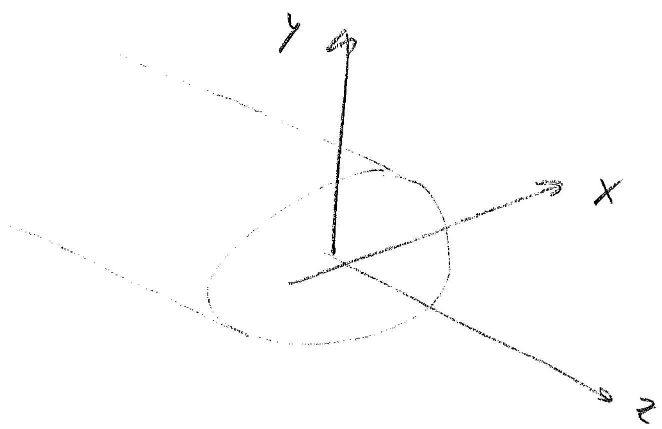
Introduction to beams and De Saint Venant Solution

Consider a 3D body, characterized by the following features:

1. the body is slender, with one dimension which is much greater in comparison to the two other directions.
2. the body is obtained by the rigid translation of the section, which is equal to itself along the beam axis.
3. the material is isotropic, elastic and homogeneous (beams with sections made of two different materials, i.e. non-homogeneous sections, are not considered)
4. loads and constraints are applied at the extremal sections.

Goal of the investigation is to determine the state of stress for a generic section, provided it is far enough from where the loads are applied (and the constraints as well, as far as they are responsible for the introduction of the reaction forces).

The 3D body, hereinafter denoted as beam, is represented as:



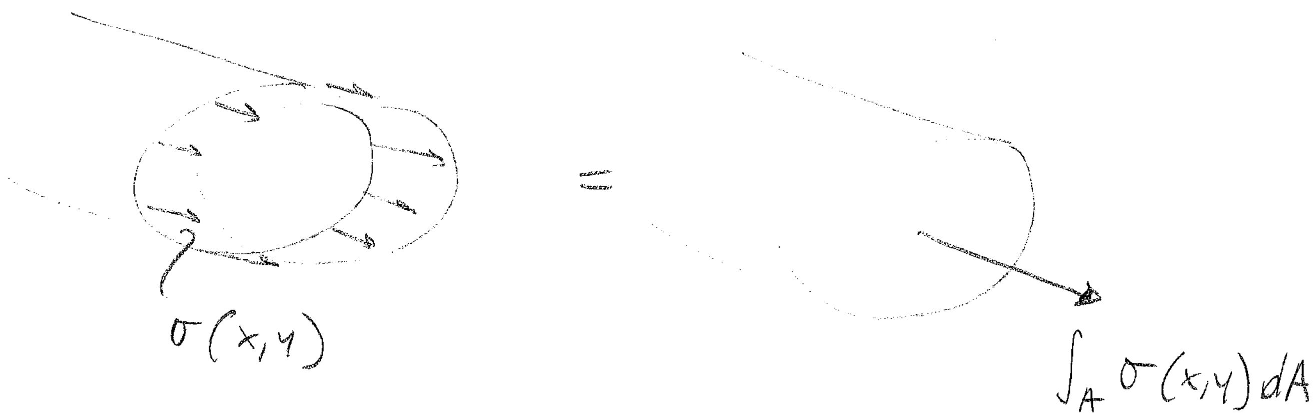
where: z: beam axis

xy: plane of the section

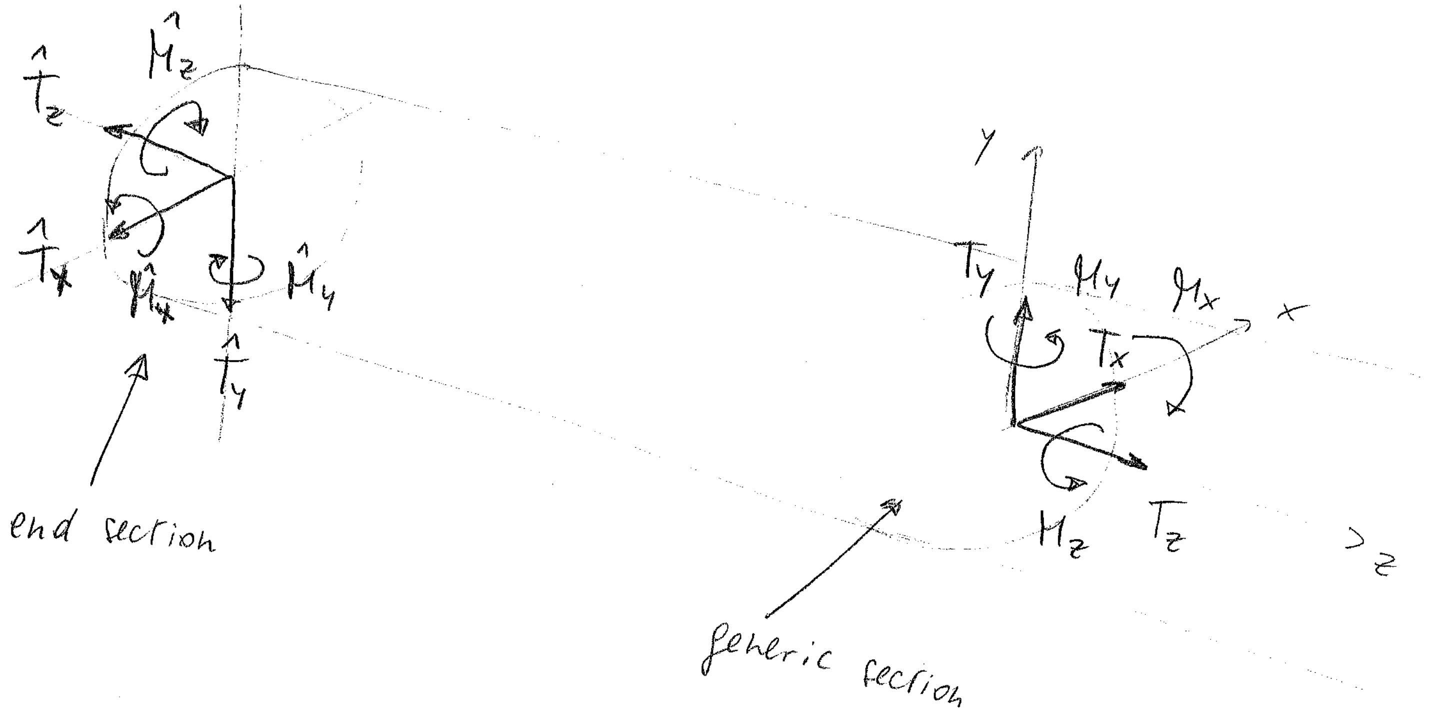
In addition, xyz is taken s.t.:

$$\left.\begin{aligned} &\int_A x\,dA = \int_A y\,dA = 0 \\ &\int_A xy\,dA = 0 \end{aligned}\right\}$$ principal centroidal axes

- It is remarked that the loads, as reported at the point 4, are applied at the ends. Notice, in addition, that the distribution of the loads over the surface is not relevant for the present investigation. The only information that is relevant is the resultant of the surface forces



Consider a generic system of applied loads, which are now considered in terms of resultants (forces and moments)



The applied loads are denoted with the ^ symbol. They are taken such that

1. The internal shear and axial forces are positive
2. The internal bending and twisting moments are positive

according to the right-hand rule convention.

