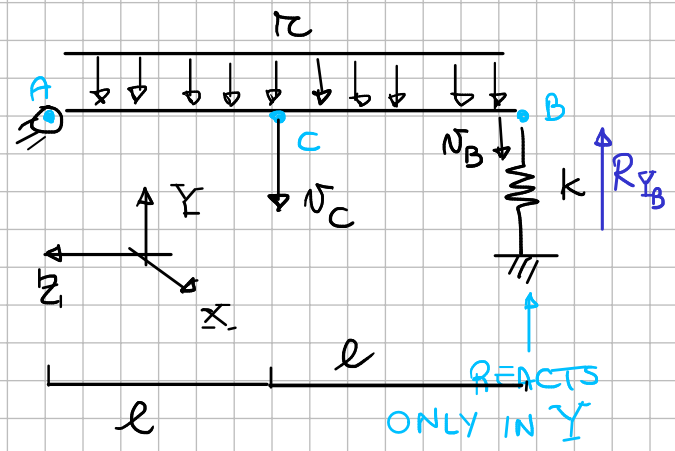
EXERCISE SESSION 5 – 18/10/22

Displacement of beam Systems

Ex 1 EXAM 06/09/2021



DATA

$l = 1200\ mm$

$r = 12\ N/mm$

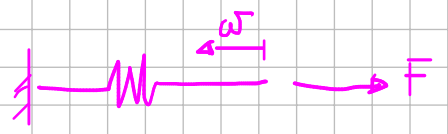
$EJ_{xx} = 10^{12}\ N\,mm^2$

$k = 750\ N/mm$

FIND

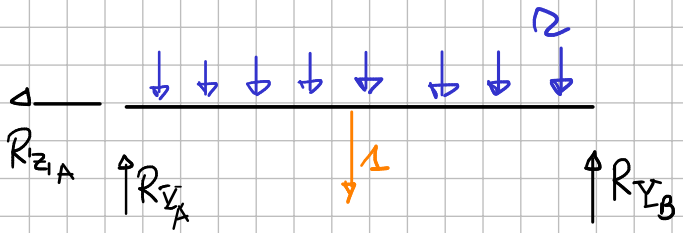
$v_C = ?$

How to treat the Spring



$F = k \cdot w$

1) REACTION FORCES



REAL

$$\begin{cases} R_{Z_A} = 0 \\ R_{Y_A} + R_{Y_B} - r \cdot 2l = 0 \\ R_{Y_B} \cdot 2l - r \cdot 2l \cdot l = 0 \end{cases}$$

