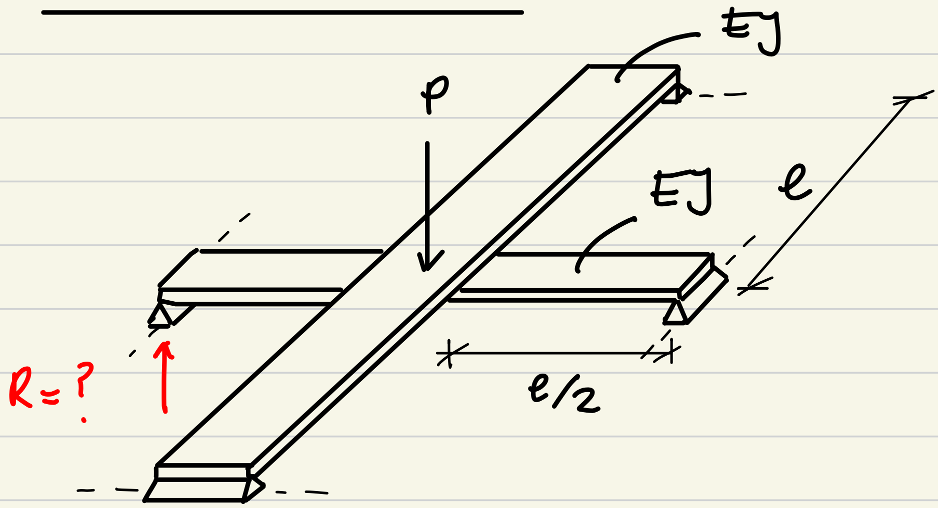
Force - based # 3



Consider the structure in the figure.
The two beams are connected at the middle and are constrained with four hinges (rotation allowed around the dashed line)
Solve the problem exactly and determine the reaction force R

Note : shearing deformation is negligible

Data

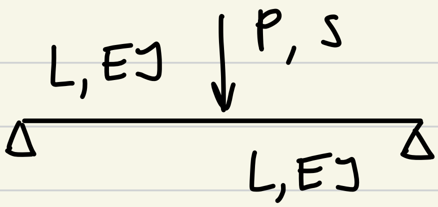
$\ell = 2000$ mm $\qquad P = 9000$ N

$EJ = 10^{11}$ N mm^2

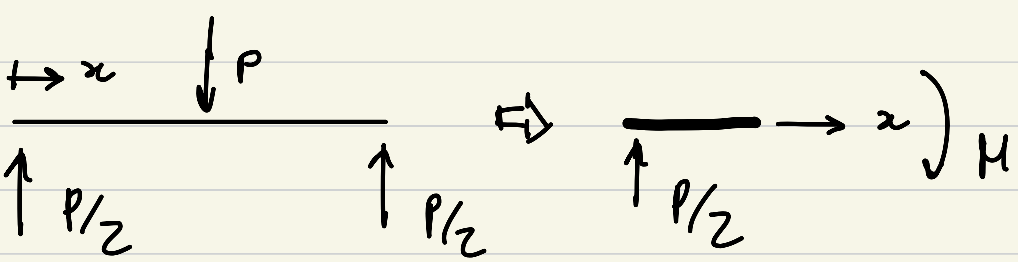
Solution

The problem can be easily solved by noting that the two beams can be represented as a system of two linear springs in parallel.

The stiffness of a simply-supported beam can be obtained following the steps below:

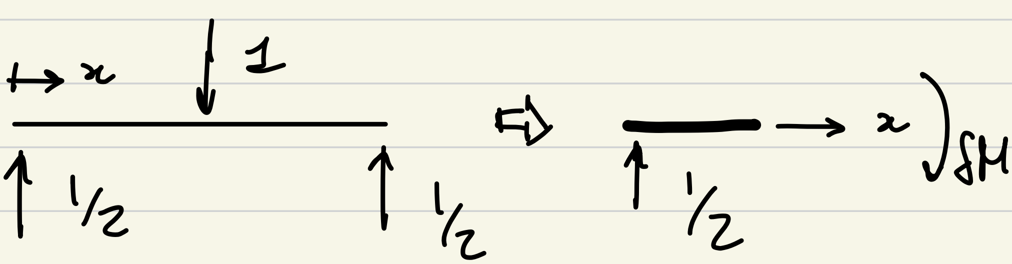


Real system



$$M = -\frac{P}{2}x$$

Dummy system



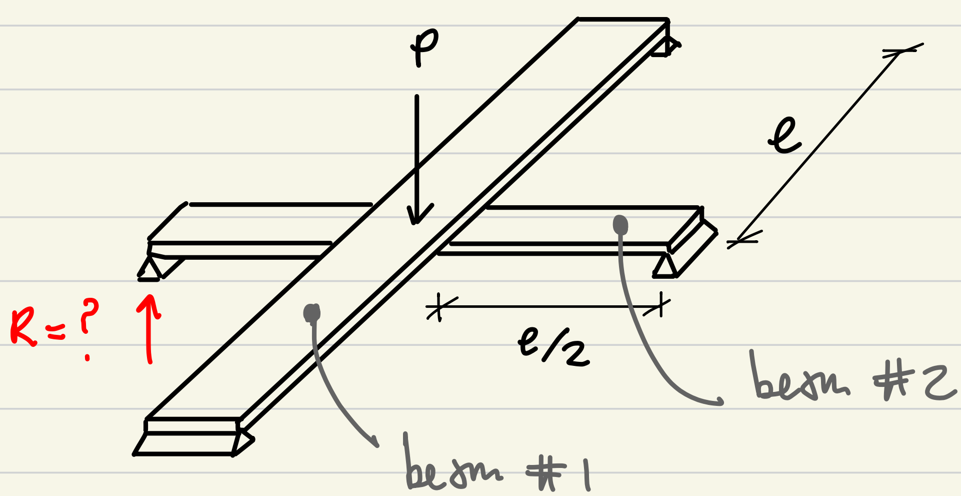
$$\delta M = -\frac{1}{2}x$$

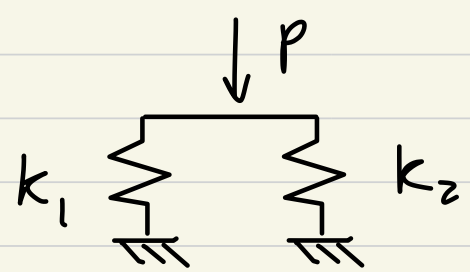
By application of the PCVW:

$$\delta = 2\int_0^L \delta M \frac{M}{EJ}\,dx = \frac{L^3}{6EJ}P \quad , \text{ from which}$$

$$\boxed{k = \frac{6EJ}{L^3}}$$

For the problem at hand, the system is:



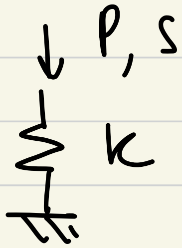


with:

$$k_1 = \frac{6EJ}{L_1^3} \quad \text{with } L_1 = \ell \quad \text{so: } k_1 = \frac{6EJ}{\ell^3}$$

$$k_2 = \frac{6EJ}{L_2^3} \quad \text{with } L_2 = \ell/2 \quad \text{so: } k_2 = \frac{48EJ}{\ell^3}$$

The total stiffness of the system is then



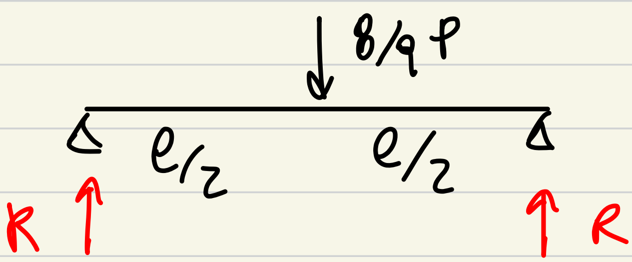
with $k = k_1 + k_2 = \dfrac{54EJ}{\ell^3}$

The displacement s is:

$$s = P/k = \frac{P\ell^3}{54EJ}$$

And so the load introduced in beam #2 reads:

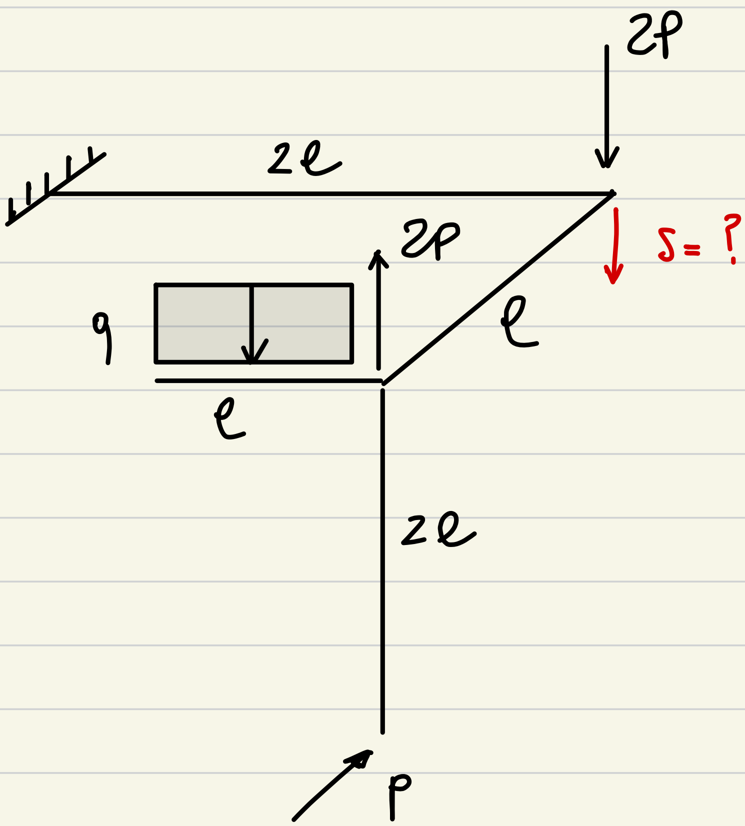
$$F_2 = k_2 s = \frac{48EJ}{\ell^3} \cdot \frac{\ell^3}{54EJ} P = 8/9\, P$$



From which the reaction force R is obtained

as $\boxed{R = 4/9\, P} = 4000\ N$

Force-based # 5



Use a force-based approach to determine the displacement δ. The contribution of shear deformability is negligible.