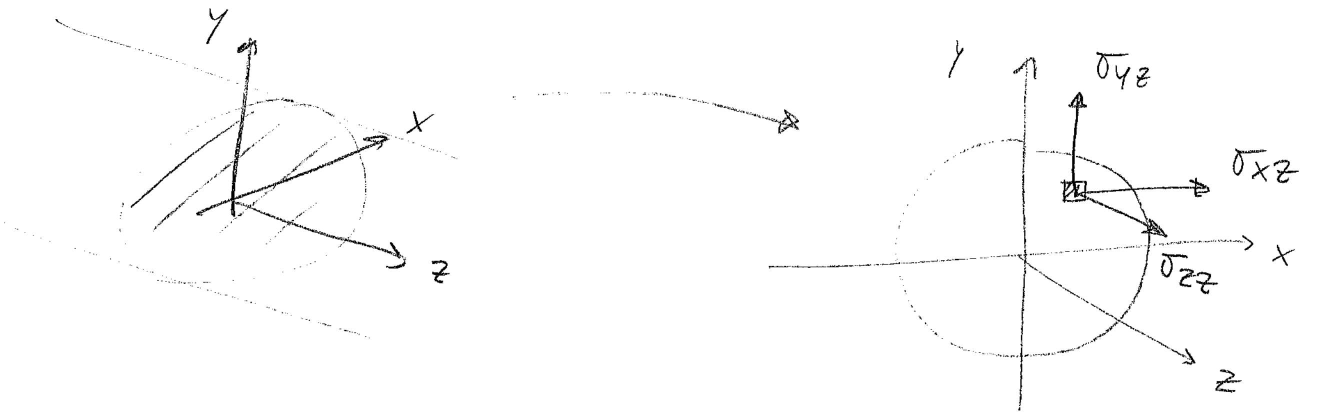
From equilibrium considerations, it is observed that

$$\begin{cases} T_x = \hat{T}_x \\ T_y = \hat{T}_y \\ T_z = \hat{T}_z \\ M_x = \hat{M}_x + z\,T_y \\ M_y = \hat{M}_y - z\,T_x \\ M_z = \hat{M}_z \end{cases}$$

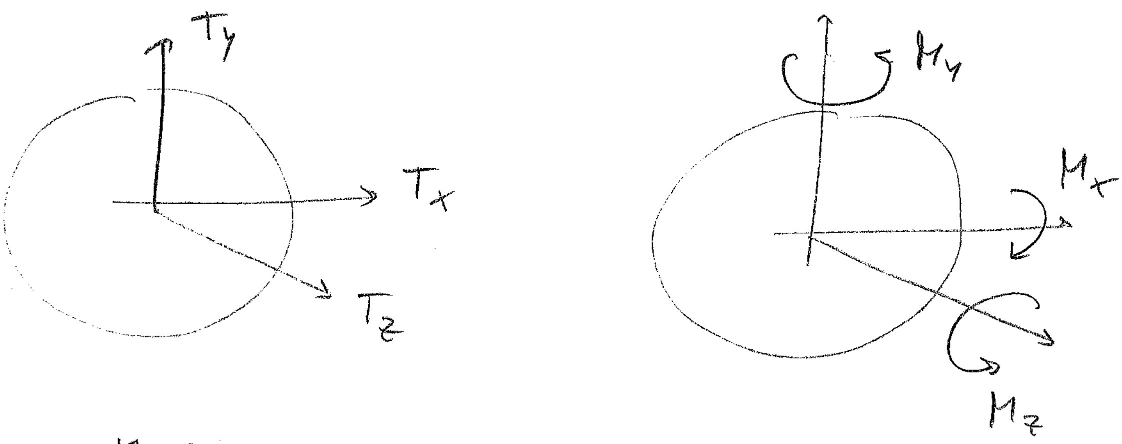
$\Rightarrow$ the shear is constant
the axial force is constant

$\Rightarrow$ the bending moments can be constant or linear

It is also remarked that the resultants are the integrals over the beam section of the stress components σ_{zz}, σ_{xz} and σ_{yz}. In particular



and recalling the convention for the positive internal actions:



It follows that:

$$T_x = \int_A \sigma_{xz}\, dA \qquad M_x = \int_A \sigma_{zz}\, y\, dA$$

$$T_y = \int_A \sigma_{yz}\, dA \qquad M_y = \int_A -\sigma_{zz}\, x\, dA$$

$$T_z = \int_A \sigma_{zz}\, dA \qquad M_z = \int_A \left(\sigma_{yz}\, x - \sigma_{xz}\, y\right) dA$$

Note that the internal actions are statically equivalent to the stress resultants, and <u>do not</u> express the equilibrium with respect to the internal state of stress.

- Formulation of the problem

The starting point are the set of equations:

$$
\begin{cases}
\operatorname{div} \underline{\underline{\sigma}} + \underline{f} = \underline{0} & \text{in } \Omega \\
\underline{\underline{\varepsilon}} - \frac{1}{2}\left(\operatorname{grad} \underline{u} + \operatorname{grad} \underline{u}^T\right) = \underline{\underline{0}} & \text{in } \Omega \\
\underline{\underline{\sigma}} \cdot \hat{\underline{n}} = \hat{\underline{t}} & \text{in } S_F \\
\underline{u} = \hat{\underline{u}} & \text{in } S_u
\end{cases}
$$

How can this problem be solved for the structure under investigation (3D slender body, ...)?

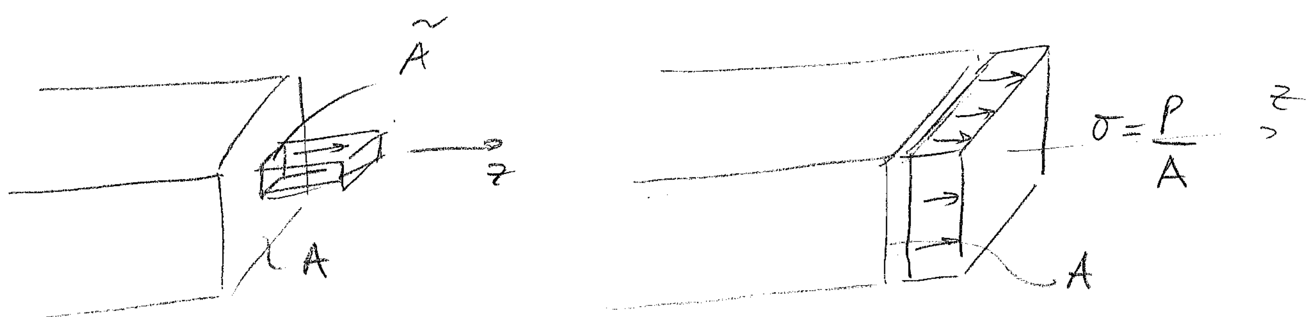
1. Instead of dealing with the compatibility requirement $\underline{\underline{\varepsilon}} - \frac{1}{2}\left(\operatorname{grad} \underline{u} + \operatorname{grad} \underline{u}^T\right) = \underline{\underline{0}}$, the compatibility equations are introduced.

2. Instead of obtaining a solution from the equations, an initial guess solution is proposed. It is checked ex-post if the solution satisfies the relevant equations. If this is the case it means that the guess solution is the exact one. This approach is called semi-inverse approach

3. Goal of the investigation is obtaining a solution far from the end sections where loads (also in the form of reaction forces) are applied.

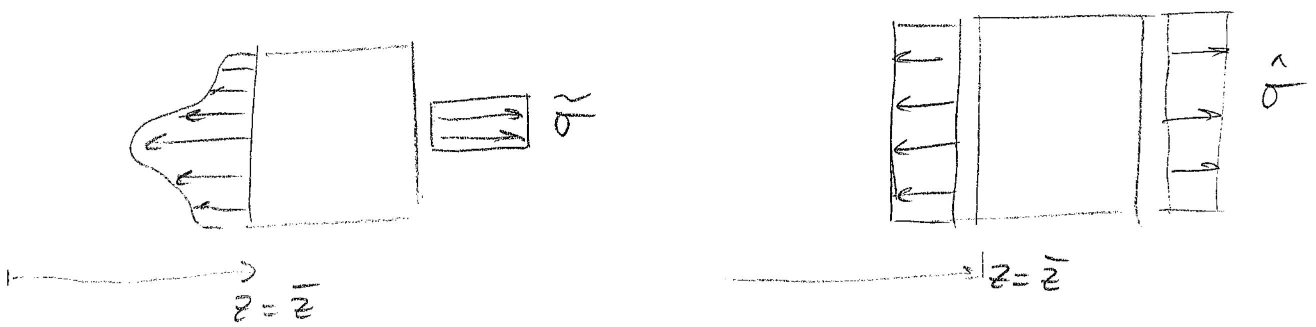
This means that the solution will be required to provide equilibrium at any section to the external loads.
No Cauchy equilibrium is involved (with regard to the sections) in the derivation of the solution.
This is, in fact, consistent with the idea of neglecting the actual shape of the applied loads and considering their resultants.