$R_{Y_B} = r \cdot l \rightarrow k v_B = rl$

$$v_B = \frac{rl}{k}$$

$R_{Y_A} = rl$

DUMMY

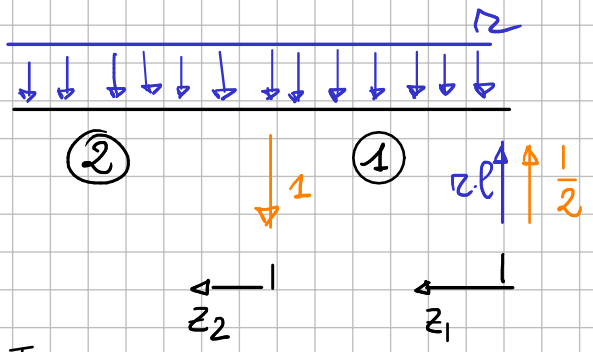
$R'_{Z_A} = 0$

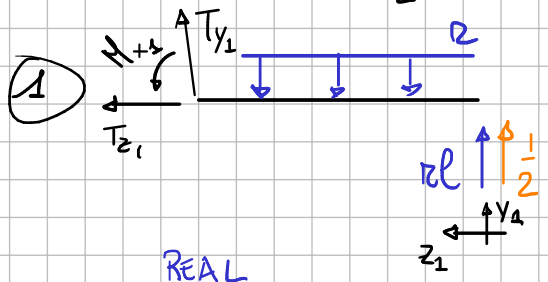
$R'_{Y_A} + R'_{Y_B} - 1 = 0$

$R'_{Y_B} \cdot 2l - 1 \cdot l = 0$

$\rightarrow R'_{Y_B} = R'_{Y_A} = \frac{1}{2}$

2) INTERNAL ACTIONS



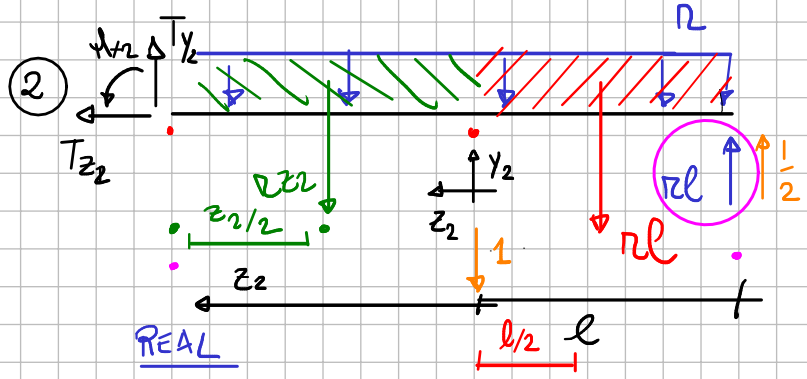


REAL

$$M_{x_1}(z_1) = q \cdot z_1 \cdot \frac{z_1}{2} - q \cdot l \cdot z_1$$

DUMMY

$$M'_{x_1}(z_1) = -\frac{1}{2} z_1$$



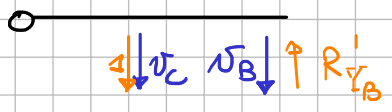
$$M_{x_2}(z_2) = \underbrace{q \cdot l \left(z_2 + \frac{l}{2}\right)}_{\text{red}} + \underbrace{q \cdot z_2 \cdot \frac{z_2}{2}}_{\text{green}} - \underbrace{q l \left(z_2 + l\right)}_{\text{pink}}$$

$$= -q \cdot l^2/2 + q \frac{z_2^2}{2}$$

DUMMY

$$M'_{x_2}(z_2) = 1 \cdot z_2 - \frac{1}{2}\left(z_2 + l\right) = \frac{1}{2} z_2 - \frac{1}{2} l$$

3) PCVW



$$\delta W_e = \underbrace{1 \cdot v_C}_{\text{disp. of C}} - R'_{Y_B} \cdot v_B$$

$$= v_C - \frac{1}{2}\frac{rl}{k}$$

$$\delta W_i = \int_0^l M'_{x_1} \cdot \frac{M_{x_1}}{EJ_{xx}}\, dz_1 + \int_0^l M'_{x_2} \frac{M_{x_2}}{EJ_{xx}}\, dz_2$$

$$= \frac{1}{EJ_{xx}}\left[-\cancel{\frac{1}{16} r z_1^4} + \frac{1}{6} r l z_1^3 + \dots \right.$$

$$\left. \dots + \cancel{\frac{1}{16} r z_2^4} - \frac{1}{8} r l^2 z_2^2 - \frac{1}{12} r l z_2^3 + \frac{1}{4} r l^3 z_2 \right]_0^l =$$

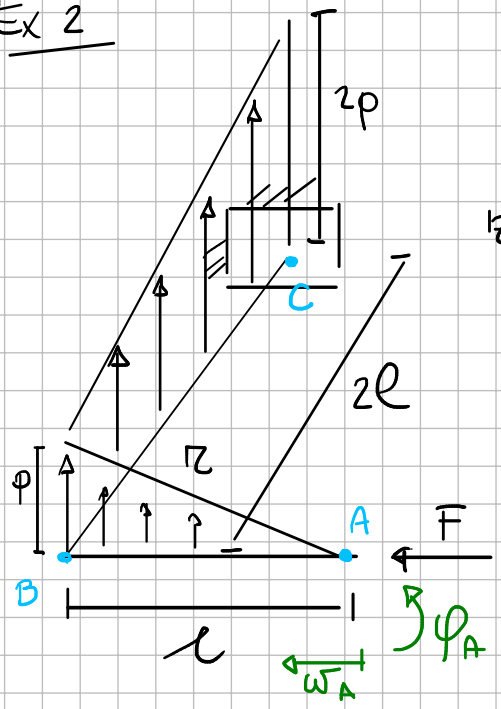
$$= \frac{1}{EJ_{xx}} \cdot \frac{5}{24} r l^4$$

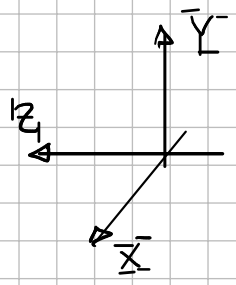
PCVW $\longrightarrow$ $\delta W_e = \delta W_i$

$$v_C - \frac{1}{2}\frac{rl}{k} = \frac{1}{EJ_{xx}} \cdot \frac{5}{24} r l^4$$

$$\Rightarrow v_C = \frac{5}{24}\frac{r l^4}{EJ_{xx}} + \frac{1}{2}\frac{rl}{k} = 14.78 \text{ mm}$$

