INTRODUCTION TO STABILITY OF STRUCTURES

Some basic ideas can be introduced by considering simple 1 dof examples.

Two preliminary considerations

1. Instability phenomena can arise in the presence of compressive loads. However, it is worth remarking that this is not the only case of practical interest: instability phenomena can be triggered also by shear loads, bending moments, ... and, in some cases, also by traction loads

2. The phenomena here considered imply the elastic behavior of the material. The source of nonlinearity, necessary to analyze the stability, is of geometrical nature.

Bifurcations

For small strains/infinitesimal displacements the uniqueness of the solution is guaranteed a priori. By removing this assumption the uniqueness is no more guaranteed (instability = loss of uniquess)

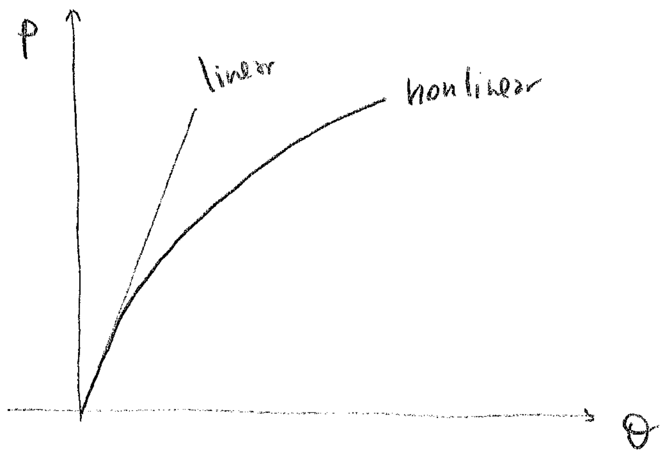
Consider a geometrically non linear system. Its response can be analyzed by "monitoring" two characteristic parameters

1. P : load parameter
2. θ : displacement parameter

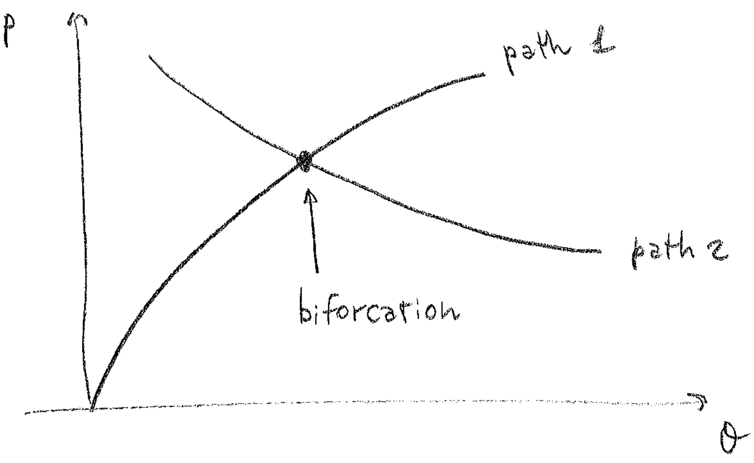
Note: 1 dof problem → P, θ are the load and the dof

N dof problem → P, θ are two scalars representative of the behaviour of the structure

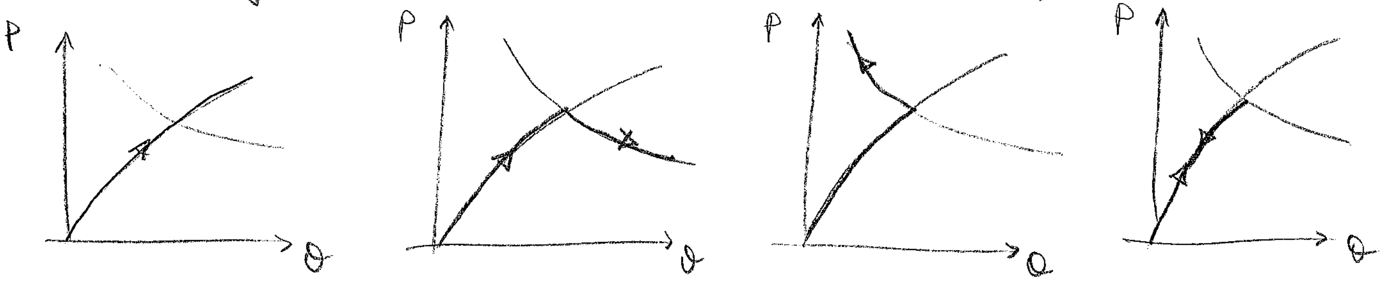
The equilibrium path in the P-θ plane is:



Due to the non-uniqueness of the solution, multiple equilibrium paths can exist

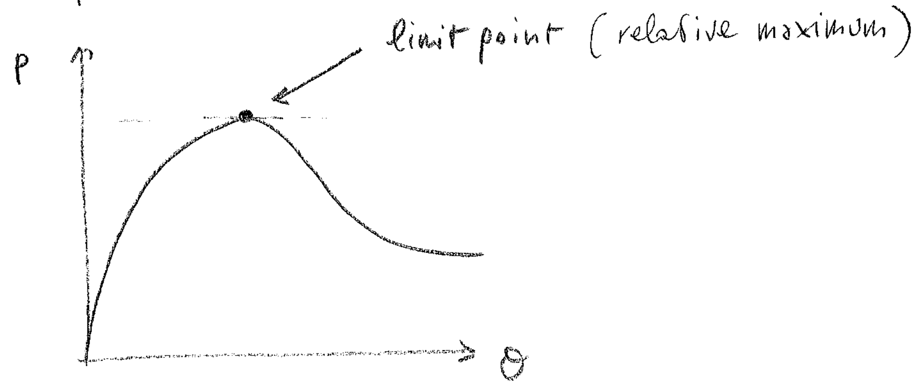


The biforcation point is the intersection between equilibrium paths
The equilibrium condition can change from stable to unstable.
After reaching the biforcation four scenarios are possible:

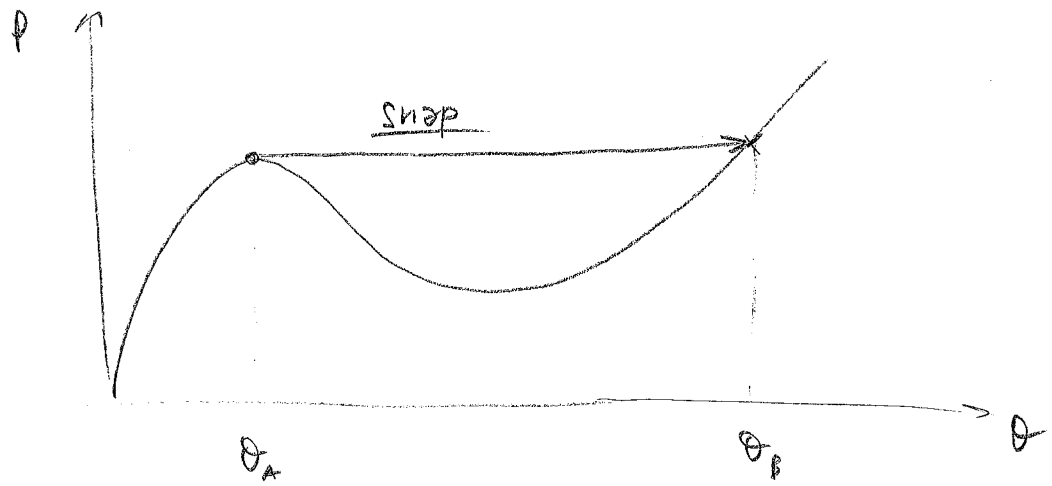


· Limit points

Equilibrium paths can be characterised by the presence of limit points



After reaching a limit point, an increase in the load parameter can determine a drastic change of configuration



