

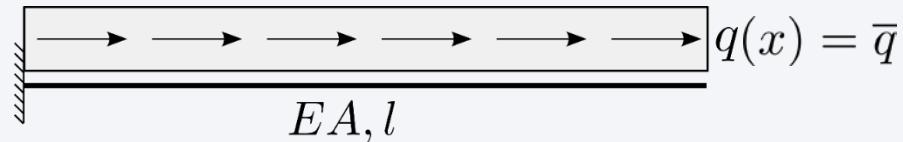
# Linear and quadratic truss elements in physical coordinates

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## Example: truss with uniform load

- Consider a truss with axial stiffness  $EA$ , constant along the axis. The bar is loaded with an axial force per unit length, constant along the truss axis. The bar is fixed at one end, and free at the other end



- The exact solution of the problem can be obtained by solving the following differential problem

$$\begin{cases} EAu_{,xx} + \bar{q} = 0 \\ u(0) = 0 \\ u_{,x}(0) = 0 \end{cases}$$

whose solution is in the form

$$u = u^h + u^p$$

and

$$u^h = Ax + B \quad u^p = -\frac{\bar{q}}{2EA}x^2$$

# Exact solutions

- After imposing the boundary conditions, the solution is obtained as:

$$u(x) = \frac{\bar{q}}{EA} \left( -\frac{x^2}{2} + lx \right)$$

- while the strains and the axial force read:

$$u_{,x} = \xi_{xx} = \frac{\bar{q}}{EA} (-x - l)$$

$$N_{xx} = EA\xi_{xx} = \bar{q}(-x - l)$$

- Note that
  - the displacements are quadratic with x
  - the strains and the axial force are linear with x

## FE formulation

- The finite element (FE) formulation relies upon the variational formulation of the problem, according to the Principle of Virtual Displacements. In particular, it is:

$$\int_l \delta \xi_{xx} N_{xx} dx = \int_l \delta u q dx$$

and recalling the strain-displacement relation

$$\xi_{xx} = u_{,x}$$

and the bar constitutive law

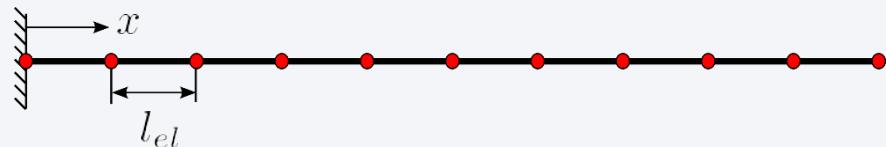
$$N_{xx} = EAu_{,x} \quad \text{where} \quad N_{xx} = \int_A \sigma_{xx} dA$$

- It is obtained the weak form formulation of the problem in terms of the displacement component  $u$

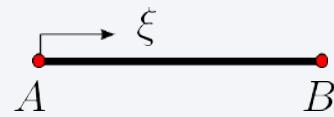
$$\int_l \delta u_{,x} EAu_{,x} dx = \int_l \delta u q dx$$

## FE formulation: linear elements

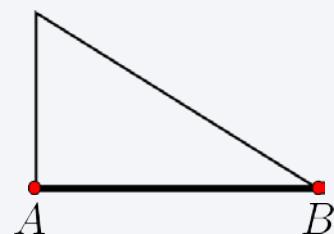
- The bar is now divided into a number  $N_{el}$  of elements of equal length  $l_{el}$ , and corresponding to a total number  $N_{el}+1$  of nodes. The global axis is denoted with  $x$ :



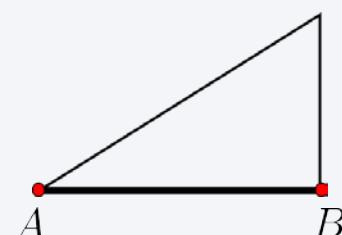
- Each element is composed of two nodes, generically denoted as A and B. A local reference system is introduced for defining the shape functions



$$u(\xi) = N_A u_A + N_B u_B = \mathbf{N} \mathbf{u}_{el} \quad \xi \in [0 \quad l_{el}]$$



$$N_A = 1 - \frac{\xi}{l_{el}} \quad N_A(x_A) = 1 \quad N_A(x_B) = 0$$



$$N_B = \frac{\xi}{l_{el}} \quad N_A(x_A) = 0 \quad N_A(x_B) = 1$$

## FE formulation: linear elements

- For each element, the internal virtual work reads:

$$\delta W_{i,el} = \int_0^{l_{el}} \delta u_{,x} E A u_{,x} d\xi$$

- The displacement field is approximated by interpolating the nodal values by means of linear shape functions:

$$u = \mathbf{N} \mathbf{u}_{el}$$

- Observing that the local axis  $\xi$  and global one  $x$  are related by a rigid translation as:

$$\xi = x - x_{el} \quad \Rightarrow \quad \begin{cases} dx = d\xi \\ u_{,x} = u_{,\xi} \\ \mathbf{N}_{,x} = \mathbf{N}_{,\xi} \end{cases}$$

- And so the internal work read:

$$\begin{aligned} \delta W_{i,el} &= \delta \mathbf{u}_{el}^T \int_0^{l_{el}} \mathbf{N}_{,\xi}^T E A \mathbf{N}_{,\xi} d\xi \mathbf{u}_{el} \\ &= \delta \mathbf{u}_{el}^T \mathbf{K}_{el} \mathbf{u}_{el} \end{aligned}$$

$$\text{where } \mathbf{K}_{el} = \int_0^{l_{el}} \mathbf{N}_{,\xi}^T E A \mathbf{N}_{,\xi} d\xi = \frac{EA}{l_{el}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

# FE formulation: linear elements

- The external virtual work is:

$$\delta W_{e,el} = \int_{l_{el}} \delta u \bar{q} dx$$

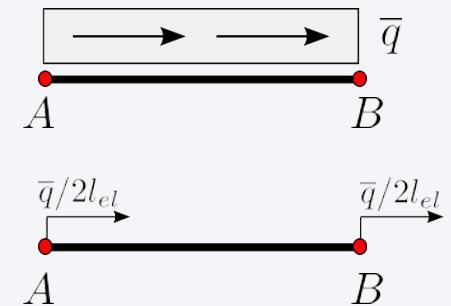
- and substituting the finite element approximation is:

$$\begin{aligned}\delta W_{e,el} &= \delta \mathbf{u}_{el}^T \bar{\mathbf{q}} \int_0^{l_{el}} \mathbf{N}^T d\xi \\ &= \delta \mathbf{u}_{el}^T \mathbf{f}_{el}\end{aligned}$$

- where, according to the uniform distribution of the load, it is:

$$\mathbf{f}_{el} = \bar{q} \int_0^{l_{el}} \mathbf{N}^T d\xi = \bar{q} l_{el} \begin{Bmatrix} 1/2 \\ 1/2 \end{Bmatrix}$$

- where the projection of the distributed load using the shape functions corresponds to reporting half of the resultant on the node A, and another half of the node B



