Variational Principles

- The solution of the elastic problem requires that both <u>equilibrium</u> and <u>compatibility</u> are fulfilled.

  It was shown that the set of governing equations is given by the PDE representing the equilibrium and the compatibility in the domain as well as at the boundaries (via boundary conditions)

- When the solution is obtained by solving exactly or approximately the governing equations, it is said that the problem is formulated in a <u>strong form manner</u>

- Another approach consists in solving the problem in <u>weak form</u>.

  The equations are multiplied with test functions and integrated over the domain. Generally this is the approach which is used in structural mechanics. The implementation into numerical technique is straightforward and the requirements on the <u>regularity</u> of the solution are relaxed.

  Among the class of weak form formulations, a special set is given by the <u>variational principles</u>, such as the Principle of Virtual Displacements (PVD), the minimum potential energy principle and their dual counterparts. (the complementary-energy related ones)

Principle of virtual works
Minimum Potential energy principles] → impose equilibrium (compatibility guaranteed a priori)

Principle of complementary virtual works
Menabrea's theorem] → impose compatibility (equilibrium should be guaranteed a priori)

This means that or the equilibrium, or the compatibility need to be satisfied a priori.
(More advanced variational principles exist where nor compatibility, nor equilibrium need to be satisfied a priori → <u>mixed approaches</u>)

- <u>Force-based</u> approaches rely upon the use of forces as problem unknown. The equilibrium can be easily established a priori and, whenever it is necessary, the compatibility can be imposed using the <u>PCVW or Menabrea</u>
- <u>Displacement-based</u> approaches are formulated by assuming a displacement field which is intrinsically compatible. The solution, in this case, is obtained by enforcing the equilibrium condition using the PVW or the Minimum Potential Energy principle

- Principle of Virtual Works (PVW)

The set of governing equations for a 3D continuum, under the assumptions of infinitesimal strain/displacements is given by:

$$\begin{cases} \operatorname{div} \underline{\underline{\sigma}} + \underline{f} = \underline{0} & \text{in } \Omega \\ \underline{\underline{\varepsilon}} - \frac{1}{2}\left(\operatorname{grad} \underline{u} + \operatorname{grad} \underline{u}^T\right) = \underline{\underline{0}} & \text{in } \Omega \\ \underline{t} = \hat{\underline{t}} & \text{in } S_F \\ \underline{u} = \hat{\underline{u}} & \text{in } S_u \end{cases}$$

Assume that compatibility conditions in Ω and S_u are satisfied.
Consider now a set of arbitrary infinitesimal displacements $\delta \underline{u}$. It follows that the problem can be formulated as:

$$\boxed{\int_V \delta \underline{u} \cdot \left(\operatorname{div} \underline{\underline{\sigma}} + \underline{f}\right) dV - \int_{S_F} \delta \underline{u} \cdot \left(\underline{t} - \hat{\underline{t}}\right) dS_F = 0 \quad \forall \delta \underline{u}}$$

- Whenever this condition is satisfied for any $\delta \underline{u}$, it means that the equilibrium conditions are identically satisfied, viz:

$$\begin{cases} \operatorname{div} \underline{\underline{\sigma}} + \underline{f} = 0 & \text{in } \Omega \\ \underline{t} - \hat{\underline{t}} = \underline{0} & \text{in } S_F \end{cases}$$

- Note that the "weak form" reported in the box is the same mathematical approach adopted in the courses of Mathematics where $\delta \underline{u}$ where denoted as test functions (or weight functions)

- Recall that the virtual displacements are compatible, meaning that $\delta \underline{u} = 0$ on S_u.

The equation in the box is not yet the expression of the PVW. To achieve the final form of the PVW it is necessary to perform integration by parts and apply the divergence theorem.

$$\int_V \delta \underline{u} \cdot \operatorname{div} \underline{\underline{\sigma}} \, dV = \int_V \delta u_i \, \sigma_{ik/k} \, dV$$

$$= \int_V (\delta u_i \sigma_{ik})_{/k} \, dV - \int_V \delta u_{i/k} \, \sigma_{ik} \, dV$$

$$= \int_V \operatorname{div}(\delta \underline{u} \cdot \underline{\underline{\sigma}}) \, dV - \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} \, dV$$

