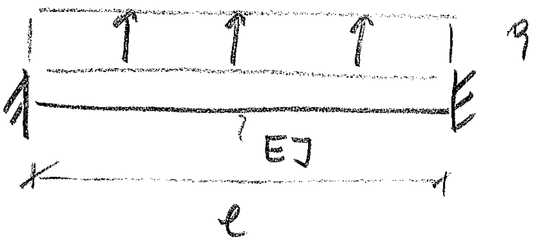
Exercise



Compare the mid-displacement obtained from:

- the exact solution
- the Ritz solution using 1, 2 and 3 trigonometric functions
- the Galerkin solution using 1 trigonometric function

1. Exact solution

$$\begin{cases} EJ\, w_{/xxxx} = q \\ w(0) = 0 \\ w_{/x}(0) = 0 \\ w(l) = 0 \\ w_{/x}(l) = 0 \end{cases}$$

$\longleftarrow$ essential conditions

The solution is given by

$$w = w^H + w^P$$

where:

$$w^P = \frac{1}{24}\frac{q}{EJ}x^4$$

and

$$w^H = A_0 + A_1 x + A_2 x^2 + A_3 x^3$$

So:

$$w = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \frac{1}{24}\frac{q}{EJ}x^4$$

Impose the boundary conditions:

$$w(0) = 0 \quad \Rightarrow \quad A_0 = 0$$

$$w_{/x}(0) = 0 \quad \Rightarrow \quad A_1 = 0$$

$$w(\ell) = 0 \quad \Rightarrow \quad A_2 + A_3 \ell + \frac{q}{24EJ} \ell^2 = 0$$

$$w_{/x}(\ell) = 0 \quad \Rightarrow \quad 2A_2 + 3A_3 \ell + \frac{q}{6EJ} \ell^2 = 0$$

which leads to:

$$A_3 = -\frac{q}{12EJ} \ell \quad ; \quad A_2 = \frac{q}{24EJ} \ell^2$$

and so:

$$w = \frac{q\ell^4}{24EJ} \left[\left(\frac{x}{\ell}\right)^2 - 2\left(\frac{x}{\ell}\right)^3 + \left(\frac{x}{4}\right)^4 \right]$$

$$w(\ell/2) = \frac{q\ell^4}{384EJ}$$

2. Ritz solution - 1 term

The trial function is taken as:

$$\phi_1 = 1 - \cos\frac{2\pi x}{\ell}$$

and $w = c_1 \phi_1$

Note that the trial function satisfies the 4 essential conditions:

$$w(0) = w_{/x}(0) = w(\ell) = w_{/x}(\ell) = 0$$

The stiffness matrix and the load vectors are:

$$K_{ij} = \int_0^\ell \phi_{i/xx} EJ \phi_{j/xx} dx$$

$$f_i = \int_0^\ell \phi_i q \, dx$$

In this case

$$K_{11} = \frac{8\pi^4 EJ}{\ell^3}$$

$$f_1 = q\ell$$

The unknown amplitude is then:

$$c_1 = f_1/K_{11} = \frac{q\ell^4}{8\pi^4 EJ}$$

The displacement field is then

$$w = \frac{q\ell^4}{8\pi^4 EJ}\left(1 - \cos\frac{2\pi x}{\ell}\right)$$

and

$$w\left(\frac{\ell}{2}\right) = \frac{q\ell^4}{4EJ\pi^4}$$

3. Ritz solution – 2 terms

$$\phi_1 = 1 - \cos\frac{2\pi x}{\ell} \qquad \phi_2 = 1 - \cos\frac{4\pi x}{\ell}$$

The stiffness matrix reads:

$$K_{11} = \int_0^\ell \phi_{1/xx}\, EJ\, \phi_{1/xx}\, dx = \frac{8\pi^4 EJ}{\ell^3} \quad \text{(already available)}$$

$$K_{12} = K_{21} = \int_0^\ell \phi_{1/xx}\, EJ\, \phi_{2/xx}\, dx = 0$$

$$K_{22} = \int_0^\ell \phi_{2/xx}\, EJ\, \phi_{2/xx}\, dx = \frac{128\,\pi^4 EJ}{\ell^3}$$