## Results: 1 (linear) element

- The simplest model consists in modeling the structure using one single element



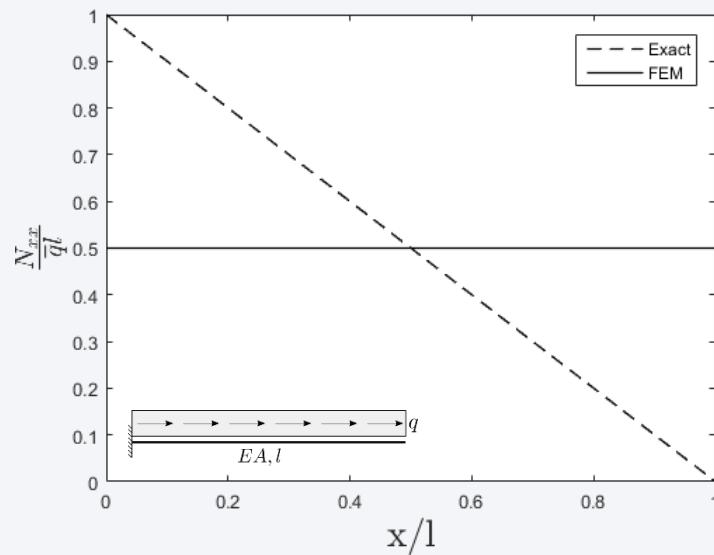
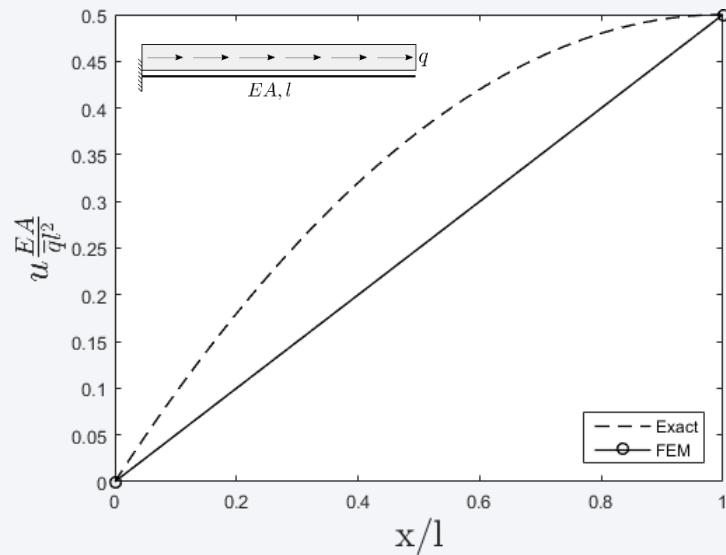
- The discrete equations expressing the equilibrium conditions of the unconstrained system are:

$$\frac{EA}{l_{el}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \bar{q}l_{el} \begin{Bmatrix} 1/2 \\ 1/2 \end{Bmatrix}$$

- The boundary condition at the node 1 can be imposed by removing the first row and the first column from the linear system, leading to the scalar problem

$$\frac{EA}{l_{el}} u_2 = \bar{q}l_{el} \frac{1}{2} \Rightarrow u_2 = \frac{\bar{q}l_{el}^2}{2EA}$$

# Results: 1 (linear) element

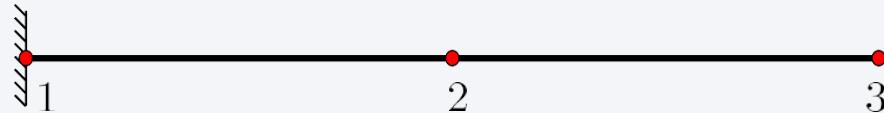


## Remarks

- The approximation using one element is rather poor. The displacement is approximated as linear over the entire domain, whilst the axial force is constant

## Results: 2 (linear) elements

- Consider a finite element model with 2 elements



- The linear system is obtained by assembling the contribution of the two elements

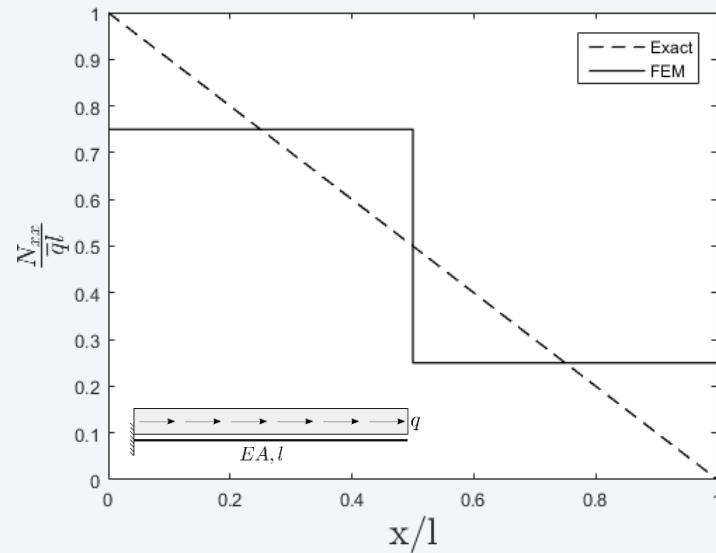
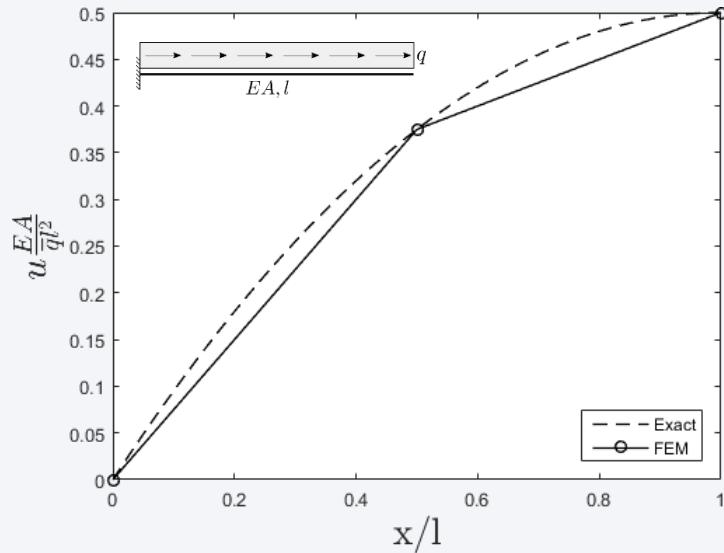
$$\frac{EA}{l_{el}} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \bar{q}l_{el} \begin{Bmatrix} 1/2 \\ 1/2 + 1/2 \\ 1/2 \end{Bmatrix}$$

- and imposing the boundary condition at the node 1:

$$\frac{EA}{l_{el}} \begin{bmatrix} 1+1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \bar{q}l_{el} \begin{Bmatrix} 1/2 + 1/2 \\ 1/2 \end{Bmatrix}$$

- from which:
$$u_2 = \frac{3}{2} \frac{\bar{q}l_{el}^2}{EA}$$
$$u_3 = 2 \frac{\bar{q}l_{el}^2}{EA}$$

# Results: 2 (linear) elements

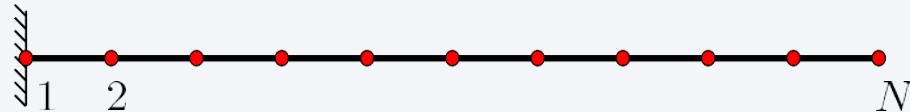


## Remarks

- The accuracy of the results improves as the number of elements is increased. The displacement field is now piecewise linear, and the axial force is piecewise constant. Note that the exact solution is never achieved (for the problem at hand, it is outside the capabilities of linear elements)