To clarify consider these two cases



The two set of loads are statically equivalent, i.e. $\tilde{\sigma}\tilde{A} = \sigma A$, however their distribution is different

Consider now the exact 3D solution in a section close to the loaded area.



Internal stress distribution (component σ_{zz}) at $z = \bar{z}$

It is clear that both the solutions satisfy the equilibrium and the compatibility equations. However they are different because the boundary conditions over the load section are different.

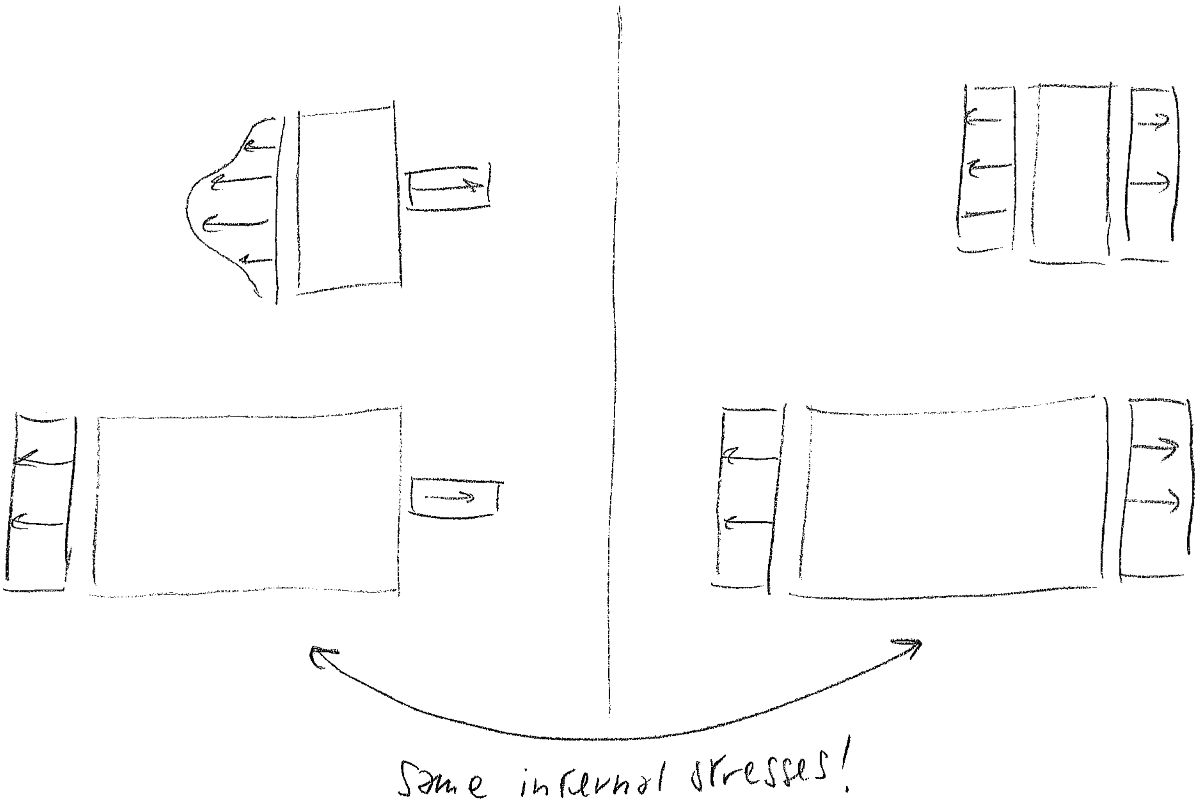
$\underline{\underline{\sigma}} \cdot \underline{n} = \tilde{\sigma} \underline{e}_3$ $\qquad\qquad$ $\underline{\underline{\sigma}} \cdot \underline{n} = \hat{\sigma} \underline{e}_3$

At the same time the resultant of the stresses in the two sections is the same (however the equilibrium would not be fulfilled)

De Saint Venant postulates that the two solutions, when calculated at a distance of the order of the maximum size of the section, are equal.

This means that <u>local effects</u> have disappeared (they have been spatially damped out).

This is known as <u>De Saint Venant principle</u>



The DSV principle is exactly the reason for having assumed at the beginning that 1. the load shape is not relevant, 2. the solution is sought for away from the extremal sections.

It follows that the boundary conditions relative to the section area, which could be imagined as $\underline{\underline{\sigma}} \cdot \underline{n} = \hat{\underline{t}}$, are replaced by a sort of "<u>integral condition</u>", in the form

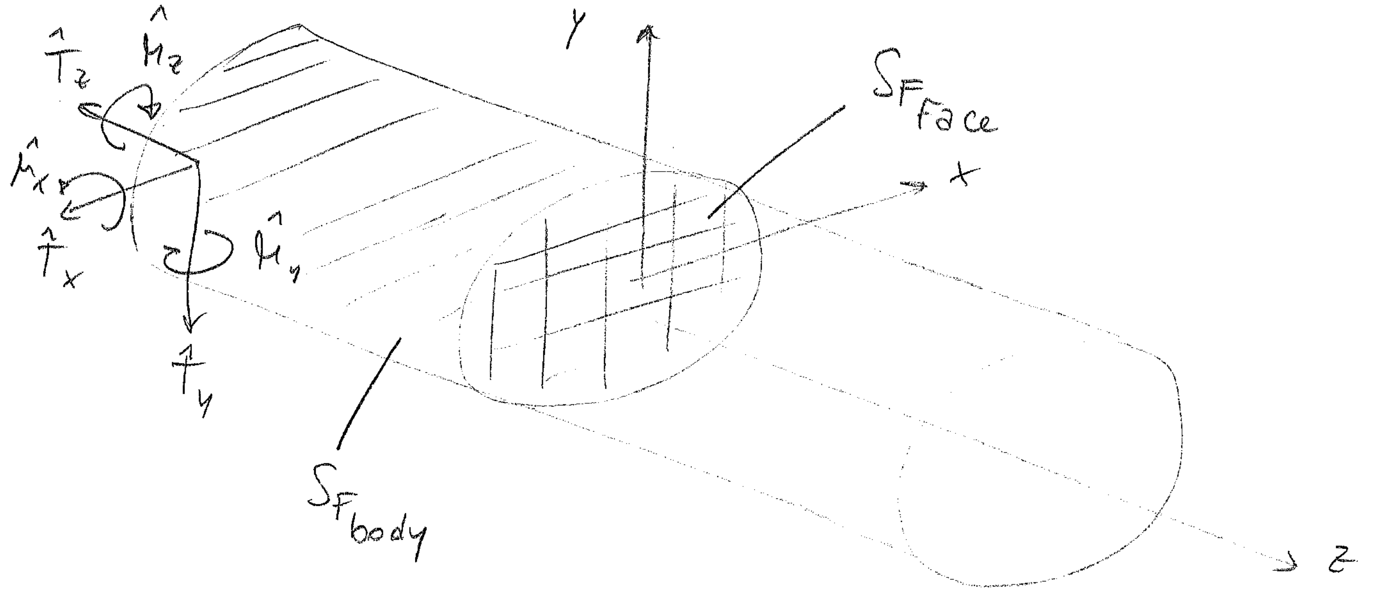
$$T_z = \int_A \sigma_{zz}\, dA \qquad M_x = \int_A \sigma_{zz}\, y\, dA$$

$$T_y = \int_A \sigma_{zy}\, dA \qquad M_y = \int_A -\sigma_{zz}\, x\, dA$$

$$T_x = \int_A \sigma_{zx}\, dA \qquad M_z = \int_A \left(\sigma_{zy}\, x - \sigma_{zx}\, y\right) dA$$

which are intrinsically independent on how the stresses are distributed, but depend only on their integrals over A.

