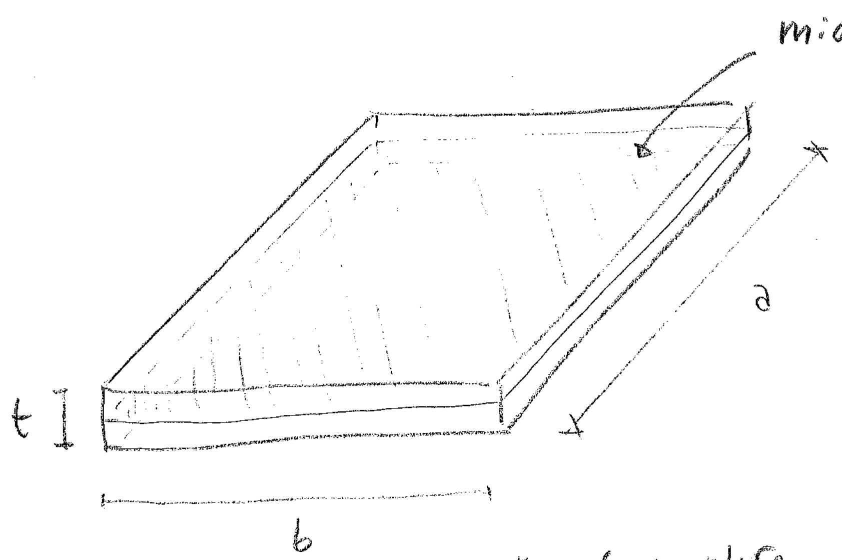
• Mindlin plate theory

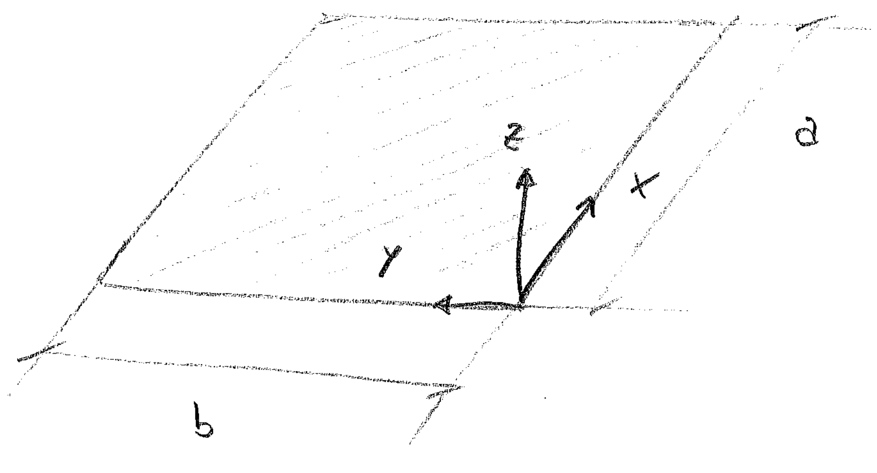
A plate idealization can be applied to those 3D solids characterized by one of the three dimensions which is much smaller in comparison to other two.
The smallest dimension is commonly denoted as thickness.
It is worth noting, since the beginning, that the Mindlin plate model (as well as any other plate model) represents a simplified description of a structure that in reality is, in any case, inherently 3D.
A plate identifies a structure characterized by a flat middle surface, whereas shells are those structures characterized by a curved middle surface (in this second case several kinds of curvatures are possible; cylindrical shells are those more closely associated with typical space applications)



t: plate thickness
a, b: plate dimensions

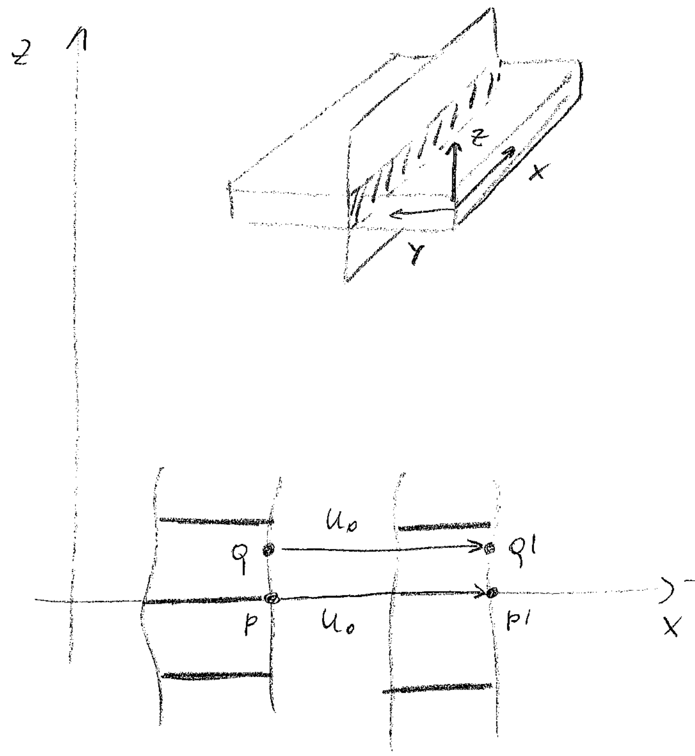
geometry of a plate and midsurface

Consider now a relatively thin panel, where $t << a$ (say in the range $a/t > 20$), characterized by thickness t, and planar dimensions a and b.
The midsurface is taken as the surface passing through the middle of the thickness (although it is not strictly necessary; any through-the-thickness position could be considered).
A Cartesian coordinate system is taken over the midsurface with xy defining the plane of the midsurface and z the thickness-wise direction.

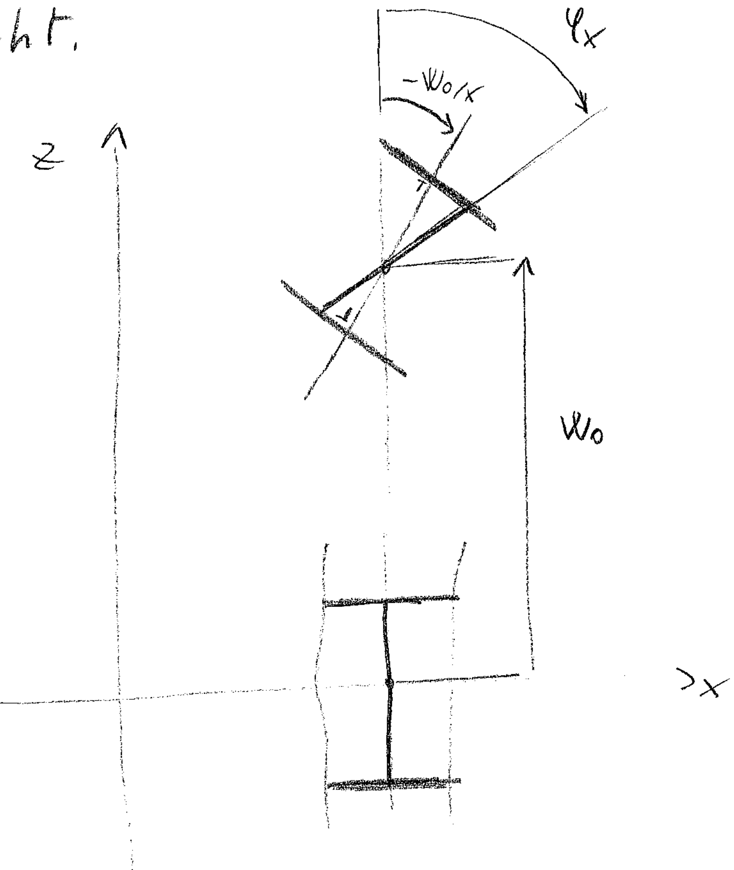


The plate model is developed in the context of a <u>kinematic formulation</u>, in analogy to the approach adopted in the case of beams. This means that the underlying idea consists in describing the displacement field of the 3D structure (recall, the plate is characterized by $t << a$ but is, in any case 3D) starting from the knowledge of generalized displacement quantities referred to the reference surface of the plate.

The Mindlin plate model, which is the 2D counterpart of the Timoshenko beam model, relies upon the assumption that sections initially normal to the reference surface can rotate during the deformation process, but are constrained to remain straight.



membrane displacement
(zx plane)



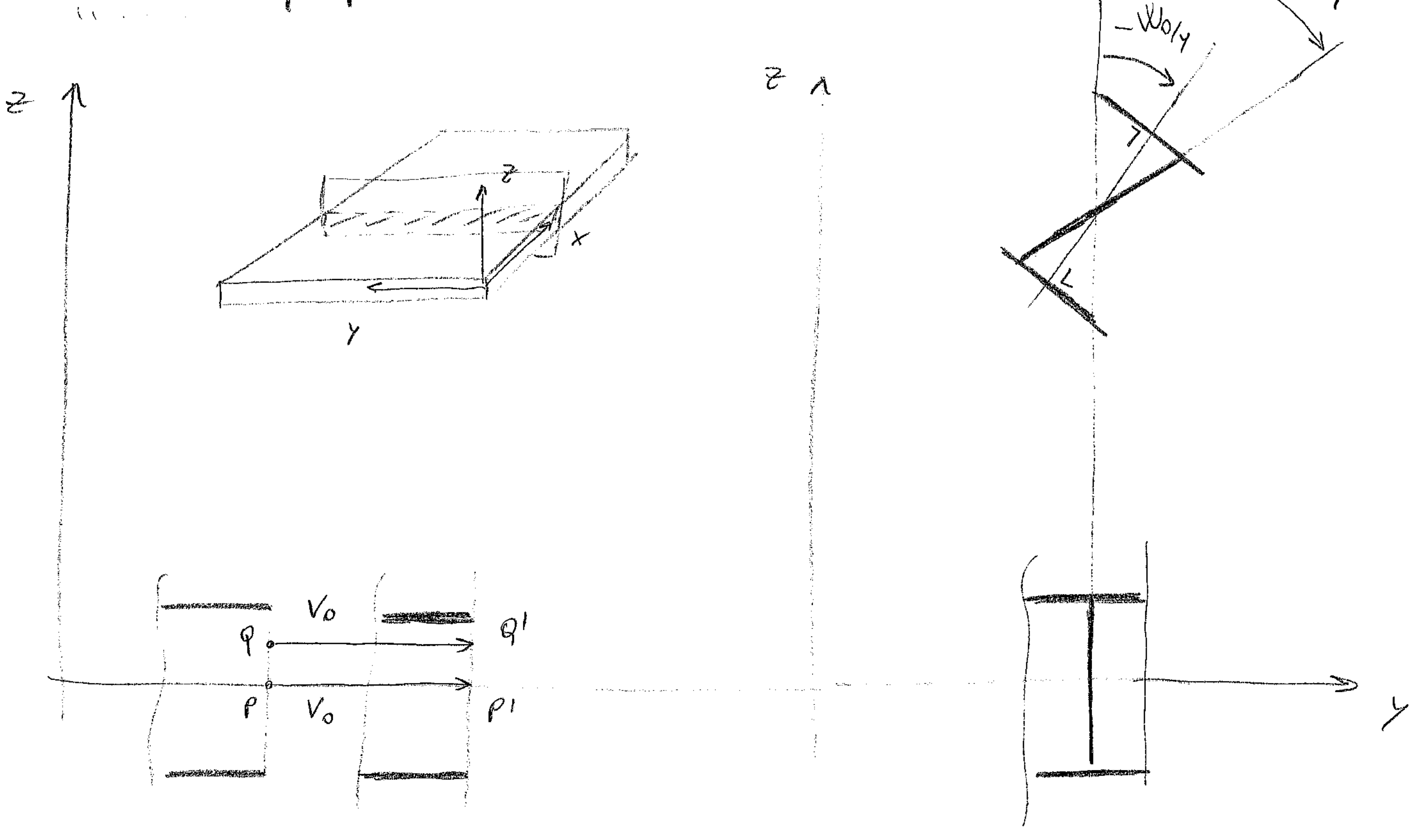
bending displacement
(zx plane)

The kinematic model, exactly as for Timoshenko beam model, assumes that

1. the membrane part of the displacement is characterized by an x-wise displacement equal for all the points belonging to the cross section

2. the bending part is given by a rotation φ_x, due to a rotation of the section

The same kinematic description is adopted with regard to the zy plane



The kinematic model is then formulated as:

$$\boxed{\begin{aligned} u(x,y,z) &= u_0(x,y) + z\,\varphi_x(x,y) \\ v(x,y,z) &= v_0(x,y) + z\,\varphi_y(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned}}$$

where $u_0, v_0, w_0, \varphi_x, \varphi_y$ are the generalized displacement components of the Mindlin plate model.

The kinematic description should be interpreted as a "rule" for describing the 3D displacement field of the plate (u, v and w are functions of x, y and z)

starting from the generalized components which are referred to the plate reference surface (u_0, v_0, w_0, φ_x and φ_y depend on x and y only).