# Results: N (linear) elements

- The generic model composed by N nodes is:



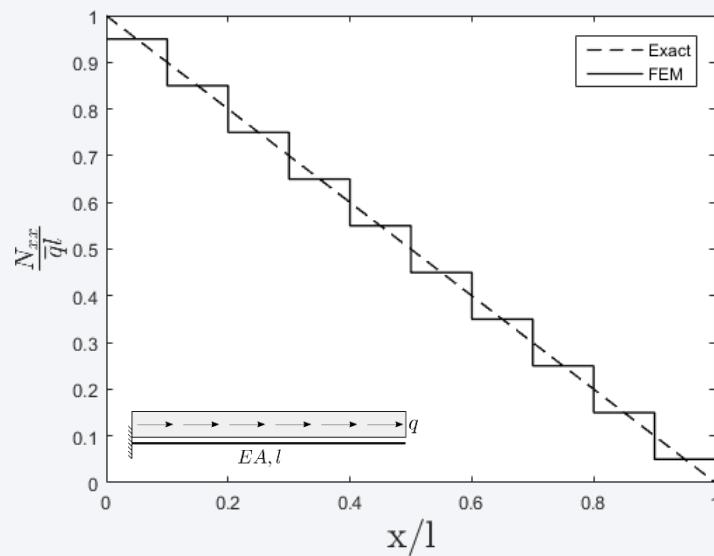
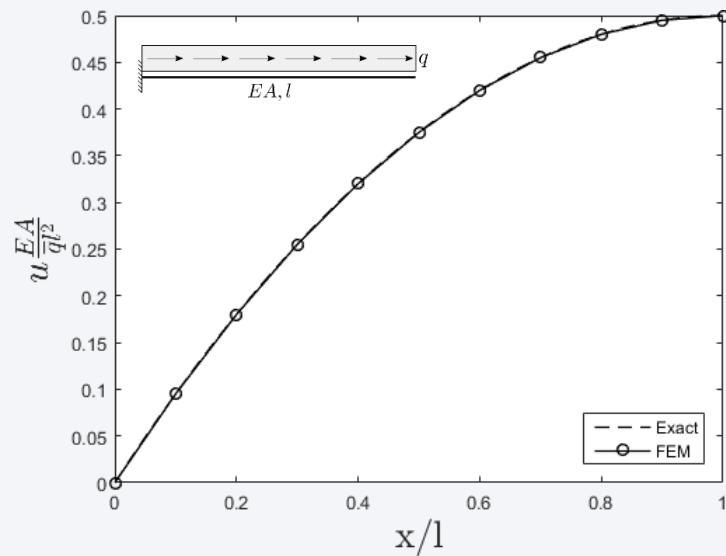
- and the relevant set of linear equations reads:

$$\frac{EA}{l_{el}} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & \dots & \dots & \dots \\ & & & & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_N \end{Bmatrix} = \bar{q}l_{el} \begin{Bmatrix} 1/2 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1/2 \end{Bmatrix}$$

- and imposing the boundary conditions:

$$\frac{EA}{l_{el}} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \dots & \dots & \dots \\ & & & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_N \end{Bmatrix} = \bar{q}l_{el} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1/2 \end{Bmatrix}$$

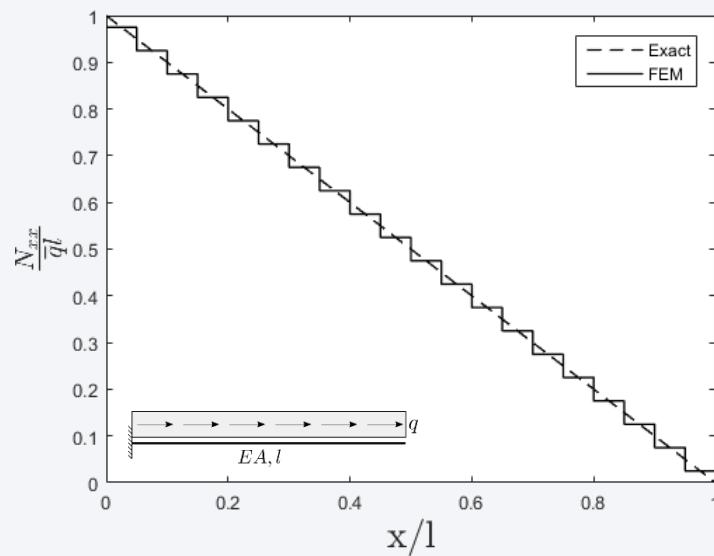
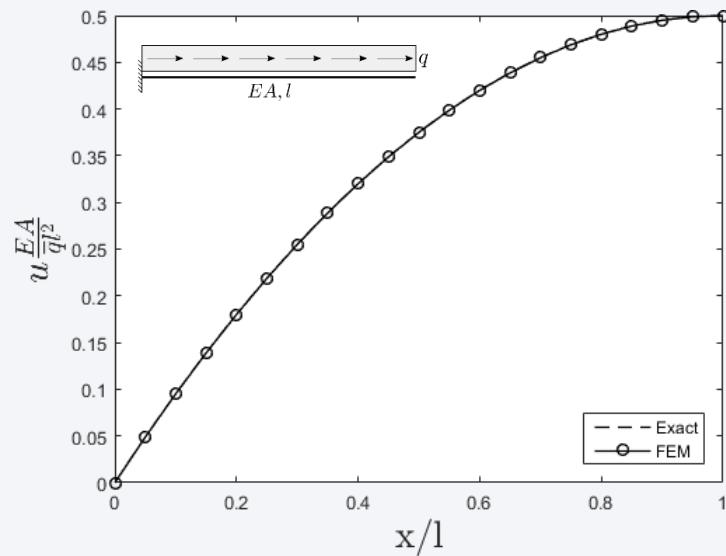
# Results: 10 (linear) elements



## Remarks

- Using 10 elements, the displacement field is almost undistinguishable from the exact one. The difference between the exact and the approximated axial force is more noticeable (indeed the convergence of the generalized stresses is slower)

# Results: 20 (linear) elements

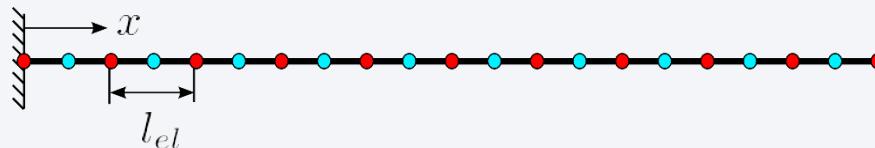


## Remarks

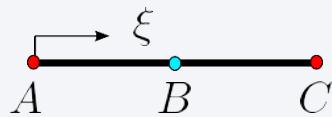
- With 20 elements, the quality of the approximation is further improved

## FE formulation: quadratic elements

- The bar is now divided into  $N_{el}$  elements of equal length  $l_{el}$ . The elements are now characterized by the presence of a third node in the middle of the element. The global axis is denoted with  $x$ :



- Defining with A and C the outer nodes and B the middle node, the displacement field is now approximated as the sum of three contributions