(having noted that $\varepsilon_{ik} = \frac{1}{2}(u_{i/k} + u_{k/i})$

$\delta \varepsilon_{ik} = \frac{1}{2}(\delta u_{i/k} + \delta u_{k/i})$

in addition $\sigma_{ik} = \sigma_{ki}$, so:

$\delta \varepsilon_{ik} \, \sigma_{ik} = \frac{1}{2}(\delta u_{i/k} + \delta u_{k/i}) \, \sigma_{ik}$

$= \delta u_{i/k} \, \sigma_{ik}$)

$$= \int_S \delta \underline{u} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \, dS - \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} \, dV$$

Observe now that $\delta \underline{u} = 0$ in S_u

$\delta \underline{u} \neq 0$ in S_F

$$= \int_{S_F} \delta \underline{u} \cdot \underline{\underline{\sigma}}\, \underline{n}\, dS_F - \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}}\, dV$$

$$= \int_{S_F} \delta \underline{u} \cdot \underline{t}\, dS_F - \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}}\, dV$$

The expression in the box can then be re-organized as:

$$\int_{S_F} \delta \underline{u} \cdot \underline{t}\, dS_F - \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}}\, dV + \int_V \delta \underline{u} \cdot \underline{f}\, dV - \int_{S_F} \delta \underline{u} \cdot (\underline{t} - \hat{\underline{t}})\, dS_F = 0$$

and simplifying:

$$\int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}}\, dV = \int_V \delta \underline{u} \cdot \underline{f}\, dV + \int_{S_F} \delta \underline{u} \cdot \hat{\underline{t}}\, dS_F$$

or

$$\delta W_i = \delta W_e$$

Principle of Virtual Works

where:

$$\delta W_i = \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}}\, dV \qquad \text{internal virtual work}$$

$$\delta W_e = \int_V \delta \underline{u} \cdot \underline{t}\, dV + \int_{S_F} \delta \underline{u} \cdot \hat{\underline{t}}\, dS_F \qquad \text{external virtual work}$$

A body is in equilibrium if and only if the virtual internal work equals the external virtual work.

Note that the equilibrium equation + BCs are identically satisfied whenever the PVW is verified for any set of infinitesimal virtual displacements.

- The PVW is a <u>variational principle</u>: Both the equilibrium equation and the equilibrium condition at the boundaries S_F are accounted for.
- The PVW is a scalar (one single!) equation
- No reference is made to the material model (the constitutive law did not enter the previous steps). It follows that the PVW holds irrespective of the material behaviour.
- It is useful to remark, once more, that compatibility requirements need to be verified a priori.

Equilibrium equations from PVW

Clearly the reverse process can be performed, and the equilibrium conditions obtained from the PVW. It is a useful exercise:

$$\delta W_i = \delta W_e$$

$$\int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} \, dV = \int_V \delta \underline{u} \cdot \underline{f} \, dV + \int_{S_F} \delta \underline{u} \cdot \hat{\underline{t}} \, dS_F$$

Consider the internal virtual work

$$\delta W_i = \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} \, dV = \int_V \delta \varepsilon_{ik} \, \sigma_{ik} \, dV$$

recalling that $\delta \varepsilon_{ik} = \frac{1}{2} (\delta u_{i/k} + \delta u_{k/i})$
(the virtual variations should be compatible)

$$\Rightarrow \quad \delta W_i = \int_V \delta u_{i/k} \, \sigma_{ik} \, dV$$

$$= \int_V (\delta u_i \, \sigma_{ik})_{/k} \, dV - \int_V \delta u_i \, \sigma_{ik/k} \, dV$$

$$= \int_V \operatorname{div} (\delta \underline{u} \cdot \underline{\underline{\sigma}}) \, dV - \int_V \delta \underline{u} \cdot \operatorname{div} \underline{\underline{\sigma}} \, dV$$

$$= \int_S \delta \underline{u} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \, dS - \int_V \delta \underline{u} \cdot \operatorname{div} \underline{\underline{\sigma}} \, dV$$

$$= \int_{S_F} \delta \underline{u} \cdot \underline{t} \, dS_F - \int_V \delta \underline{u} \cdot \operatorname{div} \underline{\underline{\sigma}} \, dV$$