Data

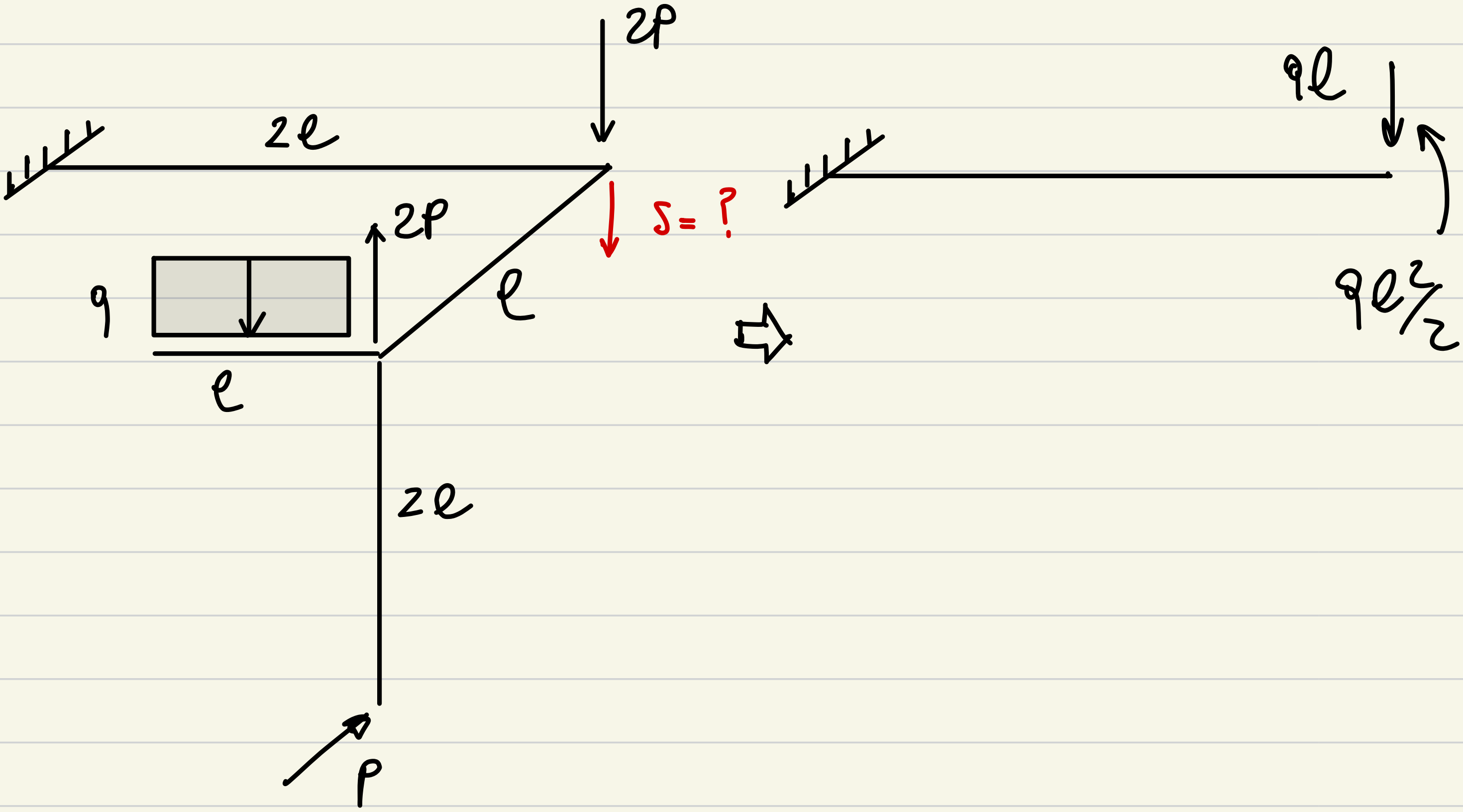
$\ell = 1200$ mm $\qquad q = 2$ N/mm

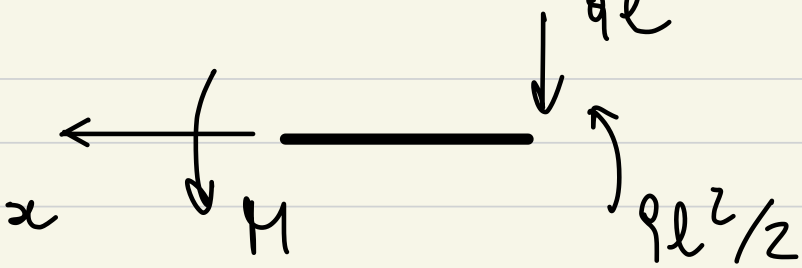
$EJ = 10^{12}$ Nmm^2 $\qquad P = 2000$ N

Solution

Real system

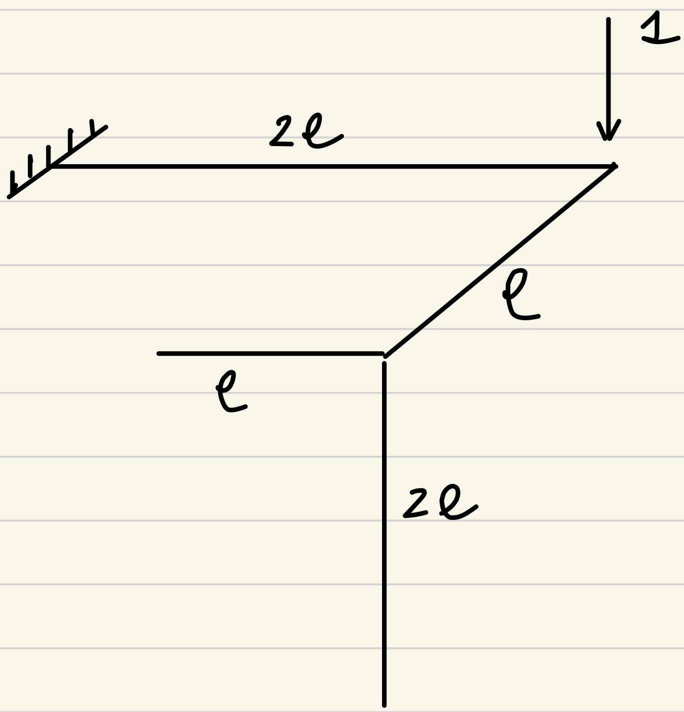
The only beam leading to a not null contribution in the PCVW is the fixed one. Then, the internal actions are reported for this beam only. Only the shear force and the bending moment lead to a not null energy contribution. Then, the other forces are not reported