1 DOF Examples

The stability is here analyzed using energy principles.

$$\Delta\Pi = \Pi(\theta + \delta\theta, P) - \Pi(\theta, P)$$

$$= \Pi'(\theta,P)\Big|_{\theta_{eq}} \delta\theta + \frac{1}{2!}\Pi''(\theta,P)\Big|_{\theta_{eq}} \delta\theta^2 + \frac{1}{3!}\Pi'''(\theta,P)\Big|_{\theta_{eq}} \delta\theta^3 + \dots$$

Equilibrium condition: $\boxed{\Pi'(\theta,P)\Big|_{\theta_{eq}} = 0} \quad \forall\, \delta\theta$

For an equilibrium configuration, the sign of $\Delta\Pi$ depends on the second order term:

Stability $\boxed{\Pi''(\theta,P)\Big|_{\theta_{eq}} \geq 0} \quad \forall\, \delta\theta$

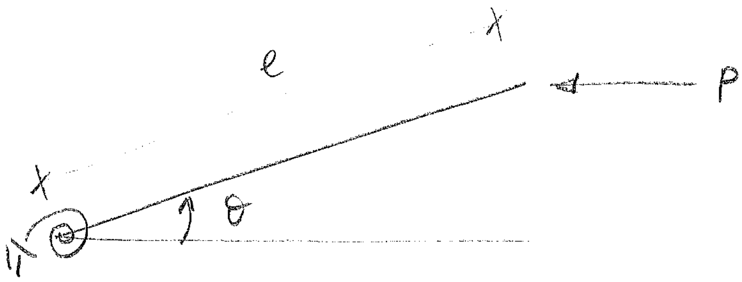
The transition from stability $\left(\Pi''(\theta,P)\Big|_{\theta_{eq}} \geq 0\right)$ to instability is represented by

$$\boxed{\Pi''(\theta,P)\Big|_{\theta_{eq}} = 0}$$

└ NEUTRAL EQUILIBRIUM METHOD

(note: this is not the only possible method; see Trefftz method later)

Example 1



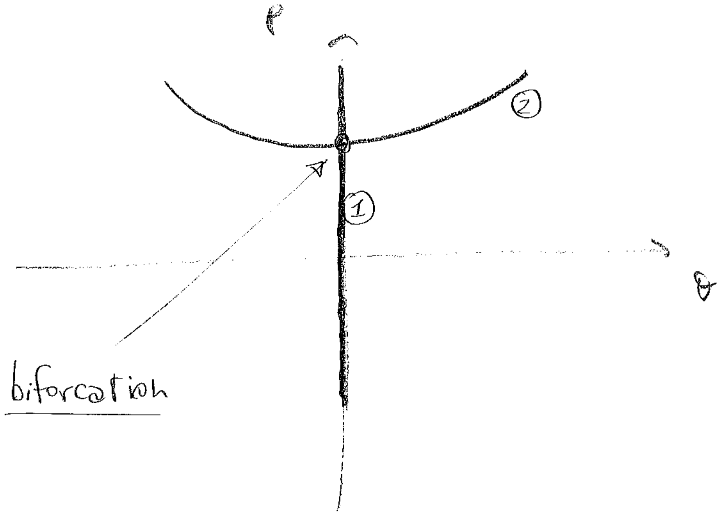
$\Pi = \frac{1}{2} K\theta^2 - Pu; \quad u = l(1-\cos\theta)$

$\Pi' = K\theta - Pl\sin\theta$

Equilibrium: $\Pi' = 0 \Rightarrow \boxed{K\theta = Pl\sin\theta}$

Two equilibrium paths can be identified as solution of the previous relation

1. $\theta = 0$
2. $P = \frac{K}{l}\frac{\theta}{\sin\theta}$



Stability of equilibrium $\quad \Pi''|_{\theta_{eq}} = 0$

$\Pi'' = K - Pl\cos\theta$

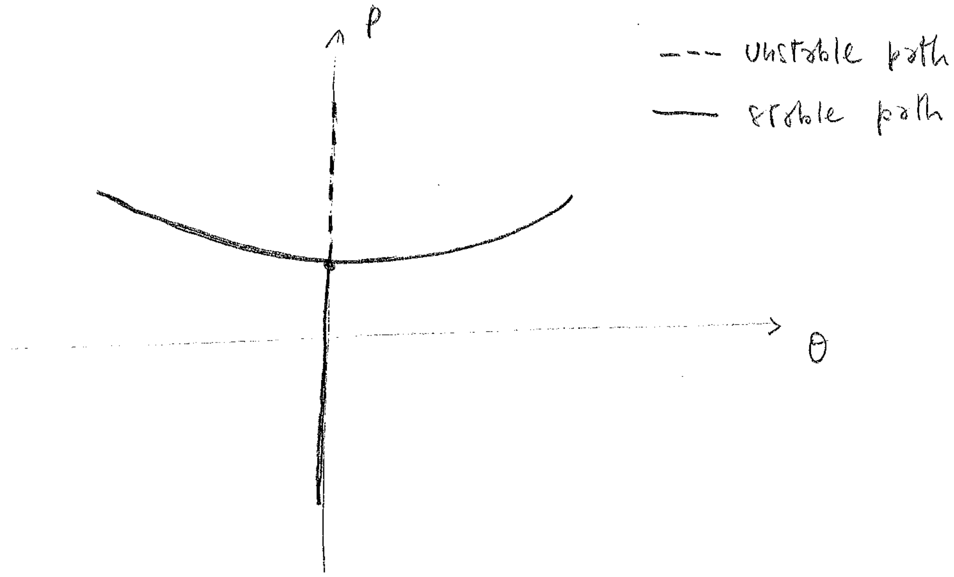
a. Equilibrium path 1 $(\theta = 0)$

$$\Pi''\Big|_{\theta=0} = k - Pl \quad \begin{cases} > 0 & \text{if } P < k/l \\ \leq 0 & \text{if } P \geq k/l \end{cases}$$

b. Equilibrium path 2 $\left(P = \frac{k}{l}\frac{\theta}{\sin\theta}\right)$

$$\Pi''\Big|_{P=\frac{k}{l}\frac{\theta}{\sin\theta}} = k - \frac{k}{l}\frac{\theta}{\sin\theta}\, l\cos\theta = k\left(1 - \frac{\theta}{\operatorname{tg}\theta}\right) > 0 \quad \forall\theta$$

Thus:



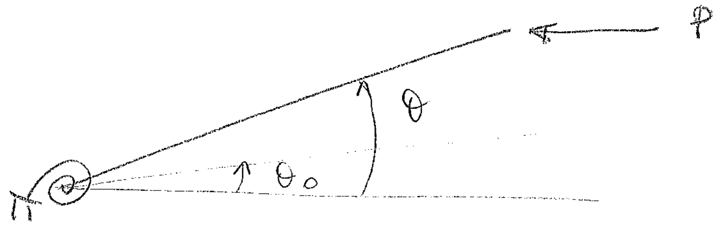
Note that:

1. The biforcation determines the passage from stable to unstable equilibrium (one of the three paths is unstable)

2. The bifurcation is, in this case, symmetric and stable

Example 1 (including imperfections)

Let's analyze the effect of a small deviation from the nominal geometry on the response of a structure characterized by a symmetric and stable bifurcation.



$$\Pi = \frac{1}{2} k (\theta - \theta_0)^2 - Pu; \qquad u = \ell \cos\theta_0 - \ell \cos\theta$$

$$= \frac{1}{2} k(\theta^2 + \theta_0^2 - 2\theta\theta_0) - P\ell \cos\theta_0 + P\ell \cos\theta$$

$$\Pi' = k(\theta - \theta_0) - P\ell \sin\theta$$

$$\Pi'' = k - P\ell \cos\theta$$

Equilibrium : $k(\theta - \theta_0) - P\ell \sin\theta = 0$

$\theta = 0$ is not a solution!

