The Ritz method

The Ritz method belongs to the class of the "direct methods", approximating techniques which are based on the variational formulation of the problem. The differential equations governing the problem at hand are not considered. This means that the functional expressing the total potential energy is considered within the context of a displacement-based formulation. On the contrary, the indefinite equilibrium equations do not need to be accounted for.

It is noted that the Ritz method is here presented within the framework of displacement-based formulations. However the method, which is an approximating strategy for imposing the stationarity of a functional, can be applied even in the context of force-based formulations (for imposing the compatibility setting to zero the first variation of the total complementary energy).

At this stage it should be recalled that a variational principle embeds in a single scalar equation both the governing equations and the natural boundary conditions (but not the essential ones).

For a displacement-based formulation, the variational principle is $\delta\Pi = 0$ (or $\delta W_i = \delta W_e$); the

governing equations are the equilibrium equations and the natural conditions express the equilibrium between internal and external forces at the boundaries.

Overview of the method

The underlying idea of the method is to transform the problem from a differential one (imposing the stationarity of a functional, $\delta\Pi = 0$, requires the solution of the Euler-Lagrange equations, which are a set of PDE whose solution can be hard to find) into an algebraic one, with clear advantages on the simplicity in obtaining a solution.

The transformation (from differential to algebraic problem) is carried out by approximating the unknown functions as a linear combination of functions, called trial functions, each multiplied with an unknow scalar amplitudes.

The scalar amplitudes then become the unknown of the problem to be evaluated by means of the variational principle.

Consider a differential problem which is governed by the equation

$$Au = f$$

where A is a differential operator, u is the unknown function and f is the forcing term

Assume now that the weak-form of the differential problem can be written as:

$$I(u) = \frac{1}{2} \mathcal{B}(u,u) - \ell(u)$$

where $I(u)$ is the functional, $\mathcal{B}(\cdot,\cdot)$ is a bilinear operator (quadratic in the unknown u) and $\ell(u)$ is a linear operator.

The functional $I(u)$ can be seen as the total potential energy Π, while the differential problem $Au = f$ the corresponding equilibrium equation.

In other words $Au = f$ is the Euler-Lagrange equation associated with the functional $I(u)$, viz. the condition that guarantees $\delta I(u) = 0$.

Setting: $\delta I(u) = 0$ implies that: $Au = f$

The Ritz method relies upon the idea of approximating the function u as:

$$\tilde{u}(x) = \sum_{i=1}^{N} c_i \phi_i(x) + \phi_0(x)$$

where:

$\phi_i(x)$, $\phi_0(x)$ are predetermined functions (thus known quantities)

c_i are the unknown amplitudes associated with the functions ϕ_i

- The functions $\phi_i(x)$ are identically zero over those portions of the boundary where the displacements are prescribed (Γ_u), while $\phi_0(x)$ is introduced in order to satisfy any non-homogeneous boundary condition.
- The unknown parameters c_i are the Lagrangian coordinates of the problem, or the generalized displacements. In this sense, the approximation $\tilde{u} = \sum_i c_i \phi_i + \phi_0$ can be interpreted as an additional kinematic assumption (indeed it is an aprioristic choice on the expected behaviour of u) which is overlaid to the kinematic assumptions of the structural model, e.g. those of the Euler-Bernoulli or Timoshenko beam model.

The steps of the Ritz method can then be summarized as:

1. Substitute the assumed description of the unknown into the functional:

$$I(u) \simeq I(\tilde{u}) = \frac{1}{2} \mathcal{B}(\tilde{u}, \tilde{u}) - \ell(\tilde{u})$$

2. Impose the stationarity condition for the approximated functional

$$\delta I(\tilde{u}) = 0$$

$$= \frac{\partial I}{\partial c_1} \delta c_1 + \frac{\partial I}{\partial c_2} \delta c_2 + \cdots \frac{\partial I}{\partial c_N} \delta c_N$$

$$= \frac{\partial I}{\partial c_i} \delta c_i \qquad (i = 1, \ldots, N)$$

The stationarity condition is then reduced to:

$$\boxed{\frac{\partial I}{\partial c_i} = 0 \qquad (i = 1, \dots N)}$$

↳ N algebraic equations

Example

Consider a truss of stiffness EA, which is loaded with a distributed load $\hat{n}$.

The differential problem reads:

$$\begin{cases} EAu_{,xx} + \hat{n} = 0 \\ u(0) = 0 \\ EAu_{,x}(\ell) = 0 \end{cases}$$

According to the notation previously introduced:

$$\mathcal{A}(\cdot) = EA(\cdot)_{,xx}$$

$$f = -\hat{n}$$

Looking at the variational principle:

$$\Pi = \frac{1}{2}\int_\ell EAu_{,x}^2\,dx - \int_\ell u\hat{n}\,dx \qquad \text{or}$$

$$I = \frac{1}{2}\mathcal{B}(u,u) - \ell(u) \qquad \text{with:}$$

$$\mathcal{B}(\cdot,\blacksquare) = \int_\ell EA(\cdot)_{,x}(\blacksquare)_{,x}\,dx \qquad \ell(\cdot) = \int_\ell \cdot\,\hat{n}\,dx$$

• Choice of the trial functions

The functions ϕ_i and ϕ_0 must be selected so that the system of solving algebraic equations is composed of N linearly independent equations (otherwise the solution cannot be obtained). In addition the functions must be chosen in order to guarantee the convergence to the exact solution, while fulfilling the compatibility conditions.
To summarize, the following requirements should be respected when choosing the trial functions:

1. The functions must satisfy the essential boundary conditions, but not necessarily the natural ones. Indeed, it is recalled that the variational principle automatically embeds the natural conditions, but not the essential ones. Trial functions satisfying both the essential and the natural conditions are called comparison functions, which are particularly effective in terms of convergence properties (not that, in general, it can be hard to identify a set of functions satisfying all of the boundary conditions).

   The displacement field is expressed as:

   $$\tilde{u}(x) = \sum_i c_i \phi_i(x) + \phi_0(x)$$

The essential conditions, whenever they are specified, will be in the general form:

$u = \hat{u}$ in Γ_u

The satisfaction of this condition is guaranteed by choosing ϕ_1 and ϕ_0 such that:

$\phi_1 = 0$ in Γ_u

$\phi_0 = \hat{u}$ in Γ_u

so that $\tilde{u} = 0 + \hat{u}$ in Γ_u

It follows that, whenever the essential conditions are of homogeneous type (no imposed displacements) then $\phi_0 = 0$ and the Ritz approximation is in the form

$\tilde{u} = \sum_i c_i \phi_i(x)$

2. The functions should be <u>continuous and differentiable up to the order required by the functional</u>. Denoting with n the minimum order of the derivative appearing in the functional, the trial functions shoul be, at least, C^{n-1}.

3. The set of function should be <u>complete</u> and the functions be <u>linearly independent</u>.

The idea of completeness can be understood by assuming a polynomial expansion, where:

$\tilde{u} = c_1 x + c_2 x^2 + \cdots c_N x^N + c_0$ (complete)

Assuming that, for instance, the quadratic term is removed from the expansion, it would be

$\tilde{u} = c_1 x + c_3 x^3 + \cdots c_N x^N + c_0$ (not complete)

This expansion would be incomplete; indeed the assumed expression of $\tilde{u}$ would be never able to come arbitrarily close to a quadratic solution.

It is then concluded that completeness implies not to avoid one or more terms up to the order of the expansion

Another example can be illustrated with regard to a trigonometric expansion:

$$\tilde{u} = \sin \pi x\, c_1 + \cos \pi x\, c_2 + \sin 2\pi x\, c_3 + \cos 2\pi x\, c_4 + \cdots \sin n\pi x\, c_{2n-1} + \cos n\pi x\, c_{2n} + c_0 \quad \text{(complete)}$$

Retaining one contribution, e.g. $\sin 2\pi x$, would lead to an incomplete set.

$$\tilde{u} = \sin \pi x\, c_1 + \cos \pi x\, c_2 + \cos 2\pi x\, c_4 + c_0 \quad \text{(not complete)}$$

The completeness of the set of functions guarantees the convergence of the method, meaning that the approximate solution will become closer and closer to the exact solution as the number of terms of the expansion is increased.

Remark

The most popular choices for the shape functions are polynomials and trigonometric functions.

- Polynomials offer the advantage of allowing the fulfillment of the essential boundary conditions in a easy way. However they suffer from a major disadvantage: the number of trial functions that can be used is limited for numerical reasons. In particular, the ill-conditioning of the stiffness matrix becomes more severe as far as the number of functions N is increased.
For some problems (for instance in the case of composite structures) it may be necessary to use several functions to achieve a good degree of accuracy. If polynomials are used this cannot be done as numerical issues typically arise for $N > 10$.

A much better choice consists in using orthogonal polynomials. (Legendre, Chebyshev, Jacobi, ...)
This kind of polynomials do not suffer from numerical issues due to the ill-conditioning of the problem and can be used up to high

number of terms.
In addition they offer very good convergence properties and the accuracy of the solution is generally much higher with respect to the accuracy achieved using different expansions (using the same number of dofs).
Another advantage given by orthogonal polynomials relies in the high degree of sparsity of the resulting matrices.

- Trigonometric functions are commonly used because they offer a serie of advantages, although their choice is not always the best one (in general orthogonal polynomials perform much better).
  In particular, trigonometric functions are C^∞, so they can be differentiated according to any need of the problem.
  In some cases, the exact solution of the problem is of trigonometric type (e.g. Navier solution for plates) thus this choice often seems the most natural one.
  For some sets of boundary conditions (typically simply-supported edges or hinges) the

sine-type trial function identically satisfies the essential and the natural conditions, so this choice is, again, the most direct one.
Another advantage relies in the orthogonality of trigonometric functions. The resulting matrices are characterized by a high-degree of sparsity (often they are diagonal). Also no ill conditioning problems exist, and the expansion can be increased up to the desired order.

Two remarks (extra)

- While the 3 previous conditions are those reported in the vast majority of the textbooks, it is worth emphasizing that a fourth condition would, in principle, need to be added.

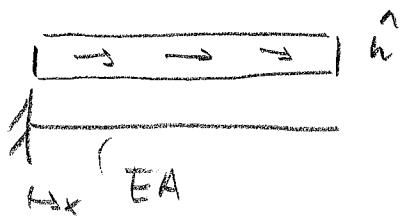
  4. The trial function should satisfy, in addition to the essential boundary conditions, also the complementary conditions.

  The complementary conditions specify that:

  a. the generalized displacement energetically conjugated with a prescribed generalized force on Γ_F is different from zero.

  b. the generalized force energetically conjugated with a prescribed generalized displacement on Γ_U is different from zero.

