Preliminaries

A generic vector $\underline{v}$ can be expressed as:

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3$$

where $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ are the unit vectors of a Cartesian coordinate system, and v_1, v_2 and v_3 are the components of $\underline{v}$ along the directions 1,2 and 3.

The following notation can be adopted:

$$\underline{v} = v_i \underline{e}_i \qquad i = 1,2,3$$

where the summation is implied with respect to the repeated indexes (in this case just i). In other words:

$$\boxed{\begin{aligned} \underline{v} &= \sum_{i=1}^{3} v_i \underline{e}_i \\ &= v_i \underline{e}_i \end{aligned}}$$

Einstein's convention

Similarly, a second-order tensor can be represented as:

$$\underline{\underline{A}} = A_{ij} \underline{e}_i \otimes \underline{e}_j$$

where summation is now implied with respect to i and j, or:

$$\boxed{\begin{aligned} \underline{\underline{A}} &= \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} \underline{e}_i \otimes \underline{e}_j \\ &= A_{ij} \underline{e}_i \otimes \underline{e}_j \end{aligned}}$$

The term A_{ij} identifies the components of the tensor $\underline{\underline{A}}$ for a given set of unit vectors $\underline{e}_i$.

The tensor itself does not depend upon the choice of the reference system, but its components A_{ij} depend on the reference system.

$$\begin{aligned} \underline{\underline{A}} &= A_{ij}\, \underline{e}_i \otimes \underline{e}_j \\ &= \hat{A}_{ij}\, \underline{\hat{e}}_i \otimes \underline{\hat{e}}_j \end{aligned} \quad \rightsquigarrow \quad A_{ij} \neq \hat{A}_{ij}$$

where $\underline{e}_i$ and $\underline{\hat{e}}_i$ are different sets of unit vectors.

Often it is useful to collect the components of vectors and second-order tensors into one- and two-dimensional arrays:

$$\underline{v} = v_i\, \underline{e}_i \quad \rightsquigarrow \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\underline{\underline{A}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j \quad \rightsquigarrow \quad \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

This is an intuitive way for "storing" the components of a tensor, making it possible to perform tensor operations by means of the techniques of matrix algebra.

However it is worth highlighting that <u>tensors are not arrays</u>!

Scalar product

The scalar product between two vectors $\underline{a}$ and $\underline{b}$ is:

$$
\begin{aligned}
c &= \underline{a} \cdot \underline{b} \\
&= \left(a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3\right) \cdot \left(b_1 \underline{e}_1 + b_2 \underline{e}_2 + b_3 \underline{e}_3\right) \\
&= a_1 b_1\, \underline{e}_1 \cdot \underline{e}_1 + a_1 b_2\, \cancel{\underline{e}_1 \cdot \underline{e}_2} + a_1 b_3\, \cancel{\underline{e}_1 \cdot \underline{e}_3} + \\
&\quad a_2 b_1\, \cancel{\underline{e}_2 \cdot \underline{e}_1} + a_2 b_2\, \underline{e}_2 \cdot \underline{e}_2 + a_2 b_3\, \cancel{\underline{e}_2 \cdot \underline{e}_3} + \\
&\quad a_3 b_1\, \cancel{\underline{e}_3 \cdot \underline{e}_1} + a_3 b_2\, \cancel{\underline{e}_3 \cdot \underline{e}_2} + a_3 b_3\, \underline{e}_3 \cdot \underline{e}_3 \\
&= a_1 b_1\, \underline{e}_1 \cdot \underline{e}_1 + a_2 b_2\, \underline{e}_2 \cdot \underline{e}_2 + a_3 b_3\, \underline{e}_3 \cdot \underline{e}_3 \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3
\end{aligned}
$$

Using a compact notation:

$$
\begin{aligned}
c &= \underline{a} \cdot \underline{b} \\
&= a_i \underline{e}_i \cdot b_j \underline{e}_j \\
&= a_i b_j\, \underline{e}_i \cdot \underline{e}_j \\
&= a_i b_j\, \delta_{ij} \\
&= a_i b_i = a_j b_j
\end{aligned}
$$