The load vector is:

$$f_1 = q\ell$$

$$f_2 = q\ell$$

The system to be solved is:

$$\begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

which leads to:

$$c_1 = \frac{q\ell^4}{8EJ\pi^4} \qquad c_2 = \frac{q\ell^4}{128\,EJ\,\pi^4}$$

The displacement at the midspan is still

$$w\left(\frac{l}{2}\right) = \frac{q l^4}{4 E J \pi^4}$$

which is not surprising as $\phi_2\left(\frac{l}{2}\right) = 0$

4. Ritz solution - 3 terms

$$\phi_1 = 1 - \cos\frac{2\pi x}{l} \; ; \; \phi_2 = 1 - \cos\frac{4\pi x}{l} \; ; \; \phi_3 = 1 - \cos\frac{6\pi x}{l}$$

Due to the orthogonality of the functions, the only terms to be computed are:

$$k_{33} = \int_0^l \phi_{3/xx}\, E J\, \phi_{3/xx}\, dx = \frac{648 E J \pi^4}{l^3}$$

$$f_3 = \int_0^l \phi_3\, q\, dx = q l$$

the final system is:

$$\begin{bmatrix} k_{11} & & \\ & k_{22} & \\ & & k_{33} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

leading to:

$$c_1 = \frac{q l^4}{8 E J \pi^4} \qquad c_2 = \frac{q l^4}{128 E J \pi^4} \qquad c_3 = \frac{q l^4}{648 E J \pi^4}$$

The mid-span displacement is then:

$$w\left(\ell/2\right) = \frac{41}{162}\,\frac{q\ell^4}{EJ\pi^4}$$

5. Galerkin solution – 1-term

The governing equation reads:

$EJ\,w_{/xxxx} = q$, so:

$$\int_0^\ell \phi_1\, EJ\, \phi_{1/xxxx}\, dx\, c_1 = \int_0^\ell \phi_1 q\, dx$$

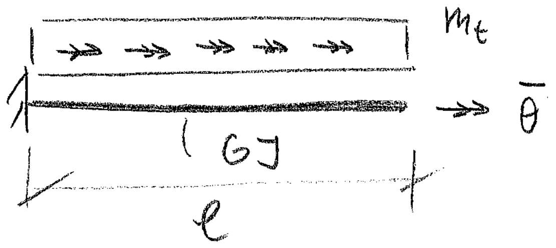
which is:

$$\frac{8\,EJ\pi^4}{\ell^3}\, c_1 = q\ell$$

$$\Rightarrow \quad c_1 = \frac{q\ell^4}{8\,EJ\pi^4}$$

(same result obtained with Ritz)

Exercise



Obtain the exact displacement field and compare it with the 1-term solution obtained using Ritz with polynomial functions

Exact solution

$$\delta\Pi = \int_0^\ell \delta\theta_{/x}\, GJ\, \theta_{/x}\, dx - \int_0^\ell \delta\theta\, m_t\, dx$$

$$= -\int_0^\ell \delta\theta\, GJ\, \theta_{/xx}\, dx + \delta\theta\, GJ\, \theta_{/x}\Big|_0^\ell - \int_0^\ell \delta\theta\, m_t\, dx$$

$$\begin{cases} GJ\theta_{/xx} + m_t = 0 \\ GJ\theta_{/x}(0) = 0 \quad \text{or} \quad \boxed{\delta\theta(0) = 0} \\ GJ\theta_{/x}(\ell) = 0 \quad \text{or} \quad \boxed{\delta\theta(\ell) = 0} \end{cases}$$

The problem is then:

$$\boxed{\begin{array}{l} \theta_{/xx} = -\dfrac{m_t}{GJ} \\ \theta(0) = 0 \\ \theta(\ell) = \bar{\theta} \end{array}}$$

The solution is found as:

$$\theta = \theta^H + \theta^P$$

with

$$\theta^P = -\frac{1}{2}\frac{m_t}{GJ}x^2$$

and

$$\theta^H = A_0 + A_1 x$$

Imposing the boundary conditions:

$$\theta(0) = 0 \Rightarrow A_0 = 0$$

$$\theta(\ell) = \bar{\theta} \Rightarrow A_1 \ell - \frac{1}{2}\frac{m_t}{GJ}\ell^2 = \bar{\theta} \Rightarrow A_1 = \frac{\bar{\theta}}{\ell} + \frac{1}{2}\frac{m_t}{GJ}\ell$$

