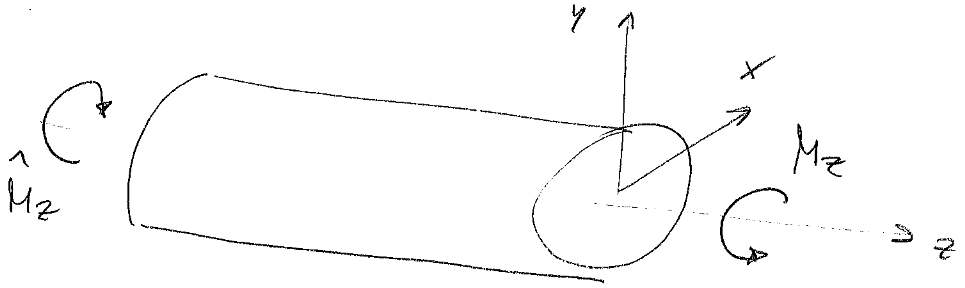
Evaluation of Shear stresses

- The DSV solution for σ_{zz} illustrates that an exact closed-form solution can be found for calculating the direct stress.
  Often this solution suffices for solving beam problems, as far as the beams are slender ($\Rightarrow$ shear contribution is negligible) and no torsion loads are applied.
  For instance, the statically undetermined reactions, as well as the displacements in a given point, can be obtained by using the solution for σ_{zz}.
- In many other cases – thin walled beams, typically used in aerospace applications, are an example – the energy contribution due to the shear cannot be neglected.
  In addition, shear stresses need to be evaluated for verifying whether the structure will yield or buckle locally.
  This is even more true when torsional loads are accounted for (which is a typical loading condition for aerospace structures)
- Overall the problem of the evaluation of the shear stresses is a complex one, and closed-form solutions can be hardly found.
- The problem will be illustrated for the torsional case first and for general loading conditions later.
  Given the difficulty of this problem, approximate solution strategies will be discussed.

- Torsion

Aeronautical structures are commonly subjected to torsional loads, which is thus an important design condition. In general torsional stiffness requirements may arise for aeroelastic considerations



The set of solving equations, available from the solution of the DSV problem for σ_{zz} is

$$\begin{cases} \sigma_{xz/z} = 0 \\ \sigma_{yz/z} = 0 \\ \sigma_{xz/x} + \sigma_{yz/y} + \sigma_{zz/z} = 0 \end{cases}$$

$\sigma_{xz/z} = 0$, $\sigma_{yz/z} = 0$ → state that $\sigma_{xz} = \sigma_{xz}(x,y)$ and $\sigma_{yz} = \sigma_{yz}(x,y)$

$$\begin{cases} \sigma_{zz/yy} = \sigma_{zz/xx} = \sigma_{zz/xy} = 0 \\ -\bar{\nu}\, \sigma_{zz/yz} = \sigma_{xz/xy} - \sigma_{yz/xx} \\ -\bar{\nu}\, \sigma_{zz/xz} = \sigma_{yz/xy} - \sigma_{xz/yy} \end{cases}$$

$\sigma_{zz/yy} = \sigma_{zz/xx} = \sigma_{zz/xy} = 0$ → identically satisfied with the DSV solution for σ_{zz}

+

$\underline{t} \cdot \underline{n} = 0$ on $S_{F\,body}$

+

$$T_x = \int_A \sigma_{xz}\, dA$$

$$T_y = \int_A \sigma_{yz}\, dA$$

$$M_z = \int_A (-\sigma_{xz}\, y + \sigma_{yz}\, x)\, dA$$

$$\boxed{\begin{aligned} T_z &= \int_A \sigma_{zz}\, dA \\ M_x &= \int_A \sigma_{zz}\, y\, dA \\ M_y &= \int_A -\sigma_{zz}\, x\, dA \end{aligned}}$$

The semi-inverse approach can be formulated for the torsion problem by assuming that:

$$\boxed{\begin{aligned} &\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0 \\ &\text{and} \\ &\sigma_{zz} = 0 \end{aligned}}$$

Among the 6 components of $\underline{\underline{\sigma}}$, the only unknowns are

$$\boxed{\begin{aligned} \sigma_{xz} &= \sigma_{xz}(x,y) \\ \sigma_{yz} &= \sigma_{yz}(x,y) \end{aligned}}$$

where the dependence on x, y descends from the two equilibrium equations.

The updated set of governing equations (assuming that $\sigma_{zz}=0$) is:

$$\sigma_{xz/x} + \sigma_{yz/y} + \cancel{\boxed{\sigma_{zz/z}}} = 0 \quad \leftarrow 3^{rd} \text{ eq. equilibrium}$$

$$\left.\begin{aligned} \cancel{\left|-\bar{\nu}\,\sigma_{zz/yz}\right|} &= \sigma_{xz/xy} - \sigma_{yz/xx} \\ \cancel{\left|-\bar{\nu}\,\sigma_{zz/xz}\right|} &= \sigma_{yz/xy} - \sigma_{xz/yy} \end{aligned}\right\} \text{ compatibility}$$

+

$\underline{t} \cdot \underline{n} = 0$ in $S_{F\,body}$

+

$$\left.\begin{aligned} M_t &= \int_A \left(-\sigma_{xz}\,y + \sigma_{yz}\,x\right) dA \\ T_x &= 0 \\ T_y &= 0 \end{aligned}\right\} \text{ in } S_{F\,face}$$

Rearrange the compatibility (for convenience) as:

$$\left.\begin{aligned} \left(\sigma_{xz/y} - \sigma_{yz/x}\right)_{/x} &= 0 \\ \left(\sigma_{yz/x} - \sigma_{xz/y}\right)_{/y} &= 0 \end{aligned}\right\} \Rightarrow \sigma_{yz/x} - \sigma_{xz/y} = C$$

$(C = \text{const})$

The final set of governing equations for the torsion problem, having assumed $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{zz} = 0$, is:

$$\boxed{\begin{array}{lll} \sigma_{xz/x} + \sigma_{yz/y} = 0 & \text{(equilibrium)} & \text{in } V \\ \sigma_{yz/x} - \sigma_{xz/y} = C & \text{(compatibility)} & \text{in } V \\ \underline{t} \cdot \underline{n} = 0 & & \text{in } S_{F\,body} \\ M_t = \int_A \left(-\sigma_{xz}\, y + \sigma_{yz}\, x \right) dA & & \text{in } S_{F\,face} \end{array}}$$

Box 1

Solution procedure (based on assumed displacements)

The guess solution is

$$\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{zz} = 0$$

From the constitutive isotropic 3D law:

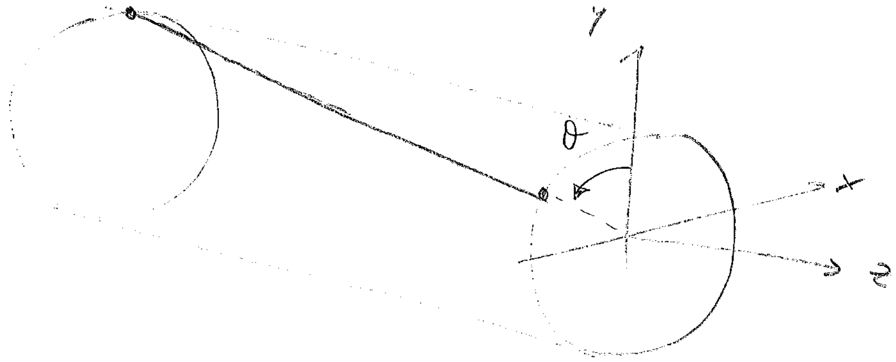
$$\begin{array}{ll} \varepsilon_{xx} = 0 & \gamma_{xy} = 0 \\ \varepsilon_{yy} = 0 & \gamma_{xz} \neq 0 \\ \varepsilon_{zz} = 0 & \gamma_{yz} \neq 0 \end{array}$$

Recalling that

$$\begin{array}{ll} \varepsilon_{xx} = u_{/x} & \gamma_{xy} = u_{/y} + v_{/x} \\ \varepsilon_{yy} = v_{/y} & \\ \varepsilon_{zz} = w_{/z} & \end{array}$$

It follows that the displacement field is

$$\begin{array}{l} u = u(\cancel{x}, y, z) \\ v = v(x, \cancel{y}, z) \\ w = w(x, y, \cancel{z}) \end{array} \Rightarrow \boxed{\begin{array}{l} u = u(y,z) \\ v = v(x,z) \\ w = w(x,y) \end{array}}$$



Based on the previous results, the displacement field can be assumed as described below.

