An introduction to the geometrically nonlinear formulation of continua

Within the framework of infinitesimal strain/displacement approximation (i.e. linear analysis) it is assumed that:

1. displacements are infinitesimal
2. strains are infinitesimal
3. material behaviour is linear

The equilibrium is imposed on the undeformed configuration (which is undistinguishable from the deformed one due to the assumption of infinitesimal displacements)

The effect of geometric nonlinearities is now accounted for by removing the assumption 2. and, to a more general extent, by considering:

1. non infinitesimal displacements
2. infinitesimal strains

Note that strains are still assumed small such that material nonlinearities do not need to be accounted for.

The equilibrium conditions are now expressed by considering the deformed configuration and, to this aim, proper stress and strain measures need to be introduced.

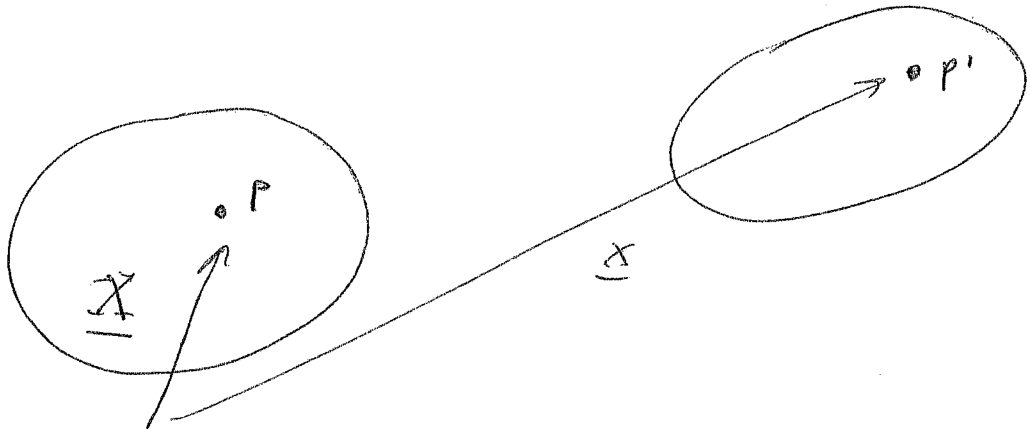
Lagrangian approach (material)

the kinematics of the body is described by referring to the initial configuration C^0. In this sense, the position of the generic point P is described as:

$$\boxed{\underline{x} = \underline{x}(\underline{X}, t)}$$

and, similarly, any other quantity (the density, for instance) is expressed as:

$$\boxed{\phi = \phi(\underline{X}, t)}$$ ϕ generic quantity

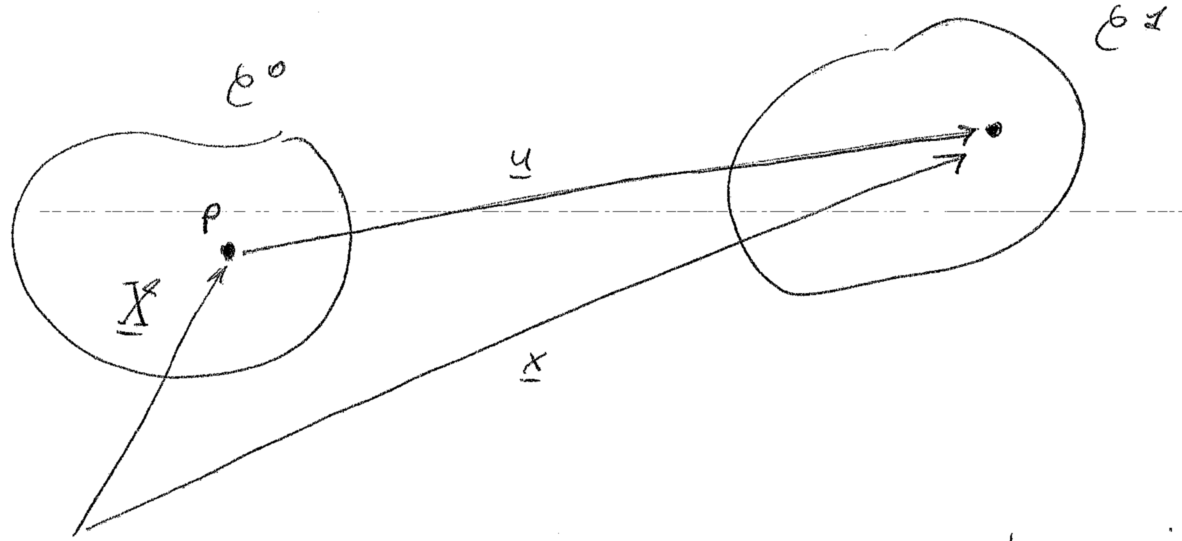


In other words, the relations above require the initial position $\underline{X}$ to obtain, as output, the final position $\underline{x}$ or the value of ϕ.
The initial coordinates $\underline{X}$, the lagrangian ones, are thus adapted for describing the overall deformation process

Kinematics

Consider a 3D body and define with:

- $\mathcal{C}^0$ the initial configuration
- $\mathcal{C}^1$ the deformed configuration



The generic point P is characterized by an initial position $\underline{X}$ and a final position $\underline{x}$

At the beginning of the deformation process $(t = t_0)$

$$\underline{x} \equiv \underline{X}$$

After the application of the external loads $(t > t_0)$

$$\underline{x} \neq \underline{X}$$

There are two possibilities for describing the motion of the body, the lagrangian or the eulerian approach.

Eulerian approach (spatial)

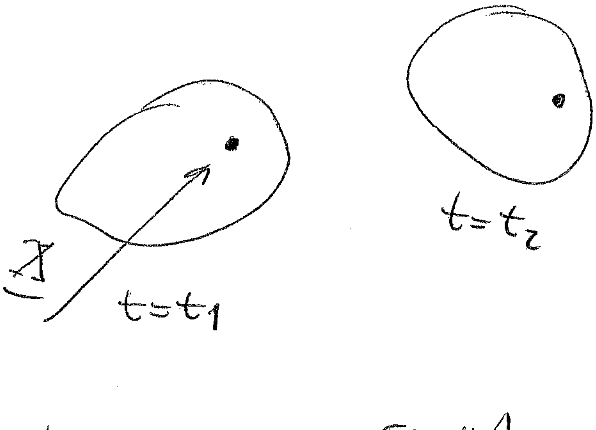
The Eulerian approach refers to the current configuration as;

$$\boxed{\begin{aligned} \underline{X} &= \underline{X}(\underline{x}, t) \\ \phi &= \phi(\underline{x}, t) \end{aligned}}$$

So the "input" of the functions describing the position and any other quantity ϕ are those of the current configuration, $\underline{x}$.