Ex 2





DATA

$\ell = 750$ mm

$p = 0.08$ N/mm

$E = 200000$ MPa

$G = 77000$ MPa

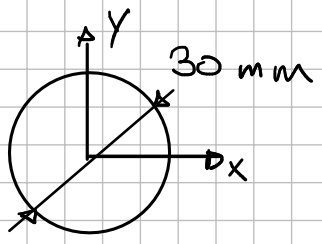
$F = 1000$ N

FIND

$\omega_A = ?$

$\varphi_A = ?$

SECTION

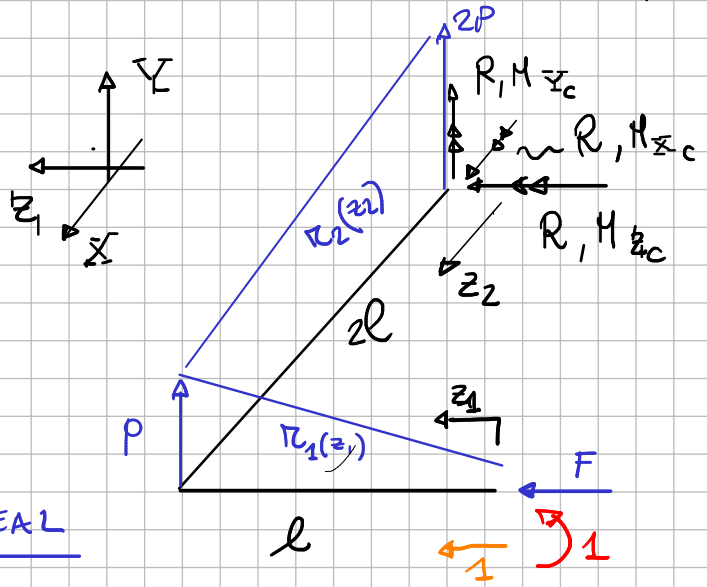


$$J_{xx} = J_{yy} = \frac{\pi r^4}{4} = 39761 \text{ mm}^4$$

$$J_p = J_{xx} + J_{yy} = 79522 \text{ mm}^4$$

$$A = \pi r^2 = 706{,}86 \text{ mm}^2$$

1) REACTION FORCES OF THE SYSTEM



REAL

$R_{Xc} = 0$

$$R_{Yc} = \underbrace{-\frac{1}{2}p\ell}_{r_1} \underbrace{- p2\ell - \frac{1}{2}p2\ell}_{e_2} = -\frac{7}{2}p\ell$$

$$R_{Zc} = -F$$

$$M_{Xc} = -\frac{1}{2}p\ell \cdot \frac{\ell}{3} = -\frac{1}{6}p\ell^2$$

$$M_{Yc} = F \cdot 2\ell$$

$$M_{Zc} = -\frac{1}{2}p\ell \cdot 2\ell - p2\ell \cdot \ell - \frac{1}{2}p2\ell \cdot \frac{2\ell}{3} = -\frac{11}{3}p\ell^2$$