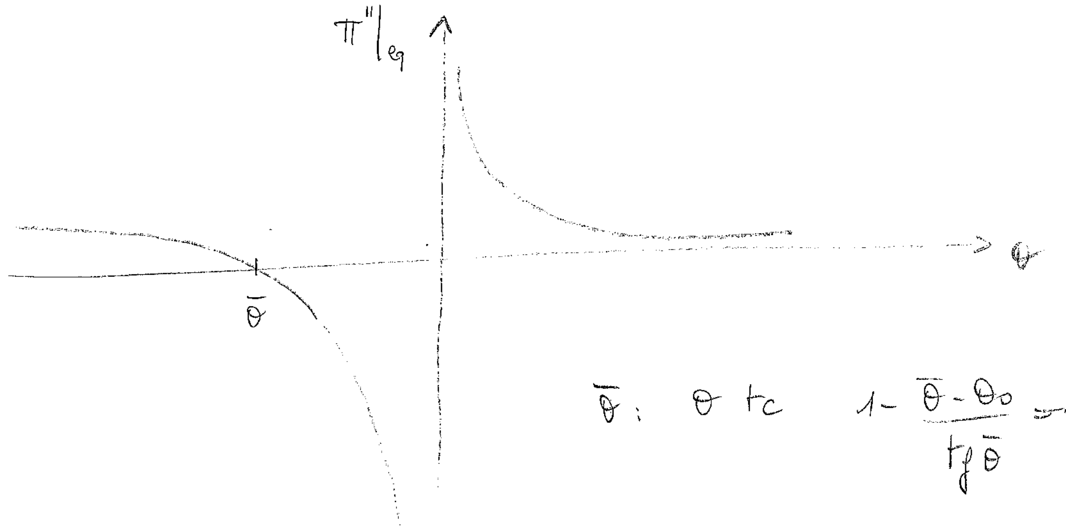
The only solution is

$$\boxed{P = \frac{k(\theta - \theta_0)}{\ell \sin\theta}}$$

Stability of equilibrium

$$\Pi''\Big|_{P=\frac{k(\theta-\theta_0)}{\ell\sin\theta}} = k - \frac{k(\theta-\theta_0)}{\ell\sin\theta}\cdot \ell\cos\theta$$

$$= k\left(1-\frac{\theta-\theta_0}{tg\,\theta}\right)$$

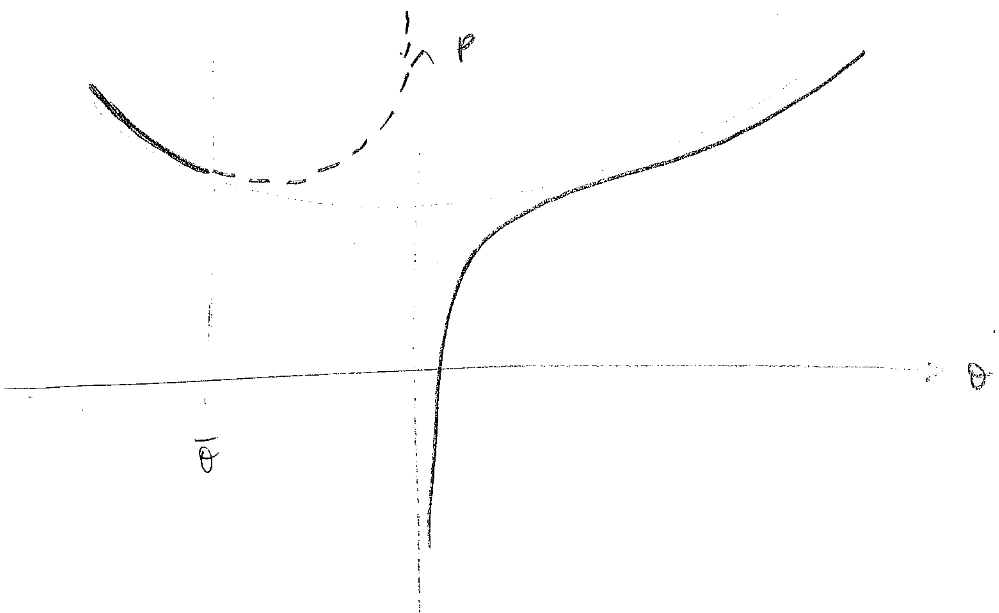


$\bar{\theta}$: θ tc $\quad 1-\frac{\bar{\theta}-\theta_0}{tg\,\bar{\theta}}=0$

$\theta_{cr} < \theta < 0$ unstable equilibrium

$\theta < \theta_{cr}$ or $\theta > 0$ stable equilibrium

The bifurcation diagram reads:

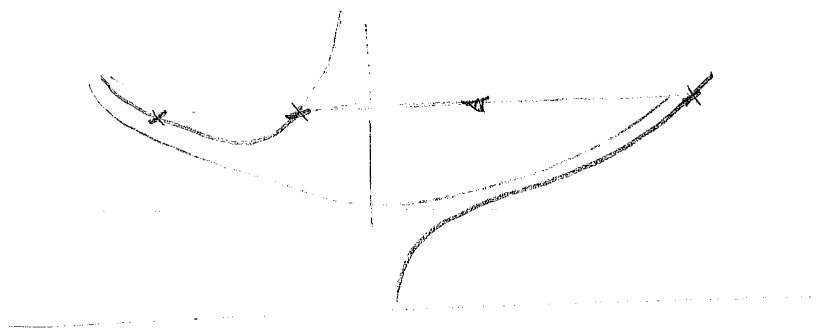


Remarks

1. No more bifurcation! As P is increased, the structure follows the equilibrium path

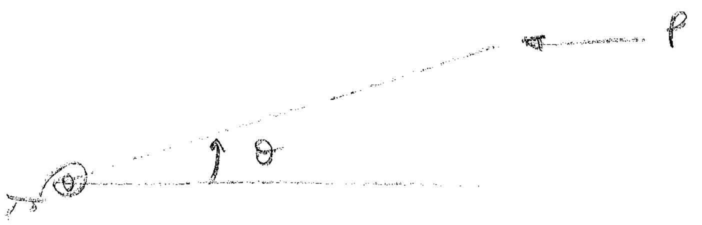
2. In the neighborhoods of the bifurcation of the "perfect structure" a sudden increase of θ is observed for small increments of P
→ the critical load of the perfect structure is a useful information.

