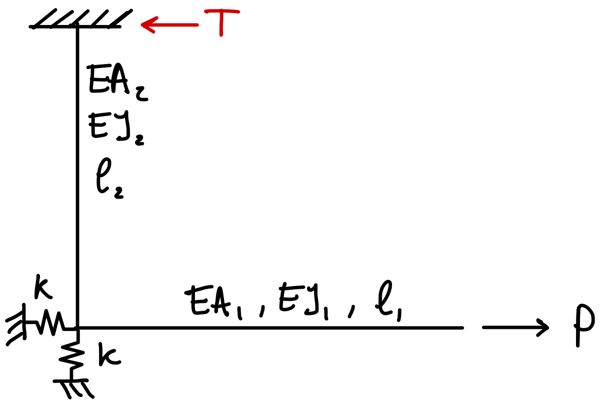
Exercise 1



Determine the shear force T at the fixed end.

Data

$\ell_1 = 1000$ mm

$\ell_2 = 1000$ mm

$EA_1 = 1 \cdot 10^6$ N

$EA_2 = 3 \cdot 10^6$ N

$EJ_1 = 2 \cdot 10^{12}$ Nmm^2

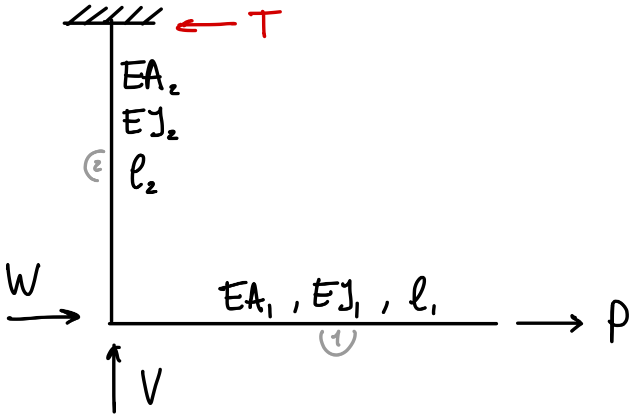
$EJ_2 = 1 \cdot 10^{12}$ Nmm^2

$k = 2000\,(1 + A/10)$

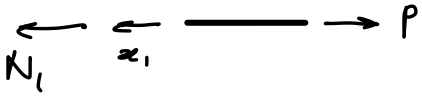
$P = 1000$ N

Solution

Real system

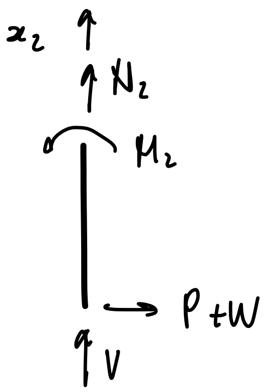


- Beam 1



$$N_1 = P$$

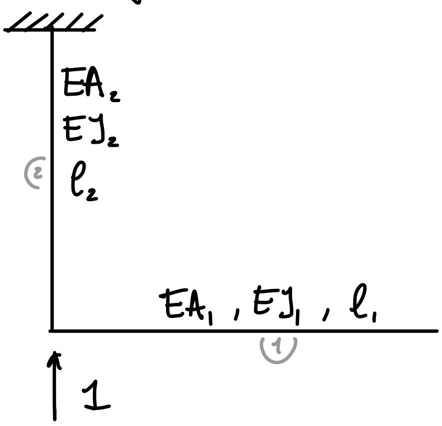
- Beam 2



$$N_2 = -V$$

$$M_2 = -Px_2 - Wx_2$$

Dummy system #1



- Beam 1

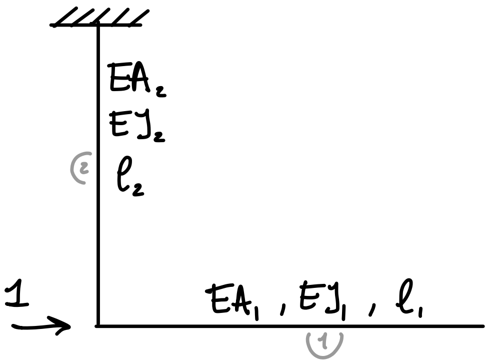
${}^1\delta N_1 = 0$

- Beam 2

${}^1\delta N_2 = -1$

${}^1\delta M_2 = 0$

Dummy system #2



- Beam 1

${}^2\delta N_1 = 0$

- Beam 2

${}^2\delta N_2 = 0$

${}^2\delta M_2 = -x_2$

By application of PVCW:

1) $\int_0^{\ell_2} \frac{N_2}{EA_2} {}^1\delta N_2 \, dx_2 + V/k = 0 \quad \Rightarrow \quad V = 0$

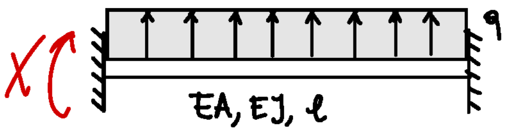
2) $\int_0^{\ell_2} \frac{M_2{}^2 \, \delta M_2}{EJ_2} dx_2 + W/k = 0$

From which: $W = -\frac{P k \ell_2^3}{k \ell_2^3 + 3EJ_2} = -400 \text{ N}$

And the shear force at the fixed end reads

$T = P + W = 600 \text{ N}$

Exercise 3



Determine the unknown reaction force X as reported in the sketch

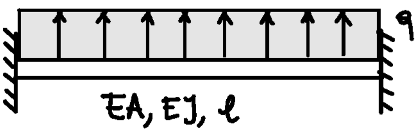
Data

$\ell = 1000\,(1 + c/10)$ $\qquad$ $q = 0.1$ N/mm

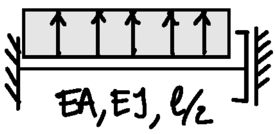
$EA = 1 \cdot 10^6$ N

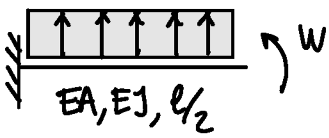
$EJ = 1 \cdot 10^9$ N mm^2

Solution

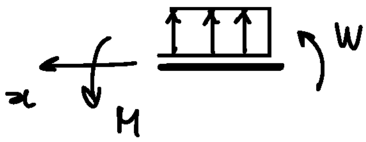


To simplify the solution, the symmetry of the problem can be exploited (clearly this is not strictly necessary).



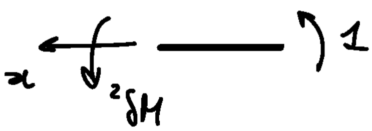


Real system



$$M = -W - qx^2/2$$

Dummy system



$$^2\delta M = -1$$

By application of the PCVW

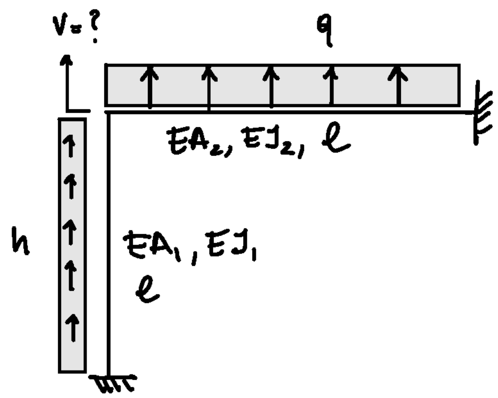
$$\int_0^{\ell/2} \frac{M\ \delta M}{EJ}\, dx = 0$$

From which: $W = -q\ell^2/24$

And the reaction force X is then

$$X = W + q\left(\ell/2\right)^2 1/2 = 1/12\, q\ell^2 = 8333 \text{ N mm}$$

Exercise 10



Determine the vertical displacement v.
Solve the problem using the Ritz method and consider the smallest possible number of dofs using a polynomial representation.