The displacement field is then:

$$\boxed{\theta(x) = \frac{1}{2}\frac{m_t}{GJ}\ell^2\left(\frac{x}{\ell} - \frac{x^2}{\ell^2}\right) + \frac{\bar{\theta}}{\ell}x}$$

<u>Ritz solution - 1 term</u>

In this case the problem is characterized by two essential conditions. However, one of the two conditions is non-homogeneous.
The expansion is then taken as:

$$\theta = \sum_{i=1}^{N} c_i \phi_i + \phi_0$$

where

$\phi_i \rightarrow$ satisfy the homogeneous part of the essential conditions

$\phi_0 \rightarrow$ satisfies the non-homogeneous essential boundary conditions.

More specifically:

$\phi_i(0) = 0$ and $\phi_0(0) = 0$

$\phi_i(\ell) = 0$ $\phi_0(\ell) = \bar{\theta}$

The simplest choice for ϕ_0 is a linear polynomial

$$\phi_0(x) = \frac{\bar{\theta}}{\ell} x$$

The functions ϕ_i should be taken as a second order polynomial, so that the two conditions can be imposed:

$$\theta^H = a_0 + a_1 x + a_2 x^2$$

$\theta(0) = 0 \Rightarrow a_0 = 0$

$\theta(\ell) = 0 \Rightarrow a_1 = -a_2 \ell$

$$\Rightarrow \theta^H = a_2 (x^2 - \ell x)$$

The Ritz functions are then taken as:

$$\boxed{\theta = c_1 (x^2 - \ell x) + \frac{\bar{\theta}}{\ell} x}$$

Due to the fact that the exact solution is quadratic, it is clear that the assumed Ritz expansion will lead to the exact solution. This is easily verified:

$$\int_0^l \delta\theta_{/x}\, GJ\, \theta_{/x}\, dx - \int_0^l \delta\theta\, m_t\, dx = 0$$

but

$$\theta = c_1 \phi_1 + \phi_0 \qquad \text{and} \qquad \delta\theta = \delta c_1 \phi_1 \qquad \text{so:}$$

$$\delta c_1 \int_0^l \phi_{1/x}\, GJ \left(\phi_{1/x}\, c_1 + \phi_{0/x}\right) dx = \delta c_1 \int_0^l \phi_1 m_t\, dx$$

or

$$\delta c_1 \underbrace{\int_0^l \phi_{1/x}\, GJ\, \phi_{1/x}\, dx}_{K_{11}}\, c_1 = \delta c_1 \underbrace{\int_0^l \left(\phi_1 m_t - \phi_{1/x}\, GJ\, \phi_{0/x}\right) dx}_{f_1}$$

where

$$K_{11} = GJ\, l^3/3 \qquad \text{and} \qquad f_1 = -\frac{m_t l^3}{6}$$

and so:

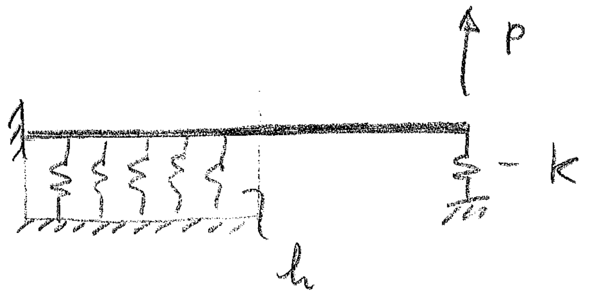
$$c_1 = -\frac{m_t}{2GJ}$$

leading to:

$$\theta = c_1\phi_1 + \phi_0 = \frac{1}{2}\frac{m_t l^2}{GJ}\left(\frac{x}{l} - \frac{x^2}{l^2}\right) + \bar{\theta}\frac{x}{l}$$

as obtained previously

Exercise



Essential conditions:

$$\begin{cases} w(0) = 0 \\ w_{,x}(0) = 0 \end{cases} \Rightarrow \quad \begin{array}{l} \phi_i(0) = 0 \\ \phi_{i,x}(0) = 0 \end{array}$$