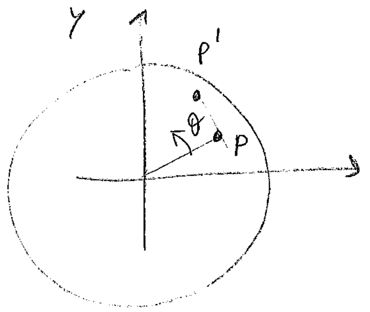
Consider θ as:

$$\boxed{\theta = \theta(z) = \theta_{/z}\, z \qquad \text{where } \theta_{/z} = \text{const}}$$

θ: rotation angle

θ': torsion

Assume that the displacement over the beam section can be represented as:



p: before deformation

p': after deformation

Note: w does not depend on z so $w = \square\, \Psi$. It is multiplied with θ' because a dependency on the "amount" of torsion has to be accounted for!

$$u = u(y,z) = -y\theta$$
$$v = v(x,z) = +x\theta$$

→ see previous figure

$$w = w(x,y) = \theta_{/z}\Psi(x,y)$$

Ψ = warping function

Recalling that $\theta = \theta_{/z}\, z = \theta' z$ (prime denotes $(\cdot)_{/z}$)

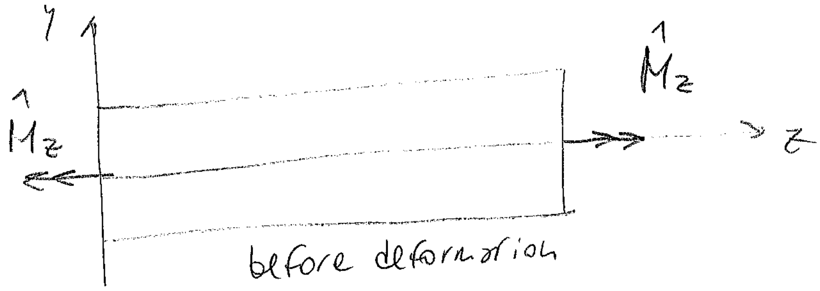
$$\boxed{\begin{aligned} u &= -y\theta' z \\ v &= x\theta' z \\ w &= \theta'\Psi \end{aligned}}$$

Box 2

→ w does not depend on z

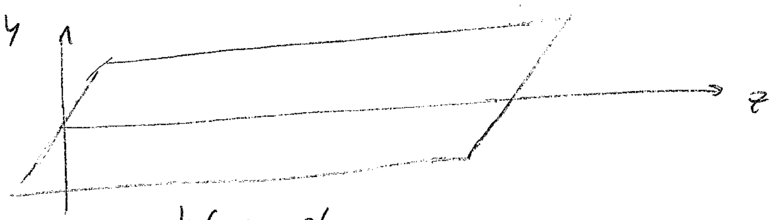
↑ assumed set of displacement components, in agreement with the guess solution here assumed

Note that $w = w(x,y) \Rightarrow$ no dependence on z.
Within the context of the DSV solution the beam is free to warp, i.e. <u>warping is not prevented</u> by the boundary conditions



before deformation

view on yz plane



deformed

<u>warping not prevented</u>

<u>Set of equations in terms of displacements</u>

The set of equations in Box 1 can be re-written in terms of displacement components using Box 2.
In particular, it is sufficient to note that:

$$\gamma_{xz} = w_{/x} + u_{/z} = \theta'(\psi_{/x} - y)$$
$$\gamma_{yz} = w_{/y} + v_{/z} = \theta'(\psi_{/y} + x)$$

and from the constitutive law:

$$\boxed{\begin{aligned}\sigma_{xz} &= G\theta'(\psi_{/x} - y)\\ \sigma_{yz} &= G\theta'(\psi_{/y} + y)\end{aligned}}$$

The governing equations of Box 1 are then:

$$\begin{cases} G\theta'(\Psi_{/xx}+\Psi_{/yy})=0 \\ G\theta' + G\theta' = C \\ \underline{t}\cdot\underline{n} = \sigma_{xz}\cdot n_x + \sigma_{yz}\, n_y = G\theta'(\Psi_{/x}-y)\,n_x + G\theta'(\Psi_{/y}+x)\,n_y = 0 \\ M_t = G\theta'\int_A \left[(-\Psi_{/x}\,y+y^2)+(\Psi_{/y}\,x+x^2)\right]dA \end{cases}$$

and, simplifying:

$$\boxed{\begin{array}{ll} \Psi_{/xx}+\Psi_{/yy}=0 & \text{in } V \\ 2G\theta' = C & \text{in } V \\ \Psi_{/n} = y\,n_x - x\,n_y & \text{in } S_{F\,body} \\ M_t = GJ\theta' & \text{in } S_{F\,face} \end{array}}$$

Box3

$$\underline{t}\cdot\underline{n} = (\sigma_{xz}\,\underline{e}_x + \sigma_{yz}\,\underline{e}_y)\cdot(n_x\,\underline{e}_x + n_y\,\underline{e}_y) = \sigma_{xz}\,n_x + \sigma_{yz}\,n_y = \dots$$

having defined $J = \int_A \left[(-\Psi_{/x}\,y+y^2)+(\Psi_{/y}\,x+x^2)\right]dA$

Remarks

1. The problem in Box3 is a Neumann problem (the boundary conditions regard the derivatives of Ψ). Sometimes the problem is referred to as Neumann–Dini problem. The problem is well-posed and a solution can be found $\Rightarrow$ the initial guess was correct

2. The solution procedure can be outlined as:

   a. Solve Box 3 and determine Ψ

   b. From Ψ determine σ_{xz}, σ_{yz}

   c. Evaluate $\theta' = \dfrac{M_t}{GJ} = \dfrac{1}{GJ}\int_A (-\sigma_{xz}\,y + \sigma_{yz}\,x)\,dA$

   d. If needed the displacement field is available as $u = -yz\theta'$; $v = xz\theta'$; $w = \Psi\theta'$

3. The compatibility condition in Box 3 is not that useful. This is clear that, having assumed the displacement field, no compatibility issues can be expected.

4. It was introduced J as

$$J = \int_A \left(x^2 + y^2 - \Psi_{/x}\, y + \Psi_{/y}\, x\right) dA$$

Thus J is not the polar moment of inertia !

$J = J_p$ iif $\Psi = 0$ (no warping)

Whenever the section warps $\Rightarrow J \neq J_p$. More specifically it can be demonstrated (although it is not to hard to accept it!) that

$J \leq J_p$

Meaning of warping

Which is the sense of warping?

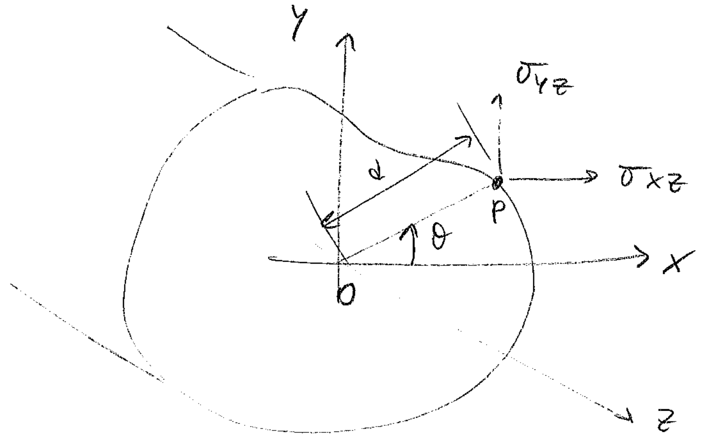
Consider the expression of the stresses:

$$\sigma_{xz} = G\theta' \left(\Psi_{/x} - y\right)$$
$$\sigma_{yz} = G\theta' \left(\Psi_{/y} + x\right)$$

Assume that the section does not warp $\Rightarrow \Psi = 0$, so:

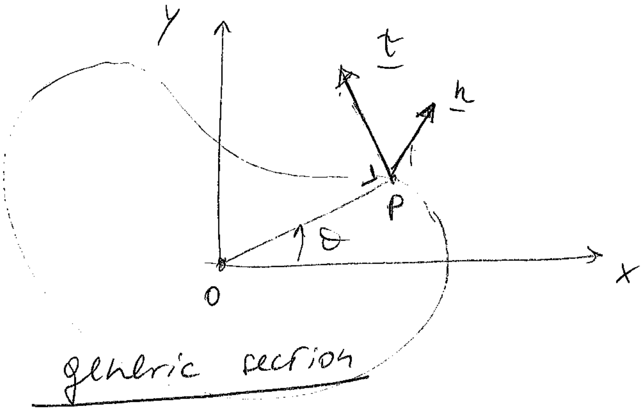
$$\boxed{\begin{aligned} \sigma_{xz} &= -G\theta' y = -ky \\ \sigma_{yz} &= G\theta' x = kx \end{aligned}} \qquad (k = \text{const})$$

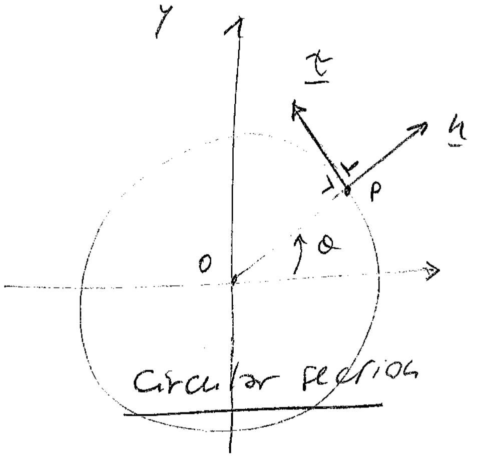
The shear stress at a generic point P of the beam along the boundary of the section is then



$$\bar{x} = d\cos\theta \qquad \Rightarrow \qquad \sigma_{xz} = -ky = -kd\sin\theta$$
$$\bar{y} = d\sin\theta \qquad\qquad \sigma_{yz} = kx = kd\cos\theta$$

meaning that the shear vector $\underline{t} = \sigma_{xz}\,\underline{e}_x + \sigma_{yz}\,\underline{e}_y$ is normal to $\overline{OP}$ $\boxed{\underline{t}\cdot\overline{OP} = 0}$





It is then clear that, unless the section is circular,

$\underline{t}\cdot\underline{n} \neq 0$ (if $\psi = 0$)

For a circular section $\quad \underline{n} = \cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_y$

So: $\quad \underline{t}\cdot\underline{n} = (-kd\sin\theta\,\underline{e}_x + kd\cos\theta\,\underline{e}_y)\cdot(\cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_y)$

$$= -kd\sin\theta\cos\theta + kd\cos\theta\sin\theta$$

$$= 0$$

$\boxed{\text{The warping can be seen as a way to restore the Cauchy equilibrium condition } \underline{t}\cdot\underline{n} = 0}$

Evaluation of the torsional stiffness

It was found that

$$M_z = GJ\theta'$$

$$J = \int_A \left(x^2 + y^2 + \Psi_{/y}\, x - \Psi_{/x}\, y\right) dA$$

- The evaluation of J requires the knowledge of the warping function Ψ (which should be determined by solving the Neumann-Dini problem)
- The solution of the problem in Box 3 is not straightforward. A useful expression for J can be alternatively obtained as it follows.

$$\int_A \left(\Psi_{/y}\, x - \Psi_{/x}\, y\right) dA$$

$$= \int_A \left[(\Psi x)_{/y} - (\Psi y)_{/x}\right] dA$$

(Recalling the gradient theorem)

$$= \int_{\partial A} \left(\Psi x\, n_y - \Psi y\, n_x\right) d\Gamma$$

$$= \int_{\partial A} \Psi \left(x n_y - y n_x\right) d\Gamma$$

Recalling, from Box 3, that $\Psi_{/n} = y n_x - x n_y$:

$$= - \int_{\partial A} \Psi\, \Psi_{/n}\, d\Gamma = - \int_{\partial A} \Psi\, \mathrm{grad}\,\Psi \cdot \underline{n}\, d\Gamma$$

and from the divergence theorem.

$$= -\int_A \operatorname{div}(\psi \operatorname{grad} \psi)\, dA$$

$$= -\int_A \left[(\psi \psi_{/x})_{/x} + (\psi \psi_{/y})_{/y}\right] dA$$

$$= -\int_A \left(\psi \psi_{/xx} + \psi_{/x}^2 + \psi \psi_{/yy} + \psi_{/y}^2\right) dA$$

$$= -\int_A \left(\psi_{/x}^2 + \psi_{/y}^2 + \underbrace{\psi\left(\psi_{/xx} + \psi_{/yy}\right)}_{=0 \text{ (from Box 3)}}\right) dA$$

Thus, it is obtained that:

$$\boxed{\int_A \left(\psi_{/y} x - \psi_{/x} y\right) dA = -\int_A \left(\psi_{/x}^2 + \psi_{/y}^2\right) dA} \quad \text{partial result}$$

Recalling now the expression of J:

$$J = \int_A \left(x^2 + y^2 + \psi_{/y} x - \psi_{/x} y\right) dA$$

$$= \int_A \left(x^2 + y^2 + \psi_{/y} x - \psi_{/x} y\right) dA + \underbrace{\int_A \left(\psi_{/y} x - \psi_{/x} y + \psi_{/x}^2 + \psi_{/y}^2\right) dA}_{=0}$$

$$= \int_A \left(x^2 + y^2 + 2\psi_{/y} x - 2\psi_{/x} y + \psi_{/x}^2 + \psi_{/y}^2\right) dA =$$

$$= \int_A \left[\left(\psi_{/x} - y\right)^2 + \left(\psi_{/y} + x\right)^2\right] dA$$

and recalling that the stresses were obtained as:

$$\tau_{xz} = G\theta' \left(\psi_{/x} - y\right) \quad \text{and} \quad \tau_{yz} = G\theta' \left(\psi_{/y} + x\right)$$

It follows that:

$$J = \frac{1}{(G\theta')^2} \int_A \left(\tau_{xz}^2 + \tau_{yz}^2 \right) dA$$

but $\quad M_z = GJ\theta' \quad \Rightarrow \quad (G\theta')^2 = M_z^2 / J^2$

and so:

$$\boxed{J = \frac{M_z^2}{\int_A \left(\tau_{xz}^2 + \tau_{yz}^2 \right) dA}}$$

Remarks

1. This expression of J is a reformulation of the original form $J = \int_A \left(x^2 + y^2 + \Psi_{/y} x - \Psi_{/x} y \right) dA$
It is useful because it makes possible the evaluation of J without the need for computing the warping function Ψ.
Clearly, the problem has been now shifted to the evaluation of the shear stresses: Ψ is not needed, but a way for obtaining τ_{xz} a τ_{yz} is necessary.

Strain energy and complementary strain energy for torsion

The previous result is useful for obtaining the expression of the strain and complementary strain energy.

$$U = \frac{1}{2}\int_V \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} \, dV = \frac{1}{2}\int_V \left(\sigma_{xz}\gamma_{xz} + \sigma_{yz}\gamma_{yz}\right) dV$$

and from the constitutive law: $= \dfrac{1}{2G}\displaystyle\int_V \left(\sigma_{xz}^2 + \sigma_{yz}^2\right) dV$

Recalling how that $J = \dfrac{M_z^2}{\int_A \left(\sigma_{xz}^2 + \sigma_{yz}^2\right) dA}$, it is obtained:

$$U = \frac{1}{2G}\int_\ell \frac{M_z^2}{J}\, dz$$ and observing that $M_z = GJ\theta'$

$$\Rightarrow \boxed{U = \frac{1}{2}\int_\ell GJ\theta'^2 \, dz}$$

The complementary energy $U^* = U$ is expressed as function of the internal action M_z recalling that $\theta' = \dfrac{M_z}{GJ}$, so:

$$\boxed{U^* = \frac{1}{2}\int_\ell M_z \frac{M_z}{GJ}\, dz}$$

Qualitative interpretation of the results by means of the hydraulic analogy

Consider the set of governing equations in Box 1, and analyze the equilibrium requirements:

$$
\boxed{\begin{aligned} &\sigma_{xz/x} + \sigma_{yz/y} = 0 && \text{in } V \\ &\underline{t} \cdot \underline{n} = 0 && \text{in } S_{F_{body}} \end{aligned}}
$$