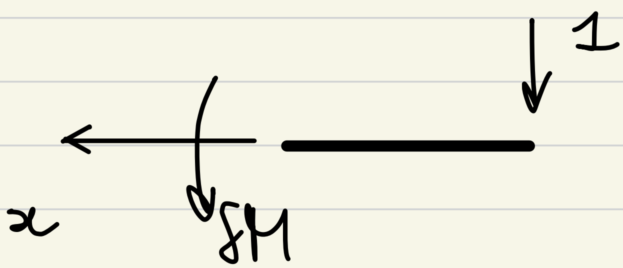




$$M = ql\,x - \frac{ql^2}{2} = ql\left(x - l/2\right)$$

Dummy system





$\delta M = x$

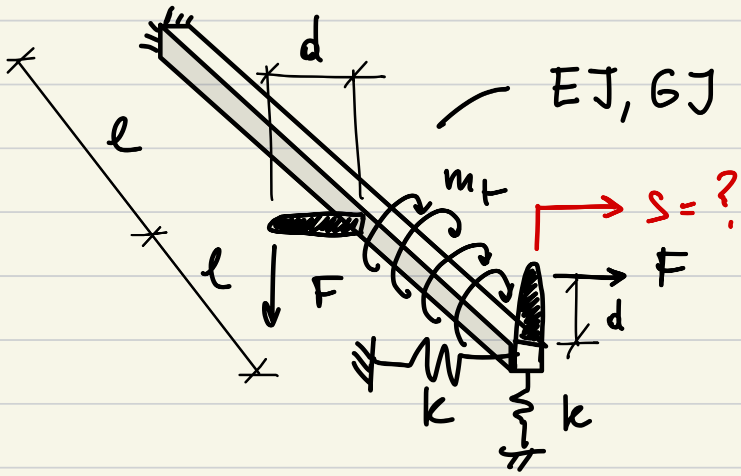
And by application of the PCVW:

$$\int_0^{2\ell} M \frac{\delta M}{EJ} dx = s$$

, one obtains

$$s = 5/3 \frac{q\ell^4}{EJ} = 6.91 \text{ mm}$$

Ritz #2



Use the method of Ritz to determine the displacement s. Use the simplest approximation of the unknown fields, using polynomial trial functions

Data

$l = 1300$ mm
$d = 60$ mm

$EJ = 10^{11}$ Nmm^2

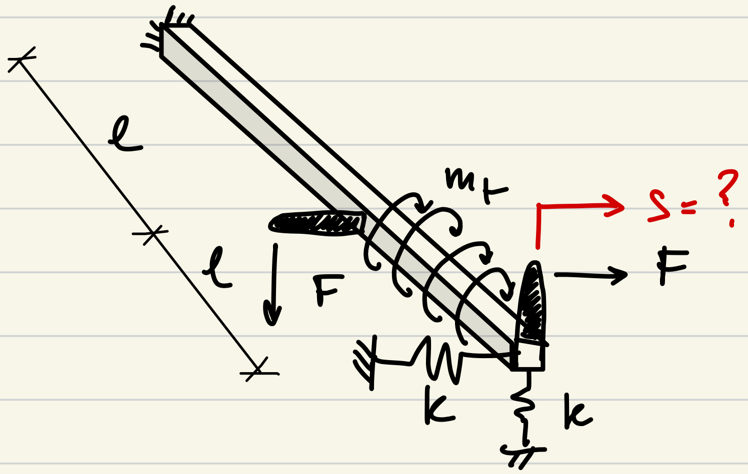
$GJ = 2 \cdot 10^9$ Nmm^2

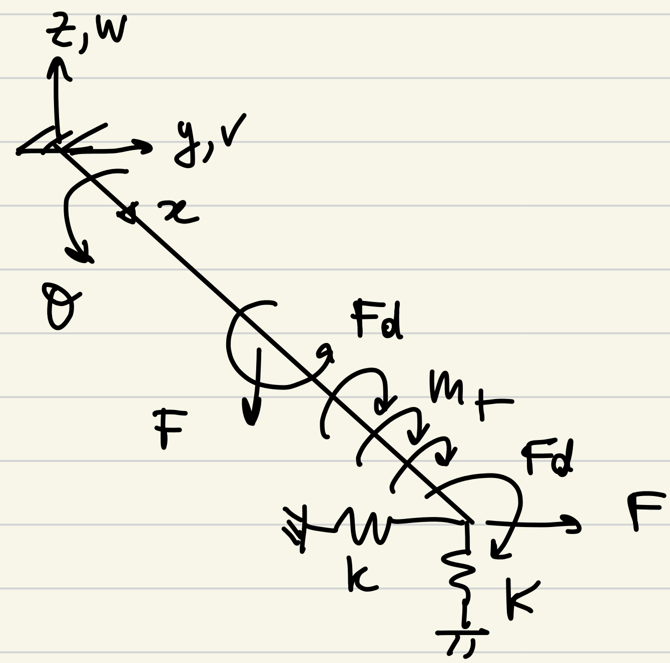
$k = 50$ N/mm

$F = 2000$ N

$m_t = 20$ N

Solution





The essential condition are:

$$w(0) = 0 \qquad v(0) = 0 \qquad \theta(0) = 0$$

$$w_{/x}(0) = 0 \qquad v_{/x}(0) = 0$$

So the displacements are expanded as:

$$w = a_1 \left(x/2l\right)^2$$

$$v = a_2 \left(x/2l\right)^2$$

$$\theta = a_3 \left(x/2l\right)$$

The PVW reads

$$\int_0^{2l} \delta w'' EJ w'' dx + \delta w(2l)\, k\, w(2l) +$$

$$+ \int_0^{2l} \delta v'' EJ v'' dx + \delta v(2l)\, k\, v(2l) +$$

$$+ \int_0^{2\ell} \delta\theta' GJ\theta' dx =$$

$$= -\delta w(\ell) F + \delta v(2\ell) F + \delta\theta(\ell) Fd - \delta\theta(2\ell) Fd +$$

$$- \int_\ell^{2\ell} \delta\theta\, m_t\, dx$$

Note that the displacement w does not provide any contribution to the displacement s.
Furthermore, there is no coupling between w and v, θ. So the problem can be solved by considering θ and v only:

$$\int_0^{2\ell} \delta v'' EJ v'' dx + \int_0^{2\ell} \delta\theta' GJ \theta' dx + \delta v(2\ell)\, k\, \delta v(2\ell) =$$

$$= \delta v(2\ell) F + \delta\theta(\ell) Fd - \delta\theta(2\ell) Fd - \int_\ell^{2\ell} \delta\theta\, m_t\, dx$$

Upon substitution of the approximation into the PVW one obtains:

$$\begin{bmatrix} k + EJ/2\ell^3 & 0 \\ 0 & GJ/2\ell \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} F \\ -Fd/2 - \frac{3}{4} m_t \ell \end{Bmatrix}$$

From which: $a_2 = 27.49$ mm

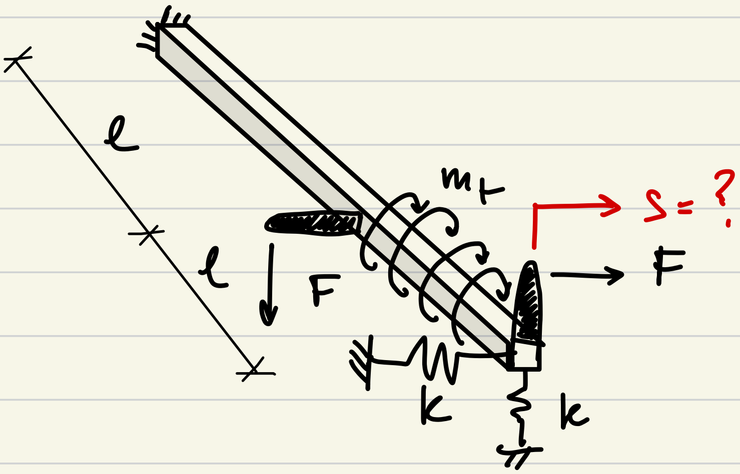
$a_3 = -0.10$ rad

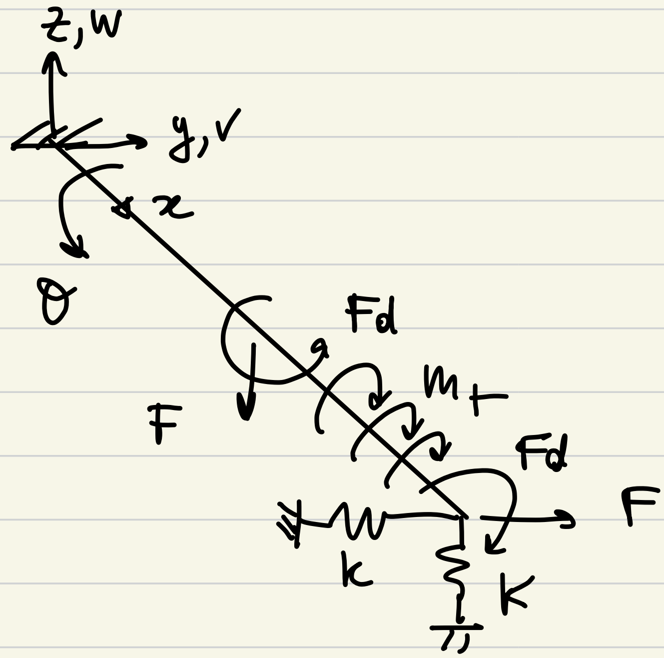
The displacement s is:

$$s = v(2\ell) - \theta(2\ell)\,d =$$

$$= a_2 - a_3 d = 33.69 \text{ mm}$$

Solution





The essential condition are:

$w(0) = 0 \qquad v(0) = 0 \qquad \theta(0) = 0$

$w_{/x}(0) = 0 \qquad v_{/x}(0) = 0$

So the displacements are expanded as:

$$w = a_1 \left(x/2l\right)^2$$
$$v = a_2 \left(x/2l\right)^2$$
$$\theta = a_3 \left(x/2l\right)$$

The PVW reads

$$\int_0^{2l} \delta w'' EJ w'' dx + \delta w(2l)\, k\, w(2l) +$$
$$+ \int_0^{2l} \delta v'' EJ v'' dx + \delta v(2l)\, k\, v(2l) +$$

$$+ \int_0^{2\ell} \delta\theta' GJ\theta' dx =$$

$$= -\delta w(\ell)F + \delta v(2\ell)F + \delta\theta(\ell)Fd - \delta\theta(2\ell)Fd +$$

$$- \int_\ell^{2\ell} \delta\theta\, m_t\, dx$$

Note that the displacement w does not provide any contribution to the displacement s.
Furthermore, there is no coupling between w and v, θ so the problem can be solved by considering θ and v only:

$$\int_0^{2\ell} \delta v'' EJ v'' dx + \int_0^{2\ell} \delta\theta' GJ\theta' dx + \delta v(2\ell)\, k\, \delta v(2\ell) =$$

$$= \delta v(2\ell)F + \delta\theta(\ell)Fd - \delta\theta(2\ell)Fd - \int_\ell^{2\ell} \delta\theta\, m_t\, dx$$