3. A transition to the secondary path is made possible if energy is introduced in the system.



Example 1 - linearized analysis

let's consider the analysis of the elastic stability in the context of a linearized analysis. This is the kind of analysis performed by the majority of finite element codes



$$\Pi = \frac{1}{2} k\theta^2 - Pl + Pl\cos\theta$$

$$\Pi \simeq \delta\Pi + \frac{1}{2!}\delta^2\Pi + \dots$$

$$\delta\Pi = k\theta\,\delta\theta - Pl\sin\theta\Big|_{\theta=0}\delta\theta$$
$$= 0$$

$$\delta^2\Pi = (k - Pl\cos\theta)\Big|_{\theta=0}\delta\theta^2$$
$$= (k - Pl)\,\delta\theta^2$$

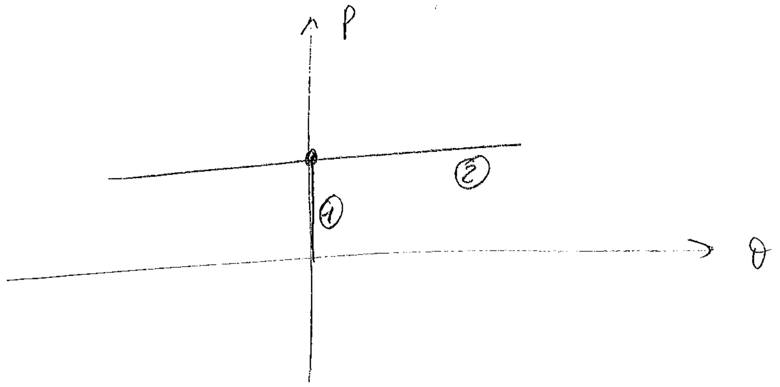
$$\Pi \simeq \frac{1}{2}(k - Pl)\theta^2$$ (quadratic Π = linear equations)

Equilibrium: $\Pi' = (k - Pl)\theta = 0$

$(k - Pl)\theta = 0$ ← 1x1 eigenvalue pb.

the equilibrium solutions are:

1. $\theta = 0$ (trivial solution)
2. $P = k/l$ (non-trivial solution)



Stability of Equilibrium

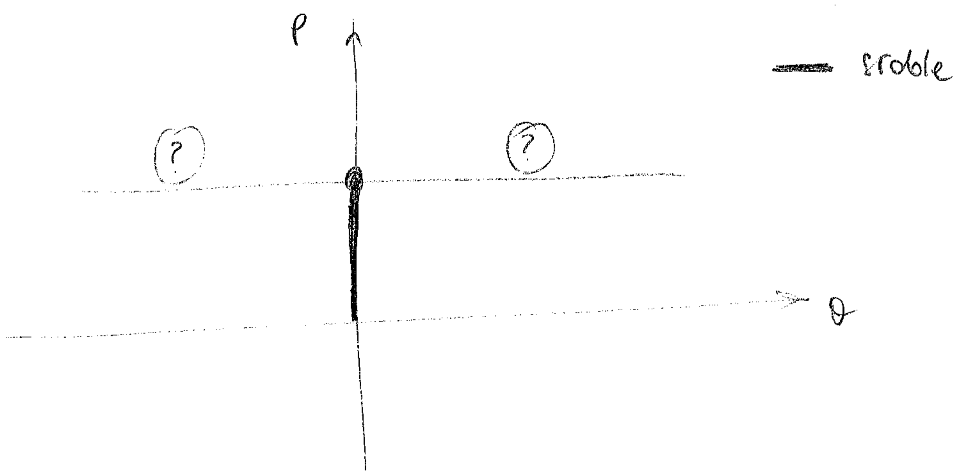
a. Equilibrium path 1

$$\Pi''\big|_{\theta=0} = k - P\ell > 0 \quad \text{if} \quad P < k/\ell$$

b. Equilibrium path 2

$$\Pi''\big|_{P=k/\ell} = k - k = 0$$

The bifurcation diagram is:



No information regarding the stability of the equilibrium beyond the bifurcation!

Example 1 - initial post-buckling analysis

- By considering a second order Taylor expansion for Π, nothing can be said on the stability of the equilibrium paths departing from the bifurcation.

- One possibility is to rise up the order of the expansion

$$\Pi = \frac{1}{2} k\theta^2 - Pl + Pl\cos\theta$$

$$\delta\Pi = 0 \cdot \delta\theta$$

$$\delta^2\Pi = (k - Pl)\,\delta\theta^2$$

$$\delta^3\Pi = Pl\sin\theta\Big|_{\theta=0} \delta\theta^3 = 0 \cdot \delta\theta^3$$

$$\delta^4\Pi = Pl\cos\theta\Big|_{\theta=0} \delta\theta^4 = Pl\,\delta\theta^4$$

So:

$$\Pi = \frac{1}{2}(k - Pl)\theta^2 + \frac{1}{4!} Pl\,\theta^4$$

Equilibrium: for small deviation with respect to $\theta=0$, the quartic term is negligible (if compared to the quadratic one)

$\Rightarrow$ 1. $\theta = 0$
2. $P = k/l$

Stability of equilibrium.

$$\Pi'' = k - Pl + \frac{1}{2} Pl\,\theta^2$$

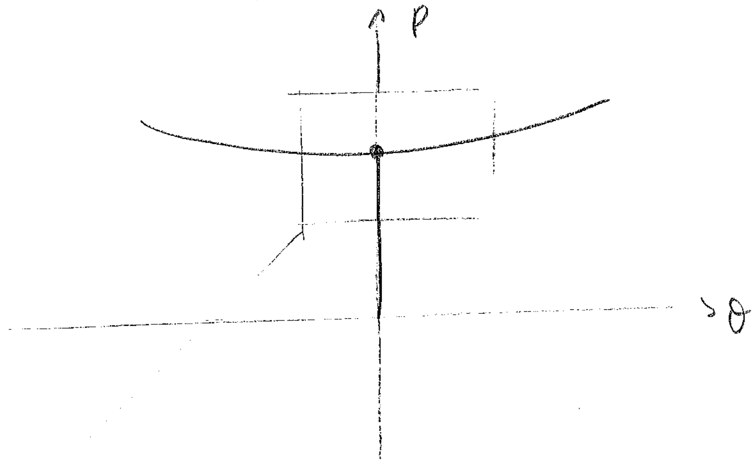
a. Equilibrium path 1 ($\theta = 0$)

$$\Pi''\Big|_{\theta=0} = k - Pl > 0 \quad \text{if} \quad P < k/l$$

b. Equilibrium path 2 ($P = k/l$)

$$\Pi''\Big|_{P=k/l} = k - k + \frac{1}{2}\frac{k}{l}\,l\theta^2 = \frac{1}{2}k\theta^2 > 0 \quad \forall\theta$$

The bifurcation diagram is:



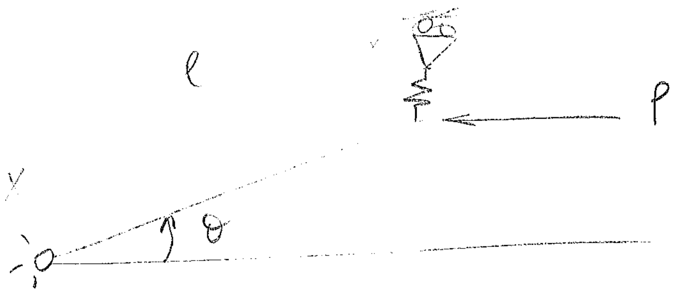
the linearization around $\theta = 0$ holds in the neighborhood of $\theta = 0$