The inherent sense of a 2D model (such as a plate model) relies in the reduction of the 3D structure to a surface (with zero thickness), where the dependence on the thicknesswise dimension is embedded into the kinematic description of the displacement field.

It can be noted that the generalized displacement components φ_x and φ_y are related to the rotations around the x and y axes (taken positive according to the right hand rule) by:

$$\boxed{\begin{aligned} \varphi_x &= \theta_y \\ \varphi_y &= -\theta_x \end{aligned}}$$

where θ_x, θ_y rotations around the x and y axes

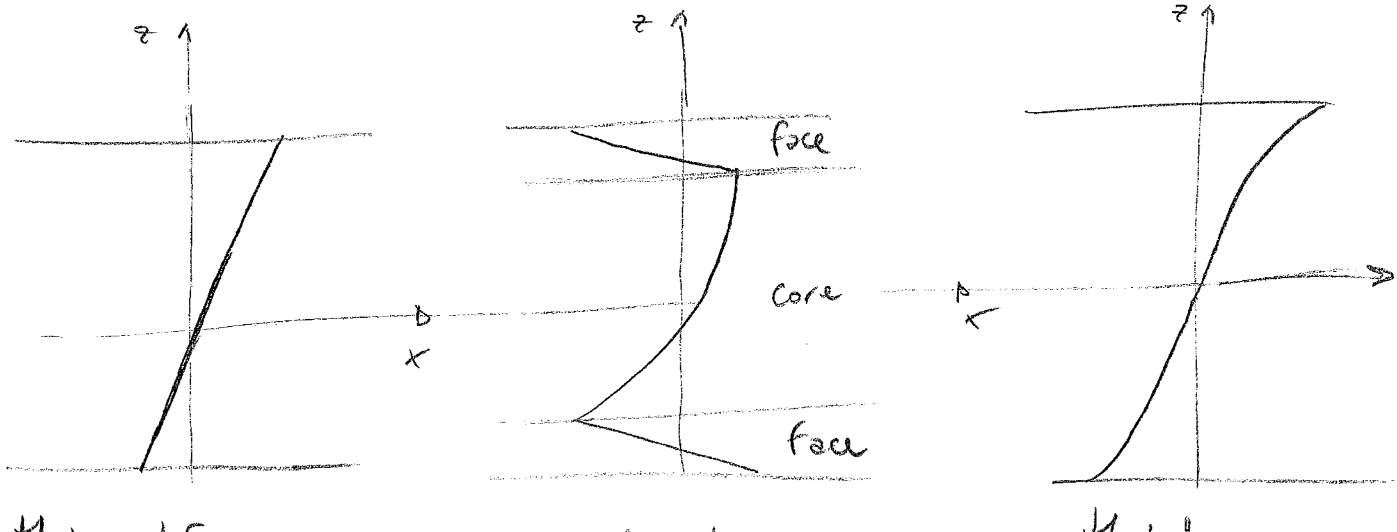
This means that φ_i are not the rotations, but they are generalized parameters associated with the rotation.

In principle the generalized parameters of a kinematic model do not even need to be associated with a clear geometric meaning of rotation or displacement.

It is more appropriate to identify the generalized parameters as Lagrangian coordinates introduced to represent the displacement field.

Note also that the assumption of sections remaining straight is, indeed, an assumption. No guarantees exist that the thicknesswise behaviour will be characterized by such behaviour. In many cases of practical interest this is a reasonable approximation leading to sufficiently accurate results. In other cases, the approximation of straight normal could be not appropriate. In particular, attention should be paid when the assumption is applied to plates characterized by

1. <u>abrupt variations</u> of mechanical properties along the thickness direction (e.g. sandwich panels)
2. relatively <u>thick</u> panels ($a/t \leq 20$) especially when the transverse shear stiffnesses are relatively weak (e.g. composite panels)



thin plate — sandwich — thick composite plate

Examples of $u(x,y,z)$

Generalized deformations

By substituting the kinematic field into the expression of the infinitesimal displacement strain tensor:

$$
\boxed{
\begin{aligned}
\varepsilon_{xx} &= u_{/x} = u_{0/x} + z\,\varphi_{x/x} \\
\varepsilon_{yy} &= v_{/y} = v_{0/y} + z\,\varphi_{y/y} \\
\gamma_{xy} &= v_{/x} + u_{/y} = v_{0/x} + u_{0/y} + z\left(\varphi_{x/y} + \varphi_{y/x}\right) \\
\gamma_{xz} &= w_{/x} + u_{/z} = w_{0/x} + \varphi_x \\
\gamma_{yz} &= w_{/y} + v_{/z} = w_{0/y} + \varphi_y \\
\left(\varepsilon_{zz} \right.&= \left. w_{/z} = 0 \right)
\end{aligned}
}
$$

or, by separating the dependence on (x,y) from the dependence on z:

$$
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}
=
\begin{Bmatrix} u_{0/x} \\ v_{0/y} \\ v_{0/x} + u_{0/x} \\ 0 \\ 0 \end{Bmatrix}
+ z
\begin{Bmatrix} \varphi_{x/x} \\ \varphi_{y/y} \\ \varphi_{x/y} + \varphi_{y/x} \\ 0 \\ 0 \end{Bmatrix}
+
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ w_{0/x} + \varphi_x \\ w_{0/y} + \varphi_y \end{Bmatrix}
$$

$$
= \underline{\varepsilon} + z\,\underline{k} + \underline{t}
$$

- $\underline{t}$: generalized <u>transverse shear</u> deformations
- $\underline{k}$: generalized <u>bending</u> deformations
- $\underline{\varepsilon}$: generalized <u>membrane</u> deformations

Indicial notation

Given the amount of not null contributions of $\underline{\underline{\varepsilon}}$ and the number of contributions characterizing the expression of the virtual work, it is convenient to adopt a compact notation. According to the indicial notation, the kinematic field can be expressed as:

$$\boxed{\begin{aligned} u_\alpha(x_\alpha, x_3) &= u_{0_\alpha}(x_\alpha) + x_3 \varphi_\alpha(x_\alpha) \\ w(x_\alpha, x_3) &= w_0(x_\alpha) \end{aligned}}$$

$\alpha = 1,2$

$x_3 \equiv z$

Accordingly the deformations can be written as:

$$\boxed{\begin{aligned} \varepsilon_{\alpha\beta} &= \overset{\varepsilon}{\delta}_{\alpha\beta} + x_3 K_{\alpha\beta} \\ \gamma_{\alpha 3} &= w_{/\alpha} + \varphi_\alpha \end{aligned}}$$

where the membrane deformations $\overset{\varepsilon}{\delta}_{\alpha\beta}$ and the curvatures $K_{\alpha\beta}$ are introduced as:

$$\boxed{\begin{aligned} \overset{\varepsilon}{\delta}_{\alpha\beta} &= \frac{1}{2}\left(u_{0\alpha/\beta} + u_{0\beta/\alpha}\right) \\ K_{\alpha\beta} &= \varphi_{\alpha/\beta} \end{aligned}}$$

- Generalized stresses

They are obtain by application of the PVW:

$$\delta W_i = \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} \, dV$$

$$= \int_V \left(\delta \varepsilon_{\alpha\beta} \, \sigma_{\alpha\beta} + \delta \gamma_{\alpha 3} \, \sigma_{\alpha 3} \right) dV$$

and by substituting the expressions of $\varepsilon_{\alpha\beta}$ and $\gamma_{\alpha 3}$ as obtained from the kinematic model:

$$= \int_V \left[\left(\delta \varepsilon^{0}_{\alpha\beta} + z \, \delta K_{\alpha\beta} \right) \sigma_{\alpha\beta} + \delta \gamma_{\alpha 3} \, \sigma_{\alpha 3} \right] dV$$

$$= \int_A \delta \varepsilon^{0}_{\alpha\beta} \int_t \sigma_{\alpha\beta} \, dx_3 \, dA + \int_A \delta K_{\alpha\beta} \int_t z \, \sigma_{\alpha\beta} \, dx_3 \, dA \; +$$

$$+ \int_A \delta \gamma_{\alpha 3} \int_t \sigma_{\alpha 3} \, dx_3 \, dA$$

Defining now:

$$N_{\alpha\beta} = \int_t \sigma_{\alpha\beta} \, dx_3$$ (membrane forces per unit length)

$$M_{\alpha\beta} = \int_t z \, \sigma_{\alpha\beta} \, dx_3$$ (moments per unit length)

$$Q_{\alpha} = \int_t \sigma_{\alpha 3} \, dx_3$$ (transverse shear forces per unit length)

It follows that:

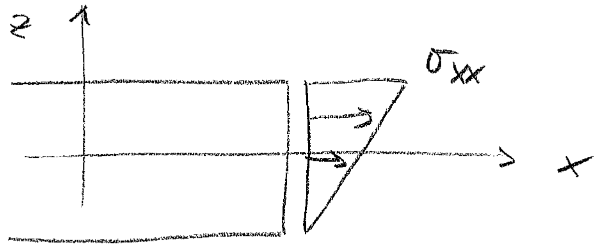
$$\boxed{\begin{aligned} \delta W_i &= \int_A \left(\delta \varepsilon_{\alpha\beta} N_{\alpha\beta} + \delta k_{\alpha\beta} M_{\alpha\beta} + \delta \gamma_{\alpha 3} Q_\alpha \right) dA \\ &= \int_A \left(\delta \underline{\underline{\varepsilon}} : \underline{\underline{N}} + \delta \underline{\underline{k}} : \underline{\underline{M}} + \delta \underline{\gamma} \cdot \underline{Q} \right) dA \end{aligned}}$$

which is the expression of internal virtual work for a plate modelled according to the Mindlin model.
The expression illustrates the work conjugacy between the stress fluxes $\underline{\underline{N}}$ and the membrane deformations $\underline{\underline{\varepsilon}}$; between the moment fluxes $\underline{\underline{M}}$ and the curvatures $\underline{\underline{k}}$; between the transverse shear fluxes $\underline{Q}$ and the transverse shear deformations $\underline{\gamma}$.

It is then concluded that $\underline{\underline{N}}$, $\underline{\underline{M}}$ and $\underline{Q}$ are the generalized stress measures to be used in the context of the Mindlin model.

The 3D structure (the panel) has been condensed into a 2D model (geometrically represented by the reference surface). Accordingly the generalized stresses are quantities associated with the internal state of stress which are reported to 2D reference surface

Consider, for instance, the plane xz



Imagine that the internal state of stress along the thickness, at a specific location (x,y), is given by σ_{xx} as represented in the figure.

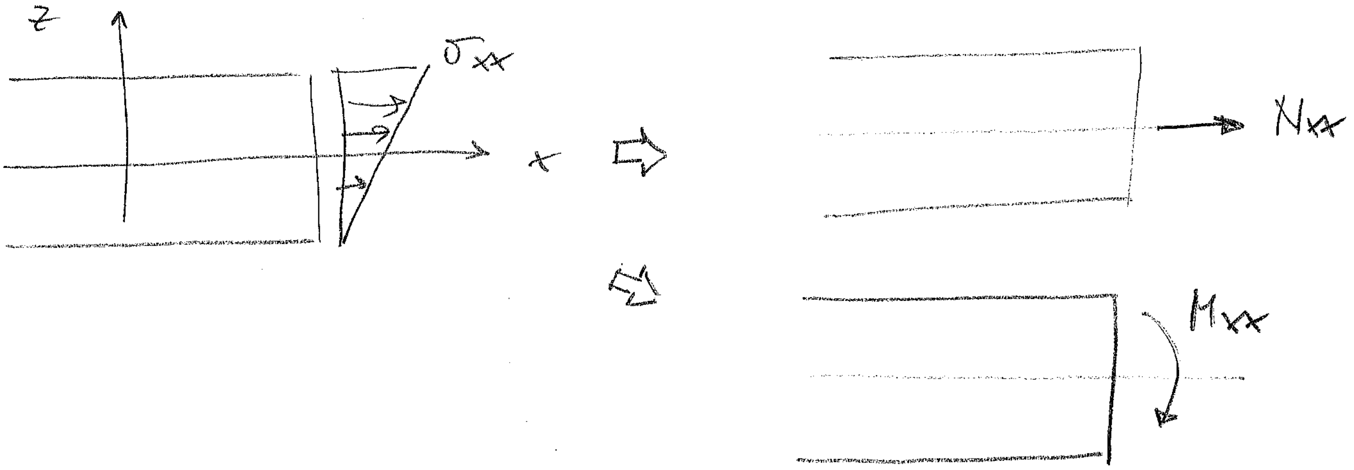
According to the generalized stress measures here introduced

$$N_{\alpha\beta} = \int_t \sigma_{\alpha\beta}\, dx_3 \qquad\qquad N_{xx} = \int_t \sigma_{xx}\, dx_3$$

$$\Rightarrow$$

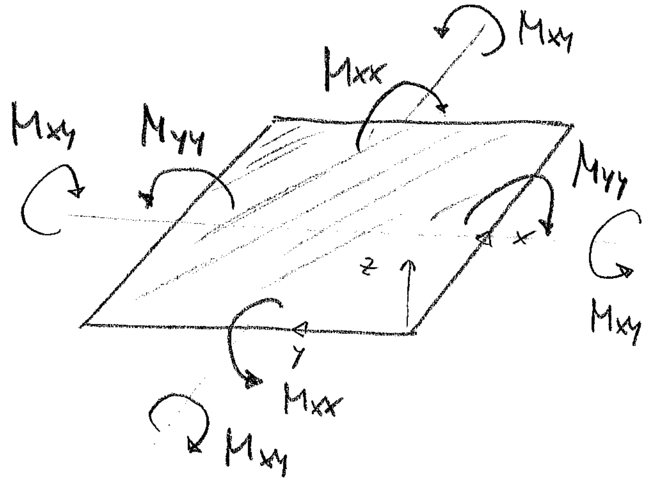
$$M_{\alpha\beta} = \int_t \sigma_{\alpha\beta}\, x_3\, dx_3 \qquad\qquad M_{xx} = \int_t \sigma_{xx}\, x_3\, dx_3$$

the corresponding generalized stress components will be:

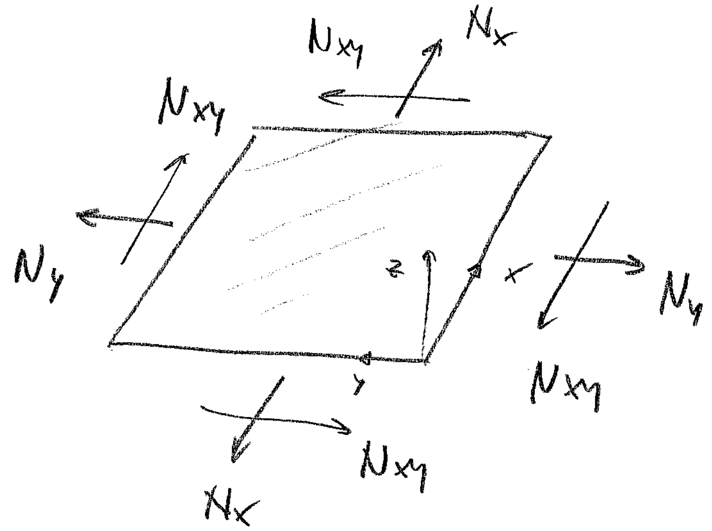


And similarly for the other components.

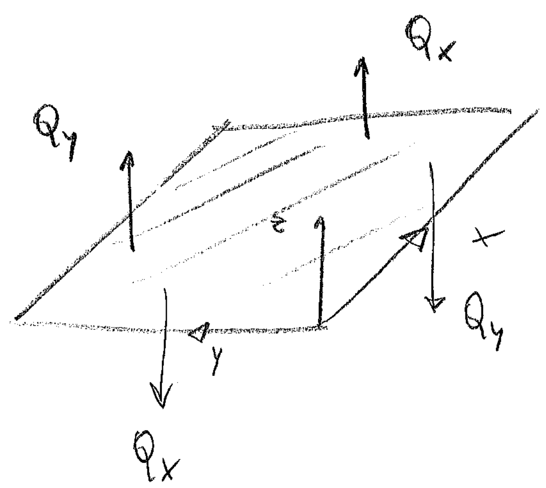
In general the internal forces will be identified as:



moments per unit length



membrane forces per unit length



transverse shear forces per unit length

Note that the moments are not defined according to the right-hand rule (e.g. M_{xx} is not the moment around the x-axis). Rather, the moments are defined as

$$M_{\alpha\beta} = \int_t \sigma_{\alpha\beta}\, x_3 \, dt$$

so, for example, M_{xx} is the moment due to the σ_{xx} components, and so on for M_{xy} and M_{yy}

Note also that M_{xy} is the torsional moment due to the thicknesswise distribution of in-plane shear stresses σ_{xy}.

• Generalized external forces

Similarly to the case of the Timoshenko beam model, the Mindlin plate allows for the description of transverse shear deformations; the external loads consistent with this kinematic model are then in the form of distributed axial forces $\hat{n}_x$, $\hat{n}_y$, transverse shear load $\hat{n}_z = p$ and, thanks to the ability of the model to account for shear deformability, the distributed couples $\hat{m}_x$ and $\hat{m}_y$

So:

$$\delta W_e = \int_A \left(\delta u_0 \hat{n}_x + \delta v_0 \hat{n}_y + \delta w_0 p + \delta \varphi_x \hat{m}_x + \delta \varphi_y \hat{m}_y \right) dA$$

or, by using the indicial notation:

$$\boxed{\delta W_e = \int_A \left(\delta u_{0\alpha} \hat{n}_\alpha + \delta w_0 p + \delta \varphi_\alpha \hat{m}_\alpha \right) dA} \qquad \alpha = 1, 2$$

where

$$\hat{n}_\alpha = \int_t F_\alpha \, dx_3$$

$$p = \int_t F_3 \, dx_3 \qquad \text{and} \quad \underline{F} : \text{volume force}$$

$$\hat{m}_\alpha = \int_t F_\alpha x_3 \, dx_3$$

(Extra)

The previous results can be obtained by adopting the same approach presented in the case of Timoshenko beam model. The idea is to consider a volume force $\underline{F}$ and evaluate how the force is "seen" by the model in the context of the kinematic assumptions adopted. This latter operation (understanding how the model "sees" the volume forces) is conducted by evaluating the external work.

$$\delta W_e = \int_V \delta \underline{u} \cdot \underline{F} \, dV = \int_V \delta u_\alpha F_\alpha \, dV + \int_V \delta w \, F_3 \, dV = \quad (\alpha = 1, 2)$$

$$= \int_V \delta(u_{0\alpha} + x_3 \varphi_\alpha) F_\alpha \, dV + \int_V \delta w_0 F_3 \, dV =$$

observing how that $u_{0\alpha}$, w_0 and φ_α do not depend on x_3:

$$= \int_A \delta u_{0\alpha} \int_t F_\alpha \, dx_3 \, dA + \int_A \delta \varphi_\alpha \int_t x_3 F_\alpha \, dx_3 \, dA +$$

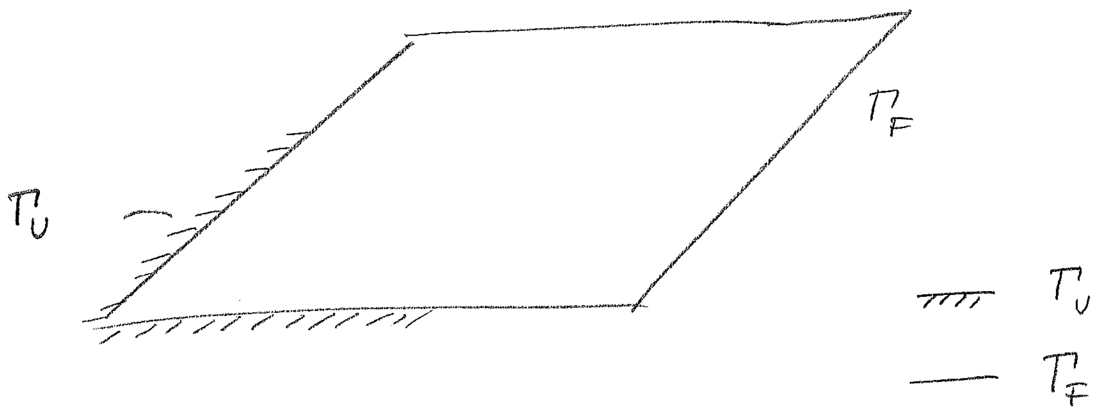
$$+ \int_A \delta w_0 \int_t F_3 \, dx_3 \, dA$$

$$= \int_A \left(\delta u_{0\alpha} \hat{n}_\alpha + \delta \varphi_\alpha \hat{m}_\alpha + \delta w_0 \, p \right) dA$$

Equilibrium conditions

The equilibrium conditions are derived by applying the PVW.

The plate is characterized by the boundary Γ which is partially subjected to imposed displacements (Γ_U) and to applied generalized forces (Γ_F)



$$\Gamma = \Gamma_F \cup \Gamma_U$$
$$\Gamma_F \cap \Gamma_U = \emptyset$$

The PVW is then written as:

$$\boxed{\begin{aligned} &\int_A \left(\delta\varepsilon_{\alpha\beta} N_{\alpha\beta} + \delta k_{\alpha\beta} M_{\alpha\beta} + \delta\gamma_{\alpha 3} Q_\alpha \right) dA = \\ &= \int_A \left(\delta u_{0\alpha} \hat{n}_\alpha + \delta\varphi_\alpha \hat{m}_\alpha + \delta w_0\, p \right) dA \; + \\ &+ \int_{\Gamma_F} \left(\delta u_{0\alpha} \hat{N}_\alpha + \delta\varphi_\alpha \hat{M}_\alpha + \delta w_0 \hat{Q} \right) d\Gamma_F \end{aligned}}$$

→ forces/moments per unit surface

→ forces/moments per unit length

The equilibrium equations are obtained by expressing the virtual variations in terms of displacement components. To this aim, it is necessary to re-arrange the expression of the internal virtual work.

$$\delta W_i = \int_A \left(\delta \varepsilon_{\alpha\beta} N_{\alpha\beta} + \delta k_{\alpha\beta} M_{\alpha\beta} + \delta \gamma_{\alpha 3} Q_\alpha \right) dA$$

$$= \int_A \left[\delta u_{\alpha/\beta} N_{\alpha\beta} + \delta \varphi_{\alpha/\beta} M_{\alpha\beta} + \left(\delta w_{o/\alpha} + \delta \varphi_\alpha \right) Q_\alpha \right] dA$$

Note that: $\delta \varepsilon_{\alpha\beta} N_{\alpha\beta} = \frac{1}{2} \left(\delta u_{\alpha/\beta} + \delta u_{\beta/\alpha} \right) N_{\alpha\beta}$

$$= \frac{1}{2} \delta u_{\alpha/\beta} N_{\alpha\beta} + \frac{1}{2} \delta u_{\beta/\alpha} N_{\alpha\beta}$$

but $N_{\alpha\beta} = \int_t \sigma_{\alpha\beta} \, dx_3 = \int_t \sigma_{\beta\alpha} \, dx_3 = N_{\beta\alpha}$

so:

$$= \frac{1}{2} \delta u_{\alpha/\beta} N_{\alpha\beta} + \frac{1}{2} \delta u_{\beta/\alpha} N_{\beta\alpha}$$

$$= \frac{1}{2} \delta u_{\alpha/\beta} N_{\alpha\beta} + \frac{1}{2} \delta u_{\alpha/\beta} N_{\alpha\beta} = \delta u_{\alpha/\beta} N_{\alpha\beta}$$

Integrating now by parts:

$$= - \int_A \delta u_\alpha N_{\alpha\beta/\beta} \, dA + \int_\Gamma \delta u_\alpha N_{\alpha\beta} n_\beta \, d\Gamma +$$

$$- \int_A \delta \varphi_\alpha M_{\alpha\beta/\beta} \, dA + \int_\Gamma \delta \varphi_\alpha M_{\alpha\beta} n_\beta \, d\Gamma +$$

$$- \int_A \delta w_o Q_{\alpha/\alpha} \, dA + \int_\Gamma \delta w_o Q_\alpha n_\alpha \, d\Gamma + \int_A \delta \varphi_\alpha Q_\alpha \, dA$$

Collecting now the virtual variations:

$$= - \int_A \left[\delta u_\alpha N_{\alpha\beta/\beta} + \delta\varphi_\alpha \left(M_{\alpha\beta/\beta} - Q_\alpha \right) + \delta w_0 \, Q_{\alpha/\alpha} \right] dA$$

$$+ \int_\Gamma \left[\delta u_\alpha N_{\alpha\beta} n_\beta + \delta\varphi_\alpha M_{\alpha\beta} n_\beta + \delta w_0 \, Q_\alpha n_\alpha \right] d\Gamma$$

and observing that the virtual variations are compatible, viz. they do respect the boundary conditions, it follows that

$$\begin{cases} \delta u_\alpha = 0 \\ \delta\varphi_\alpha = 0 & \text{in } \Gamma_v \\ \delta w_0 = 0 \end{cases}$$

So:

$$\delta W_i = - \int_A \left[\delta u_\alpha N_{\alpha\beta/\beta} + \delta\varphi_\alpha \left(M_{\alpha\beta/\beta} - Q_\alpha \right) + \delta w_0 \, Q_{\alpha/\alpha} \right] dA$$

$$+ \int_{\Gamma_F} \left[\delta u_\alpha N_{\alpha\beta} n_\beta + \delta\varphi_\alpha M_{\alpha\beta} n_\beta + \delta w_0 \, Q_\alpha n_\alpha \right] d\Gamma_F$$