The idea is rather simple and can be easily understood by means of an example.



$$\delta\Pi = \int_0^\ell \left(\delta u_{,x}\, EA\, u_{,x} - \delta u \hat{n} \right) dx = 0$$

$$= -\int_0^\ell \delta u \left(EAu_{,xx} + \hat{n}\right) dx + \delta u\, EAu_{,x}\Big|_0^\ell = 0$$

which leads to:

$$EAu_{,xx} + \hat{n} = 0$$

$$EAu_{,x}(\ell) = 0 \quad \text{or} \quad \delta u(\ell) = 0$$

$$EAu_{,x}(0) = 0 \quad \text{or} \quad \delta u(0) = 0$$

For the problem at hand the conditions would be:

$EAu_{,x}(\ell) = 0$ — natural

$u(0) = 0$ — essential

The complementary boundary conditions are,

$u(\ell) \neq 0$ ↑

$EAu_{,x}(0) \neq 0$ ↓ — complementary

The complementary conditions, for this problem, state that:

- the displacement u in $x = \ell$ cannot be zero. At $x = \ell$ the displacement is unknown, thus its value cannot be specified a priori.
- the reaction force in $x = 0$ cannot be zero ($EAu_{,x}$ is the axial force). Indeed the truss is fixed so the exact solution of the problem requires a not null reaction force at the fixed end.

Based on these considerations it is clear that the approximation

$$\tilde{u} = c_1 \sin\frac{\pi x}{\ell}$$

would be respectful of the essential condition but not of the complementary one $u(\ell) \neq 0$

- The completeness of the set of functions is commonly associated with the ability of the approximating functions to describe any arbitrary curve. This implies that the expansion $\tilde{u} = \sum_i c_i \phi_i$ should be able to get arbitrarily close to any function. However it is more correct to intend the completeness as the ability to get arbitrarily close to any value of the potential energy Π. In some cases a set of function could be capable of getting close to any function but not to the potential energy Π.
In other words <u>completeness has to be intended energetically</u>.

Convergence of the method

The method of Ritz guarantees the monotonic convergence of the total potential energy Π. As far as the set of trial functions is enriched with more and more functions, the value of Π approaches the exact one. In this sense:

$$\Pi^1 \geq \Pi^2 \geq \Pi^3 \cdots \geq \Pi^N \geq \Pi$$

where Π^i: total potential energy using i trial functions.

Π: " " " associated with the exact solution

This property is easily understood by recalling that the equilibrium is associated with $\delta\Pi = 0$, corresponding to the minimum value of Π. It is then clear that the minimization of Π using 1 single dof (Π^1) will lead to a higher value of the minimization of the problem using 2 dof (Π^2) and so forth.

With this regard it can be stated that the stiffness of the model is progressively reduced as the number of dofs is increased. The total potential energy provides a global measure of the stiffness; the idea that a model with few dofs is stiffer than a model with more dofs should then be

intended in a <u>global sense</u>; locally it is possible that the displacement of a model with few dofs is higher in comparison to the one obtained using more dofs.

It is then necessary to distinguish between two different measures of the convergence:

* see next page

$$\lim_{N\to+\infty} \|e_N^u\|_\infty = 0 \quad \text{with} \quad e_N^u = \max_{x\in\Omega} |u - \tilde{u}_N|^2 \qquad \leftarrow \underline{\text{local}} \ (\text{uniform})$$

$$\lim_{N\to+\infty} \|e_N^u\|_{L^2} = 0 \quad \text{with} \quad e_N^u = \sqrt{\int_\Omega (u-\tilde{u}_N)^2 \, d\Omega} \qquad \leftarrow \underline{\text{global}}$$

The local convergence guarantees that the maximum difference between the approximate solution and the exact one gets smaller as N is increased.
This convergence is guaranteed if and only if the solution is continuous up to the order $n-1$, where n is the highest derivative in Π.
In structural mechanics this is typically verified for the displacement function, which is always a continuous function.
Thus <u>displacements converge uniformly</u>.

Often it is useful to normalize the errors in order to facilitate the comparison between different solutions. The two norms are then taken as:

$$\| \bar{e}_N^u \|_\infty = \frac{\max_{x \in \Omega} | u - \tilde{u}_N |}{| u(x_{max}) |}$$

where $u(x_{max})$ is the exact value in correspondence of the position where the max error is achieved.

$$\| \bar{e}_N^u \|_{L^2} = \frac{\sqrt{\int_\Omega (u - \tilde{u}_N)^2 d\Omega}}{\sqrt{\int_\Omega u^2 \, d\Omega}}$$

The values of $\| \bar{e}_N \|_\infty$ and $\| \bar{e}_N \|_{L^2}$, multiplied by 100, provide the percent difference of the approximate solution of the maximum and the average error, respectively.

On the contrary, the generalized stress may be discontinuous functions; thus convergence of the generalized stresses is, in general, of global type but not local.

It is also noted that the rate of convergence is lower when derivatives are of concern. This means that strains and stresses, which are associated with the derivatives of the displacement functions, are characterized by a slower convergence with respect to the displacement function.

How can the convergence of the solution be monitored to decide how many terms N consider?
In general, it is dependent on the kind of analysis and the goals of the analysis itself.
For a static analysis, aimed at the evaluation of the stress distribution, one possibility would be to check the natural boundary conditions and set a tolerance value to consider as a stopping criterion over the number of the functions N.
Another possibility would be to monitor the stress in some spots of interest and check their convergence as N is increased.

In the context of free-vibration or buckling analysis, the response of interest would be a global quantity (the natural frequencies or the buckling load). It would be a proper choice checking the convergence of the solution with respect to the eigenvalues of the problem or by monitoring the value of π.

Some features of the Ritz method

1. Consider the stiffness matrix obtained using 2 trial functions: (The problem is in the form $\underline{\underline{K}}\,\underline{c} = \underline{f}$, where $\underline{c}$ are the Ritz coefficients)

$$\underset{2\times 2}{\underline{\underline{K}}} = \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}$$

Increasing now the number of trial functions from 2 to 4 leads to a stiffness matrix of dimension 4×4, whose coefficients are partially known. In particular:

$$\underset{4\times 4}{\underline{\underline{K}}} = \begin{bmatrix} \underset{2\times 2}{\underline{\underline{K}}} & \begin{matrix} \times & \times \\ \times & \times \end{matrix} \\ \begin{matrix} \times & \times \\ \times & \times \end{matrix} & \begin{matrix} \times & \times \\ \times & \times \end{matrix} \end{bmatrix}$$

When the number of functions is increased, it is then not necessary to recompute all the terms of $\underline{\underline{K}}$. A sub-portion of them can be inherited from the previous discretization obtained with less dofs.

2. Whenever it is possible, the adoption of <u>orthogonal trial functions</u> is a <u>very good choice</u>. In this case $\mathcal{B}(\phi_i, \phi_j) = \delta_{ij}$ and the stiffness matrix is a diagonal matrix:

$$\underline{\underline{K}} = \begin{bmatrix} \bullet & & & \\ & \bullet & & \\ & & \times & \\ & & & \times \end{bmatrix}$$

The efficiency of the solution is, in this case, drastically improved. The system of equations $\underline{\underline{K}}\,\underline{c} = \underline{f}$ can be solved quickly thanks to the diagonal pattern of $\underline{\underline{K}}$; furthermore $\underline{\underline{K}}$ can be stored as a sparse matrix with consequent benefits on the amount of memory needed by the program implementing the method.

3. It is worth noting that the stiffness matrix $\underline{\underline{K}}$ is

   a. in general, <u>fully populated</u> (unless the trial functions are orthogonal)

   b. <u>symmetric and semi-positive definite</u>

The symmetry can be exploited to quickly construct $\underline{\underline{K}}$ by computing just $N\frac{(N+1)}{2}$ terms instead of $N \times N$

4. The natural conditions are satisfied in a weak form sense. As the number of trial functions is increased the solution converges to the exact one, which implies better and better satisfaction of the equilibrium equations and natural boundary conditions

Additional remarks

1. The method is highly efficient as accurate results can be obtained with few dofs. The ratio accuracy to problem size is then particularly advantageous.
However it is worth noting that the method suffers from two main drawbacks
   a. the handling of domains with complex shapes is not always easy. The method works very fine for simple shapes (eg. circular, rectangular, .. plates)

b. When boundary conditions are modified, the entire matrices ($\underline{\underline{K}}$, $\underline{\underline{M}}$, ...) have to be recalculated.
This is a major drawback when comparing Ritz to the FEM where a change of the boundary conditions does not imply the re-evaluation of the matrices.

The Galerkin method

The method commonly referred to as Galerkin method is sometimes denoted as Bubnov-Galerkin method.

The method is applied to the governing differential equations of the problem, and not to the variational principle as the method of Ritz.
The starting point is then given by the equilibrium equations.
The method is based on the approximation of the unknown functions using global functions (as the Ritz method). The approximation is then substituted into the governing equations; as far as the approximation is not the exact solution, a residual is obtained. The method of Galerkin is based on the idea of setting to zero the weighted residual over the domain, by considering weight functions equal to the functions used for the approximation of the unknown field.

Consider the differential problem expressed as:

$$\mathcal{A}u = f \quad \Rightarrow \quad \mathcal{A}u - f = 0$$

1. Approximate the unknown function as:

$$\tilde{u} = \sum_{i=1}^{N} c_i \phi_i(\underline{x}) + \phi_0(\underline{x})$$

2. Substitute the approximation into the governing equation

$$\mathcal{A}\tilde{u} - f \neq 0$$

The quantity on the left-hand side is the <u>residual</u>, which is a function of the problem coordinates x_i expressing the error due to the approximation $\tilde{u}$

$$R = \mathcal{A}\tilde{u} - f \neq 0 \qquad \text{residual}$$

3. Set to zero the weighted residual over the domain, using the same functions adopted for the expansion of u:

$$\boxed{\int_\Omega \phi_i R = 0 \qquad i = 1, \dots, N}$$

and so:

$$R = \mathcal{A}\tilde{u} - f$$

$$= \mathcal{A}\left(\sum_i c_i \phi_i + \phi_0\right) - f$$

$$\int_\Omega \phi_i \, R \, d\Omega = \int_\Omega \phi_i \, \mathcal{A}\left(\sum_{j=1}^{N} \phi_j \, c_j\right) d\Omega$$

$$+ \int_\Omega \phi_i \left(\mathcal{A}\phi_0 - f\right) d\Omega$$

which leads to a discrete set of equations in the form:

$$\underline{\underline{K}}\,\underline{c} = \underline{f}$$

with

$$K_{ij} = \int_\Omega \phi_i \, \mathcal{A}\phi_j \, d\Omega$$

$$f_i = -\int_\Omega \phi_i \left(\mathcal{A}\phi_0 - f\right) d\Omega$$

Observe that:

1. the global functions ϕ_i should now satisfy both the essential and the natural boundary conditions.