The trial functions are taken as:

$$\phi_i = x^{i+1} \qquad i = 1, \dots N$$

Considering one single term:

$$\phi_1 = x^2$$

The PVW reads:

$$\int_0^{\ell} \delta w_{,xx} EJ \, w_{,xx} \, dx + \int_0^{\ell/2} \delta w \, h \, w \, dx + \delta w(\ell) \, k \, w(\ell)$$

$$= \delta w(\ell) \, P$$

Upon substitution of the Ritz expansion it is

obtained:

$$\delta c_1\left[\int_0^{\ell} \phi_{1/xx}\, EJ\, \phi_{1/xx}\, dx + \int_0^{\ell/2} \phi_1\, h\, \phi_1\, dx + \phi_1(\ell)\, k\, \phi_1(\ell)\right] c_1$$

$$= \delta c_1\, \phi_1(\ell)\, P$$

which leads to:

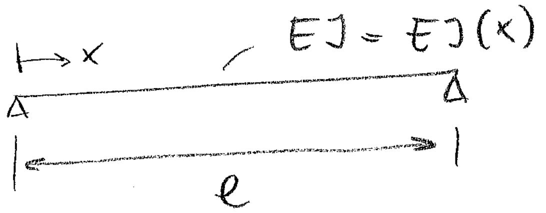
$$\left(4EJ\ell + \frac{h\ell^5}{160} + \ell^4 k\right) c_1 = P\ell^2$$

from which:

$$c_1 = \frac{160\, P\ell}{640\, EJ + h\ell^4 + 160\,\ell^3 k}$$

Exercise

Determine the first bending natural frequency using 1 trigonometric function and the method of Ritz



$$EJ = EJ_0 \sin \frac{\pi x}{\ell}$$

Ritz solution - 1 term

The essential conditions are:

$w(0) = 0$

$w(\ell) = 0$

The Ritz approximation is then taken as:

$$w = C_1 \sin \frac{\pi x}{\ell}$$

To evaluate the free vibrations, the external work is written as:

$$\delta W_e = -\int_V \left(\delta u \, \rho \ddot{u} + \delta w \, \rho \ddot{w}\right) dV$$

$$= -\int_V \left[\left(\delta u_0 - z\,\delta w_{0/x}\right)\rho\left(\ddot{u}_0 - z\,\ddot{w}_{0/x}\right) + \delta w_0 \, \rho \ddot{w}_0\right] dV$$

$$= -\int_V \left[\delta u_0\left(\rho \ddot{u}_0 - z\rho \ddot{w}_{0/x}\right) - \delta w_{0/x}\left(\rho z \ddot{u}_0 - \rho z^2 \ddot{w}_{0/x}\right) + \delta w_0 \, \rho \ddot{w}_0\right] dV$$

$$= -\int_0^{\ell} \left[\delta u_0\left(I_0 \ddot{u}_0 - I_1 \ddot{w}_{0/x}\right) - \delta w_{0/x}\left(I_1 \ddot{u}_0 - I_2 \ddot{w}_{0/x}\right) + \delta w_0 \, \rho \ddot{w}_0\right] dV$$

The term I_1 is zero; the rotary inertia I_2 is negligible so:

$$\delta W_e = -\int_0^\ell \left(\delta u_o\, I_o\, \ddot{u}_o + \delta w_o\, I_o\, \ddot{w}_o\right) dx$$

The first contribution is associated with the axial vibrations. The second one contributes to the bending vibrations. So.

$\delta W_i = \delta W_e$ is:

$$\int_0^\ell \delta w_{o/xx}\, EJ\, w_{o/xx}\, dx + \int_0^\ell \delta w_o\, I_o\, \ddot{w}_o\, dx = 0$$

$$w(x,t) = c_1\, e^{i\omega t} \sin\frac{\pi x}{\ell} = c_1\, \phi_1\, e^{i\omega t}$$

So:

$$\Big[\delta c_1 \underbrace{\int_0^\ell \phi_{1/xx}\, EJ\, \phi_{1/xx}\, dx}_{K_{11}}\, c_1 - \delta c_1\, \omega^2 \underbrace{\int_0^\ell \phi_1\, I_o\, \phi_1\, dx}_{M_{11}}\, c_1\Big] e^{i\omega t} = 0$$

