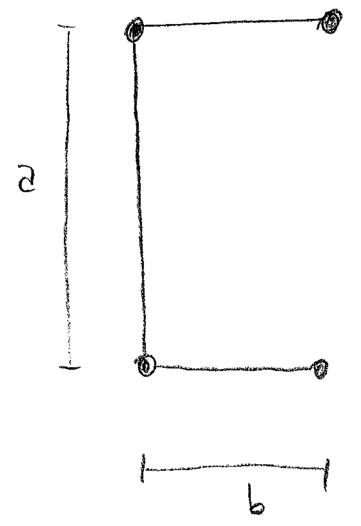
Evaluation of section properties - exercises -

Determine the stiffnesses $EA, EJ_{xx}, EJ_{yy}, GA_x^*, GA_y^*$ and GJ for the beam sections reported in the following exercises. (The semi-monoque scheme is used for solving the section)

Exercise 1



$a = 600$ mm

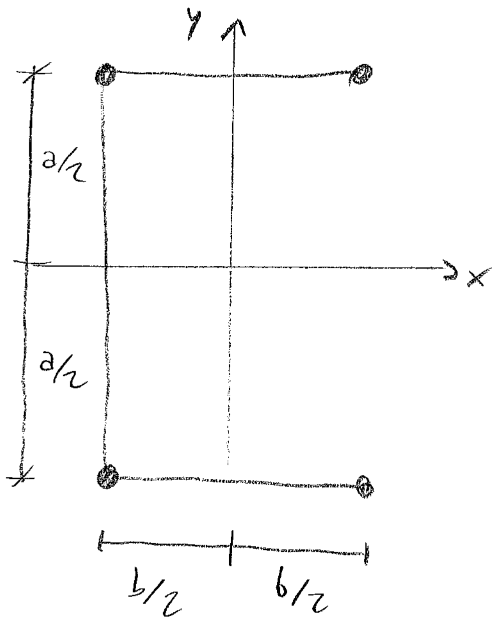
$b = 200$ mm

$A = 300$ mm^2

$t = 1.5$ mm

• **Solution**

The principal axes are available from the symmetry of the section



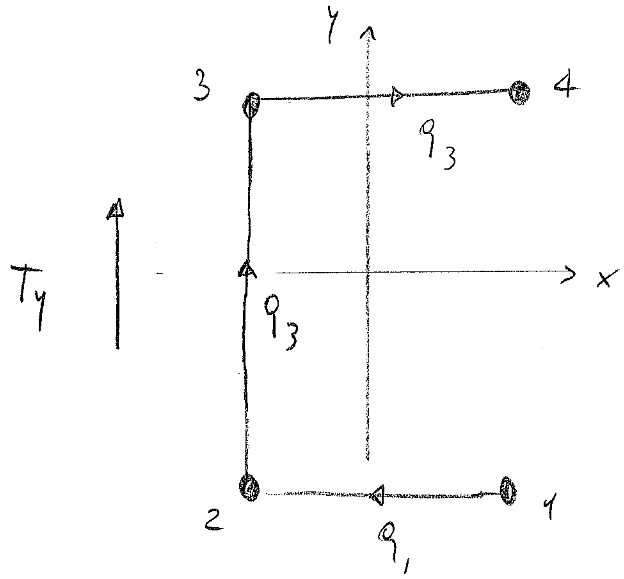
$$J_{xx} = 2A\left(\frac{a}{2}\right)^2 \cdot 2 = Aa^2$$

$$J_{yy} = Ab^2$$

$$\boxed{\begin{aligned} EJ_{xx} &= EAa^2 \\ EJ_{yy} &= EAb^2 \\ (EA) &= E\sum_{i=1}^{4} A_i = 4EA \end{aligned}}$$

Shear stiffness GA_y^*

The evaluation of GA_y^* requires a previous analysis of the shear flows for an arbitrary force T_y.
As far as the section is open, it is assumed that the shear force is applied on the shear center



$$S'_{x_1} = -A\,a/2$$
$$S'_{x_2} = -A\,a$$
$$S'_{x_3} = -A\,a/2$$

$$J_{xx} = A\,a^2$$

$$q_1 = -\frac{T_y S'_{x_1}}{J_{xx}} = +\frac{T}{2a}$$

$$q_2 = -\frac{T_y S'_{x_2}}{J_{xx}} = +\frac{T}{a}$$

$$q_3 = -\frac{T_y S'_{x_3}}{J_{xx}} = +\frac{T}{2a}$$

The shear stiffness GA_y^* is obtained by the energy equivalence

$$\frac{T^2}{GA_y^*} = \Sigma_i \frac{q_i^2 l_i}{G t_i} \Rightarrow A_y^* = \frac{T^2}{\Sigma_i \frac{q_i^2 l_i}{t_i}}$$

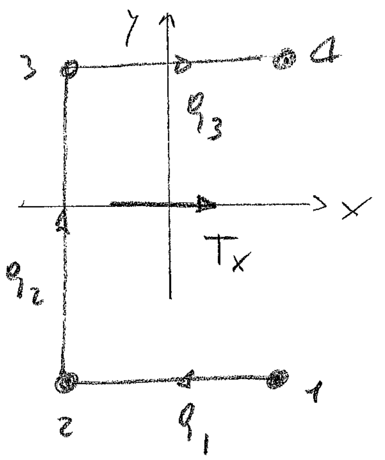
$$\sum_i \frac{q_i^2 l_i}{t_i} = \left[\left(\frac{T}{2a}\right)^2 b + \left(\frac{T}{a}\right)^2 \cdot a + \left(\frac{T}{2a}\right)^2 b\right] \frac{1}{t}$$

$$= \frac{T^2}{t}\left(\frac{b}{4a^2} + \frac{1}{a} + \frac{b}{4a^2}\right)$$

$$= \frac{T^2}{t}\left(\frac{2b+4a}{4a^2}\right)$$

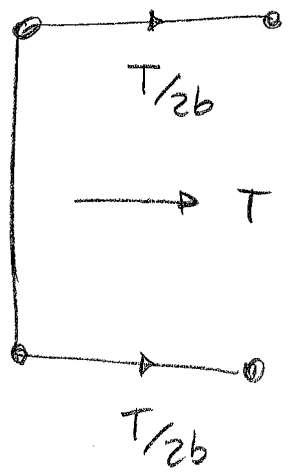
$$A_y^* = \frac{T^2}{\sum_i \frac{q_i^2 l_i}{t_i}} = \frac{4a^2 t}{2b+4a} \qquad \boxed{GA_y^* = G\,\frac{4a^2 t}{2b+4a}}$$

· <u>Shear stiffness GA_x^*</u>



T_x is applied on the shear center (the y position is available from the symmetry of the section)

The solution is readily found from the symmetry of the section with respect to the x-axis.



$$q_3 = q_1 = T/2b$$

The energy equivalence is:

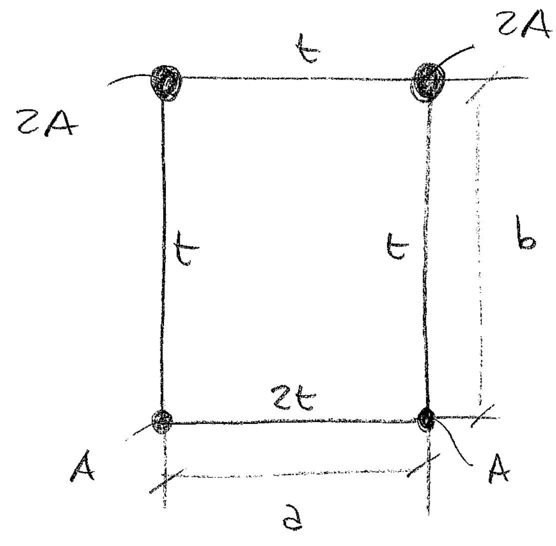
$$A_x^* = \frac{T^2}{\sum_i \frac{q_i^2 l_i}{t_i}} = \frac{t}{\frac{1}{4b} + \frac{1}{4b}} = 2bt$$

$$\boxed{GA_x^* = G\,2bt}$$

Torsional stiffness GJ

No torsional stiffness is associated with the semi-monocoque approximation of the section. Note that $GJ = 0$ is not an intrinsic property of an open section, but is associated with the model considered for describing the section behaviour.

Exercise



$a = 150$ mm

$b = 200$ mm

$t = 1$ mm

$A = 200$ mm^2

$E = 72$ GPa

$\nu = 0.3$

<u>Note</u>: A includes the contribution of the panels

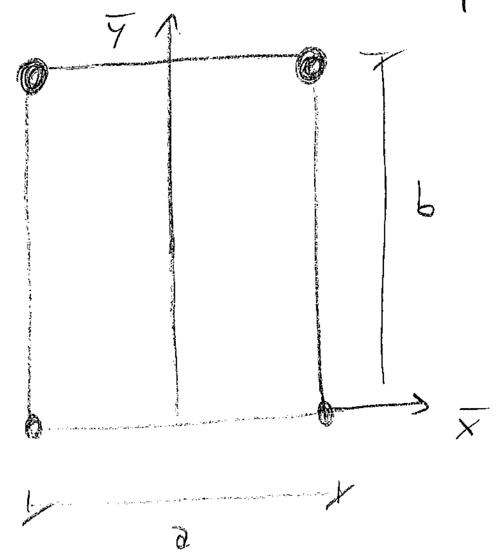
- ## Solution

Axial stiffness

$$(EA) = E \sum_i A_i = 6EA = 86.40 \cdot 10^6 \text{ N}$$

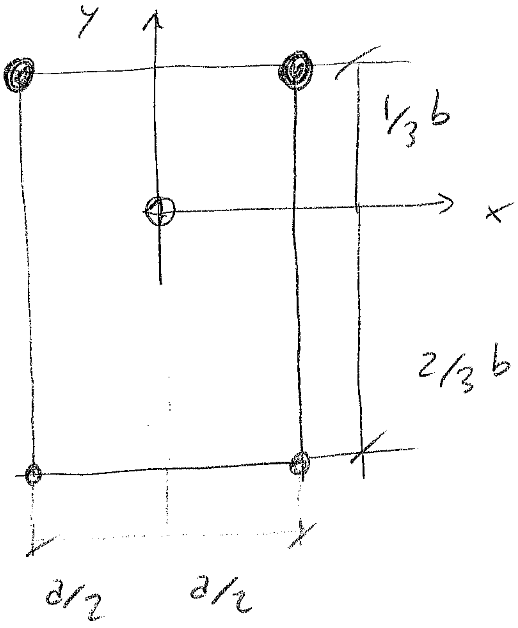
Bending stiffness

Evaluate first the principal axes



$x_{CG} = 0$ (from the symmetry)

$$y_{CG} = \frac{4Ab^2}{6A} = \frac{2}{3} b = 133.33 \text{ mm}$$



$$J_{xx} = 4A\left(\frac{1}{3}b\right)^2 + 2A\left(\frac{2}{3}b\right)^2$$

$$= \left(\frac{4}{9} + \frac{8}{9}\right) Ab^2 = \frac{4}{3} Ab^2$$

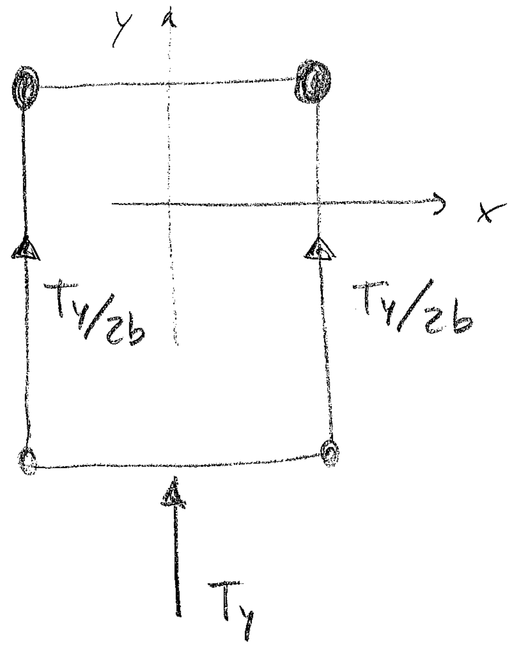
$$J_{yy} = 6A\left(\frac{a}{2}\right)^2 = \frac{3}{2} Aa^2$$

$$EJ_{xx} = \frac{4}{3} EAb^2 = 7.68 \cdot 10^{11} \ Nmm^2$$

$$EJ_{yy} = \frac{3}{2} EAa^2 = 4.86 \ 10^{11} \ Nmm^2$$

Shear stiffness GA_y^*

The x position of the shear center is available from the symmetry of the section.



Even the shear flows are readily available from the symmetry of the section.

The area A_y^* is then:

$$A_y^* = \frac{T_y^2}{\sum_i \frac{q_i^2 l_i}{t_i}} = \frac{T_y^2}{T_y^2\left(\frac{1}{4b^2}\cdot b + \frac{1}{4b^2} b\right)\frac{1}{t}} = 2bt$$

Note that $A_y^* = 2bt$ is, in fact, the total area of the vertical webs. As far as the shear flow is constant along the webs, the portion of area contributing to the shear resistance of the section is equal to the areas of the webs, i.e.

$$\left(A_y^* = \tilde{\chi} A_{webs} \quad \text{and} \quad \chi = 1\right)$$

To evaluate the shear stiffness, the shear factor is introduced with respect to the total area of the section.

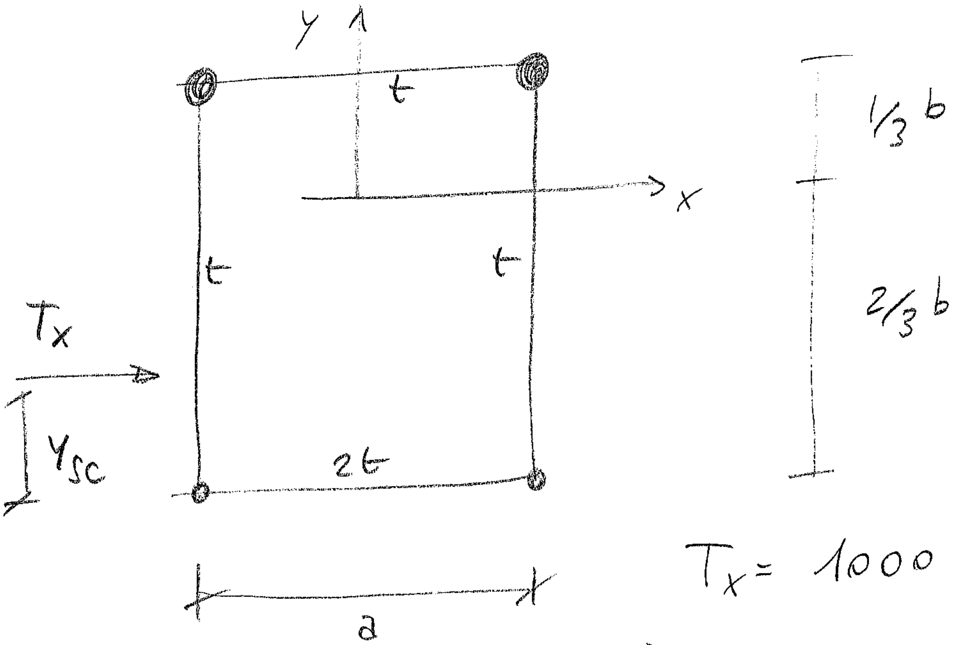
$$\chi = \frac{A_y^*}{A_{TOT}} = \frac{2bt}{6A} = 0.33$$

Note that χ is $<< 1$, meaning that a small fraction of the total area is contributing to the shear stiffness of the section $\Rightarrow$ the section is characterized by a non-negligible shear deformability $\Rightarrow$ the shear deformability has to be taken into account when applying the PCVW, Menabrea's th, PVW, ..

$$GA^* = \chi G A_{TOT} = 1.11 \cdot 10^7 \text{ N}$$

Shear stiffness GA_x^*

The position of the shear center along the y-axis is not available from the symmetry of the section.



