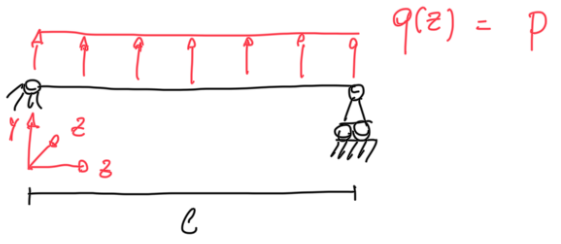
EX 11 - Displacement Methods I

We assume an arbitrary COMPATIBLE displacement function, then we solve the problem using PVW.

1)



(I) POLYNOMIAL APPROXIMATION OF THE DISPLACEMENT IN Y

$$v(z) = C_0 + C_1 \cdot z + C_2 z^2 + C_3 z^3 + \dots$$

Essential BC
$$\begin{cases} v(\emptyset) = \emptyset & \rightarrow \quad C_0 = \emptyset \\ v(l) = \emptyset & \rightarrow \quad C_1 = -(C_2 l + C_3 l^2 + \dots) \end{cases}$$

$$v(z) = C_2(z^2 - lz) + C_3(z^3 - l^2 z) + C_4(z^4 - l^3 z) + \dots$$

$$= \sum_{i=2}^{\infty} C_i \cdot \phi_i \quad \text{where} \quad \underline{\phi_i = (z^i - l \cdot z^{(i-1)})}$$

shape function

$$= \underline{\phi} \cdot \underline{C} = [\phi_1, \phi_2, \phi_3, \dots, \phi_\infty] \cdot \begin{bmatrix} C_1 \\ \dots \\ C_\infty \end{bmatrix}$$

$$\delta v(z) = \sum_{i=2}^{\infty} \delta C_i \cdot \phi_i$$

- One term approximation

$$v(z) = C_2 \cdot \phi_2 = C_2(z^2 - lz)$$

PVW Internal Virtual Work = External Virtual Work

$$\delta W_i = \int_0^\ell \delta v_{/zz}\, EJ\, v_{/zz}\, dz$$

$$v_{/z} = 2C_2 \cdot z - C_2 \cdot \ell \qquad \delta v = \delta C_2 \,(z^2 - \ell z)$$

$$v_{/zz} = 2C_2 \qquad \delta v_{/z} = 2 \cdot \delta C_2 \cdot z - \delta C_2 \cdot \ell$$

$$\delta v_{/zz} = 2 \cdot \delta C_2$$

$$\delta W_e = \int_0^\ell \delta v \cdot q(z)\, dz$$

$$\int_0^\ell 2 \cdot \delta C_2 \cdot EJ \cdot 2C_2\, dz = \int_0^\ell \delta C_2 \cdot (z^2 - \ell z) \cdot p \cdot dz$$

$$\cancel{\delta C_2} \cdot \underbrace{C_2}_{\underline{c}} \underbrace{\int_0^\ell 4EJ\, dz}_{\underline{K}} = \cancel{\delta C_2} \underbrace{\int_0^\ell (z^2 - \ell z) \cdot p \cdot dz}_{\underline{f}}$$

$$C_2 = -\frac{p \cdot \ell}{24EJ} \longleftarrow \text{to be checked}$$

- Two Terms Approximation

$$v(z) = \begin{bmatrix} z^2 - \ell z & z^3 - \ell^2 z \end{bmatrix} \cdot \begin{bmatrix} C_2 \\ C_3 \end{bmatrix} = \underline{\phi} \cdot \underline{c}$$

$$v_{/zz} = \begin{bmatrix} 2 & 6z \end{bmatrix} \begin{bmatrix} C_2 \\ C_3 \end{bmatrix} = \underline{\phi}_{/zz} \cdot \underline{c}$$