The problem can thus imagined as it follows:



where: • $S_{F_{body}}$ is the external surface of the body. No loads are applied here (thus the prescribed force is zero, while the displacements are part of the solution)

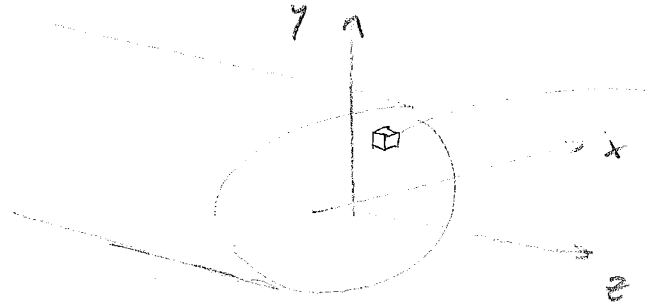
• $S_{F_{face}}$ is the beam cross-section. Here the DSV hypothesis takes place, and the equivalence with the internal actions is imposed. These latest can be seen as "integral" conditions, as they regard the integrals of the stresses over the section.

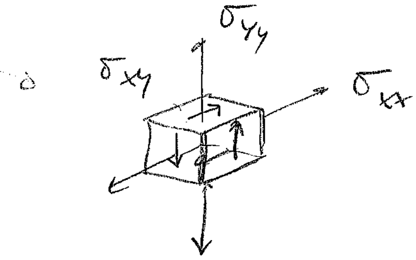
The problem is then formulated as:

$\operatorname{div} \underline{\underline{\sigma}} = 0$	$(\underline{f} = 0)$	in Ω
compatibility eqs		in Ω
$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0}$		in $S_{F_{body}}$
equivalence with internal actions		in $S_{F_{face}}$

• Solution via semi-inverse method

The guess solution is taken as $\boxed{\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0}$



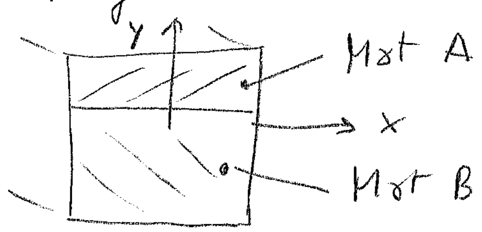


• Clarly it is not hard to understand that this is a reasonable guess!

• The procedure will be the following: if the guess solution satisfies the differential problem in the box, then it is the exact solution (unicity of the solution is guaranteed in linear elasticity).

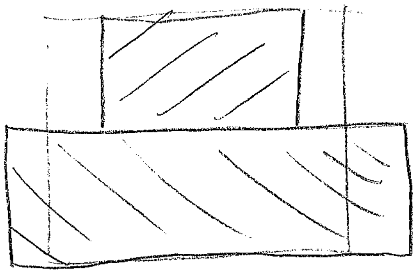
• Important remark: the approach here outlined relies upon the assumptions of isotropic material and homogeneous section. If this is not the case, as it happens in the case of composite materials, the stress state will be in general 3D. ($\Rightarrow \sigma_{xx} \neq 0;\ \sigma_{yy} \neq 0;\ \sigma_{xy} \neq 0$). In this case the classical DSV solution cannot be achieved in the analytical form here presented.

Imagine a cross section composed of two different materials:

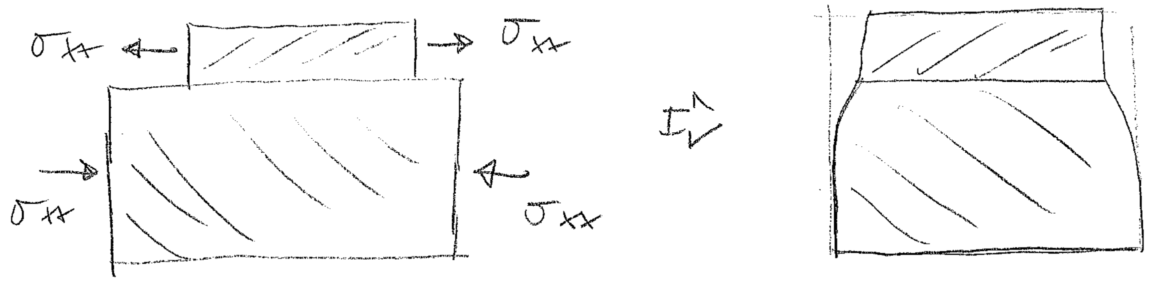


The different elastic properties will determine the onset of σ_{xx}, σ_{yy} and σ_{xy} to restore the compatibility conditions between the two parts

Different Poisson's ratio would, in fact, determine a deformed cross-section as



Proper internal stress must then arise to restore the compatibility



uncompatible

compatibility restored $\Rightarrow \sigma_{xx} \neq 0$

This simple example illustrate the concept by highlighting the contribution σ_{xx}. However it should be clear that in general the state of stress will be <u>3D</u> for a non homogeneous and/or non isotropic section.

Just to avoid misunderstandings: for a composite beam the DSV solution can still be obtained, but the approach to be used is different from the semi-inverse strategy here illustrated.

Equilibrium equations

Begin with the equilibrium equations

$$\begin{cases} \boxed{\sigma_{xx/x}} + \boxed{\sigma_{xy/y}} + \sigma_{xz/z} = 0 \\ \boxed{\sigma_{xy/x}} + \boxed{\sigma_{yy/y}} + \sigma_{yz/z} = 0 \\ \sigma_{xz/x} + \sigma_{yz/y} + \sigma_{zz/z} = 0 \end{cases}$$

$\boxed{}$: $= 0$ as the guess solution is $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$

$$\Downarrow$$

$$\boxed{\begin{cases} \sigma_{xz/z} = 0 \\ \sigma_{yz/z} = 0 \\ \sigma_{xz/x} + \sigma_{yz/y} + \sigma_{zz/z} = 0 \end{cases}}$$

Result R1

- The first two equations reveal that:

$$\sigma_{xz} = \sigma_{xz}(x,y)$$
$$\sigma_{yz} = \sigma_{yz}(x,y)$$

(they do not depend on z)

- The last equation reveals that:

σ_{zz} can be up to a <u>linear</u> function of z

$$\Big(\underbrace{\sigma_{xz/x} + \sigma_{yz/y}}_{\text{depend on } x,y} + \underbrace{\sigma_{zz/z}}_{} = 0$$

depend on x, y

$\sigma_{zz/z}$ depends on x, y

$\Rightarrow \sigma_{zz}$ can be linear in z

Constitutive law

Recall that the solid under investigation is a 3D solid. As such, consider the 3D elastic law:

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}} \quad \rightsquigarrow \quad \underset{6\times1}{\{\sigma\}} = \underset{6\times6}{[C]}\underset{6\times1}{\{\varepsilon\}} \quad \Rightarrow \quad \{\varepsilon\} = [S]\{\sigma\}$$

for a 3D isotropic hyperelastic material, the strain-stress relation reads:

$$\varepsilon_{xx} = \frac{1}{E}\left(\boxed{\sigma_{xx} - \nu\sigma_{yy}} - \nu\sigma_{zz}\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\boxed{-\nu\sigma_{xx} + \sigma_{yy}} - \nu\sigma_{zz}\right)$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\boxed{-\nu\sigma_{xx} - \nu\sigma_{yy}} + \sigma_{zz}\right)$$