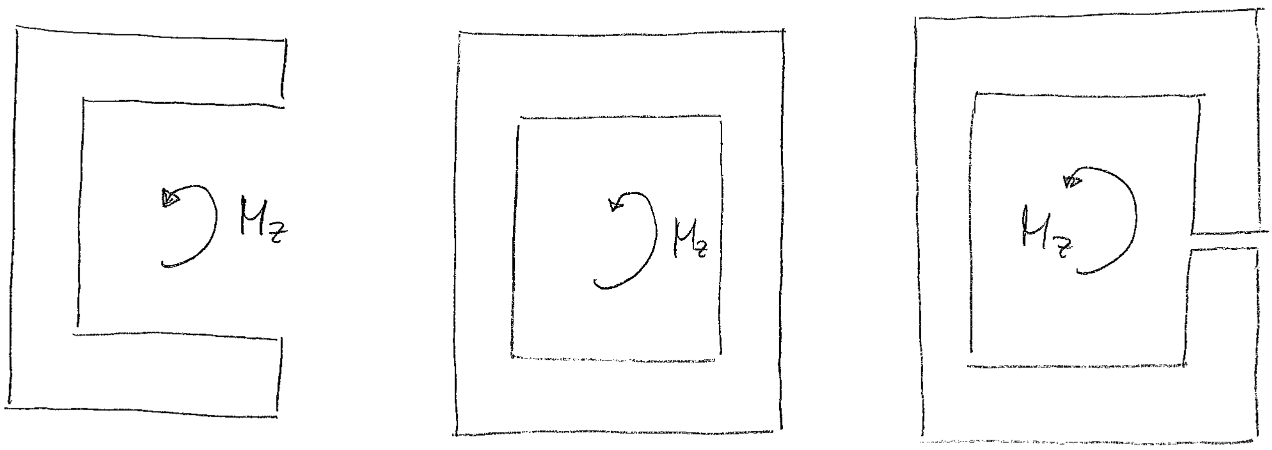
(but σ_{xz} and σ_{yz} are functions of x, y so the equation holds in S)

or $\begin{cases} \operatorname{div} \underline{t} = 0 \\ \underline{t} \cdot \underline{n} = 0 \end{cases}$ by replacing $\underline{t}$ with the velocity vector $\underline{v}$, the equations of equilibrium are

$\begin{cases} \operatorname{div} \underline{v} = 0 \\ \underline{v} \cdot \underline{n} = 0 \end{cases}$ which are the equations describing the mass balance and the tangency condition for an incompressible inviscid flow.

The analogy with the motion of a fluid allows to qualitatively understand how the shear stress will flow in the section.

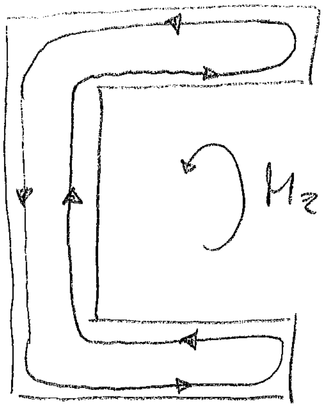
Consider, for instance, the three examples here below



Section 1 (open) — Section 2 (closed) — Section 3 (open)

The distribution of the shear stresses can be understood referring to the hydraulic analogy.

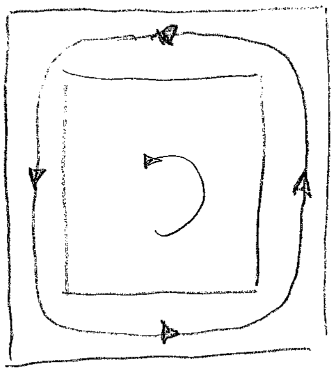
Consider <u>section 1</u>:



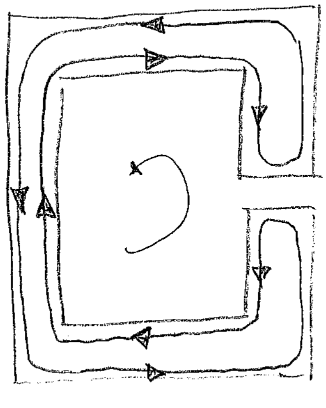
- M_z is the internal action, thus the shear stresses should satisfy the <u>equivalence</u> of the resultant with M_z (not the equilibrium!)

  $\Rightarrow$ the direction of the flow is easily obtained
- They should respect the tangency condition at the boundaries.

Consider now <u>section 2 & 3</u>



- The two previous consideration still hold
- For section 2 the tangency conditions do not impose the need for a reversal of the flow along the thickness-wise direction. The flow simply circulates inside the "loop"

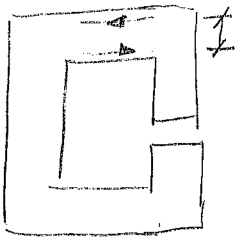


- Section 3 is very similar to section 2. However, the presence of a "cut" introduces novel tangency condition along the cut itself.

  The flow is thus forced to reverse its direction along the thickness coordinate

This comparison reveals an important distinction between open- and closed-section profiles:

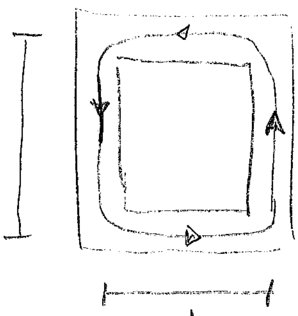
1. open-section profiles can react a twisting moment with an internal shear stress distribution characterized by a <u>small arm</u>.



d = arm between shear flows

It follows that, for a given moment, the shear stresses will be higher than in a similar closed-section profile

2. closed-section profiles are characterized by higher arms between the shear flows



d_1

d_2 = arm

For reacting a given twisting moment, the shear stresses will be then smaller

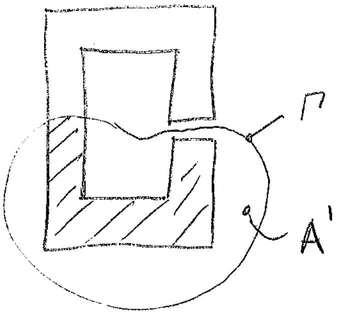
3. • open-section ⇒ higher stresses ⇒ higher deformation ⇒ smaller rigidity

   • closed-section ⇒ smaller stresses ⇒ lower deformation ⇒ higher rigidity

From a mathematical standpoint, the previous considerations can be formalized as it follows:

$$\begin{cases} \operatorname{div} \underline{t} = 0 & \text{in } A \\ \underline{t} \cdot \underline{n} = 0 & \text{in } \Gamma \end{cases},$$

Integrate over a generic portion of the section A':



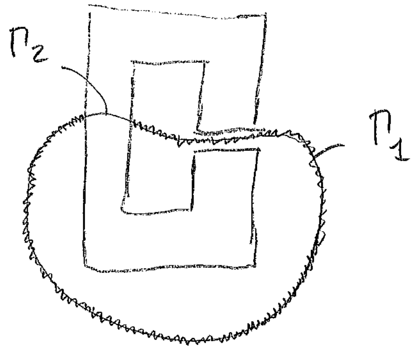
$$\int_{A'} \operatorname{div} \underline{t} \, dA' = 0$$ apply the divergence th.:

$$\oint_{\Gamma} \underline{t} \cdot \underline{n} \, d\Gamma = 0$$

Denoting with ϕ the quantity $\phi = \oint_{\Gamma} \underline{t} \cdot \underline{n} \, d\Gamma$, it follows that

$\boxed{\phi = 0}$ where ϕ denotes the net flow of shear stresses outflowing from Γ (from the definition of ϕ)

Consider the previous example



$\Gamma = \Gamma_1 \cup \Gamma_2$

— Γ_2

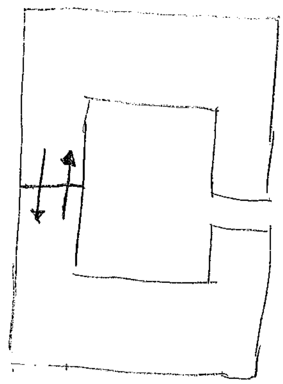
~~ Γ_1

It is clear that no flow is crossing the portion of the boundary denoted with Γ_1

$$\oint_{\Gamma} \underline{t} \cdot \underline{n} \, d\Gamma = \int_{\Gamma_1} \underline{t} \cdot \underline{n} \, d\Gamma_1 + \int_{\Gamma_2} \underline{t} \cdot \underline{n} \, d\Gamma_2 = 0$$