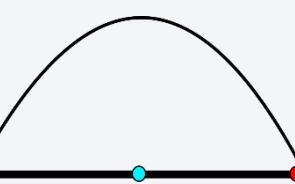
$$u(\xi) = N_A u_A + N_B u_B + N_C u_C = \mathbf{N} \mathbf{u}_{el}$$



$$N_A = 1 - 3\frac{\xi}{l_{el}} + 2\frac{\xi^2}{l_{el}^2}$$

$$N_B = 4\frac{\xi}{l_{el}} - 4\frac{\xi^2}{l_{el}^2}$$

$$N_C = -\frac{\xi}{l_{el}} + 2\frac{\xi^2}{l_{el}^2}$$



- The shape functions are thus quadratic

# FE formulation: quadratic elements

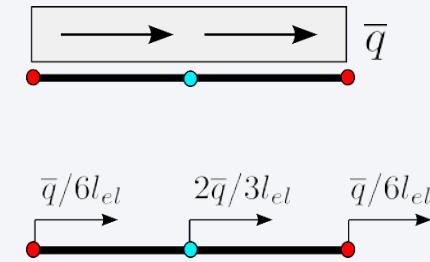
- The element stiffness matrix is obtained by recalling the expression available from the internal virtual work, and considering now the vector of the quadratic shape functions:

$$\mathbf{K}_{el} = \int_0^{l_{el}} \mathbf{N}_{,\xi}^T E A \mathbf{N}_{,\xi} d\xi = \frac{EA}{l_{el}} \begin{bmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{bmatrix}$$

- In a similar fashion, the vector of the applied loads is available from the expression of the external virtual work, leading to:

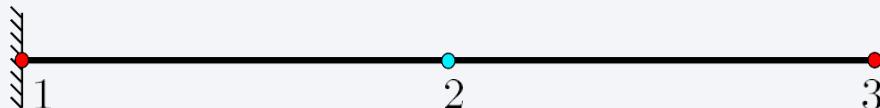
$$\mathbf{f}_{el} = \bar{q} \int_0^{l_{el}} \mathbf{N}^T d\xi = \bar{q} l_{el} \begin{Bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{Bmatrix}$$

- Note that the projection of the load onto the quadratic shape functions (consistent nodal loads) leads to a somewhat counterintuitive distribution of the nodal loads. The two outer loads carry  $1/6$  of the total load acting on the element, while the mid-node carries  $2/3$  of the total load



# Results: 1 (quadratic) element

- The model with 1 element is now characterized by three nodes (and thus three dofs)



- The governing equations are:

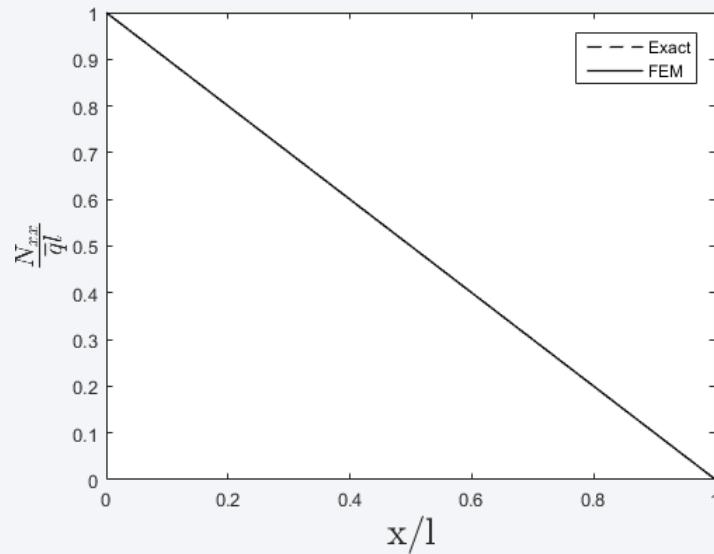
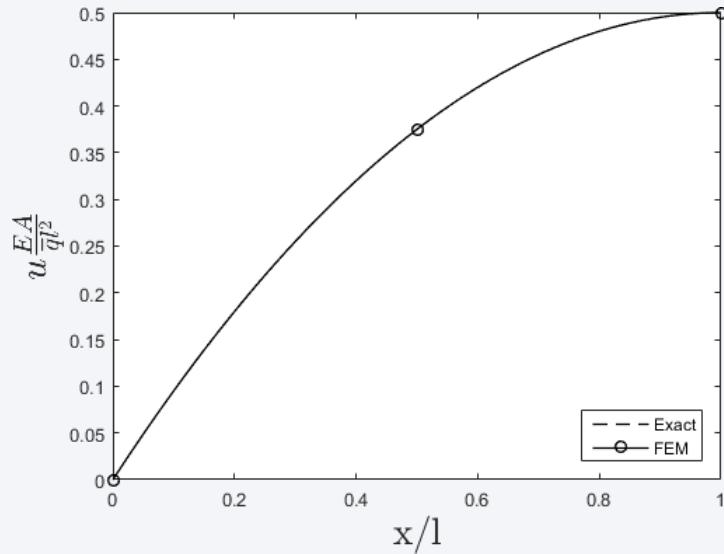
$$\frac{EA}{l_{el}} \begin{bmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \bar{q}l_{el} \begin{Bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{Bmatrix}$$

- and after imposing the boundary condition at the node 1:

$$\frac{EA}{l_{el}} \begin{bmatrix} 16/3 & -8/3 \\ -8/3 & 7/3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \bar{q}l_{el} \begin{Bmatrix} 2/3 \\ 1/6 \end{Bmatrix}$$

- and so:  
$$u_2 = \frac{3}{8} \frac{\bar{q}l_{el}^2}{EA}$$
  
$$u_3 = \frac{1}{2} \frac{\bar{q}l_{el}^2}{EA}$$
- $$\Rightarrow u = N_2 u_2 + N_3 u_3 = \frac{\bar{q}}{EA} \left( -\frac{x^2}{2} + l_{el}x \right)$$
- which is the exact solution of the problem

# Results: 1 (quadratic) element

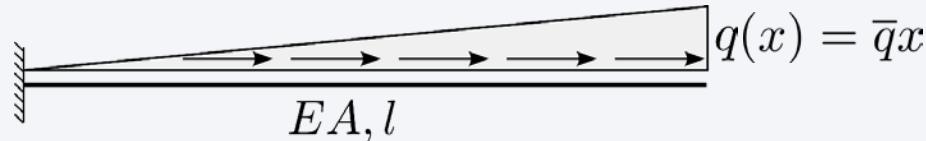


## Remarks

- The FEM solution converges to the exact one, using one single element. This is not surprising as three-node elements allow for a quadratic description of the displacement field, and a linear description of the axial force. For the problem at hand, the exact displacement is linear, and the axial force is constant. It follows that the exact solution can be correctly represented by the FEM approximation

## Example: truss with linear load

- Consider now a truss with constant axial stiffness  $EA$ , and subjected to a linear load. The bar is fixed at one end, and free at the other end