$$\gamma_{xy} = \frac{1}{G}\boxed{\sigma_{xy}}$$

$$\gamma_{yz} = \frac{1}{G}\sigma_{yz}$$

$$\gamma_{xz} = \frac{1}{G}\sigma_{xz}$$

$\boxed{} : = 0$ as $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$

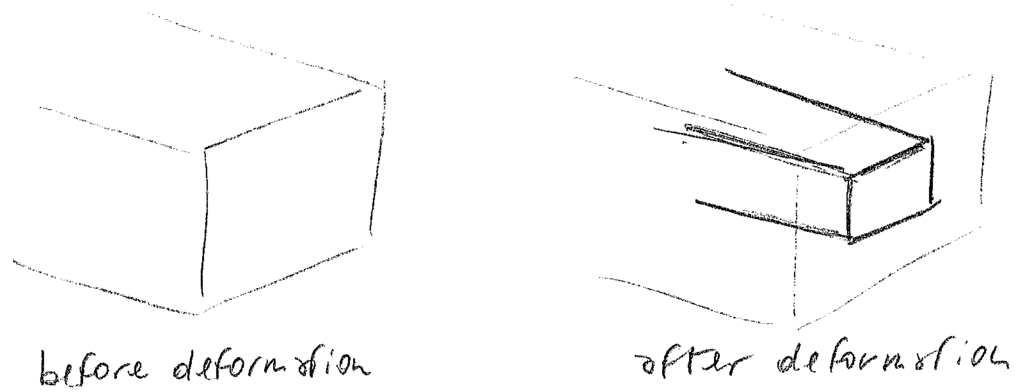
It follows that:

$$\boxed{\begin{aligned} \varepsilon_{xx} &= -\frac{\nu}{E}\sigma_{zz} & \gamma_{xy} &= 0 \\ \varepsilon_{yy} &= -\frac{\nu}{E}\sigma_{zz} & \gamma_{yz} &= \frac{1}{G}\sigma_{yz} \\ \varepsilon_{zz} &= \frac{\sigma_{zz}}{E} & \gamma_{xz} &= \frac{1}{G}\sigma_{xz} \end{aligned}}$$

Result R2

Note that, according to the guess solution (which will be demonstrated to be the exact one) the not null components of ε_{ik} are 5.

More specifically, the presence of an axial stress σ_{zz} implies that $\varepsilon_{xx} \neq 0$ and $\varepsilon_{yy} \neq 0$



Thus the section changes its original shape (in contrast to the common wrong assumption that the section remains equal to itself after the deformation process)

Compatibility equations

$$
\left\{
\begin{aligned}
&\varepsilon_{xx/yy} + \varepsilon_{yy/xx} = \boxed{\gamma_{xy/xy}} \\
&\boxed{\varepsilon_{yy/zz}} + \varepsilon_{zz/yy} = \gamma_{yz/yz} \\
&\varepsilon_{zz/xx} + \boxed{\varepsilon_{xx/zz}} = \gamma_{zx/zx} \\
&2\varepsilon_{xx/yz} = \boxed{\gamma_{xy/xz}} + \gamma_{xz/xy} - \gamma_{yz/xx} \\
&2\varepsilon_{yy/zx} = \gamma_{yz/yx} + \boxed{\gamma_{yx/yz}} - \gamma_{zx/yy} \\
&2\varepsilon_{zz/xy} = \gamma_{zx/zy} + \gamma_{zy/zx} - \boxed{\gamma_{xy/zz}}
\end{aligned}
\right.
$$

where:

$\boxed{}$: $= 0$ as $\gamma_{xy} = 0$ (from R2)

: $= 0$ as $\sigma_{xz} = \sigma_{xz}(x,y)$, $\sigma_{yz} = \sigma_{yz}(x,y)$ (from R1) $\Rightarrow$ $\gamma_{xz/z} = 0$, $\gamma_{yz/z} = 0$ (from R2)

: $= 0$ as $\sigma_{zz/zz} = 0$ (from R1) $\Rightarrow$ $\varepsilon_{xx/zz} = 0$, $\varepsilon_{yy/zz} = 0$, $\varepsilon_{zz/zz} = 0$ (from R2) and so:

$$\begin{cases} \varepsilon_{xx/yy} + \varepsilon_{yy/xx} = 0 \\ \varepsilon_{zz/yy} = 0 \\ \varepsilon_{zz/xx} = 0 \\ 2\varepsilon_{xx/yz} = \gamma_{xz/xy} - \gamma_{yz/xx} \\ 2\varepsilon_{yy/zx} = \gamma_{yz/yx} - \gamma_{zx/yy} \\ 2\varepsilon_{zz/xy} = 0 \end{cases}$$

and expressing the compatibility condition in terms of stresses using the constitutive law (Result R2):

$$\begin{cases} \sigma_{zz/yy} + \sigma_{zz/xx} = 0 \\ \sigma_{zz/yy} = 0 \\ \sigma_{zz/xx} = 0 \\ -2\frac{\nu}{E}\sigma_{zz/yz} = \frac{1}{G}\left(\sigma_{xz/xy} - \sigma_{yz/xx}\right) \\ -2\frac{\nu}{E}\sigma_{yy/zx} = \frac{1}{G}\left(\sigma_{yz/yx} - \sigma_{zx/yy}\right) \\ \sigma_{zz/xy} = 0 \end{cases}$$

or, rearranging:

$$\boxed{\begin{array}{l} \sigma_{zz/yy} = 0 \\ \sigma_{zz/xx} = 0 \\ -\bar{\nu}\,\sigma_{zz/yz} = \sigma_{xz/xy} - \sigma_{yz/xx} \\ -\bar{\nu}\,\sigma_{yy/zx} = \sigma_{yz/xy} - \sigma_{zx/yy} \\ \sigma_{zz/xy} = 0 \end{array}}$$

Result R3

Summary of the governing equations

$$
\begin{cases}
\sigma_{xz/z} = 0 \\
\sigma_{yz/z} = 0 \\
\sigma_{xz/x} + \sigma_{yz/y} + \sigma_{zz/z} = 0
\end{cases}
$$

equilibrium

$$
\begin{cases}
\sigma_{zz/yy} = 0 \\
\sigma_{zz/xx} = 0 \\
\sigma_{zz/xy} = 0 \\
-\bar{\nu}\,\sigma_{zz/yz} = \sigma_{xz/xy} - \sigma_{yz/xx} \\
-\bar{\nu}\,\sigma_{yy/zx} = \sigma_{yz/xy} - \sigma_{zx/yy}
\end{cases}
$$

compatibility

+

Equivalence with internal actions in $S_{F_{face}}$

$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0}$ in $S_{F_{body}}$

Note that the constitutive relation has been used for obtaining the current expression of the compatibility equation.

Solution

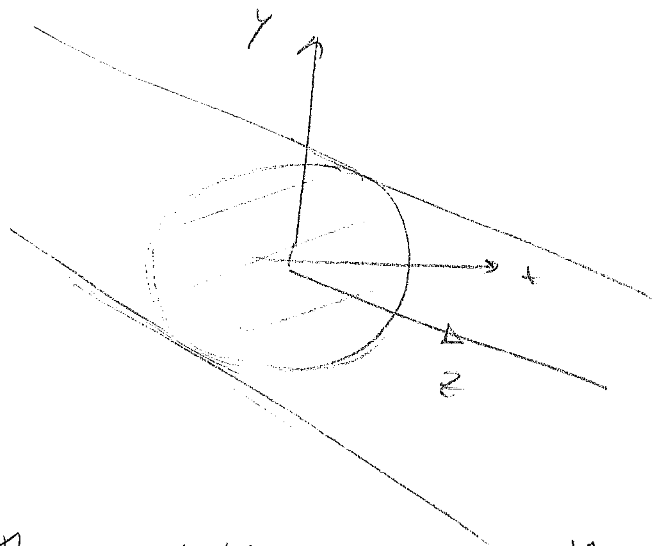
The solution is sought starting from the equilibrium

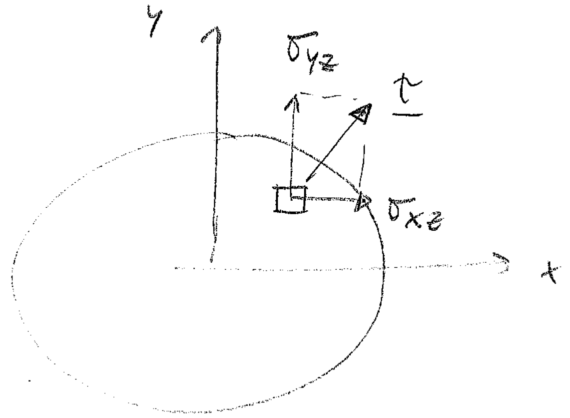
$$\begin{cases} \sigma_{xz/x} + \sigma_{yz/y} + \sigma_{zz/z} = 0 & \text{in } \Omega \\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} & \text{in } S_{F_{face}} \end{cases}$$

(For now only the equilibrium along the beam axis is considered.)