<u>Note</u>: the indexes used for $\underline{a}$ and $\underline{b}$ must be different(!) to avoid any misunderstanding on the repeated indexes.

where $\delta_{ij} \begin{cases} = 0 & \text{if } i \neq j \\ = 1 & \text{if } i = j \end{cases}$

Consider now the scalar product between a second order tensor and a vector:

$$\underline{\underline{A}} \cdot \underline{b} = \underline{\underline{A}}\,\underline{b}$$ (the dot can be explicitly reported or not)

$$= A_{ij}\, \underline{e}_i \otimes \underline{e}_j \cdot b_k\, \underline{e}_k$$

$$= A_{ij}\, b_k\, \underline{e}_i \otimes \underbrace{\underline{e}_j \cdot \underline{e}_k}$$

$$= A_{ij}\, b_k\, \underline{e}_i\, \delta_{jk}$$

$$= A_{ij}\, b_k\, \delta_{jk}\, \underline{e}_i$$

$$= A_{ij}\, b_j\, \underline{e}_i = A_{ik}\, b_k\, \underline{e}_i$$

The result is then a vector $\underline{c}$, whose component c_i is equal to $A_{ik} b_k$.

Expanding the previous expression:

$$\underline{\underline{A}} \cdot \underline{b} = A_{ik}\, b_k\, \underline{e}_i$$

$$= A_{1k}\, b_k\, \underline{e}_1 + A_{2k}\, b_k\, \underline{e}_2 + A_{3k}\, b_k\, \underline{e}_3$$

$$= (A_{11} b_1 + A_{12} b_2 + A_{13} b_3)\, \underline{e}_1 + (A_{21} b_1 + A_{22} b_2 + A_{23} b_3)\, \underline{e}_2 + (A_{31} b_1 + A_{32} b_2 + A_{33} b_3)\, \underline{e}_3$$

Double scalar product (contraction)

When dealing with second- (or higher-) order tensors it is possible to introduce a "double" scalar product, which is commonly denoted with a double dot.

$$\underline{a} \cdot \underline{b} = c$$

(scalar product)

A scalar quantity is obtained from two vectors.

$$\underline{\underline{A}} : \underline{\underline{B}} = c$$

(contraction)

A scalar quantity is obtained from two second-order tensors.

$$\underline{\underline{A}} : \underline{\underline{B}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j : B_{rs}\, \underline{e}_r \otimes \underline{e}_s$$

$$= A_{ij} B_{rs}\, \underline{e}_i \otimes \underline{e}_j : \underline{e}_r \otimes \underline{e}_s$$

It is important to establish a convention to define how the double scalar product is taken (this choice is not unique, and different conventions are found in the literature)

$$= A_{ij} B_{rs} \, \underline{e}_i \otimes \underline{e}_j : \underline{e}_r \otimes \underline{e}_s$$

The convention adopted here considers the scalar products according to the graphical representation above, meaning that:

$$= A_{ij} B_{rs} \left(\underline{e}_i \cdot \underline{e}_r\right)\left(\underline{e}_j \cdot \underline{e}_s\right)$$

$$= A_{ij} B_{rs} \, \delta_{ir} \, \delta_{js}$$

$$= A_{ij} B_{ij} = A_{rs} B_{rs}$$

The expanded form is then:

$$\underline{\underline{A}} : \underline{\underline{B}} = A_{ij} B_{ij}$$

$$= A_{1j} B_{1j} + A_{2j} B_{2j} + A_{3j} B_{3j}$$

$$= A_{11} B_{11} + A_{12} B_{12} + A_{13} B_{13} +$$
$$A_{21} B_{21} + A_{22} B_{22} + A_{23} B_{23} +$$
$$A_{31} B_{31} + A_{32} B_{32} + A_{33} B_{33}$$