The difference between the two approaches can be clarified by referring to the description of the generic quantity ϕ in two different instants of time t_1 and t_2.

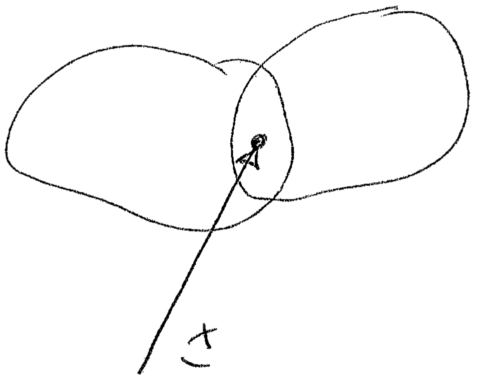
$$\phi = \phi(\underline{X}, t) \begin{cases} \xrightarrow{t = t_1} \phi(\underline{X}, t_1) \\ \xrightarrow{t = t_2} \phi(\underline{X}, t_2) \end{cases}$$



Lagrangian

In this case (Lagrangian) the same material particle, i.e. the one associated with the coordinate $\underline{X}$, is monitored at different steps of the deformation process

$$\phi = \phi(\underline{x}, t) \begin{cases} \xrightarrow{t=t_1} \phi(\underline{x}, t_1) \\ \xrightarrow{t=t_2} \phi(\underline{x}, t_2) \end{cases}$$



<u>Eulerian</u>

In the Eulerian case, the input of the function ϕ is the position $\underline{x}$, i.e. the spatial coordinate. The function $\phi = \phi(\underline{x}, t)$ is thus providing a description of how ϕ varies with for a fixed position in the space.

- <u>Time derivatives</u>

It is necessary to clarify what is intended with time-derivative. Indeed it must be distinguished the case where $\underline{x}$ is kept fixed with the case where $\underline{X}$ is fixed.

$$\left.\frac{d}{dt}\phi\right|_{\underline{X}=\text{fixed}} = \frac{D\phi}{Dt}$$ material derivative

In the context of a Lagrangian description, $\phi = \phi(\underline{X}, t)$ so:

$$\boxed{\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t}}$$

In the Eulerian case $\phi = \phi(\underline{x}, t)$ so:

$$\frac{d\phi(\underline{x},t)}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial t}$$

$$= \frac{\partial \phi}{\partial t} + \underline{v} \cdot \operatorname{grad} \phi$$

where $\underline{v} = \frac{\partial \underline{x}}{\partial t} = \left. \frac{\partial \underline{x}(\underline{X}, t)}{\partial t} \right|_{X = \text{fixed}}$

and so:

$$\boxed{\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \underline{v} \cdot \operatorname{grad} \phi}$$

convective part

Example

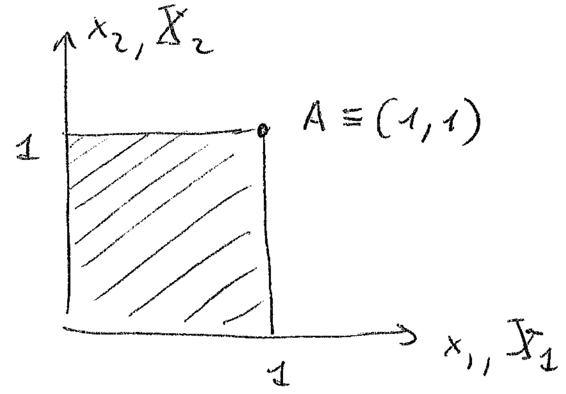
Assume that the relation $\underline{x} = \underline{x}(\underline{X})$ is available in closed-form as:

$$\begin{cases} x_1 = X_1 + a X_2 \\ x_2 = X_2 + a X_1 \end{cases} \quad \leftarrow \underline{x} = \underline{x}(\underline{X})$$

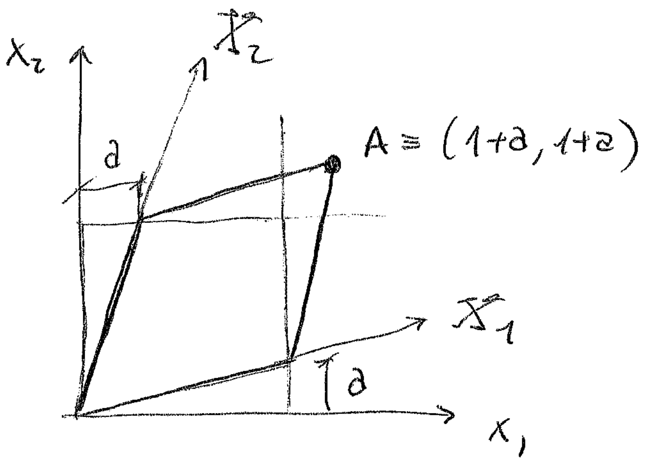
~~So the inverse is:~~

$$\begin{cases} X_1 = \dfrac{1}{1-a^2} x_1 - \dfrac{a}{1-a^2} x_2 \\[2ex] X_2 = -\dfrac{a}{1-a^2} x_1 + \dfrac{1}{1-a^2} x_2 \end{cases} \quad \leftarrow \underline{X} = \underline{X}(\underline{x})$$

Consider now the body in the figure



before deformation, $\underline{x} \equiv \underline{X}$



after deformation

Lagrangian point of view:

the point A is initially in the position $X_1 = 1$, $X_2 = 1$

After the deformation, the point A goes to:

$x_1 = 1 + a$

$x_2 = 1 + a$

(using the relation $\underline{x} = \underline{s}(\underline{X}, t)$)