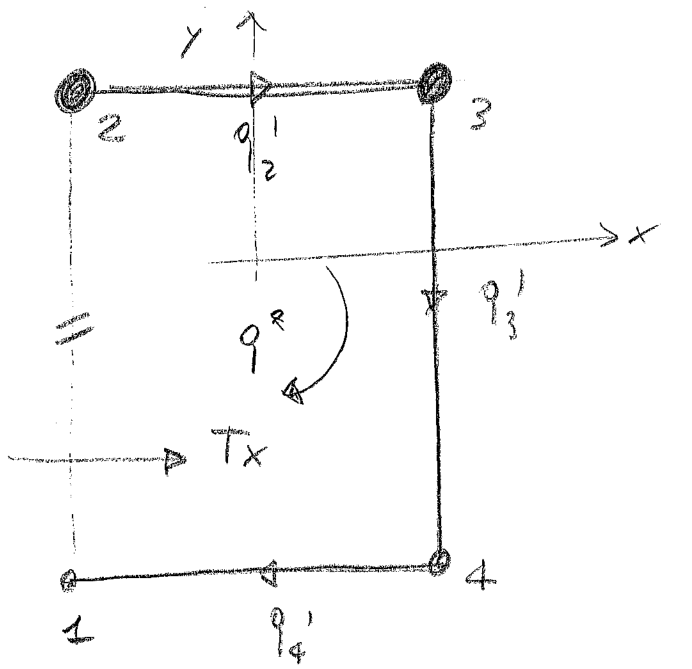
$T_x = 1000$ N (arbitrary)

The shear center (y-position) is obtained by applying:

a. 3 shear flow equations

b. 1 equivalence with internal moment

c. 1 compatibility condition ($\theta_i' = 0$)



$$S_{y_2}' = -Aa$$

$$S_{y_3}' = 0$$

$$S_{y_4}' = A\,a/2$$

$$q_2' = -T_x \frac{S_{y2}'}{J_{yy}} = 4.44 \text{ N/mm}$$

$$q_3' = 0$$

$$q_4' = -T_x \frac{S_{y4}'}{J_{yy}} = -2.22 \text{ N/mm}$$

Equivalence with internal moment (with respect to 1)

$$2\Omega_2 q_2' + \cancel{2\Omega_3 q_3'} + 2\Omega_{c_1} q^* = +T_x Y_{sc}$$

with:

$$\Omega_2 = \Omega_3 = ab/2$$

$$\Omega_{c_1} = ab$$

$$\boxed{q^* - 0.0167\, Y_{sc} = -2.2222}$$

Compatibility

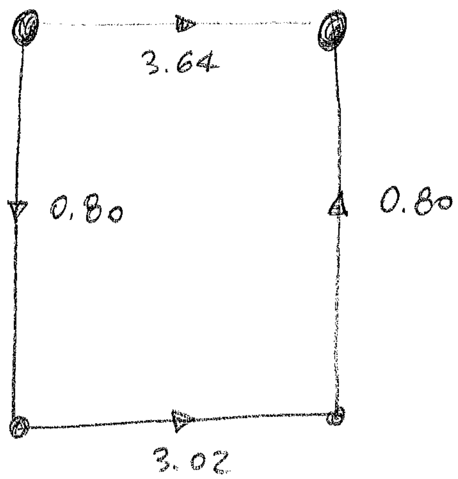
$$\theta' = \frac{1}{2\Omega_{c_1} G t}\left(q_2' a + q_4' \frac{a}{2} + q^* \left(2b + \frac{3}{2}a\right)\right) = 0$$

$$\boxed{q^* = -0.80}$$

The system of equations is then:

$$\begin{cases} q^* - 0.0167\, Y_{sc} = -2.2222 \\ q^* = -0.80 \end{cases} \Rightarrow \begin{array}{l} Y_{sc} = 85.16 \text{ mm} \\ q^* = -0.80 \text{ N/mm} \end{array}$$

The shear flows are then:



$q_1 = -0.80$ N/mm

$q_2 = 3.64$ N/mm

$q_3 = -0.80$ N/mm

$q_4 = -3.02$ N/mm

The area A_x^* is then:

$$A_x^* = \frac{T_x^2}{\sum_i \frac{q_i^2 \ell_i}{t_i}} = \frac{T_x^2}{q_1^2 \frac{b}{t} + q_3^2 \frac{b}{t} + q_2^2 \frac{a}{t} + q_4^2 \frac{a}{2t}} = 341.59 \text{ mm}^2$$

Corresponding to a shear factor

$$\chi = \frac{A_x^*}{A_{TOT}} = 0.28$$

The shear stiffness is

$$GA^* = \chi \, GA_{TOT} = 9.46 \cdot 10^6 \text{ N}$$

- <u>Torsional stiffness GJ</u>

The torsional constant J is obtained from

$$J = \frac{4\Omega^2}{\oint_\Gamma \frac{1}{t(s)} d\Gamma}$$

Where:

$$\oint_{\Gamma} \frac{1}{t(s)} = \frac{1}{t}(2b+a) + \frac{1}{2t}a = \frac{1}{t}\left(2b+\frac{3}{2}a\right) = \frac{1}{t}\left(\frac{4b+3a}{2}\right)$$

$$\Omega^2 = (ab)^2$$

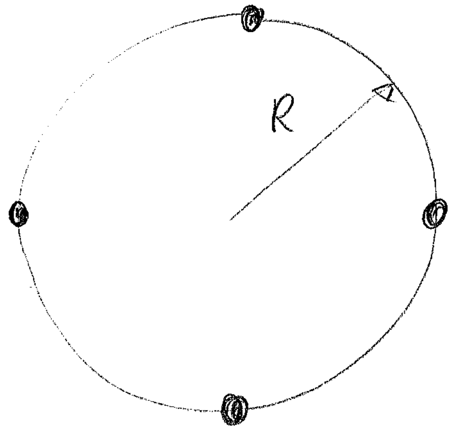
So:

$$J = \frac{8a^2b^2t}{4b+3a} = 5.76 \cdot 10^6 \ mm^4$$

and

$$GJ = 1.60 \cdot 10^{11} \ N mm^2$$

Exercise



$R = 100$ mm

$A = 500$ mm^2

$t = 1.2$ mm

$E = 72$ GPa

$\nu = 0.3$

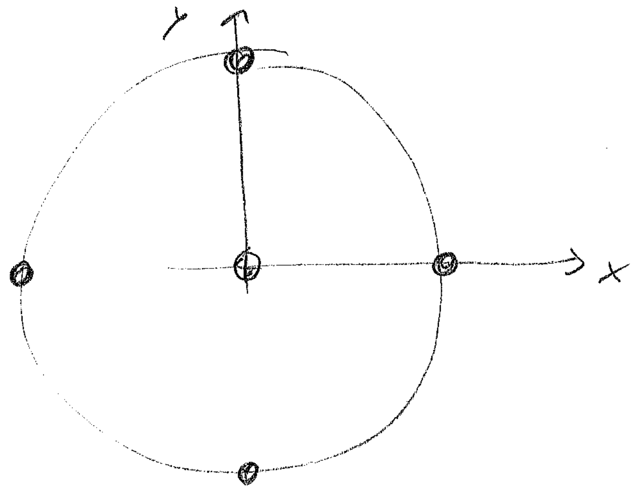
- Solution

Axial stiffness

$$(EA) = E \Sigma_i A_i = 4EA = 1.44 \cdot 10^8 \text{ N}$$

Bending stiffness

Thanks to the double symmetry of the section, the principal axes are readily available



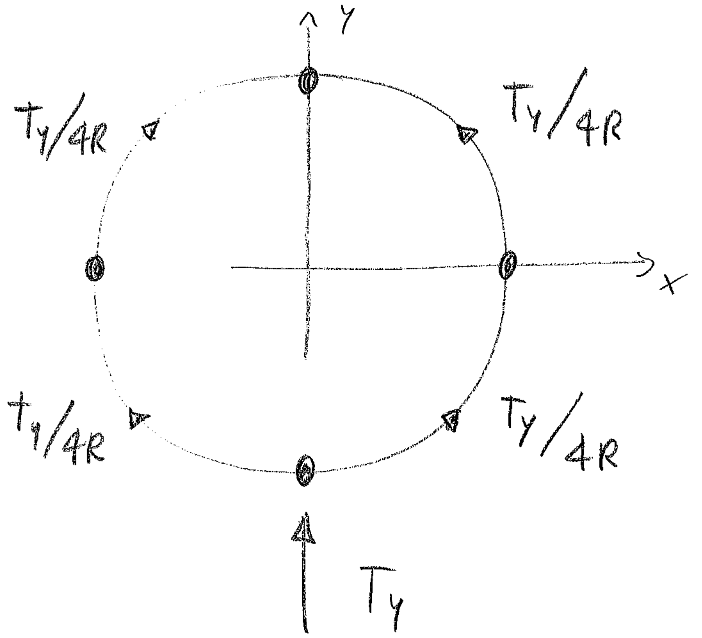
$$J_{xx} = 2AR^2$$

$$J_{yy} = J_{xx}$$

$$EJ_{xx} = EJ_{yy} = 1.0 \cdot 10^7 \text{ N mm}^2$$

Shear stiffness GA_x^* and GA_y^*

The symmetry of the section is such that $GA_x^* = GA_y^*$.
The position of the shear center is available from the double symmetry of the section, and coincides with the centroid.



Even the shear flows are available by exploiting the symmetry.

$$\left(q_i = \frac{T_y}{4R} \right)$$

The area A_y^* is then

$$A_y^* = \frac{T_y^2}{\sum_i \frac{q_i^2 l_i}{t_i}} = \frac{T_y^2}{\frac{T_y^2\, 2\pi R}{16 R^2 t}} = \frac{8Rt}{\pi} = 305.58\ mm^2$$

The shear factor is then

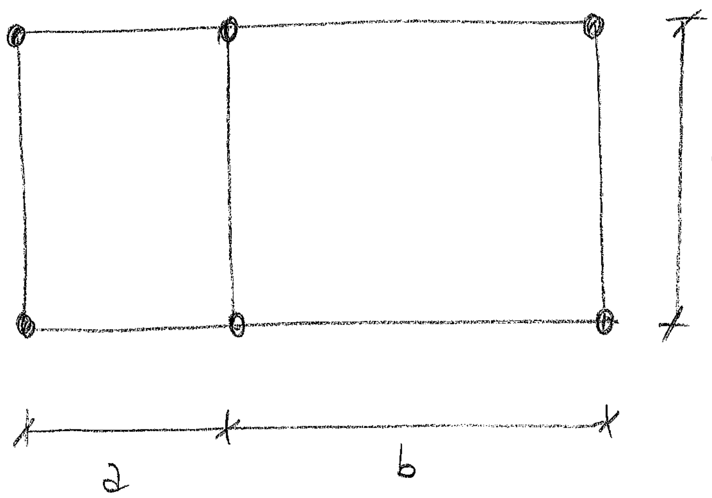
$$\chi_y = \frac{A_y^*}{A_{TOT}} = 0.15$$

and the shear stiffness

$$GA_y^* = \chi_y G A_{TOT} = 8.46 \cdot 10^6\ N$$

$$GA_x^* = GA_y^*$$

Exercise



$A = 500\ mm^2$

$a = 100\ mm$

$b = 300\ mm$

$c = 200\ mm$

$t = 1\ mm$

$E = 72\ GPa$

$\nu = 0.3$

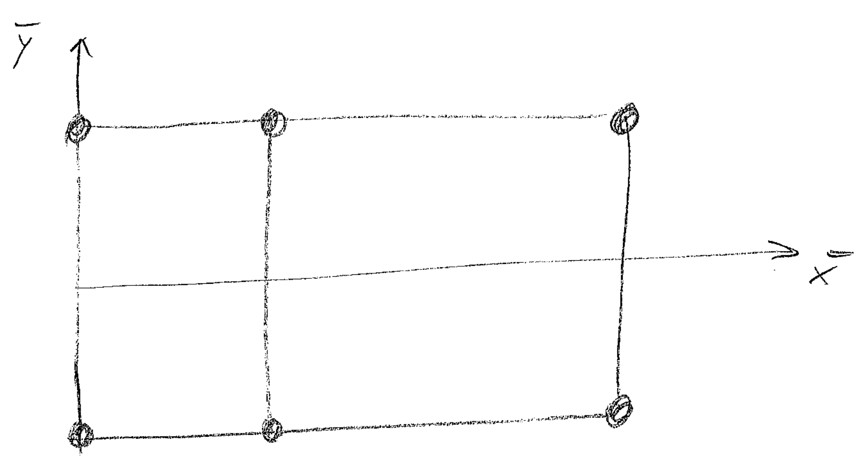
- Solution

Axial stiffness

$$(EA) = E \sum_i A_i = 6EA$$

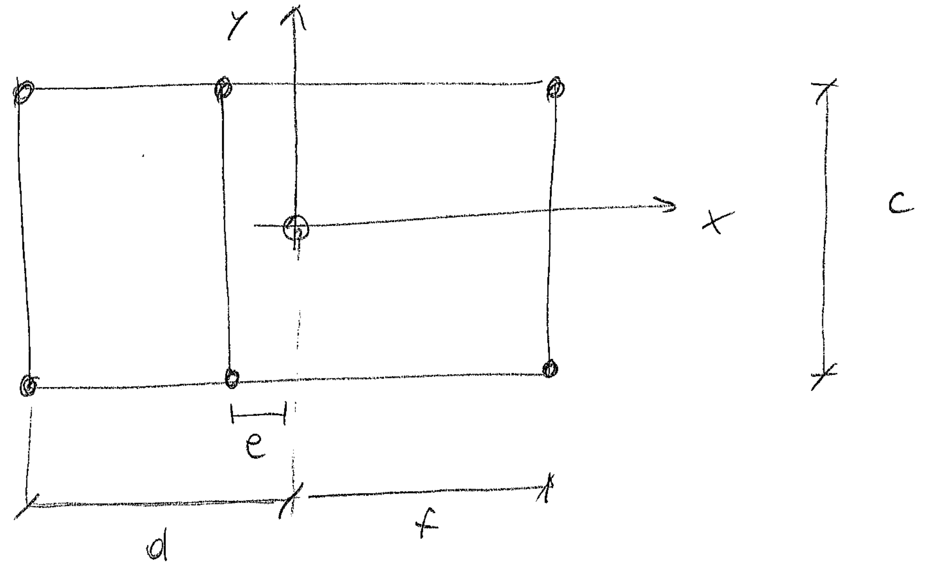
Bending stiffness

The principal axes are obtained as:



$$x_{CG} = \frac{2A(a+b) + 2A(a)}{6A} = \frac{2a+b}{3} = 166.67\ mm$$

$$y_{CG} = 0$$



$d = x_{CG} = 166.67$ mm $\qquad f = a + b - d = 233.33$ mm

$e = x_{CG} - a = 66.67$ mm

$$J_{xx} = 6A\left(\frac{c}{2}\right)^2 = \frac{3}{2}Ac^2$$

$$J_{yy} = 2Af^2 + 2Ae^2 + 2Ad^2$$

$$EJ_{xx} = \frac{3}{2}EAc^2 = 2.16 \cdot 10^{12} \text{ Nmm}^2$$

$$EJ_{yy} = 2EA\left(f^2 + e^2 + d^2\right) = 6.24 \cdot 10^{12} \text{ Nmm}^2$$

Shear stiffness GA_y^*

The position of the shear center has to be evaluated.

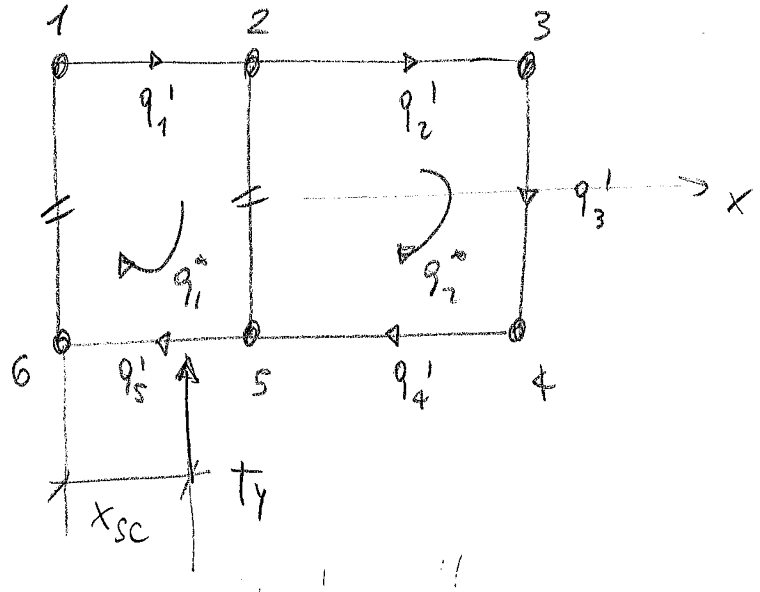
To this aim, the equations to be applied are

a. 5 shear flow equations

b. 1 equivalence with internal moment

c. 1 compatibility condition $(\theta'_{c1} = \theta'_{c2})$

d. 1 compatibility condition $(\theta'_{c_i} = 0)$



$T_y = 1000$ N (arbitrary)

$S_{x_1}' = A\,c/2$

$S_{x_2}' = A\,c$

$S_{x_3}' = \frac{3}{2} A\,c$

$S_{x_4}' = A\,c$

$S_{x_5}' = A\,c/2$

$$q_i' = -T_y \frac{S_{x_i}'}{J_{xx}} \qquad i = 1, \dots 5$$

$q_1' = -1.67$ N/mm

$q_2' = -3.33$ N/mm

$q_3' = -5.00$ N/mm

$q_4' = -3.33$ N/mm

$q_5' = -1.67$ N/mm

<u>Equivalence with internal moment (with respect to 1)</u>

$$2\Omega_3 q_3' + 2\Omega_4 q_4' + 2\Omega_5 q_5' + 2\Omega_{c_1} q_1^* + 2\Omega_{c_2} q_2^* + T_y x_{sc} = 0$$

with

$\Omega_3 = (a+b)\,c/2$ $\qquad \Omega_5 = a\,c/2$ $\qquad \Omega_{c_2} = b\,c$

$\Omega_4 = b\,c/2$ $\qquad \Omega_{c_1} = a\,c$

$$q_1^* + \frac{\Omega_{c_2}}{\Omega_{c_1}} q_2^* + \frac{T_y}{2\Omega_{c_1}} x_{sc} = -\frac{1}{\Omega_{c_1}} \left(\Omega_3 q_3' + \Omega_4 q_4' + \Omega_5 q_5' \right)$$

$$\boxed{q_1^* + 3 q_2^* + 0.025\, x_{sc} = 15.8333}$$

Compatibility

$$\theta_1' = \frac{1}{2G\Omega_{c_1} t} \left(q_1' a + q_5' a + q_1^* (2a + 2c) - q_2^* c \right)$$

$$\theta_2' = \frac{1}{2G\Omega_{c_2} t} \left(q_2' b + q_4' b + q_2^* (2b + 2c) - q_1^* c + q_3' c \right)$$

Imposing $\theta_1' = \theta_2'$.

$$q_1' a + q_5' a + (2a + 2c) q_1^* - q_2^* c = \frac{\Omega_{c_1}}{\Omega_{c_2}} \left(q_2' b + q_4' b + \right.$$
$$\left. + q_2^* (2b + 2c) - q_1^* c + q_3' c \right)$$

which leads to:

$$\left(2a + 2c + \frac{ac}{b} \right) q_1^* - \left(2a + \frac{2ac}{b} + c \right) q_2^* = a \left(q_2' + q_4' - q_1' - q_5' \right) + \frac{ac}{b} q_3'$$

or

$$\boxed{q_1^* - 0.8\, q_2^* = -1}$$

The second compatibility requirement is:

$\theta_1' = 0$, so:

$$(2a+2c)\, q_1^* - c\, q_2^* = -q_1' a - q_5' a \quad , \quad \text{or}$$

$$\boxed{q_1^* - 0.3333 q_2^* = 0.5556}$$

Solving system:

$$\begin{cases} q_1^* + 3 q_2^* + 0.025\, x_{sc} = 15.8333 \\ q_1^* - 0.8\, q_2^* = -1 \\ q_1^* - 0.3333\, q_2^* = 0.5556 \end{cases}$$

which leads to

$q_1^* = 1.6666$ N/mm

$q_2^* = 3.3332$ N/mm

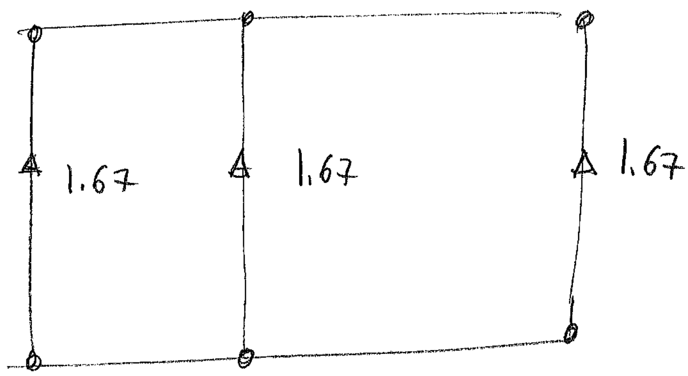
$x_{sc} = 166.68$ mm

and so:

$q_1 = q_2 = q_4 = q_5 = 0$

$q_3 = -1.67$ N/mm

$q_6 = q_7 = 1.67$ N/mm



The area A_y^* is then:

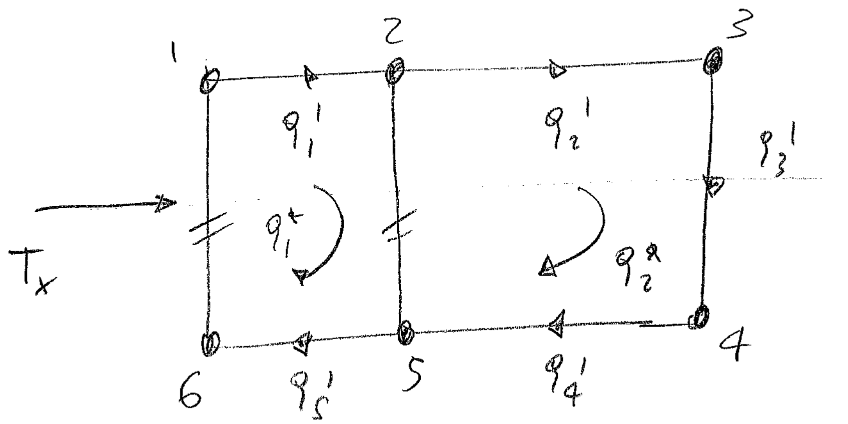
$$A_y^* = \frac{T_y^2}{\sum_i \frac{q_i^2 l_i}{t_i}} = 600 \ mm^2$$

$$\chi_y = \frac{A_y^*}{A_{TOT}} = 0.2$$

$$GA_y^* = \chi_y \, GA_{TOT} = 1.66 \cdot 10^7 \ N$$

Shear stiffness GA_x^*

The y-position of the shear center is available from the symmetry of the section



$T_x = 1000 \ N$
(arbitrary)

The solving equations are

a. 5 shear flow equations
b. 1 equivalence with internal moment
c. 1 compatibility condition

$$S'_{y_1} = -Ad \qquad S'_{y_4} = Af + S'_{y_3}$$

$$S'_{y_2} = -Ae + S'_{y_1} \qquad S'_{y_5} = -Ae + S'_{y_4}$$

$$S'_{y_3} = Af + S'_{y_2}$$

and

$$q'_i = -T_x \frac{S'_{y_i}}{J_{yy}} \qquad \text{with} \quad i = 1, .. 5$$

$$q'_1 = 0.96 \ N/mm \qquad q'_4 = -1.35 \ N/mm$$

$$q'_2 = 1.35 \ N/mm \qquad q'_5 = -0.96 \ N/mm$$

$$q'_3 = 0$$

<u>Equivalence with internal moment</u> (with respect to 1)

Obtained by modifying the previous one:

$$q_1^* + 3q_2^* = -\frac{1}{\Omega_{c_1}}\left(\Omega_3 q'_3 + \Omega_4 q'_4 + \Omega_5 q'_5\right) - \frac{T_x}{2\Omega_{c_1}} \frac{c}{2}$$

$$q_1^* + 3q_2^* = 0$$

<u>Compatibility</u>

$$q_1^* - 0.8 q_2^* = \left[a\left(q'_2 + q'_4 - q'_1 - q'_5\right) + \frac{ac}{b} q'_3\right] \frac{1}{2a + 2c + \frac{ac}{b}}$$

$$q_1^* - 0.8 q_2^* = 0$$

Solution:

$q_1^* = q_2^* = 0$ and so:

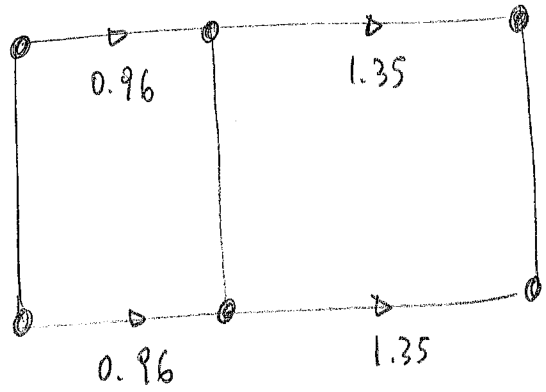
$q_3 = q_6 = q_7 = 0$

$q_1 = 0.96$

$q_2 = 1.35$

$q_4 = -1.35$

$q_5 = -0.96$



The area A_x^* is then

$$A_x^* = \frac{T_x^2}{\sum_i \frac{q_i^2 l_i}{t_i}} = 786.04 \text{ mm}^2$$

The shear factor χ_x is.

$$\chi_x = \frac{A_x^*}{A_{TOT}} = 0.26$$

and the shear stiffness is:

$$GA_x^* = \chi_x GA_{TOT} = 2.17 \cdot 10^7 \text{ N}$$

Torsional stiffness

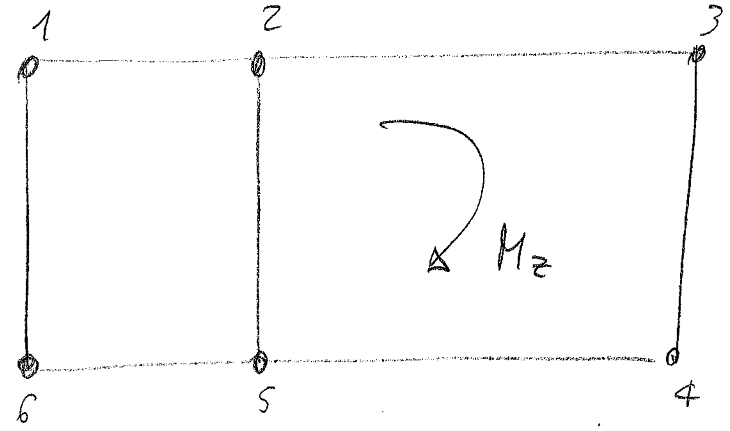
The torsional stiffness cannot be evaluated using the formula $J = 4\Omega^2 / \oint_\Gamma \frac{1}{t(s)} d\Gamma$, which holds for sections characterized by one single cell.

It is then necessary to recall the definition of torsional constant from:

$$M_z = GJ\theta' \Rightarrow J = M_z / G\theta'$$

where θ' is the torsion due to an arbitrary torsion M_z.

Consider then:



and apply:

a. (5 shear flow equations)

b. 1 equivalence with internal moment

c. 1 compatibility condition ($\theta_1' = \theta_i'$)

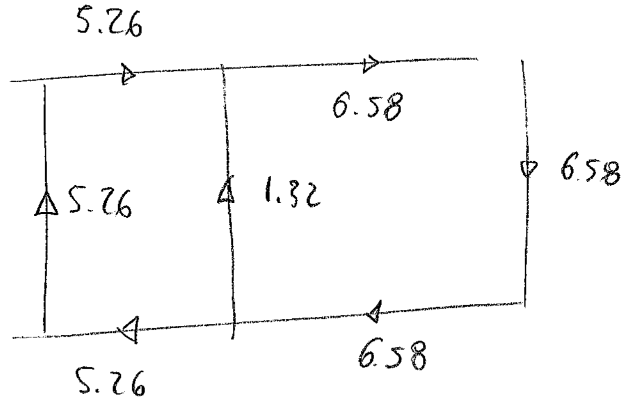
Most of the equations are available from the previous results

The shear flow equations simply state that $q_i' = 0$

The two other equations are:

$$\begin{cases} q_1^* + 3q_2^* = \dfrac{M_z}{2\Omega_{c_1}} \\ q_1^* - 0.8\, q_2^* = 0 \end{cases} \Rightarrow \quad \begin{aligned} q_1^* &= 5.26 \\ q_2^* &= 6.58 \end{aligned}$$

and so:



$q_1 = q_5 = q_6 = 5.26$ N/mm

$q_2 = q_3 = q_4 = 6.58$ N/mm

$q_7 = 1.32$ N/mm

Recalling that

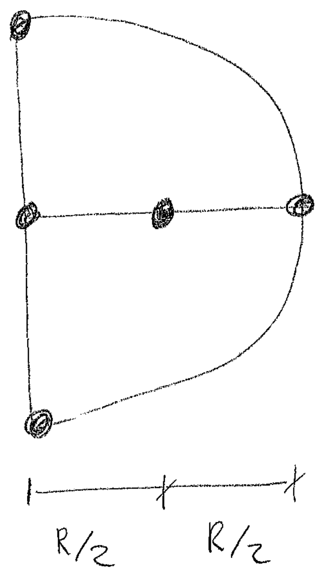
$$\theta_1' = \frac{1}{2G\Omega_{c_1} t}\left(q_1 \cdot 2a + q_6 \cdot c - q_7 \cdot c\right) = 1.663 \cdot 10^{-6} \text{ rad}$$

$$J = M_z / G\theta' = 2.17 \cdot 10^{7} \text{ mm}^4$$

and finally:

$$GJ = 6.01 \cdot 10^{11} \text{ N} \cdot \text{mm}^2$$

Exercise



$R = 100$ mm

$A = 500$ mm

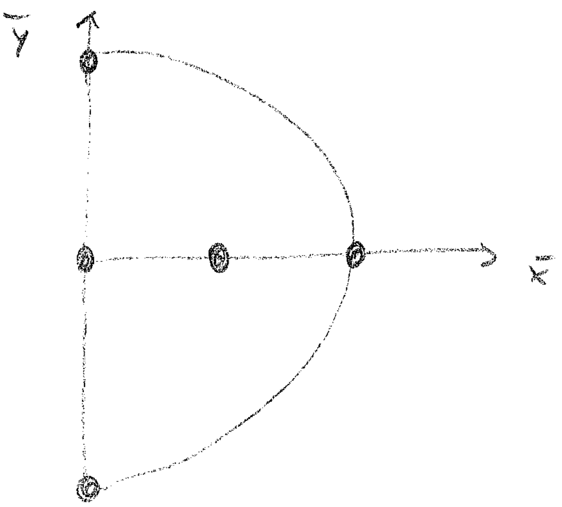
$t = 1$ mm

- ## Solution

Axial stiffness

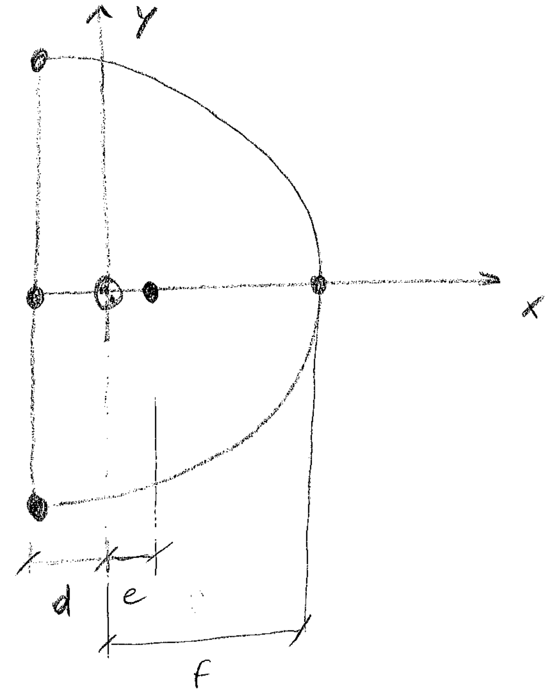
$$(EA) = E\sum_i A_i = 5EA = 1.8 \cdot 10^8 \text{ N}$$

Bending stiffness



$$y_{CG} = 0$$

$$x_{CG} = \frac{AR + AR/2}{5A} = \frac{3}{10}R = 30 \text{ mm}$$



$d = \frac{3}{10} R = 30$ mm

$e = R/2 - d = 20$ mm

$f = R/2 + e = 70$ mm

$J_{xx} = 2AR^2$

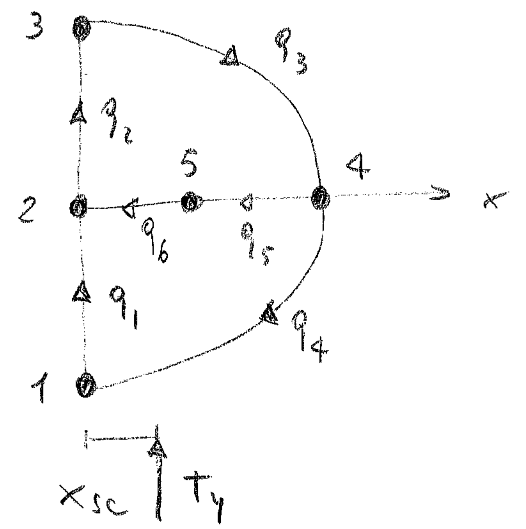
$J_{yy} = 3Ad^2 + Ae^2 + Af^2 =$

$EJ_{xx} = 7.20 \cdot 10^{11}$ N mm^2

$EJ_{yy} = 2.88 \cdot 10^{11}$ N mm^2

Shear stiffness GA_y^*

The x-position of the shear center is not know a priori, and has to be determined.



$T_y = 1000$ N
(arbitrary)

The results are obtained by applying

a. 4 shear flow equations

b. 1 equivelence with internal moment

c. 1 compatibility condition $(\theta_1' = \theta_2')$

d. 1 " " $(\theta_1' = 0)$

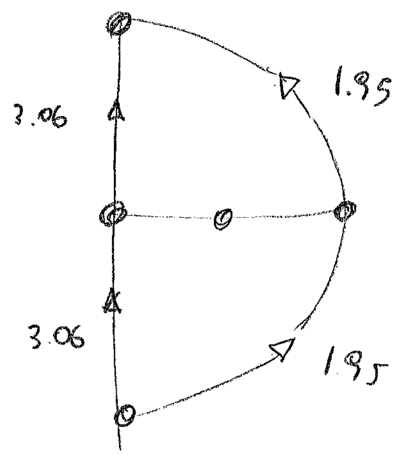
The results are (verify as exercise)

$q_1 = q_2 = 3.06$ N/mm

$q_3 = q_4 = -1.95$ N/mm

$q_5 = q_6 = 0$

$x_{sc} = 61.10$ mm



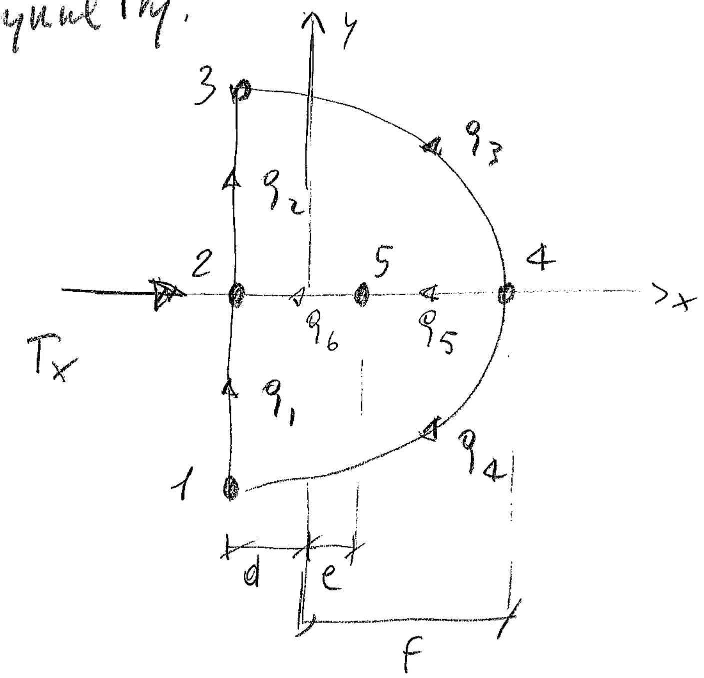
The area A_y^* is obtained as:

$$A_y^* = \frac{T_Y^2}{\sum_i \frac{q_i^2 l_i}{t_i}} = 326 \text{ mm}^2$$

$\chi_y = 0.13$, and $GA_y^* = \chi GA_{tot} = 9.03 \cdot 10^6$ N

Shear stiffness GA_x^*

The y position of the shear center is known from the symmetry.