   This is a distinction of fundamental

importance with respect to Ritz.
The Galerkin method relies upon an approximation of the governing equations, thus no information regarding the natural conditions is contained in the discrete set of equations $\underline{\underline{K}}\,\underline{q} = \underline{f}$.

2. The requirements on the regularity of the functions ϕ_i is stronger than in Ritz. In particular the functions should now be $\phi^{h=1}$ where n is the maximum order of the derivative in the strong form (which is different from the maximum order of the weak form)

The modified Galerkin method

Another approximating technique based on the strong form formulation of the problem is the so-called "modified Galerkin method". The main motivation of this method is to relax the requirements on the functions ϕ_i. In particular, the method is useful whenever it is hard to find a set of functions ϕ_i satisfying both the essential and the natural conditions (as required by Galerkin). In this sense, the modified Galerkin method tries to recover the advantage of Ritz in the context of a method based on the strong form formulation of the problem.

Consider the problem in the form:

$$\begin{cases} \mathcal{A}u = f & \text{in } \Omega \\ u = \hat{u} & \text{in } \Gamma_u \\ \mathcal{B}u = \hat{f} & \text{in } \Gamma_f \end{cases} \quad \left.\begin{matrix} \text{(essential)} \\ \rightarrow \text{boundary conditions} \\ \text{(natural)} \end{matrix}\right.$$

The residual is constructed by accounting for a domain and a boundary-related contribution

$$R_\Omega = \mathcal{A}\tilde{u} - f$$

$$R_\Gamma = \mathcal{B}\tilde{u} - \hat{f}$$

and the weighted-residual formulation is then:

$$\int_\Omega \phi_i (\mathcal{A}\tilde{u} - f)\, d\Omega + \int_{S_F} \phi_i (B\tilde{u} - \hat{f})\, dS_F = 0$$

and recalling that:

$$\tilde{u} = \sum_j c_j \phi_j + \phi_0$$

$$\int_\Omega \phi_i \left[\mathcal{A}\left(\sum \phi_j c_j + \phi_0\right) - f\right] + \int_{S_F} \phi_i \left[B\left(\sum \phi_j c_j + \phi_0\right) - \hat{f}\right] dS_F$$

$$\Rightarrow \quad K_{ij} c_j = f_i \qquad \text{where:}$$

$$K_{ij} = \int_\Omega \phi_i \mathcal{A} \phi_j \, d\Omega + \int_{S_F} \phi_i B \phi_j$$

$$f_i = -\int_\Omega \phi_i (\mathcal{A}\phi_0 - f)\, d\Omega - \int_{S_F} \phi_i (B\phi_0 - \hat{f})\, dS_F$$

The two contributions obtained by integrating over S_F are those associated with the natural boundary conditions.

The Petrov-Galerkin method and other weighted residual methods

The (Bubnov)-Galerkin method can be interpreted as a special case of the more general class of the weighted residual methods.

In particular the Petrov-Galerkin method is obtained by weighting the residual R with a set of functions different from those used for approximating the solution, so:

$$\tilde{u} = \sum_i c_i \phi_i + \phi_0$$

$$R = A\tilde{u} - f$$

$$\int_\Omega \psi_i R \, d\Omega = 0 \qquad \text{with} \quad \psi_i \neq \phi_i$$

The collocation method consists in setting to zero the residual in a discrete number of points in the domain Ω:

$$R_i = A u(x_i) - f = 0 \qquad i = 1, \dots N$$

This corresponds to the weighted-residual formulation where the weighting function is

taken as a Dirac's delta:

$$\int_{\Omega} \delta(\underline{x} - \underline{x}_i)\, R \, d\Omega = 0$$

Also the <u>least-square method</u> can be interpreted in the framework of weighted residual methods. Indeed:

$$\min \quad I = \int_{\Omega} R^2 \, d\Omega$$

This formulation corresponds to imposing:

$$\delta I = 0 \quad \rightarrow \quad \delta I = \int_{\Omega} \delta R \, R \, d\Omega$$

where $\delta R = \dfrac{\partial R}{\partial c_i} \delta c_i$ and recalling that

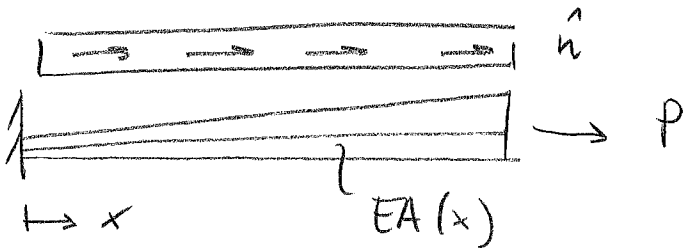
$$R = A\tilde{u} - f = A(\phi_i c_i + \phi_0) - f$$

it is: $\dfrac{\partial R}{\partial c_i} = A\phi_i$ and so:

$$\delta I = \int_{\Omega} A\phi_i \, R \, d\Omega$$

Examples

Consider a bar characterized by axial stiffness $EA = EA(x)$ and subjected to a distributed axial load, and a concentrated force at the end.



1. Ritz formulation

$$\Pi = \frac{1}{2}\int_0^\ell u_{/x}\, EA\, u_{/x}\, dx - \int_0^\ell u\,\hat{n}\, dx - u(\ell)\,P$$

The Ritz functions ϕ_i are taken s.t.:

$u(0) = 0$ (essential condition)

is identically satisfied.

The axial displacement is then:

$$u = \sum_{i=1}^{N} c_i\, \phi_i(x)$$

and so:

$$\Pi = \frac{1}{2}\sum_{i,j}^{N,N} c_i \int_0^\ell \phi_{i/x}\, EA\, \phi_{j/x}\, dx\, c_j - \sum_{i=1}^{N} c_i \int_0^\ell \phi_i\, \hat{n}\, dx - \sum_{i=1}^{N} c_i\, \phi_i(\ell)\, P$$

Setting now $\delta\Pi = 0$ leads to:

$$\delta\Pi = \sum_{i,j}^{N,N} \delta c_i \int_0^\ell \phi_{i/x}\, EA\, \phi_{j/x}\, dx\, c_j - \sum_{i=1}^{N} \delta c_i \int_0^\ell \phi_i \hat{n}\, dx$$

$$- \sum_{i=1}^{N} \delta c_i\, \phi_i(\ell)\, P = 0$$

and so $\delta\Pi = \dfrac{\partial \Pi}{\partial c_i} = 0 \qquad i = 1, \ldots, N$:

$$\sum_{j=1}^{N} \int_0^\ell \phi_{i/x}\, EA\, \phi_{j/x}\, dx\, c_j = \int_0^\ell \phi_i \hat{n}\, dx + \phi_i(\ell)\, P$$

or

$$k_{ij}\, c_j = f_i$$

where

$$k_{ij} = \int_0^\ell \phi_{i/x}\, EA\, \phi_{j/x}\, dx$$

$$f_i = \int_0^\ell \phi_i \hat{n}\, dx + \phi_i(\ell)\, P$$

2. Galerkin formulation

The differential problem reads:

$$\begin{cases} (EAu_{/x})_{/x} + \hat{n} = EA_{/x}\, u_{/x} + EAu_{/xx} + \hat{n} = 0 \\ EAu_{/x} = P \quad \text{in } x = \ell \\ u = 0 \quad \text{in } x = 0 \end{cases}$$

The displacement is expanded as:

$$u = \sum_{i=1}^{N} c_i \phi_i(x) \qquad \text{(formally identical to Ritz)}$$

The functions ϕ_i must now be taken such that:

$$u_{/x}(\ell) = P/EA$$

$$u(0) = 0$$

are identically satisfied.

The residual is:

$$R = EA_{/x} \sum_{i=1}^{N} c_i \phi_{i/x} + EA \sum_{i=1}^{N} c_i \phi_{i/xx} + \hat{n}$$

The weighted residual is now set to zero over the domain $x \in [0 \quad \ell]$:

$$\int_0^{\ell} \phi_i \left[EA_{/x} \sum_{j=1}^{N} \phi_{j/x} c_j + EA \sum_{j=1}^{N} \phi_{j/xx} c_j + \hat{n} \right] dx = 0$$

and so:

$$K_{ij} c_j = f_i$$

where:

$$K_{ij} = \int_0^\ell \left(\phi_i EA/x \, \phi_{j/x} + \phi_i EA \, \phi_{j/xx} \right) dx$$

$$f_i = - \int_0^\ell \phi_i \hat{n} \, dx$$

Note that the expression of K_{ij} contains the second derivative of ϕ_j with respect to x, while in the case of Ritz only the first derivative is present. For this reason the order on the regularity of ϕ is higher when Galerkin is applied, as it is now necessary that the second derivative exists.

3. Modified Galerkin

The effect due to the natural conditions is now considered:

$$R_\Omega = EA_{/x} \sum_{i=1}^{N} c_i \phi_{i/x} + EA \sum_{i=1}^{N} c_i \phi_{i/xx} + \hat{n}$$

$$R_F = EA \sum_{i=1}^{N} c_i \phi_{i/x}(\ell) - P$$

and the weighted residual is:

$$\int_0^\ell \phi_i R_\Omega \, dx + \phi_i(\ell) R_F = 0 \qquad i = 1, \ldots, N$$

which leads to:

$$\int_0^\ell \phi_i \left[EA_{/x} \sum_{j=1}^{N} \phi_{j/x} + EA \sum_{j=1}^{N} \phi_{j/xx} \right] c_j \, dx$$

$$+ \int_0^\ell \phi_i \hat{n} \, dx + \phi_i(\ell) EA \sum_{j=1}^{N} \phi_{j/x}(\ell) \, c_j$$

$$- \phi_i(\ell) P = 0$$

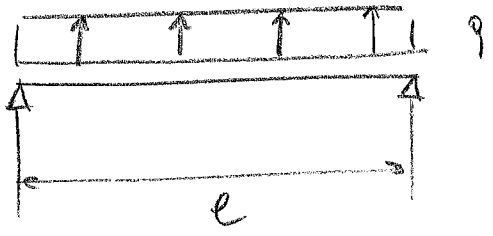
or

$$k_{ij} c_j = f_i \qquad \text{where:}$$

$$k_{ij} = \int_0^\ell \phi_i \left[EA_{/x} \phi_{j/x} + EA \phi_{j/xx} \right] dx + \phi_i(\ell) EA \sum_{j=1}^{N} \phi_{j/x}(\ell)$$

$$f_i = - \int_0^\ell \phi_i \hat{n} \, dx + \phi_i(\ell) P$$

Example



Determine the deflected pattern using a Ritz approximation with 1, 2 and 3 polynomial terms