Data

$\ell = 1300\,(1 + A/10)$

$EA_1 = 3 \cdot 10^6$ N

$EJ_1 = 4 \cdot 10^{12}$ Nmm^2

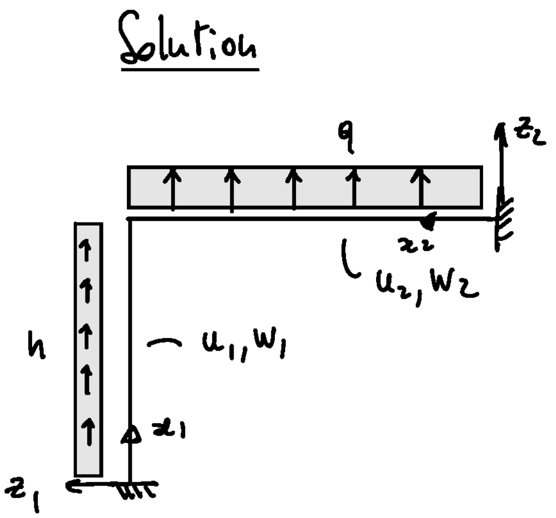
$EA_2 = 6 \cdot 10^6$ N

$EJ_2 = 8 \cdot 10^{12}$ Nmm^2

$q = 100$ N/mm

$h = 70$ N/mm

Solution



$d_2 = -c_1 \ell$

$d_1 = -c_1$

$u_1(\ell) = a_1$

$w_2(\ell) = d_2$

The following set of trial functions is considered:

$$u_1 = \boxed{a_0} + a_1 \left(x_1/\ell \right)$$
$$w_1 = \boxed{b_0} + \boxed{b_1} \left(x_1/\ell \right) + b_2 \left(x_1/\ell \right)^2$$
$$u_2 = \boxed{c_0} + c_1 \left(x_2/\ell \right)$$
$$w_2 = \boxed{d_0} + \boxed{d_1} \left(x_2/\ell \right) + d_2 \left(x_2/\ell \right)^2$$

10 dof

- Essential boundary conditions (due to constraints) $\boxed{1}$

$u_1(0) = 0 \quad \Rightarrow \quad u_1 = a_1 \left(x_1/\ell \right)$

$w_1(0) = 0$

$w_1'(0) = 0 \quad \Rightarrow \quad w_1 = b_2 \left(x_1/\ell \right)^2$

6 constr

$$u_2(0)=0 \quad \Rightarrow \quad u_2 = c_1 (x_2/\ell)$$
$$\left.\begin{array}{l} w_2(0)=0 \\ w_2'(0)=0 \end{array}\right\| \Rightarrow \quad w_2 = d_2 (x_2/\ell)^2$$

- Essential boundary conditions at the interface

$$u_1(\ell) = w_2(\ell) \quad \Rightarrow \quad a_1 = d_2$$
$$w_1(\ell) = u_2(\ell) \quad \Rightarrow \quad b_2 = c_1$$
$$w_1'(\ell) = -w_2'(\ell) \quad \Rightarrow \quad b_2 = -d_2$$

3 constr

So the unknown amplitudes can be expressed as a function of one single amplitude, e.g. a_1:

$$\boxed{\begin{array}{l} d_2 = a_1 \\ c_1 = -a_1 \\ b_2 = -a_1 \end{array}}$$

By application of the PVW:

$$\delta W_i = \int_0^\ell \left(\delta u_1' EA_1 u_1' + \delta w_1'' EJ_1 w_1'' \right) dx_1 + $$
$$+ \int_0^\ell \left(\delta u_2' EA_2 u_2' + \delta w_2'' EJ_2 w_2'' \right) dx_2$$

$$\delta W_e = \int_0^l \delta u_1 \, n \, dx_1 + \int_0^l \delta w_2 \, q \, dx_2$$

From which:

$$K = \frac{EA_1}{l} + \frac{EA_2}{l} + \frac{4EJ_1}{l^3} + \frac{4EJ_2}{l^3}$$

$$F = nl/2 + ql/3$$

The linear static problem is then:

$$K a_1 = F$$

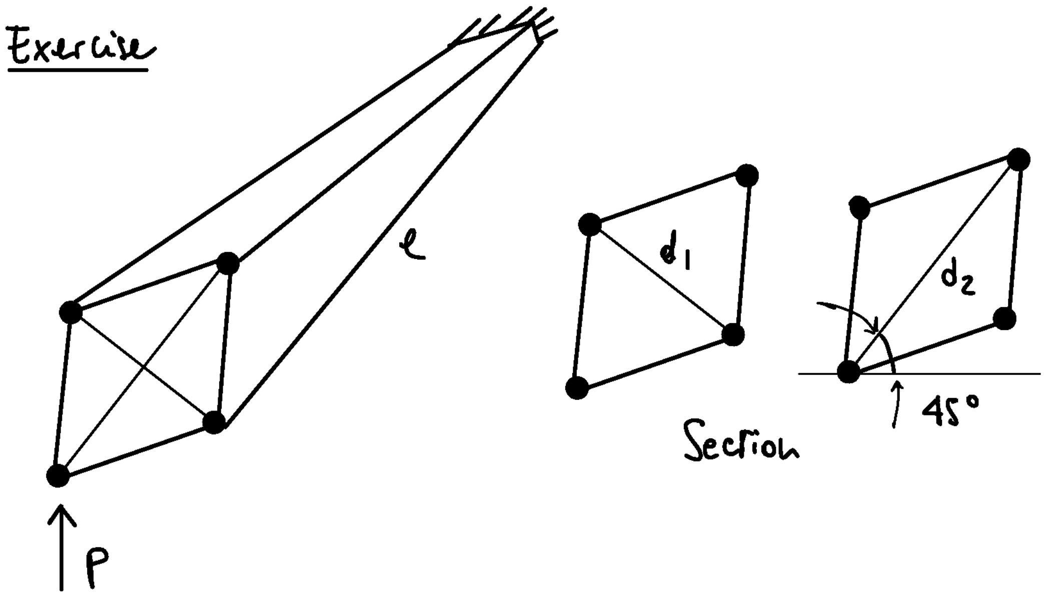
From which:

$$a_1 = 3.0876 \text{ mm}$$

The vertical displacement v is then:

$$u_1(l) = a_1 = 3.0876$$

Exercise



Compute the rotation of the loaded end

Data

$l = 5000$ mm $\qquad t = 0.8$ mm

$d_1 = 500\ (1+F/10)$ mm

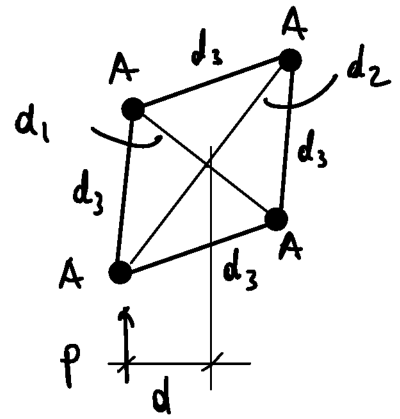
$d_2 = 250\ (1+E/6)$ mm

$A = 500$ mm^2

$P = 8000$ N

$G = 27000$ MPa

Solution



$$\Omega = d_1 d_2 / 2$$

$$d_3 = \sqrt{(d_1/2)^2 + (d_2/2)^2}$$

$$d = d_2/2 \cos 45$$

The torsional constant of the section is evaluated referring to the Bredt's formula:

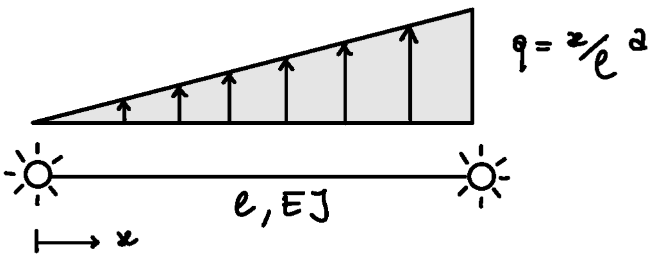
$$J = \frac{4\Omega^2}{\oint_p \frac{1}{t}\, d\Gamma} = \frac{4\Omega^2}{4 d_3 / t} = \frac{\Omega^2 t}{d_3}$$

but $M_t = Pd = GJ\theta'$, so:

$$\theta' = \frac{Pd}{GJ}$$

And so: $\theta = \theta' l = 0.6711$ deg

Exercise 20



Compute the vertical displacement at $x = \ell/2$ using Ritz and choosing between the approximations below:

$w = c \cos \frac{\pi x}{\ell}$ $\qquad$ $w = c\, x^2 (x - \ell)^2$

$w = c\, x$ $\qquad$ $w = c\, x (x - \ell)$

$w = c\, x^2 (x - \ell)$

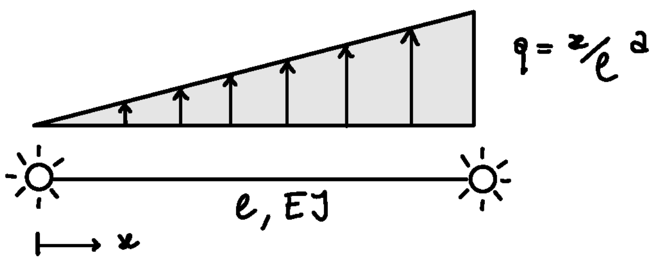
Data

$\ell = 1500\,(1 + A/10)$ mm

$EJ = 10^{10}$ Nmm²

$a = 1.0$ N/mm

Solution



Consider the approximation

$$w = c\,x(x-\ell)$$

So:

$$w' = c\,2x - c\,\ell\,;$$
$$w'' = c\,2$$

The PVW reads:

$$\int_0^\ell \delta w''\, EJ\, w''\, dx = \int_0^\ell \delta w \, \frac{x}{\ell} a \, dx$$