Setting now $\delta W_i = \delta W_e$, it is obtained:

$$\boxed{\begin{aligned} & - \int_A \Big[\delta u_\alpha \left(N_{\alpha\beta/\beta} + \hat{n}_\alpha \right) + \delta\varphi_\alpha \left(M_{\alpha\beta/\beta} - Q_\alpha + \hat{m}_\alpha \right) \\ & \qquad + \delta w_0 \left(Q_{\alpha/\alpha} + p \right) \Big] dA \; + \\ & + \int_{\Gamma_F} \Big[\delta u_\alpha \left(N_{\alpha\beta} n_\beta - \hat{N}_\alpha \right) + \delta\varphi_\alpha \left(M_{\alpha\beta} n_\beta - \hat{M}_\alpha \right) \\ & \qquad + \delta w_0 \left(Q_\alpha n_\alpha - \hat{Q} \right) \Big] d\Gamma_F = 0 \end{aligned}}$$

Due to the arbitrarity of the variations δu_α, $\delta \varphi_\alpha$ and δw_0 the equilibrium conditions are obtained as:

$N_{\alpha\beta/\beta} + \hat{n}_\alpha = 0$	(membrane equilibrium, 2 eqs)	equilibrium equations in Ω
$M_{\alpha\beta/\beta} - Q_\alpha + \hat{m}_\alpha = 0$	(moment equilibrium, 2 eqs)	
$Q_{\alpha/\alpha} + p = 0$	(shear equilibrium, 1 eq)	
$N_{\alpha\beta} n_\beta = \hat{N}_\alpha$	or $\delta u_\alpha = 0$	
$M_{\alpha\beta} n_\beta = \hat{M}_\alpha$	or $\delta \varphi_\alpha = 0$	
$Q_\alpha n_\alpha = \hat{Q}$	or $\delta w_0 = 0$	
↑ natural boundary conditions (in Γ_F)	↑ essential boundary conditions (in Γ_u)	

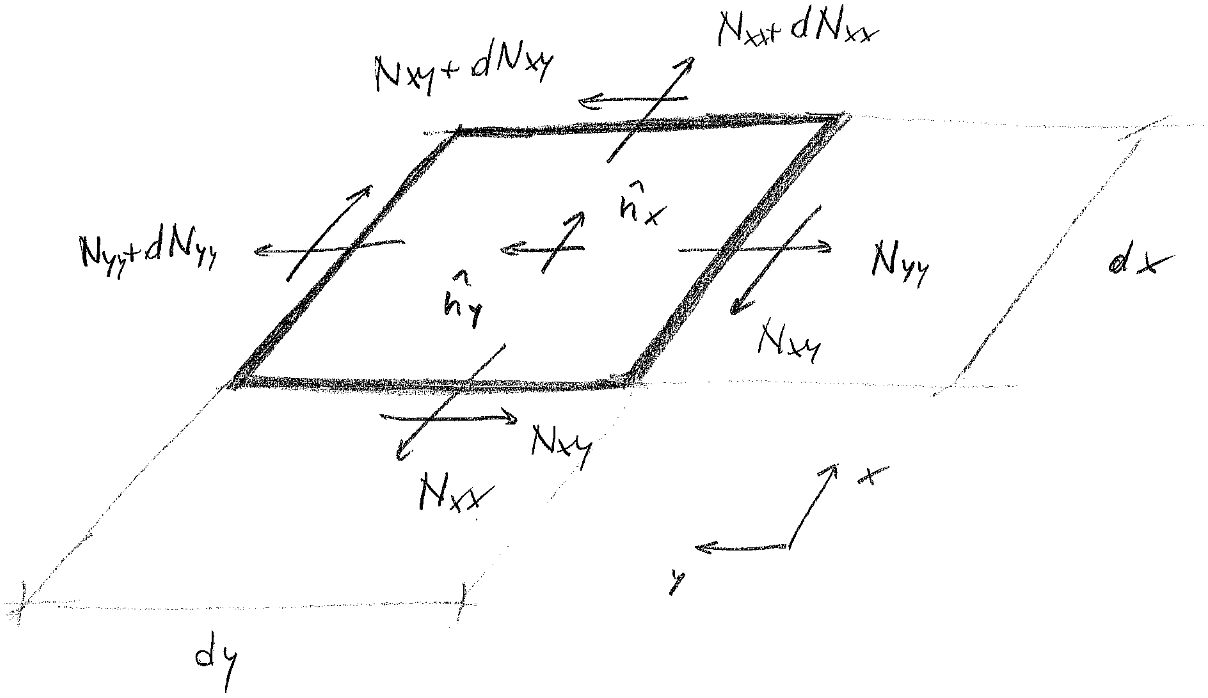
- This is the strong-form formulation of the problem for the Mindlin plate model.
- The set of equations and relevant boundary conditions is obtained automatically by substituting the kinematic assumptions into the PVW.
- The problem is well-posed and a solution can be obtained
- The set of equations obtained is given by 5 PDE in the 5 unknowns $u_{0\alpha}$, w_0, φ_α

- The equilibrium equations could have been obtained by imposing the equilibrium of an infinitesimal plate element.

Consider, for example, the first two equations:

$$N_{\alpha\beta/\beta} + \hat{n}_\alpha = 0 \quad \begin{cases} \xrightarrow{\alpha = x} & N_{xx/x} + N_{xy/y} + \hat{n}_x = 0 \\ \xrightarrow{\alpha = y} & N_{yx/x} + N_{yy/y} + \hat{n}_y = 0 \end{cases}$$

These two equations express the membrane behaviour in terms of equilibrium along x and y.



Equilibrium conditions

a) Along x: $(N_{xx} + dN_{xx} - N_{xx})\, dy + (N_{xy} + dN_{xy} - N_{xy})\, dx$

$$+ \hat{n}_x \, dx \, dy = 0$$

$$dN_{xx}\,dy + dN_{xy}\,dx + \hat{n}_x\,dx\,dy = 0$$

and dividing by $dx\,dy$:

$$\boxed{N_{xx/x} + N_{xy/y} + \hat{n}_x = 0}$$

b) Along y: $(N_{yy} + dN_{yy} - N_{yy})\,dx + (N_{xy} + dN_{xy} - N_{xy})\,dy + \hat{n}_y\,dx\,dy = 0$

and so:

$$\boxed{N_{yy/y} + N_{xy/x} + \hat{n}_y = 0}$$

The remaining equations can be obtained with a similar approach. Apart from the potential difficulties in realizing a nice sketch of the infinitesimal element, the problem would then be shifted to the obtainment of the boundary conditions. In other words: provided the equilibrium equations can be found by considering the differential element, which are the boundary conditions allowing for a proper set-up of the differential problem?

The PVW provides an automatical answer, furnishing both the equilibrium conditions <u>and</u> <u>the boundary conditions consistent with the kinematic model assumed.</u>

- Clarification (extra)

In the previous description it was assumed that

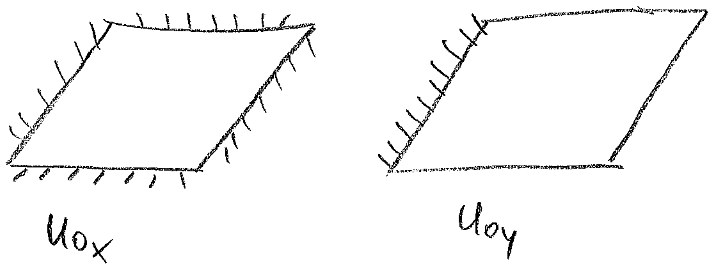
$\Gamma = \Gamma_U \cup \Gamma_F$ and $\Gamma_U \cap \Gamma_F = \emptyset$

For simplicity no distinction was made between the different components of the generalized displacement field. ($u_{0\alpha}$, φ_α, w_0).

At this stage it is worth noting that a more precise way of representing the different parts of the boundary would have been:

$\Gamma_{U_i} \cup \Gamma_{F_i} = \Gamma$ and $\Gamma_{U_i} \cap \Gamma_{F_i} = \emptyset$ $i = 1, \dots, 5$

This means that the boundary conditions can be different for the different components of the generalized displacement. For example



Γ_U

Γ_F

The component u_{0x} could be specified along all the boundary, so: $\Gamma = \Gamma_{U_1}$ $\Gamma_{F_1} = \emptyset$

while the component u_{0y} could be specified along a part of the boundary, so:

$$\Gamma = \Gamma_{U_2} \cup \Gamma_{F_2} \qquad \Gamma_{U_2} \cap \Gamma_{F_2} = \emptyset$$

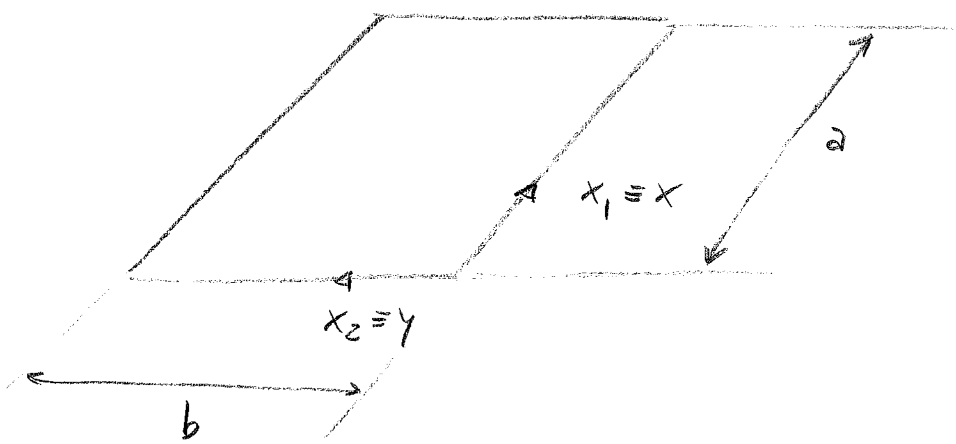
with $\quad \Gamma_{U_1} \neq \Gamma_{U_2}$

$$\Gamma_{F_1} \neq \Gamma_{F_2}$$

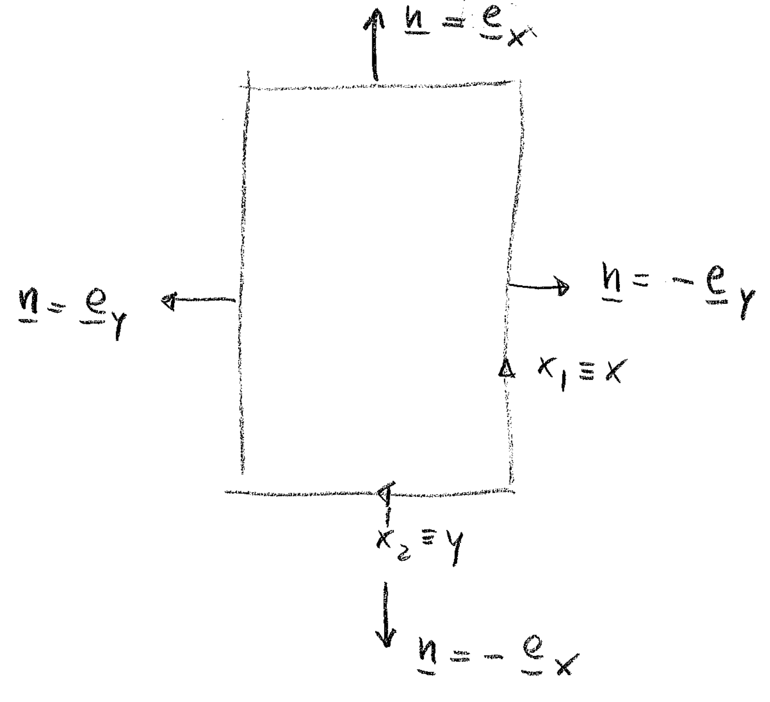
- Rectangular plates

The set of governing equations previously derived can be specialized to the case of rectangular plates, which represent one of the most common cases within the typical applications of plate structures.