Eulerian point of view:

the point in the space identified by

$x_1 = 1 + a$

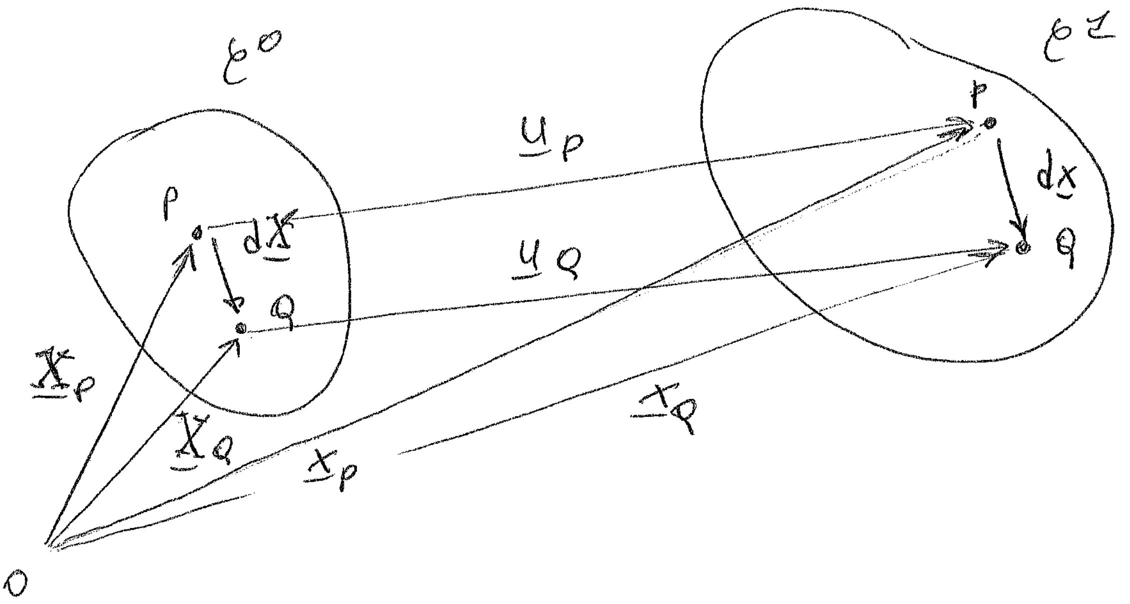
$x_2 = 1 + a$

is the one initially occupying the position:

$$\left.\begin{aligned} X_1 &= \frac{1}{1-a^2}(1+a) - \frac{a}{1-a^2}(1+a) = 1 \\ X_2 &= -\frac{a}{1-a^2}(1+a) + \frac{1}{1-a^2}(1+a) = 1 \end{aligned}\right\} \quad \underline{X} = \underline{X}(\underline{x})$$

Measure of deformation

Consider now a body in two successive instants (time dependency is here considered in the sense of pseudo-time instants describing the various steps of the deformation process), and define with P and Q two generic points infinitesimally close each other.



$\underline{u}_P$: displacement of P $= \underline{x}_P - \underline{X}_P$

$\underline{u}_Q$: displacement of Q $= \underline{x}_Q - \underline{X}_Q$

$\underline{X}_P$: initial position of P

$\underline{X}_Q$: initial position of Q

$\underline{x}_P$: final position of P

$\underline{x}_Q$: final position of Q

By adopting a lagrangian point of view:

$$\underline{u} = \underline{u}(\underline{X}, t) = \underline{x}(\underline{X}, t) - \underline{X}$$

It is clear that there is no deformation whenever there is no relative motion between P and Q.
On the contrary, a relative motion between P and Q (and, to a more general extent, of the points composing the continuum) determines the onset of a deformation state.
How to measure the deformation?
A key element for defining the deformation is the deformation gradient tensor $\underline{\underline{F}}$. It is defined as:

$$\boxed{d\underline{x} = \underline{\underline{F}} \cdot d\underline{X}}$$

or

$$dx_i = F_{ik}\, dX_k \quad , \quad \text{where:}$$

$$\boxed{\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{X}} = \mathrm{Grad}\, \underline{x}}$$

where the gradient is expressed with respect to the lagrangian coordinates $\underline{X}$.

Recalling that $\underline{x} = \underline{X} + \underline{u}$, it follows that:

$$\underline{\underline{F}} = \frac{\partial}{\partial \underline{X}} \left(\underline{X} + \underline{u} \right) = \underline{\underline{I}} + \text{Grad}\, \underline{u}$$

$$\boxed{\underline{\underline{F}} = \underline{\underline{I}} + \text{Grad}\, \underline{u}}$$

From the initial definition of $\underline{\underline{F}}$, $d\underline{x} = \underline{\underline{F}}\, d\underline{X}$, one can note that the deformation gradient cannot be used as a measure of the strain.
If a rigid rotation $\underline{\underline{R}}$ is considered, then:

$d\underline{x} = \underline{\underline{R}}\, d\underline{X} \Rightarrow \underline{\underline{F}} = \underline{\underline{R}} \Rightarrow \underline{\underline{F}}$ is not null even if the solid undergoes a rigid rotation.

The deformation can be characterized by looking at the quadratic distance between P and Q.

$$\boxed{\begin{array}{l} dS^2 = d\underline{X} \cdot d\underline{X} \\ ds^2 = d\underline{x} \cdot d\underline{x} \end{array}}$$

The difference between the quadratic distance before and after the deformation is then:

$$ds^2 - dS^2 = d\underline{x} \cdot d\underline{x} - d\underline{X} \cdot d\underline{X}$$

(and recalling that: $d\underline{x} = \underline{\underline{F}}\, d\underline{X}$, it is:)

$$= (\underline{\underline{F}}\, d\underline{X}) \cdot (\underline{\underline{F}}\, d\underline{X}) - d\underline{X} \cdot d\underline{X}$$

$$= d\underline{X} \cdot (\underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{I}}) \cdot d\underline{X}$$

$$\overset{def}{=} d\underline{X} \cdot 2\underline{\underline{\epsilon}} \cdot d\underline{X} \quad , \text{ where:}$$

$$\boxed{\underline{\underline{\epsilon}} = \frac{1}{2}(\underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{I}})}$$ Green-Lagrange strain tensor

The tensor $\underline{\underline{\epsilon}}$ can be expressed in terms of displacements by recalling that:

$\underline{\underline{F}} = \underline{\underline{I}} + \operatorname{Grad} \underline{u}$, so:

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left[(\underline{\underline{I}} + \operatorname{Grad}\underline{u})^T (\underline{\underline{I}} + \operatorname{Grad}\underline{u}) - \underline{\underline{I}}\right] \quad \text{and:}$$

$$\boxed{\underline{\underline{\epsilon}} = \frac{1}{2}\left(\operatorname{Grad}\underline{u} + \operatorname{Grad}\underline{u}^T + \operatorname{Grad}\underline{u}^T \cdot \operatorname{Grad}\underline{u}\right)}$$

Consider again the case of a rigid rotation:

$$d\underline{x} = \underline{\underline{R}}\, d\underline{X}$$

$$d\underline{X} \xrightarrow{\underline{\underline{R}}} d\underline{x}$$

with $\underline{\underline{R}} = \underline{\underline{F}}$

The deformation, according to the Green-Lagrange strain tensor, reads:

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\underline{\underline{F}}^T\underline{\underline{F}} - \underline{\underline{I}}\right) = \frac{1}{2}\left(\underline{\underline{R}}^T\underline{\underline{R}} - \underline{\underline{I}}\right) =$$

and recalling that, for a rigid rotation, $\underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{I}}$, it follows that:

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\underline{\underline{I}} - \underline{\underline{I}}\right) = \underline{\underline{0}}$$ $\Rightarrow$ $\underline{\underline{\epsilon}}$ is an adequate choice for measuring the strain. Any rigid body translation or rotation leads to zero deformation.