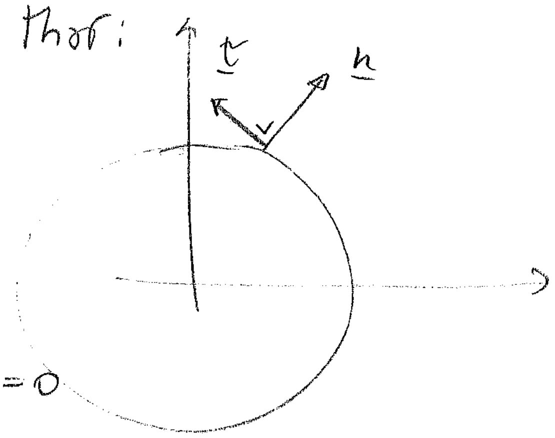
It is worth observing that the Cauchy condition $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0}$, expressing the equilibrium at the boundaries between the internal stresses and the applied ones (none), is here written as:

$$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} \quad \Rightarrow \quad \underline{t} \cdot \underline{n} = \underline{0} \quad \text{with } \underline{t} = \sigma_{xz}\, \underline{e}_1 + \sigma_{yz}\, \underline{e}_2$$





The condition requires that:



$$\underline{t} = \sigma_{xz}\, \underline{e}_1 + \sigma_{yz}\, \underline{e}_2$$

$$\underline{n} = n_x\, \underline{e}_1 + n_y\, \underline{e}_2$$

$$\underline{t} \cdot \underline{n} = 0 \quad \Rightarrow \quad \underline{t} \perp \underline{n} = 0$$

Integrate the equilibrium equation over the cross section:

$$\int_A \left(\sigma_{xz/x} + \sigma_{yz/y} + \sigma_{zz/z} \right) dA =$$

$$= \int_A \left(\operatorname{div} \underline{t} + \sigma_{zz/z} \right) dA =$$

$$= \oint_{\partial A} \underline{t} \cdot \underline{n} \, ds + \int_A \sigma_{zz/z} \, dA = 0$$

$$\Rightarrow \quad \boxed{\int_A \sigma_{zz/z} \, dA = 0}$$

Considering now the following results:

$$\begin{cases} \int_A \sigma_{zz/z} \, dA = 0 & \text{(equilibrium along beam axis)} \\ \sigma_{zz/yy} = \sigma_{zz/xx} = \sigma_{zz/xy} = 0 & \text{(compatibility)} \end{cases}$$

it is clear that σ_{zz} is expressed as:

$$\boxed{\sigma_{zz} = A_0 + A_1 x + A_2 y + \left(A_3 x + A_4 y \right) z}$$

Note that $\sigma_{zz/z} = A_3 x + A_4 y$ and

$$\int_A \sigma_{zz/z} \, dA = \int_A \left(A_3 x + A_4 y \right) dA$$

$$= A_3 \int_A x \, dA + A_4 \int_A y \, dA = 0$$ as the axes are principal centroidal

Internal actions

According to the definition of the internal actions:

- $T_z = \int_A \sigma_{zz}\, dA$

$$= A_0\, A \qquad \text{(as the axes are principal centroidal)}$$

- $M_x = \int_A \sigma_{zz}\, y\, dA$

$$= \int_A \left(\cancel{A_0 y} + \cancel{A_1 xy} + A_2 y^2 + \cancel{A_3 xyz} + A_4 y^2 z\right) dA$$

$$= J_{xx}\,(A_2 + z A_4)$$

- $M_y = -\int_A \sigma_{zz}\, x\, dA$

$$= -\int_A \left(\cancel{A_0 x} + A_1 x^2 + \cancel{A_2 xy} + A_3 x^2 z + \cancel{A_4 xyz}\right) dA$$

$$= -J_{yy}\,(A_1 + A_3 z)$$

Recalling that

$$\begin{cases} T_z = \text{const} = A_0 A \\ M_x = \hat{M}_x + z\, T_y = J_{xx}\,(A_2 + z A_4) \\ M_y = \hat{M}_y - z\, T_x = -J_{yy}\,(A_1 + A_3 z) \end{cases}$$

and equating the constant and the linear part:

$$A_0 = T_z / A \quad \Rightarrow \quad A_0 = T_z / A$$

$$T_y = J_{xx} A_4 \quad \Rightarrow \quad A_4 = T_y / J_{xx}$$

$$\hat{M}_x = J_{xx} A_2 \quad \Rightarrow \quad A_2 = \hat{M}_x / J_{xx}$$

$$T_x = J_{yy} A_3 \quad \Rightarrow \quad A_3 = T_x / J_{yy}$$

$$\hat{M}_y = - J_{yy} A_1 \quad \Rightarrow \quad A_1 = - \hat{M}_y / J_{yy}$$

It follows that:

$$\sigma_{zz} = A_0 + A_1 x + A_2 y + (A_3 x + A_4 y) z$$

$$= \frac{T_z}{A} - \frac{\hat{M}_y}{J_{yy}} x + \frac{\hat{M}_x}{J_{xx}} y + \left(\frac{T_x}{J_{yy}} x + \frac{T_y}{J_{xx}} y \right) z$$

$$= \frac{T_z}{A} - \frac{\hat{M}_y - T_x z}{J_{yy}} x + \frac{\hat{M}_x + T_y z}{J_{xx}} y$$

$$\boxed{\sigma_{zz} = \frac{T_z}{A} - \frac{M_y}{J_{yy}} x + \frac{M_x}{J_{xx}} y}$$

where $M_y = M_y(z)$

$M_x = M_x(z)$

- **Remarks**

1. The problem has been solved for σ_{zz}.
Starting from the guess solution, it was verified that equilibrium and compatibility conditions can be identically fulfilled if σ_{zz} has the expression here obtained.
The solution derived is the <u>exact</u> one. No approximations are involved in this procedure.
A stress field characterized by

$$\boxed{\begin{array}{l} \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0 \\ \sigma_{zz} = \dfrac{T_z}{A} - \dfrac{M_y}{J_{yy}}\,x + \dfrac{M_x}{J_{xx}}\,y \end{array}}$$

identically satisfies the equations highlighted below:

$$\left\{\begin{array}{ll}
\sigma_{xz/z} = 0 & \\
\sigma_{yz/z} = 0 & \text{in } \Omega \\
\boxed{\sigma_{xz/x} + \sigma_{yz/y} + \sigma_{zz/z} = 0} & \\
\boxed{\sigma_{zz/xx} = \sigma_{zz/yy} = \sigma_{zz/xy} = 0} & \\
-\bar{\nu}\,\sigma_{zz/yz} = \sigma_{xz/xy} - \sigma_{yz/xx} = 0 & \text{in } \Omega \\
-\bar{\nu}\,\sigma_{yy/zx} = \sigma_{yz/xy} - \sigma_{zx/yy} = 0 & \\
\boxed{\underline{\underline{\sigma}} \cdot \underline{n} = 0 \quad \text{in } S_{F\,body}} & \\
\boxed{T_z = \int_A \sigma_{zz}\,dA;\; M_x = \int_A \sigma_{zz}\,y\,dA;\; M_y = \int_A -\sigma_{zz}\,x\,dA} & \text{in } S_{F\,face} \\
T_y = \int_A \sigma_{zy}\,dA;\; T_x = \int_A \sigma_{zx}\,dA;\; M_z = \int_A \left(\sigma_{yz}\,x - \sigma_{xz}\,y\right) dA &
\end{array}\right.$$

2. If the shear stresses are to be evaluated, the problem to be solved is:

$$
\begin{cases}
\sigma_{xz/z} = 0 & \\
\sigma_{yz/z} = 0 & \text{in } \Omega \\
\sigma_{xz/x} + \sigma_{yz/y} + \sigma_{zz/z} = 0 & \\
-\bar{\nu}\, \sigma_{zz/yz} = \sigma_{xz/xy} - \sigma_{yz/xx} = 0 & \text{in } \Omega \\
-\bar{\nu}\, \sigma_{yy/zx} = \sigma_{yz/xy} - \sigma_{zx/yy} = 0 & \\
\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} & \text{in } S_{F\,body} \\
T_y = \int_A \sigma_{zy}\, dA ;\ T_x = \int_A \sigma_{zx}\, dA ,\ M_z = \int_A \left(\sigma_{yz} x - \sigma_{xz} y \right) dA & \text{in } S_{F\,face}
\end{cases}
$$

This problem is a complex one, and some approximations are generally needed to achieve a solution.