Consider a rectangular configuration with a and b representing the longitudinal and transverse direction respectively



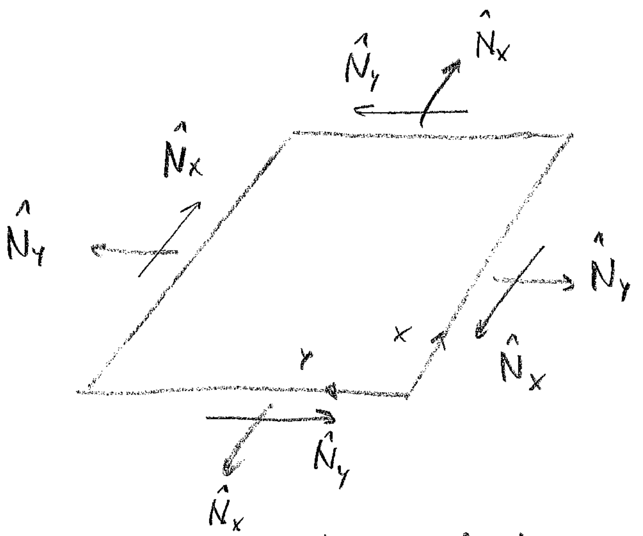
The normal vectors to the 4 edges are reported below:



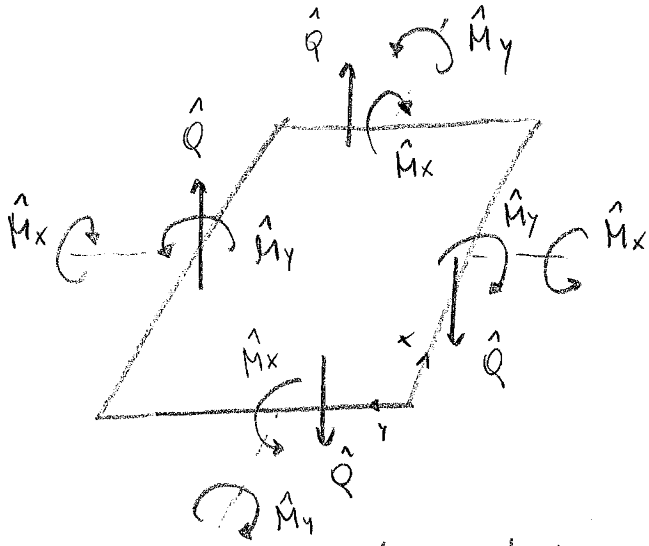
The boundary conditions, in their expanded form (by taking the double summatory over α and β, are:

$$\left.\begin{aligned} N_{xx} n_x + N_{xy} n_y &= \hat{N}_x \quad \text{or} \quad \delta u_x = 0 \\ N_{xy} n_x + N_{yy} n_y &= \hat{N}_y \quad \text{or} \quad \delta u_y = 0 \end{aligned}\right\} \text{membrane part}$$

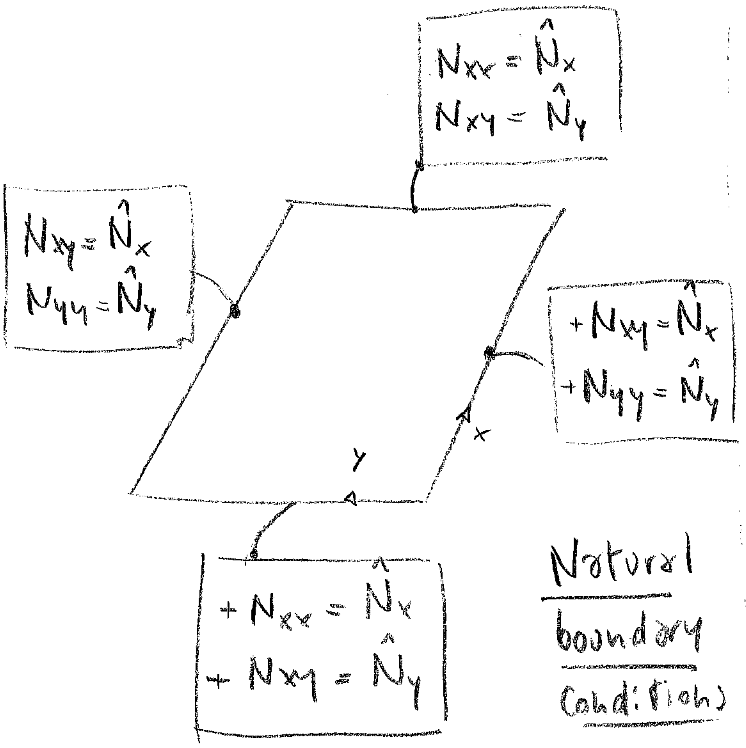
$$\left.\begin{aligned} M_{xx} n_x + M_{xy} n_y &= \hat{M}_x \quad \text{or} \quad \delta \varphi_x = 0 \\ M_{xy} n_x + M_{yy} n_y &= \hat{M}_y \quad \text{or} \quad \delta \varphi_y = 0 \\ Q_x n_x + Q_y n_y &= \hat{Q} \quad \text{or} \quad \delta w_0 = 0 \end{aligned}\right\} \text{bending and transverse shear part}$$

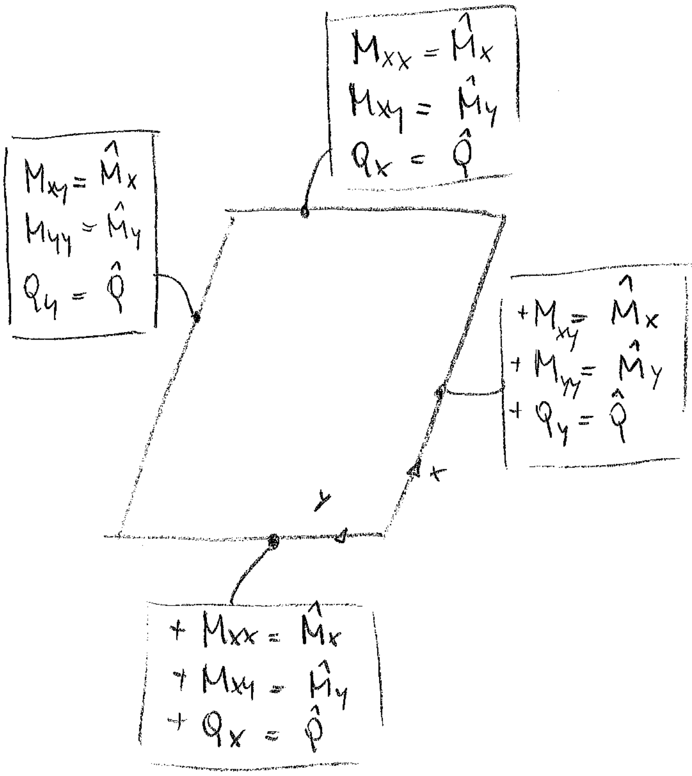


applied membrane loads



applied bending/shear loads





Natural boundary conditions

The boundary conditions are thus expressing the equilibrium between the internal (generalized) forces and the applied loads.

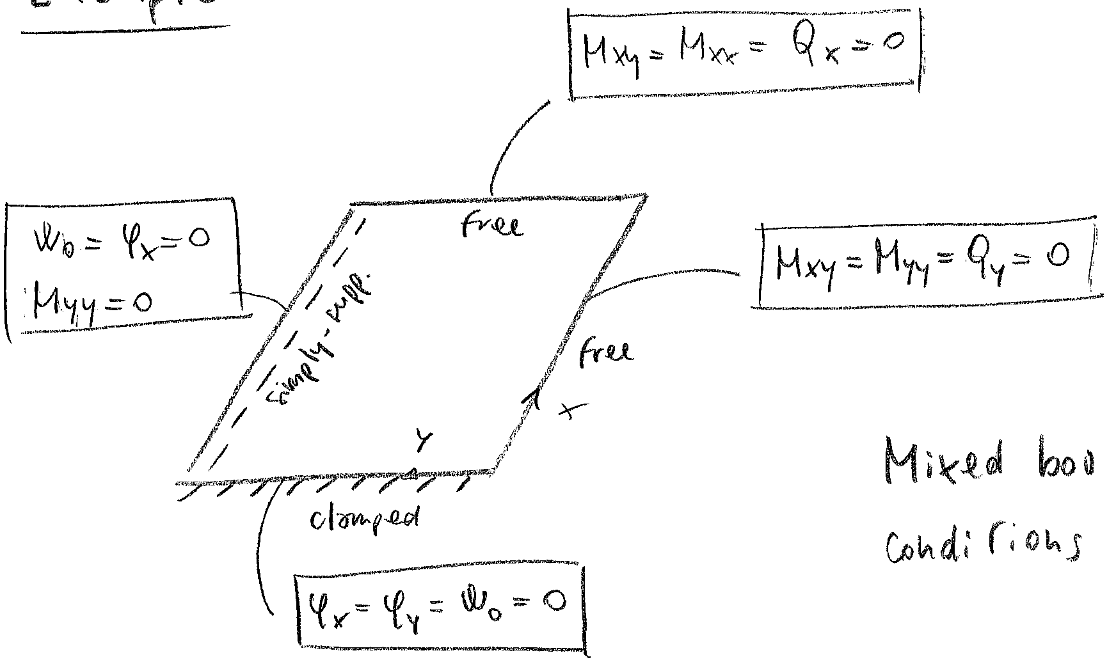
In any portion of the boundary either a natural condition or an essential condition has to be specified for all of the sets of relevant conditions derived.

The common sets of boundary conditions, restricting now the focus to the bending part, are:

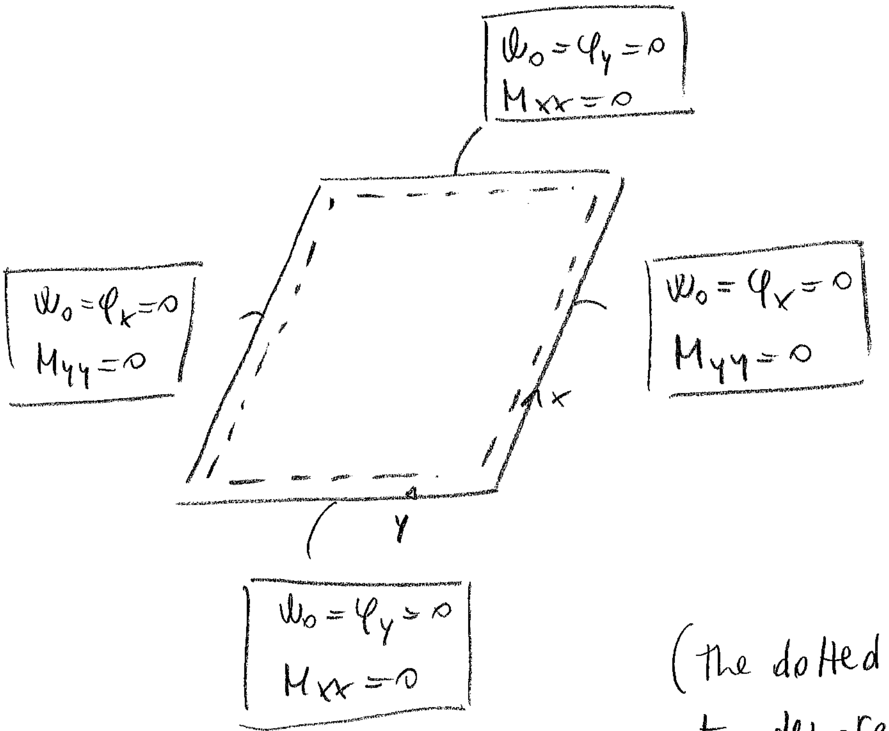
1. Free edge: no kinematic condition is specified. All the conditions are of natural type
   $\Rightarrow$ 3 natural conditions
2. Simply-supported edge: the transverse displacement and the rotation around the normal to the edge are prevented.
   The rotation around the tangent to the edge is free.
   $\Rightarrow$ 2 essential conditions
   1 natural condition

3. Clamped edge: all the displacement components are prevented
   $\Rightarrow$ 3 essential conditions

Examples



Mixed boundary conditions



Fully simply-supported

(The dotted line is commonly used to denote a simple-support)

Plate constitutive law

In order to express the equations in terms of the generalized displacement components, it is necessary to introduce the plate constitutive law.
This is the relation establishing a link between the generalized stresses and the generalized deformations of the kinematic model.

