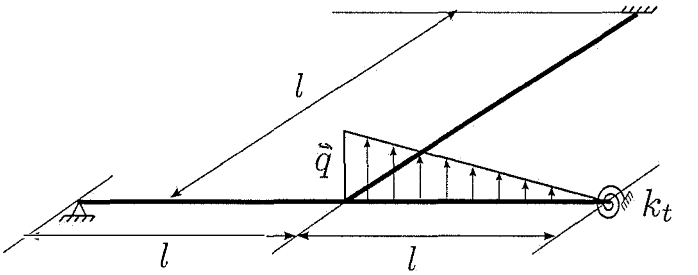
Course of Spacecraft Structures

Written test, September 11^{th}, 2017

Exercise 1

The structure in the figure is composed of three beams of length l, which are charaterized by the same elastic properties. In particular, the bending and torsional stiffnesses are denoted with EJ and GJ, respectively. The structure is elastically supported by a torsional spring of stiffness k_t at one end. A linearly varying force per unit length is applied along one of the beams, as illustrated in the sketch. The force per unit length is equal to zero at one end, and is equal to $\overline{q}$ at the other end. Determine the reaction forces and plot the internal actions.



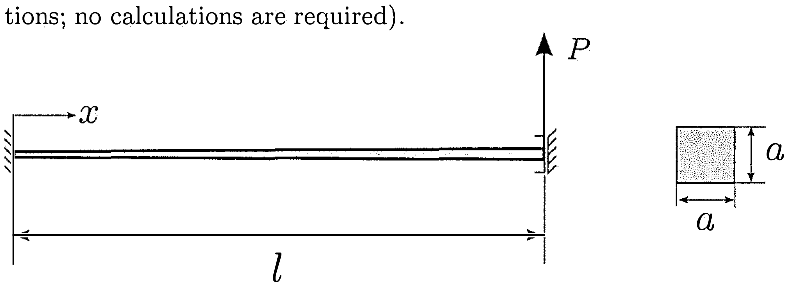
Data
$EJ = 72\text{e}6\ N\,mm^2$
$GJ = EJ/2$
$k_t = 72\ Nmm$
$l = 1000\ mm$
$\overline{q} = 100\ N/mm$

Exercise 2

The beam in the figure is made of an isotropic material with modulus E and Poisson's coefficient ν. It is characterized by a square cross section of dimension a, where $a(x) = \gamma \sqrt[4]{a_0 + (a_1 - a_0)\frac{x}{l}}$, and $x \in [0, l]$. The first end of the beam is fixed; the second one is free to translate along the transverse direction, while the rotation is prevented. A concentrated load of magnitude P is applied at $x = l$.

Assume a Euler-Bernoulli beam model, and determine: the transverse displacement at $x = l$; the maximum stress at $x = l/4$. To this aim, apply the method of Ritz and, using polynomial functions, and solve the problem by considering one single shape function.

Discuss how the problem can be formulated if two shape functions are adopted (report the relevant equations; no calculations are required).

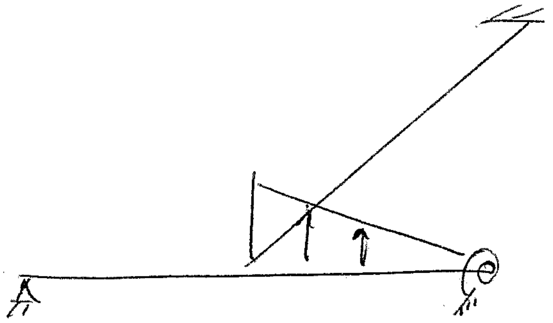


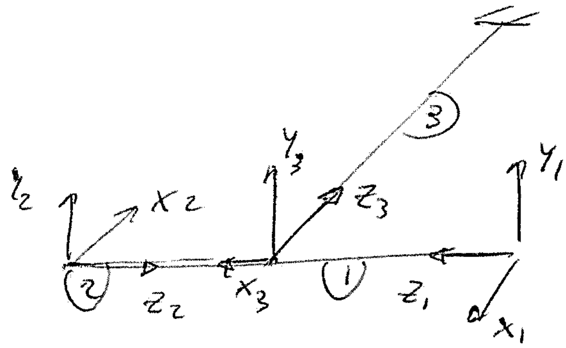
Data
$E = 72000\ N\,mm^2$
$l = 1000\ mm$
$\gamma = 10\ mm$
$a_0 = 1$
$a_1 = 16$

Question 1

Illustrate the relation between the stress measures according to Cauchy, Piola-Kirchhoff I and Piola-Kirchhoff II.

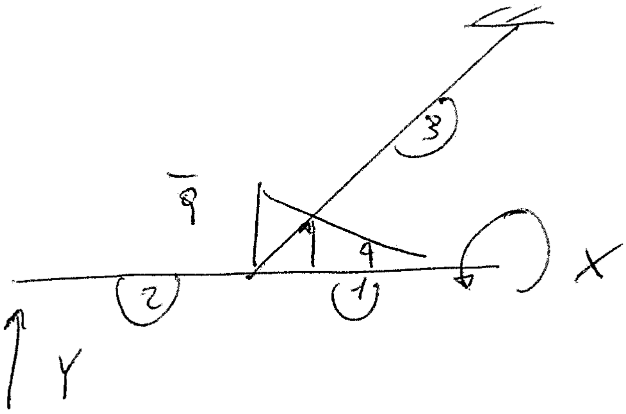
Exercise 1





Ref. systems

Real system

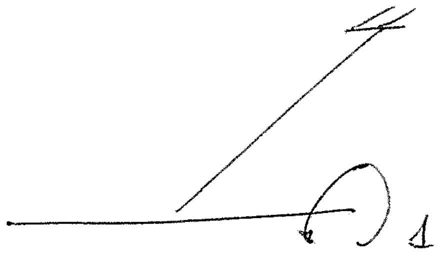


Note: X is a torsional moment

$$q(z_1) = \frac{\bar{q}}{\ell} z_1$$

$M_x^{(1)} = -\dfrac{\bar{q}}{6\ell} z_1^3$ $M_z^{(1)} = X$	beam 1
$M_x^{(2)} = -Y z_2$	beam 2
$M_x^{(3)} = X - \left(\bar{q}\ell/2 + Y\right) z_3$ $M_z^{(3)} = \dfrac{\bar{q}\ell^2}{6} - Y\ell$	beam 3