Alternatively it is possible to consider the expression of $\underline{\underline{\epsilon}}$ in terms of displacements:

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\text{Grad}\,\underline{u} + \text{Grad}\,\underline{u}^T + \text{Grad}\,\underline{u}^T\,\text{Grad}\,\underline{u}\right)$$

and $\underline{\underline{F}} = \underline{\underline{I}} + \text{Grad}\,\underline{u}$. In this case

$$\underline{\underline{R}} = \underline{\underline{I}} + \text{Grad}\,\underline{u} \Rightarrow \text{Grad}\,\underline{u} = \underline{\underline{R}} - \underline{\underline{I}}$$

So:

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\cancel{\underline{\underline{R}}} - \cancel{\underline{\underline{I}}} + \cancel{\underline{\underline{R}}^T} - \underline{\underline{I}} + \underbrace{\underline{\underline{R}}^T\underline{\underline{R}}}_{\underline{\underline{I}}} - \cancel{\underline{\underline{R}}^T} - \cancel{\underline{\underline{R}}} + \underline{\underline{I}}\right)$$

$$\Rightarrow \quad \underline{\underline{\epsilon}} = \underline{\underline{0}}$$

If the non linear contribution $\mathrm{Grad}\,\underline{u}^T\,\mathrm{Grad}\,\underline{u}$ is neglected

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\mathrm{Grad}\,\underline{u} + \mathrm{Grad}\,\underline{u}^T\right) \quad , \text{ then:}$$

$$= \frac{1}{2}\left(\underline{\underline{R}} - \underline{\underline{I}} + \underline{\underline{R}}^T - \underline{\underline{I}}\right)$$

$$= \frac{1}{2}\left(\underline{\underline{R}} + \underline{\underline{R}}^T - 2\,\underline{\underline{I}}\right) \neq 0$$

$\Rightarrow$ the small displacement strain tensor $\underline{\underline{\varepsilon}}$, which is linear in $\underline{u}$, is not a suitable choice for analyzing non infinitesimal displacements. Indeed a rigid rotation leads to a non null measure of the strain.

Measures of stress

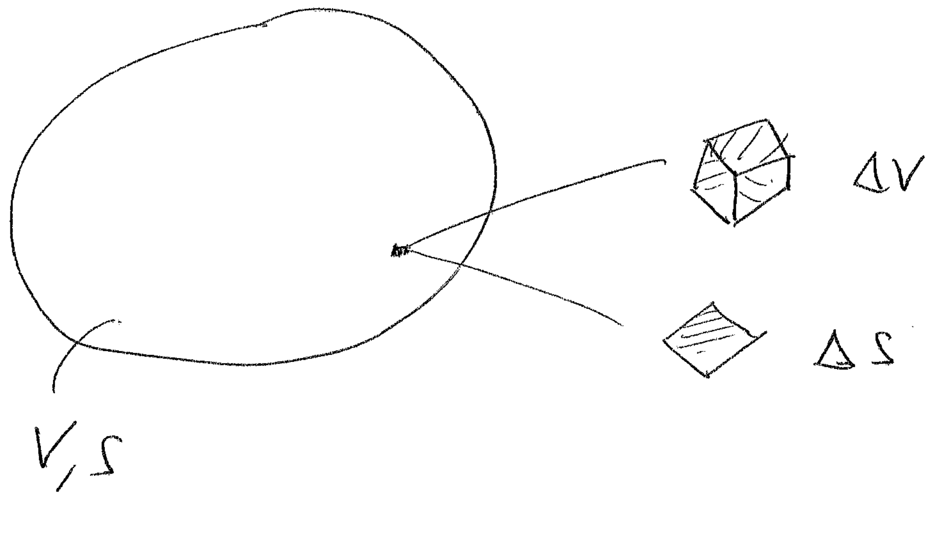
It is firstly important to clarify what is the meaning of what we define stress. An insightful description is reported here below and taken from the Feynman lectures on Physics.

"Consider a body of some elastic material – say a block of jello. If we make a cut through the block, the material on each side of the cut will, in general, get displaced by the internal forces. Before the cut was made, there must have been forces between the two parts of the block that kept the material in place. We can define the stresses in terms of these forces".

The above discussion clarifies that the stresses are not to be intended just as "forces per unit surface". Of course they are, dimensionally, forces per unit surface. The meaning of stresses is, however, deeper than their dimensional analysis and is inherently associated with the idea of exchange of internal forces.

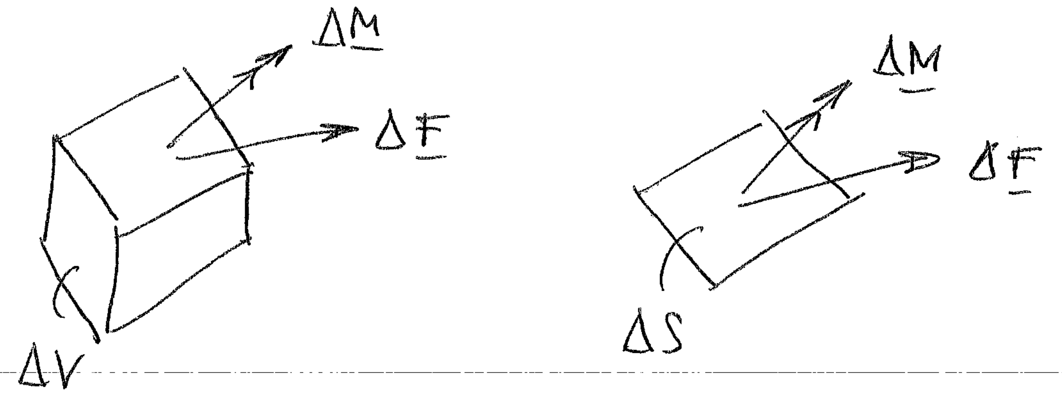
We define Cauchy continuum a continuum whose infinitesimal elements are capable of exchanging forces per unit volume and surface, but not moments per unit volume and surface. This is clearly a modeling assumption, which is a good one for a wide class of continua. Other models are obviously possible, and the continuum problem can be formulated by assuming that also moments per unit volume and surface can be exchanged. This approach leads to the Cosserat continuum model which is not covered in this course.