Consider the case of homogeneous isotropic material:
Under the assumption of state of plane stress (viz. $\sigma_{zz}=0$):

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

while the transverse shear stress components are available from the 3D constitutive law:

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \chi \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

where χ is the shear factor, taken equal to $\chi = 5/6$, which is introduced to reduce the intrinsic excess of stiffness of the model (exactly as done for the

Timoshenko beam)

Recalling now the expression of the generalized stresses:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_t \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dx_3 = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \int_t \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} dx_3$$

Recalling now the expression of the generalized strains:

$$\varepsilon_{xx} = \varepsilon^0_{xx} + x_3 \varphi_{x/x}$$

$$\varepsilon_{yy} = \varepsilon^0_{yy} + x_3 \varphi_{y/y}$$

$$\gamma_{xy} = 2\varepsilon^0_{xy} + x_3 (\varphi_{x/y} + \varphi_{y/x})$$

It is obtained:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \int_t \begin{Bmatrix} \varepsilon^0_{xx} + x_3 \varphi_{x/x} \\ \varepsilon^0_{yy} + x_3 \varphi_{y/y} \\ 2\varepsilon^0_{xy} + x_3 (\varphi_{x/y} + \varphi_{y/x}) \end{Bmatrix} dx_3$$

The linear contributions vanish as:

$$\int_{-t/2}^{t/2} x_3 \, dx_3 = 0$$ (the reference surface is taken as the mid-surface)

So:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ 2\varepsilon^0_{xy} \end{Bmatrix}$$

Similarly:

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_t x_3 \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dx_3 = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \int_t \begin{Bmatrix} x_3 \varepsilon_{xx} + x_3^2 \varphi_{x/x} \\ x_3 \varepsilon_{yy} + x_3^2 \varphi_{y/y} \\ 2x_3 \varepsilon_{xy} + x_3^2 (\varphi_{x/y} + \varphi_{y/x}) \end{Bmatrix}$$

defining: $D = \dfrac{Et^3}{12(1-\nu^2)}$ (flexural stiffness)

and performing the integration $\int_t = \int_{-t/2}^{t/2}$, it is obtained:

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varphi_{x/x} \\ \varphi_{y/y} \\ \varphi_{x/y} + \varphi_{y/x} \end{Bmatrix}$$

The transverse shear forces:

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \int_t \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} dx_3 = \chi \begin{bmatrix} Gt & 0 \\ 0 & Gt \end{bmatrix} \begin{Bmatrix} \varphi_y + w_{0/y} \\ \varphi_x + w_{0/x} \end{Bmatrix}$$

where the integral is simply carried out by multiplying the constitutive law with t as none of the terms is dependent on x_3

Summarizing, the constitutive relation is found in the form:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{array}\right] \begin{Bmatrix} u_{0/x} \\ v_{0/y} \\ u_{0/y} + v_{0/x} \\ \varphi_{x/x} \\ \varphi_{y/y} \\ \varphi_{x/y} + \varphi_{y/x} \end{Bmatrix}$$

and:

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} A_{44}^* & 0 \\ 0 & A_{55}^* \end{bmatrix} \begin{Bmatrix} \varphi_y + w_{0/y} \\ \varphi_x + w_{0/x} \end{Bmatrix}$$

↳ the notation A_{in}^* has been introduced to remind that a shear factor has been introduced.

- The relations reported here above hold for an isotropic plate (by replacing the various terms with the coefficients obtained in the previous page). However the expression of the constitutive law has the same pattern also for <u>orthotropic</u> plates.

- The case of generically layered composite plates can be considered by adopting a similar approach. In this case the only difference would be that, in general, the matrices expressing the constitutive law would be fully populated.

Equilibrium equations in terms of displacements

Having introduced the constitutive relation of the plate, it is possible to express the equilibrium equation in terms of displacement components.

The equilibrium equations, in their extended form, are:

$$\begin{cases} N_{xx/x} + N_{xy/y} + \hat{n}_x = 0 & (1) \\ N_{yy/y} + N_{xy/x} + \hat{n}_y = 0 & (2) \\ M_{xx/x} + M_{xy/y} - Q_x + \hat{m}_x = 0 & (3) \\ M_{yy/y} + M_{xy/x} - Q_y + \hat{m}_y = 0 & (4) \\ Q_{x/x} + Q_{y/y} + p = 0 & (5) \end{cases}$$

and, introducing the constitutive law:

$$(1):\ \left[A_{11} u_{o/x} + A_{12} v_{o/y}\right]_{/x} + \left[A_{66}\left(u_{o/y} + v_{o/x}\right)\right]_{/y} + \hat{n}_x = 0$$

$$(2):\ \left[A_{12} u_{o/x} + A_{22} v_{o/y}\right]_{/y} + \left[A_{66}\left(u_{o/y} + v_{o/x}\right)\right]_{/x} + \hat{n}_y = 0$$

$$(3):\ \left[D_{11} \varphi_{x/x} + D_{12} \varphi_{y/y}\right]_{/x} + \left[D_{66}\left(\varphi_{x/y} + \varphi_{y/x}\right)\right]_{/y} +$$

$$- A^*_{55}\left(\varphi_x + w_{o/x}\right) + \hat{m}_x = 0$$

$$(4):\ \left[D_{12} \varphi_{x/x} + D_{22} \varphi_{y/y}\right]_{/y} + \left[D_{66}\left(\varphi_{x/y} + \varphi_{y/x}\right)\right]_{/x} +$$

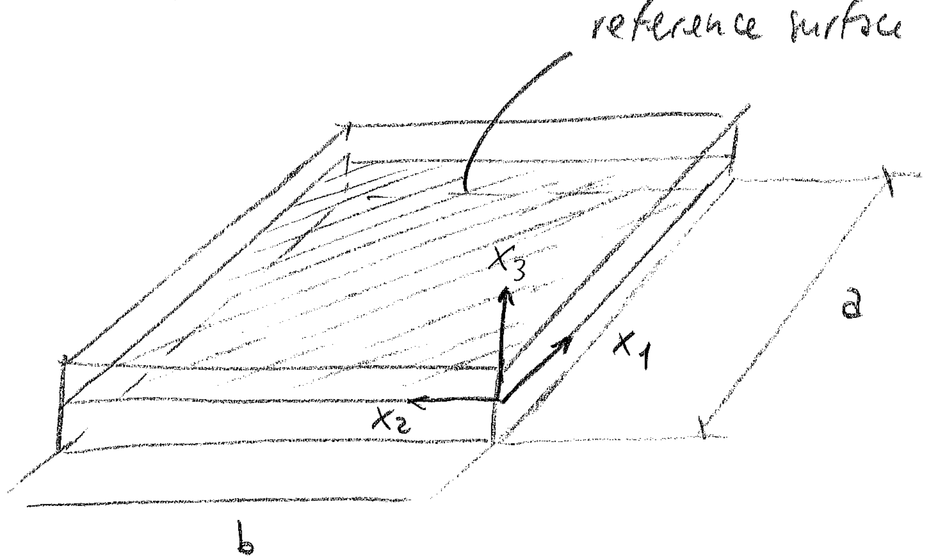
$$- A^*_{44}\left(\varphi_y + w_{o/x}\right) + \hat{m}_y = 0$$

$$(5):\ \left[A^*_{55}\left(\varphi_x + w_{o/x}\right)\right]_{/x} + \left[A^*_{44}\left(\varphi_y + w_{o/y}\right)\right]_{/y} + p = 0$$

- Kirchhoff plate theory

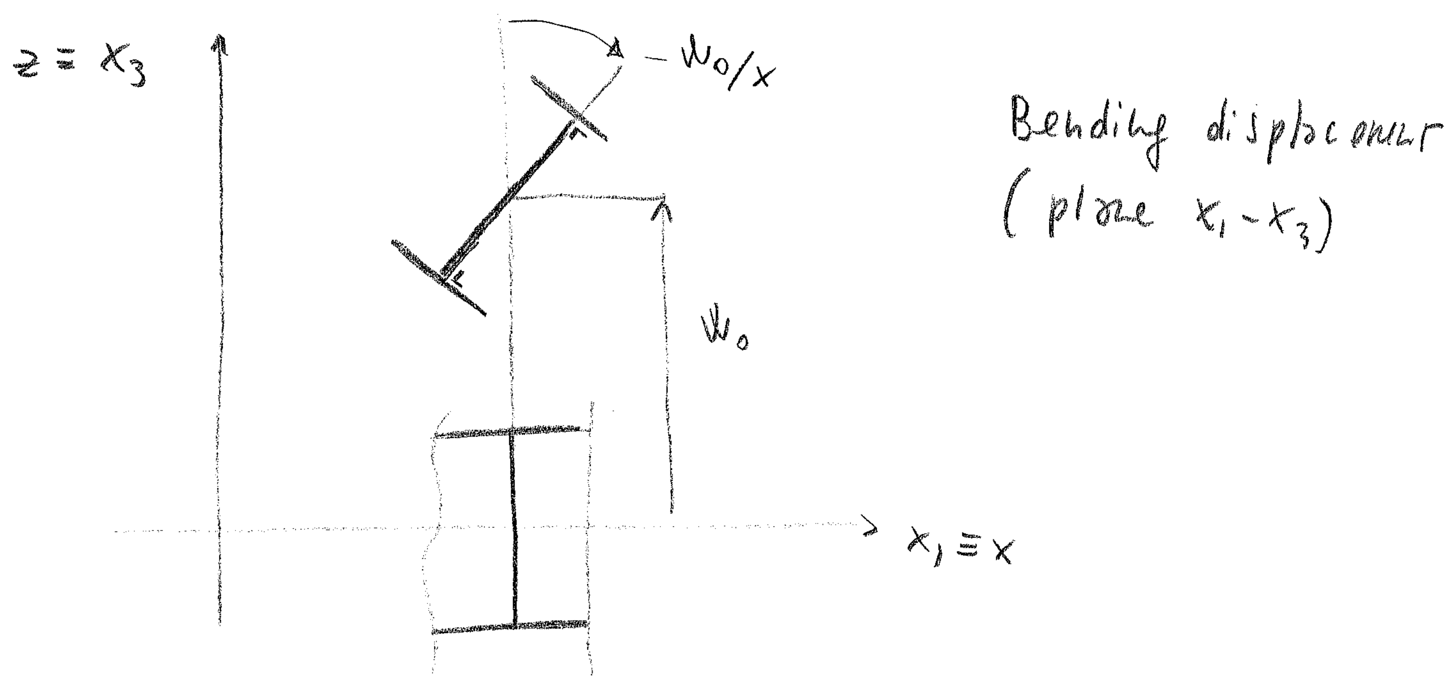
The plate model due to Kirchhoff can be seen as the 2D counterpart of the Euler-Bernoulli model. Indeed, the Kirchhoff plate model is based upon a kinematic field which leads to null transverse shear strains.
This means that Kirchhoff model is a suitable plate model for those cases where the energy contribution due to the transverse shear can be reasonably neglected.
This is the case of thin isotropic plates and thin composite plates. The range of applicability for isotropic plates is generally wider in comparison to the case of composite materials. Indeed the transverse shearing stiffness of typical aerospace composite materials (G_{23} and G_{13} shear moduli) is much lower in comparison to the isotropic ones, and the role played by shear deformability is then higher for some values of relative thickness a/h.
While a geometrical ratio $a/h = 1000$ is certainly a situation that can be analyzed referring to the Kirchhoff model, independently on the material type, when the plate gets thicker, say $a/h = 50$, it could be worth checking the appropriateness of the Kirchhoff model, also in relation to the material.

Consider a plate of dimensions a, b, thickness t and a Cartesian reference system where $x_3 \equiv z$ is the thicknesswise coordinate, while the reference surface is identified by $x_1 - x_2$



The assumption introduced by Kirchhoff is that sections normal to the plate midsurface remain normal after the deformation process.

This means that, in addition to the assumption introduced by Mindlin of section remaining straight, it is now assumed that they remain normal.



The same description is introduced with regard to the plane $x_2 - x_3$.

Note that the difference with respect to Mindlin deals with the description of the bending part of the displacement field whilst the membrane behaviour is represented in the same manner.

According to the above mentioned assumptions, the displacement field is expressed as:

$$\boxed{\begin{aligned} u_\alpha(x_\alpha, x_3) &= u_{0\alpha} - x_3 w_{0/\alpha} \\ w_0(x_\alpha, x_3) &= w_0 \end{aligned}} \qquad \alpha = 1, 2$$

where $u_{0\alpha}$, $w_{0/\alpha}$ and w_0 are the generalized displacement components of the Kirchhoff plate model.

As already observed for the other kinematic models, the generalized displacement components depend only on the position on the reference surface, i.e.

$$u_{0\alpha} = u_{0\alpha}(x_\alpha)$$

$$w_0 = w_0(x_\alpha)$$

$$w_{0/\alpha} = w_{0/\alpha}(x_\alpha)$$

It can be observed that, in the context of the Kirchhoff model, the rotation coincides with the first derivatives of the out-of-plane displacement field.

The strains are obtained as:

$$\boxed{\begin{aligned} \varepsilon_{\alpha\beta} &= \overset{\circ}{\varepsilon}_{\alpha\beta} + x_3 K_{\alpha\beta} \\ \gamma_{\alpha 3} &= 0 \end{aligned}} \qquad \alpha, \beta = 1, 2$$

where the membrane strains and the curvatures $K_{\alpha\beta}$ are

$$\boxed{\begin{aligned} \overset{\circ}{\varepsilon}_{\alpha\beta} &= \frac{1}{2}\left(u_{o\alpha/\beta} + u_{o\beta/\alpha}\right) \\ K_{\alpha\beta} &= - w_{o/\alpha\beta} \end{aligned}}$$

The differences with the Mindlin model are then

1. the curvatures $K_{\alpha\beta}$ are associated with the second derivative of the out of plane displacement (on the contrary the membrane deformations are still equal. Indeed the difference between the Mindlin and Kirchhoff model relies on the description of the bending behaviour)

2. the transverse shear deformations $\gamma_{\alpha 3}$ are null. The Kirchhoff plate model is thus uncapable of accounting for transverse shear deformability. In other words the Kirchhoff model relies upon the assumption that the strain energy contribution due to the transverse shear deformability is negligible with respect to the bending one.