3. The solution $\sigma_{zz} = \frac{T_z}{A} - \frac{M_y}{J_y} x + \frac{M_x}{J_x} y$:

(a.) is obtained assuming principal centroidal axes

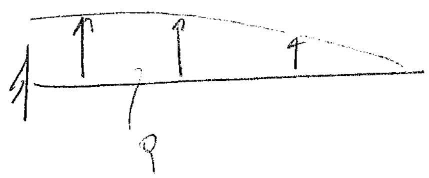
(b.) is an exact solution under the DSV hypothesis (constant section, material..., loads at the extreme sections)

(c.) what is useful is the possibility of obtaining the stress state once the internal actions are available.

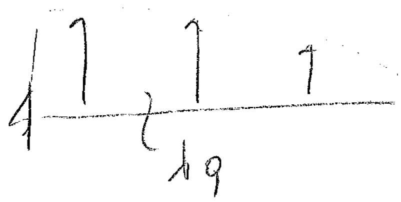
4. It is better to remark that:

- in the presence of M_x and/or M_y and/or T_z $\longrightarrow$ $\underline{\underline{\sigma}} \rightsquigarrow \sigma_{zz}$
- in the presence of T_x and/or T_y $\rightarrow \underline{\underline{\sigma}} \rightsquigarrow \sigma_{zz}$ and $\sigma_{zx} / \sigma_{zy}$
- in the presence of $M_z \rightarrow \underline{\underline{\sigma}} \rightsquigarrow \sigma_{zx} / \sigma_{zy}$

5. This result is obtained in the context of linearity assumption. This means that by amplifying the loading conditions of a factor λ, the stresses will be amplified by the same factor λ



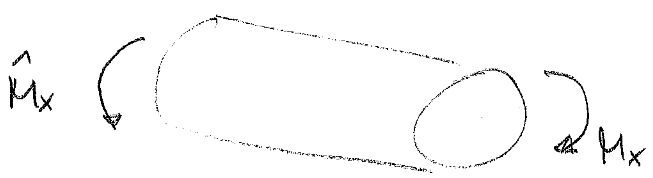
$\sigma_{zz} = \hat{\sigma}_{zz}$



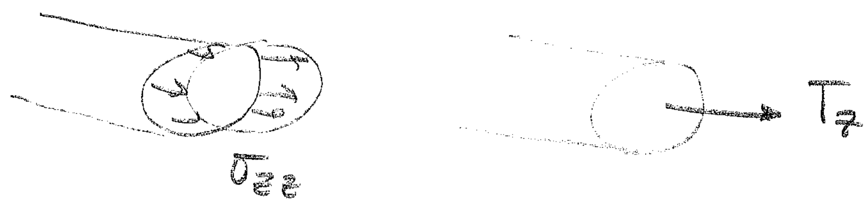
$\sigma_{zz} = \lambda \hat{\sigma}_{zz}$

6. Notice the distinction between equilibrium and equivalence

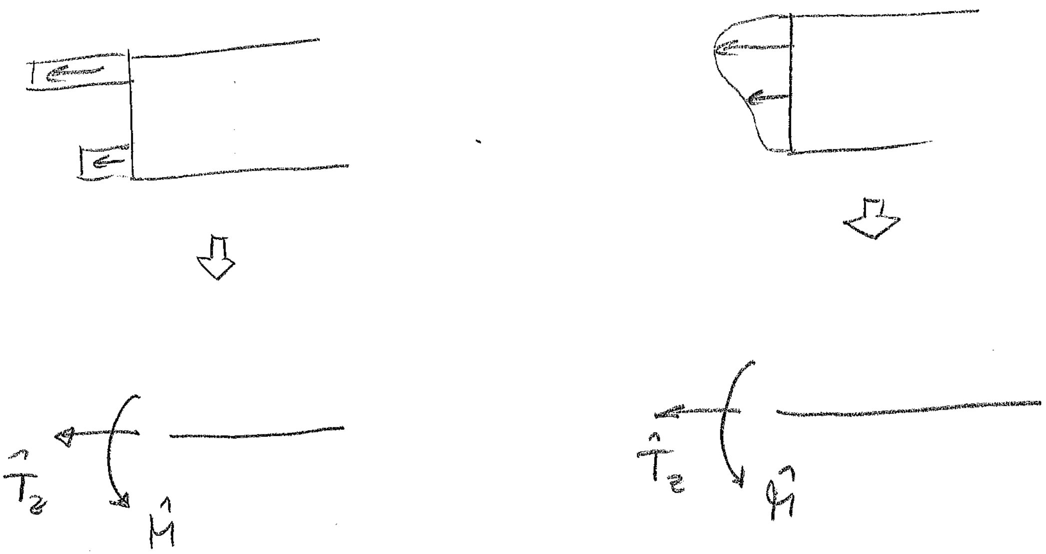
a. The internal actions equilibrate the external loads



6. the internal actions are statically equivalent to the cross section state of stress



7. The beam model, consistently with the DSV assumptions, cannot account for the differences between applied loads characterized by same resultants



8. Note that $\sigma_{zz} = \frac{T_z}{A} - \frac{M_y}{J_y} x + \frac{M_x}{J_x} y$

does not depend on the section elastic properties

σ_{zz} is just function of the internal actions.

This means that for a given set of internal actions T_z, M_y and M_x the state of stress σ_{zz} will be identical for two beams made of different materials, say a beam of steel and another of rubber

The internal stress arises to equilibrate the external loads.

The difference between the steel and the rubber beam will regard the deformation!

DSV solution for axial force

Consider the case where the only internal action is given by T_z

The DSV solution is then

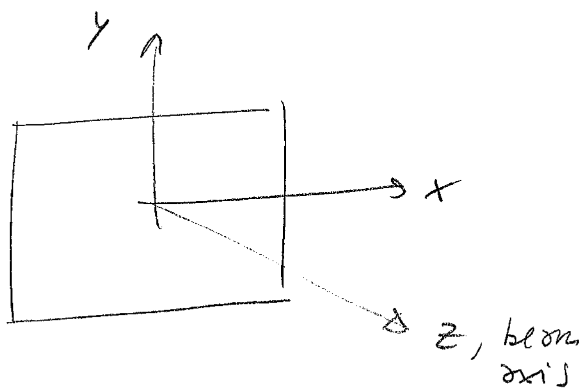
$$\sigma_{zz} = \frac{T_z}{A} + \frac{M_{xx}}{J_{xx}} y - \frac{M_{yy}}{J_{yy}} z \implies \boxed{\sigma_{zz} = \frac{T_z}{A}}$$

The displacement field can be obtained after

1. introducing the constitutive law
2. integrating the strain components.

