- <u>Strain energy and complementary strain energy in beams</u>

The DSV solution allows to express the strain energy and the complementary one as function of the internal actions.

Recall that:

$\underline{\underline{\sigma}} = u_{/\underline{\underline{\varepsilon}}}$ and

$$\boxed{u = \frac{1}{2}\, \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} = \frac{1}{2}\, \underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}) : \underline{\underline{\varepsilon}}}$$ strain energy density

$\underline{\underline{\varepsilon}} = u^{*}_{/\underline{\underline{\sigma}}}$ and

$$\boxed{u^{*} = \frac{1}{2}\, \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}(\underline{\underline{\sigma}})}$$ Complementary strain energy density

where $u^{*} = \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} - u$

- Note that u has to be intended as function of $\underline{\underline{\varepsilon}}$ while u^{*} as function of $\underline{\underline{\sigma}}$.
- In the <u>linear</u> case $u = u^{*}$
- Recall that u is used in the context of displacement based approaches (to impose equilibrium via Minimum potential energy principle); u^{*} is used in force-based approaches to impose compatibility (via Menabrea's theorem)

Strain energy (axial force and bending)

The strain energy is now specialized to case of axial force and bending, so:

$$u = \frac{1}{2}\left(\sigma_{zz}\,\varepsilon_{zz} + \underbrace{\sigma_{xz}\,\gamma_{xz} + \sigma_{yz}\,\gamma_{yz}}_{\text{neglect for now these contributions}}\right)$$

The two contributions $\sigma_{xz}\gamma_{xz}$ and $\sigma_{yz}\gamma_{yz}$ are now neglected by assuming that the beam is <u>slender</u> (axial force and/or bending moment, constant or linear, are the only internal actions; no twisting moment.)

The DSV solution was found to be:

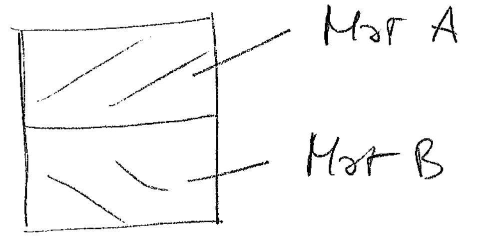
$$\sigma_{zz} = \frac{T_z}{A} + \frac{M_x}{J_{xx}}\,y - \frac{M_y}{J_{yy}}\,x$$

and

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} = \frac{T_z}{EA} + \frac{M_x}{EJ_{xx}}\,y - \frac{M_y}{EJ_{yy}}\,x$$

<u>Note</u>: for the case of homogeneous and isotropic beam the contributions EA and $EJ_{\alpha\alpha}$ are the product between the elastic modulus and the area or the second moment of inertia, respectively. However, it is more appropriate to intend EA as a single contribution expressing the axial stiffness of the beam. Similarly $EJ_{\alpha\alpha}$ should be intended as the bending stiffness, not necessarily the product between E and $J_{\alpha\alpha}$.

This generalization of the idea of axial and bending stiffness is useful when dealing with non-homogeneous beams. Consider, for instance, a beam made of two materials

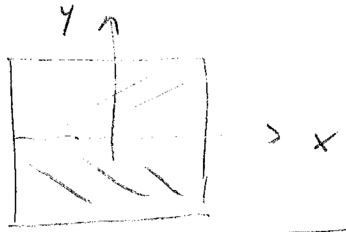


denote with E_A and E_B the elastic moduli of the two materials and A_A and A_B the corresponding areas.

Clearly $A_A + A_B = A$, where A is the section area.

Which is the axial stiffness of the area?

$$EA = \int_A E \, dA \quad \text{where } E = E(x,y)$$



In this example $E = E_A$ if $y > 0$
$E = E_B$ if $y < 0$

thus: $\boxed{EA = E_A A_A + E_B A_B}$

It is clear that the axial stiffness cannot be seen as the product between an elastic modulus (which one? E_A or E_B?) and an area (A_A or A_B?)

The symbol EA stands for the axial stiffness and highlights the dependence on the elastic properties and the area of the section.

