l} \qquad i = 2 \dots N$$

Example 4: solution of the paradox

- Results using functions with quadratic term

	$\overline{w}(l/2) \cdot 10^2$	$\overline{M}(l/2) \cdot 10$	$\overline{\Pi} \cdot 10^3$
N = 1	2.0833	-1.6667	-13.8889
N = 3	4.3932	-1.1601	-24.7965
N = 10	4.4261	-1.2621	-24.9960
N = 25	4.4270	-1.2516	-24.9998
exact	4.4271	-1.2500	-25.0000

	$\|\|e^{\overline{w}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{M}}\|\|_{L_2} 10^2$	$\|\|e^{\overline{Q}}\|\|_{L_2} 10^2$
N = 1	41.9314	66.6667	100.0000
N = 3	0.7091	9.0224	48.9993
N = 10	0.0129	1.2603	25.2869
N = 25	0.0007	0.3054	15.7590

Note

- Convergence of the solution is then achieved in all the relevant quantities (displacements, moments, shear), as initially expected. The trust in the Ritz method can been restored!

Example 4: solution of the paradox