Remarks

1. Higher order terms make it possible the investigation of the stability of the equilibrium paths beyond the bifurcation

2. Far away from $\theta = 0$, the results here derived could be wrong

Example 2



$$\Pi = \frac{1}{2} k \left(l \sin\theta \right)^2 - Pu \qquad u = l\,(1-\cos\theta)$$

$$= \frac{1}{2} k l^2 \sin\theta^2 - Pl + Pl\cos\theta$$

Equilibrium: $\Pi' = k l^2 \sin\theta\cos\theta - Pl\sin\theta = 0$

1. $\theta = 0$

2. $P = kl\cos\theta$

Stability of equilibrium:

$$\Pi'' = k l^2 \left(\cos\theta^2 - \sin\theta^2 \right) - Pl\cos\theta$$

a. Equilibrium path 1 $(\theta = 0)$

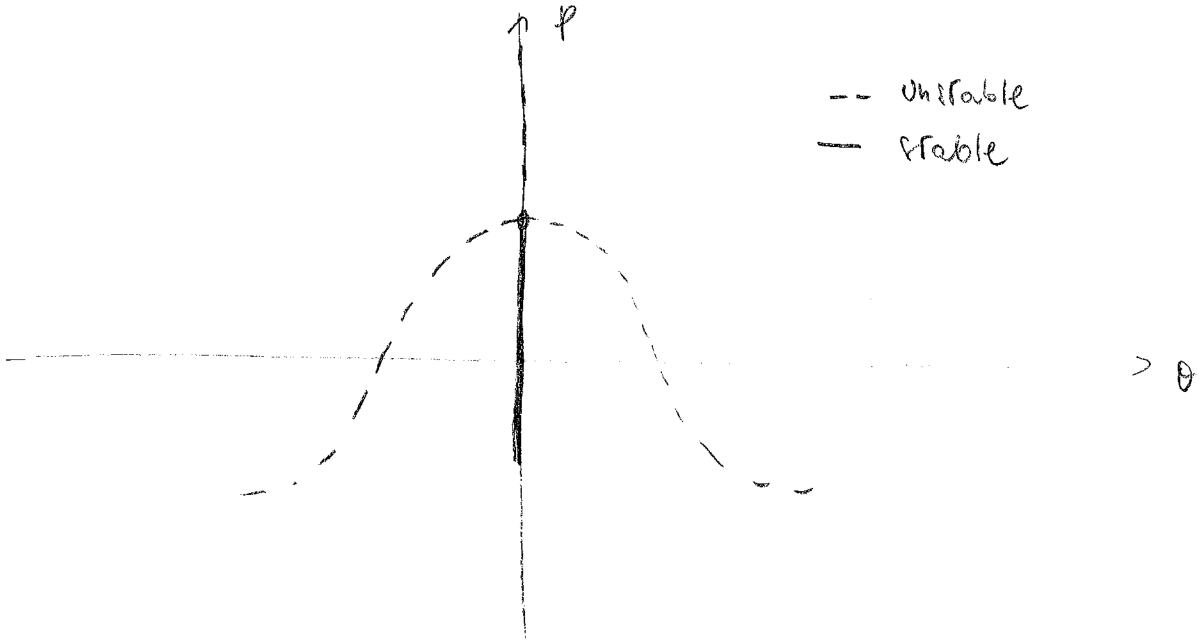
$$\Pi'' = kl^2 - Pl > 0 \quad \text{if} \quad P < k/l$$

b. Equilibrium path 2 $(P = kl\cos\theta)$

$$\Pi'' = kl^2(\cos\theta^2 - \sin\theta^2) - kl^2\cos\theta^2$$

$$= -kl^2\sin\theta^2 < 0 \quad \forall\theta$$

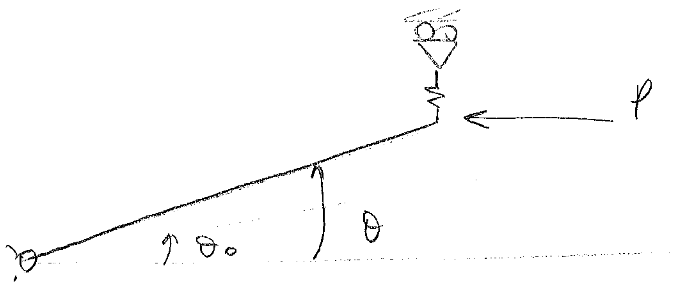
Bifurcation diagram



Remarks

1. The bifurcation is now symmetric and unstable
2. After the bifurcation the structure undergoes a catastrophic failure (according to this model
3. All the equilibrium paths beyond the bifurcation are unstable

Example 2 - with imperfections



$$\Pi = \frac{1}{2} k \left(l \sin\theta - l \sin\theta_0 \right)^2 - P \left(l \cos\theta_0 - l \cos\theta \right)$$

$$= \frac{1}{2} k l^2 \left(\sin\theta^2 - 2 \sin\theta \sin\theta_0 \right) + P l \cos\theta$$

Equilibrium $\quad \Pi' = k l^2 \cos\theta \left(\sin\theta - \sin\theta_0 \right) - P l \sin\theta$

1. $P = k l \cos\theta \left(1 - \frac{\sin\theta_0}{\sin\theta} \right)$

$\theta = 0$ is not a solution!

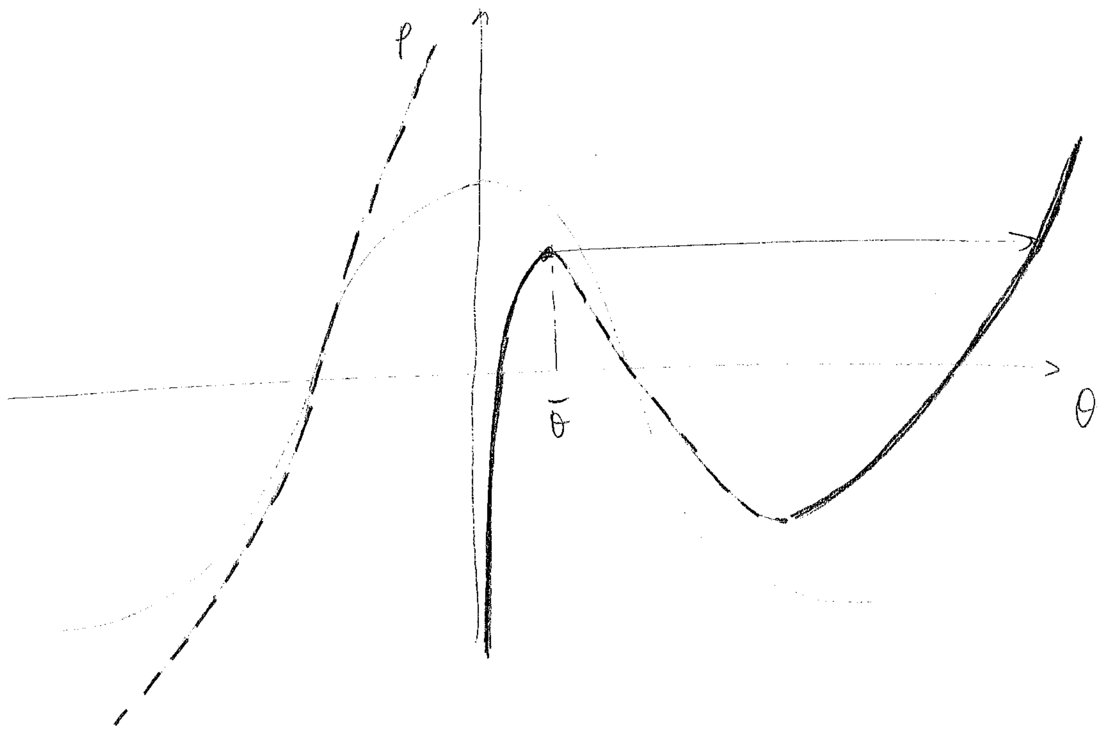
Stability of equilibrium

$$\Pi'' = k l^2 \left(- \sin\theta^2 + \cos\theta^2 + \sin\theta \sin\theta_0 \right) - P l \cos\theta$$

$$\Pi'' \Big|_{P = k l \cos\theta \left(1 - \frac{\sin\theta_0}{\sin\theta} \right)} = k l^2 \left(- \sin\theta^2 + \frac{\sin\theta_0}{\sin\theta} \right)$$