DUMMY 1

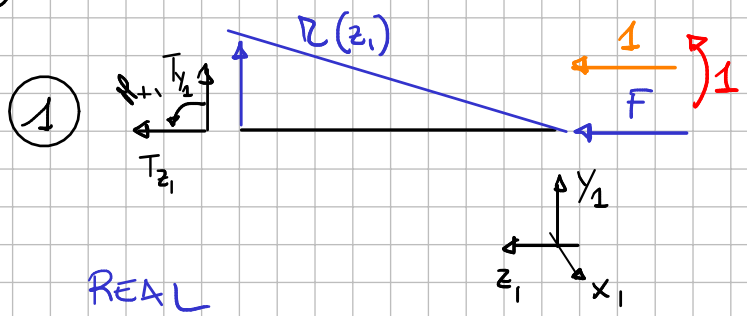
$R'_{z_1 C} = -1$

$M'_{Y_C} = 1 \cdot 2\ell$

DUMMY 2

$M''_{X_C} = -1$

2) INTERNAL ACTIONS



$r_1(z_1) = p \frac{z_1}{\ell}$

REAL

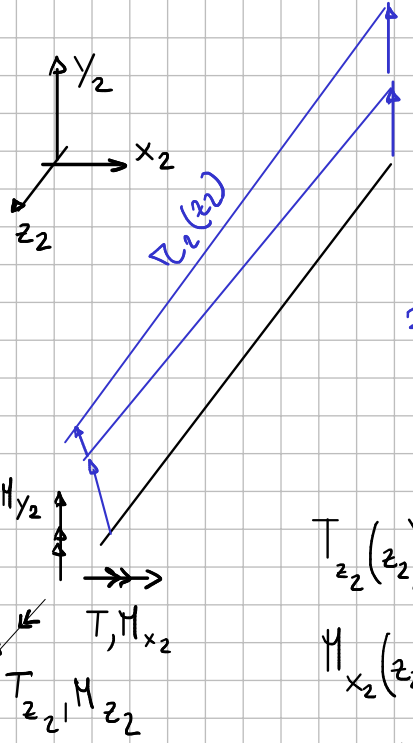
$$\begin{cases} T_{z_1}(z_1) = -F \\ T_{y_1}(z_1) = -\frac{1}{2} r_1(z_1) \cdot z_1 = -\frac{1}{2} \frac{p z_1^2}{\ell} \\ M_{x_1}(z_1) = -\frac{1}{2} r_1(z_1) \cdot z_1 \cdot \frac{z_1}{3} = -\frac{1}{6} \frac{p z_1^3}{\ell} \end{cases}$$

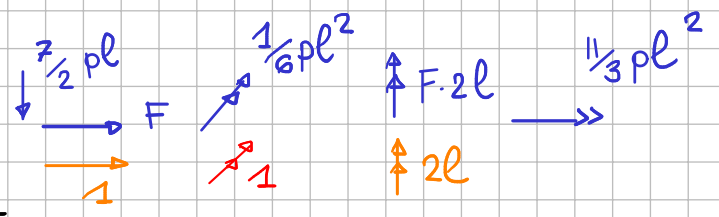
DUMMY 1

$T'_{z_1}(z_1) = -1$

DUMMY 2

$M''_{x_1}(z_1) = -1$





REAL $\quad R_2(z_2) = \underbrace{p}_{R_{21}} + \underbrace{p\left(1 - \frac{z_2}{2l}\right)}_{R_{22}}$

$$T_{z_2}(z_2) = 0$$

$$H_{x_2}(z_2) = +\frac{7}{2}pl \cdot z_2 - R_{21} \cdot z_2 \cdot \frac{z_2}{2} - \int_0^{z_2} R_{22} z_2 \cdot dz_2 - \frac{11}{3}pl^2$$

$$H_{y_2}(z_2) = -2Fl + F \cdot z_2$$

$$H_{z_2}(z_2) = \frac{1}{6}pl^2$$

DUMMY 1 $\qquad$ DUMMY 2

$$H'_{y_2}(z_2) = -2l + 1 \cdot z_2 \qquad H''_{z_2} = 1$$

3) PCVW

REAL $\quad$ DUMMY

$$\delta W_e = 1 \cdot w_A$$

$$\delta W_i = \int_0^l T'_{z_1} \frac{T_{z_1}}{EA} dz_1 + \int_0^{2l} H'_{y_2} \frac{H_{y_2}}{EJ_{yy}} dz_2$$

$$= \int_0^l \frac{1}{EA}(-1 \cdot -F) dz_1 + \int_0^{2l} (-2l + z_2)(-2lF + z_2 F) \frac{1}{EJ_{yy}} dz_2$$

$$= \frac{Fl}{EA} + \left[\frac{F4l^2 z_2}{EJ_{yy}} - \frac{1}{2} \frac{F 2l z_2^2}{EJ_{yy}} - \frac{1}{2} \frac{Fl z_2^2 \cdot 2}{EJ_{yy}} + \frac{1}{3} F z_2^3 \frac{1}{EJ_{yy}} \right]_0^{2l}$$

$$= \frac{Fl}{EA} + \frac{8}{3} \frac{Fl^2}{EJ_{yy}}$$

PCVW $\rightarrow \quad w_A = \delta W_i = \frac{Fl}{EA} + \frac{8}{3} \frac{Fl^2}{EJ_{yy}} = 0{,}194 \text{ mm}$