$T_x = 1000$ N
(arbitrary)

The shear flows are obtained as:

$q_1 = 0.74$ N/mm

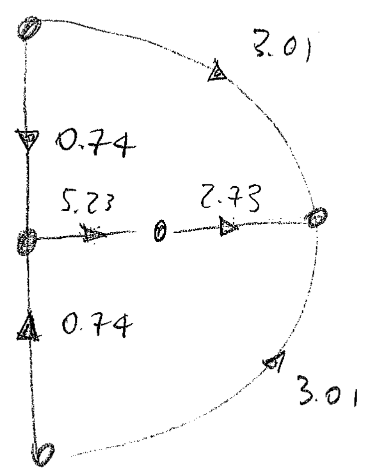
$q_2 = -0.74$ N/mm

$q_3 = 3.01$ N/mm

$q_4 = -3.01$ N/mm

$q_5 = -2.73$ N/mm

$q_6 = -5.23$ N/mm



Accordingly, the area A_x^* reads:

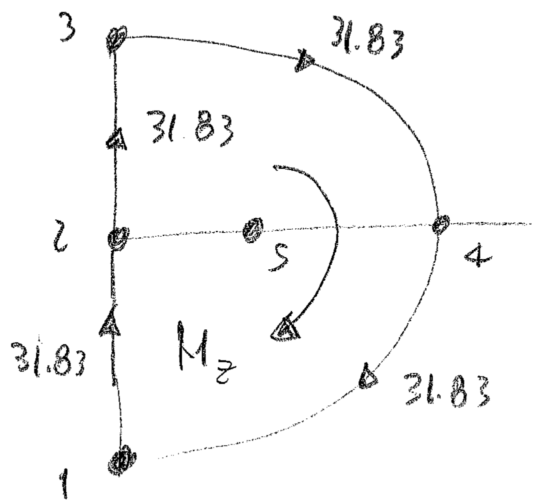
$$A_x^* = \frac{T_x^2}{\sum_i \frac{q_i^2 l_i}{t_i}} = 213 \text{ mm}^2 \quad \text{and } \chi_x = \frac{A_x^*}{A_{TOT}} = 0.09$$

The shear stiffness is then $GA_x^* = 5.90 \cdot 10^6$ N

Torsional stiffness GJ

The closed form expression $J = \dfrac{4\Omega^2}{\oint_\Gamma \frac{1}{t(s)} d\Gamma}$ cannot be used, as the section is composed of two cells.

To this aim, an arbitrary torsional moment is considered:



with:

$M_z = 10^6$ N·mm

(arbitrary)

$\theta_1' = \theta_2' = 1.88 \cdot 10^{-5}$ rad

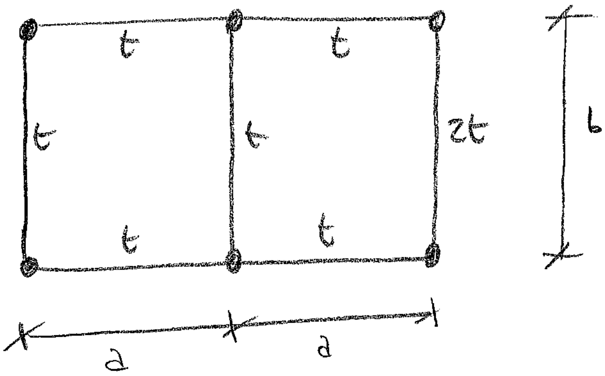
and

$$J = \frac{M_z}{G\theta_1'} = 1.92 \cdot 10^6 \text{ mm}^4$$

and:

$GJ = 5.32 \cdot 10^{10}$ N mm^2

Exercise



$A = 400\ mm^2$

$a = 110\ mm$

$b = 70\ mm$

$t = 1.3\ mm$

$E = 72 GPa$

$\nu = 0.3$

Note: A denotes the lumped area (the contribution of the panels is already included)

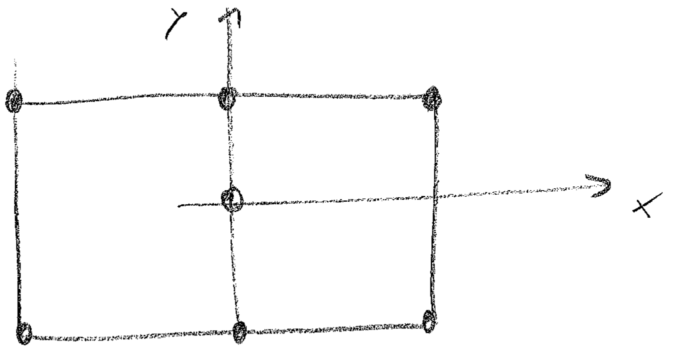
- Solution

Axial stiffness EA

$$(EA) = 6EA = 1.73 \cdot 10^8\ N$$

Bending stiffness EJ_{xx} and EJ_{yy}

The principal axes are available from the symmetry:



$$J_{xx} = 6A\left(\frac{b}{2}\right)^2 = \frac{3}{2}Ab^2$$

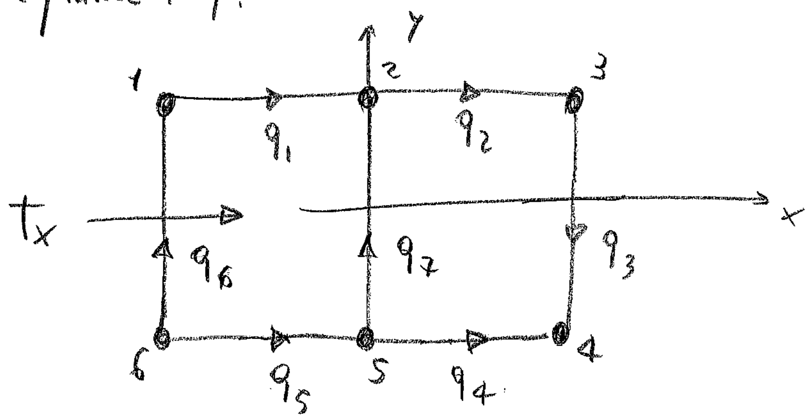
$$J_{yy} = 4Aa^2$$

$$EJ_{xx} = 2.12 \cdot 10^{11}\ N\,mm^2$$

$$EJ_{yy} = 1.40 \cdot 10^{12}\ N\,mm^2$$

Shear stiffness GA_x^*

The y-position of the shear center is available from the symmetry.



$T_x = 1000$ N (arbitrary)

The solution can be found easily by noting that:

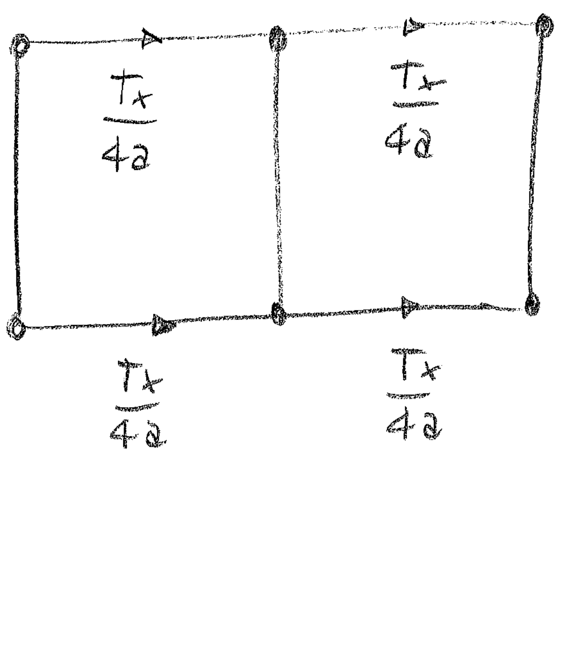
1. a symmetry with respect to x exists, so:

   $q_3 = q_6 = 0$

   $q_1 = q_5$ and $q_4 = q_2$

2. the stringers 2 and 5 are located in correspondence of the neutral axis $\Rightarrow$ $q_1 = q_2$ and $q_4 = q_5$

It follows that



The shear flows are then constant along the y-direction.
It follows that

$$A_x^* = A_{webs} = 4at$$

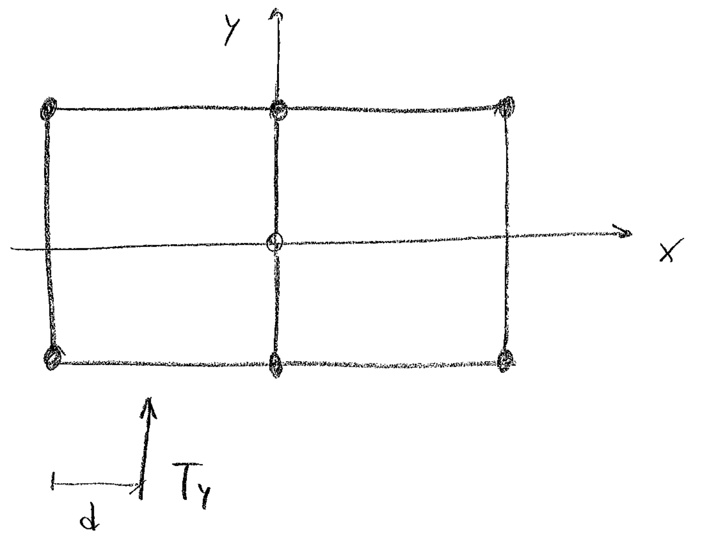
and so:

$$GA_x^* = 1.58 \cdot 10^7 \text{ N}$$

<u>Shear stiffness GA_y^*</u>

The x-position of the shear center cannot be determined from symmetry considerations. Indeed the panel 3 has thickness $2t$, while the others t.

(<u>Note</u>: the position of the section centroid depends on the lumped areas only. Thus symmetry has to be checked with respect to lumped areas.
The position of the shear center depends also on the panel thicknesses / shape, as they affect the equivalence and compatibility equations.
In this sense, the shear center position, in order to be available from symmetry considerations, requires the symmetry with respect to lumped areas <u>and</u> panels geometry)

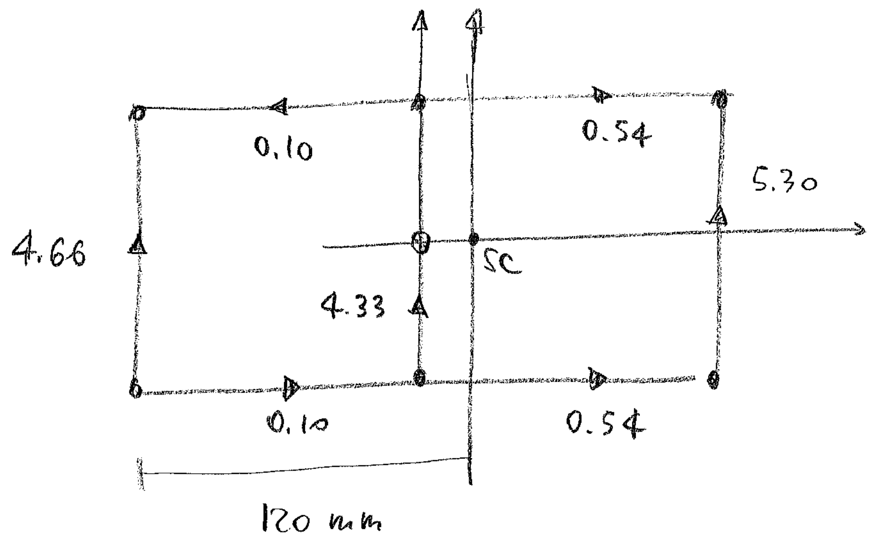


$T_y = 1000$ N
(arbitrary)

The problem can be solved by applying

1. 5 shear flow equations
2. 1 equivalence condition
3. $\theta_1' = \theta_2'$
4. $\theta_1' = 0$

and the results are found (verify!) as:



$A_y^* = 33.5\ mm^2$ (Note: A_y^* is smaller than the total area of the vertical webs $A_{web} = tb + tb + 2tb$; when the shear flows are constant and equal on the webs then $A^* = A_{web}$, otherwise $A^* < A_{web}$)

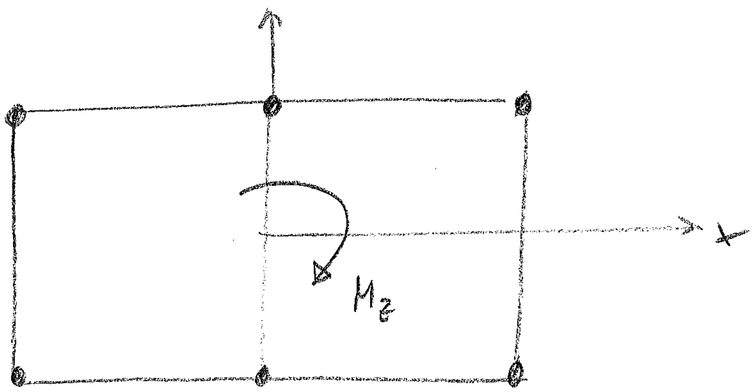
and so

$$GA_y^{*} = 9.28 \cdot 10^6 \ N$$

Torsional stiffness GJ

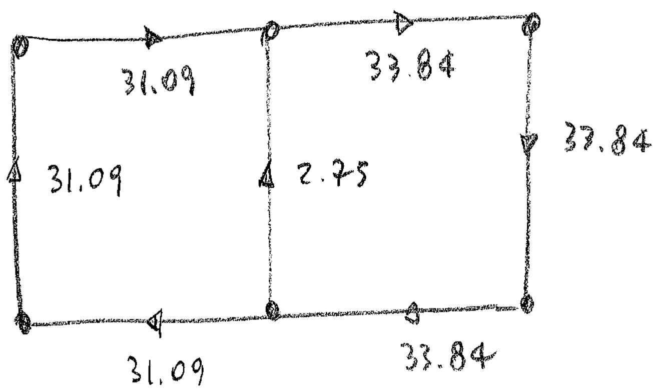
This is a multicell configuration that requires the evaluation of

$$GJ = \frac{M_z}{\theta'}$$ with M_z arbitrary, θ' torsion due to M_z



$M_z = 1 \cdot 10^6$ Nmm (arbitrary)

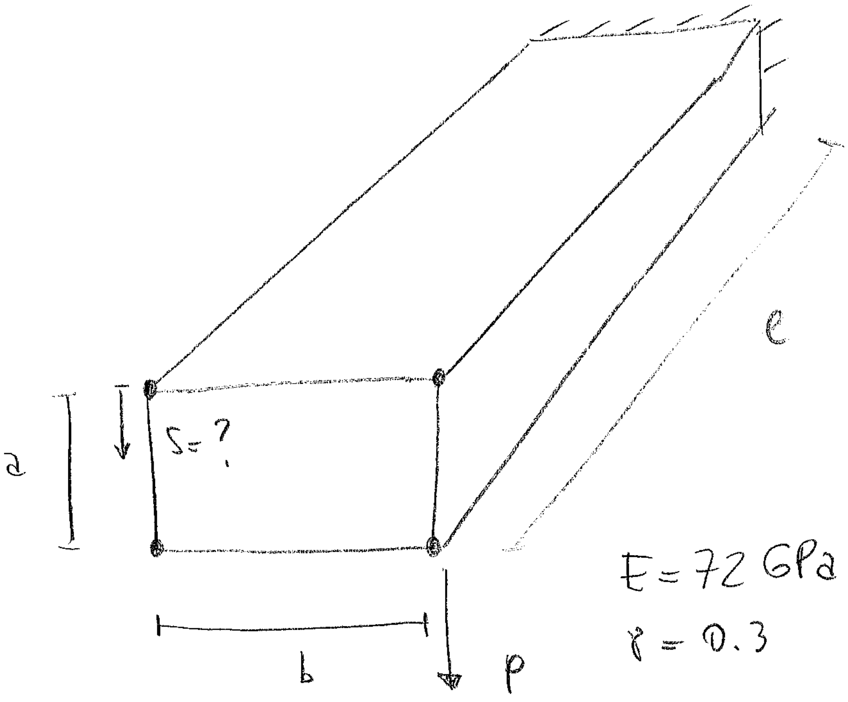
The shear flows are determined as:



$$\theta'_1 = \theta'_2 = 1.59 \cdot 10^{-5} \ \text{rad}$$

$$GJ = \frac{M_z}{\theta'} = 6.28 \cdot 10^{10} \ N \cdot mm^2$$

Exercise



Data

$\ell = 1800$ mm

$a = 150$ mm

$b = 250$ mm

$t = 1.5$ mm

$P = 3000$ N

$A = 300$ mm^2

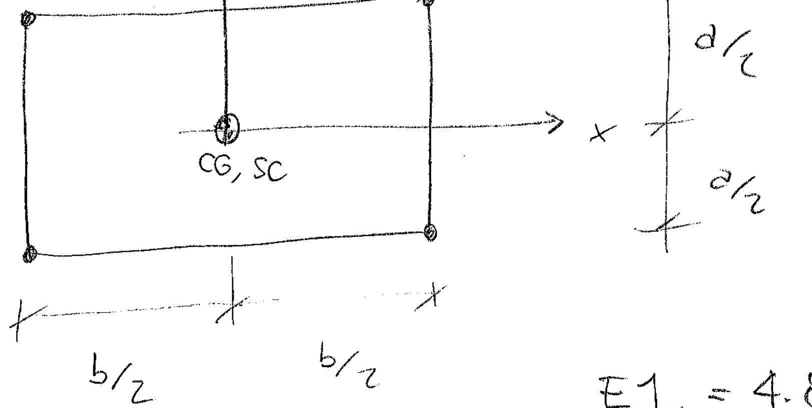
$E = 72$ GPa

$\nu = 0.3$

Evaluate the displacement s (vertical) due to applied force P. Quantify the contribution due to the shear deformability

Solution

The section properties are firstly evaluated.
Due to the symmetry, the position of the centroid and the shear center are coincident and readily available:



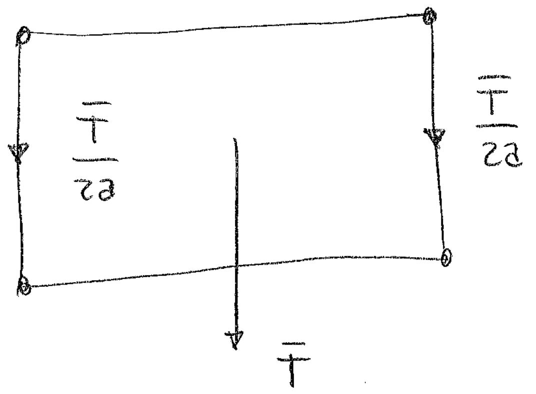
$$J_{xx} = A a^2$$

$$EJ_{xx} = 4.86 \cdot 10^{11} \text{ N mm}^2$$

To solve the problem, it is necessary to evaluate the shear and the torsional stiffnesses GA_y^* and GJ, respectively.