Define now $\bar{\theta}$: $\quad - \sin\bar{\theta}^2 + \frac{\sin\theta_0}{\sin\bar{\theta}} = 0$

The bifurcation diagram is:

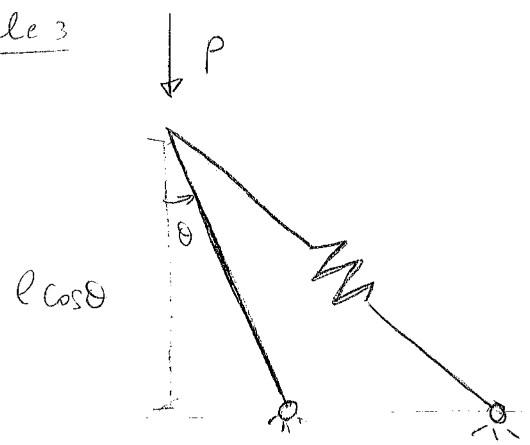


Remarks

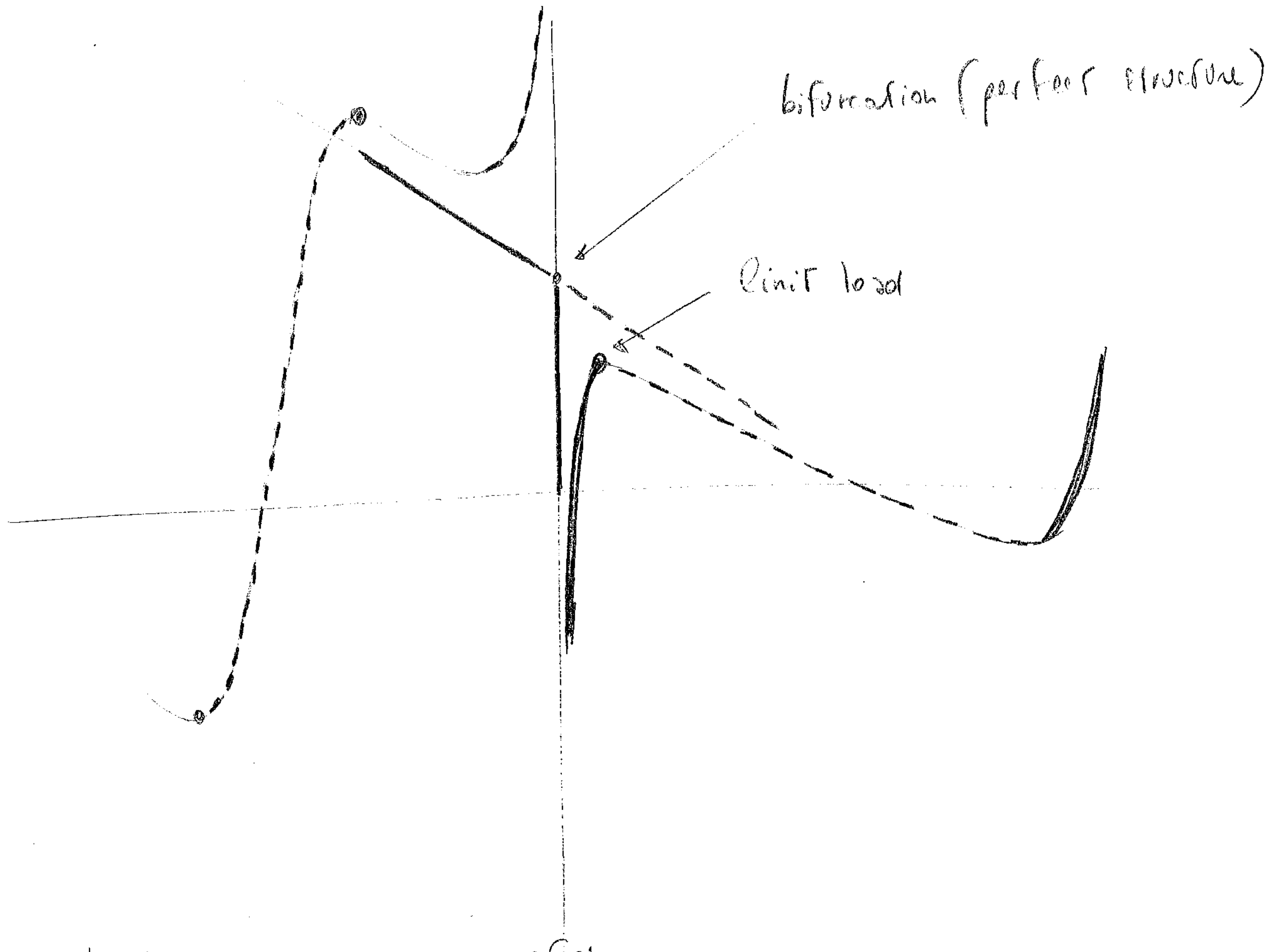
1. The perfect structure is characterized by an unstable postbuckling response. As a consequence, the introduction of geometric imperfections determines a drastic reduction of the max load carried by the structure.

2. Note the presence of a limit load, followed by a snap

Example 3

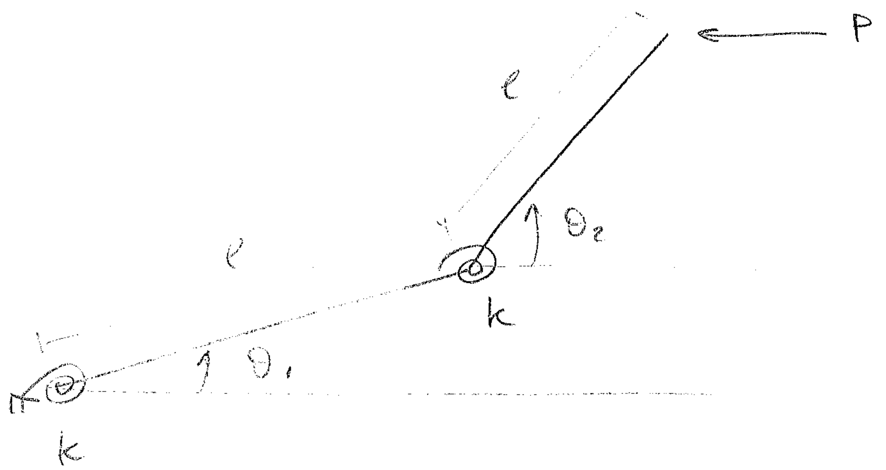


The following diagram can be obtained



- The bifurcation is asymmetric
- The perfect structure is characterized by the presence of one stable path and an unstable one (after the bifurcation)
- The presence of an unstable path determine a drastic reduction of the max load carried by the imperfect structure