Solution

$$\Pi = \frac{1}{2}\int_0^\ell EJ\, w_{,xx}^2\, dx - \int_0^\ell w q\, dx$$

$$\left.\begin{aligned} w(0) &= 0 \\ w(\ell) &= 0 \end{aligned}\right\} \text{ essential}$$

$$\left.\begin{aligned} w_{,xx}(0) &= 0 \\ w_{,xx}(\ell) &= 0 \end{aligned}\right\} \text{ natural}$$

The Ritz approximation is taken as:

$$w = \sum_{i=1}^{N} c_i\, \phi_i(x)$$

chosen to satisfy the essential conditions (but not necessarily the natural ones)

The variational principle is then:

$$\Pi = \frac{1}{2}\int_0^\ell \sum_{i,j=1}^{N,N} c_i\, \phi_{i,xx}\, EJ\, \phi_{j,xx}\, c_j\, dx - \int_0^\ell \sum_{i=1}^{N} c_i\, \phi_i\, q\, dx$$

Setting now $\delta\Pi = 0$ leads to:

$$\delta\Pi = \frac{\partial \Pi}{\partial c_i}\,\delta c_i = 0 \quad \Rightarrow \quad \frac{\partial \Pi}{\partial c_i} = 0 \qquad i = 1, \dots, N$$

So:

$$\frac{\partial \Pi}{\partial c_i} = \sum_{j=1}^{N} \int_0^{\ell} \phi_{i/xx}\, EJ\, \phi_{j/xx}\, dx\, c_j - \int_0^{\ell} \phi_i\, q\, dx$$

$$= k_{ij}\, c_j - f_i = 0$$

With:

$$k_{ij} = \int_0^{\ell} \phi_{i/xx}\, EJ\, \phi_{j/xx}\, dx = EJ \int_0^{\ell} \phi_{i/xx}\, \phi_{j/xx}\, dx$$

$$f_i = \int_0^{\ell} \phi_i\, q\, dx = q \int_0^{\ell} \phi_i\, dx$$

Selection of the trial functions

A polynomial expansion is adopted up to the fourth order. The general expression is then:

$$w = \tilde{c}_0 + \tilde{c}_1 x + \tilde{c}_2 x^2 + \tilde{c}_3 x^3 + \tilde{c}_4 x^4$$

To guarantee the fulfillment of the essential conditions it is imposed:

$$\begin{cases} w(0) = 0 \\ w(\ell) = 0 \end{cases} \Rightarrow \begin{cases} \tilde{c}_0 = 0 \\ \tilde{c}_1 \ell + \tilde{c}_2 \ell^2 + \tilde{c}_3 \ell^3 + \tilde{c}_4 \ell^4 = 0 \end{cases}$$

From the second relation:

$$\tilde{C}_1 = -\tilde{C}_2 \ell - \tilde{C}_3 \ell^2 - \tilde{C}_4 \ell^3 \quad , \text{ and so:}$$

$$w = \left(-\tilde{C}_2 \ell - \tilde{C}_3 \ell^2 - \tilde{C}_4 \ell^3\right) x + \tilde{C}_2 x^2 + \tilde{C}_3 x^3 + \tilde{C}_4 x^4$$

and re-arranging the terms (and re-naming the coefficients):

$$w = C_1 (x^2 - \ell x) + C_2 (x^3 - \ell^2 x) + C_3 (x^4 - \ell^3 x)$$

1. Solution using one single term

Consider first the case where w is approximated as:

$$w = C_1 (x^2 - \ell x) \qquad \text{so} \qquad \phi_1 = x^2 - \ell x$$

It is obtained:

$$k = \int_0^\ell \phi_{1/xx} \, \phi_{1/xx} \, dx \; EJ = 4 EJ \ell$$

$$f = q \int_0^\ell \phi_{1/xx} \, dx = -\frac{1}{6} q \ell^3$$

and so $$C_1 = -\frac{q \ell^2}{24 EJ}$$

2. Solution using two terms

$$w = C_1 (x^2 - \ell x) + C_2 (x^3 - \ell^2 x)$$

$$= C_1 \phi_1 + C_2 \phi_2$$

$$k_{11} = EJ \int_0^\ell \phi_{1/xx}\, \phi_{1/xx}\, dx = 4EJ\ell \qquad \text{(already obtained)}$$

$$k_{12} = EJ \int_0^\ell \phi_{1/xx}\, \phi_{2/xx}\, dx = 6EJ\ell^2$$

$$k_{21} = k_{12} \quad \text{(from symmetry)}$$

$$k_{22} = EJ \int_0^\ell \phi_{2/xx}\, \phi_{2/xx}\, dx = 12EJ\ell^3$$

$$f_1 = q \int_0^\ell \phi_{1/xx}\, dx = -\frac{1}{6} q\ell^3$$

$$f_2 = q \int_0^\ell \phi_{2/xx}\, dx = -\frac{q\ell^4}{4}$$

and so:

$$\begin{bmatrix} 4EJ\ell & 6EJ\ell^2 \\ 6EJ\ell^2 & 12EJ\ell^3 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} -q\ell^3/6 \\ -q\ell^4/4 \end{Bmatrix}$$

From which:

$$c_1 = -\frac{q\ell^2}{24EJ} \qquad c_2 = 0$$

The result is thus identical to the previous one as $w = \phi_1 c_1$

This is quite intuitive as the deflected pattern is expected to be symmetric with respect to the

midspan. The function ϕ_2 is not symmetric so the contribution is null.

3. Solution using three terms

The displacement is now expanded as:

$$w = c_1 (x^2 - \ell x) + c_2 (x^3 - \ell^2 x) + c_3 (x^4 - \ell^3 x)$$

The novel contributions to be determined are

$$k_{13} = EJ \int_0^\ell \phi_{1/xx} \phi_{3/xx} dx = 8 EJ \ell^3$$

$$k_{23} = EJ \int_0^\ell \phi_{2/xx} \phi_{3/xx} dx = 18 EJ \ell^4$$

$$k_{33} = EJ \int_0^\ell \phi_{3/xx} \phi_{3/xx} dx = \frac{144}{5} EJ \ell^5$$

and

$$f_3 = q \int_0^\ell \phi_3 dx = -\frac{3}{10} q \ell^5$$

and the solution of the problem

$$\underset{3\times3}{\underline{\underline{k}}} \; \underset{3\times1}{\underline{c}} = \underset{3\times1}{\underline{f}} \quad \text{is:}$$

$$c_1 = 0$$

$$c_2 = -\frac{q\ell}{12EJ}$$

$$c_3 = \frac{q}{24EJ}$$

and so:

$$w = -\frac{ql}{12EJ}\left(x^3 - \ell^2 x\right) + \frac{q}{24EJ}\left(x^4 - \ell^3 x\right)$$

$$\boxed{w = \frac{q\ell^4}{24EJ}\left(\frac{x}{\ell} - 2\frac{x^3}{\ell^3} + \frac{x^4}{\ell^4}\right)}$$

This solution is, in fact, the exact solution of the problem. It can be easily verified by integrating the EDO governing the differential problem:

$$\begin{cases} EJ\, w_{,xxxx} = q \\ w(0) = 0 \\ w(\ell) = 0 \\ w_{,xx}(0) = 0 \\ w_{,xx}(\ell) = 0 \end{cases}$$

(alternatively: it is easier to check that the Ritz solution obtained with three dofs identically satisfies the differential problem)

Final remarks

1. In this example the Ritz solution with 3 dof is the exact solution. However

it is not a rule to obtain the exact solution.
In this special case the exact solution is polynomial so a polynomial expansion is capable of capturing the exact solution.
With this regard it is worth noting that the expansion could be increased using a number of terms $N>3$.
The result would be:

$$C_1 = 0$$

$$C_2 = -\frac{ql}{12EJ}$$

$$C_3 = \frac{q}{24EJ}$$

$$\left.\begin{aligned} C_4 &= 0 \\ &\vdots \\ C_N &= 0 \end{aligned}\right\} \quad N>3$$

Clearly the quality of the solution can not improve as the solution found with $N=3$ is exact.
Thus, there not exist a combination of the Ritz amplitudes which is better than the one obtained with $N=3$.

2. Another possibility for defining the trial function would be to build the expansion as:

$$w = c_1 x(x-\ell) + c_2 x^2(x-\ell) + c_3 x^3(x-\ell)$$

which clearly satisfies the two essential boundary conditions.
In this case the expression of the stiffness matrix and the load vector would be different, but the deflected shapes would be equal to the ones previously obtained.
Indeed the same kind of expansion (polynomial), up to the same degree, has the same representation properties (more precisely the vector space spanned by the trial functions is the same)

3. It can be useful to compare the results in terms of displacements and energies.

Considering:

$\ell = 100$

$EJ = \frac{1}{24} 10^8$

$q = 1$

It is obtained

	W_{mid}	$M_{x,mid} \cdot 10^3$	π
N = 1,2	0.2500	0.8333	- 8.3333
N = 3	0.3125	1.2500	- 10.000

The errors are:

	$\% W_{mid}$	$\|e^W\|_{L^2} \cdot 10^2$	$\% M_{x,mid}$	$\|e^{M_x}\|_{L^2} \cdot 10^2$
N = 1,2	- 20.00	17.96	- 33.33	40.8250
N = 3	0.00	0.00	0.00	0.00

The solution obtained using one single term is rather poor and the errors on the displacement W and the bending moment are 20 % and 33 %, respectively.

The convergence of the solution is, in this case, very fast as the expansion up to N=3 can represent the exact solution.