So, the variational statement reads:

$$\int_{S_F} \delta \underline{u} \cdot \underline{t} \, dS_F - \int_V \delta \underline{u} \cdot \operatorname{div} \underline{\underline{\sigma}} \, dV = \int_V \delta \underline{u} \cdot \underline{f} \, dV + \int_{S_F} \delta \underline{u} \cdot \hat{\underline{t}} \, dS_F$$

or

$$\int_V \delta \underline{u} \cdot \left(\operatorname{div} \underline{\underline{\sigma}} + \underline{f} \right) dV + \int_{S_F} \delta \underline{u} \cdot \left(\hat{\underline{t}} - \underline{t} \right) dS_F = 0$$

if this equation is verified for $\forall \delta \underline{u}$, then:

$$\operatorname{div} \underline{\underline{\sigma}} + \underline{f} = \underline{0} \qquad \text{in } \Omega$$

$$\underline{t} = \hat{\underline{t}} \qquad \text{in } S_F$$

$+$

$$\left(\begin{array}{ll} \underline{\underline{\varepsilon}} = \frac{1}{2} \left(\operatorname{grad} \underline{u} + \operatorname{grad} \underline{u}^T \right) & \text{in } \Omega \\ \underline{u} = \hat{\underline{u}} & \text{in } S_U \end{array} \right) \quad \text{satisfied a priori}$$

Note that, in spite of the fact that compatibility conditions are satisfied a priori, the virtual variations satisfy homogeneous boundary conditions.

$\underline{u}$ satisfies

$$\underline{\underline{\varepsilon}} = \frac{1}{2} \left(\operatorname{grad} \underline{u} + \operatorname{grad} \underline{u}^T \right) \qquad \text{in } \Omega$$

$$\underline{u} = \hat{\underline{u}} \qquad \text{in } S_u$$

The perturbed configuration

$\underline{u} + \delta \underline{u}$ will lead to:

$$\cancel{\underline{\underline{\varepsilon}}} + \delta \underline{\underline{\varepsilon}} = \frac{1}{2} \left(\cancel{\operatorname{grad} \underline{u} + \operatorname{grad} \underline{u}^T} + \operatorname{grad} \delta \underline{u} + \operatorname{grad} \delta \underline{u}^T \right)$$

$$\cancel{\underline{u}} + \delta \underline{u} = \cancel{\hat{\underline{u}}}$$

and so:

$$\delta \underline{\underline{\varepsilon}} = \frac{1}{2} \left(\operatorname{grad} \delta \underline{u} + \operatorname{grad} \delta \underline{u}^T \right) \qquad \text{in } \Omega$$

$$\delta \underline{u} = \underline{0} \qquad \text{in } S_U$$

- Principle of Complementary Virtual Works (PCVW)

Consider again the set of governing equations in terms of equilibrium and compatibility

$$\begin{cases} \operatorname{div}\underline{\underline{\sigma}} + \underline{F} = \underline{0} & \text{in } \Omega \\ \underline{\underline{\varepsilon}} - \frac{1}{2}\left(\operatorname{grad}\underline{u} + \operatorname{grad}\underline{u}^T\right) = \underline{\underline{0}} & \text{in } \Omega \\ \underline{t} = \hat{\underline{t}} & \text{in } S_F \\ \underline{u} = \hat{\underline{u}} & \text{in } S_u \end{cases}$$

Assume now that the equilibrium conditions are identically satisfied in Ω and S_F.

The weak form of the compatibility conditions is then obtained as:

$$\boxed{\int_V \delta\underline{\underline{\sigma}} : \left(\underline{\underline{\varepsilon}} - \frac{1}{2}\left(\operatorname{grad}\underline{u} + \operatorname{grad}\underline{u}^T\right)\right) dV + \int_{S_u} \delta\underline{t}\cdot\left(\underline{u} - \hat{\underline{u}}\right) dS_u = 0}$$

The variations $\delta\underline{\underline{\sigma}}$ and $\delta\underline{t}$ are <u>infinitesimal</u> and in <u>equilibrium</u>

$$\int_V \left[\delta\underline{\underline{\sigma}} : \frac{1}{2}\left(\operatorname{grad}\underline{u} + \operatorname{grad}\underline{u}^T\right)\right] dV =$$

$$= \int_V \delta\sigma_{ik} \frac{1}{2}\left(u_{i/k} + u_{k/i}\right) dV = \int_V \delta\sigma_{ik} u_{i/k}\, dV$$