In this example it could be possible to write

$$EA = E_A \left(A_A + \frac{E_B}{E_A} A_B \right) = E_A \cdot \tilde{A}$$

with $\tilde{A} = A_A + \dfrac{E_B A_B}{E_A}$

Note that, in any case, $\tilde{A}$ depends on the elastic properties thus a description as the product between E and A is no more possible.
The same considerations hold for the bending stiffness (and, even more, for the torsional one)

From the DSV solution the strain energy reads:

$$U = \int_V u\, dV = \frac{1}{2}\int_V \sigma_{zz}\,\varepsilon_{zz}\, dV$$

$$= \frac{1}{2}\int_V \left(\frac{T_z}{A} + \frac{M_x}{J_{xx}}y - \frac{M_y}{J_{yy}}x\right)\left(\frac{T_z}{EA} + \frac{M_x}{EJ_{xx}}y - \frac{M_y}{EJ_{yy}}x\right) dV$$

$$= \frac{1}{2}\int_0^\ell \int_A \left(\frac{T_z}{A}\frac{T_z}{EA} + \frac{M_x}{J_{xx}}\frac{M_x}{EJ_{xx}}y^2 + \frac{M_y}{J_{yy}}\frac{M_y}{EJ_{yy}}x^2 + \right.$$
$$+ \frac{T_z}{A}\frac{M_x}{EJ_{xx}}y - \frac{T_z}{A}\frac{M_y}{EJ_{yy}}x + \frac{M_x}{J_{xx}}\frac{T_z}{EA}y - \frac{M_x}{J_{xx}}\frac{M_y}{EJ_{yy}}x +$$
$$\left. - \frac{M_y}{J_{yy}}\frac{T_z}{EA}x - \frac{M_y}{J_{yy}}\frac{M_x}{EJ_{xx}}xy\right) dA\, dz$$

Recalling that for principal centroidal axes
$\int_A x\, dA = \int_A y\, dA = \int_A xy\, dA = 0$, it follows that

$$U = \frac{1}{2}\int_0^\ell \left(T_z\frac{T_z}{EA} + M_x\frac{M_x}{EJ_{xx}} + M_y\frac{M_y}{EJ_{yy}}\right) dz$$

From the DSV solution for axial and bending moments it is known that:

$$\varepsilon_{zz} = \frac{T_z}{EA} \quad \text{and} \quad K_x = \frac{M_x}{EJ_{xx}} \quad ; \quad K_y = \frac{M_y}{EJ_{yy}}$$

K_x, K_y : curvatures

The strain energy U is then written as:

$$\boxed{U = \frac{1}{2}\int_0^{\ell}\left(EA\,\varepsilon_{zz}^2 + EJ_{xx}\,k_x^2 + EJ\,k_y^2\right)dz} \quad *$$

having highlighted the displacement-related components ε_{zz} and k_α as far as the strain energy is intended as

$U = U(\underline{\underline{\varepsilon}})$

A few remarks:

- Alternatively U can be written as

$$U = \frac{1}{2}\int_0^{\ell}\left(T_z\,\varepsilon_{zz} + M_x\,k_x + M_y\,k_y\right)dz$$

the quantities multiplied in the expression are energetically-conjugated. The axial force is energy conjugated with the deformation ε_{zz}, while the bending moments M_α are energy conjugated with the curvature k_α.

- For this reason the quantities ε_{zz} and k_α can be referred to as generalized displacements.

- Remind the the expression in the box does not account for the shear contribution. For slender beams this contribution is, in fact, negligible.
Whenever a torsional rigidity is accounted for, the contribution associated with the torsion must be included

* or, alternatively as

$$\boxed{U = \frac{1}{2}\int_0^{\ell}\left(EA\,w_{/z}^2 + EJ_{xx}\,v_{/zz}^2 + EJ_{yy}\,u_{/zz}^2\right)dz}$$

Complementary strain energy (axial force and bending)

The expression of the complementary strain energy is already available from the strain energy U.

Indeed, for a linear hyperelastic material, $U = U^*$

(however, it is important to remark, one more time, that U and U^* are associated with different physical meanings. U^* is the Legendre transform of U and has to be intended as a scalar function which depends on the stresses).