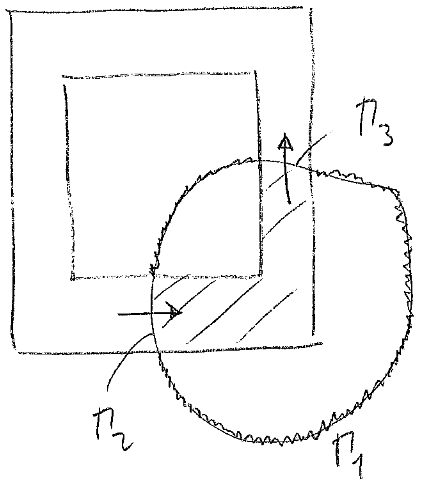
The net flow across Γ_2 is then null

$$\int_{\Gamma_2} \underline{t} \cdot \underline{n} \, d\Gamma_2 = 0$$



To satisfy this condition, part of the shear stresses need to be directed along the downward direction, part along the upward direction.

For the same reasons, if a closed-section is considered



$$\oint_{\Gamma} \underline{t} \cdot \underline{n} = \int_{\Gamma_1} \underline{t} \cdot \underline{n} \, d\Gamma_1 + \int_{\Gamma_2} \underline{t} \cdot \underline{n} \, d\Gamma_2 + \int_{\Gamma_3} \underline{t} \cdot \underline{n} \, d\Gamma_3 = 0$$

but $\int_{\Gamma_1} \underline{t} \cdot \underline{n} \, d\Gamma_1 = 0$

$$\Rightarrow \boxed{\int_{\Gamma_2} \underline{t} \cdot \underline{n} \, d\Gamma_2 = -\int_{\Gamma_3} \underline{t} \cdot \underline{n} \, d\Gamma_3}$$

The inflow of shear stresses in Γ_2 is equal to the outflow in Γ_2 and viceversa

Approximate solution for thin walled profiles

In most cases, typical aerospace constructions are characterized by thin walled sections, meaning that thicknesses are small in relation to the typical dimensions of the section.

In these cases the solution, i.e. the evaluation of the shear stresses, can be obtained by introducing proper approximation related with the exiguity of the thicknesses

Recall the set of equations for the torsional case

$$\boxed{\begin{aligned} &\sigma_{xz/x} + \sigma_{yz/y} = 0 \\ &\sigma_{yz/x} - \sigma_{xz/y} = 2G\theta' = 2M_z/J \\ &\underline{\tau} \cdot \underline{n} = 0 \\ &M_t = GJ\theta' \end{aligned}}$$

(equilibrium)

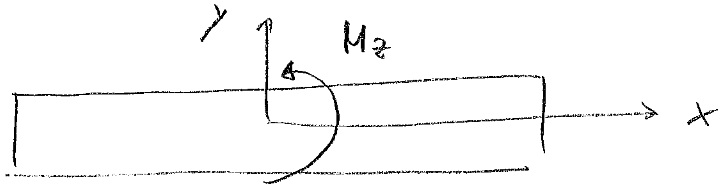
(compatibility)

equilibrium boundary

equiv. internal action

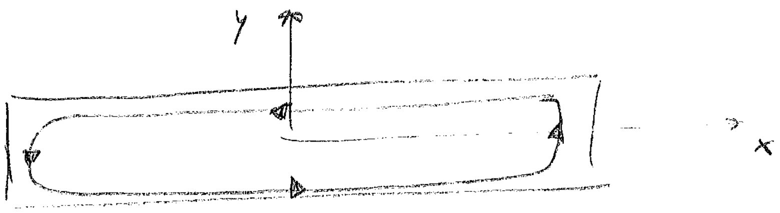
(From Box 1 and Box 3 for compatibility)

Open profiles



Narrow strip subjected to torsion M_z

A qualitative analysis reveals that the shear stresses will be directed as:

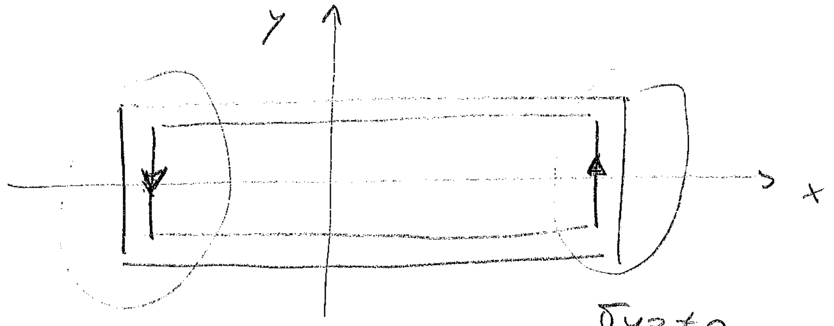


qualitative analysis

It is then clear that

$$\boxed{\sigma_{yz} \simeq 0}$$

This is a good approximation in most of the section, apart from the external portions where the shear flow is directed along the vertical direction, so $\sigma_{yz} \neq 0$

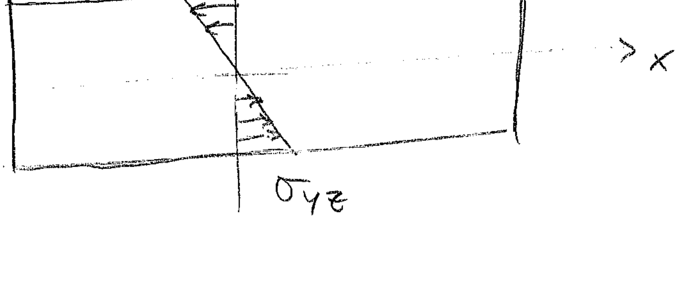


Substitute now the approximation σ_{yz} in the set of governing equations to obtain:

$$\boxed{\begin{array}{ll} \sigma_{xz/x} = 0 & \text{equilibrium} \\ -\sigma_{xz/y} = 2M_z/J & \text{compatibility} \end{array}}$$

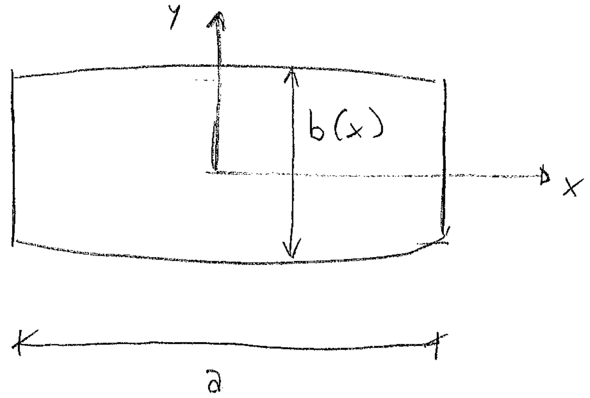
$$\left.\begin{array}{lll} \sigma_{xz/x} = 0 & \Rightarrow & \sigma_{xz} = Ay \\ -\sigma_{xz/y} = 2M_z/J & \Rightarrow & A = -\dfrac{2M_z}{J} \end{array}\right\} \Rightarrow \boxed{\sigma_{xz} = -\frac{2M_z}{J}y}$$

The approximate solution for thin open profiles state that $\sigma_{yz} \simeq 0$ and σ_{xz} is linear along the thickness



The value of J is readily obtained recalling that:

$$J = \frac{M_t^2}{\int_A (\tau_{xz}^2 + \tau_{yz}^2)\, dA} = \frac{J^2}{\int_A 4y^2\, dA} \Rightarrow J = \int_A 4y^2\, dA$$



Assume, for generality, that the thickness b is not constant, but depends on x

$$\int_A 4y^2\, dA = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} 4y^2\, dy\, dx = \int_{-a/2}^{a/2} \frac{1}{3} b^3(x)\, dx$$

$$\boxed{J = \frac{1}{3} \int_{-a/2}^{a/2} b^3(x)\, dx \qquad J = \frac{1}{3} a b^3}$$