To formalize the assumptions associated with the Cauchy continuum, it is possible to consider a generic body with volume V and surface S



and extract a small volume ΔV and a small area ΔS from it.

In general, the force and moment resultants will be acting over the volume ΔV and the surface ΔS.



The Cauchy continuum relies on the assumptions that:

$$\lim_{\Delta V \to 0} \frac{\Delta \underline{M}}{\Delta V} = \underline{0}$$ volume moments

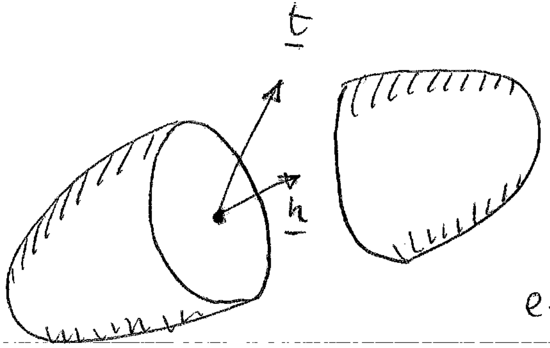
$$\lim_{\Delta S \to 0} \frac{\Delta \underline{M}}{\Delta S} = \underline{0}$$ surface moments

and

$$\lim_{\Delta V \to 0} \frac{\Delta \underline{F}}{\Delta V} = \underline{f}$$ volume forces

$$\lim_{\Delta S \to 0} \frac{\Delta \underline{F}}{\Delta S} = \underline{t}$$ surface forces

Consider now a body, which is divided (virtually) into two parts.



In order to guarantee the equilibrium, it is necessary that an exchange of internal forces exists between the two "virtual" surfaces (recall the block of jello!)

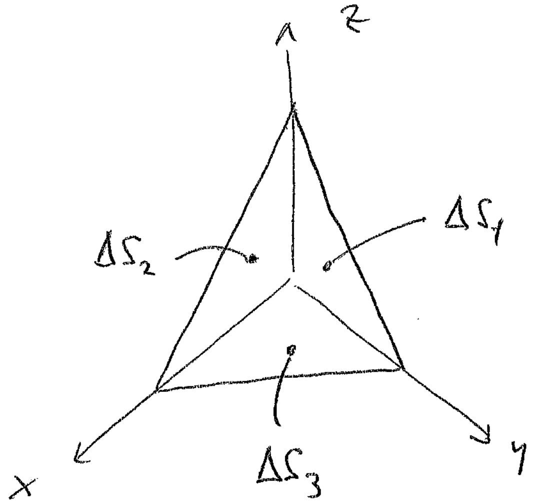
It is worth noting that:

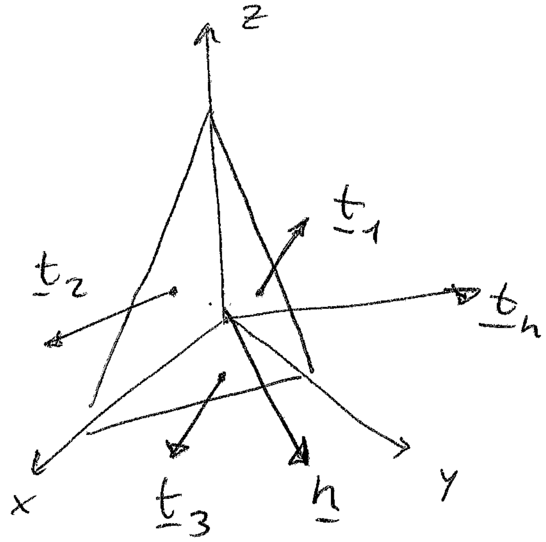
1. in general $\underline{t}$ is not directed as $\underline{n}$
   They are vectors with different direction!
2. the component of $\underline{t}$ along $\underline{n}$ defines the <u>normal</u> stress
   the component of $\underline{t}$ along the normal to $\underline{n}$ defines the <u>shearing</u> stresses

   Principal stresses are those associated with the planes of cut such that $\underline{t} \parallel \underline{n}$.

3. From the third Newton's law, it is:
   $\underline{t}(\underline{n}) = -\underline{t}(-\underline{n})$
   or the stress acting on the other plane of cut is equal and opposite.

The last remarks highlights also the dependency of $\underline{t}$ on $\underline{n}$. In general, different planes of cut are associated with different vectors $\underline{t}$.
The link between $\underline{t}$ and $\underline{n}$ is furnished by the stress tensor and can be illustrated by considering the Cauchy tetrahedron





$\underline{n}$: normal to the oblique surface
$\underline{t}_n$: stress vector acting over the surface with normal $\underline{n}$

$\underline{t}_1, \underline{t}_2, \underline{t}_3$: stress vectors acting on the surfaces of area $\Delta S_1, \Delta S_2, \Delta S_3$

<u>Note</u>: $\underline{t}_i$ are <u>not</u> normal to the corresponding oriented surfaces

The force equilibrium is then:

$$\underline{t}\,\Delta S - \underline{t}_1\,\Delta S_1 - \underline{t}_2\,\Delta S_2 - \underline{t}_3\,\Delta S_3 + \Delta V\,\underline{f} = 0$$

which corresponds to one vector equation or three scalar equations (expressing the equilibrium along x, y and z)

~~From the gradient theorem:~~

$$\int_V \nabla\phi\, dV = \oint_S \phi\,\underline{n}\, dS$$

and so, taking $\phi = 1$, one obtains:

$$\boxed{\underline{0} = \oint_S \underline{n}\, dS}$$

which corresponds to three scalar equations expressing the fact that the vector area of a closed surface is zero.