It was found that

$$U = \frac{1}{2}\int_0^{\ell}\left(T_z \frac{T_z}{EA} + M_x \frac{M_x}{EJ_{xx}} + M_y \frac{M_y}{EJ_{yy}}\right) dz$$

This expression is already written as function of the stress-related quantities T_z, M_x and M_y. Thus:

$$\boxed{U^* = \frac{1}{2}\int_0^{\ell}\left(T_z \frac{T_z}{EA} + M_x \frac{M_x}{EJ_{xx}} + M_y \frac{M_y}{EJ_{yy}}\right) dz}$$

where the internal actions T_z, M_x and M_x are denoted as generalized forces.

PCVW / Menabrea's th. for evaluating displacements and solving statically undetermined beam systems

- The variational tools previously developed are now applied to analyze beams and systems of beams. With regard to
  1. Evaluation of displacements/rotations in parts of the structure
  2. Solution of statically undetermined problems

- These two set of problems are solved in the context of force-based approaches ($\Rightarrow$ PCVW and Menabrea's th are used).
  It is useful to remark that force-based approaches are particularly useful for solving relatively simple problems, where hand-like calculations can be performed. While it is possible to extend the use of force-based approaches even for problems with several dofs, in these cases it is generally simpler to deal with displacement-based approaches; their implementation in computer problems is, in fact, substantially easier.

- Within the context of linear hyperelastic materials the use of PCVW and Menabrea's th. is interchangeable.

$$
\boxed{
\begin{aligned}
\delta\Pi^* &= \delta(U^* + V^*) = 0 \\
&= \delta\Big(\frac{1}{2}\int_0^l \Big(T_z \frac{T_z}{EA} + M_x \frac{M_x}{EJ_{xx}} + M_y \frac{M_y}{EJ_{yy}} \Big) dz \\
&\quad + F_i \hat{u}_i + M_j \hat{\theta}_j + \hat{F}_r u_r + \hat{M}_s \theta_s \Big) = 0
\end{aligned}
}
$$

or, similarly:

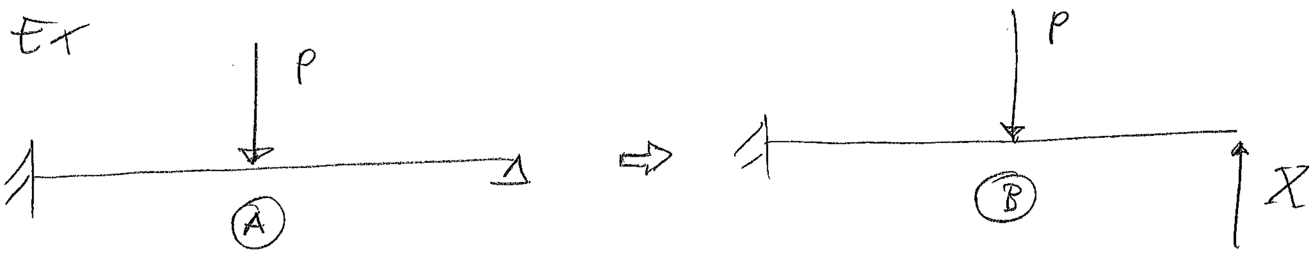
$$\boxed{\begin{aligned} &\delta W_i^* = \delta W_e^* \\ &\int_0^\ell \left(\delta T_z \frac{T_z}{EA} + \delta M_x \frac{M_x}{EJ_{xx}} + \delta M_y \frac{M_y}{EJ_{yy}} \right) dz = \\ &= \underbrace{\delta F_i \hat{u}_i + \delta M_j \hat{\theta}_j}_{\text{imposed displ./rotations}} + \underbrace{\delta \hat{F}_r u_r + \delta \hat{M}_s \theta_s}_{\text{variation of forces/moments}} \end{aligned}}$$

Solution of statically undetermined structures

The procedure is summarized as:

1. Remove the overconstraints and replace them with the reacting force
2. Impose equilibrium conditions
3. Apply PCVW/Menabrea's th. to obtain, among the class of equilibrium solutions, the one which is also compatible.

Ex



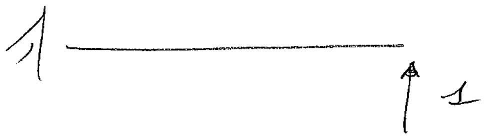
Equilibrium conditions are imposed in system B where everything is expressed as function of the unknown reaction X.