- PVW

$$\int_0^\ell \delta v_{/zz} \cdot EJ\, v_{/zz}\, dz = \int_0^\ell \delta v^T \cdot q(z)\, dz$$

$$\delta \underline{c}^T \int_0^\ell \underline{\phi}_{/zz}^T\, EJ\, \underline{\phi}_{/zz}\, dz \cdot \underline{c} = \delta \underline{c}^T \int_0^\ell \underline{\phi}^T \cdot p\, dz$$

$$EJ \cdot \int_0^\ell \begin{bmatrix} \boxed{\phi_{2/zz} \cdot \phi_{2/zz}} & \phi_{2/zz} \cdot \phi_{3/zz} \end{bmatrix} dz \cdot \underline{c} = \int_0^\ell \begin{bmatrix} \phi_2 \end{bmatrix} \cdot p\, dz$$

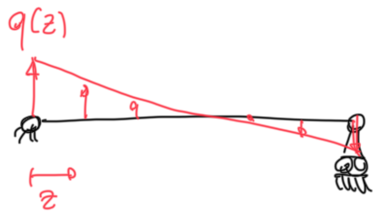
$$\underbrace{{}_0 \left[\phi_{3/zz}\cdot\phi_{2/zz} \quad \phi_{3/zz}\cdot\phi_{3/zz}\right]}_{\underline{\underline{K}}} \quad \underbrace{\quad}_{\underline{c}} \quad \underbrace{{}_0 \left[\ \phi_3\ \right]}_{\underline{f}}$$

if we move to a 3 Terms approx

$$v(z) = \left[\phi_2 \quad \phi_3 \quad \phi_4\right] \cdot \begin{bmatrix} c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$\underline{\underline{K}} = EJ \cdot \int_0^\ell \begin{bmatrix} \begin{bmatrix} 2 \text{ terms} \\ \underline{\underline{K}} \end{bmatrix} & \begin{matrix} \phi_{2/zz}\cdot\phi_{4/zz} \\ \phi_{3/zz}\cdot\phi_{4/zz} \end{matrix} \\ \text{SYMM} & \phi_{4/zz}\cdot\phi_{4/zz} \end{bmatrix} dz \qquad \underline{f} = \int_0^\ell \begin{bmatrix} \begin{bmatrix} 2 \\ \text{term} \end{bmatrix} \\ \phi_4 \end{bmatrix} \cdot p \, dz$$

- linear load



$$q(z) = p\left(-\frac{z}{\ell} + \frac{1}{2}\right)$$

$$RHS = \int_0^\ell \delta \underline{v}^T \cdot q(z)\, dz$$

$$= \delta \underline{c}^T \cdot \int_0^\ell \underline{\phi}^T \cdot p\left(-\frac{z}{\ell} + \frac{1}{2}\right) dz$$

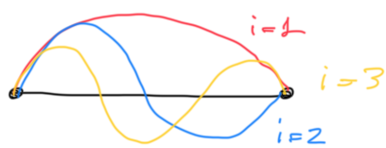
II TRIGONOMETRIC APPROXIMATION

$q(z) = p$ constant distributed load

number of the shape function

$$\phi_i(z) = \sin\left(\frac{i \pi z}{\ell}\right) \longrightarrow \text{it respect the BC !!!}$$

$$v(z) = \sum_{i=1}^{\infty} c_i \cdot \sin\left(\frac{i \pi z}{\ell}\right) = \underline{\phi} \cdot \underline{c}$$



$$\phi_{i/z} = \frac{\pi i}{\ell} \cdot \cos\left(\frac{i \pi z}{\ell}\right)$$

$$\phi_{i/zz} = \left(\frac{\pi i}{\ell}\right)^2 \cdot \left(-\sin\left(\frac{i \pi z}{\ell}\right)\right)$$

$$RHS = \int^\ell \ldots = \int^\ell \ldots$$

PVW $\int_0 \delta v_{/zz} \, EJ \, v_{/zz} \, dz = \int_0 \delta v \cdot q(z) \, dz$