- 2 dof example



$$\Pi = \frac{1}{2} k \theta_1^2 + \frac{1}{2} k (\theta_2 - \theta_1)^2 - P u$$

$$u = 2l - l\cos\theta_1 - l\cos\theta_2$$

$$\Pi = k\theta_1^2 + \frac{1}{2} k\theta_2^2 - k\theta_1\theta_2 + Pl\cos\theta_2 + Pl\cos\theta_1$$

Epuilibrium. $\delta\Pi = \frac{\partial \Pi}{\partial \theta_1}\delta\theta_1 + \frac{\partial \Pi}{\partial \theta_2}\delta\theta_2 = 0 \quad \forall\, \delta\theta_1, \delta\theta_2$

$$\frac{\partial \Pi}{\partial \theta_1} = 2k\theta_1 - k\theta_2 - Pl\sin\theta_1$$

$$\frac{\partial \Pi}{\partial \theta_2} = k\theta_2 - k\theta_1 - Pl\sin\theta_2$$

The non linear epuilibrium epuations are then:

$$\boxed{\begin{aligned} 2k\theta_1 - k\theta_2 - Pl\sin\theta_1 &= 0 \\ -k\theta_1 + k\theta_2 - Pl\sin\theta_2 &= 0 \end{aligned}}$$

Equilibrium : 1. $\theta_1 = \theta_2 = 0$ (trivial solution)

2. $\theta_1 \neq 0$, $\theta_2 \neq 0$

(can be computed using Newton-Raphson)

Stability of equilibrium

$$\Delta\Pi = \delta\Pi + \frac{1}{2!}\delta\Pi^2 + \cdots$$

$$\delta^2\Pi = \frac{\partial^2\Pi}{\partial\theta_1^2}\delta\theta_1^2 + \frac{\partial^2\Pi}{\partial\theta_2^2}\delta\theta_2^2 + 2\frac{\partial^2\Pi}{\partial\theta_1\partial\theta_2}\delta\theta_1\delta\theta_2$$

a. Equilibrium path $\theta_1 = \theta_2 = 0$

$$\frac{\partial^2\Pi}{\partial\theta_1^2} = 2k - Pl\cos\theta_1 \Big|_{\theta_1=\theta_2=0} = 2k - Pl$$

$$\frac{\partial^2\Pi}{\partial\theta_2^2} = k - Pl\cos\theta_2 \Big|_{\theta_1=\theta_2=0} = k - Pl$$

$$\frac{\partial^2\Pi}{\partial\theta_1\partial\theta_2} = -k$$

So:

$$\delta^2\Pi = \delta\underline{\theta}^T \begin{bmatrix} 2k - Pl & -k \\ -k & k - Pl \end{bmatrix} \delta\underline{\theta}$$

$$= \delta\underline{\theta}^T \underline{\underline{H}}\, \delta\underline{\theta} \qquad \underline{\underline{H}} : \text{Hessian matrix}$$

The solution is stable iif $\delta^2\Pi > 0 \quad \forall\ \delta\theta_1, \delta\theta_2$

$\Rightarrow$ the eigenvalues of $\underline{\underline{H}}$ are > 0.

The solution loses its stability when $\delta^2\Pi = 0$ for a set

$\delta\underline{\theta} \neq 0$

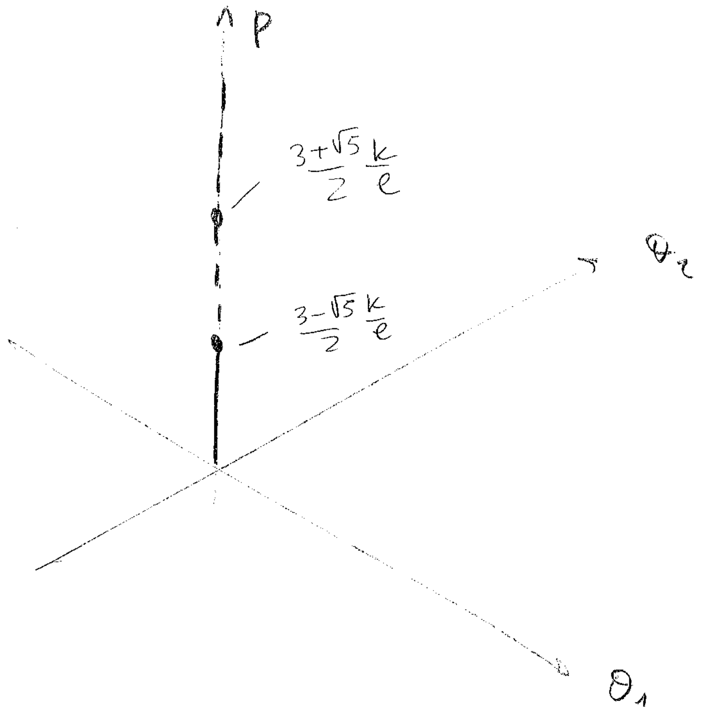
The eigenvalues of $\underline{\underline{H}}$ are.

$$\det\left(\underline{\underline{H}} - \lambda \underline{\underline{I}}\right) = \lambda^2 + 2\left(Pl - 3k\right)\lambda + k^2 + P^2 l^2 - 3Plk = 0$$

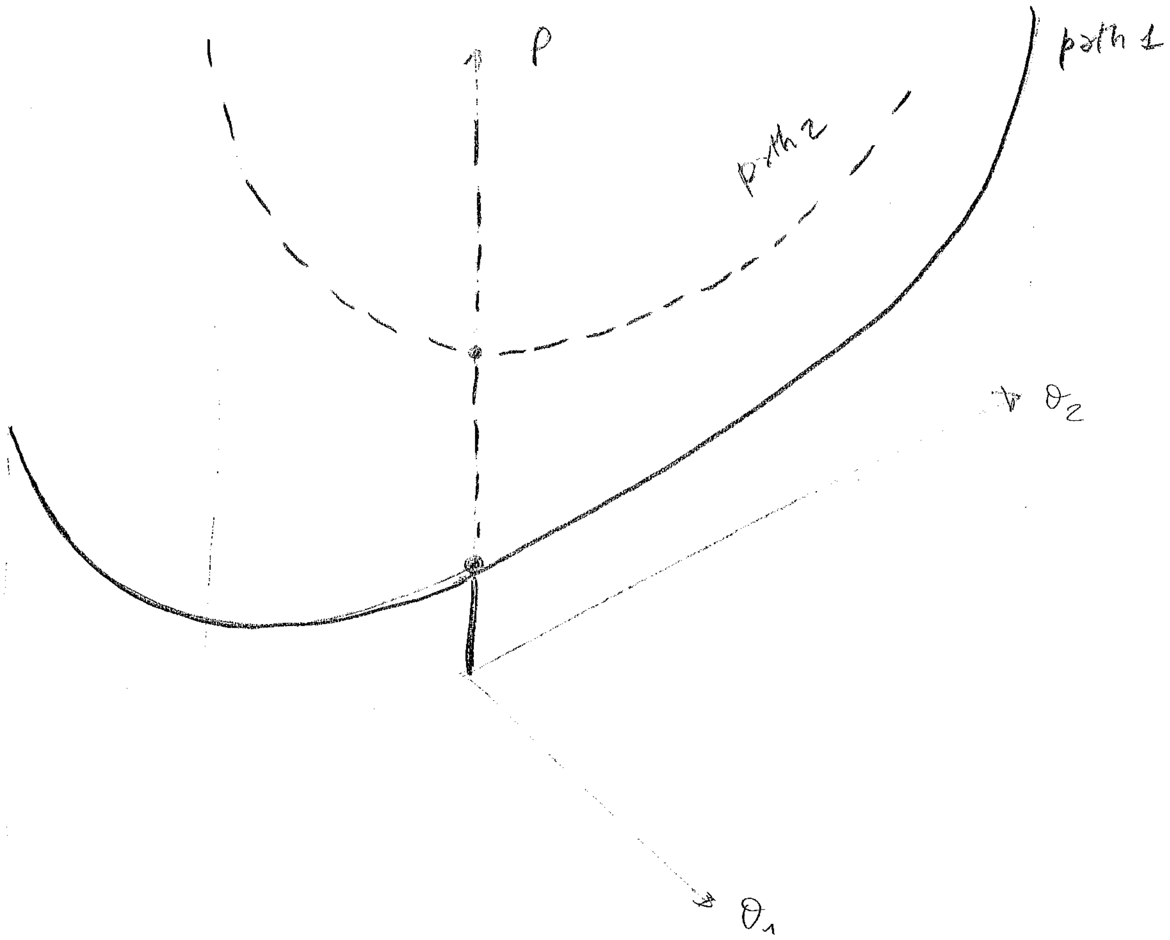
$$\Rightarrow \boxed{\begin{aligned} \lambda_1 &= \frac{3-\sqrt{5}}{2}k - Pl \\ \lambda_2 &= \frac{3+\sqrt{5}}{2}k + Pl \end{aligned}}$$