For the case at hand:

$$\oint_S \underline{n}\, dS = \Delta S\,\underline{n} + \Delta S_1\,\underline{n}_1 + \Delta S_2\,\underline{n}_2 + \Delta S_3\,\underline{n}_3$$

$$= \Delta S\,\underline{n} - \Delta S_1\,\underline{e}_1 - \Delta S_2\,\underline{e}_2 - \Delta S_3\,\underline{e}_3 = \underline{0}$$

From which, by multiplying with $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$:

$$\Delta S_1 = (\underline{n}\cdot\underline{e}_1)\,\Delta S$$
$$\Delta S_2 = (\underline{n}\cdot\underline{e}_2)\,\Delta S$$
$$\Delta S_3 = (\underline{n}\cdot\underline{e}_3)\,\Delta S$$

Furthermore:

$$\Delta V = \Delta S \frac{\Delta h}{3}$$ where Δh is the height from base to apex

The equilibrium equation becomes:

$$\underline{t}\cancel{\Delta S} - \underline{t}_1(\underline{n}\cdot\underline{e}_1)\cancel{\Delta S} - \underline{t}_2(\underline{n}\cdot\underline{e}_2)\cancel{\Delta S} - \underline{t}_3(\underline{n}\cdot\underline{e}_3)\cancel{\Delta S} +$$

$$+ \cancel{\Delta S}\frac{\Delta h}{3}\underline{f} = 0$$

The last contribution goes to zero as $\Delta h \to 0$, so:

$$\begin{aligned}
\underline{t} &= \underline{t}_1(\underline{n}\cdot\underline{e}_1) + \underline{t}_2(\underline{n}\cdot\underline{e}_2) + \underline{t}_3(\underline{n}\cdot\underline{e}_3) \\
&= \underline{t}_1(\underline{e}_1\cdot\underline{n}) + \underline{t}_2(\underline{e}_2\cdot\underline{n}) + \underline{t}_3(\underline{e}_3\cdot\underline{n}) \\
&= \underline{t}_1\otimes\underline{e}_1\cdot\underline{n} + \underline{t}_2\otimes\underline{e}_2\cdot\underline{n} + \underline{t}_3\otimes\underline{e}_3\cdot\underline{n} \\
&= \underbrace{\left(\underline{t}_1\otimes\underline{e}_1 + \underline{t}_2\otimes\underline{e}_2 + \underline{t}_3\otimes\underline{e}_3\right)}\cdot\underline{n} \\
&= \underline{\underline{\sigma}}\cdot\underline{n}
\end{aligned}$$

where $\boxed{\underline{\underline{\sigma}} = \underline{t}_i\otimes\underline{e}_i}$ Cauchy stress tensor

and the "link" between $\underline{t}$ and $\underline{n}$ is thus given by $\underline{\underline{\sigma}}$ as:

$$\boxed{\underline{t} = \underline{\underline{\sigma}}\cdot\underline{n}}$$

The meaning of the components of $\underline{\underline{\sigma}}$ can now be illustrated by observing that:

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n} \Rightarrow \begin{array}{l} \underline{t}_1 = \underline{\underline{\sigma}} \cdot \underline{e}_1 \\ \underline{t}_2 = \underline{\underline{\sigma}} \cdot \underline{e}_2 \\ \underline{t}_3 = \underline{\underline{\sigma}} \cdot \underline{e}_3 \end{array} \rightsquigarrow \underline{t}_k = \underline{\underline{\sigma}} \cdot \underline{e}_k$$

Observe now that:

$$\underline{\underline{\sigma}} = \sigma_{ij} \, \underline{e}_i \otimes \underline{e}_j$$

and so:

$$\underline{t}_k = \underline{\underline{\sigma}} \cdot \underline{e}_k$$

$$= \sigma_{ij} \, \underline{e}_i \otimes \underline{e}_j \cdot \underline{e}_k$$

$$\Rightarrow \boxed{\underline{t}_k = \sigma_{ik} \, \underline{e}_i}$$

σ_{ik} identifies:

1. the stress along i
2. over a face with normal parallel to the direction k

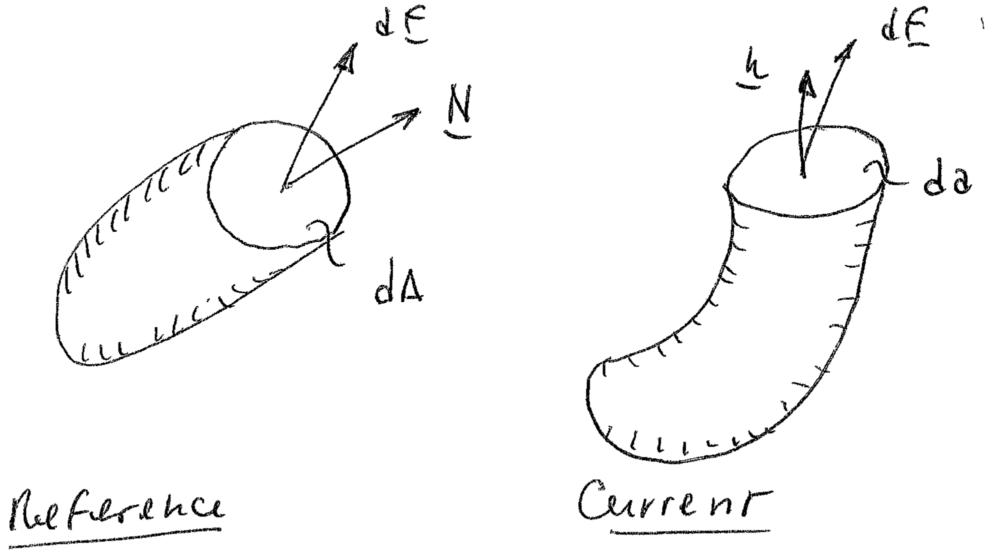
For instance σ_{12} is the stress component directed along 1 acting over the face with normal parallel to 2.

First and second Piola Kirchhoff stress tensors

The Cauchy stress tensor is a natural choice for representing the state of internal tension in a body: it refers to the current, deformed configuration both in terms of area and normal.

In the context of a Lagrangian formulation it is natural to refer the description of the continuum to the reference configuration. For this reason, other descriptions of the state of stress are then possible.

→ **First P-K stress tensor**



$\underline{N}$, dA ; normal and infinitesimal area in the reference config.

$\underline{n}$, da : normal and infinitesimal area in the current config.