The generalized deformation can be obtained as

$\frac{T_z}{EA}$, $\frac{M_x}{EJ_{xx}}$ and $\frac{M_y}{EJ_{yy}}$

The variational principle is then imposed after evaluating the variations δT_z, δM_x and δM_y in a properly choosen "dummy system".

In the previous case:



Note that

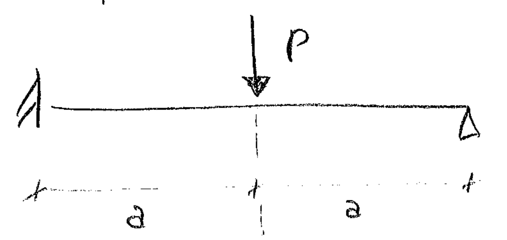
1. the dummy system should be taken such that the external virtual work can be easily expressed

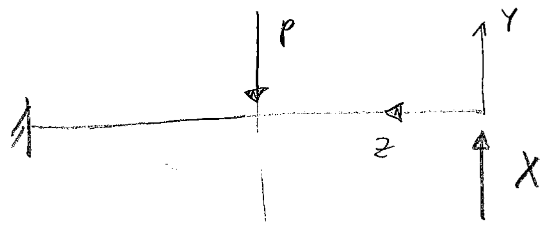
2. the variations δT_z, δM_x and δM_y are required to be in equilibrium; compatibility is not a constraint! Any choice of the variations is fine, provided they are in equilibrium.

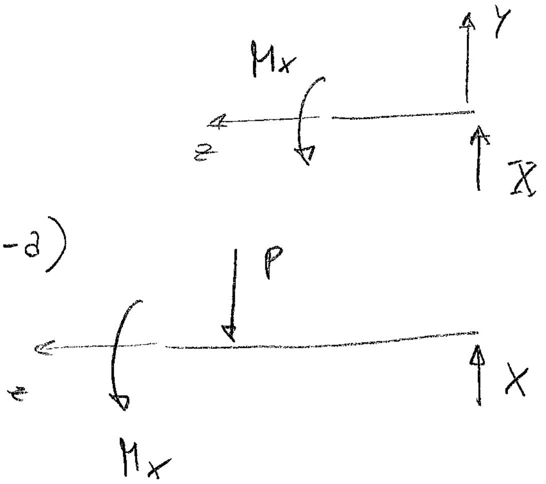
3. for a statically determined problem (say a cantilever beam) the internal actions are readily available from equilibrium considerations. Compatibility is intrinsically satisfied.

4. The evaluation of the unknown reactions (in this example X) requires the application of a variational principle, whose expression is function of the stiffnesses of the beam. In turn, the reaction forces are function of the stiffnesses or, in other words, they depend upon the internal load paths.

The previous example is then solved as:







$0 \leq z \leq a^-$: $\quad M_x = -Xz$

$a^+ \leq z \leq 2a$: $\quad M_x = -Xz + P(z-a)$

The real deformations are then:

$$\frac{M_x}{EJ} = -\frac{Xz}{EJ} \qquad 0 \leq z \leq a^-$$

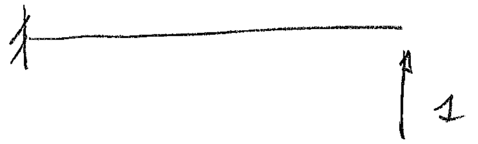
$$\frac{M_x}{EJ} = \frac{P(z-a) - Xz}{EJ} \qquad a^+ \leq z \leq 2a$$

Apply now the PCVW:

$$\int_0^{2a} \delta M_x \frac{M_x}{EJ}\, dz = \delta X \hat{v}$$

imposed displacement at $z=0$
$(\hat{v} = 0)$

The dummy system is then taken as



$\delta M_x = -z$

equilibrated variation

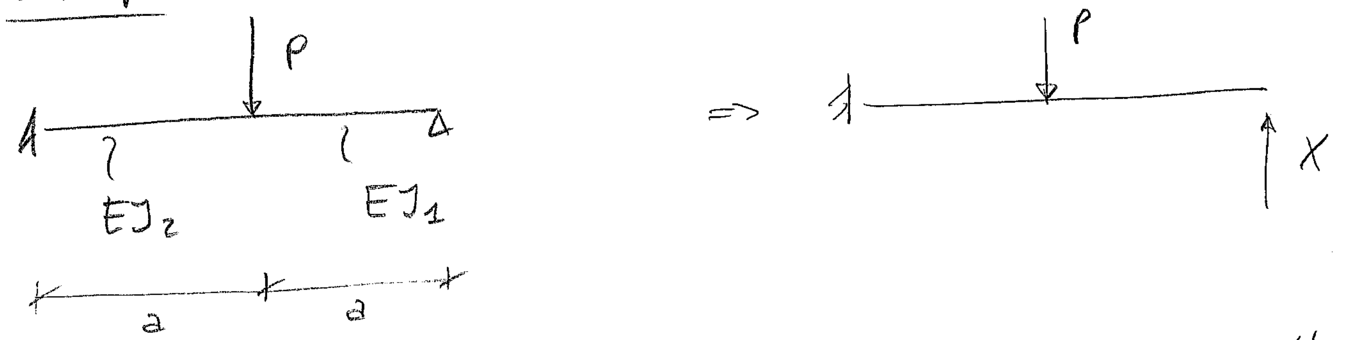
and finally:

$$\int_0^{2a} \delta M_x \frac{M_x}{EJ}\, dz = \int_0^{a} \frac{X z^2}{EJ}\, dz + \int_a^{2a} \frac{-(P-X)z^2 + Paz}{EJ}\, dz$$

$$= \frac{1}{EJ}\left(\frac{1}{3}Xa^3 - \frac{1}{3}(P-X)\,7a^3 + P\frac{3}{2}a^3\right) = 0$$

$$\Rightarrow \boxed{X = \frac{5}{16}P}$$

Example



The internal actions, as long as the dummy system, are readly available from the previous example

$$\int_0^a \frac{Xz^2}{EJ_1}\,dz + \int_a^{2a} \frac{(P-X)z^2 - Paz}{EJ_2}\,dz = 0$$

$$\frac{1}{EJ_1}\frac{a^3}{3}X + \frac{1}{EJ_2}\frac{7}{3}a^3 X - \frac{1}{EJ_2}\frac{5}{6}Pa^3 = 0$$

$$\left(\frac{\alpha}{3} + \frac{7}{3}\right)X = \frac{5}{6}P \qquad \text{con} \quad \alpha = \frac{EJ_2}{EJ_1}$$

$$\boxed{X = \frac{5}{2}\,\frac{P}{\alpha+7}}$$

Note that X depends on the stiffnesses EJ_1 and EJ_2.

For increasing values of α, the value of X is progressively reduced. The internal load path is in fact modified and higher values of EJ_2 tend to unload the beam of stiffness EJ_1. The load flows where the structure is stiffer.

Extensions of beam model

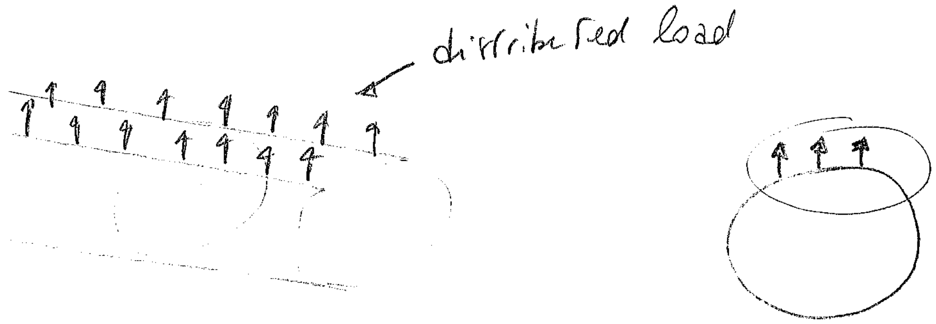
For several cases of practical interest, it is necessary to relax the hypothesis involved in the derivation of the DSV solution. In particular the DSV solutions will be assumed to be a good approximation when

1. The beam is slightly curved
2. The beam is slightly twisted
3. The beam is slightly tapered
4. The load are applied not only at the ends of the beam, but they are introduced also at the outer surface δF_{body} (concentrated forces can be seen as Dirac's functions)

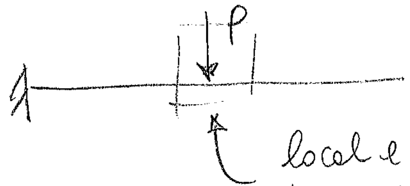
Observe that

1. for those cases the DSV solution is an <u>approximation</u>
2. the higher the violation of the hypothesis, the higher the level of approximation.
3. the application of the DSV results is reasonable provided the region of interest is not drastically affected by local effects.