(due to the symmetry of $\underline{\underline{\sigma}}$)

So:

$$\int_V \left(\delta\sigma_{ik}\varepsilon_{ik} - \delta\sigma_{ik} u_{i/k}\right) dV + \int_{S_u} \delta t_i \left(u_i - \hat{u}_i\right) dS_u = 0$$

$$\int_V \left[\delta\sigma_{ik}\varepsilon_{ik} - \left(\delta\sigma_{ik} u_i\right)_{/k} + \delta\sigma_{ik/k} u_i\right] dV + \int_{S_u} \delta t_i \left(u_i - \hat{u}_i\right) dS_u = 0$$

$$\int_V \left[\delta\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} - \operatorname{div}(\underline{u} \cdot \delta\underline{\underline{\sigma}}) + \underline{u} \cdot \operatorname{div} \delta\underline{\underline{\sigma}} \right] dV + \int_{S_U} \delta\underline{t} \cdot (\underline{u} - \hat{\underline{u}})\, dS_U = 0$$

Applying the divergence theorem:

$$\int_V \left(\delta\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} + \underline{u} \cdot \operatorname{div} \delta\underline{\underline{\sigma}} \right) dV - \int_S \underline{u} \cdot \delta\underline{\underline{\sigma}} \cdot \underline{n}\, dS + \int_{S_U} \delta\underline{t} (\underline{u} - \hat{\underline{u}})\, dS_U = 0$$

Recall now that the variations $\delta\underline{\underline{\sigma}}$ and $\delta\underline{t}$ should respect the equilibrium conditions:

$$\begin{cases} \operatorname{div} \delta\underline{\underline{\sigma}} + \delta\underline{F} = \underline{0} & \text{in } \Omega \\ \delta\underline{t} = \delta\underline{\underline{\sigma}} \cdot \underline{n} = \delta\hat{\underline{t}} & \text{in } S_F \end{cases}$$

So:

$$\int_V \underline{u} \cdot \operatorname{div} \delta\underline{\underline{\sigma}}\, dV = \int_V - \delta\underline{F} \cdot \underline{u}\, dV$$

$$\int_S (\delta\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{u}\, dS = \int_S \delta\underline{t} \cdot \underline{u}\, dS =$$

$$= \int_{S_F} \delta\underline{t} \cdot \underline{u}\, dS_F + \int_{S_U} \delta\underline{t} \cdot \underline{u}\, dS_U \qquad (S_F \cup S_U = S)$$

$$= \int_{S_F} \delta\hat{\underline{t}} \cdot \underline{u}\, dS_F + \int_{S_U} \delta\underline{t} \cdot \underline{u}\, dS_U$$

The variational statement is then written as:

$$\int_V \left(\delta\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} - \delta\underline{F} \cdot \underline{u} \right) dV - \int_{S_F} \delta\hat{\underline{t}} \cdot \underline{u}\, dS_F - \int_{S_U} \delta\underline{t} \cdot \underline{u}\, dS_U$$

$$+ \int_{S_U} \delta\underline{t} \cdot (\underline{u} - \hat{\underline{u}})\, dS_U = 0$$

and so:

$$\boxed{\int_V \delta\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}\, dV = \int_V \delta\underline{f}\cdot\underline{u}\, dV + \int_{S_F} \delta\hat{\underline{t}}\cdot\underline{u}\, dS_F + \int_{S_u} \delta\underline{t}\cdot\hat{\underline{u}}\, dS_u}$$

or

$$\boxed{\delta W_i^* = \delta W_{ext}^*}$$

- The internal virtual work obtained from an infinitesimal and equilibrated variation of the stress components is equal to the external complementary virtual work.
- The PCVW, if verified for $\forall\, \delta\underline{\underline{\sigma}}$, is identical to the compatibility conditions on Ω and S_u. This means that if $\delta W_i^* = \delta W_{ext}^*$ then the compatibility conditions are fulfilled.

Compatibility from PCVW

As done for the PVW, it is instructive to illustrate that the compatibility conditions can be derived from the PCVW.