- 1 term approximation

$$\delta c_1 \cdot \int_0^\ell \phi_{1/zz} \cdot EJ \cdot \phi_{1/zz} \cdot dz \cdot c_1 = \delta c_1 \cdot \int_0^\ell \phi_1 \cdot p \cdot dz$$

$$EJ \left(\frac{\pi}{\ell}\right)^4 \cdot \underbrace{\int_0^\ell \sin^2\left(\frac{\pi z}{\ell}\right) dz}_{\ell/2} \cdot c_1 = \underbrace{\int_0^\ell \sin\left(\frac{\pi z}{\ell}\right) dz}_{2\ell/\pi} \cdot p$$

$$c_1 = \frac{4\ell^4}{\pi^5 EJ} \cdot p$$

Take $n \to \infty$

PVW $$\cancel{\delta \underline{c}^T} \cdot \underbrace{\int_0^\ell \underline{\phi}^T_{/zz} \, EJ \, \underline{\phi}_{/zz} \, dz}_{\underline{K}} \cdot \underline{c} = \cancel{\delta \underline{c}^T} \cdot \int_0^\ell \underline{\phi} \cdot p \, dz$$

$$K_{ij} = EJ \cdot \int_0^\ell \phi_{i/zz} \cdot \phi_{j/zz} \, dz = EJ \cdot \underbrace{\int_0^\ell \sin\left(\frac{i\pi z}{\ell}\right) \cdot \sin\left(\frac{j\pi z}{\ell}\right) \cdot \left(\frac{i\pi}{\ell}\right)^2 \left(\frac{j\pi}{\ell}\right)^2}$$

↳ $i = j \to \frac{\ell}{2}$

↳ $i \neq j \to \emptyset$

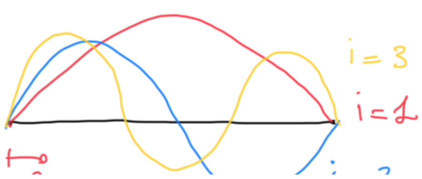
$\underline{K}$ is diagonal

$$f_i = \int_0^\ell \phi_i \, dz \cdot p = \underbrace{\int_0^\ell \sin\left(\frac{i\pi z}{\ell}\right) dz}$$

$\to \emptyset$ if i is EVEN

$\to \frac{2\ell}{\pi i}$ if i is ODD

this is the area of the shape function



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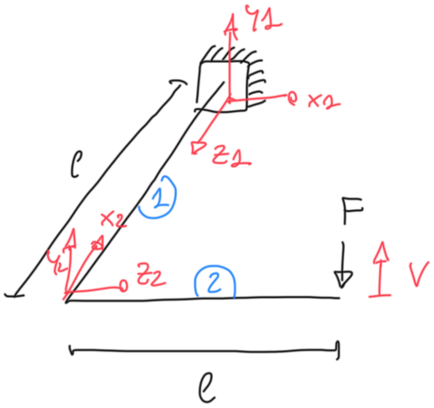
EX 2 EXAM 13/06/2023

occhio al segno della rotazione



DATA

$l = 1000$ mm

$F = 1000$ N

$GJ = 10^{10}$ Nmm^2

$EJ = 10^{10}$ Nmm

- Polynomial Approx

– VERTICAL DISPLACEMENT 2nd Order

$$V_1(z_1) = a_0 + a_1 \cdot z_1 + a_2 \cdot z_1^2$$
$$V_2(z_2) = b_0 + b_1 \cdot z_2 + b_2 \cdot z_2^2$$

– z – ROTATION 1st Order

$$\theta_1(z_1) = C_0 + C_1 \cdot z \qquad \theta_2(z_2) = \emptyset$$

- Compatibility

BEAM 1 – Clamp

$V_1(\emptyset) = \emptyset$

x-rotation

$V_{1/z}(\emptyset) = \emptyset$

$\Big\}$ $a_0 = a_1 = \emptyset$ $\qquad V_1(z) = a_2 \cdot z_1^2$

z-rotation

$\theta_1(\emptyset) = \emptyset$ $\quad \}$ $C_0 = \emptyset$ $\qquad \theta_1(z) = C_1 \cdot z_1$