The problem is then in the form

$$\left(K_{11} - \omega^2 M_{11}\right) c_1 = 0$$

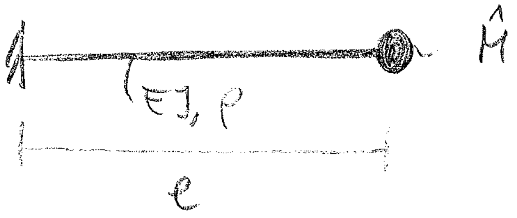
where:

$$K_{11} = \frac{4\pi^3}{3\ell^3} EJ_o \qquad M_{11} = I_o\, \frac{\ell}{2}$$

The first circular frequency is then obtained as:

$$\omega_1 = \sqrt{\frac{K_{11}}{M_{11}}} = \sqrt{\frac{EJ_0}{I_0} \frac{8\pi^3}{3\ell^4}}$$

Exercise



Evaluate the first natural frequency considering a polynomial expansion.
Evaluate the possibility of considering one-dof approximations using static deflections.

Weak-form

$$\delta W_i = \int_0^l \delta w_{/xx} \, EJ \, w_{/xx} \, dx$$

$$\delta W_e = -\int_0^l \delta w \, I_0 \, \ddot{w} \, dx - \delta w(l) \, \hat{M} \, \ddot{w}(l)$$

Ritz functions

The simplest choice is to consider a polynomial expansion where:

$$w = c_1 x^2 \quad \Rightarrow \quad \phi_1 = x^2$$

(the constant and linear contributions are not accounted for as they are not respectful of the boundary conditions in $x=0$)

$$\begin{cases} w(0) = 0 \\ w_{/x}(0) = 0 \end{cases} \quad \leftarrow \text{essential conditions}$$

Upon substitution in the weak form, it is obtained:

$$\delta c_1 \int_0^\ell \phi_{1/xx} \, EJ \, \phi_{1/xx} \, dx \; c_1 = -\delta c_1 \int_0^\ell \phi_1 \, I_0 \, \phi_1 \, dx \; \ddot{c}_1$$
$$- \delta c_1 \; \phi_1(\ell) \, \hat{M} \, \phi_1(\ell) \; \ddot{c}_1$$

or

$$\delta c_1 \, k_{11} \, c_1 = - \delta c_1 \, M \ddot{c}_1 \qquad \forall \, \delta c_1$$

where

$$k_{11} = \int_0^\ell \phi_{1/xx} \, EJ \, \phi_{1/xx} \, dx = 4 EJ \ell$$

$$M_{11} = \int_0^\ell \phi_1 \, I_0 \, \phi_1 \, dx + \phi_1(\ell) \, \hat{M} \, \phi_1(\ell) = \ell^4 \left(\frac{1}{5} I_0 \, \ell + \hat{M} \right)$$

The equation of motion is:

$$M_{11} \ddot{c}_1 + k_{11} c_1 = 0$$

and setting $c_1 = \hat{c}_1 \, e^{i \omega t}$ it is

$$\left(-\omega^2 M_{11} + k_{11} \right) c_1 = 0$$

and so:

$$\omega^2 = \frac{k_{11}}{M_{11}}$$

Ritz functions - 2 terms

Expand now up to the 3^{rd} order:

$$W = C_1 X^2 + C_2 X^3 \implies \phi_1 = x^2 ; \ \phi_2 = x^3$$

The expression of the stiffness and mass matrices is still in the form:

$$K_{ij} = \int_0^\ell \phi_{i/xx} \, EJ \, \phi_{j/xx} \, dx$$

$$M_{ij} = \int_0^\ell \phi_i \, I_0 \, \phi_j \, dx + \phi_i(\ell) \, \hat{M} \, \phi_j(\ell)$$