In other words, what happens if the PCVW is required to hold for any arbitrary set of admissible virtual variations?

$$\int_V \delta\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}\, dV = \int_V \delta\underline{f} \cdot \underline{u}\, dV + \int_{S_F} \delta\hat{\underline{t}} \cdot \underline{u}\, dS_F + \int_{S_U} \delta\underline{t} \cdot \hat{\underline{u}}\, dS_U$$

To account for the fact that the variations are in equilibrium, the variational statement can be modified by introducing those conditions via Lagrange multipliers

$$\delta W_i^* = \delta W_{ext}^* - \int_V \underline{\lambda} \cdot \left(\operatorname{div} \delta\underline{\underline{\sigma}} + \delta\underline{f}\right) dV + \int_{S_F} \underline{\mu} \cdot \left(\delta\underline{t} - \delta\hat{\underline{t}}\right) dS_F$$

where $\underline{\lambda}, \underline{\mu}$ are vectors of Lagrange multipliers (it is clear that the expression of the PCVW is unchanged whenever $\operatorname{div} \delta\underline{\underline{\sigma}} + \delta\underline{f} = 0$ and $\delta\underline{t} = \delta\hat{\underline{t}}$)

Rearrange now the first contribution

$$\int_V \underline{\lambda} \cdot \left(\operatorname{div} \delta\underline{\underline{\sigma}} + \delta\underline{f}\right) dV =$$

$$= \int_V \left(\lambda_i\, \delta\sigma_{ik/k} + \lambda_i\, \delta f_i\right) dV$$

$$= \int_V \left[\left(\lambda_i \delta\sigma_{ik}\right)_{/k} - \lambda_{i/k}\, \delta\sigma_{ik} + \lambda_i\, \delta f_i\right] dV$$

$$= \int_V \left[\left(\lambda_i \delta\sigma_{ik}\right)_{/k} - \delta\sigma_{ik}\, \frac{1}{2}\left(\lambda_{i/k} + \lambda_{k/i}\right) + \lambda_i\, \delta f_i\right] dV$$

$$= \int_V \left[\operatorname{div}(\underline{\lambda} \cdot \delta \underline{\underline{\sigma}}) - \delta \underline{\underline{\sigma}} : \frac{1}{2}\left(\operatorname{grad} \underline{\lambda} + \operatorname{grad} \underline{\lambda}^T\right) + \delta \underline{f} \cdot \underline{\lambda} \right] dV$$

$$= \int_S \underline{\lambda} \cdot \delta \underline{\underline{\sigma}} \cdot \underline{n} \, dS - \int_V \left[\delta \underline{\underline{\sigma}} : \frac{1}{2}\left(\operatorname{grad} \underline{\lambda} + \operatorname{grad} \underline{\lambda}^T\right) - \delta \underline{F} \cdot \underline{\lambda} \right] dV$$

$$= \int_{S_F} \delta \underline{t} \cdot \underline{\lambda} \, dS_F + \int_{S_u} \delta \underline{t} \cdot \underline{\lambda} \, dS_u - \int_V \left[\delta \underline{\underline{\sigma}} : \frac{1}{2}\left(\operatorname{grad} \underline{\lambda} + \operatorname{grad} \underline{\lambda}^T\right) + \right.$$

$$\left. - \delta \underline{f} \cdot \underline{\lambda} \right] dV$$

The expression of the PCVW (augmented using the Lagrange multipliers) reads:

$$\int_V \delta \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} \, dV = \int_V \delta \underline{f} \cdot \underline{u} \, dV + \int_{S_F} \delta \underline{\hat{t}} \cdot \underline{u} \, dS_F + \int_{S_u} \delta \underline{t} \cdot \underline{\hat{u}} \, dS_u$$

$$- \int_{S_F} \delta \underline{t} \cdot \underline{\lambda} \, dS_F - \int_{S_u} \delta \underline{t} \cdot \underline{\lambda} \, dS_u + \int_V \left[\delta \underline{\underline{\sigma}} : \frac{1}{2}\left(\operatorname{grad} \underline{\lambda} + \operatorname{grad} \underline{\lambda}^T\right) \right.$$

$$\left. - \delta \underline{f} \cdot \underline{\lambda} \right] dV + \int_{S_F} \delta \underline{t} \cdot \underline{\mu} \, dS_F - \int_{S_F} \delta \underline{\hat{t}} \cdot \underline{\mu} \, dS_F$$