1. Shear stiffness GA_y^*

As the position of the shear center is known, the evaluation is straightforward:



$\bar{T}$ = arbitrary

And GA_y^* is:

$$\frac{\bar{T}^2}{GA_y^*} = \frac{\sum_i q_i^2 l_i}{Gt} \Rightarrow A_y^* = \frac{\bar{T}^2}{\sum_i \frac{q_i^2 l_i}{t}} =$$

$$= \frac{1}{\frac{1}{2at}} \Rightarrow \boxed{A_y^* = 2at}$$

Note: the result could have been immediately obtained by observing that q is constant along the two vertical webs $\Rightarrow$ A_y^* is total area of the vertical webs

$$GA_y^* = 1.25 \cdot 10^7 \text{ N}$$

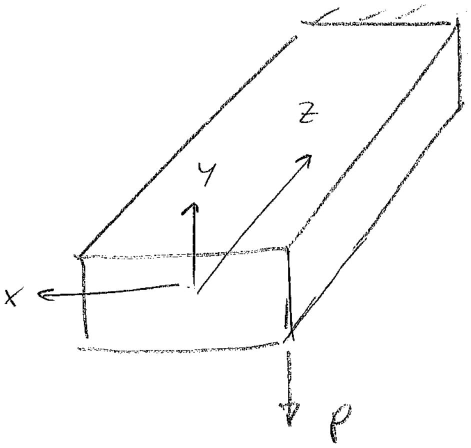
2. Torsional stiffness

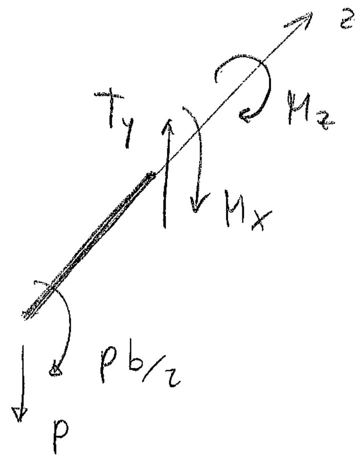
$$J = \frac{4\Omega^2}{\oint_\Gamma \frac{1}{t(s)}\,ds} = \frac{2t\,a^2b^2}{a+b}$$

$$GJ = 2.92 \cdot 10^{11}\ \text{Nmm}^2$$

PCVW

Real system



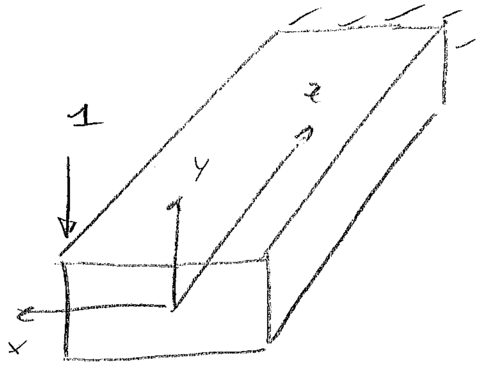


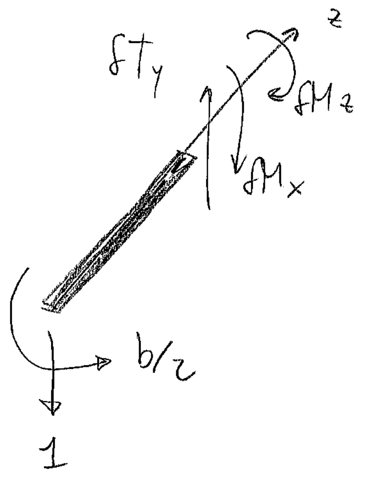
$T_y = P$

$M_x = Pz$

$M_z = -Pb/2$

Dummy system





$\delta T_y = 1$

$\delta M_x = z$

$\delta M_z = b/2$

The PCVW is then applied as:

$$\int_0^l \left(\delta M_x \frac{M_x}{EJ} + \delta T_y \frac{T_y}{GA_y^*} + \delta M_z \frac{M_z}{GJ} \right) dz = 1 \cdot s$$

($s > 0$ in the downward direction)

$$\int_0^l \left(\frac{Pz^2}{EJ} + \frac{P}{GA_y^*} - P\frac{b^2}{4GJ} \right) dz = s$$

$$\delta = P\left(\frac{\ell^3}{3EJ} + \frac{\ell}{GA_y^*} - \frac{b^2\ell}{4GJ}\right) = 12.14 \text{ mm}$$

The contribution due to shear is

$$\delta^{shear} = \frac{\ell}{GA_y^*} P = 0.43 \text{ mm}$$

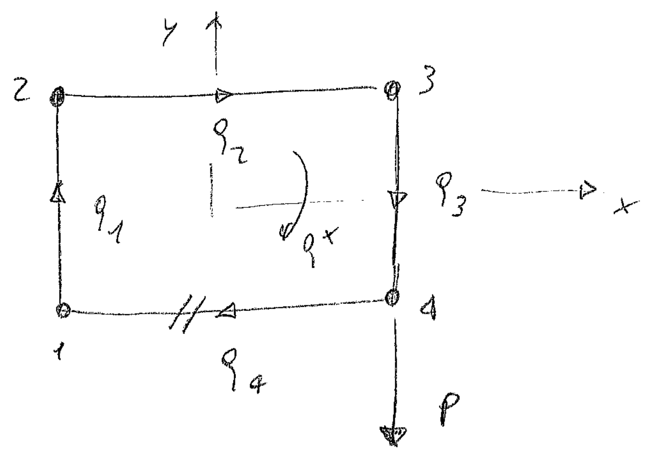
$$\delta^{bend} = P\frac{\ell^3}{3EJ} = 12 \text{ mm}$$

Alternative procedure

If there is no need for separating the energy contribution due to shear and torsion, another approach is possible.

- This strategy is more convenient if the shear center position is not known a priori (e.g. from symmetry considerations)
- The shear flows associated with the real and dummy system are directly evaluated without passing through the evaluation of GA_y^* and GJ.

Real system



$S'_{x_1} = -Aa/2$

$S'_{x_2} = 0$

$S'_{x_3} = Aa/2$

$$q_1' = -T_y \frac{S'_{x_1}}{J_{xx}} = -P \frac{Aa}{2} \frac{1}{Aa^2} = -\frac{P}{2a}$$

$$q_2' = 0$$

$$q_3' = +\frac{P}{2a}$$

Equivalence (wrt to 4):

$$2q_1'\Omega_1 + 2q^*\Omega = 0 \qquad \Omega = ab;\ \Omega_1 = ab/2$$

$$\cancel{2}q_1'\cancel{\Omega_1} + \cancel{2}q^* 2\cancel{\Omega_1} = 0$$

$$q^* = -\frac{q_1'}{2} = \frac{P}{4a}$$

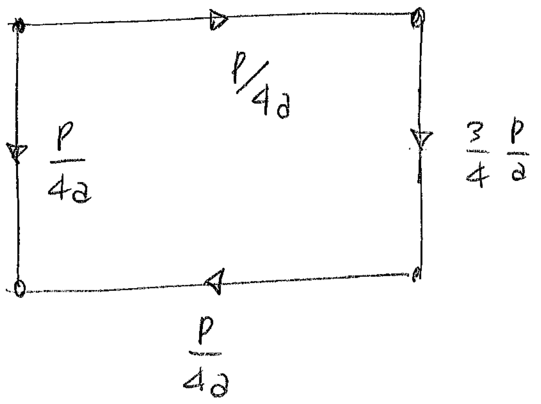
So:

$$q_1 = q_1' + q^* = -\frac{P}{4a}$$

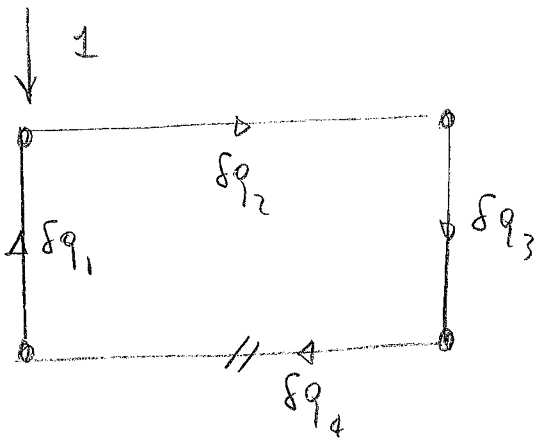
$$q_2 = q^* = \frac{P}{4a}$$

$$q_3 = q_3' + q^* = \frac{3}{4}\frac{P}{a}$$

$$q_4 = \frac{P}{4a}$$



Dummy system



$$\delta q_1' = -\frac{1}{2a}$$

$$\delta q_2' = 0$$

$$\delta q_3' = \frac{1}{2a}$$

Equivalence (wrt 1)

$$2\Omega_3 \delta q_3' + 2\Omega \delta q^* = 0 \qquad \Omega_3 = \frac{ab}{2}; \quad \Omega = ab$$

$$\delta q_3' + 2\delta q^* = 0$$

$$\delta q^* = -\frac{1}{2}\delta q_3' = -\frac{1}{4a}$$

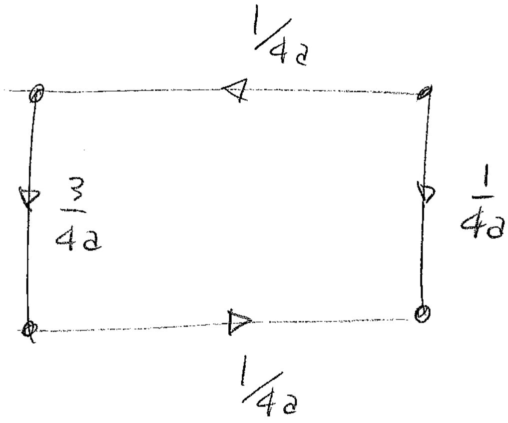
So:

$$\delta q_1 = -\frac{3}{4a}$$

$$\delta q_2 = -\frac{1}{4a}$$

$$\delta q_3 = \frac{1}{4a}$$

$$\delta q_4 = -\frac{1}{4a}$$



The PCVW is now applied as:

$$\int_0^l \delta M_x \frac{M_x}{EJ}\, dz + \int_0^l \Sigma_i\, \delta q_i \frac{l_i}{Gt_i} q_i\, dz = \delta$$

where

$M_x = Pz$ and $\delta M_x = z$

It follows that:

$$\int_0^l \delta M_x \frac{M_x}{EJ}\, dz = \frac{Pl^3}{3EJ}$$

$$\int_0^l \Sigma_i\, \delta q_i \frac{l_i}{Gt_i} q_i\, dz = \quad \left(q \text{ is constant on } z \Rightarrow \int_0^l (\cdot)\, dz = l\,(\cdot) \right)$$

$$= \frac{l}{Gt} \Sigma_i\, \delta q_i\, l_i\, q_i = \frac{l}{Gt} \left(\delta q_1 q_1 a + \delta q_2 q_2 b + \delta q_3 q_3 a + \delta q_4 q_4 b \right)$$

$$= \frac{l}{Gt} P \left(\frac{3a}{16a^2} - \frac{b}{16a^2} + \frac{3a}{16a^2} - \frac{b}{16a^2} \right) = \frac{Pl}{Gt} \left(\frac{3a-b}{8a^2} \right)$$

And so:

$$\delta = P\left(\frac{\ell^3}{3EJ} + \underbrace{\frac{\ell}{Gt}\,\frac{3a-b}{8a^2}}\right)$$

↳ This contribution does not distinguish between torsion and shear contribution!

As a verification compare with the previous result

$$\delta = P\left(\frac{\ell^3}{3EJ} + \frac{\ell}{GA^*_y} - \frac{b^2\ell}{4GJ}\right)$$

where $A^*_y = 2at$

$$J = \frac{2t\,a^2 b^2}{a+b}$$

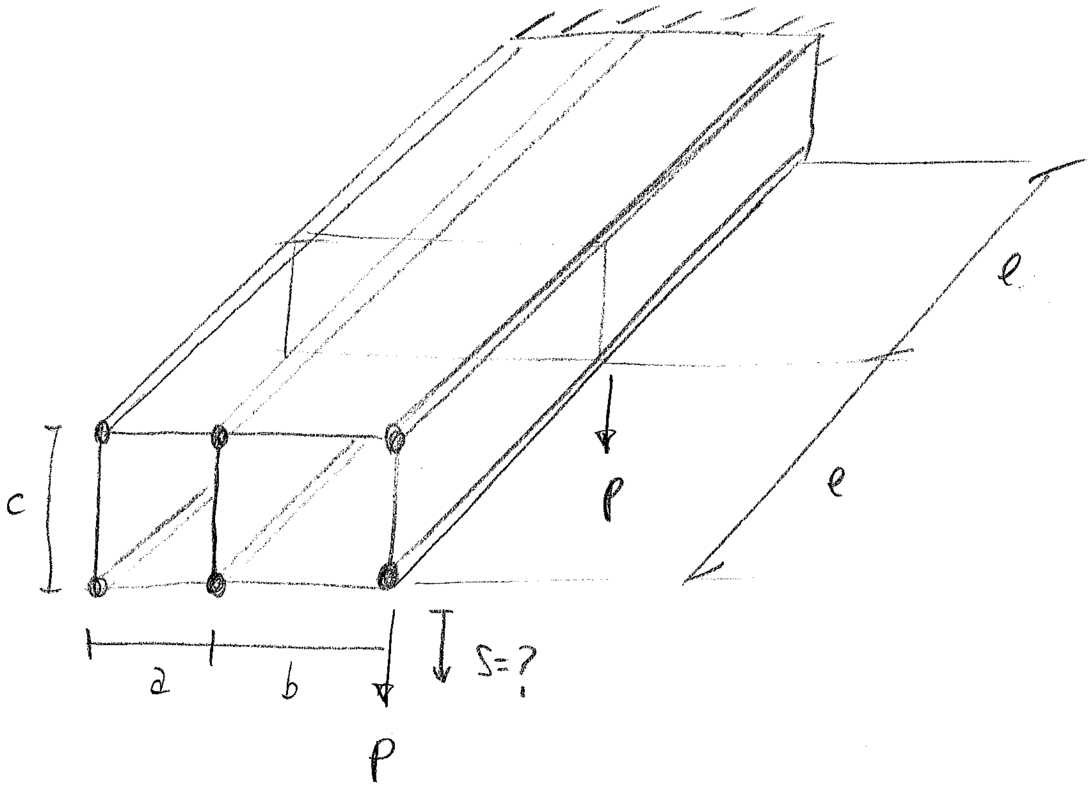
So:

$$\frac{\ell}{GA^*_y} - \frac{b^2\ell}{4GJ} = \frac{\ell}{2Gat} - \frac{b^2\ell}{4G}\,\frac{a+b}{2ta^2b^2}$$

$$= \frac{\ell}{2Gat} - \frac{\ell(a+b)}{8Ga^2t}$$

$$= \frac{4a\ell - a\ell - \ell b}{8Ga^2t} = \frac{\ell(3a-b)}{Gt\,8a^2} \quad ✓$$

Exercise



$a = 100$ mm

$b = 300$ mm

$c = 200$ mm

$t = 1$ mm

$A = 500 \text{ mm}^2$

$l = 900$ mm

$P = 1000$ N

Evaluate the displacement S and assess the contribution of shear deformability in relation to the bending one.

Solution

The section properties have been already determined. In particular, they are:

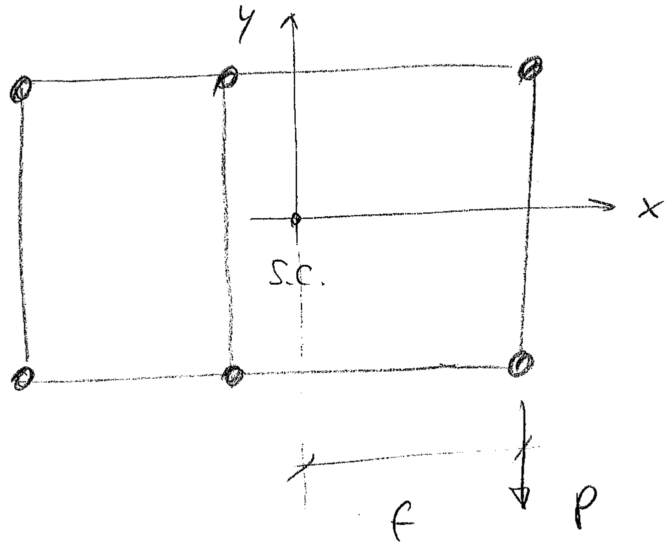
$$EJ_{xx} = 2.16 \cdot 10^{12} \text{ Nmm}^2$$

$$GA_y^* = 1.66 \cdot 10^7 \text{ N}$$

$$GJ = 6.01 \cdot 10^{11} \text{ Nmm}^2$$

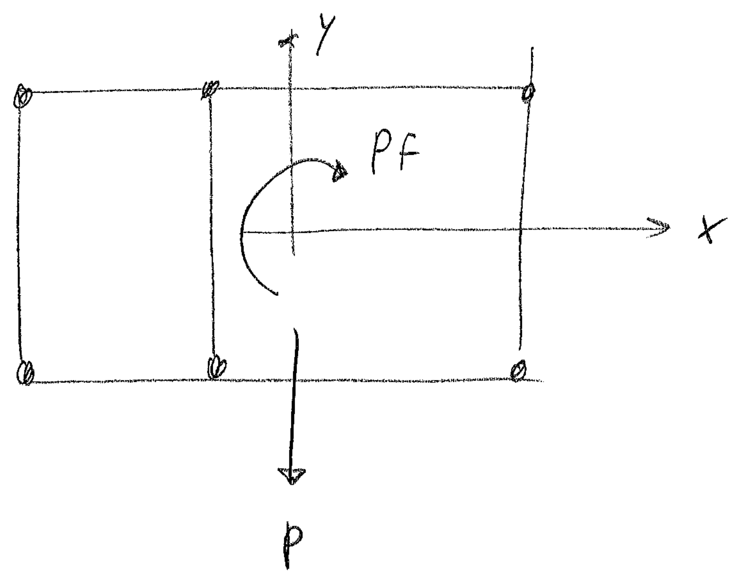
The solution of the problem can be obtained by applying the PCVW.