The terms K_{11} and M_{11} are already available.
The novel contributions to be evaluated are:

$$K_{12} = \int_0^\ell \phi_{1/xx} \, EJ \, \phi_{2/xx} \, dx$$

$$= 6EJ \, \ell^2$$

$$M_{12} = \int_0^\ell \phi_1 \, I_0 \, \phi_2 \, dx + \phi_1(\ell) \, \hat{M} \, \phi_2(\ell)$$

$$= \ell^5 \left(\frac{1}{6} I_0 \ell + \hat{M} \right)$$

and

$$K_{22} = \int_0^\ell \phi_{2/xx} \, EJ \, \phi_{2/xx} \, dx$$

$$= 12 EJ \, \ell^3$$

$$M_{22} = \int_0^{\ell} \phi_2 \, I_0 \, \phi_2 \, dx + \phi_2(\ell) \, \hat{M} \, \phi_2(\ell)$$

$$= \ell^6 \left(\frac{1}{7} I_0 \, \ell + \hat{M} \right)$$

Ritz functions - static deflections

The accuracy to dof ratio can be improved if the shape functions are properly selected.
While a polynomial expansion in the form
$w = c_i x^{i+1}$ with $i = 1 \dots N$ certainly guarantees the convergence (the trial functions are a complete set of functions), it can be necessary to increase the number of dofs in order to achieve the desired level of accuracy.
A good approach for obtaining a good solution using one single dof consists in selecting a static deflected shape. How is it chosen?
One possibility is to consider a cantilever beam with a concentrated force at the tip. It is easy to understand that this kind of shape would be appropriate when the tip mass $\hat{M}$ is much greater with respect to the mass associated with the beam itself, i.e.
$\hat{M} \gg I_o l$

A second possibility would be to consider the deflected shape associated with a uniformly distributed load.
This case would represent the effect of distributed inertial forces whenever they are much greater with

respect to the inertial force due to the tip mass; in other words, this choice would be the most natural one whenever

$$I_0 \ell >> \overset{1}{M}$$

Consider now the two cases:

1. <u>$\overset{1}{M} >> I_0 \ell$</u>

The deflected shape is selected by solving the problem:

$$\begin{cases} EJ\, w_{/xxxx} = 0 \\ w(0) = 0 \\ w_{/x}(0) = 0 \\ w_{/xx}(\ell) = 0 \\ w_{/xxx}(\ell) = -\dfrac{P}{EJ} \end{cases}$$

$w(0)=0$, $w_{/x}(0)=0$ → essential conditions

$w_{/xx}(\ell)=0$, $w_{/xxx}(\ell)=-P/EJ$ → natural conditions

The solution of the problem is obtained after integrating the equilibrium equation and imposing the boundary conditions. It is obtained as:

$$w = \frac{P}{6EJ}\left(3x^2\ell - x^3\right)$$

The Ritz function is then taken as:

$$w = q_1 \phi_1 \quad \text{with} \quad \phi_1 = 3x^2 l - x^3$$

Note that the static analysis allows to obtain one single dof expansion where both quadratic and cubic terms are accounted for. Their relative weight is indeed given by the solution of the static problem.

The stiffness and the mass matrix are now obtained as:

$$K_{11} = \int_0^l \phi_{1/xx} EJ \phi_{1/xx} \, dx$$

$$= 12 EJ l^3$$

$$M_{11} = \int_0^l \phi_1 I_0 \phi_1 \, dx + \phi_1(l) \hat{M} \phi_1(l)$$

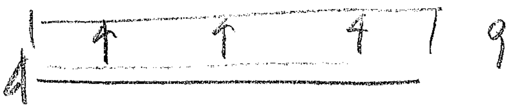
$$= l^6 \left(\frac{33}{35} I_0 l + 4 \hat{M} \right)$$

and so:

$$\omega^2 = \frac{K_{11}}{M_{11}}$$

2. $I_0 \, l \gg \hat{M}$

The deflected pattern is now taken by considering the following problem



$$\begin{cases} EJ \, w_{,xxxx} = q \\ w(0) = 0 \\ w_{,x}(0) = 0 \\ w_{,xx}(0) = 0 \\ w_{,xxx}(0) = 0 \end{cases}$$

$w(0)=0$, $w_{,x}(0)=0$ ← essential conditions

$w_{,xx}(0)=0$, $w_{,xxx}(0)=0$ ← natural conditions

The problem is solved as:

$w = w^H + w^P$

w^P — particular integral

w^H — homogeneous integral

where $w^P = \frac{1}{24} \frac{q}{EJ} x^4$

$$w^H = A_0 + A_1 x + A_2 x + A_3 x^3$$

Imposing the boundary conditions it is:

$$w = \frac{1}{24} \frac{q}{EJ} \left(x^4 - 4x^3 l + 6x^2 l^2 \right)$$

The single-dof Ritz representation is then

$$W = c_1 \phi_1 \quad \text{with} \quad \phi_1 = x^4 - 4x^3 l + 6x^2 l$$

where ϕ_1 is now including the polynomial terms up to the fourth-order. Note that the presence of the fourth-order contribution does not imply better accuracy with respect to the previous choice of the shape (that was including the quadratic and cubic terms only). Indeed the higher order contribution is related to the other one by pre-defined weighting factors (x^4, x^3 and x^2 are not independent each other). These weighting factors are going to be a proper choice when $I_o l >> \hat{M}$. On the contrary the weighting factors associated with the previous choice are going to be the best choice when $\hat{M} >> I_o l$.

The stiffness and mass contributions are now obtained as:

$$k_{11} = \frac{144}{5} EJ l^5$$

$$M_{11} = l^8 \left(\frac{104}{45} I_o l + 9M \right)$$

and so:

$$\omega_1 = k_{11}/M_{11}$$

Comparison of results

The results obtained with the different approaches are now compared in terms of predicted circular frequencies ω. More specifically the comparison is conducted by expressing ω in the nondimensional form:

$$\bar{\omega} = \omega \sqrt{\frac{\ell^3 M}{EJ}}$$

The adimensionalization is introduced in order to make the results independent on the specific values chosen for M, ℓ and EJ.

Consider now two cases: one where $M >> I_0 \ell$ and another one where $M << I_0 \ell$

1. Case 1: $M = 10\, I_0 \ell$

(Consider $I_0 = 1$, $\ell = 1$; $EJ = 1$; $M = 10$)

	polynomial	tip load	uniform load
N = 1	1.9803	1.7120	1.7663
N = 2	1.7120		
N = 3	1.7120		
N = 4	1.7120		

Values of $\bar{\omega}$

estimate of the exact frequency value.

The convergence is from above as it is a ratio between the stiffness and the mass properties of the structure (recall, for the 1 dof case, $\omega = \sqrt{k/m}$): as the number of dof is increased, the stiffness of the structure gets smaller and, consequently, the value of the frequency reduces.

2. Case 2: $M = \frac{1}{1000} I_0 \ell$

(Consider $I_0 = 1$; $\ell = 1$; $EJ = 1$; $M = \frac{1}{1000}$)

	polynomial	Tip load	uniform
N = 1	1.4107	1.1258	1.1141
N = 2	1.1149		
N = 3	1.1100		
N = 4	1.1096		

Values of $\overline{w} \cdot 10$

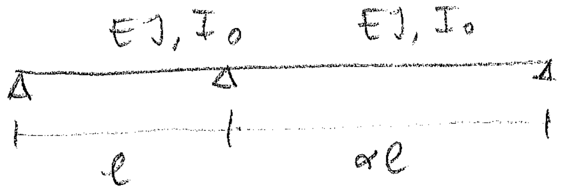
<u>Comments to the results</u>

1 In the first case the choice of the tip load static deflection turns out to be the most efficient one. The one-dof approximation leads to better results with respect to the uniform load deflection as well as the polynomial one.

2. In the second case, as expected, the uniform load static deflection is the most appropriate one. The one-dof solution is more accurate with respect to the tip-load approximation and the polynomial expansion with N = 1 and 2.