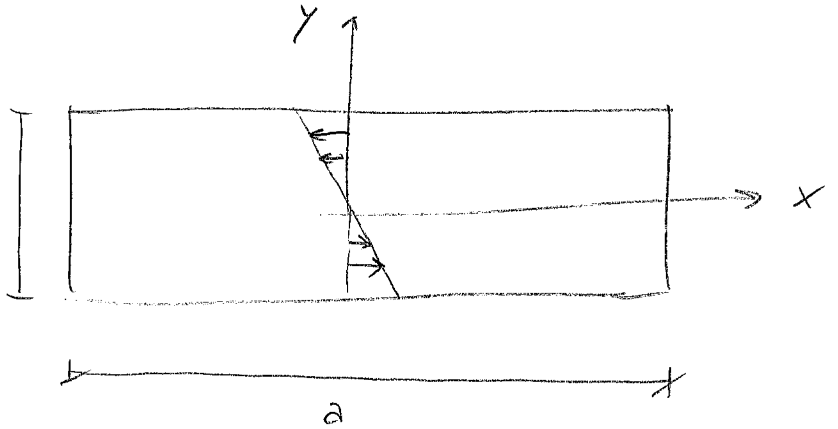
variable thickness

constant thickness

Remark

It is interesting to evaluate the resultant over the section associated with the solution here derived

$$\tau_{yz} = 0 \, ; \quad \tau_{xz} = -\frac{2M_t}{J} y$$



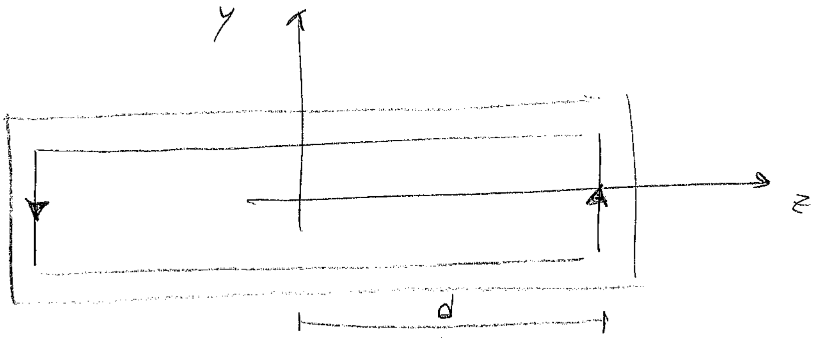
$$-\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} \sigma_{xz}\, y\, dy\, dx$$

$$= a\cdot\int_{-b/2}^{b/2} \frac{2M_z}{J}\, y^2\, dy = \frac{2aM_z}{J}\left(\frac{1}{3}\frac{b^3}{8}+\frac{1}{3}\frac{b^3}{8}\right)$$

$$= \frac{2aM_z}{J}\,\frac{b^3}{12}$$ and recalling that $J=\frac{1}{3}ab^3$

$$= \frac{M_z}{2} \quad \Rightarrow \quad \boxed{-\int_{a/2}^{a/2}\int_{-b/2}^{b/2} \sigma_{xz}\, y\, dy\, dx = \frac{M_z}{2}}$$

The result illustrates that the integral over the section of the approximate solution leads to a moment $M_z/2$, just half of the internal moment M_z.
This means that the equivalence condition with the internal action is not respected (indeed it was not applied throughout the solution process). How is it explained? Is something wrong?
No, everything is fine within the context of the approximation here introduced. Recall that σ_{yz} was assumed $\simeq 0$

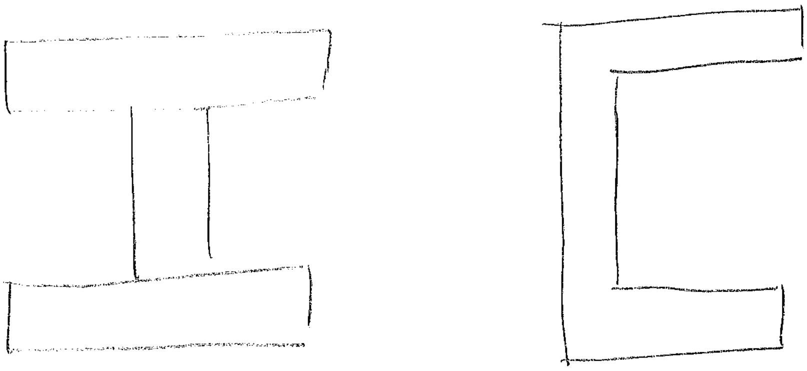


The arm of the contribution σ_{yz} is large!

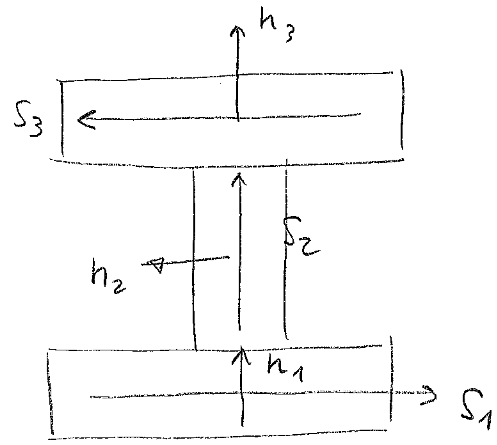
Neglecting σ_{yz} means neglecting a large part of the contribution to the internal twisting moment

Open sections with generic shape

The previous results can be readily extended to the case of open sections with generic shape, e.g.



These shapes can be analyzed as an assembly of narrow strips

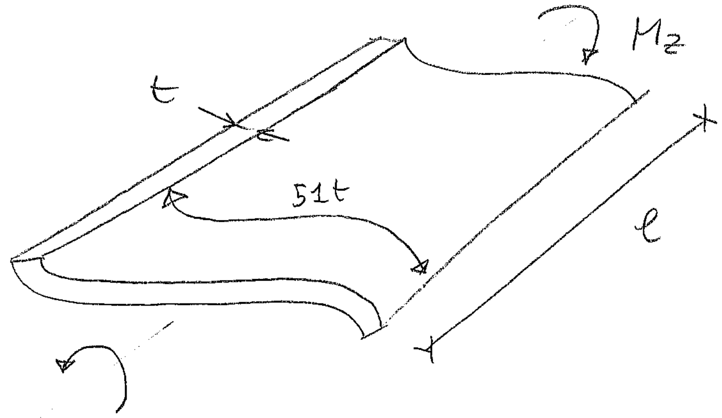


n_i : axis normal to i-th strip
s_i : axis tangent to i-th strip

$$\boxed{\begin{aligned}&\sigma_{nz} = 0\\ &\sigma_{sz} = -\frac{2M_t}{J}n\\ &\text{with } J = \frac{1}{3}\sum_i a_i b_i^3\end{aligned}}$$

Exercise

Consider the beam in the figure

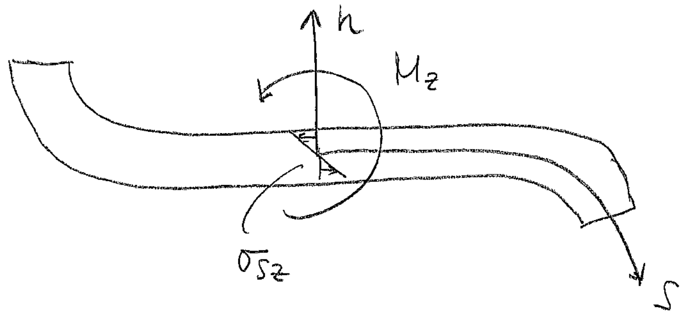


Evaluate:

1. the maximum stress over the section
2. the relative rotation between the two outer section

Solution

The internal stress state is



$$\sigma_{sz} = -\frac{2M_z}{J} n \quad \text{with} \quad J = \frac{1}{3} a b^3 = \frac{1}{3} 51t \cdot t^3 = 17 t^4$$

$$\Rightarrow \boxed{\sigma_{sz} = -\frac{2M_z}{17t^4} n}$$

Clearly the maximum stress is at the top and bottom, i.e.