- The exact solution of the problem can be obtained by solving the following differential problem

$$\begin{cases} EAu_{,xx} + \bar{q}x = 0 \\ u(0) = 0 \\ u_{,x}(0) = 0 \end{cases}$$

whose solution is in the form

$$u = u^h + u^p$$

and

$$u^h = Ax + B \quad u^p = -\frac{\bar{q}}{6EA}x^3$$

# Exact solutions

- After imposing the boundary conditions, the solution is obtained as:

$$u(x) = \frac{\bar{q}}{EA} \left( \frac{l^2 x}{2} - \frac{x^3}{6} \right)$$

- while the strains and the axial force read:

$$u_{,x} = \xi_{xx} = \frac{\bar{q}}{EA} \left( \frac{l^2}{2} - \frac{x^2}{2} \right)$$

$$N_{xx} = EA\xi_{xx} = \bar{q} \left( \frac{l^2}{2} - \frac{x^2}{2} \right)$$

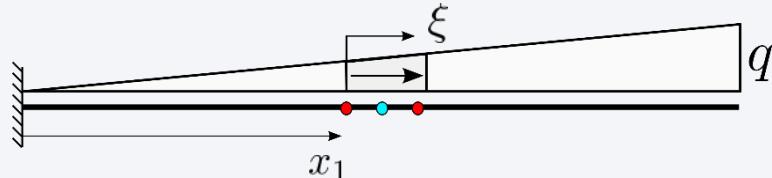
- Note that:
  - the displacements are now cubic with x
  - the strains and the axial force are quadratic with x
- Nor the two-node nor the three-node elements can exactly describe this behaviour; it is then interesting to compare the performance of the two elements

# FE formulation

- The stiffness matrix of the linear and quadratic elements are the same obtained in the previous example
- The external virtual work modifies as:

$$\begin{aligned}\delta W_{e,el} &= \int_{l_{el}} \delta u q dx = \bar{q} \int_{l_{el}} \delta u x dx \\ &= \bar{q} \int_0^{l_{el}} \delta u (\xi + x_1) d\xi\end{aligned}$$

and recalling that  $\xi = x - x_1$



- Substituting the finite element approximation is:

$$\delta W_{e,el} = \delta \mathbf{u}_{el}^T \bar{q} \int_0^{l_{el}} \mathbf{N}^T (\xi + x_1) d\xi = \delta \mathbf{u}_{el}^T \mathbf{f}_{el}$$

- and for the two- and three-node elements becomes:

$$\mathbf{f}_{el} = \bar{q} l_{el} \left\{ \begin{array}{l} \frac{l_{el}}{6} + \frac{x_1}{2} \\ \frac{l_{el}}{3} + \frac{x_1}{2} \end{array} \right\}$$

$$\mathbf{f}_{el} = \bar{q} l_{el} \left\{ \begin{array}{l} \frac{1}{6} x_1 \\ \frac{1}{3} l_{el} + \frac{2}{3} x_1 \\ \frac{1}{6} l_{el} + \frac{1}{6} x_1 \end{array} \right\}$$

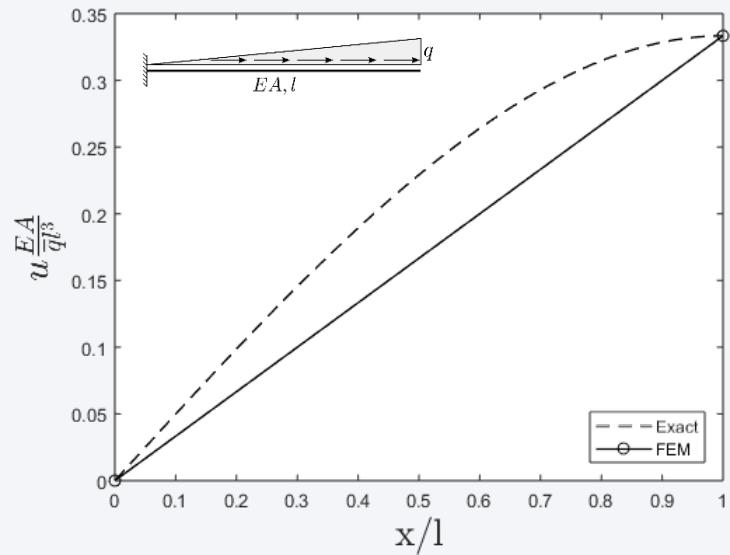
## FE formulation

- If one single element is considered, the linear system expressing the equilibrium for the unconstrained problem is:

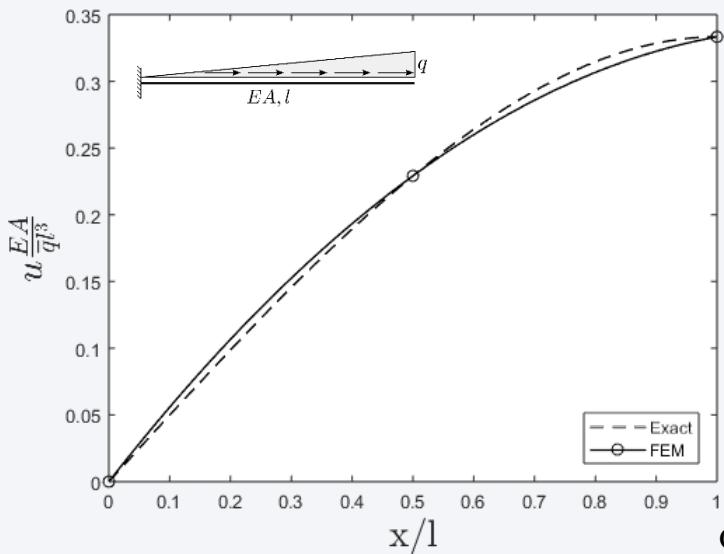
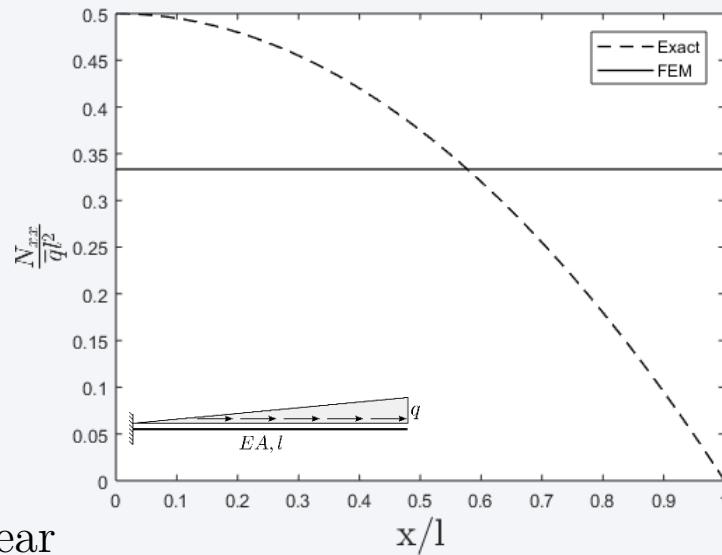
$$\frac{EA}{l_{el}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \bar{q}l_{el} \begin{Bmatrix} \frac{l_{el}}{6} + \frac{x_1}{2} \\ \frac{l_{el}}{3} + \frac{x_1}{2} \end{Bmatrix} \quad (\text{linear element})$$

$$\frac{EA}{l_{el}} \begin{bmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \bar{q}l_{el} \begin{Bmatrix} \frac{1}{6}x_1 \\ \frac{1}{3}l_{el} + \frac{2}{3}x_1 \\ \frac{1}{6}l_{el} + \frac{1}{6}x_1 \end{Bmatrix} \quad (\text{quadratic element})$$