Example:

Repeat the previous example by considering trigonometric functions

Solution

The trigonometric description of the out-of-plane displacement is taken as:

$$w = \sum_{m=1,3,5}^{N} c_m \sin \frac{m\pi x}{\ell}$$

- Note that the assumed displacement field satisfies both the essential and the natural conditions. Thus they are comparison functions

- Note also that the even contributions can be removed from the expansion. Indeed the even terms are anti-symmetric and the net work associated with external load is zero. They do not provide any contribution to the description of the solution (whose anti-symmetric contributions are zero)

In this case the stiffness matrix becomes:

$$k_{ij} = \delta_{ij} \; EJ \left(\frac{i\pi}{\ell}\right)^4 \frac{\ell}{2}$$

$$f_i = \frac{2\ell}{i\pi} \qquad \text{with } i \text{ odd}$$

The results are obtained as:

	w_{mid}	$M_{x,mid} \cdot 10^3$	Π
N = 1	0.3137	1.2901	− 9.9856
N = 3	0.3124	1.2423	− 9.9992
N = 5	0.3125	1.2526	− 9.9999
⋮			
N = 23	0.3125	1.2500	− 10.0000

and the errors:

	% w_{mid}	$\|e^w\|_{L^2} \cdot 10^2$	% $M_{x,mid}$	$\|e^{M_x}\|_{L^2} \cdot 10^2$
N = 1	0.3857	0.4128	3.2049	3.8013
N = 3	−0.0274	0.0326	−0.6175	0.8673
N = 5	0.0047	0.0062	0.2081	0.3364
⋮				
N = 23	0.0000	0.0000	−0.0037	0.0112

It can be noted that:

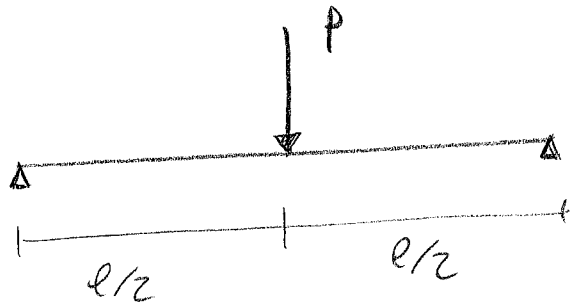
1. The displacement in the middle obtained with $N=1$ is higher with respect to the exact value and to the values obtained with $N>1$. One could be tempted to conclude that the models associated with $N>1$ are stiffer with respect to the model with $N=1$. This conclusion would be <u>wrong</u>. Indeed the stiffness of the model, as a whole, has to be intended from a global standpoint: the correct metric to be considered is the <u>total potential energy</u>. With this regard it can be seen that the value of Π gets smaller as N increases, thus the models become more and more compliant by increasing the number of dofs.

2. Despite the oscillations in the convergence of W, the percent difference, in terms of absolute values, becomes smaller and smaller as N is increased. The convergence of W is then obtained in the norm $\|e^W\|_\infty$.

3. It is important to remark that the convergence of the bending moment M_x is slower in comparison to the displacements.
Indeed $M_x = -EJ\, w_{0,xx}$, thus it is associated with the derivatives of the displacement field.
This operation has a negative effect on the convergence, as seen from the results.

4. Note the better accuracy of the 1-term solution with respect to the polynomial one with $N=1$.
The sine-type deflection is, indeed, a good representation of the exact pattern.
The quality of this solution is not surprising by recalling also that the natural conditions are identically satisfied.

Example



$\ell = 100$

$P = 1$

$EJ = \frac{1}{48} P \ell^3$

The expression of the internal work is the same of the previous example, and so the expression of the stiffness matrix is unchanged.

$$K_{ij} = EJ \int_0^{\ell} \phi_{i/xx} \phi_{j/xx} \, dx$$

The only difference relies on the applied load:

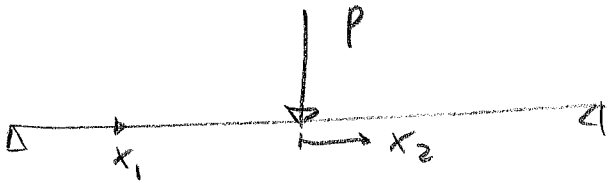
$$\delta W_e = -\delta u(\ell/2) P$$

and so, after introducing the Ritz approximation it is:

$$f_i = -\phi_i(\ell/2) P$$

Exact solution

The exact solution of the problem can be derived by integrating the governing differential equation. It is:



$$w_1 = \frac{P\ell^3}{48EJ}\left(4\frac{x_1^3}{\ell^3} - 3\frac{x_1}{\ell}\right) \qquad 0 \le x_1 \le \ell/2$$

$$w_2 = \frac{P\ell^3}{48EJ}\left(-4\frac{x_2^3}{\ell^3} + 6\frac{x_2^2}{\ell^2} - 1\right) \qquad 0 \le x_2 \le \ell/2$$

and so the bending moments are

$$M = -\frac{P}{2}x_1 \qquad 0 \le x_1 \le \ell/2; \quad T = -\frac{P}{2}$$

$$M = \frac{P(2x_2 - \ell)}{4} \qquad 0 \le x_2 \le \ell/2; \quad T = \frac{P}{2}$$

These results will be used for checking the quality of the Ritz approximations.

Solution using polynomial functions

The trial functions can be taken as:

$$w = c_1 (x^2 - \ell x) + c_2 (x^3 - \ell^2 x) + c_3 (x^4 - \ell^3 x) + \cdots$$

or

$$w = \sum_{i=1}^{N} c_i \left(x^{i+1} - \ell^i x \right)$$

The stiffness matrix is available from the previous example. It is here expanded to consider $N=5$ terms

$$\underline{\underline{k}} = EJ \begin{bmatrix} 4\ell & 6\ell^2 & 8\ell^3 & 10\ell^4 & 12\ell^5 \\ & 12\ell^3 & 18\ell^4 & 24\ell^5 & 30\ell^6 \\ & & \frac{144}{5}\ell^5 & 40\ell^6 & \frac{360}{7}\ell^7 \\ & & & \frac{400}{7}\ell^7 & 75\ell^8 \\ & & & & 100\ell^9 \end{bmatrix}$$

$$\underline{f} = P \begin{Bmatrix} \ell^2/4 \\ 3/8\,\ell^3 \\ \frac{7}{16}\ell^4 \\ 15/32\,\ell^5 \\ 31/64\,\ell^6 \end{Bmatrix}$$

The results are obtained as:

	w_{mid}	$M_{x,mid}$	Π
N = 1	-0.7500	-12.5000	-0.3750
N = 2	-0.7500	-12.5000	-0.3750
N = 3	-0.9844	-20.3125	-0.4922
N = 4	-0.9844	-20.3125	-0.4922
N = 5	-0.9961	-22.0703	-0.4980

and the errors are

	% w_{mid}	$\|e^w\|_{L^2} \cdot 10^2$	% $M_{x,mid}$	$\|e^{M_x}\|_{L^2} \cdot 10^2$
N = 1	-25.00	21.8619	-50.0000	50.0000
N = 2	-25.00	21.8619	-50.0000	50.0000
N = 3	-1.5625	1.0719	-18.7500	12.5000
N = 4	-1.5625	1.0719	-18.7500	12.5000
N = 5	-0.3906	0.2413	-11.7188	6.2500

Again note:

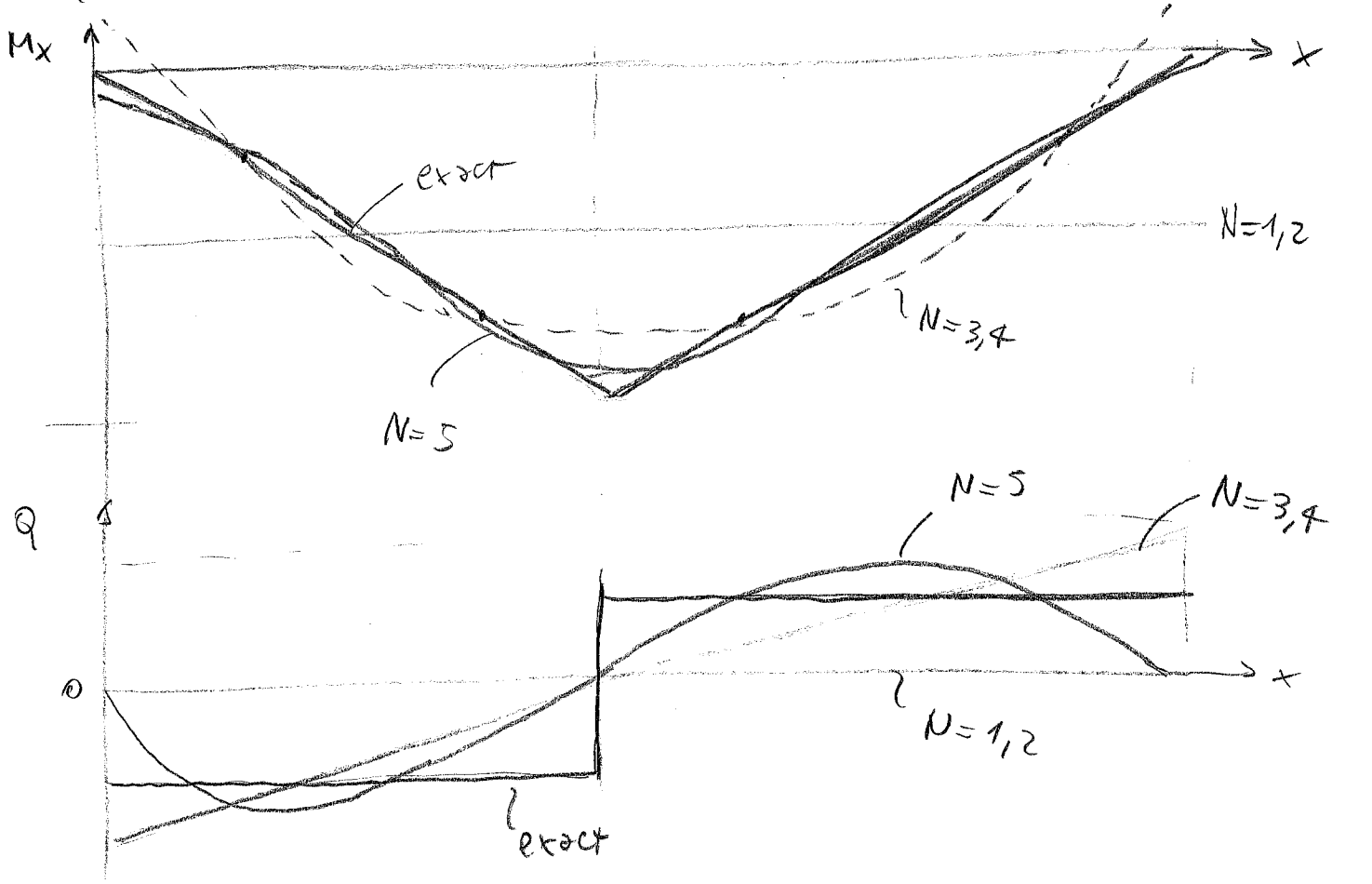
1. The monotonic convergence of Π
2. The uniform convergence of w
3. The highest rate of convergence of w with respect to M_x

With respect to the case of uniform load, the polynomial solution does not lead to the exact solution.

Indeed the exact solution is piecewise cubic, and is characterized by a discontinuity of the third derivative (the shear) at the mid-span. Clearly this discontinuity cannot be captured by the Ritz functions, which are defined at global level.

This is clearly seen by plotting the internal bending moment and the internal shear force.

(recall that $M_x = -EJ w_{0,xx}$ and $Q = M_{x,x} = -EJ w_{0,xxx}$)



As far as the number of terms is increased, the solution gets closer to the exact behaviour but in a global sense.