Upon substitution of the approximation into the PVW one obtains:

$$\begin{bmatrix} k + EJ/2\ell^3 & 0 \\ 0 & GJ/2\ell \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} F \\ -Fd/2 - \frac{3}{4} m_t \ell \end{Bmatrix}$$

From which: $a_2 = 27.49$ mm

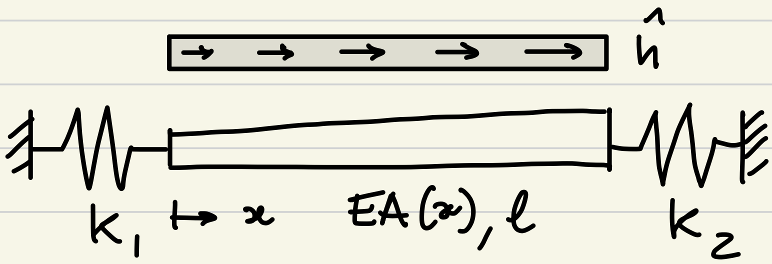
$a_3 = -0.10$ rad

The displacement s is:

$$s = v(2\ell) - \vartheta(2\ell)\, d =$$

$$= a_2 - a_3 d = 33.69 \text{ mm}$$

Ritz #3



The axial stiffness varies linearly between EA_0 at $x=0$ and EA_1 at $x=\ell$

Use a two-term polynomial approximation and the Ritz method to determine the axial displacement at $x = \ell/2$

Data

$\ell = 1500$ mm $\qquad \hat{n} = 10$ N/mm

$EA_0 = 7 \cdot 10^6$ N
$EA_1 = 12 \cdot 10^6$ N

$k_1 = 2 \cdot 10^3$ N

$k_2 = 2 \cdot 10^2$ N

Solution

The axial stiffness is:

$$EA(x) = EA_0 + (EA_1 - EA_0)(x/\ell)$$

The boundary conditions are of natural type, so the displacement is expanded as:

$$u = a_0 + a_1\, x/\ell$$

From which:

$$u' = a_1/\ell$$

The PVW reads: $\delta W_i = \delta W_e$ with:

$$\delta W_i = \int_0^{\ell} \delta u' \, EA(x)\, u' \, dx + \delta u(0)\, k_1 u(0) + \\ + \delta u(\ell)\, k_2 u(\ell)$$

$$\delta W_e = \int_0^{\ell} \delta u \, \hat{n} \, dx$$

And upon substitution of the polynomial expansion:

$$\delta W_i = \delta a_1 \frac{1}{\ell^2}\int_0^\ell EA(x)\,dx\, a_1 + \delta a_0 k_1 a_0 +$$

$$+(\delta a_0 + \delta a_1)\, k_2 (a_0 + a_1)$$

$$= \delta a_0 \left[(k_1+k_2) a_0 + k_2 a_1 \right]$$

$$+ \delta a_1 \left[k_2 a_0 + \left(k_2 + \frac{EA_0 + EA_1}{2\ell}\right) a_1 \right]$$

$$\delta W_e = \int_0^\ell \delta u\, \hat{n}\, dx =$$

$$= \int_0^\ell \left[\delta a_0 + \delta a_1 \left(x/\ell\right)\right] \hat{n}\, dx$$

$$= \delta a_0 \hat{n} \ell + \delta a_1 \hat{n}\, \ell/2$$

The equilibrium equations are then:

$$\begin{bmatrix} k_1+k_2 & k_2 \\ k_2 & \frac{EA_0+EA_1}{2\ell} + k_2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \hat{n}\ell \begin{Bmatrix} 1 \\ 1/2 \end{Bmatrix}$$