$n = \pm t/2$

and so:

$$\boxed{|\sigma_{sz}| = \frac{M_z}{17t^3}}$$

the relative rotation is obtained recalling that:

$$M_z = GJ\theta' \Rightarrow \theta' = \frac{M_z}{GJ} \quad \text{but} \quad J = 17t^4$$

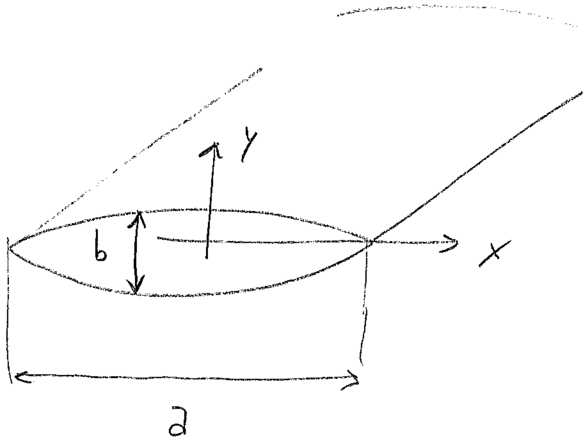
$$\theta' = \frac{M_z}{17Gt^4}$$

as far as θ is linear with the beam axis z,

$$\boxed{\Delta\theta = \theta' \cdot l = \frac{M_z l}{17Gt^4}}$$

Exercise

Evaluate the torsional stiffness of a wing profile which is approximated as



$$b = b(x) = b_0 \left[1 - \left(\frac{2x}{a} \right)^2 \right]$$

Determine also the maximum stress assuming that the internal torsional moment is M_z.

Solution

$$J = \frac{1}{3} a \int_{-b/2}^{b/2} b(x)^3 dx = \frac{1}{3} a \int_{-b/2}^{b/2} \left\{ b_0 \left(1 - \left(\frac{2x}{a} \right) \right) \right\}^3 dy$$

$$= \frac{16}{105} a b_0^3$$

The torsional stiffness is then $\boxed{GJ = G \frac{16}{105} a b_0^3}$

The maximum stress is obtained by recalling that:

$$\sigma_{xz} = - \frac{2M_t}{J} z = - \frac{2M_t}{16 a b_0^3} \cdot 105 M_z \cdot z$$

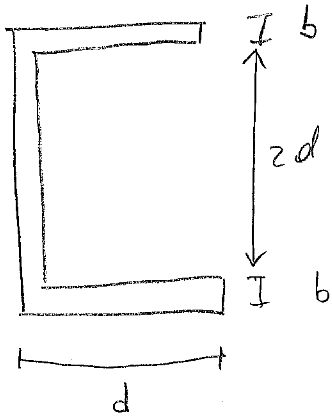
$$|\sigma_{xz\,max}| = \sigma_{xz}(0, \pm b/2)$$

The thickness b in $x=0$ is $b = b_0$, so:

$$\boxed{|\sigma_{xz\,max}| = + \frac{105}{16 a b_0^2} M_z}$$

Exercise

Evaluate the torsional stiffness of the beam in the figure



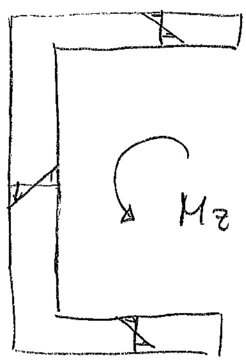
Evaluate then the maximum stress when a torsion moment M_z is applied

Solution

$$J = \frac{1}{3}\sum_{i=1}^{3} a_i b_i^3 = \frac{1}{3} d b^3 + \frac{1}{3} 2 d b^3 + \frac{1}{3} d b^3$$

$$= \frac{4}{3} d b^3$$

The internal stresses are directed as



$$\sigma_{sz} = -\frac{2M_z}{J} n$$

$$= -\frac{2M_z \, 3}{4 \, d b^3} n$$

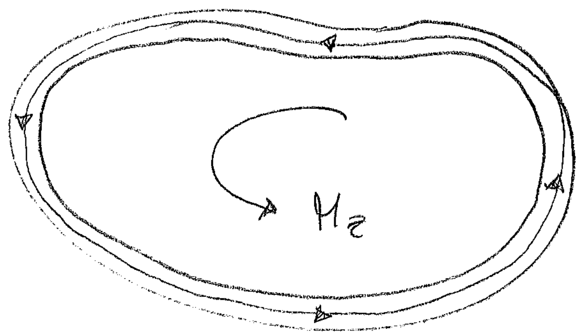
$$\left|\sigma_{sz\,max}\right| = \sigma_{sz}\Big|_{n=b/2}$$

$$\boxed{\left|\sigma_{sz\,max}\right| = \frac{M_z}{d b^2}\frac{3}{4}}$$

Closed thin-walled profiles

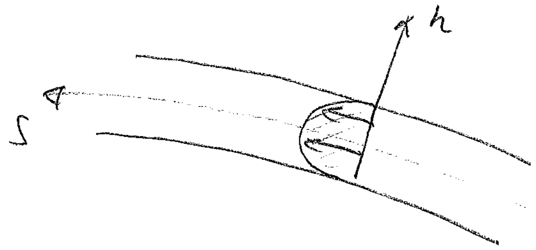
Whenever the torsional stiffness is relevant for design purposes - as it often happens for aeronautical beams -, the adoption of a closed-profile is clearly advantageous. Indeed, it was illustrated, at least from a qualitative point of view, that the arm of the internal shear stresses is much higher in comparison to open-sections.

It was found that the internal shear stresses are directed as in the figure

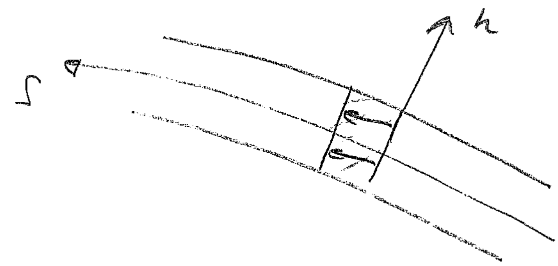


qualitative behaviour

For thin-walled sections, it can be reasonably assumed that the shear stress component σ_{sz} is constant along the thickness and parallel to the midline



exact



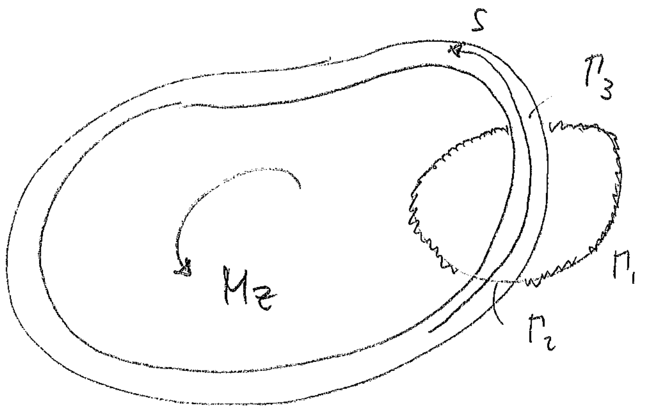
approximated

This assumption, which is more and more realistic as the the thickness gets smaller, simplifies the derivation of a closed-form solution.

Recall the equilibrium condition:

$$\operatorname{div} \underline{t} = 0 \quad \Rightarrow \quad \int_{A'} \operatorname{div} \underline{t}\, dA' = \oint_{\Gamma} \underline{t}\cdot\underline{n}\, d\Gamma = 0$$

$$\Rightarrow \quad \phi = 0$$

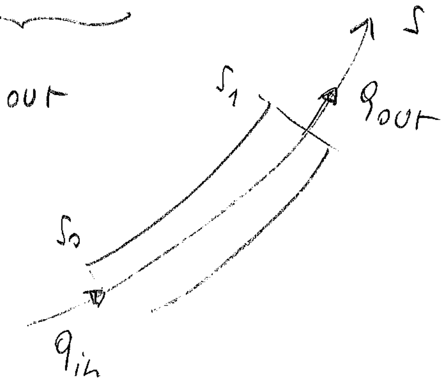


$$\oint_{\Gamma} \underline{t}\cdot\underline{n}\, d\Gamma = \oint_{\Gamma_1} \underline{t}\cdot\underline{n}\, d\Gamma_1 + \oint_{\Gamma_2} \underline{t}\cdot\underline{n}\, d\Gamma_2 + \oint_{\Gamma_3} \underline{t}\cdot\underline{n}\, d\Gamma_3$$