BEAM 2 — BEAM 1

$$\left.\begin{aligned} v_2(\emptyset) &= v_1(\ell) \rightarrow b_0 = a_2 \cdot \ell^2 \\ v_{2/z_2}(\emptyset) &= \theta_1(\ell) \rightarrow b_1 = c_1 \cdot \ell \end{aligned}\right\} \quad v_2(z) = a_2 \cdot \ell^2 + c_1 \cdot \ell \cdot z + b_2 \cdot z_2^2 \quad *$$

↑ if $\theta_1(\ell) > 0 \rightarrow v_{2/z_2} > 0$ because of how I defined $\vec{v}$

- PVW

$$\delta W_i = \overbrace{\int_0^\ell \delta v_{1/z_2} \, EJ \, v_{1/zz} \cdot dz_1}^{\text{BENDING BEAM 1}} + \overbrace{\int_0^\ell \delta\theta_{1/z} \cdot GJ \cdot \theta_{1/z} \, dz_1}^{\text{TORSION BEAM 1}} + \overbrace{\int_0^\ell \delta v_{2/z_2} \cdot EJ \, v_{2/z_2} \, dz_2}^{\text{BENDING BEAM 2}}$$

$$\delta W_e = -\delta v_2(\ell) \cdot F \overset{*}{=} -(\delta a_2 + \delta c_1 + \delta b_2) \cdot \ell^2 \cdot F$$

We can rewrite the PLV as a system of three equations because the three components of internal work are energetically decoupled:

	beam 1	beam 2
a_2	bending	rigid translation
		no work
c_1	torsion	rigid rotation
b_2	—	bending

- $\delta a_2 \cdot a_2 \cdot \int_0^\ell 2EJ \cdot 2 \, dz_1 = -\delta a_2 \cdot \ell^2 F \qquad a_2 = -\frac{F\ell}{4EJ}$
- $\delta c_1 \cdot c_1 \cdot \int_0^\ell GJ \, dz_1 = -\delta c_1 \cdot \ell^2 \cdot F \qquad c_1 = -\frac{F\ell}{GJ}$
- $\delta b_2 \cdot b_2 \cdot \int_0^\ell 2 \cdot EJ \cdot 2 \, dz_2 = -\delta b_2 \cdot \ell^2 \cdot F \qquad b_2 = -\frac{F\ell}{4EJ}$

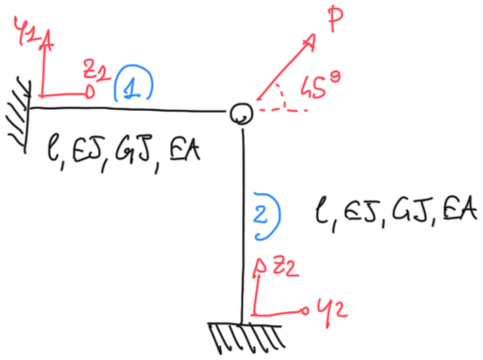
$v_2(\ell) = -150$ mm

$$\delta \begin{bmatrix} a_2 \\ c_1 \end{bmatrix}^T \cdot \begin{bmatrix} \int_0^\ell 4EJ \, dz_1 & 0 & 0 \\ 0 & \int_0^\ell GJ \, dz_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ c_1 \end{bmatrix} = \delta \begin{bmatrix} a_2 \\ c_1 \end{bmatrix}^T \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \ell^2 \cdot F$$

$\lfloor b_2 \rfloor \quad \lfloor \quad 0 \quad - \quad 0 \quad \int_0 4EJ\, dz_2 \rfloor \quad \lfloor b_2 \rfloor \qquad \lfloor b_2 \rfloor \quad \lfloor 1 \rfloor$