Solution using trigonometric functions

The trial functions are taken as

$$w = \sum_i c_i \sin \frac{i\pi x}{\ell} \qquad i = 1, 3, \ldots \quad \text{(odd values)}$$

The stiffness matrix reads:

$$k_{ij} = \delta_{ij} EJ \left(\frac{i\pi}{\ell}\right)^4 \frac{\ell}{2}$$

$$f_i = -P \sin \frac{i\pi}{2}$$

The results are:

	w_{mid}	$M_{x,mid}$	Π
N = 1	-0.9855	-20.2642	-0.4928
N = 3	-0.9977	-22.5158	-0.4989
N = 5	-0.9993	-23.3264	-0.4996
⋮			
N = 23	-1.0000	-24.5781	-0.5000

The exact solution is characterized by

$w_{mid} = -1.0000$; $M_{x,mid} = -25.0000$ and $\Pi = -0.5000$

Note that Π converges at the exact value quickly, whilst $M_{x,mid}$ is still different from the exact value

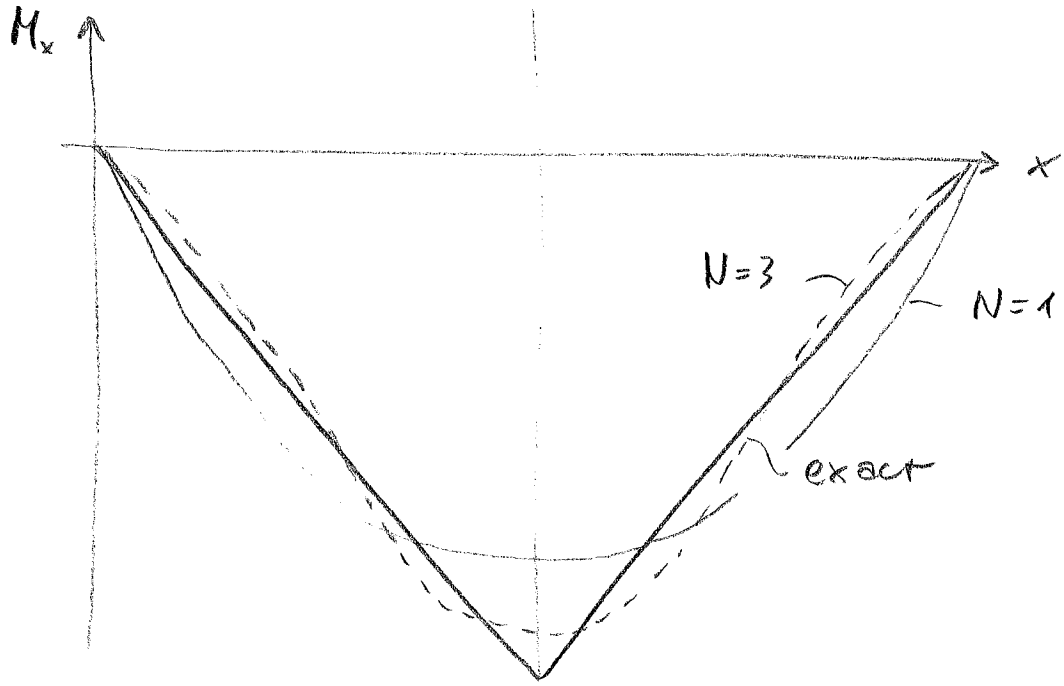
The errors are

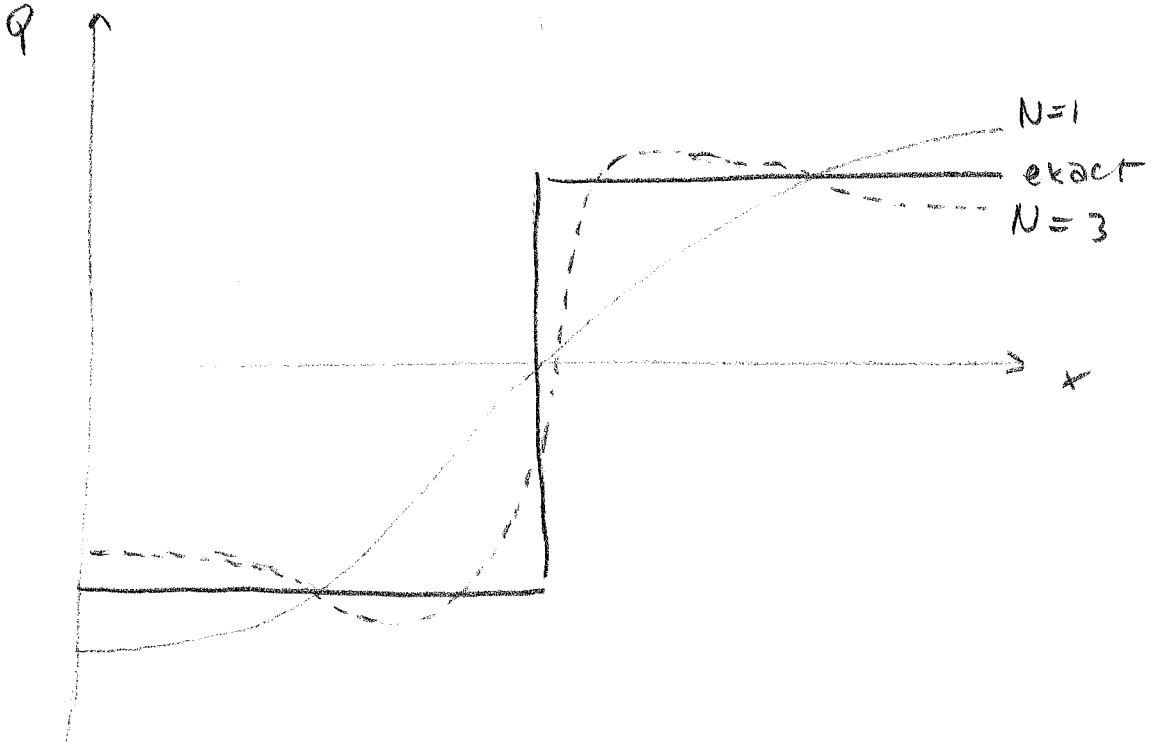
	$\%\, w_{mid}$	$\|\|e^w\|\|_{L^2} \cdot 10^2$	$\%\, M_{x,mid}$	$\|\|e^{M_x}\|\|_{L^2} \cdot 10^2$
N = 1	-1.4466	1.2456	-18.9431	12.0275
N = 3	-0.2299	0.1662	-9.9367	4.7949
N = 5	-0.0722	0.0451	-6.6944	2.6868
⋮				
N = 23	-0.0012	0.0004	-1.6877	0.3446

Notice the different rate of convergence between the displacements (both locally, i.e $\%\, w_{mid}$, and globally, i.e. $\|\|e^w\|\|_{L^2}$) with respect to the internal action M_x.

The results illustrate that M_x converge, although slowly, to the exact solution. However it should be noted that also the trigonometric functions cannot represent the discontinuity of the third derivative and the convergence of the internal actions will be characterized by oscillating behaviour, approximating the response in a global

sense


M_x
x
N=3
N=1
exact


Q
N=1
exact
N=3
x

Some admissible functions

1. Cantilever beam

$\hookrightarrow x$

$w(0) = 0$
$w_{,x}(0) = 0$ ← essential conditions

Consider a polynomial expansion:

$$w = a_0 + a_1 x + a_2 x^2 + \cdots a_N^N$$

$w(0) = 0 \Rightarrow a_0 = 0$
$w_{,x}(0) = 0 \Rightarrow a_1 = 0$

Thus:

$$w = a_2 x^2 + a_3 x^3 + \cdots a_N x^N$$

It can be useful to express the trial functions in nondimensional form so:

$$w = a_2 \frac{x^2}{\ell^2} + a_3 \frac{x^3}{\ell^3} + \cdots a_N \frac{x^N}{\ell^4}$$

$$\Rightarrow \boxed{\phi_i = \left(\frac{x}{\ell}\right)^i \qquad i = 2, \ldots N}$$

2 Fixed simply-supported beam

$w(0) = 0$

$w_x(0) = 0$ ← essential conditions

$w(\ell) = 0$

Considering a polynomial expansion:

$$w = a_0 + a_1 x + a_2 x^2 + \cdots a_N x^N$$

It is obtained:

$w(0) = 0 \Rightarrow a_0 = 0$

$w_x(0) = 0 \Rightarrow a_1 = 0$

$w(\ell) = 0 \quad a_2 \ell^2 + a_3 \ell^3 + \cdots a_N \ell^N = 0$

$$w = \left(-a_3 \ell - a_4 \ell^2 - \cdots a_N \ell^{N-2}\right) x^2 + a_3 x^3 + \cdots a_N x^N$$

and so:

$$w = a_3 (x^3 - \ell x^2) + a_4 (x^4 - \ell^2 x^2) + \cdots$$

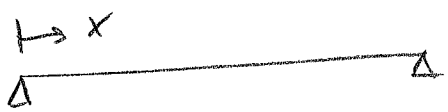
$$= a_3 x^2 (x - \ell) + a_4 x^2 (x^2 - \ell^2) + \cdots$$

and expressing the functions in a non-dimensional form:

$$w = a_3 \frac{x^2}{\ell^2}\left(\frac{x}{\ell} - 1\right) + a_4 \frac{x^2}{\ell^2}\left(\frac{x^2}{\ell^2} - 1\right) + \cdots$$

So $\boxed{\phi_i(x) = \left(\frac{x}{\ell}\right)^2 \left(\left(\frac{x}{\ell}\right)^i - 1\right) \quad i = 1, \ldots N}$

3. Simply-supported beam



$w(0) = 0$
$w(\ell) = 0$ ← essential conditions

a. Polynomial expansion

$$w = a_0 + a_1 x + \cdots a_N x^N$$

$w(0) = 0 \Rightarrow a_0 = 0$

$w(\ell) = 0 \Rightarrow a_1 \ell + a_2 \ell^2 + \cdots a_N \ell^N = 0$

$$a_1 = -a_2 \ell - a_3 \ell^2 - \cdots a_N \ell^{N-1}$$

$$w = \left(-a_2 \ell - a_3 \ell^2 - \cdots\right) x + a_2 x^2 + \cdots$$

$$= a_2 \left(x^2 - \ell x\right) + a_3 \left(x^3 - \ell^2 x\right) + \cdots$$

In nondimensional form:

$$w = a_2 \left(\frac{x^2}{\ell^2} - \frac{x}{\ell}\right) + a_3 \left(\frac{x^3}{\ell^3} - \frac{x}{\ell}\right) + \cdots$$

From which:

$$\phi_i(x) = \left(\frac{x}{\ell}\right)^i - \frac{x}{\ell} \qquad i = 2, \dots N$$

b. Trigonometric functions

$$\phi_i(x) = \sin \frac{i\pi x}{\ell} \qquad i = 1, \dots N$$

4. Clamped - clamped beam

$w(0) = 0$

$w_x(0) = 0$ ← essential conditions

$w(\ell) = 0$

$w_x(\ell) = 0$

a. Polynomial functions

Can be found by imposing the fulfillment of the boundary conditions. The expression is in this case a little bit more involved:

$$\phi_i(x) = \left(\frac{x}{\ell}\right)^i - (i-2)\left(\frac{x}{\ell}\right)^3 + (i-3)\left(\frac{x}{\ell}\right)^2$$

with $i = 4, \ldots N$

b. Trigonometric functions

$$\phi_i(x) = 1 - \cos\frac{2i\pi x}{\ell} \qquad i = 1, \ldots N$$

Admissible functions using boundary functions (extra)

Another possibility for constructing admissible functions consists in writing the generic Ritz function as:

$$\phi_i(x) = b(x)\,\chi_i(x)$$

where $b(x)$ is the boundary function

$\chi_i(x)$ is the generic function of the expansion

The boundary functions are obtained such that their value is zero in correspondence of the boundary where an essential condition is specified.