If this scalar equation is required to hold for any variation $\delta \underline{\underline{\sigma}}$, then it is equivalent to the following set of equations:

$$\begin{cases} \underline{\underline{\varepsilon}} = \frac{1}{2}\left(\operatorname{grad} \underline{\lambda} + \operatorname{grad} \underline{\lambda}^T\right) & \text{in } \Omega & (1) \\ \underline{u} = \underline{\lambda} & \text{in } \Omega & (2) \\ \underline{u} = \underline{\mu} & \text{in } S_F & (3) \\ \underline{\hat{u}} = \underline{\lambda} & \text{in } S_u & (4) \\ \underline{\lambda} = \underline{\mu} & \text{in } S_F & (5) \end{cases}$$

From the conditions (2) and (4) it is clear that the Lagrange multiplier $\underline{t}$ is the displacement field $\underline{u}$. Thus, the equations can be expressed by replacing $\underline{t}$ with $\underline{u}$:

$$\begin{cases} \underline{\underline{\varepsilon}} = \frac{1}{2} \left(\operatorname{grad} \underline{u} + \operatorname{grad} \underline{u}^T \right) & \text{in } \Omega \\ \underline{u} = \hat{\underline{u}} & \text{in } S_u \end{cases}$$

It can be observed, in addition, that $\underline{\mu}$ is the displacement in S_F

Minimum potential energy principle

It is the energetic counterpart of the PVW. While, as seen, the PVW holds irrespective of the constitutive law, it is now introduced the assumption of hyperelastic material

$$\underline{\underline{\sigma}} = u_{/\underline{\underline{\varepsilon}}} \qquad \left(\text{or} \quad \sigma_{ik} = u_{/\varepsilon_{ik}} \right)$$

Consider the PVW: $\delta W_i = \delta W_e$

$$\delta W_i = \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} \, dV$$

$$\delta W_e = \int_V \delta \underline{u} \cdot \underline{f} \, dV + \int_{S_F} \delta \underline{u} \, \hat{\underline{t}} \, dS_F$$

According to the assumption of hyperelasticity, the internal virtual work reads:

$$\delta W_i = \int_V \delta \underline{\underline{\varepsilon}} : u_{/\underline{\underline{\varepsilon}}} \, dV$$

$$= \int_V \delta u \, dV \qquad \left(\text{as} \quad u = u(\underline{\underline{\varepsilon}}) \Rightarrow \delta u = u_{/\underline{\underline{\varepsilon}}} : \delta \underline{\underline{\varepsilon}} \right)$$

It follows that

$$\delta W_i = \delta W_e \quad \Rightarrow \quad \boxed{\delta (U + V) = \delta \Pi = 0}$$

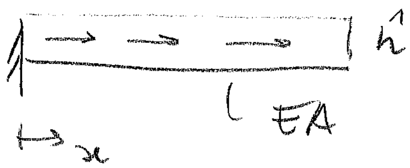
where: $\Pi = U + V$

- V: potential of external forces
- U: strain energy
- Π: total potential energy

- The Minimum Potential Energy Principle is restricted to the case of hyperelastic materials (not necessarily linear)
- Similarly to the PVW it is a variational tool for imposing the equilibrium conditions. The equilibrium conditions are obtained in terms of displacements as far as the constitutive equation is introduced.

Contrarily, the equations obtained via PVW, still representing the equilibrium, are obtained in terms of forces (as no use is made of the constitutive equation)

Consider this simple example



$$\delta W_i = \int_0^l \delta u_{/x} N_x \, dx$$

$$\delta W_e = \int_0^l \delta u \, \hat{n} \, dx$$

$\delta W_i = \delta W_e$ leads to:

$$N_{x/x} + \hat{n} = 0$$

$$U = \frac{1}{2}\int_0^l EA u_{/x}^2 \, dx$$

$$V = \int_0^l \hat{n} u \, dx$$

$\delta(U+V) = 0$ leads to

$$EA u_{/xx} + \hat{n} = 0$$

Minimum complementary potential energy principle

(Menabrea's theorem)