Recall that the stiffnesses GA_y^* and GJ were obtained by referring to the system with z-axis passing through the shear center. Then, the internal actions have to be now referred to the same system.

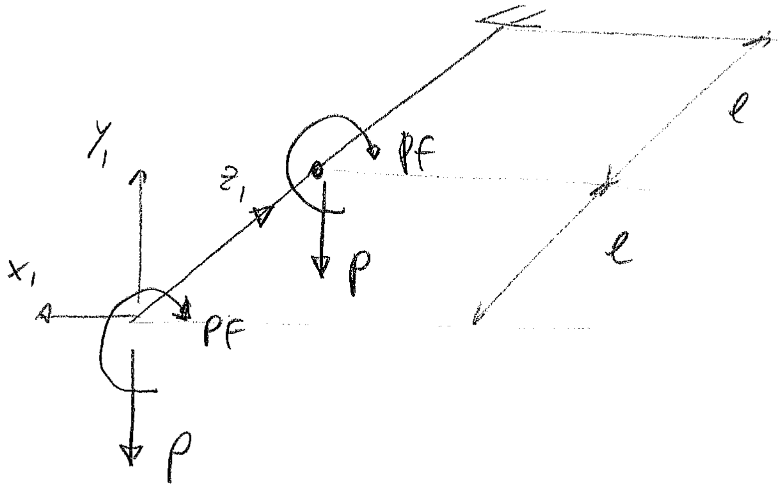


$f = 233.33$ mm
(obtained previously)

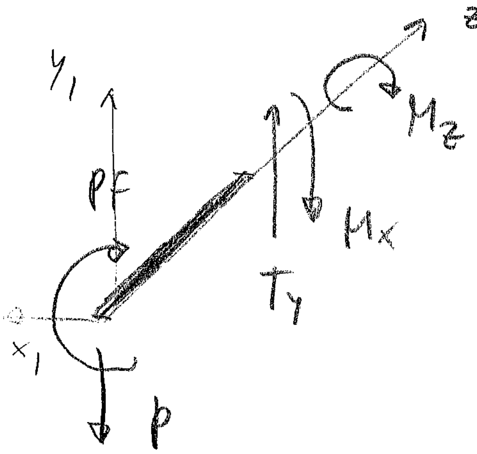
and so:



Real system

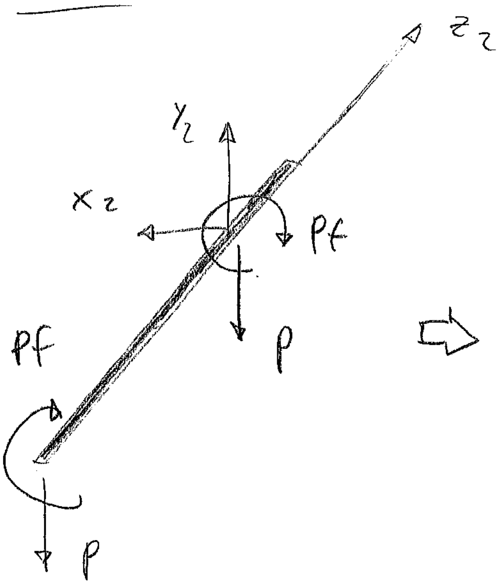


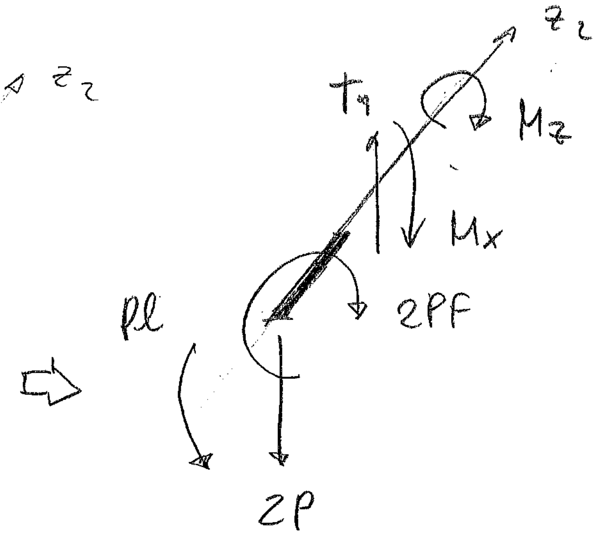
Beam 1



$$T_y^{(1)} = P$$
$$M_x^{(1)} = Pz_1$$
$$M_z^{(1)} = -Pf$$

Beam 2

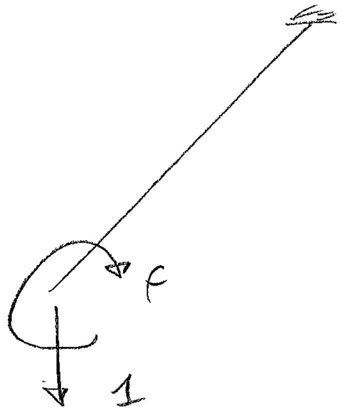




$$T_y^{(2)} = 2P$$
$$M_x^{(2)} = P\ell + 2Pz_2$$
$$M_z^{(2)} = -2Pf$$

Dummy system

The dummy system is



Note: pay attention not to forget the torsional moment f!

Beam 1

$\delta T_y^{①} = 1$

$\delta M_x^{①} = z_1$

$\delta M_z^{①} = -f$

Beam 2

$\delta T_y^{②} = +1$

$\delta M_x^{②} = \ell + z_2$

$\delta M_z^{②} = -f$

PCVW

$$\int_0^\ell \left(\frac{T_y^{①} \delta T_y^{①}}{GA_y^*} + \frac{M_x^{①} \delta M_x^{①}}{EJ_{xx}} + \frac{M_z^{①} \delta M_z^{①}}{GJ} \right) dz_1 +$$

$$\int_0^\ell \left(\frac{T_y^{②} \delta T_y^{②}}{GA_y^*} + \frac{M_x^{②} \delta M_x^{②}}{EJ_{xx}} + \frac{M_z^{②} \delta M_z^{②}}{GJ} \right) dz_2 = S$$

$$\int_0^{\ell}\left(\frac{P}{GA_y^*}+\frac{Pz_1^2}{EJ_{xx}}+\frac{Pf^2}{GJ}\right)dz+\int_0^{\ell}\left(\frac{2P}{GA_y^*}+\frac{P\ell^2+P\ell z_2+2P\ell z_2+2Pz_2^2}{EJ_{xx}}+\right.$$

$$\left.+\frac{2Pf^2}{GJ}\right)dz =$$

$$P\left(\frac{\ell}{GA_y^*}+\frac{\ell^3}{3EJ}+\frac{f^2\ell}{GJ}+\frac{2\ell}{GA_y^*}+\frac{\ell^3}{EJ_{xx}}+\frac{3\ell^3}{2EJ_{xx}}+\frac{2\ell^3}{3EJ_{xx}}+\frac{2f^2\ell}{GJ}\right)$$

$$P\left(\frac{21}{6}\frac{\ell^3}{EJ}+\frac{3\ell}{GA_y^*}+\frac{3f^2\ell}{GJ}\right)=S$$

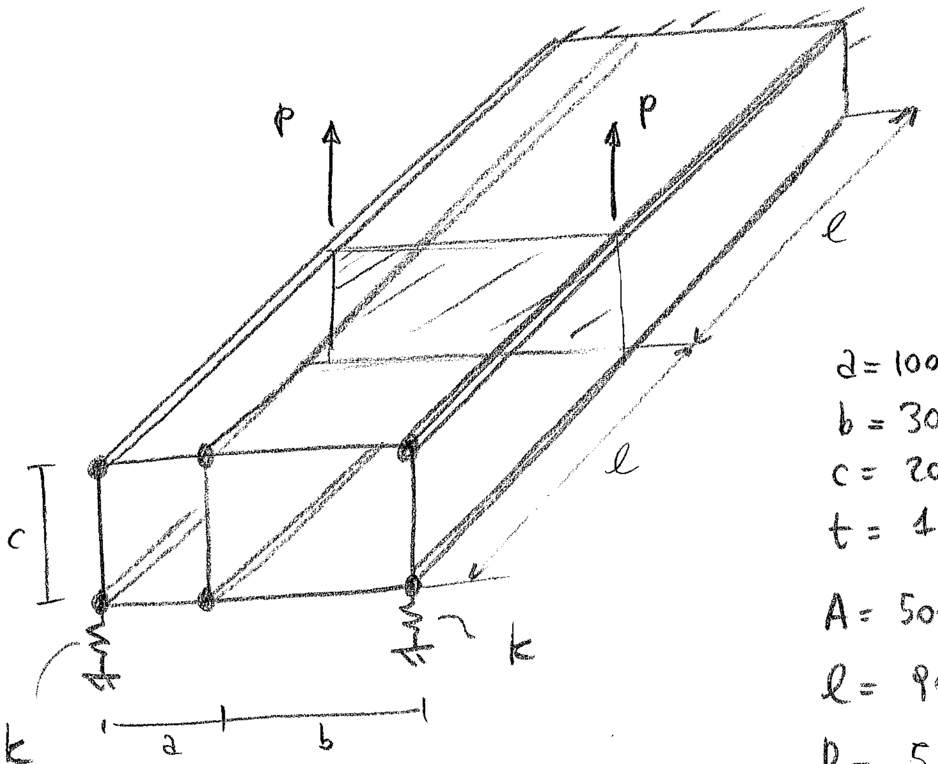
Thus:

$$S=\left(\frac{7}{2}\frac{\ell^3}{EJ}+\frac{3\ell}{GA_y^*}+\frac{3f^2\ell}{GJ}\right)P = 1.59 \text{ mm}$$

$$S_{shear}=\frac{3\ell}{GA_y^*}P = 0.16 \text{ mm}$$

$$S_{bending}=\frac{7}{2}\frac{\ell^3}{EJ}P = 1.18 \text{ mm}$$

Exercise



$a = 100$ mm

$b = 300$ mm

$c = 200$ mm

$t = 1$ mm

$A = 500$ mm^2

$l = 900$ mm

$P = 5000$ N

$k = 10^6$ N/mm

$E = 72$ GPa

$\nu = 0.3$

Determine the reaction forces in correspondence of the linear springs.

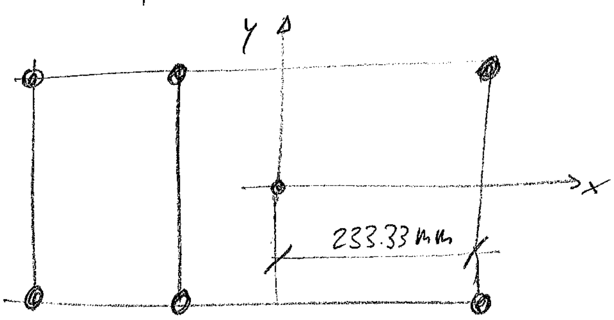
Solution

The section properties have been already evaluated

$EJ_{xx} = 2.16 \cdot 10^{12}$ Nmm^2

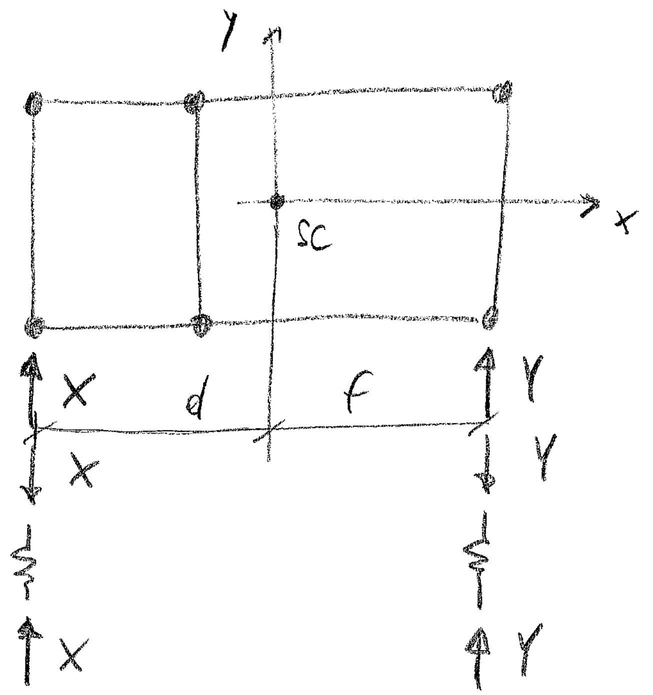
$GA_y^* = 1.66 \cdot 10^7$ N

$GJ = 6.01 \cdot 10^{11}$ Nmm^2



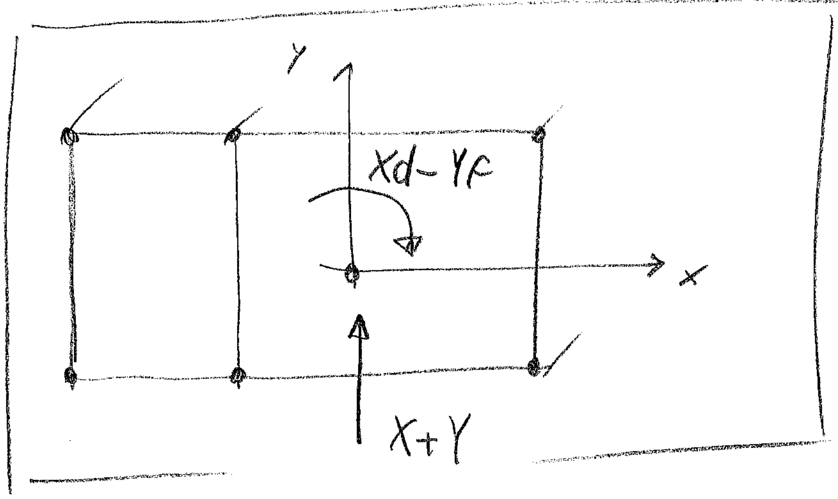
The position of the shear center is coincident with the centroid of the section

The beam is two times statically indeterminate.
The problem is thus solved by applying two times the PCVW.

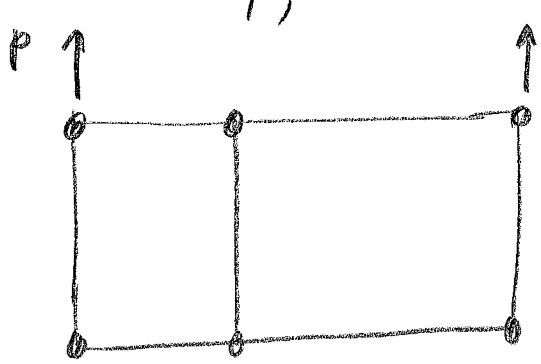


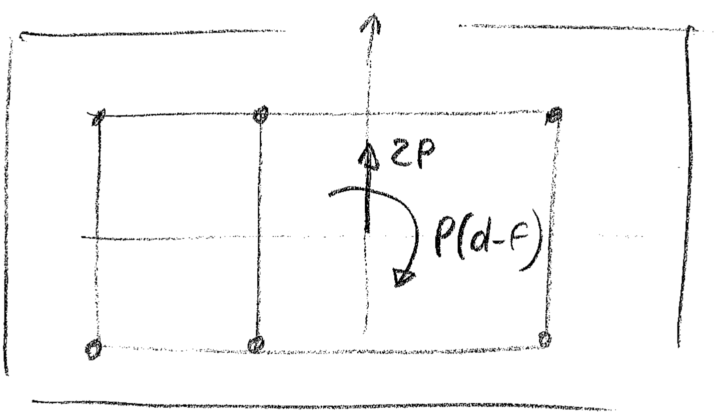
end section

The internal actions of shear and torsion are now referred to the system passing through the shear center.