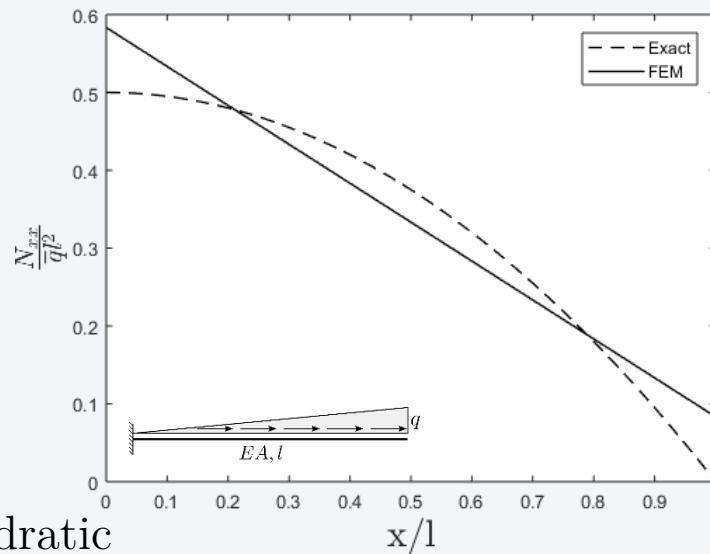
# Results: 1 element



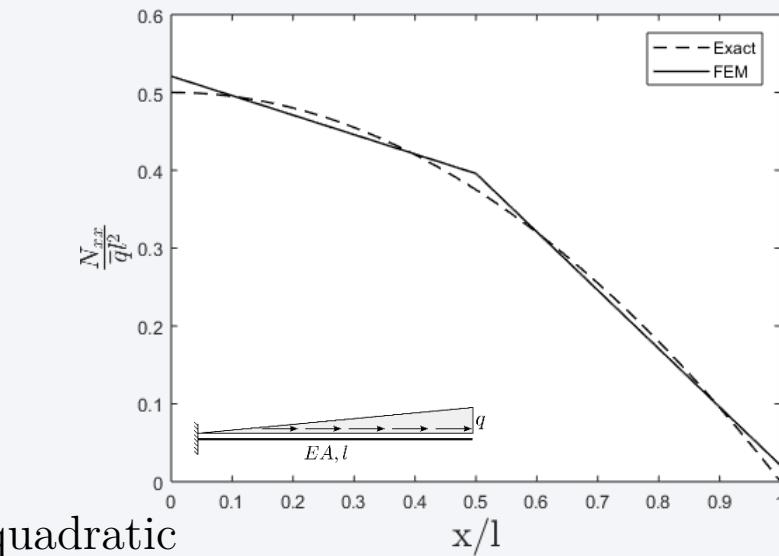
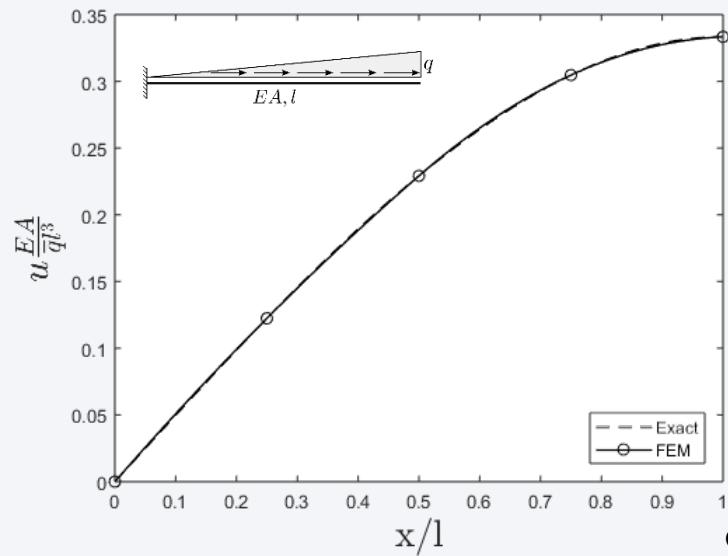
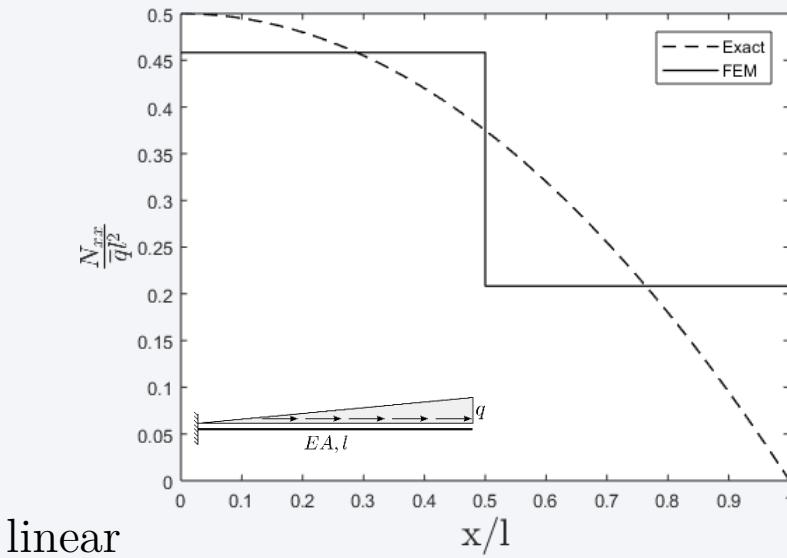
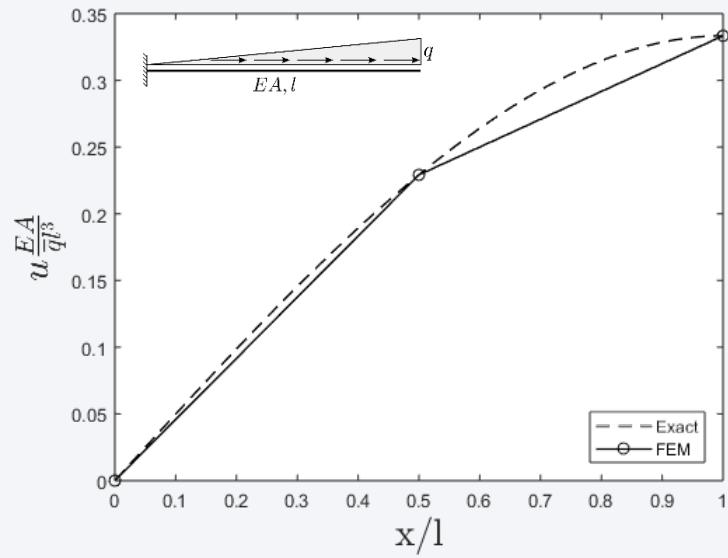
linear