$d\underline{f}$: local elementary force, i.e. $d\underline{f} = \underline{t}\, da$

where $\underline{t}$ is the stress vector

In the current configuration: $\underline{df} = \underline{t}\, da$

$$= \underline{\underline{\sigma}} \cdot \underline{n}\, da$$

In the reference configuration: $\underline{df} = \boxed{?}\, \underline{N}\, dA$

The still unknown tensor $\boxed{?}$ establishes a link between $\underline{df}$ and $\underline{N}\, dA$ (the oriented area in the reference configuration) and is called <u>First P-K stress tensor</u>

The expression of the first P-K stress tensor can be found as follows:

$$\underline{df} = \underline{\underline{\sigma}} \cdot \underline{n}\, da = \underline{\underline{P}} \cdot \underline{N}\, dA$$

$$\left(\text{where } \boxed{?} = \underline{\underline{P}}\right)$$

recalling the Nanson's formula, $\underline{n}\, da = J\, \underline{\underline{F}}^{-T}\, \underline{N}\, dA$

$$= \underline{\underline{\sigma}} \cdot J\, \underline{\underline{F}}^{-T}\, \underline{N}\, dA = \underline{\underline{P}} \cdot \underline{N}\, dA$$

$$\Rightarrow \boxed{\underline{\underline{P}} = J\, \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-T}}$$

A few remarks:

1. $$\left.\begin{aligned} \underline{df} &= \underline{\underline{\sigma}} \cdot \underline{n}\, da \\ \underline{df} &= \underline{\underline{P}} \cdot \underline{N}\, dA \end{aligned}\right\}$$ $\underline{df}$ is the <u>same</u> ~~stress~~ force vector

2. $\underline{\underline{P}}$ is used for expressing $d\underline{f}$ in terms of reference surface dA and normal $\underline{N}$ of the reference configuration

$\underline{\underline{\sigma}}$, on the contrary, refers to the current deformed surface da and the normal $\underline{n}$ of the current configuration

→ Second P-K stress tensor

The Cauchy stress tensor $\underline{\underline{\sigma}}$ is a symmetric tensor, whilst the first P-K stress tensor $\underline{\underline{P}}$ is not. Indeed, the relation $\underline{\underline{P}} = J \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-T}$ cannot be seen as a transformation of $\underline{\underline{\sigma}}$ which preserves the symmetry of the tensor.
To recover the symmetry of the stress measure, it is then introduced the second P-K stress tensor $\underline{\underline{S}}$.
In this case it is hard to clearly establish a physical interpretation of the tensor, which is nothing but the transformation of $\underline{\underline{\sigma}}$ by means of the deformation gradient $\underline{\underline{F}}$.
It can be seen starting from the elementary force $d\underline{f}$, and considering that:

$$d\underline{f} = \underline{\underline{F}}\, d\hat{\underline{f}} \qquad \text{or} \qquad d\hat{\underline{f}} = \underline{\underline{F}}^{-1} d\underline{f}$$

where $d\hat{\underline{f}}$ is the transformation of $d\underline{f}$ by means of the deformation gradient tensor.

The second P-K stress tensor is the link between $d\hat{\underline{f}}$ and $\underline{N}dA$ or:

$$d\hat{\underline{f}} = \underline{\underline{S}}\,\underline{N}\,dA$$

the relation between the first and second P-K stress tensor is then obtained as:

$$d\hat{\underline{f}} = \underline{\underline{F}}^{-1} d\underline{f} = \underline{\underline{S}}\,\underline{N}\,dA$$

$$= \underline{\underline{F}}^{-1}\underline{\underline{P}}\,\underline{N}\,dA = \underline{\underline{S}}\,\underline{N}\,dA$$

from which:

$$\boxed{\underline{\underline{P}} = \underline{\underline{F}}\,\underline{\underline{S}}} \quad \text{or} \quad \boxed{\underline{\underline{S}} = \underline{\underline{F}}^{-1}\underline{\underline{P}}}$$

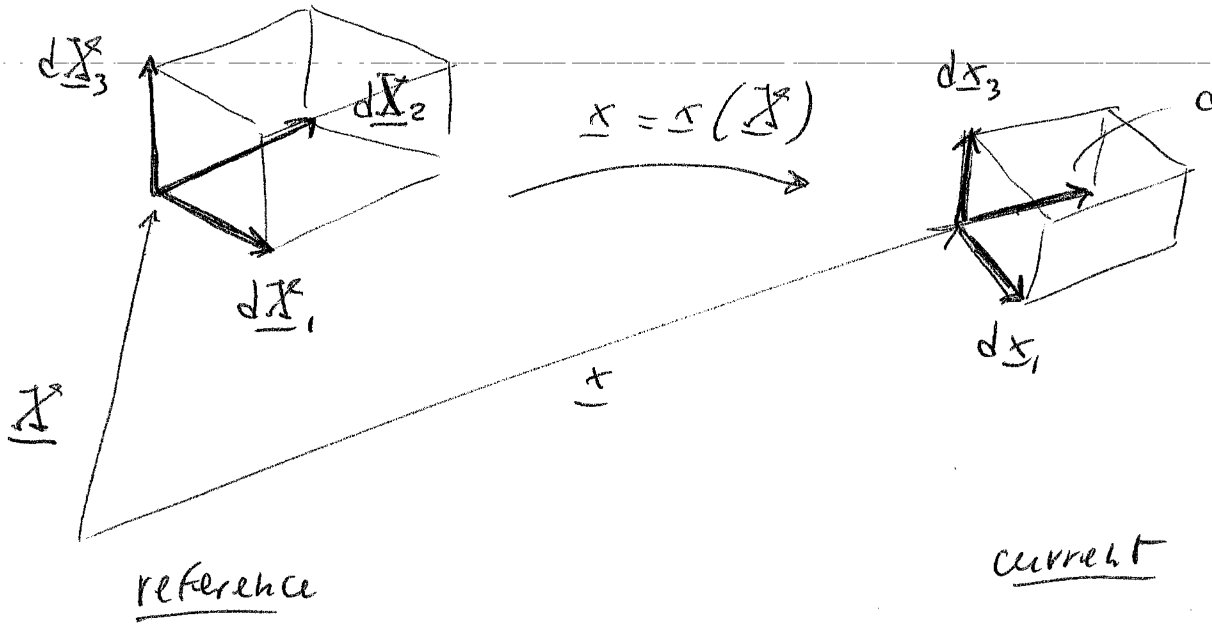
The relation with the Cauchy stress tensor $\underline{\underline{\sigma}}$ clarifies why $\underline{\underline{P}}$ is a symmetric tensor too:

$$\underline{\underline{S}} = \underline{\underline{F}}^{-1}\underline{\underline{P}}$$

$$= \underline{\underline{F}}^{-1} J\,\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T} \Rightarrow \boxed{\underline{\underline{S}} = J\,\underline{\underline{F}}^{-1}\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T}}$$