For instance



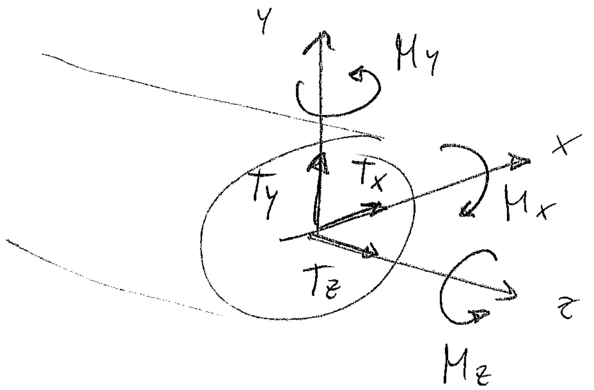
the stress state here will be intrinsically 3D



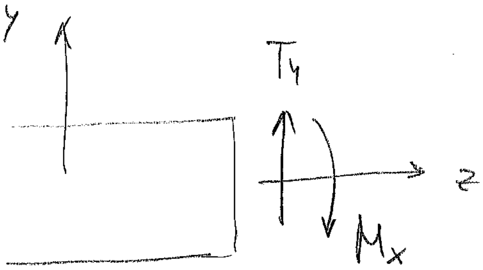
local effects could be relevant here. If an accurate description of the internal stress is sought, a 3D analysis is needed (accordingly the idealization of the load as a concentrated force could be questionable)

Sign conventions

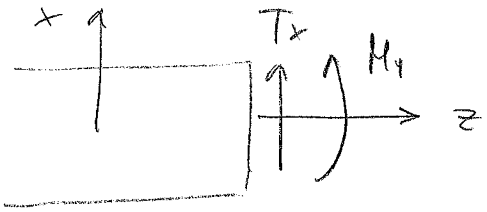
Having extended the analysis range to the case of distributed loads, it is useful to define sign conventions and corresponding differential relations.



- Forces > 0 if directed along coordinate axis
- Moments > 0 according to right hand rule