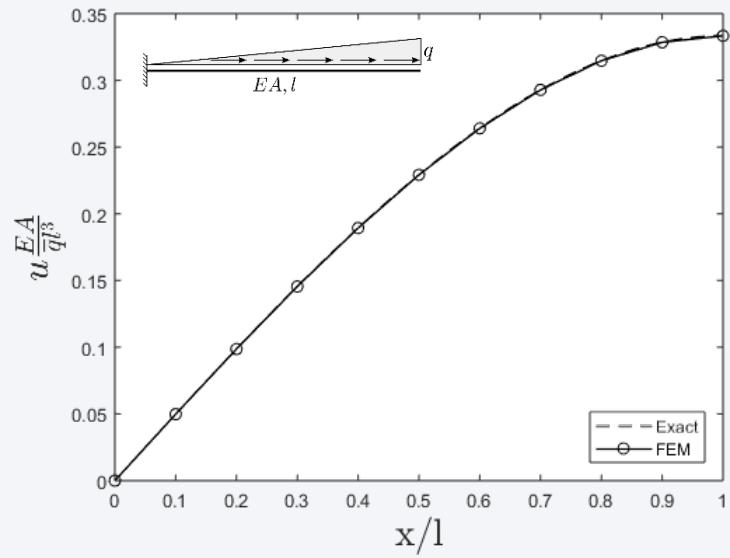
quadratic



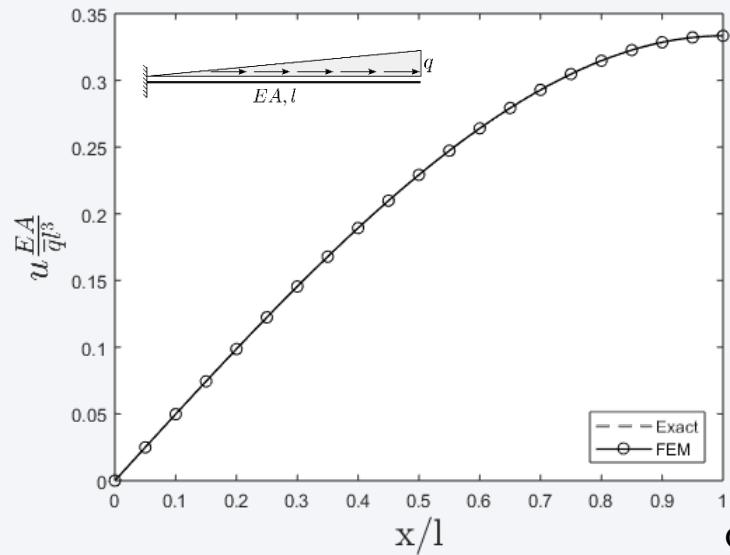
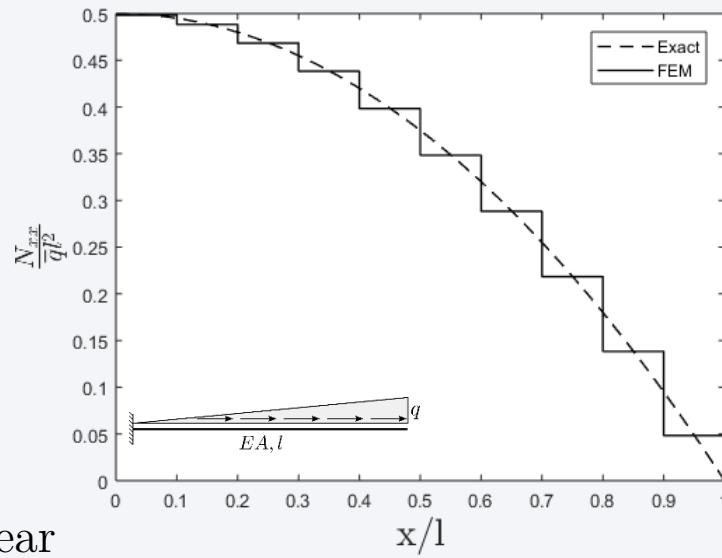
# Results: 2 elements



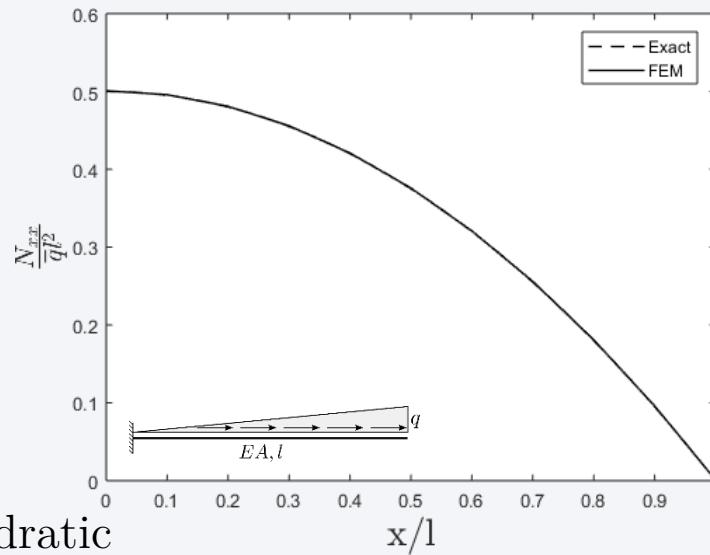
# Results: 10 (linear) elements



linear



quadratic



# Convergence of the results

- In analogy with the method of Ritz, the finite element solution is characterized by a monothonic convergence of the total potential energy, which becomes smaller and smaller as the number of dofs is increased

| $\bar{\Pi} = \frac{EA}{\bar{q}^2 l^3} \Pi$ | 2-node elements |         | 3-node elements |         |
|--|-----------------|---------|-----------------|---------|
|  | N = 1           | -5.5556 | N = 1           | -6.5972 |
| N = 2                                      | -6.3368         |         | -6.6623         |         |
| N = 5                                      | -6.6116         |         | -6.6666         |         |
| N = 10                                     | -6.6528         |         | -6.6667         |         |
| N = 100                                    | -6.6665         |         | -6.6667         |         |
| exact                                      | -6.6667         |         | -6.6667         |         |

## Convergence of the results

- The analysis of the convergence is performed by introducing two norms for measuring the error of the approximate solution (in this simple case the error can be obtained referring to available exact solution; to a more general extent, the reference solution could be taken as the one calculated with a refined mesh)

$$\|e^u\|_{L_2} = \frac{\sqrt{\int_0^l (u - u_h)^2 dx}}{\int_0^l u dx} \quad L_2 \text{ norm}$$

$$\|e^u\|_{en} = \frac{\sqrt{\int_0^l (u_{,x} - u_{h,x})^2 dx}}{\int_0^l u_{,x} dx} \quad \text{energy norm}$$

- Note that the energy norm contains the derivatives up to the order contained in the weak-form formulation of the problem. Accordingly, this norm provides a measure of the error in evaluating the derivative-related quantities of the unknown displacement field: (generalized) strains and (generalized) stresses