So:

$$\boxed{\begin{aligned} \varepsilon_{xx} &= -\frac{\nu}{E}\sigma_{zz} = -\nu\frac{T_z}{EA} \\ \varepsilon_{yy} &= -\frac{\nu}{E}\sigma_{zz} = -\nu\frac{T_z}{EA} \\ \varepsilon_{zz} &= \frac{\sigma_{zz}}{E} = \frac{T_z}{EA} \end{aligned}}$$



$$\varepsilon_{xx} = u_{/x} \implies u = -\nu\frac{T_z}{EA}x$$

$$\varepsilon_{yy} = v_{/y} \implies v = -\nu\frac{T_z}{EA}y$$

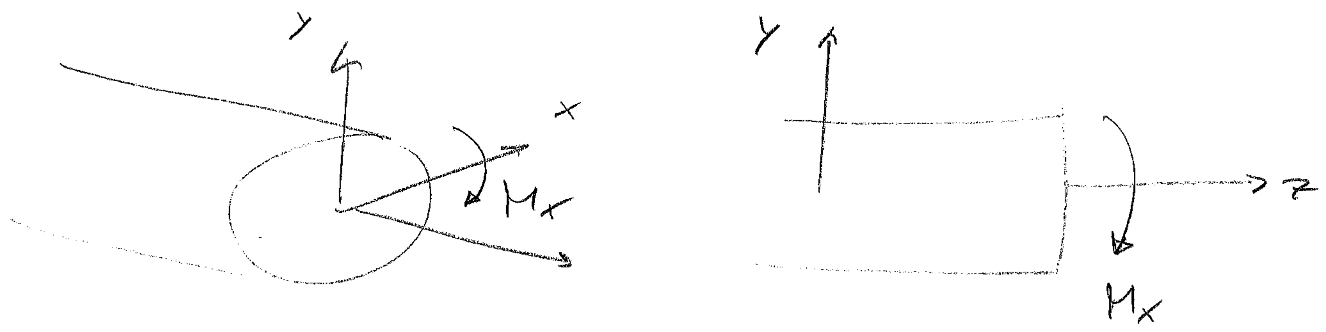
Change of shape of the section due to Poisson's effect

$$\varepsilon_{zz} = w_{/z} \qquad w = \frac{T_z}{EA}z$$

→ linear displacement along the beam axis

DSV solution for bending

Consider the case of pure bending (constant along the beam axis). No shear-induced bending contribution.



The DSV simplifies to:

$$\sigma_{zz} = \frac{M_x}{J_{xx}} y \qquad \text{where } M_x = \text{const} = \hat{M}_x + \cancel{\boxed{z \hat{T}_y}}$$

The strains are thus:

$$\varepsilon_{xx} = -\frac{\nu}{E}\sigma_{zz} = -\nu \frac{M_x}{EJ_{xx}} y$$

$$\varepsilon_{yy} = \quad = -\nu \frac{M_x}{EJ_{xx}} y$$

$$\varepsilon_{zz} = \quad = \frac{M_x}{EJ_{xx}} y$$

$$\gamma_{xy} = 0$$

$$\gamma_{xz} = 0$$

$$\gamma_{yz} = 0$$

The displacement field is:

$$\varepsilon_{xx} = u_{/x} \Rightarrow u = -\nu \frac{M_x}{EJ_{xx}} xy + f_u(y,z)$$

$$\varepsilon_{yy} = v_{/y} \Rightarrow v = -\nu \frac{M_x}{EJ_{xx}} \frac{1}{2} y^2 + f_v(x,z)$$

$$\varepsilon_{zz} = w_{/z} \Rightarrow w = \frac{M_x}{EJ_{xx}} zy + f_w(x,y)$$

It can be shown that a solution is found by taking $\boxed{f_w = f_u = 0}$

How to determine f_v?

$$\left.\begin{aligned} \gamma_{xy} &= u_{/y} + v_{/x} = -\nu \frac{M_x}{EJ_{xx}} x + f_{v/x} = 0 \\ \gamma_{yz} &= v_{/z} + w_{/y} = f_{v/z} + \frac{M_x}{EJ_{xx}} z = 0 \end{aligned}\right\} \Rightarrow f_v = \nu \frac{M_x}{EJ_{xx}} \frac{x^2}{2} - \frac{M_x}{EJ_{xx}} \frac{z^2}{2}$$

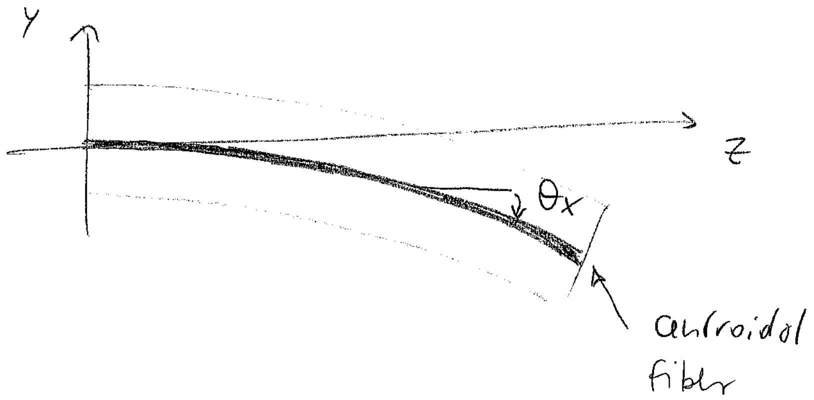
$$\Rightarrow \boxed{f_v = \frac{1}{2} \frac{M_x}{EJ_{xx}} \left(\nu x^2 - z^2\right)}$$

It follows that:

$$\boxed{\begin{aligned} u &= -\nu \frac{M_x}{EJ_{xx}} xy \\ v &= -\frac{M_x}{2EJ_{xx}} \left(z^2 + \nu\left(y^2 - x^2\right)\right) \\ w &= \frac{M}{EJ} zy \end{aligned}}$$

The displacement field of the centroidal fiber ($x=y=0$) is thus

$$\boxed{\begin{aligned} u &= 0 \\ v &= -\frac{M_x}{2EJ_{xx}} z^2 \\ w &= 0 \end{aligned}}$$



The rotation θ_x, at the generic point z is:

$$\boxed{\theta_x \simeq -v_{/z} = \frac{M_x}{EJ_{xx}} z}$$

It follows that:

$$\theta_{x/z} \simeq -v_{/zz} = \frac{M_x}{EJ_{xx}} \qquad \text{(constant second derivative)}$$

Denote the second derivative $v_{/zz}$ with $-k_x$ (curvature)

$-k_x = v_{/zz}$. It follows that

$$\boxed{k_x = \frac{M_x}{EJ_{xx}} \quad \text{or} \quad M_x = EJ\,k_{xx}}$$

Bernoulli-Navier law

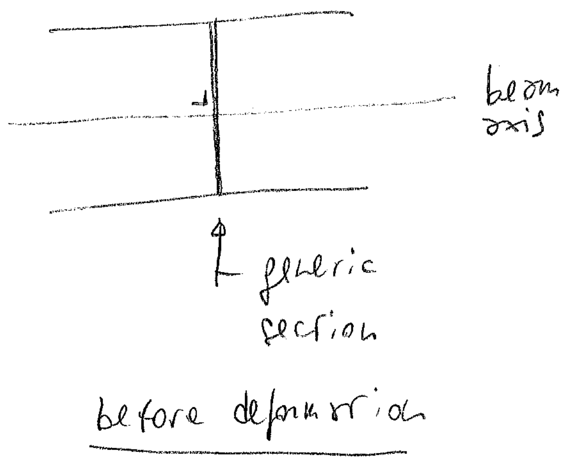
Remarks

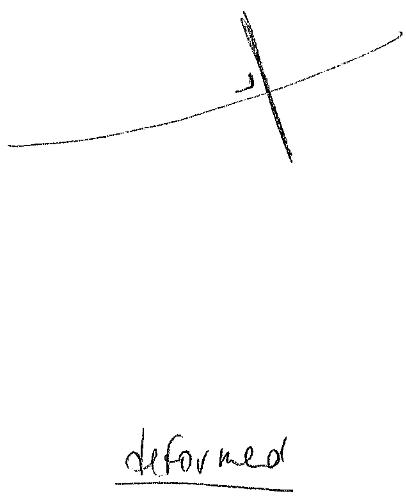
1. For constant bending moment, the curvature is constant along the beam axis

2. For constant bending moment, $\sigma_{zz} \neq 0$ while

$\sigma_{xz} = \sigma_{yz} = 0 \implies \boxed{\gamma_{xz} = \gamma_{yz} = 0}$

No shearing deformation implies that the section remains orthogonal to the beam axis after deformation happens





These observations (the fact that sections remain normal to the beam axis, i.e. $\gamma_{xz} = \gamma_{yz} = 0$) will be the ingredients for formulating the beam problem from a kinematic point of view, within the context of the so called <u>Euler-Bernoulli model</u> (see displacement-based approaches later)