REAL DUMMY 2

$$\delta W_e = 1 \cdot \varphi_A$$

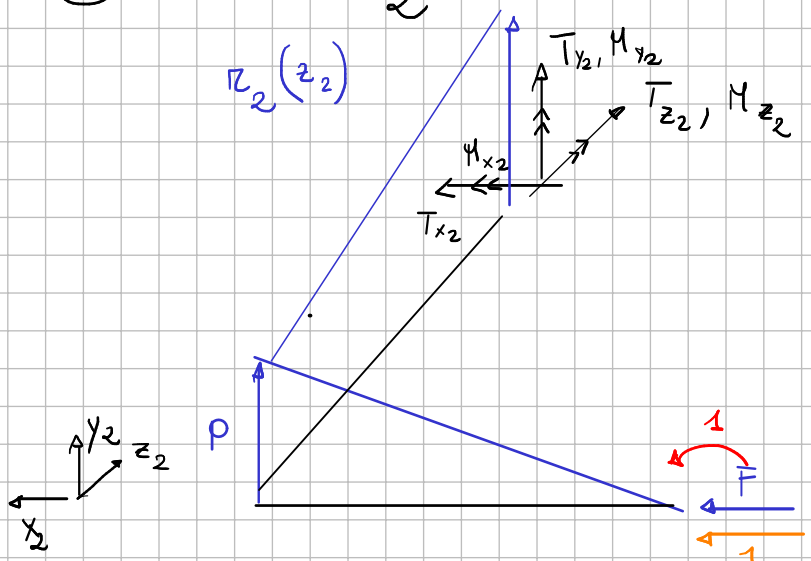
$$\delta W_i = \int_0^{\ell} M'_{x_1} \frac{M_{x_1}}{EJ_{xx}} dz_1 + \int_0^{2\ell} M'_{z_2} \frac{M_{z_2}}{GJ_p} dz_2$$

$$= \frac{1}{24} \frac{p\ell^3}{EJ_{xx}} + \frac{1}{3} \frac{p\ell^3}{GJ_p}$$

PCVW $\rightarrow$ $\varphi_A = \frac{1}{24} \frac{p\ell^3}{EJ_{xx}} + \frac{1}{3} \frac{p\ell^3}{GJ_p} = 2.014 \cdot 10^{-3}$ rad

<<

② WITH z_2 DIFFERENT



$$\pi_2(z_2) = \underbrace{p}_{\pi_{11}} + \underbrace{p\frac{z_2}{2\ell}}_{\pi_{22}}$$

REAL

$$T_{z_2}(z_2) = 0$$

$$M_{x_2}(z_2) = -\frac{1}{2}p\ell z_2 - p\cdot z_2\cdot\frac{z_2}{2} - \frac{1}{2}\pi_{22}(z_2)\cdot z_2\cdot\frac{1}{3}z_2$$

$$= -\frac{1}{2}p\ell z_2 - pz_2\cdot\frac{z_2}{2} - \frac{1}{2}p\frac{z_2}{2\ell}\cdot z_2\cdot\frac{1}{3}z_2$$

$$= -\frac{1}{2}p\ell z_2 - \frac{1}{2}pz_2^2 - \frac{1}{6}\frac{p}{2\ell}z_2^3$$

$$M_{y_2}(z_2) = +\bar{F}\cdot z_2$$

$$M_{z_2}(z_2) = \frac{1}{2}p\cdot\ell\cdot\frac{1}{3}\ell = \frac{1}{6}p\ell^2$$

DUMMY 1

$$M'_{y_2}(z_2) = +z_2$$

DUMMY 2

$$M''_{z_2}(z_2) = 1$$

3) PCVW

REAL DUMMY 1

$$\delta W_e = 1\cdot w_A$$

$$\delta W_i = \int_0^\ell \frac{1}{EA}(-1\cdot -F)\,dz_1 + \int_0^{2\ell}\frac{1}{EJ_{xx}}(z_2)\cdot(\bar{F}z_2)\,dz_2 = \frac{F\ell}{EA} + \frac{8}{3}\frac{F\ell^3}{EJ_{xx}}$$

REAL DUMMY 2

$$\delta W_e = 1 \cdot \varphi_A$$

$$\delta W_i = \int_0^{\ell} M_{x_1}^{1} \cdot \frac{M_{x_1}}{EJ_{xx}} dz_1 + \int_0^{2\ell} M_{z_2}^{1} \cdot \frac{M_{z_2}}{GJ_p} dz_2$$

$$= \int_0^{\ell} -1 \cdot \left(-\frac{1}{6} \frac{p z_1^3}{\ell EJ_{xx}} \right) dz_1 + \int_0^{2\ell} +1 \cdot \frac{1}{6} p\ell^2 \cdot \frac{1}{GJ_p} dz_2$$

$$= \frac{1}{24} \frac{p\ell^3}{EJ_{xx}} + \frac{1}{3} \frac{p\ell^3}{GJ_p}$$