Note that by means of this approach some of the admissible functions are different from the ones previously derived. Their ability to represent the unknown field is, in any case, unchanged as they span the same vector space

1. Cantilever beam

$w(0) = 0$

$w_{/x}(0) = 0$

The boundary function has to satisfy two conditions, so it is taken as a second order polynomial:

$$b(x) = a_0 + a_1 x + a_2 x^2$$

Imposing the boundary conditions it is obtained:

$$\begin{matrix} w(0)=0 & a_0 = 0 \\ w_{/x}(0)=0 & a_1 = 0 \end{matrix} \Rightarrow b(x) = x^2$$

or, taking the nondimensional form:

$$\boxed{b(x) = \left(\frac{x}{\ell}\right)^2}$$

The set of trial functions are now constructed by using a complete set of functions, irrespective of the essential conditions (which are identically satisfied thanks to the boundary function)

Thus, for a polynomial expansion

$$\chi_1 = 1 \qquad \chi_2 = \frac{x}{\ell} \qquad \chi_3 = \left(\frac{x}{\ell}\right)^2 \quad \ldots$$

The trial functions are then written as:

$$\phi_i(x) = \left(\frac{x}{\ell}\right)^2 \left(\frac{x}{\ell}\right)^{i-1} \qquad i = 1, \ldots N$$

$$\boxed{\phi_i(x) = \left(\frac{x}{\ell}\right)^{i+1}} \qquad i = 1, \ldots N$$

2 Fixed- simply-supported

$w(0) = 0$

$w_{/x}(0) = 0$

$w(\ell) = 0$

There are three conditions to be satisfied by the boundary functions, so a polynomial of third order is taken.

$$b(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$w(0) = 0 \quad \Rightarrow a_0 = 0$

$w_{/x}(0) = 0 \quad \Rightarrow a_1 = 0$

$w(\ell) = 0 \quad \Rightarrow a_2 + a_3 \ell = 0 \quad \Rightarrow a_2 = -a_3 \ell$

$b(x) = (x^3 - x^2 \ell)$ or, in non-dimensional form:

$$b(x) = \left(\frac{x}{\ell}\right)^2 \left(\frac{x}{\ell} - 1\right)$$

The trial functions are then:

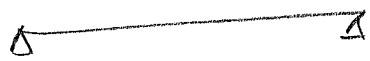
$$\phi_i(x) = b(x)\,\chi_i(x)$$

with $\chi_1 = 1 \quad \chi_2 = \frac{x}{\ell} \quad \chi_3 = \left(\frac{x}{\ell}\right)^2 \cdots$

$$\phi_i = \left(\frac{x}{\ell}\right)^2 \left(\frac{x}{\ell} - 1\right)\left(\frac{x}{\ell}\right)^{i-1} \qquad i = 1 \ldots N$$

$$\boxed{\phi_i = \left(\frac{x}{\ell} - 1\right)\left(\frac{x}{\ell}\right)^{i+1}} \qquad i = 1, \ldots N$$

3. Simply-supported beam



Proceeding in the same manner, it is obtained

$$b(x) = \frac{x}{\ell}\left(\frac{x}{\ell} - 1\right)$$

and

$$\boxed{\phi_i = \left(\frac{x}{\ell}\right)^i \left(\frac{x}{\ell} - 1\right)} \qquad i = 1, \ldots N$$

4. Clamped beam

In this case the boundary function reads:

$$b(x) = \left(\frac{x}{\ell}\right)^4 - 2\left(\frac{x}{\ell}\right)^3 + \left(\frac{x}{\ell}\right)^2$$

and so:

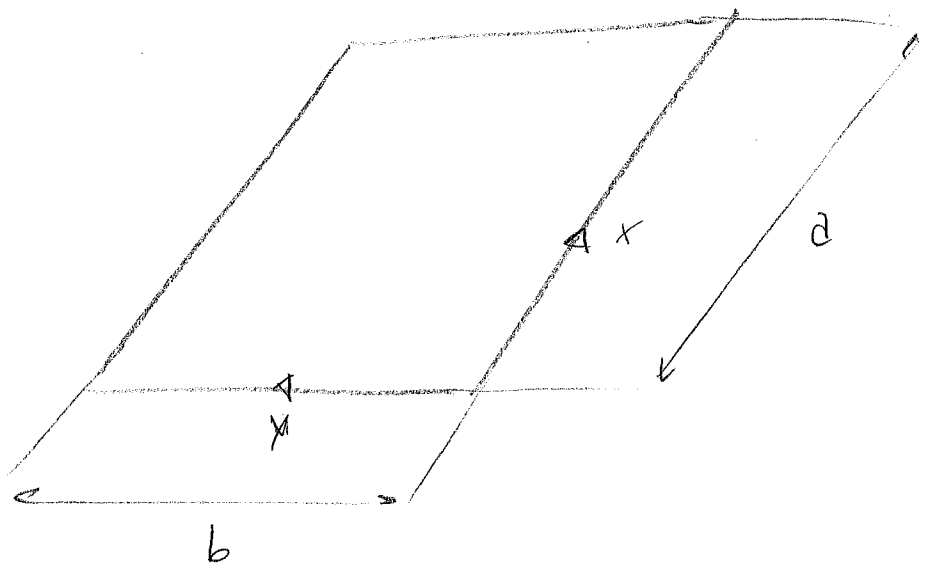
$$\phi_i(x) = b(x)\, x_i(x) \quad \text{with} \quad x_i = \left(\frac{x}{\ell}\right)^{i-1} \quad i = 1, \dots N$$

$$\boxed{\phi_i(x) = \left(\frac{x}{\ell}\right)^{i+3} - 2\left(\frac{x}{\ell}\right)^{i+2} + \left(\frac{x}{\ell}\right)^{i+1} \qquad i = 1, \dots N}$$

The Ritz method for the analysis of thin plates

The method of Ritz is particularly useful in the context of plate analysis. Indeed exact solutions are available only for a few set of boundary conditions and elastic properties. The need for obtaining approximate solutions is thus clear, especially when dealing with composite materials whose elastic couplings may make unavailable the exact solutions even for the cases commonly faced with the Navier-type approach.

Consider the plate in the figure:



The thickness is denoted with t.

Boundary conditions and loading conditions are not specified for now.

A Kirchhoff plate model is considered.

The constitutive law of the plate, by restricting the analysis to the bending behaviour, reads:

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & \\ D_{12} & D_{22} & \\ & & D_{66} \end{bmatrix} \begin{Bmatrix} -w_{0,xx} \\ -w_{0,yy} \\ -2w_{0,xy} \end{Bmatrix}$$

with:

$$\left.\begin{aligned} D_{11} = D_{22} = D &= \frac{Et^3}{12(1-\nu^2)} \\ D_{12} &= \nu D \\ D_{66} &= \frac{1-\nu}{2} D \end{aligned}\right\} \text{ for isotropic materials}$$

The constitutive law reported above hold also for composite plates with cross-ply stacking sequences ($[0/90/0]$; $[0/90/90/0]$, ...), with symmetric lay-up. In this case the values of D_{ik} have to be computed using the Classical Lamination Theory

Bending analysis

Consider now the plate subjected to a transverse force per unit surface p (pressure).
The problem is formulated by writing the total potential energy:

$$\Pi = \frac{1}{2}\int_A K_{\alpha\beta} M_{\alpha\beta}\, dA - \int_A w_0 p\, dA \qquad \alpha, \beta = x, y$$

and $K_{\alpha\beta} = -w_{0,\alpha\beta}$

Expanding the contributions it is:

$$\Pi = \frac{1}{2}\int_A \left(-w_{0,xx} M_{xx} - 2 w_{0,xy} M_{xy} - w_{0,yy} M_{yy}\right) dA$$

$$- \int_A w_0 p\, dA$$

and introducing the constitutive law:

$$\Pi = \frac{1}{2}\int_A \left(D_{11} w_{0,xx}^2 + 2 D_{12} w_{0,xx} w_{0,yy} + D_{22} w_{0,yy}^2 + \right.$$

$$\left. + 4 D_{66} w_{0,xy}^2\right) dA - \int_A w_0 p\, dA$$

The out-of-plane displacement is now expanded by means of Ritz trial functions:

$$\boxed{w_0 = \sum_{i,j}^{MN} c_{ij}\, \phi_i(x)\, \psi_j(y)}$$

where M and N are the number of functions along the directions x and y, respectively.

Note that separation of variables is assumed in the expansion. This is generally a good choice, offering the advantage of leading to relatively simple expressions. (in principle one could use an expansion in the form $w_0 = \sum_i c_i\, \phi_i(x,y)$; this is generally avoided)

The functions ϕ_i and ψ_j have to be chosen according to the boundary conditions, i.e. they must satisfy the essential conditions

The derivatives are then:

$$w_{0,xx} = \sum_{ij} c_{ij}\, \phi_{i,xx}\, \psi_j = \sum_{ij} c_{ij}\, \phi_i''\, \psi_j$$

$$w_{0,yy} = \sum_{ij} c_{ij}\, \phi_i\, \psi_{j,yy} = \sum_{ij} c_{ij}\, \phi_i\, \psi_j''$$

$$w_{0,xy} = \sum_{ij} c_{ij}\, \phi_{i,x}\, \psi_{j,y} = \sum_{ij} c_{ij}\, \phi_i'\, \psi_j'$$

where the derivative has been denoted with a prime as no ambiguity exists: ϕ can be derived only with

respect to x, and ψ can be derived only with respect to y.