This is in analogy with the assumption adopted in the case of slender beams, where it was observed that the bending energy is much higher in comparison to the shear one. In this sense, thin plates can be - in a slightly informal way - the 2D counterpart of slender beams.

It is worth noting that, in analogy with Mindlin, the normal deformation $\varepsilon_{33} \equiv \varepsilon_{zz}$ is equal to zero.
As observed for the Timoshenko model, this result represents a static inconsistency of the model, as the result $\varepsilon_{zz} = 0$ is in contrast with the assumption of plane stress condition (which, on the contrary is based on the assumption of $\sigma_{zz} = 0$, implying that $\varepsilon_{zz} \neq 0$). For this reason the evaluation of ε_{zz} should not be performed on a kinematic basis, viz calculating the derivatives of the assumed kinematic field. Whenever the value of ε_{zz} is needed, it should be evaluated by inverting the constitutive law.
It is finally noted that the effect of the inconsistency is energetically not relevant as the internal work $\sigma_{zz} \varepsilon_{zz}$ is null

- Generalized stresses and internal work

The expression of the internal work is formally equal to the one obtained for the Mindlin model.
Indeed, the expression of $\varepsilon_{\alpha\beta}$ is equal and the difference is nested in the expression of the curvatures $k_{\alpha\beta}$. Given the fact that $\gamma_{\alpha 3}=0$, the internal work due to the transverse shear is null. It follows that:

$$\delta W_i = \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} \, dV =$$

$$\boxed{\delta W_i = \int_A \left(\delta \varepsilon^{0}_{\alpha\beta} N_{\alpha\beta} + \delta k_{\alpha\beta} M_{\alpha\beta} \right) dA}$$

where the definition of $N_{\alpha\beta}$ and $M_{\alpha\beta}$ (forces and moments per unit length) is identical to the ones reported in the context of the Mindlin model.

- Generalized external forces

The evaluation of the external forces consistent with the kinematic model can be performed by writing the external virtual work due to the vector of volume forces $\underline{F}$. These steps can be easily verified, and lead to:

$$\boxed{\delta W_e = \int_A \left(\delta u_{0\alpha}\, \hat{n}_\alpha + \delta w_0\, p \right) dA}$$

(with respect to the Mindlin model, the distributed moments $\hat{m}_\alpha$ are not considered due to the inability of the model to account for transverse shear deformation effects).

Equilibrium conditions

The equilibrium conditions - viz. the strong form formulation of the problem - are derived by applying the PVW.
The plate is characterized by a boundary Γ, and
$\Gamma_F \cup \Gamma_U = \Gamma$, $\Gamma_F \cap \Gamma_U = \emptyset$.
In addition to the external forces per unit surface, the presence of forces/moments per unit length along the boundary Γ_F is then accounted for.
The PVW reads:

$$\boxed{\begin{aligned}\int_A \left(\delta \varepsilon_{\alpha\beta} N_{\alpha\beta} + \delta k_{\alpha\beta} M_{\alpha\beta} \right) dA &= \\ &= \int_A \left(\delta u_{0\alpha} \hat{n}_\alpha + \delta w_0 p \right) dA + \\ &+ \int_{\Gamma_F} \left(\delta u_{0\alpha} \hat{N}_\alpha - \delta w_{0/\alpha} \hat{M}_\alpha + \delta w_0 \hat{Q} \right) d\Gamma_F\end{aligned}}$$

where $\hat{N}_\alpha$, $\hat{M}_\alpha$ and $\hat{Q}$ are the external membrane forces, moments and transverse shear, respectively, acting over the portion of boundary Γ_F.

Note that the contribution $-\delta w_{0/\alpha} \hat{M}_\alpha$ is characterized by a negative sign as the rotation is the opposite of the first derivative of w_0.

The membrane part of the PVW is identical to the Mindlin case, so it leads to the same equilibrium conditions and boundary conditions previously obtained.
The analysis is then restricted to the bending part of the PVW, which is rewritten as:

$$\boxed{\int_A \delta k_{\alpha\beta} M_{\alpha\beta}\, dA = \int_A \delta w_0\, p\, dA + \int_{\Gamma_F} \left(- \delta w_{0/\alpha} \hat{M}_\alpha + \delta w_0 \hat{q} \right) d\Gamma_F}$$

- $\delta W_i = \int_A \delta k_{\alpha\beta} M_{\alpha\beta}\, dA = -\int_A \delta w_{0/\alpha\beta} M_{\alpha\beta}\, dA$

Integrating by parts:

$$= \int_A \delta w_{0/\alpha} M_{\alpha\beta/\beta}\, dA - \int_\Gamma \delta w_{0/\alpha} M_{\alpha\beta} n_\beta\, d\Gamma$$

The first contribution can be integrated by parts again:

$$= -\int_A \delta w_0 M_{\alpha\beta/\alpha\beta}\, dA + \int_\Gamma \delta w_0 M_{\alpha\beta/\beta} n_\alpha\, d\Gamma$$
$$- \int_\Gamma \delta w_{0/\alpha} M_{\alpha\beta} n_\beta\, d\Gamma$$

Observing that the virtual variations are compatible with the constraints:

$$\begin{cases} \delta w_0 = 0 & \text{in } \Gamma_u \\ \delta w_{0/\alpha} = 0 & \text{in } \Gamma_u \end{cases}$$

It follows that the boundary contribution is restricted to Γ_F, or:

$$\boxed{\delta W_i = -\int_A \delta w_0 \, M_{\alpha\beta/\alpha\beta} \, dA + \int_{\Gamma_F} \delta w_0 \, M_{\alpha\beta/\beta} \, n_\alpha \, d\Gamma_F - \int_{\Gamma_F} \delta w_{0/\alpha} \, M_{\alpha\beta} \, n_\beta \, d\Gamma_F}$$

Applying now the PVW, $\delta W_i = \delta W_e$ it is obtained:

$$\boxed{-\int_A \delta w_0 \left(M_{\alpha\beta/\alpha\beta} + p \right) dA + \int_{\Gamma_F} \delta w_0 \left(\underbrace{M_{\alpha\beta/\beta} \, n_\alpha}_{Q} - \hat{Q} \right) d\Gamma_F - \int_{\Gamma_F} \delta w_{0/\alpha} \left(M_{\alpha\beta} \, n_\beta - \hat{M}_\alpha \right) d\Gamma_F = 0}$$

and, from the arbitrarity of δw_0 and $\delta w_{0/\alpha}$, the condition expressed by the PVW is equivalent to:

$M_{\alpha\beta/\alpha\beta} + p = 0$	<u>equilibrium equation</u> in Ω	
$Q = \hat{Q}$	or $\delta w_0 = 0$	(1 eq.)
$M_{\alpha\beta} \, n_\beta = \hat{M}_\alpha$	or $\delta w_{0/\alpha} = 0$	(2 eqs.: $\alpha = 1,2$)
↑ natural boundary conditions (in Γ_F)	↑ essential boundary conditions (in Γ_u)	

This the <u>temporary</u> (see next page why) <u>strong form expression</u> <u>of the equilibrium conditions</u> (recall that the focus is here restricted to the bending case; the membrane part is identical to Mindlin)

The Kirchhoff condition (extra)

The differential problem defined by the equilibrium condition and the relevant boundary conditions is ill-posed, i.e. it cannot be solved in the form that has been obtained.

The reason can be easily understood by observing that:

a. The equilibrium equation is a fourth-order PDE. Indeed $M_{\alpha\beta}$, when expressed in terms of displacement components, depends upon $K_{\alpha\beta} = -w_{0/\alpha\beta}$ which is associated with the second derivative of w_0. $M_{\alpha\beta/\alpha\beta}$ leads to the fourth derivative of w_0.

b. the number of boundary conditions is 6 (3 natural and 3 essential), whereas a number of 4 is needed for a fourth-order differential problem.

It is then necessary to reduce the number of unknowns.

Clearly this operation cannot be conducted by arbitrarily removing 2 of the 6 boundary conditions, but a physically sound strategy is needed. The strategy is based on the application of the PVW.

From an historical perspective this was the first case where an equilibrium condition was obtained by making use of the PVW as an "automatic" mean for deriving equilibrium conditions.

Recall now the expression of the PVW previously obtained and consider the contribution due to $\delta w_{0/\alpha}$:

$$-\int_{\Gamma_F} \delta w_{0/\alpha} \left(M_{\alpha\beta} n_\beta - \hat{M}_\alpha \right) d\Gamma_F =$$

The components can now be expanded in a reference system sn, where the axes s and n are tangent and normal, respectively, to a generic point $P \in \Gamma_F$.

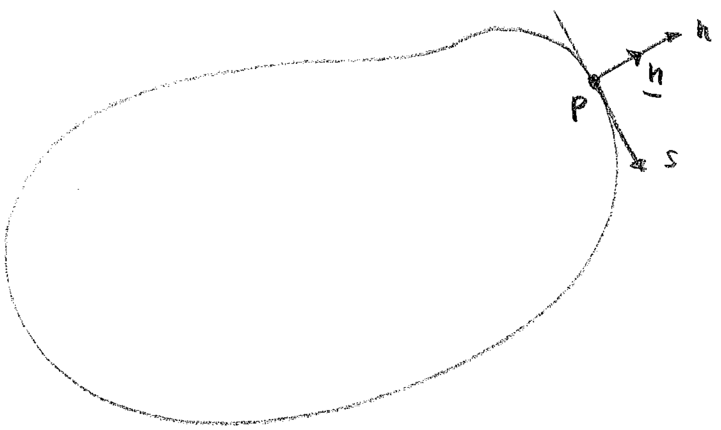


plate with generic shape and reference system sn

In the system sn, the normal vector $\underline{n}$ is written as:

$$\underline{n} = n_s \underline{e}_s + n_n \underline{e}_n$$

with $n_s = 0$
$n_n = 1$

$$= -\int_{\Gamma_F} \left[\delta w_{0/s} \left(M_{s\beta} n_\beta - \hat{M}_s \right) + \delta w_{0/n} \left(M_{n\beta} n_\beta - \hat{M}_n \right) \right] d\Gamma_F$$

$$= -\int_{\Gamma_F} \left[\delta w_{0/s} \left(\cancel{M_{ss} n_s} + M_{sn} n_n - \hat{M}_s \right) + \delta w_{0/n} \left(\cancel{M_{ns} n_s} + M_{nn} n_n - \hat{M}_n \right) \right] d\Gamma_F$$

$$= -\int_{\Gamma_F} \left[\delta w_{0/s} \left(M_{sn} - \hat{M}_s \right) + \delta w_{0/n} \left(M_{nn} - \hat{M}_n \right) \right] d\Gamma_F =$$

It is now observed that the first term can still be integrated by parts:

$$= \int_{\Gamma_F} \delta w_0 \left(M_{sn/s} - \hat{M}_{s/s} \right) d\Gamma_F - \delta w_0 \left(M_{sn} - \hat{M}_s \right)\Big|_{\Gamma_F}$$

$$- \int_{\Gamma_F} \delta w_{0/n} \left(M_{nn} - \hat{M}_n \right) d\Gamma_F$$

The boundary contribution is null:

$$\delta w_0 \left(M_{sn} - \hat{M}_s \right)\Big|_{\Gamma_F} = \delta w_0 \left(M_{sn} - \hat{M}_s \right)\Big|_{\Gamma_{start}}^{\Gamma_{end}} = 0 \qquad \Gamma_{end} \equiv \Gamma_{start}$$

and so:

$$- \int_{\Gamma_F} \delta w_{0/\alpha} \left(M_{\alpha\beta} n_\beta - \hat{M}_\alpha \right) d\Gamma_F =$$

$$= \int_{\Gamma_F} \left[\delta w_0 \left(M_{sn/s} - \hat{M}_{s/s} \right) - \delta w_{0/n} \left(M_{nn} - \hat{M}_n \right) \right] d\Gamma_F$$

The PVW can then be re-written as:

$$- \int_A \delta w_0 \left(M_{\alpha\beta/\alpha\beta} + p \right) dA + \int_{\Gamma_F} \delta w_0 \left(Q - \hat{Q} + M_{sn/s} - \hat{M}_{s/s} \right) d\Gamma_F$$

$$- \int_{\Gamma_F} \delta w_{0/n} \left(M_{nn} - \hat{M}_n \right) d\Gamma_F = 0$$

Kirchhoff free-edge condition

and so:

$M_{\alpha\beta/\alpha\beta} + p = 0$		equilibrium equation	
$Q + M_{sn/s} = \hat{Q} + \hat{M}_{sn/s}$	or	$\delta w_0 = 0$	(1 eq)
$M_{nn} = \hat{M}_n$	or	$\delta w_{0/n} = 0$	(1 eq)
natural boundary conditions (in Γ_F)		essential boundary conditions (in Γ_u)	

- The balance between the order of the equation and the number of boundary conditions is now satisfied.