$$= \int_{t_2} \sigma_{sz}(s_0)\, dt - \int_{t_3} \sigma_{sz}(s_1)\, dt$$

$$= \underbrace{\sigma_{sz}(s_0)\, t_2}_{q_{in}} - \underbrace{\sigma_{sz}(s_1)\, t_3}_{q_{out}}$$

$$\Rightarrow \quad \boxed{q_{in} = q_{out}}$$

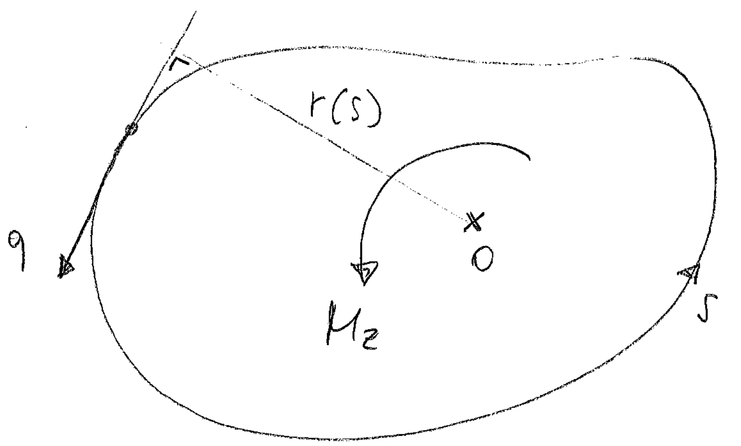


where

q: shear flow (integral of the stress component σ_{sz} along the thickness)

The relation between the internal shear flow and the torsional moment is obtained by imposing the equivalence with M_z

As far as the shear flow is assumed constant along the thickness direction, the section is represented by considering the <u>mid-line</u> only



- Take a generic pole O
- Consider the shear flow in a generic point of the section. q is tangent to the mid line

The torsional moment due to a portion ds is then:

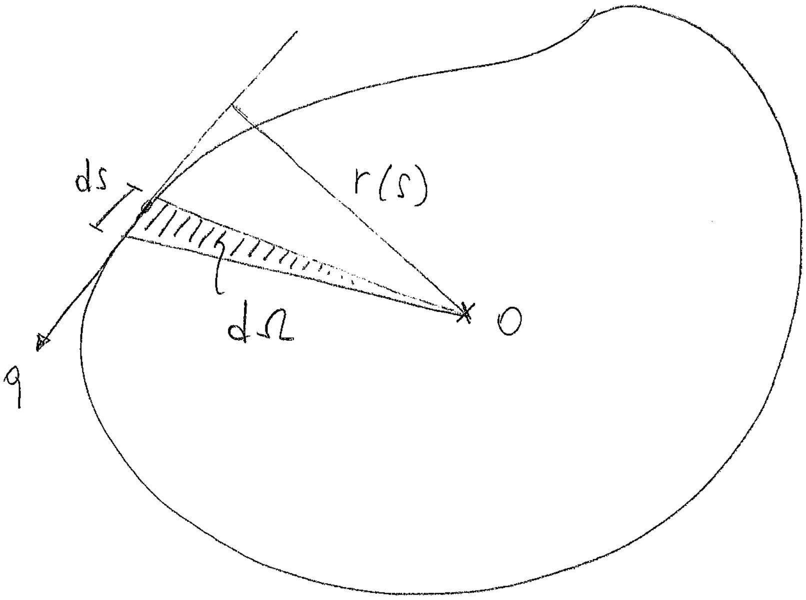
$$dM = q\,ds \cdot r(s) = q\,r(s)\,ds$$

r = arm with respect to O (it is a function of s!)

The total moment is then:

$$\oint_\Gamma dM = M = \oint_\Gamma q\,r(s)\,ds = q \oint_\Gamma r(s)\,ds \quad (\text{because } \phi = 0)$$

The integral $\oint_\Gamma r(s)\,ds$ can be easily evaluated by graphically inspecting it.



The area of the triangle $d\Omega$ is: $d\Omega = r(s)\, ds/2$

$$\Rightarrow \quad 2\,d\Omega = r(s)\,ds$$

$$\Rightarrow \quad \oint_\Gamma 2\,d\Omega = \oint_\Gamma r(s)\,ds$$

$$2\Omega = \oint_\Gamma r(s)\,ds$$

The previous expression is then:

$$M_z = q \oint_\Gamma r(s)\,ds \quad \Rightarrow \quad \boxed{M_z = 2q\Omega}$$

Bredt formula

The evaluation of the torsional stiffness is straightforward:

$$J = \frac{M_z^2}{\int_A (\tau_{xz}^2 + \tau_{yz}^2)\,dA} = \frac{M_z^2}{\int_A \tau_{sz}^2\,dA}$$

but $\tau_{sz} = \dfrac{q}{t} = \dfrac{M_z}{2\Omega t}$

and so:

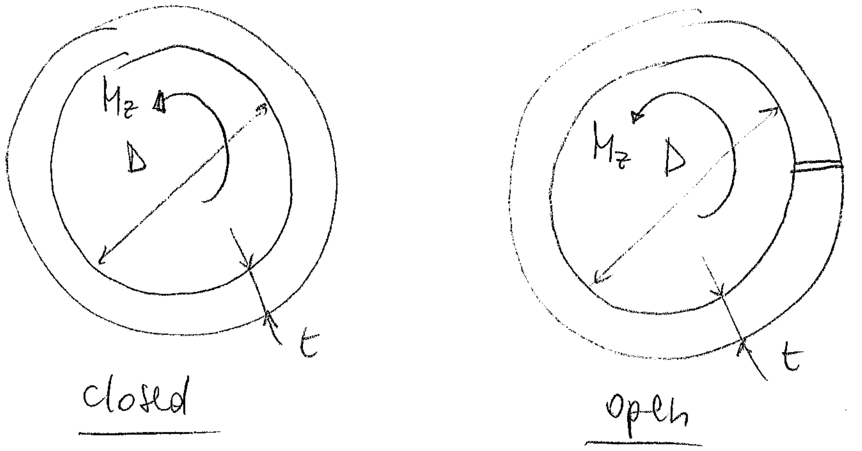
$$J = \frac{M_z^2}{\int_t \oint_\Gamma \frac{M_z^2}{4\Omega^2 t^2}\, d\Gamma\, dt} = \frac{4\Omega^2}{\oint_\Gamma \frac{1}{t(s)}\, d\Gamma}$$

and so:

$$\boxed{J = \frac{4\Omega^2}{\oint_\Gamma \frac{1}{t(s)}\, d\Gamma}}$$

Exercise

Consider two tubes with thickess t and diameter D. The first is open, the second is closed. Compare the torsional stiffness and the maximum stress for an internal twisting moment M_z.



$D = 16t$

Solution

Open tube: $J = \frac{1}{3} \pi D t^3$

$$\sigma_{sz_{max}} = \frac{2M_t}{J} \cdot \frac{t}{2} = \frac{2M_t \cdot 3}{\pi D t^3} \cdot \frac{t}{2}$$

$$= \frac{3}{\pi} \frac{M_t}{D t^2}$$

Closed tube: $$J = \frac{4\Omega^2}{\oint_\Gamma \frac{1}{t(s)}\, d\Gamma} = \frac{4\left(\pi D^2/4\right)^2}{\frac{1}{t} \cdot \pi D}$$

$$= \frac{4\pi^2 D^4}{16\, \pi D} t = \frac{\pi D^3 t}{4}$$

$$\sigma_{sz_{max}} = \frac{q}{t} \qquad \text{where} \qquad q = \frac{M_z}{2\Omega}$$

$$= \frac{M_z}{2\Omega t} = \frac{M_z}{2\pi \frac{D^2}{4} t} = \frac{2M_z}{\pi D^2 t}$$

Comparison:

$$\frac{J_{closed}}{J_{open}} = \frac{\pi D^3 t}{4} \cdot \frac{3}{\pi D t^3} = \frac{3D^2}{4t^2} = \underline{192}$$

$$\frac{\sigma_{open}}{\sigma_{closed}} = \frac{3}{\cancel{\pi}} \frac{\cancel{M_z}}{D t^2} \cdot \frac{\cancel{\pi} D^2 t}{2\cancel{M_z}} = \frac{3}{2}\frac{D}{t} = \underline{24}$$

The superiority of the closed section, both in terms of stiffness and strenght, is then clear