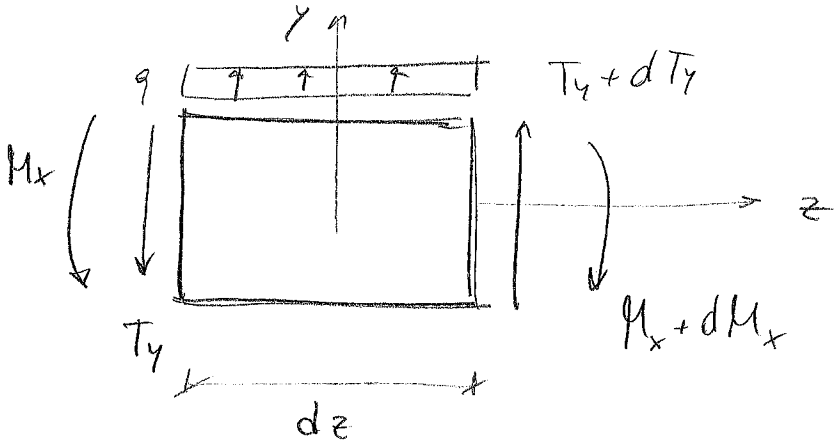


zy plane



zx plane

- Differential relations



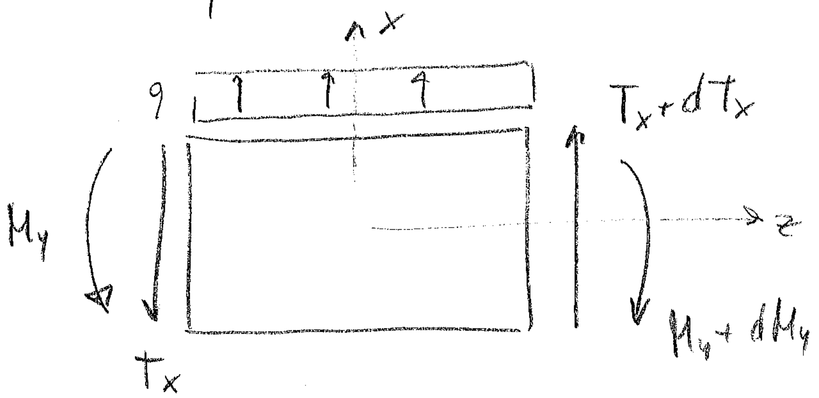
zy plane

$$dT_y = -\,q\,dz$$
$$dM_x = T_y\,dz$$

$\Rightarrow$

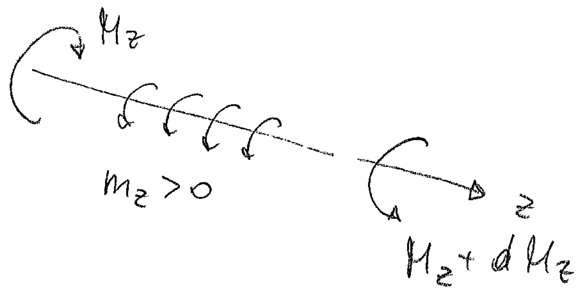
$$\boxed{\begin{aligned} T_y(z) &= -\int_0^z q(\xi)\,d\xi + T_y(0) \\ M_x(z) &= \int_0^z T_y(\xi)\,d\xi + M_x(0) \end{aligned}}$$

Similarly



$$dT_x = -q\,dz \qquad\qquad \Rightarrow \qquad \boxed{\begin{aligned} T_x(z) &= -\int_0^z q(\xi)\,d\xi + T_x(0) \\ M_y(z) &= -\int_0^z T_x(\xi)\,d\xi + M_y(0) \end{aligned}}$$
$$dM_y = -T_x\,dz$$

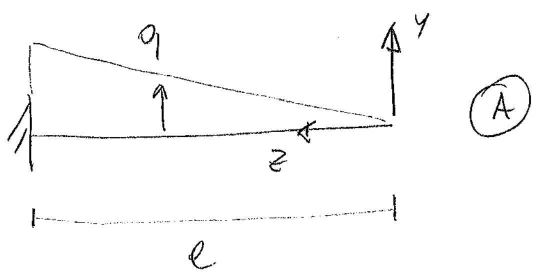
Whenever a distributed twisting moment is considered:

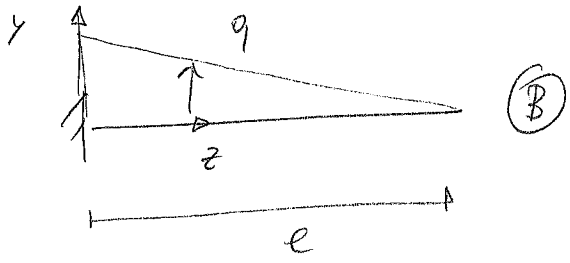


$$dM_z = -m_z\,dz \qquad \Rightarrow \qquad \boxed{M_z(z) = -\int_0^z m_z(\xi)\,d\xi + M_z(0)}$$

Example

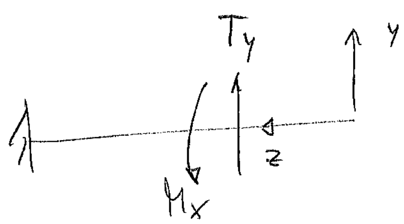
Consider a cantilever beam with a linearly distributed load.





Evaluate the internal actions in $z = l/2$ to verify the independency on the choice of the reference system.