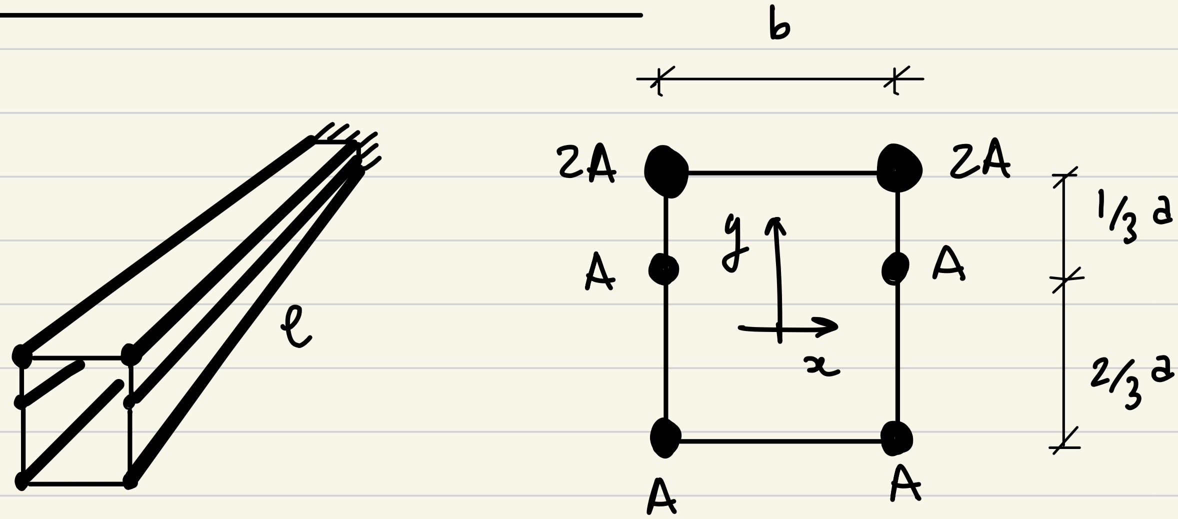
From which: $a_0 = 6.73$ mm

$a_1 = 0.94$ mm

And then

$$u(\ell/2) = a_0 + a_1/2 = 7.20 \text{ mm}$$

Semi #3 – Exercise 13



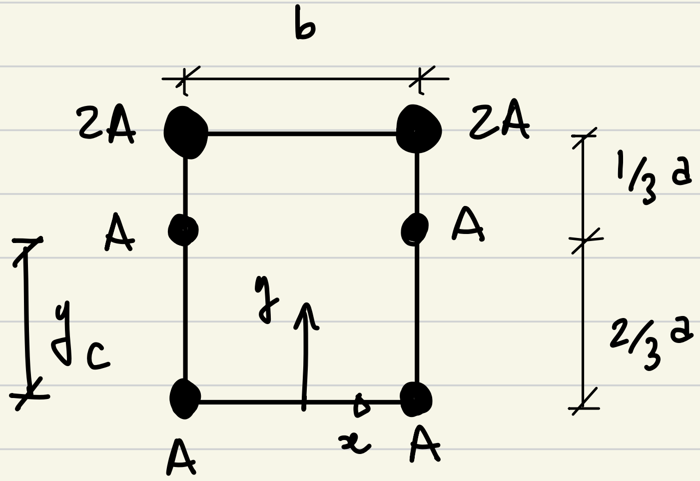
Determine the shear stiffness GA^*_y for the beam section in the figure.
All the panels have thickness t.

Data

$a = 300$ mm
$b = 160$ mm
$t = 1.5$ mm
$A = 450$ mm^2
$\ell = 3000$ mm

$E = 72$ GPa
$\nu = 0.3$

Solution



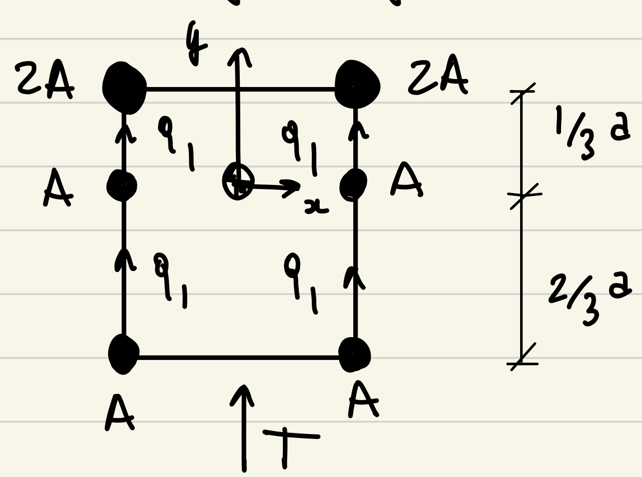
The coordinates of the centroid are:

$x_c = 0$

$$y_c = \frac{4Aa + 4/3\,Aa}{8A} = \frac{2}{3}a$$

The horizontal position of the shear center is available from the symmetry of the section.

The shear flows can be evaluated by exploiting the symmetry of the section.



Note, the jump of shear flow is zero in correspondence of the two stringers lying on the section neutral plane.

The equivalence between internal action and shear flows requires that:

$$q_1 2a = T \implies q_1 = T/2a$$

From which:

$$A^* = \frac{T^2}{\sum_i \frac{q_i^2 l_i}{t_i}} \quad \text{where:}$$

$$\sum_i \frac{q_i^2 l_i}{t_i} = \frac{T^2}{4a^2} 2a \frac{1}{t} = T^2 \frac{1}{2at}$$

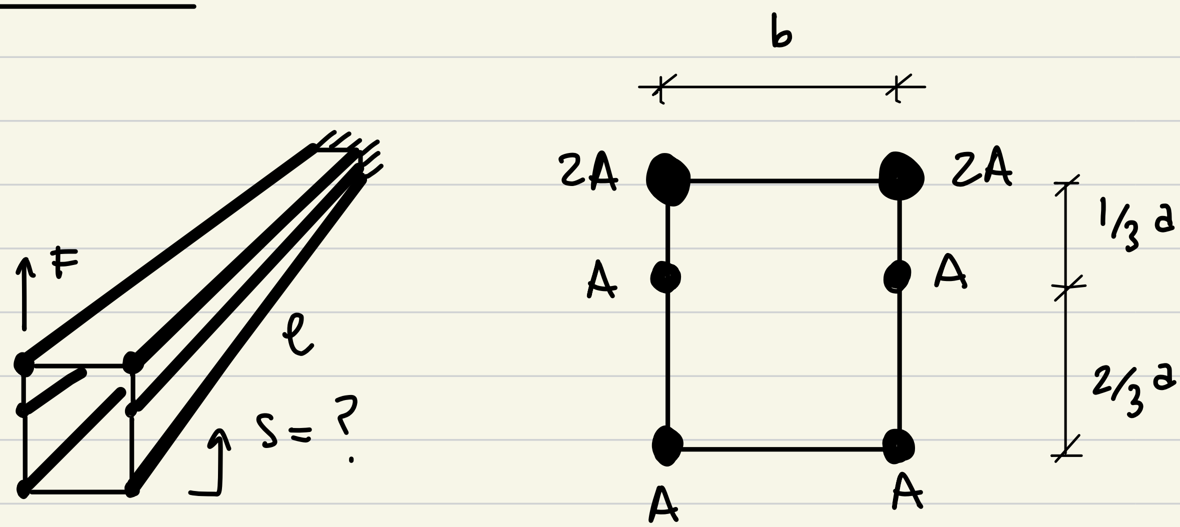
And so:

$$A^* = 2at = 900 \text{ mm}^2$$

$$G = E/2(1+\nu) = 27692 \text{ MPa}$$

$$GA^* = 2.49 \cdot 10^7 \text{ N}$$

Semi #4



Starting from the previous exercise, determine the displacement s.