And, upon substitution of the approximation:

$$\delta c \int_0^\ell 4EJ\, dx \; c = \delta c \int_0^\ell x(x-\ell)\frac{x}{\ell}\, dx$$

So:

$$(4EJ\ell)\,c = -a\ell^3/12$$

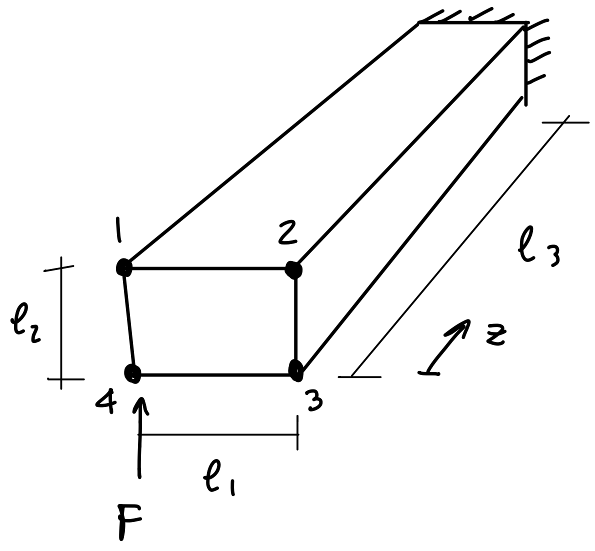
And then:

$$c = -\frac{a l^2}{48 EJ}$$

the displacement at $x = l/2$ is then

$$w = c\,x(x-l)\Big|_{x=l/2} = 2.64\ \text{mm}$$

Exercise 23



Determine the axial stress σ_{zz} in the stringer #4 at $z = \ell_3/2$

Data

$\ell_1 = 400$ mm

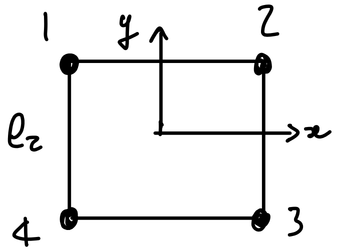
$\ell_2 = 250\,(1 + B/10)$ mm

$\ell_3 = 2000$ mm

$A = 500$ mm^2

$F = 8000$ N

Solution



$$J_{xx} = 4A\left(\frac{l_2}{2}\right)^2 = A l_2^2$$

The bending moment reads:

$M(z) = -Fz$, so:

$M(l_3/2) = -F\,l_3/2$

From which:

$$\sigma_{zz} = -\frac{F l_3}{2}\left(-\frac{l_2}{2}\right)\frac{1}{J_{xx}} = \frac{F l_3}{4 A l_2} = 32\ \mathrm{MPa}$$

- In the finite element method, the analysis of a statically indetermined structure:
  - is done with no differences with the case of a statically determined one
  - ~~requires special compatibility requirements to be added to the solving equations~~
  - ~~cannot be performed due to the overconstraints~~
- The rotation of a multi-cell thin walled cross section with N cells:
  - ~~can be computed using Bredt's formula~~
  - can be computed by solving a system of equations with N-1 compatibility equations an 1 equilibrium equation
  - ~~can be computed by finding the location of the shear center~~
- The buckling load of a compressed beam is function:
  - of the cross-section bending stiffness
  - ~~of the cross-section torsional stiffness~~
  - ~~of the cross-section axial stiffness~~
  - ~~of the cross-sections shear stiffness~~
- The internal forces in a statically determined structure depend on the material elastic properties
  - ~~True~~
  - False
- The Timoshenko beam model does not account for transverse shear deformability
  - ~~True~~
  - False
- The position of the shear center of a thin-walled beam depends on the loading conditions
  - ~~True~~
  - False