Case A



$$q(z) = q_0 \frac{z}{l}$$

$$T_y(z) = -\int_0^z q(\xi)\, d\xi$$

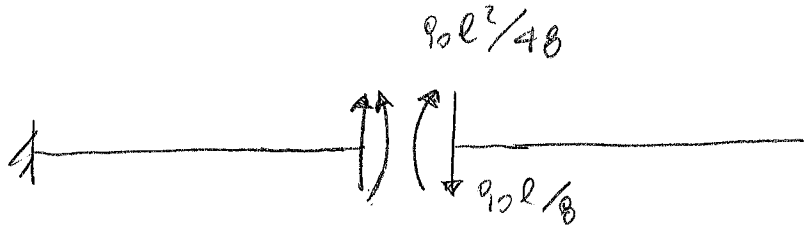
$$= -\int_0^z q_0 \frac{\xi}{l}\, d\xi = -\frac{q_0 z^2}{2l}$$

$$M_x(z) = \int_0^z T_y(\xi)\, d\xi + \cancel{M_x(0)}$$

$$= -\int_0^z \frac{q_0 \xi^2}{2l}\, d\xi = -\frac{q_0 z^3}{6l}$$

Thus:

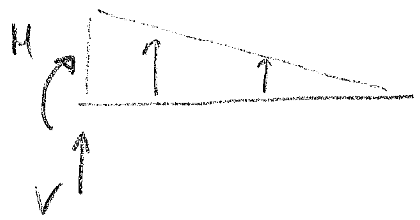
$$\boxed{\begin{aligned} T_y(l/2) &= -q_0 l/8 \\ M_x(l/2) &= -\frac{q_0 l^2}{48} \end{aligned}}$$



Case (B)

$$q(z) = q_0 (1 - z/l)$$

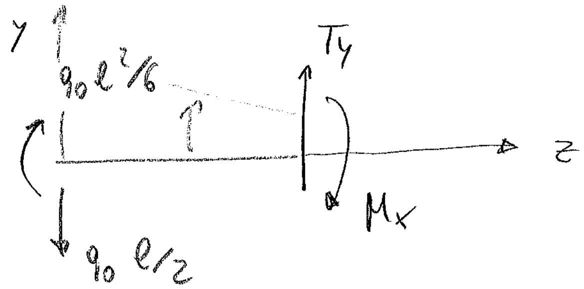
Reaction forces



$$V = - q_0 \, l/2$$

$$M = q_0 \, l^2/6$$

Internal actions



$$T_y(0) = q_0 \, l/2$$

$$M_x(0) = - q_0 \, l^2/6$$

$$T_y(z) = -\int_0^z q_0 (1 - \xi/l) \, d\xi + T_y(0)$$

$$= - q_0 z + q_0 \frac{z^2}{2l} + q_0 \frac{l}{2}$$

$$M_x(z) = \int_0^z T_y(\xi) \, d\xi + M_x(0)$$

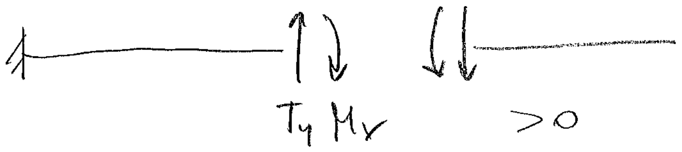
$$= - q_0 \frac{z^2}{2} + q_0 \frac{z^3}{6l} + q_0 \frac{l}{2} z - q_0 \frac{l^2}{6}$$

It follows that

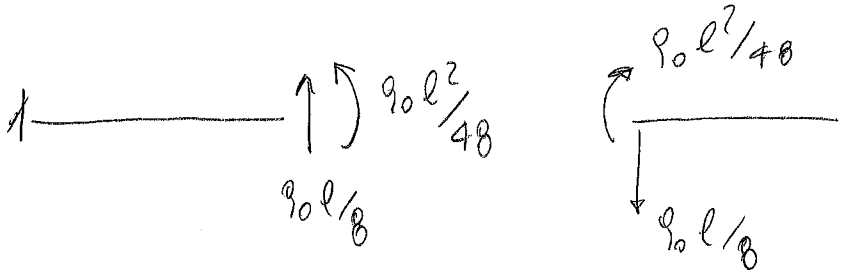
$$T_y(l/2) = q_0 \, l/8$$

$$M_x(l/2) = - q_0 \, l^2/48$$

And, according to the conventions adopted:



It follows that



- If the differential relations between the internal forces are applied, the evaluation of the internal forces is straightforward.

- It is not a surprise to verify that the internal actions do not depend on the reference system.