Data

$a = 300$ mm

$b = 160$ mm

$t = 1.5$ mm

$A = 450$ mm^2

$\ell = 3000$ mm

$E = 72$ GPa

$\nu = 0.3$

$F = 12$ kN

Solution

The solution of the problem requires the evaluation of the section properties GA^*, EJ, GJ.
The first one is available from the previous exercise

1. Bending stiffness EJ

$$J_{xx} = 4A\left(\tfrac{1}{3}a\right)^2 + 2A\left(\tfrac{2}{3}a\right)^2 = \tfrac{4}{3}Aa^2 = 5.4 \cdot 10^7 \text{ mm}^4$$

$$EJ_{xx} = 3.89 \cdot 10^{12} \text{ N mm}^2$$

2. Torsional stiffness GJ

The Bredt formula can be applied to determine the torsional constant.

$$J = \frac{4\Omega^2}{\oint_\Gamma \frac{1}{t}\, d\Gamma}$$

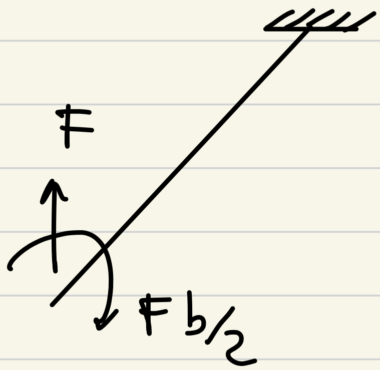
with: $\Omega = ab \Rightarrow \Omega^2 = a^2 b^2$

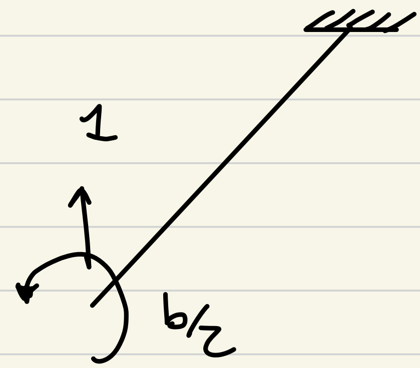
$$\oint_\Gamma \frac{1}{t}\, d\Gamma = \frac{2(a+b)}{t}$$

So: $$J = \frac{4a^2b^2}{2(a+b)}t = \frac{2a^2b^2}{a+b}t = 1.50 \cdot 10^7 \text{ mm}^4$$

and then: $GJ = 4.16 \cdot 10^{11}\ Nmm^2$

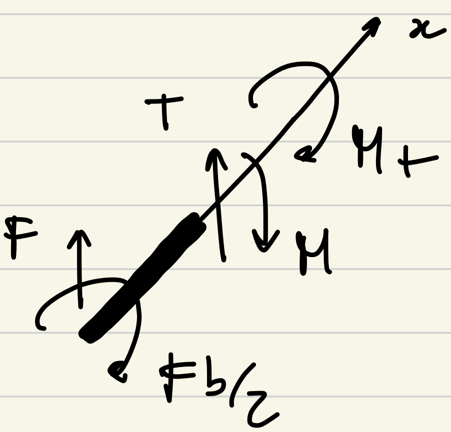
The evaluation of the displacement is done by application of the PCVW

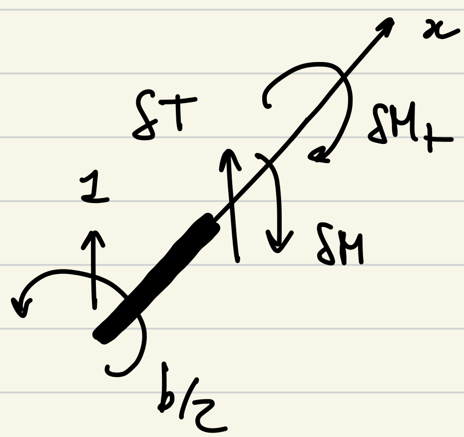




Real system

Dummy system





$$T = -F$$
$$M = -Fx$$
$$M_t = -Fb/2$$

$$\delta T = -1$$
$$\delta M = -x$$
$$\delta M_t = b/2$$

the PCVW reads:

$$\int_0^l \left(\delta T \frac{T}{GA^*} + \delta M \frac{M}{EJ} + \delta M_t \frac{M_t}{GJ} \right) dx = s$$

and then

$$s = \frac{Fl}{GA^*} + \frac{Fl^3}{3EJ} - \frac{Fb^2 l}{4GJ} = 28.67 \text{ mm}$$

- Any structure with one dimension much larger than the other two can likely be modeled as a beam
  - True
- According to the Timoshenko beam model, the transverse shear stresses are linear on the cross section
  - False
- The position of the shear center of a closed-cell section can be evaluated using the shear flow equations and the equivalence to internal moment
  - False
- A system of slender beams can be modeled by beams finite elements
  - never
  - if the structure can sustain the loads through an internal axial load path
  - whenever the shear deformability is negligible
  - always
- An Euler-Bernoulli cantilever beam with uniform stiffness is clamped at one extremity and loaded with a concentrated force at the tip. The solution obtained using a displacement-based method based on polynomial functions with two unknown coefficients is
  - exact
  - an approximation of the exact solution with errors below 10 \%
  - an approximation of the exact solution with errors depending on the problem data
- When the shear force is applied at the shear center
  - the shear flows are null
  - the torsion is null
  - the torsion can be different from zero only if the torsional moment, computed with respect to the shear center, is not null