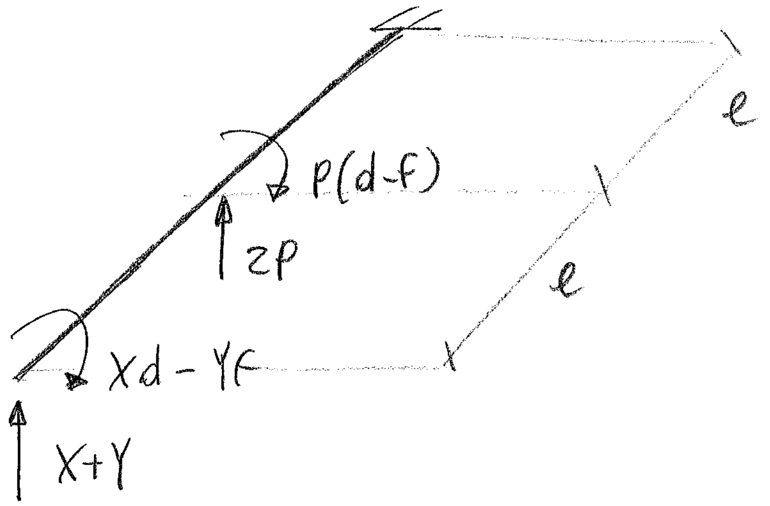


Similarly, the section where the load P is introduced



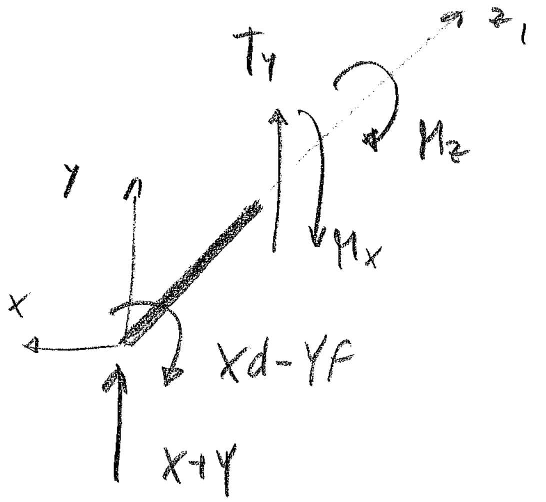


Apply now the PCVW:



Real system

Beam 1

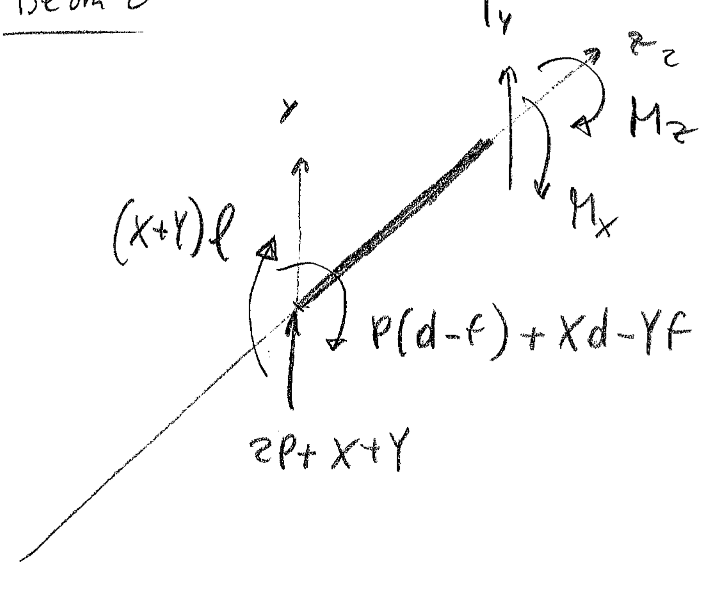


$$T_y^{(1)} = -(X+Y)$$

$$M_x^{(1)} = -(X+Y)\,z_1$$

$$M_z^{(1)} = -(Xd - Yf)$$

Beam 2

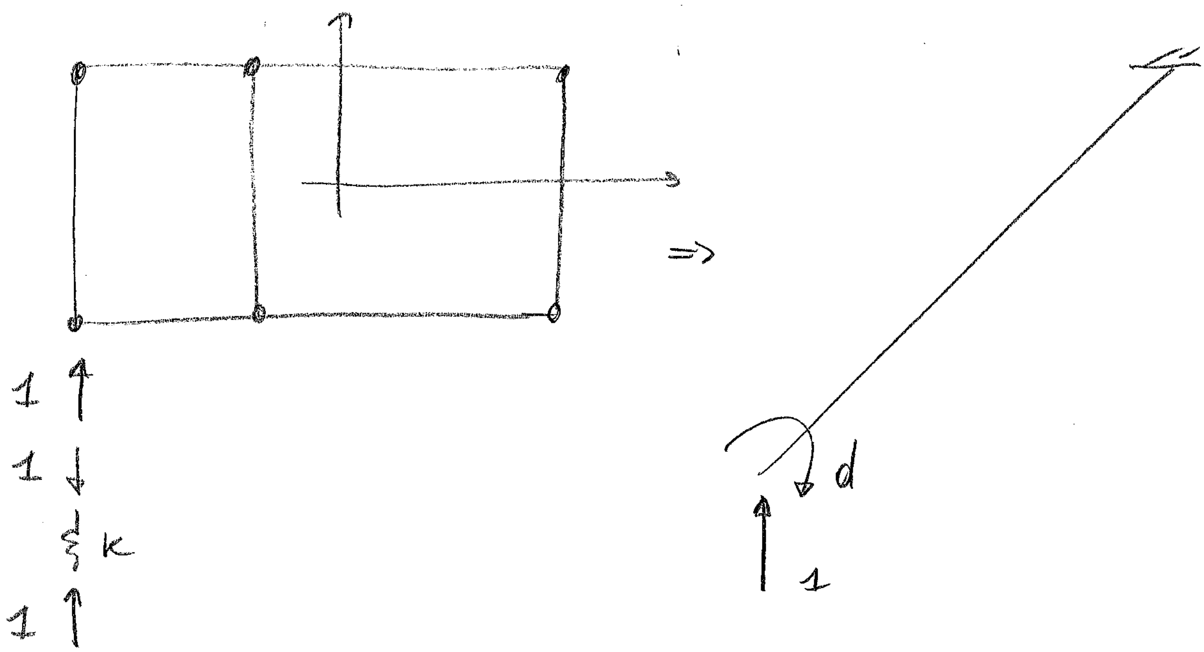


$$T_y^{(2)} = -(2P+X+Y)$$

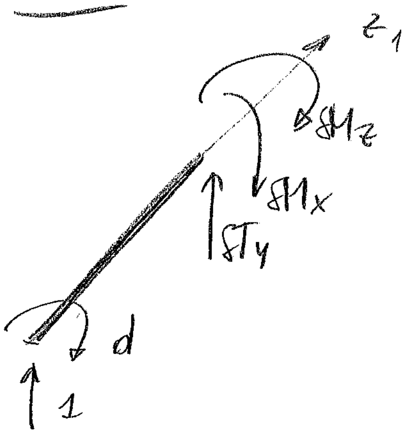
$$M_x^{(2)} = -(2P+X+Y)\,z_2 - (X+Y)\,\ell$$

$$M_z^{(2)} = -(Xd - Yf) - P(d-f)$$

Dummy system 1



Beam 1

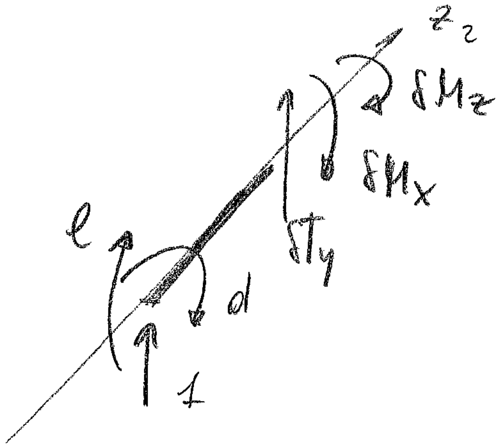


$${}^{1}\delta T_y^{(1)} = -1$$

$${}^{1}\delta M_x^{(1)} = -z_1$$

$${}^{1}\delta M_z^{(1)} = -d$$

Beam 2

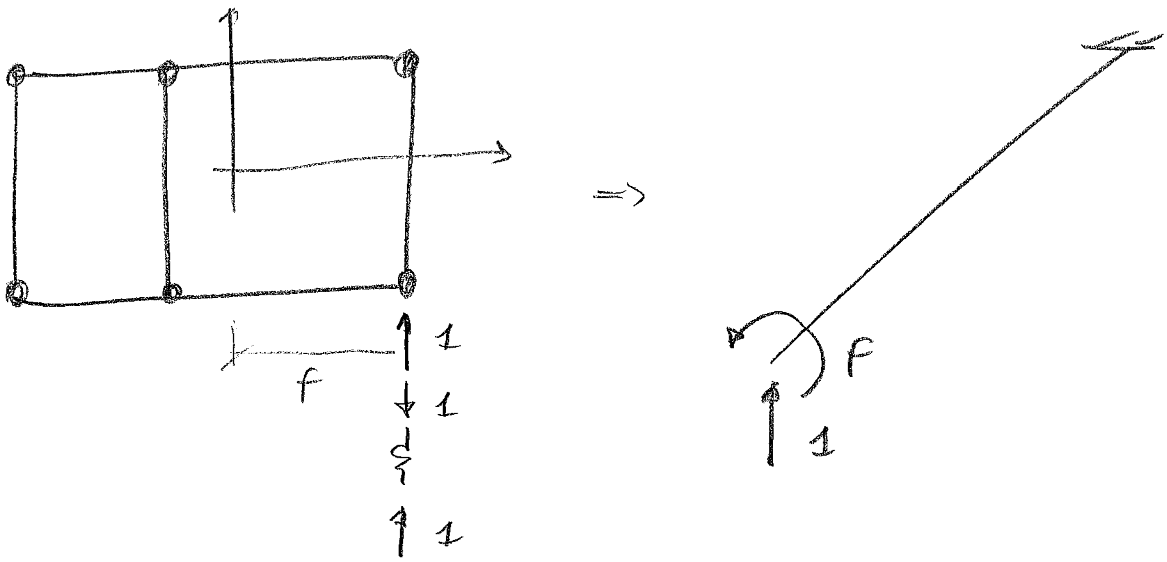


$${}^{1}\delta T_y^{(2)} = -1$$

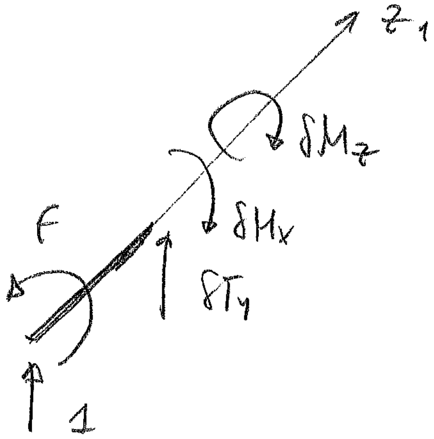
$${}^{1}\delta M_x^{(2)} = -(\ell + z_2)$$

$${}^{1}\delta M_z^{(2)} = -d$$

Dummy system 2



Beam 1

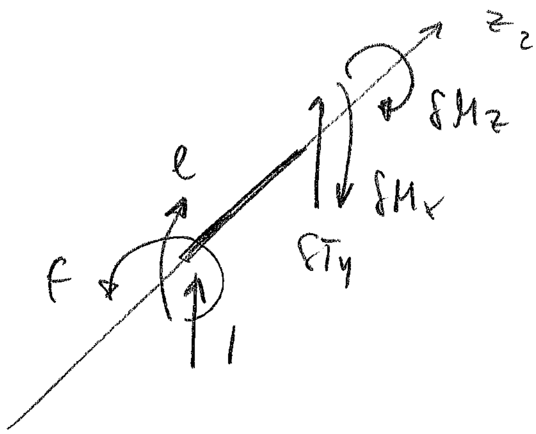


$$ {}^{2}\delta T_y^{(1)} = -1 $$

$$ {}^{2}\delta M_x^{(1)} = -z_1 $$

$$ {}^{2}\delta M_z^{(1)} = f $$

Beam 2



$$ {}^{2}\delta T_y^{(2)} = -1 $$

$$ {}^{2}\delta M_x^{(2)} = -l - z_2 $$

$$ {}^{2}\delta M_z^{(2)} = f $$

Summarizing

	Real	dummy 1	dummy 2
Beam 1	$T_y^{①} = -(X+Y)$ $M_x^{①} = -(X+Y)z$ $M_z^{①} = -(Xd - Yf)$	${}^1\delta T_y^{①} = -1$ ${}^1\delta M_x^{①} = -z$ ${}^1\delta M_z^{①} = -d$	${}^2\delta T_y^{①} = -1$ ${}^2\delta M_x^{①} = -z$ ${}^2\delta M_z^{①} = +f$
Beam 2	$T_y^{②} = -(2P+X+Y)$ $M_x^{②} = -(2P+X+Y)z - (X+Y)l$ $M_z^{②} = -(Xd - Yf) + - P(d-f)$	${}^1\delta T_y^{①} = -1$ ${}^1\delta M_x^{①} = -(z+l)$ ${}^1\delta M_z^{①} = -d$	${}^2\delta T_y^{②} = -1$ ${}^2\delta M_x^{②} = -(z+l)$ ${}^2\delta M_z^{②} = +f$

Compatibility equations

① $$\int_0^l \left({}^1\delta T_y^{①} \frac{T_y^{①}}{GA_x^*} + {}^1\delta M_x \frac{M_x^{①}}{EJ_{xx}} + {}^1\delta M_z \frac{M_z^{①}}{GJ} \right) dz_1 +$$

$$+ \int_0^l \left({}^1\delta T_y^{②} \frac{T_y^{②}}{GA_x^*} + {}^1\delta M_x^{②} \frac{M_x^{②}}{EJ_{xx}} + {}^1\delta M_z^{②} \frac{M_z^{②}}{GJ} \right) dz_2 +$$

$$+ \delta W^*_{i,spring} = 0$$

where $\delta W^*_{i,spring} = 1 \cdot \frac{X}{k}$

② $$\int_0^l \left({}^2\delta T_y^{①} \frac{T_y^{①}}{GA_x^*} + {}^2\delta M_x^{①} \frac{M_x^{①}}{EJ_{xx}} + {}^2\delta M_z^{①} \frac{M_z^{①}}{GJ} \right) dz_1 +$$

$$+ \int_0^l \left({}^2\delta T_y^{②} \frac{T_y^{②}}{GA_x^*} + {}^2\delta M_x^{②} \frac{M_x^{②}}{EJ_{xx}} + {}^2\delta M_z^{②} \frac{M_z^{②}}{GJ} \right) dz_2 + \frac{Y}{k} = 0$$

After taking the integrals, the compatibility equations are obtained as:

$$① \left(\frac{8}{3}\frac{\ell^3}{EJ_{xx}}+\frac{2\ell}{GA_y^*}+\frac{2d^2\ell}{GJ}+\frac{1}{k}\right)X+\left(\frac{8}{3}\frac{\ell^3}{EJ_{xx}}+\frac{2\ell}{GA_y^*}-\frac{2df\ell}{GJ}\right)Y$$

$$=\left(-\frac{5}{3}\frac{\ell^3}{EJ_{xx}}-\frac{2\ell}{GA_y^*}+\frac{d(f-d)\ell}{GJ}\right)P$$

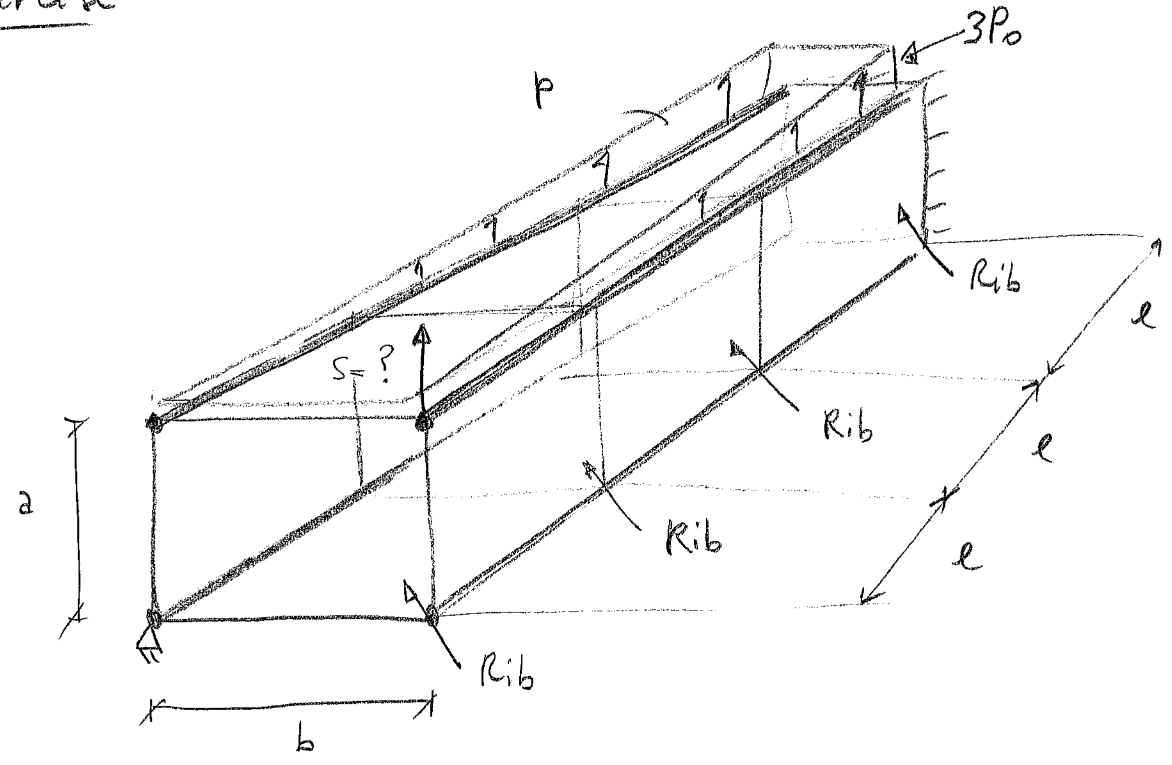
$$② \left(\frac{8}{3}\frac{\ell^3}{EJ_{xx}}+\frac{2\ell}{GA_y^*}-\frac{2df\ell}{GJ}\right)X+\left(\frac{8}{3}\frac{\ell^3}{EJ_{xx}}+\frac{2\ell}{GA_y^*}+\frac{2f^2\ell}{GJ}+\frac{1}{k}\right)Y$$

$$=\left(-\frac{5}{3}\frac{\ell^3}{EJ_{xx}}-\frac{2\ell}{GA_y^*}-\frac{f(f-d)\ell}{GJ}\right)P$$

and solving the 2x2 linear system it is obtained:

$$\boxed{\begin{aligned} X &= -1105\ N \\ Y &= -1200\ N \end{aligned}}$$

Exercise



Evaluate the displacements due to the pressure load p (constant along the chordwise direction and linear along the spanwise direction)

$a = 100$ mm $\qquad l = 400$ mm

$b = 150$ mm $\qquad E = 72$ GPa

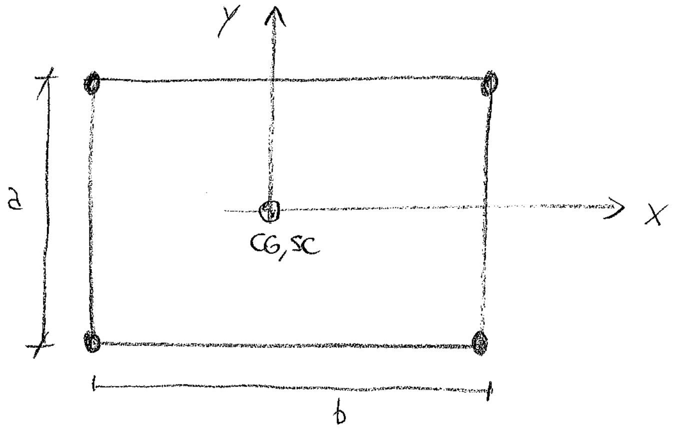
$t = 1$ mm $\qquad \nu = 0.3$

$A = 300$ mm^2 $\qquad p_0 = 0.02$ N/mm^2

Solution

The first step consists in evaluating the stiffnesses EJ_{xx}, GA_y^* and GJ.