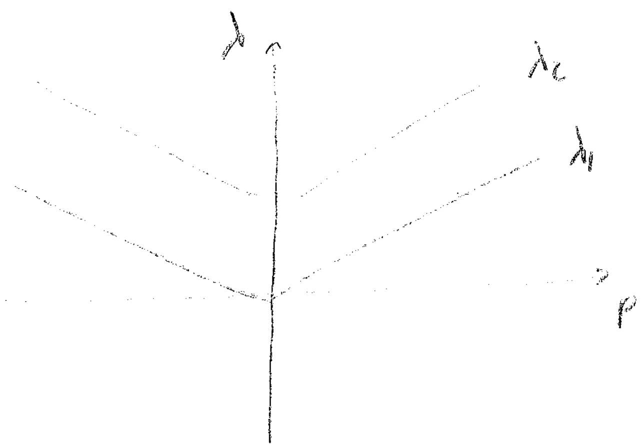
- $P < \dfrac{3-\sqrt{5}}{2}\dfrac{k}{l} \Rightarrow \lambda_1 > 0,\ \lambda_2 > 0$ (stable)

- $P > \dfrac{3-\sqrt{5}}{2}\dfrac{k}{l} \Rightarrow \lambda_1 < 0,\ \lambda_2 > 0$ (unstable)

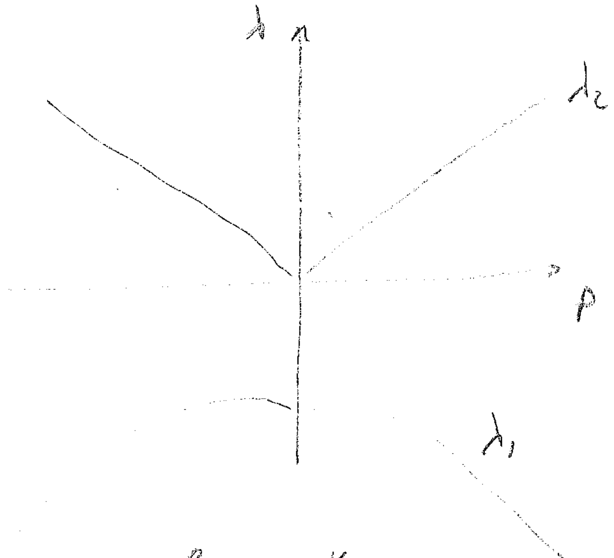


The analysis of the stability of the post-bifurcational paths can be performed numerically by solving the non linear equilibrium equations and evaluating the eigenvalues for each equilibrium solution





along path 1



along path 2

• 2 DOF example – linearized

$$\Pi = k\theta_1^2 + \frac{1}{2}k\theta_2^2 - k\theta_1\theta_2 + \underbrace{Pl\cos\theta_2 + Pl\cos\theta_1}$$

L contributions to linearize

$$\Delta\Pi = \delta\Pi + \frac{1}{2!}\delta^2\Pi + \dots$$

$$\delta\Pi = \frac{\partial\Pi}{\partial\theta_1}\delta\theta_1 + \frac{\partial\Pi}{\partial\theta_2}\delta\theta_2$$

$$= \left(2k\theta_1 - k\theta_2 - Pl\sin\theta_1\right)\delta\theta_1 + \left(k\theta_2 - k\theta_1 - Pl\sin\theta_2\right)\delta\theta_2$$

$$\delta^2\Pi = \frac{\partial^2\Pi}{\partial\theta_1^2}\delta\theta_1^2 + \frac{\partial^2\Pi}{\partial\theta_2^2}\delta\theta_2^2 + \frac{\partial^2\Pi}{\partial\theta_1\partial\theta_2}\delta\theta_1\delta\theta_2$$

$$= \left(2k - Pl\cos\theta_1\right)\delta\theta_1^2 + \left(k - Pl\cos\theta_2\right)\delta\theta_2^2 - 2k\,\delta\theta_1\delta\theta_2$$

Linearize around $\theta_1 = 0$, $\theta_2 = 0$:

$$\delta\Pi\Big|_{\theta_1=\theta_2=0} = 0\cdot\delta\theta_1 + 0\cdot\delta\theta_2 \quad\rightarrow\quad \theta_1 = 0 \quad \theta_2 = 0$$ is an equilibrium solution

$$\delta^2\Pi\Big|_{\theta_1=\theta_2=0} = \left(2k - Pl\right)\delta\theta_1^2 + \left(k - Pl\right)\delta\theta_2^2 - 2k\,\delta\theta_1\delta\theta_2$$

$$= \begin{Bmatrix}\delta\theta_1 & \delta\theta_2\end{Bmatrix}^T \begin{bmatrix} 2k - Pl & -k \\ -k & k - Pl \end{bmatrix} \begin{Bmatrix}\delta\theta_1 \\ \delta\theta_2\end{Bmatrix}$$

$$= \delta\underline{\theta}^T \underline{\underline{H}}\, \delta\underline{\theta}$$

The eigenvalues of $\underline{\underline{H}}$ are obtained as:

$$\begin{vmatrix} 2k - P\ell - \lambda & -k \\ -k & k - P\ell - \lambda \end{vmatrix} = 0$$

$$\hookrightarrow \quad \lambda_1 = \frac{3-\sqrt{5}}{2} k - P\ell$$

$$\lambda_2 = \frac{3+\sqrt{5}}{2} k + P\ell$$

(as obtained previously)

As usual, the linearized approach provides information regarding the bifurcation(s), but not on the stability of the post-bifurcation path.