It is the energy counterpart of the PCVW. Thus, by assuming that the strains can be derived as:

$$\underline{\underline{\varepsilon}} = U^*{/}_{\underline{\underline{\sigma}}} \quad \text{or} \quad \varepsilon_{in} = U^*{/}_{\sigma_{in}}$$

with U^* complementary energy density
and recalling the PCVW:

$$\delta W_i^* = \int_V \delta\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}\, dV = \int_V \delta U^* dV = \delta U^*$$

$$\delta W_e^* = \int_V \delta\underline{F} \cdot \underline{u}\, dV + \int_{S_F} \delta\hat{\underline{t}} \cdot \underline{u}\, dS_F + \int_{S_U} \delta\underline{t} \cdot \hat{\underline{u}}\, dS_U = \delta V^*$$

$$\delta W_i^* = \delta W_e^* \Rightarrow \boxed{\delta(U^* + V^*) = \delta\Pi^* = 0}$$

Hamilton's principle

Consider the displacement field $\underline{u} = \underline{u}(\underline{x}, t)$
and consider a virtual variation $\delta \underline{u}$ s.t.:

$$\boxed{\begin{aligned} &\delta \underline{u}(\underline{x}, t) = \underline{0} && \text{in } S_u, \ \forall t \\ &\delta \underline{u}(\underline{x}, t_1) = \delta \underline{u}(\underline{x}, t_2) = 0 && \text{in } \Omega, \ \forall \underline{x} \end{aligned}}$$

- The first condition is identical to the compatibility requirement used in the context of the PVW.
- The second condition states that the perturbed configuration at $t = t_1$ and $t = t_2$ is equal to the actual configuration $\underline{u}$.

$$\underline{u} + \delta \underline{u} \longrightarrow = \underline{u} \qquad \text{in } t = t_1, t_2$$

Consider now the PVW:

$$\delta W_i = \int_V \delta \underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} \, dV$$

$$\delta W_e = \int_V \delta \underline{u} \cdot \underline{t} \, dV + \int_{S_F} \delta \underline{u} \cdot \hat{\underline{t}} \, dS_F \underbrace{- \int_V \delta \underline{u} \cdot \rho \ddot{\underline{u}} \, dV}_{\text{Contribution of the inertial forces}}$$

Note that the inertial forces are naturally expressed as function of the displacements. This is why the variational approach is formulated starting from the PVW.

Integrate now between t_1 and t_2:

$$\int_{t_1}^{t_2}\left[\int_V \delta\underline{\underline{\varepsilon}} : \underline{\underline{\sigma}}\, dV\right]dt = \int_{t_1}^{t_2}\left[\int_V \delta\underline{u}\cdot\underline{f}\, dV + \int_{S_F} \delta\underline{u}\cdot\hat{\underline{t}}\, dS_F \right.$$
$$\left. - \int_V \rho\, \delta\underline{u}\cdot\ddot{\underline{u}}\, dV\right]dt$$

The last contribution can be integrated by parts:

$$\int_{t_1}^{t_2}\int_V \rho\,\delta\underline{u}\cdot\ddot{\underline{u}}\, dV = -\int_{t_1}^{t_2}\int_V \rho\,\delta\dot{\underline{u}}\,\dot{\underline{u}}\, dV dt + \int_V \rho\,\delta\underline{u}\,\dot{\underline{u}}\, dV\Big|_{t_1}^{t_2}$$
$$= -\int_{t_1}^{t_2}\int_V \rho\,\delta\dot{\underline{u}}\cdot\dot{\underline{u}}\, dV dt \qquad \left(\text{as } \delta\underline{u}(\underline{x},t_1)=0,\ \delta\underline{u}(\underline{x},t_2)=0\right)$$

Then:

$$\int_{t_1}^{t_2}\left[\int_V \delta\underline{\underline{\varepsilon}} : \underline{\underline{\sigma}}\, dV - \int_V \delta\underline{u}\cdot\underline{f}\, dV - \int_{S_F} \delta\underline{u}\cdot\hat{\underline{t}}\, dS_F - \int_V \rho\,\delta\dot{\underline{u}}\,\dot{\underline{u}}\, dV\right]dt = 0$$

$$\int_{t_1}^{t_2} \delta(U+V-K)\, dt = 0$$

or

$$\boxed{\delta\int_{t_1}^{t_2} L\, dt = 0 \qquad \text{with} \quad L = K-(U+V)}$$

L: Lagrangian

K: kinetic energy