$\underline{k} \qquad\qquad \underline{c}$

EX 3



axial displacement 1 $\quad w_1 = a_0 + a_1 \cdot z_1$ *

" 2 $\quad w_2 = b_0 + b_1 \cdot z_2$

transversal disp 1 $\quad v_1 = c_0 + c_1 \cdot z_1 + c_2 \cdot z_1^2$

" 2 $\quad v_2 = d_0 + d_1 \cdot z_2 + d_2 \cdot z_2^2$

- Compatibility

translation at the clamp $\qquad a_0 = b_0 = c_0 = d_0 = \emptyset$

rotation at the clamp $\qquad c_1 = d_1 = \emptyset$

$v_1(\ell) = w_2(\ell) \quad \rightarrow \quad b_1 = c_2 \ell \qquad c = c_2$

$v_2(\ell) = w_1(\ell) \quad \rightarrow \quad a_1 = d_2 \ell \qquad d = d_2$

*
$$w_1 = d \cdot \ell \cdot z_1$$
$$w_2 = c \cdot \ell \cdot z_2$$
$$v_1 = c \cdot z_1^2$$
$$v_2 = d \cdot z_2^2$$

- PVW

$$\int_0^\ell \delta w_{1/z_1}\, EA\, w_{1/z_1}\, dz_1 + \int_0^\ell \delta w_{2/z_2}\, EA\, w_{2/z_2}\, dz_2 +$$

$$\int^\ell \delta v_{1/}\ \ EJ\ v_{1/}\ \ dz_1 + \int^\ell \delta v_{2/}\ \ EJ\ v_{2/}\ \ =$$

$$= \left(\delta w_1(\ell) + \delta w_2(\ell)\right) \frac{P}{\sqrt{2}}$$

Knowing that

$$
\begin{aligned}
w_{1/z_1} &= d \cdot \ell & \delta w_{1/z_1} &= \delta d \cdot \ell \\
w_{2/z_2} &= c \cdot \ell & \delta w_{2/z_2} &= \delta c \cdot \ell \\
v_{1/z_1 z_1} &= 2c & \delta v_{1/z_1 z_1} &= 2 \cdot \delta c \\
v_{2/z_2 z_2} &= 2d & \delta v_{2/z_2 z_2} &= 2 \cdot \delta d
\end{aligned}
$$

$$\underline{\delta d \cdot d \cdot \int_0^\ell \ell^2 EA \, dz_1} + \underline{\delta c \cdot c \cdot \int_0^\ell \ell^2 EA \, dz_2} +$$

$$+ \underline{\delta c \cdot c \cdot \int_0^\ell 4 EJ \, dz_1} + \underline{\delta d \cdot d \cdot \int_0^\ell 4 EJ \, dz_2} =$$

$$= (\underline{\delta c} + \underline{\delta d}) \cdot \frac{\ell^2 \cdot P}{\underline{\underline{\sqrt{2}}}}$$

$$\delta \begin{bmatrix} c \\ d \end{bmatrix}^T \cdot \begin{bmatrix} \int_0^\ell \ell^2 EA \, dz_2 + \int_0^\ell 4EJ \, dz_2 & \emptyset \\ \emptyset & \int_0^\ell \ell^2 EA \, dz_1 + \int_0^\ell 4EJ \, dz_1 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \delta \begin{bmatrix} c \\ d \end{bmatrix}^T \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{P \ell^2}{\sqrt{2}}}$$

$$\begin{array}{ccccc} \delta \underline{C}^T & \underline{\underline{K}} & \underline{C} & \delta \underline{C}^T & \underline{f} \\ 1 \times 2 & 2 \times 2 & 2 \times 1 & 1 \times 2 & 2 \times 1 \end{array}$$

$$(EA \ell^3 + 4EJ\ell) \cdot d = \ell^2 \frac{P}{\sqrt{2}}$$

$$(EA \ell^3 + 4EJ\ell) \cdot c = \ell^2 \frac{P}{\sqrt{2}}$$

$$c = d = \frac{P\ell}{\sqrt{2}(EA\ell^2 + 4EJ)}$$

$$w = w_1(\ell) = \frac{P\ell^3}{\sqrt{2}(EA\ell^2 + 4EJ)}$$