- It is noted that $\hat{M}_s$ has been renamed as $\hat{M}_{sn}$ to highlight that it is a twisting moment. Indeed this moment is energetically conjugate with $\delta w_{0/s}$.

Interpretation of the Kirchhoff condition

The boundary conditions (natural) obtained are:

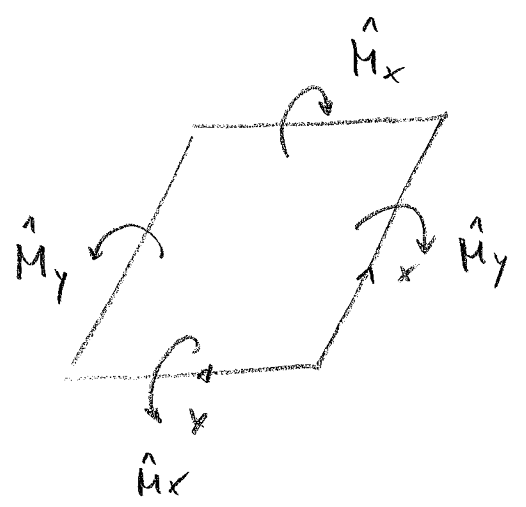
- $q + M_{sn/s} = \hat{q} + \hat{M}_{sn/s}$ ← shear equilibrium
- $M_{nn} = \hat{M}_n$ ← bending equilibrium

In contrast to the case of Mindlin, the equilibrium to the twisting moment is no longer available. The effect of an applied twisting moment $\hat{M}_{sn}$ is then reported into the shear condition. The applied twisting moment is then transformed into a statically equivalent shear.

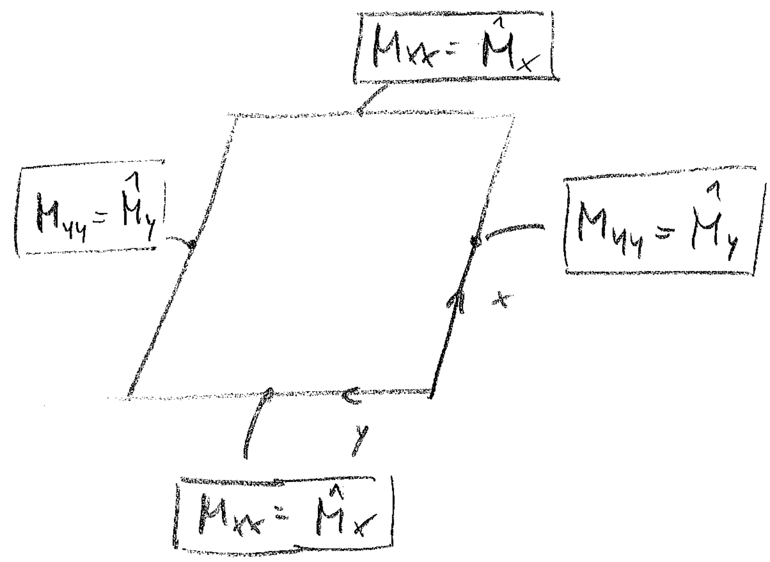
Rectangular plate

Consider now a rectangular plate. For simplicity, assume that none of the edges is free to undergo out-of-plane displacements. (So $w_0 = 0$ for all the edges)

The boundary conditions are thus:



applied loads



Natural boundary conditions

Similarly to the case of the Mindlin plate, the classification of the boundary conditions is:

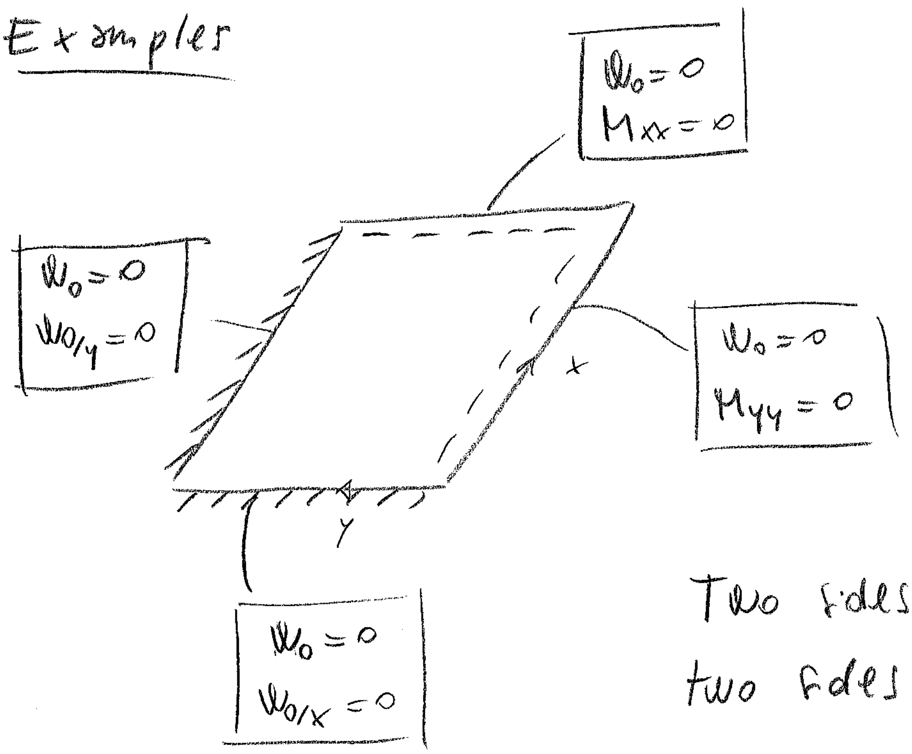
1. Free edge: no kinematic conditions specified
   $\Rightarrow$ 2 natural conditions
   Note: this case is not here considered!

2. Simply-supported edge: the transverse displacement is zero. The rotation around the tangent to the edge is free
   $\Rightarrow$ 1 essential, 1 natural condition

3. <u>Clamped edge</u>: the transverse displacement and the rotation around the tangent to the edge are prevented.

$\Rightarrow$ 2 essential conditions

<u>Examples</u>



Two sides clamped and two sides simply-supported

Plate constitutive law

The relation between generalized stresses and deformations is found in analogy to the procedure illustrated in the case of the Mindlin model. It is then assumed that the plate is subjected to a plane state of stress.

For the Kirchhoff plate model the transverse shear forces do not enter the equilibrium equations. The constitutive law is then restricted to the components:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & & & & \\ A_{12} & A_{22} & & & & \\ & & A_{66} & & & \\ \hline & & & D_{11} & D_{12} & \\ & & & D_{12} & D_{22} & \\ & & & & & D_{66} \end{array}\right] \begin{Bmatrix} u_{0/x} \\ v_{0/y} \\ u_{0/y} + v_{0/x} \\ -w_{0/xx} \\ -w_{0/yy} \\ -2w_{0/xy} \end{Bmatrix}$$

which holds for an orthotropic plate and for an isotropic plate. In this second case, as already obtained:

$$A_{11} = A_{22} = \frac{Et}{1-\nu^2} \qquad D_{11} = D_{22} = D$$

$$A_{12} = \nu \frac{Et}{1-\nu^2} \qquad D_{12} = \nu D$$

$$A_{66} = \frac{Et}{2(1+\nu)} \qquad D_{66} = \frac{1-\nu}{2} D$$

Equilibrium equation in terms of displacement

Observing that the membrane equations are identical to the ones obtained for the Mindlin model, the only equation here considered is the one relative to the bending equilibrium:

$$M_{\alpha\beta/\alpha\beta} + p = 0$$

which, in its expanded form, is:

$$M_{xx/xx} + 2 M_{xy/xy} + M_{yy/yy} + p = 0$$

Substituting now the constitutive equation:

$$- D_{11} w_{0/xxxx} - D_{12} w_{0/xxyy} - 4 D_{66} w_{0/xxyy} - D_{12} w_{0/xxyy} - D_{22} w_{0/yyyy} + p =$$

$$\boxed{D_{11} w_{0/xxxx} + 2(D_{12} + 2 D_{66}) w_{0/xxyy} + D_{22} w_{0/yyyy} = p}$$

which is the equilibrium equation governing the bending of an orthotropic plate subjected to a pressure p.

Under the assumption of isotropic material, the coefficients D_{ik} can be expressed as function of D and ν:

$$D w_{0/xxxx} + 2\left(\nu D + 2 \frac{1-\nu}{2} D\right) w_{0/xxyy} + D\, w_{0/yyyy} - p = 0$$

or

$$\boxed{\Delta\Delta w_0 = \frac{p}{D}}$$

where the operator Δ is defined as

$$\Delta(\cdot) = (\cdot)_{/xx} + (\cdot)_{/yy} \quad (\text{Laplacian})$$

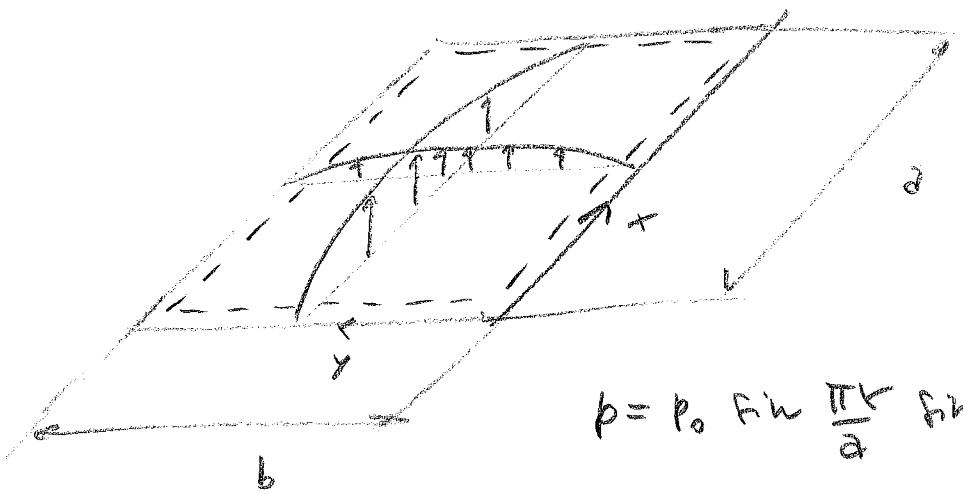
and so:

$$\Delta\Delta(\cdot) = (\cdot)_{/xxxx} + 2(\cdot)_{/xxyy} + (\cdot)_{/yyyy}$$

Exact solution for a simple case

The strong form formulation for the bending of a thin plate allows an exact solution for the case of sinusoidal pressure distribution.
This initial simple case serves also as introduction to the Navier-type solution.

Consider a plate of dimensions a and b, subjected to a simply-supported boundary condition along the four edges and loaded with a sinusoidal pressure p



$$p = p_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

The strong form formulation of the problem reads:

$$\begin{cases} \Delta\Delta w_0 = p/D & \text{in } \Omega \\ w_0 = 0 & \text{in } \partial\Omega \text{ (all the four edges)} \\ M_{xx}(0,y) = M_{xx}(a,y) = 0 & \\ M_{yy}(x,0) = M_{yy}(x,b) = 0 & \end{cases}$$

$\Big] \rightarrow$ natural conditions

The natural conditions can be expressed in terms of displacement components by introducing the constitutive law, leading to:

$$
\begin{cases}
\Delta\Delta w_0 = p/D \\
w_0(0,y) = w_0(a,y) = w_0(x,0) = w_0(x,b) = 0 & \leftarrow \text{essential} \\
w_{0/xx}(0,y) = w_{0/xx}(a,y) = 0 & \\
w_{0/yy}(x,0) = w_{0/yy}(x,b) = 0 & \text{natural}
\end{cases}
$$

It is straightforward to verify that the solution (guess)

$$w_0 = \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} A_0 \quad (\text{with } A_0 = \text{scalar to be determined})$$

identically satisfies the boundary conditions (both the essential and the natural).

After substituting the (guess) solution into the equilibrium equation it is obtained:

$$\left[\left(\frac{\pi}{a}\right)^4 + 2\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{b}\right)^4\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} A_0 = \frac{p_0}{D}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

The solution is then:

$$\boxed{A_0 = \frac{p_0}{\pi^4 D\left[\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2\right]^2}}$$

- The displacement field is then obtained as:

$$w_0 = A_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$ with A_0 previously obtained

- The stress field can be obtained as:

$$\begin{Bmatrix} \sigma_{xx}(x,y,z) \\ \sigma_{yy}(x,y,z) \\ \sigma_{xy}(x,y,z) \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} -z\,w_{0,xx} \\ -z\,w_{0,yy} \\ -2z\,w_{0,xy} \end{Bmatrix}$$

where:

$$w_{0,xx} = -A_0 \left(\frac{\pi}{a}\right)^2 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$w_{0,yy} = -A_0 \left(\frac{\pi}{b}\right)^2 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$w_{0,xy} = A_0 \frac{\pi^2}{ab} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b}$$