Change of volume

Consider the infinitesimal volume obtained by the mixed product of three infinitesimal vectors $d\underline{X}_i$ and the transformation $\underline{x} = \underline{x}(\underline{X})$



In the reference configuration, the infinitesimal volume reads:

$$dV = d\underline{X}_1 \cdot d\underline{X}_2 \times d\underline{X}_3$$

the generic vector $d\underline{X}_i$ can be written in terms of unitary vectors $\underline{E}_i$, so:

$$d\underline{X}_1 = dX_1 \, \underline{E}_1 \quad ; \quad d\underline{X}_2 = dX_2 \, \underline{E}_2 ; \quad d\underline{X}_3 = dX_3 \, \underline{E}_3$$

such that:

$$dV = \left(\underline{E}_1 \cdot \underline{E}_2 \times \underline{E}_3 \right) dX_1 \, dX_2 \, dX_3$$

$$= \quad 1 \quad dX_1 \, dX_2 \, dX_3 = dX_1 \, dX_2 \, dX_3$$

According to the transformation $\underline{x} = \underline{x}(\underline{X})$ from the reference to the current configuration:

$$d\underline{x}_i = \underline{\underline{F}}\, d\underline{X}_i \quad (i \text{ not repeated})$$

The volume in the current configuration is:

$$dv = d\underline{x}_1 \cdot d\underline{x}_2 \times d\underline{x}_3$$

$$= \left(\underline{\underline{F}}\, d\underline{X}_1\right) \cdot \left(\underline{\underline{F}}\, d\underline{X}_2\right) \times \left(\underline{\underline{F}}\, d\underline{X}_3\right)$$

$$= \left(\underline{\underline{F}}\, \underline{\underline{E}}_1\right) \cdot \left(\underline{\underline{F}}\, \underline{\underline{E}}_2\right) \times \left(\underline{\underline{F}}\, \underline{\underline{E}}_3\right) dX_1\, dX_2\, dX_3$$

or:

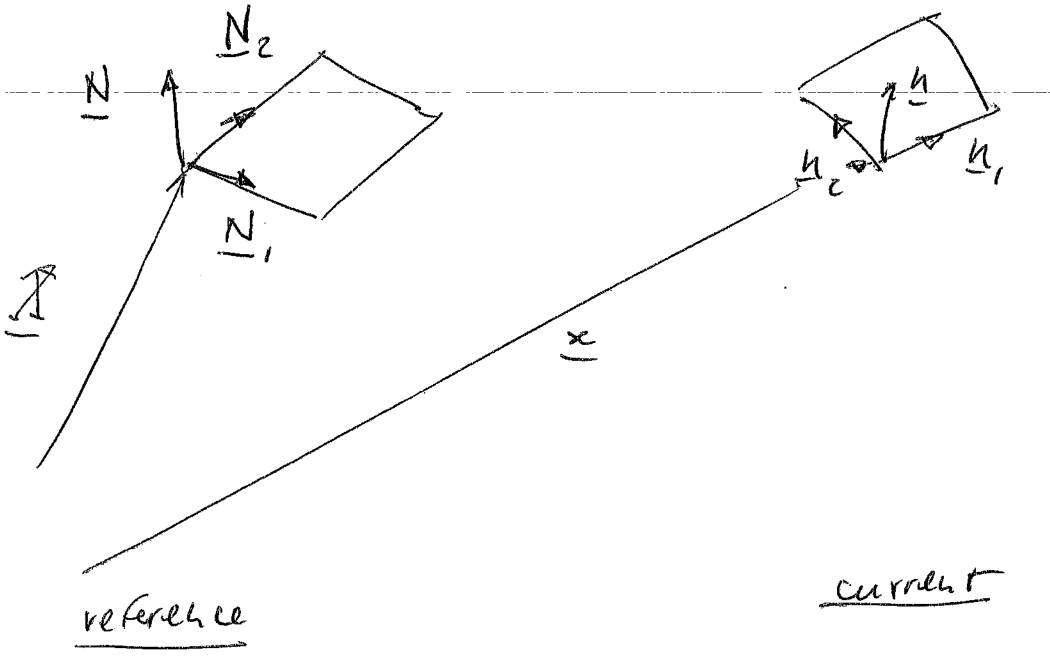
$$dv = \underbrace{\left(\underline{\underline{F}}\, \underline{\underline{E}}_1\right) \cdot \left(\underline{\underline{F}}\, \underline{\underline{E}}_2\right) \times \left(\underline{\underline{F}}\, \underline{\underline{E}}_3\right)}\, dV$$

↳ this is exactly the definition of determinant of $\underline{\underline{F}}$

$$\boxed{\begin{aligned} dv &= \det \underline{\underline{F}}\, dV \\ &= J\, dV \end{aligned}}$$

Nanson's formula

The Nanson's formula allows to establish a link between the oriented area in the reference and current configuration.



The oriented area and the volumes in the two configurations are:

$$d\underline{A} = \underline{N}\, dA \qquad\qquad d\underline{a} = \underline{n}\, da$$

$$dV = d\underline{X} \cdot d\underline{A} \qquad\qquad dv = d\underline{x} \cdot d\underline{a}$$

$$= d\underline{X} \cdot \underline{N}\, dA \qquad\qquad = d\underline{x} \cdot \underline{n}\, da$$

Recall now that $dv = J\, dV$ and $d\underline{x} = \underline{F}\, d\underline{X}$, so:

$$dv = d\underline{x} \cdot \underline{n}\, da = J\, dV$$

$$= J\, d\underline{X} \cdot \underline{N}\, dA$$

$$= J \, \underline{\underline{F}}^{-1} d\underline{x} \cdot \underline{N} \, dA$$

$$= J \, d\underline{x} \cdot \underline{\underline{F}}^{-T} \underline{N} \, dA$$

$$= d\underline{x} \cdot J \underline{\underline{F}}^{-T} \underline{N} \, dA$$, so:

$$d\underline{x} \cdot \underline{n} \, da = d\underline{x} \cdot J \underline{\underline{F}}^{-T} \underline{N} \, dA$$, from which:

$$\boxed{\underline{n} \, da = J \underline{\underline{F}}^{-T} \underline{N} \, dA}$$ Nanson's formula