The total potential energy can then be approximated after substituting the Ritz functions. Note that Π is quadratic in w_0, and the quadratic terms are expressed in the form:

$$w_{0xx} = \sum_{ij} c_{ij} \phi_i'' \psi_j \cdot \sum_{rs} c_{rs} \phi_r'' \psi_s$$

$$= \sum_{ijrs} c_{ij} \phi_i'' \phi_r'' \psi_j \psi_s c_{rs}$$

and similarly for the other contributions, so:

$$\Pi = \frac{1}{2}\Bigg[D_{11} \sum_{ijrs} c_{ij} \int_0^a \phi_i'' \phi_r'' dx \int_0^b \psi_j \psi_s dy \, c_{rs} +$$

$$+ D_{12} \sum_{ijrs} \Big(c_{ij} \int_0^a \phi_i'' \phi_r dx \int_0^b \psi_j \psi_s'' dy \, c_{rs} +$$

$$c_{ij} \int_0^a \phi_i \phi_r'' dx \int_0^b \psi_j'' \psi_s dy \, c_{rs} \Big) +$$

$$+ D_{22} \sum_{ijrs} c_{ij} \int_0^a \phi_i \phi_r dx \int_0^b \psi_j'' \psi_s'' dy \, c_{rs} +$$

$$+ 4 D_{66} \sum_{ijrs} c_{ij} \int_0^a \phi_i' \phi_r' dx \int_0^b \psi_j' \psi_s' dy \, c_{rs} \Bigg] +$$

$$- p \sum_{ij} c_{ij} \int_0^a \phi_i dx \int_0^b \psi_j dy$$

Consider now a generic contribution, the first one for example, and expand the summatory (for simplicity take $M=N=2$

$$\sum_{ijrs}^{2222} C_{ij} \int_0^a \phi_i'' \phi_r'' \, dx \int_0^b \psi_j \psi_s \, dy \, C_{rs} =$$

$$= \begin{Bmatrix} C_{11} \\ C_{12} \\ C_{21} \\ C_{22} \end{Bmatrix}^T \int_A \begin{bmatrix} \phi_1''\phi_1''\psi_1\psi_1 & \phi_1''\phi_1''\psi_1\psi_2 & \phi_1''\phi_2''\psi_1\psi_1 & \phi_1''\phi_2''\psi_1\psi_2 \\ \phi_1''\phi_1''\psi_2\psi_1 & \phi_1''\phi_1''\psi_2\psi_2 & \phi_1''\phi_2''\psi_2\psi_1 & \phi_1''\phi_2''\psi_2\psi_2 \\ \phi_2''\phi_1''\psi_1\psi_1 & \phi_2''\phi_1''\psi_1\psi_2 & \phi_2''\phi_2''\psi_1\psi_1 & \phi_2''\phi_2''\psi_1\psi_2 \\ \phi_2''\phi_1''\psi_2\psi_1 & \phi_2''\phi_1''\psi_2\psi_2 & \phi_2''\phi_2''\psi_2\psi_1 & \phi_2''\phi_2''\psi_2\psi_2 \end{bmatrix} \begin{Bmatrix} C_{11} \\ C_{12} \\ C_{21} \\ C_{22} \end{Bmatrix}$$

(r=1 s=1 | r=1 s=2 | r=2 s=1 | r=2 s=2; i=1 j=1 | i=1 j=2 | i=2 j=1 | i=2 j=2)

From the "pattern" of the contributions it can be seen that the term can be re-written as:

$$= \begin{Bmatrix} C_{11} \\ C_{12} \\ C_{21} \\ C_{22} \end{Bmatrix}^T \underline{\underline{I}}_x^{22} \otimes \underline{\underline{I}}_y^{00} \begin{Bmatrix} C_{11} \\ C_{12} \\ C_{21} \\ C_{22} \end{Bmatrix}$$

where:

$$\underline{\underline{I}}_x^{22} = \int_0^a \begin{bmatrix} \phi_1''\phi_1'' & \phi_1''\phi_2'' \\ \phi_2''\phi_1'' & \phi_2''\phi_2'' \end{bmatrix} dx \; ; \quad \underline{\underline{I}}_y^{00} = \int_0^b \begin{bmatrix} \psi_1\psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{bmatrix} dy$$

and the following notation is adopted:

$\underline{\underline{I}}^{(nm)}_{(\alpha)}$ → order of derivatives in the first and second function, respectively

↑ direction of integration

and the Kronecker product $\otimes$ is:

$$\underline{\underline{A}} \otimes \underline{\underline{B}} = \begin{bmatrix} A_{11}\,\underline{\underline{B}} & A_{12}\,\underline{\underline{B}} & \cdots \\ \vdots & & \\ A_{N1}\,\underline{\underline{B}} & \cdots & A_{NN}\,\underline{\underline{B}} \end{bmatrix}$$

The total potential energy is then written as:

$$\Pi = \frac{1}{2}\,\underline{c}^T\,\underline{\underline{K}}\,\underline{c} - \underline{c}^T\,\underline{f} \qquad \text{where:}$$

$$\underline{\underline{K}} = D_{11}\left(\underline{\underline{I}}_x^{22} \otimes \underline{\underline{I}}_y^{00}\right) + D_{12}\left(\underline{\underline{I}}_x^{20} \otimes \underline{\underline{I}}_y^{02} + \underline{\underline{I}}_x^{02} \otimes \underline{\underline{I}}_y^{20}\right) +$$

$$D_{22}\left(\underline{\underline{I}}_x^{00} \otimes \underline{\underline{I}}_y^{22}\right) + 4\,D_{66}\left(\underline{\underline{I}}_x^{11} \otimes \underline{\underline{I}}_y^{11}\right)$$

$$\underline{f} = p \begin{Bmatrix} \int_0^a \phi_1\,dx \int_0^b \psi_1\,dy \\ \int_0^a \phi_1\,dx \int_0^b \psi_2\,dy \\ \vdots \\ \int_0^a \phi_H\,dx \int_0^b \psi_N\,dy \end{Bmatrix}$$

and the set of solving equations reads:

$$\delta \Pi = \delta \underline{c}^T \underline{\underline{K}}\, \underline{c} - \delta \underline{c}^T \underline{f} = 0 \qquad \forall\, \delta \underline{c}$$

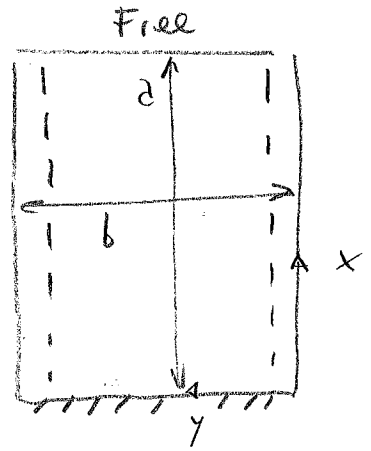
$$\Rightarrow \boxed{\underline{\underline{K}}\, \underline{c} = \underline{f}}$$

As usual the solution of the problem leads to the obtainment of $\underline{c}$, so the displacement field is available. To determine the strains and the stresses it is sufficient to take the derivatives of the displacement field.

Selection of the trial functions

Having assumed the separation of variables, the trial functions can be taken from the expansions used in 1D.

Some examples;

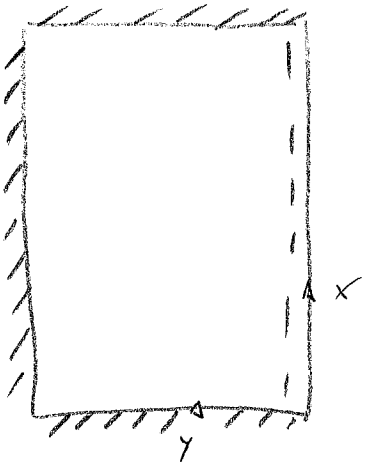


$$\phi_i = \left(\frac{x}{a}\right)^{i+1} \qquad i = 1, \ldots N$$

(as for the cantilever beam)

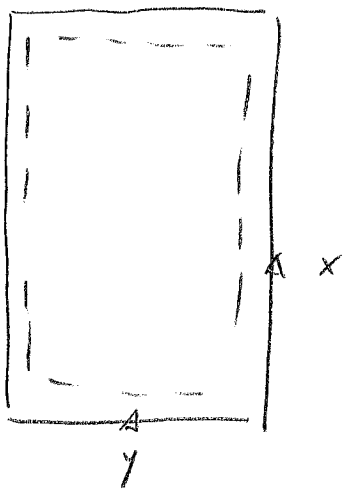
$$\psi_j = \left(\frac{y}{b}\right)^{j}\left(\frac{y}{b} - 1\right) \qquad j = 1, \ldots N$$

(as for the simply supported beam)



$$\phi_i = \left(\frac{x}{a}\right)^{i+3} - 2\left(\frac{x}{a}\right)^{i+2} + \left(\frac{x}{a}\right)^{i+1} \qquad i = 1, \ldots N$$

$$\psi_j = \left(\frac{y}{b}\right)^{j+1}\left(\frac{y}{b} - 1\right) \qquad j = 1, \ldots N$$



$$\phi_i = \sin\frac{i\pi x}{a} \qquad i = 1, \ldots N$$

$$\psi_j = \sin\frac{j\pi y}{b} \qquad j = 1, \ldots N$$

Free - vibrations

The extension to the free vibration problem is straightforward.

If the inertial contributions I_1 and I_2 are negligible (thin plate with symmetric distribution of the density) then:

$$\delta W_e = - \int_A \delta w_0 \, I_0 \, \ddot{w}_0 \, dA$$

and substituting the Ritz approximation:

$$\delta W_e = -I_0 \sum_{ijrs} \delta c_{ij} \int_0^a \phi_i \phi_r \, dx \int_0^b \psi_j \psi_s \, dy \; \ddot{c}_{rs}$$

$$= - I_0 \, \delta \underline{c}^T \left(\underline{\underline{I}}_x^{00} \otimes \underline{\underline{I}}_y^{00} \right) \ddot{\underline{c}} =$$

$$= - \delta \underline{c}^T \, \underline{\underline{M}} \, \ddot{\underline{c}}$$

with $$\boxed{\underline{\underline{M}} = - I_0 \; \underline{\underline{I}}_x^{00} \otimes \underline{\underline{I}}_y^{00}}$$

The variational principle is then:

$$\delta W_i - \delta W_e = \delta \Pi = 0 \qquad \text{so:}$$

$$\delta \underline{c}^T \left(\underline{\underline{M}} \, \ddot{\underline{c}} + \underline{\underline{K}} \, \underline{c} \right) = 0 \qquad \forall \; \delta \underline{c}$$

and so

$$\underline{\underline{M}}\,\ddot{\underline{c}} + \underline{\underline{K}}\,\underline{c} = 0$$

Assuming harmonic motion in the form:

$\underline{c} = \underline{C}\, e^{i\omega t}$, it is obtained the eigenvalue problem in the form:

$$\boxed{\left(-\omega^2 \underline{\underline{M}} + \underline{\underline{K}}\right)\underline{C} = \underline{0}}$$