3 The convergence of the solution is <u>from above</u> and the finite approximation is an <u>upper bound</u>

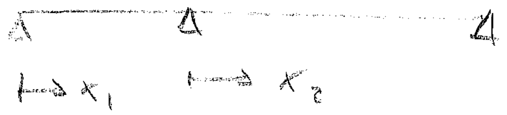
Exercise



Determine the first natural frequency by using a single-dof approximation

Weak form

Introduce two reference systems as illustrated in the figure

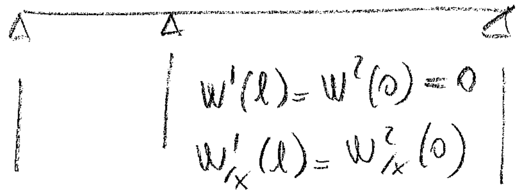


$$\delta W_i = \int_0^{l} \delta w^1_{/xx}\, EJ\, w^1_{/xx}\, dx_1 + \int_0^{\alpha l} \delta w^2_{/xx}\, EJ\, w^2_{/xx}\, dx_2$$

$$\delta W_e = -\int_0^{l} \delta w^1 I_0\, \ddot{w}^1 dx_1 - \int_0^{\alpha l} \delta w^2 I_0\, \ddot{w}^2 dx_2$$

Ritz functions

The essential conditions of the problem are summarized here below



$$w^1(l) = w^2(0) = 0$$
$$w^1_{/x}(l) = w^2_{/x}(0)$$

$$w^1(0) = 0 \qquad\qquad w^2(\alpha l) = 0$$

A possible choice is given by the adoption of sine-type functions in the form

$$w^1 = c_1 \phi_1 \quad \text{with} \quad \phi_1 = \sin \frac{\pi x_1}{\ell}$$

$$w^2 = c_2 \phi_2 \quad \text{with} \quad \phi_2 = \sin \frac{\pi x_2}{\alpha \ell}$$

The two functions clearly satisfy the essential conditions at the two outer simply-supported conditions:

$$w^1(0) = c_1 \sin(0) = 0 \quad \checkmark$$

$$w^2(\alpha\ell) = c_2 \sin\pi = 0 \quad \checkmark$$

The inner condition regarding the transverse deflection is satisfied as well:

$$w^1(\ell) = w^2(0) = 0 \quad \checkmark$$

The essential condition on the rotation is not satisfied yet, and has to be imposed. In particular:

$$w^1_{/x} = \frac{\pi}{\ell} \cos \frac{\pi x_1}{\ell} \, c_1$$

$$w^2_{/x} = \frac{\pi}{\alpha\ell} \cos \frac{\pi x_2}{\alpha\ell} \, c_2$$

Imposing now:

$$w^1_{/x}(\ell) = w^2_{/x}(0)$$, it is obtained

$$\frac{\pi}{\ell} \cos\pi \, c_1 = \frac{\pi}{\alpha\ell} \cos(0) \, c_2$$

and so:

$$-\frac{\pi}{\ell} c_1 = \frac{\pi}{\alpha \ell} c_2$$

or

$$c_2 = -\alpha c_1$$

The one-dof representation of the deflected patterns is then:

$$w^1 = c_1 \sin\frac{\pi x_1}{\ell} \quad \text{and} \quad w^2 = -c_1 \alpha \sin\frac{\pi x_2}{\alpha \ell}$$

or $$\phi_1 = \sin\frac{\pi x_1}{\ell} \quad \text{and} \quad \phi_2 = -\alpha \sin\frac{\pi x_2}{\alpha \ell}$$

where c_1 is the degree of freedom associated with the expansion of w_1 and w_2.

The stiffness matrix is obtained as:

$$k_{11} = \int_0^{\ell} \phi_{1/xx} EJ \phi_{1/xx} \, dx_1 + \int_0^{\alpha\ell} \phi_{2/xx} EJ \phi_{2/xx} \, dx_2$$

$$= \frac{EJ \pi^4 (\alpha+1)}{2\alpha \ell^3}$$

whilst the mass matrix reads

$$M_{11} = \int_0^{\ell} \phi_1 I_0 \phi_1 \, dx_1 + \int_0^{\alpha\ell} \phi_2 I_0 \phi_2 \, dx_2$$

$$= \frac{1}{2} I_0 \ell \left(1+\alpha^3\right)$$

and then:

$$\omega_1^2 = \frac{K_{11}}{M_{11}} = \frac{EJ\pi^4}{(\alpha^2-\alpha+1) I_0 \ell^4 \alpha}$$

when $\alpha = 1$ (the two beams have equal length),
the square of the circular frequency is:

$$\omega_1^2 = \frac{EJ\pi^4}{I_0 \ell^4}$$

corresponding to the value of ω_1^2 obtained for
one single simply-supported beam