# Convergence of the results

- Errors using 2- and 3-node elements

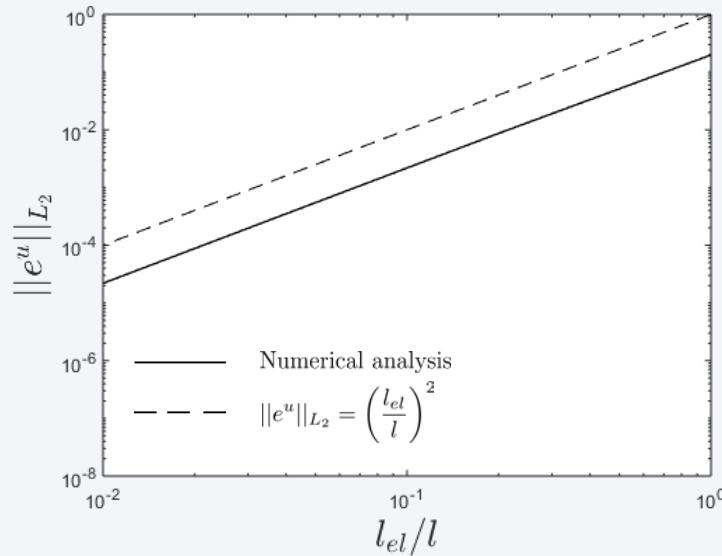
|               | 2-node elements      |                     | 3-node elements      |                     |
|---------------|----------------------|---------------------|----------------------|---------------------|
|               | $\ e^u\ _{L_2} 10^2$ | $\ e^u\ _{en} 10^2$ | $\ e^u\ _{L_2} 10^2$ | $\ e^u\ _{en} 10^2$ |
| $N_{el} = 1$  | 19.8009              | 41.4491             | 2.4714               | 10.8135             |
| $N_{el} = 5$  | 0.8672               | 8.6931              | 0.0218               | 0.4686              |
| $N_{el} = 10$ | 0.2183               | 4.3709              | 0.0027               | 0.1180              |
| $N_{el} = 20$ | 0.0547               | 2.1897              | 0.0003               | 0.0296              |

## Remarks

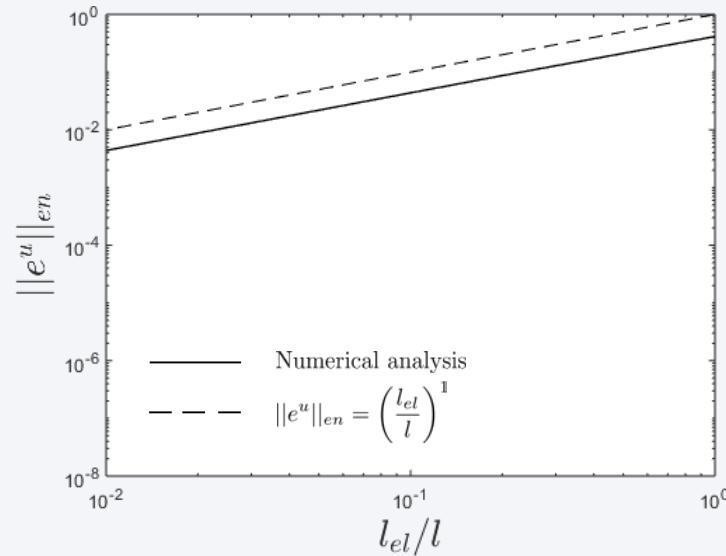
- The convergence of quadratic elements is faster with respect to the linear ones, both in terms of displacements and axial force
- In both cases, it can be noted that the convergence of displacements is faster with respect to the axial force (which is associated with the first derivative of the displacements)

# Convergence of the solution

- The order of convergence can be assessed by plotting the results in logarithmic scale. The slope of the curve provides the order of convergence



linear elements,  $L_2$  norm



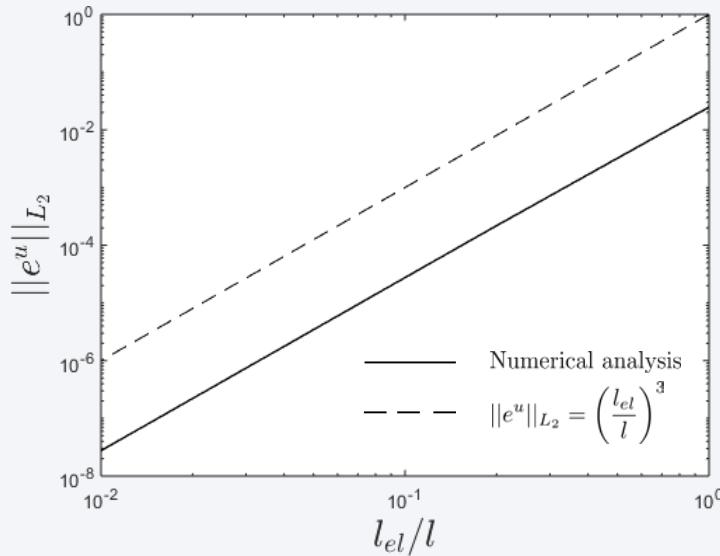
linear elements, energy norm

## Remarks

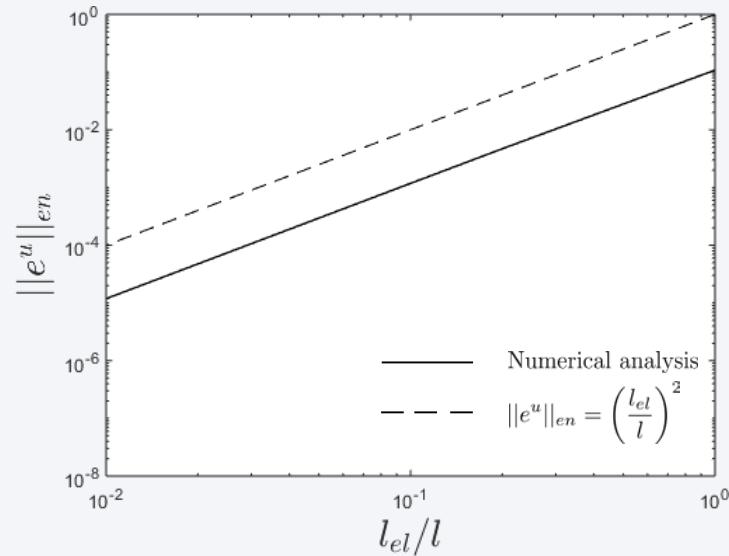
- The convergence in the  $L_2$  norm has order 2
- The convergence in the energy norm (thus the convergence of strains, stresses and axial force) has order 1

# Convergence of the solution

- The order of convergence can be assessed by plotting the results in logarithmic scale. The slope of the curve provides the order of convergence



quadratic elements,  $L_2$  norm



quadratic elements, energy norm

## Remarks

- The convergence in the  $L_2$  norm has order 3
- The convergence in the energy norm (thus the convergence of strains, stresses and axial force) has order 2

## Convergence of the solution

- The results here presented illustrate a well-known result of the finite element theory, which can be generalized as:

$$\|e\|_{L_2} = Ch^{p+1}$$

$$\|e\|_{en} = Ch^p$$

- where  $C$  is a constant, while  $p$  defines the order of completeness of the shape functions (for the linear element  $p=1$ , and for the quadratic one  $p=2$ )
- The refinement of the solution can thus be obtained by reducing the size of the element  $h$ , or by increasing the order  $p$  of the polynomial expansion
- If the size of the elements is halved, then the error decreases by a factor of  $2^{p+1}$  and  $2^p$  in the  $L_2$  and energy norm, respectively. If linear elements are considered, then the factors are 4 and 2, whilst they are 8 and 4 in the case of quadratic elements
- For the above mentioned reasons, quadratic elements usually provide a good tradeoff between number of dofs and accuracy