The double symmetry of the section facilitates this task.



The centroid and the shear center are found from symmetry considerations (and are coincident)

Thus.

$$J_{xx} = 4A\left(\frac{a}{2}\right)^2 = Aa^2$$

$$EJ_{xx} = 2.16 \cdot 10^{11} \quad N\,mm^2$$

The shear area A_y^* is readily found as:

$$A_y^* = 2at = 200 \quad mm^2$$

(Note that from the symmetry of the section the shear flows due to a generic load T_y are zero on the horizontal panels, and $T_y/2a$ on the vertical webs)

And so:

$$GA_y^* = 5.54 \cdot 10^6 \; N$$

The torsional stiffness can be obtained as:

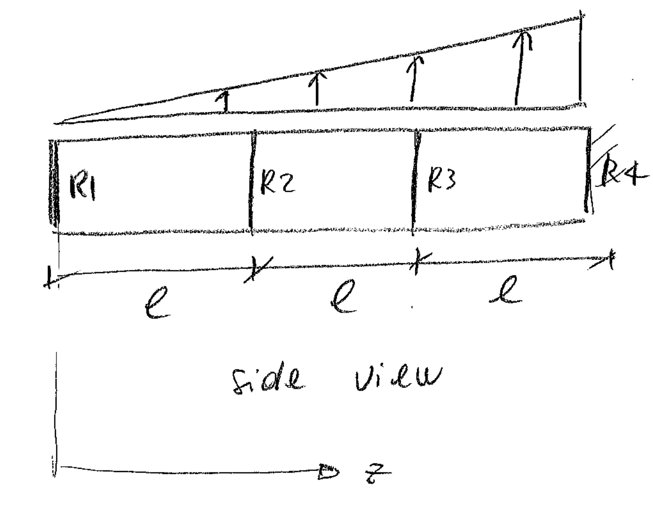
$$J = \frac{4\Omega^2}{\oint_p \frac{1}{t(s)}\,dp} = \frac{2a^2b^2t}{a+b}$$

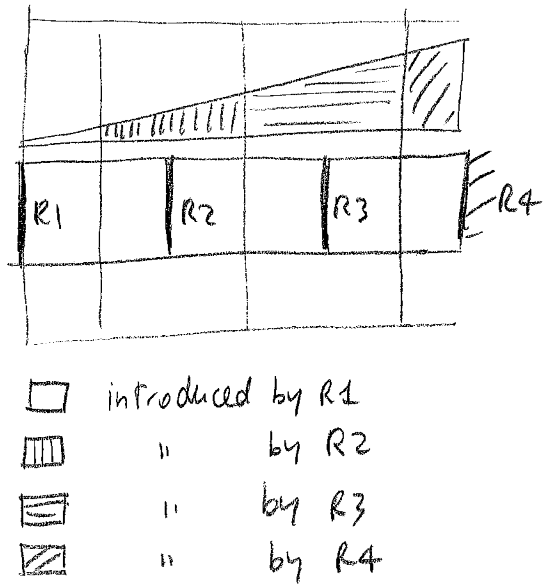
and so:

$$GJ = 4.98 \cdot 10^{10} \; N$$

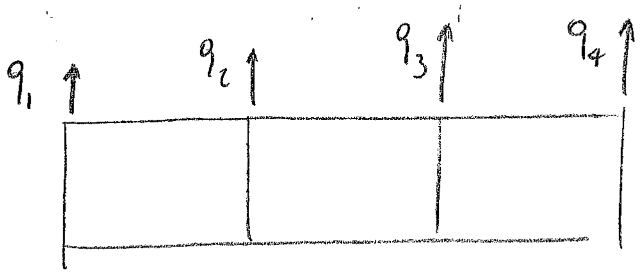
- The structure is 1 time statically indeterminate, so the evaluation of s requires a previous evaluation of the statically indeterminate reaction force.

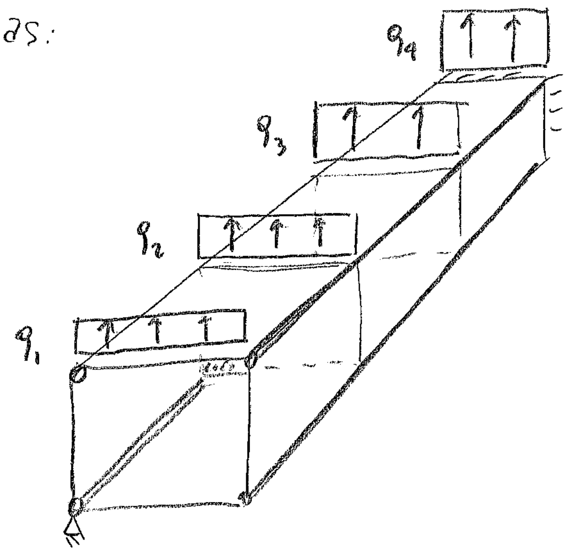
- The beam is transversally stiffened with 4 ribs which are responsible for the load introduction. The loads are then represented as:





Thus the loads are introduced as:





$$p(z) = \frac{3p_0}{3l} z = \frac{p_0}{l} z$$

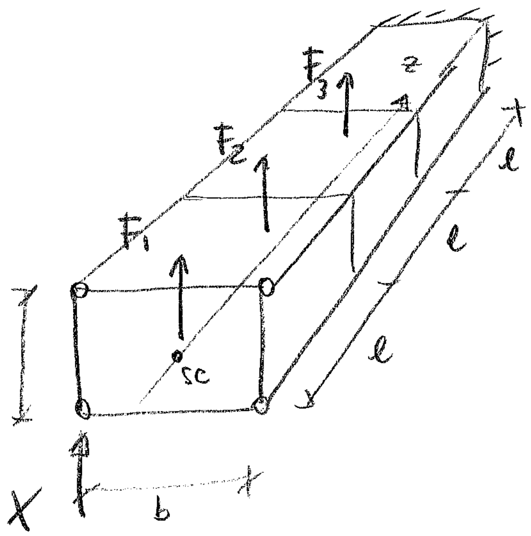
$$q_1 = \int_0^{l/2} p(z) = \frac{p_0}{l}\frac{l^2}{8} = \frac{p_0 l}{8}$$

$$q_2 = \int_{l/2}^{3/2\, l} p(z) = p_0 l$$

$$q_3 = \int_{3/2\, l}^{5/2\, l} p(z)\, dz = 2 p_0 l$$

$$q_4 = \int_{5/2\, l}^{3l} p(z)\, dz = \frac{11}{8} p_0 l$$

Evaluation of the reaction force



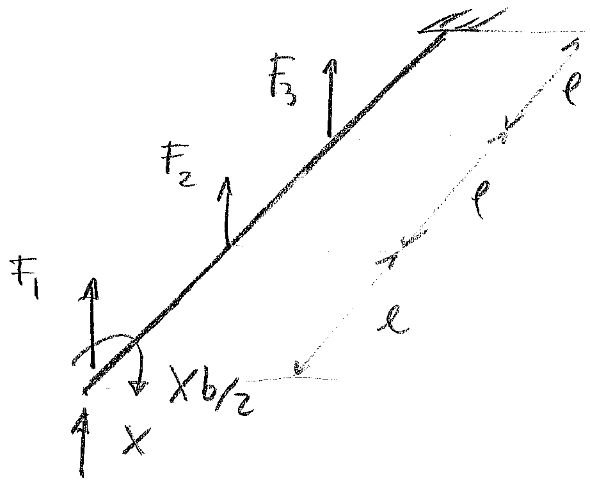
$F_1 = q_1 b = 150\ N$

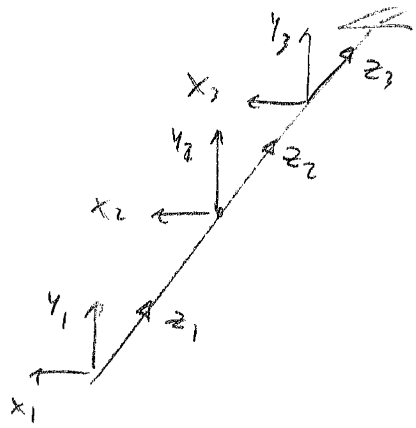
$F_2 = q_2 b = 1200\ N$

$F_3 = q_3 b = 2400\ N$

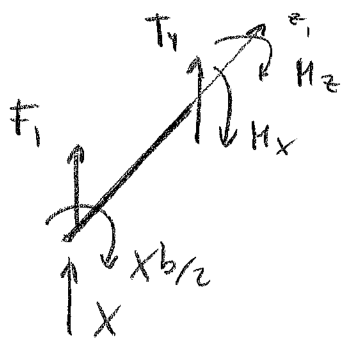
Note that F_i are shear forces applied in the shear center.

- Real system





Beam #1

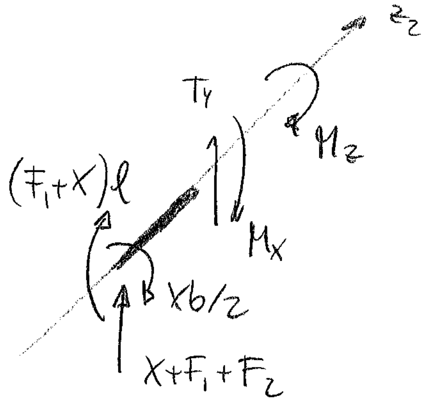


$T_y^{(1)} = -(X + F_1)$

$M_x^{(1)} = -(X + F_1)\, z_1$

$M_z^{(1)} = -X b/2$

Beam #2

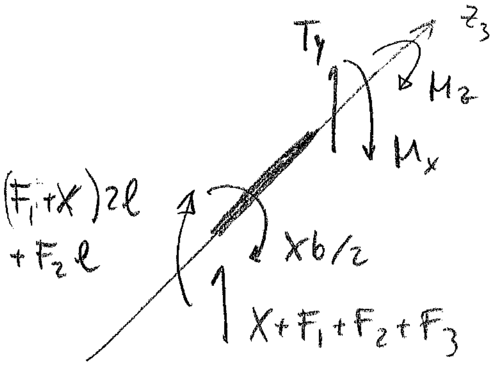


$$T_y^{(2)} = -(X+F_1+F_2)$$

$$M_x^{(2)} = -(F_1+X)\ell - (X+F_1+F_2)\, z_2$$

$$M_z^{(2)} = -X b/2$$

Beam #3

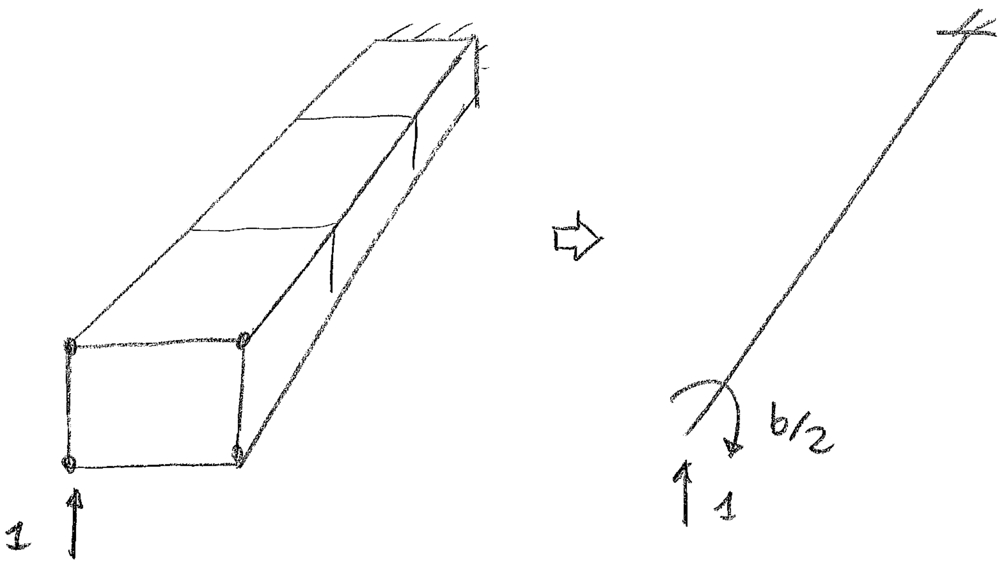


$$T_y^{(3)} = -(X+F_1+F_2+F_3)$$

$$M_x^{(3)} = -(X+F_1+F_2+F_3)\, z_3 - (F_1+X)2\ell - F_2\ell$$

$$M_z^{(3)} = -X b/2$$

- Dummy system



Beam #1

$$\delta T_y^{①} = -1$$
$$\delta M_x^{①} = -z_1$$
$$\delta M_z^{①} = -b/2$$

Beam #2

$$\delta T_y^{②} = -1$$
$$\delta M_x^{②} = -(\ell + z_2)$$
$$\delta M_z^{②} = -b/2$$

Beam #3

$$\delta T_y^{③} = -1$$
$$\delta M_x^{③} = -(2\ell + z_3)$$
$$\delta M_z^{③} = -b/2$$

The compatibility condition is found as:

$$\sum_{i=1}^{3} \int_0^{\ell} \left(\delta T_y^{(i)} \frac{T_y}{GA_y^*} + \delta M_x^{(i)} \frac{M_x}{EJ_{xx}} + \delta M_z^{(i)} \frac{M_z}{GJ} \right) dz_i = 0$$

which leads to:

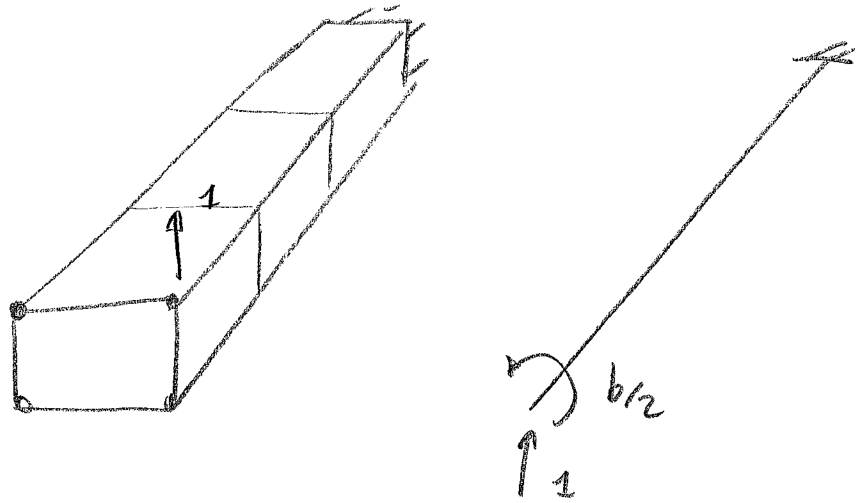
$$\left(\frac{13}{2} \frac{\ell^3}{EJ} + \frac{3\ell}{GA} + \frac{3b^2\ell}{4GJ} \right) X = -\frac{\ell}{GA_y^*} (3F_1 + 2F_2 + F_3) - \frac{\ell^3}{EJ} \left(\frac{13}{2} F_1 + \frac{13}{6} F_2 \right)$$

$$X = -631 \text{ N}$$

Evaluation of the displacement s

The reaction force X is now available, thus the internal actions of the real system are function of known quantities.

The dummy system for evaluating s is:



And so:

$\delta T_y^{(1)} = -1$ $\qquad$ $\delta T_y^{(2)} = -1$ $\qquad$ $\delta T_y^{(3)} = -1$

$\delta M_x^{(1)} = -z_1$ $\qquad$ $\delta M_x^{(2)} = -(\ell + z_2)$ $\qquad$ $\delta M_x^{(3)} = -(2\ell + z_3)$

$\delta M_z^{(1)} = b/2$ $\qquad$ $\delta M_z^{(2)} = b/2$ $\qquad$ $\delta M_z^{(3)} = b/2$

The compatibility condition reads:

$$\sum_{i=1}^{3} \int_0^{\ell} \left(\delta T_y^{(i)} \frac{T_y^{(i)}}{G A_y^*} + \delta M_x^{(i)} \frac{M_x^{(i)}}{E J_{xx}} + \delta M_z^{(i)} \frac{M_z^{(i)}}{G J} \right) dz_i = s$$

The calculations do not need to be repeated as the dummy system differs from the pevious one only for the sign of $\delta M_z^{(1)}$

Then:

$$S = \frac{l}{GA_y^*}\left(3F_1 + 3X + 2F_2 + F_3\right) + \frac{l^3}{EJ}\left(\frac{13}{2}(F_1 + X) + \frac{13}{6}F_2\right) - \frac{3}{4}\frac{b^2 l X}{GJ}$$

$$= 0.17 \quad mm$$