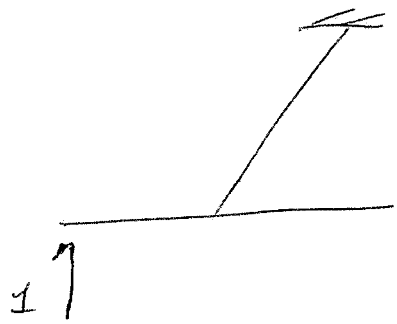
Dummy #1



$^1\delta M_z^{(1)} = 1$

$^1\delta M_x^{(3)} = 1$

Dummy #2



$^2\delta M_x^{(2)} = -z_2$

$^2\delta M_x^{(3)} = -z_3$

$^2\delta M_z^{(3)} = -\ell$

PCVW

$$\left(\frac{\ell}{GJ}+\frac{\ell}{EJ}+\frac{1}{k_t}\right)X - \frac{\ell^2}{2EJ}Y = \frac{\bar{p}\ell^3}{4EJ}$$

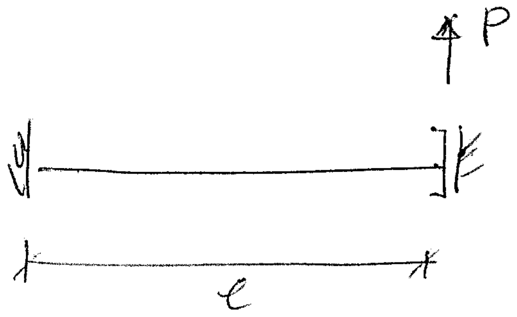
$$\frac{\ell^2}{2EJ}X - \left(\frac{2\ell^3}{3EJ}+\frac{\ell^3}{GJ}\right)Y = \frac{\ell^3}{3EJ}\frac{\bar{p}\ell}{2} - \frac{\ell^4\bar{p}}{6GJ}$$

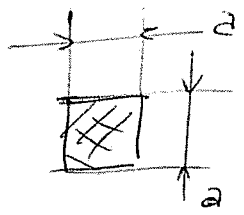
Da cui

$X = 28043.5$ Nmm

$Y = 6255.3$ N

Exercise 2





$$a(x) = \gamma \sqrt[4]{a_0 + (a_1 - a_0)\, x/\ell}$$

$$EJ = \frac{1}{12} a^4 \cdot E = \frac{1}{12} E \left(\gamma^4 \; a_0 + (a_1 - a_0)\, x/\ell \right)$$

$$EJ = EJ_0 + EJ_1 \left(x/\ell \right)$$

with $\quad EJ_0 = \frac{1}{12} E \gamma^4 a_0$

$$EJ_1 = \frac{1}{12} E \gamma^4 \left(a_1 - a_0 \right)$$

Trial functions

$$w = C_0 + C_1 x + C_2 x^2 + C_3 x^3$$

The essential conditions are:

$$\begin{cases} w(0) = 0 \\ w_{/x}(0) = 0 \\ w_{/x}(\ell) = 0 \end{cases}$$

From which:

$$w = C_3 \left(x^3 - \frac{3}{2} \ell x^2 \right)$$

The expression is re-organized to obtain a non-dimensional trial function: (not strictly necessary!)

$$w = c\left(\left(\frac{x}{\ell}\right)^3 - \frac{3}{2}\left(\frac{x}{\ell}\right)^2\right)$$

So $w = c_\phi \phi$ with $\phi = \left(\frac{x}{\ell}\right)^3 - \frac{3}{2}\left(\frac{x}{\ell}\right)$

Stiffness matrix

$$\delta W_i = \int_0^\ell \delta w_{,xx} \, EJ \, w_{,xx} \, dx$$

$$= \delta c \, k \, c$$

with $k = \int_0^\ell \left[EJ_0 + EJ_1\left(x/\ell\right)\right] \phi_{,xx}^2 \, dx$

$$= \frac{1}{\ell^3}\left(3EJ_0 + \frac{3}{2}EJ_1\right)$$

External loads

$$\delta W_e = \delta w(\ell) \, P$$

$$= \delta c \, f$$

with $f = -\frac{1}{2}P$

Solution

$$c = f/k = -\frac{\ell^3}{2}\frac{P}{3EJ_0 + \frac{3}{2}EJ_1} = -32.68 \text{ mm}$$

- Displacement at $x = \ell$:

$$w = -32.68\,\phi(\ell) = 16.34 \text{ mm}$$

- Stress at $x = \ell/4$

$$\sigma = E\varepsilon = E(-y\,w_{/xx})$$

$$= E(-y\,\phi_{/xx})\,c$$

$$\sigma(\ell/4) = -E\,y\,\phi_{/xx}(\ell/4)\,c$$

the maximum stress is obtained at $y = a/2$, so:

$$\sigma(\ell/4) = -\frac{Ea}{2}\phi_{/xx}(\ell/2)\,c =$$

$$= 26.05 \text{ MPa}$$

where $a = 14.76$ mm