One example of application of the contraction operation is the evaluation of the strain energy density:

$$u = \frac{1}{2} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}$$

where $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$ are proper measures of stresses and strains. (which are second order tensors)

One can note that if $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$ are symmetric then the contraction reads:

$$\begin{aligned}
\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} = {} & \sigma_{11} \varepsilon_{11} + \sigma_{12} \varepsilon_{12} + \sigma_{13} \varepsilon_{13} + \\
& \sigma_{12} \varepsilon_{12} + \sigma_{22} \varepsilon_{22} + \sigma_{23} \varepsilon_{23} + \\
& \sigma_{13} \varepsilon_{13} + \sigma_{23} \varepsilon_{23} + \sigma_{33} \varepsilon_{33} \\
= {} & \sigma_{11} \varepsilon_{11} + \sigma_{22} \varepsilon_{22} + \sigma_{33} \varepsilon_{33} + \\
& \sigma_{12} \, 2 \varepsilon_{12} + \sigma_{13} \, 2 \varepsilon_{13} + \sigma_{23} \, 2 \varepsilon_{23}
\end{aligned}$$

For this reason, stress and strain components can be collected into one-dimensional arrays as:

$$\underline{\underline{\sigma}} \rightsquigarrow \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \qquad \underline{\underline{\varepsilon}} \rightsquigarrow \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

and the operation of contraction is then represented as:

$$\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} \rightsquigarrow \{ \varepsilon_{11} \; \varepsilon_{22} \; \varepsilon_{33} \; 2\varepsilon_{23} \; 2\varepsilon_{13} \; 2\varepsilon_{12} \} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

$$= \underline{\varepsilon}^T \underline{\sigma} = \underline{\sigma}^T \underline{\varepsilon}$$

This notation, generally denoted as Voigt-notation, is based on the representation of second-order tensor components by means of one-dimensional arrays. While it is useful for performing several algebraic manipulations, it is important to bear in mind that $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$ <u>are not</u> vectors

For instance, the rotation of $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$ follows the rules of the rotation of a second-order tensor, not those of a vector!

- Gradient and divergence

The gradient and divergence operators are defined as:

$$\boxed{\begin{aligned} \nabla(\cdot) &= \text{grad}\,(\cdot) = \frac{\partial}{\partial x_i}(\cdot)\,\underline{e}_i \\ \nabla\cdot(\cdot) &= \text{div}\,(\cdot) = \frac{\partial}{\partial x_i}(\cdot)\cdot\underline{e}_i \end{aligned}}$$

gradient

divergence

According to the above definitions, the gradient of a vector $\underline{a}$ is:

$$\text{grad}\,\underline{a} = \frac{\partial \underline{a}}{\partial x_i}\,\underline{e}_i$$

$$= \frac{\partial a_j \underline{e}_j}{\partial x_i}\,\underline{e}_i$$

$$= \frac{\partial a_j}{\partial x_i}\,\underline{e}_j \otimes \underline{e}_i$$

$$= a_{j/i}\,\underline{e}_j \otimes \underline{e}_i = a_{i/j}\,\underline{e}_i \otimes \underline{e}_j$$

Similarly, the divergence is:

$$\text{div}\,\underline{a} = \frac{\partial \underline{a}}{\partial x_i}\cdot\underline{e}_i$$

$$= \frac{\partial a_j}{\partial x_i}\,\underline{e}_j\cdot\underline{e}_i$$

$$= \frac{\partial a_j}{\partial x_i}\,\delta_{ij} = a_{i/i}$$

The divergence of a second-order tensor is:

$$\operatorname{div} \underline{\underline{A}} = \frac{\partial A_{rs}}{\partial x_i}\, \underline{e}_r \otimes \underline{e}_s \cdot \underline{e}_i$$

$$= A_{rs/i}\, \delta_{si}\, \underline{e}_r$$

$$= A_{ri/i}\, \underline{e}_r = A_{ir/r}\, \underline{e}_i$$