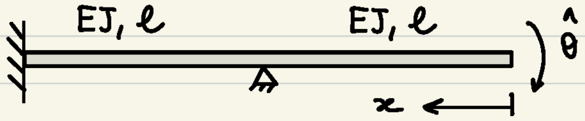
Force-based #1



Determine the bending moment at $x = l/2$

Solve the problem using:

(a) a displacement-based approach

(b) a force-based approach

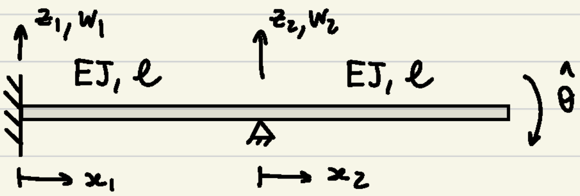
Assume that shearing deformation is negligible

Data

$l = 1000$ mm $\qquad \hat{\theta} = 10$ deg

$EJ = 10^{12}$ Nmm^2

Solution (displacement - based)



$$EJ\, w_1^{IV} = 0 \qquad\qquad EJ\, w_2^{IV} = 0$$

$$w_1(0) = 0 \quad (1) \qquad\qquad w_2(0) = 0 \quad (4)$$

$$w_1'(0) = 0 \quad (2) \qquad\qquad w_2'''(\ell) = 0 \quad (5)$$

$$w_1(\ell) = 0 \quad (3) \qquad\qquad w_2'(\ell) = -\hat{\theta} \quad (6)$$

$$w_1''(\ell) = w_2''(0) \quad (7)$$

$$w_1'(\ell) = w_2'(0) \quad (8)$$

The solution of the ODEs reads:

$$w_1 = A_0 + A_1 x_1 + A_2 x_1^2 + A_3 x_1^3$$

$$w_2 = B_0 + B_1 x_2 + B_2 x_2^2 + B_3 x_2^3$$

From which:

$w_1' = A_1 + 2A_2 x_1 + 3A_3 x_1^2$	$w_2' = B_1 + 2B_2 x_2 + 3B_3 x_2^2$
$w_1'' = 2A_2 + 6A_3 x_1$	$w_2'' = 2B_2 + 6B_3 x_2$
$w_1''' = 6A_3$	$w_2''' = 6B_3$

(1)-(2) : $A_0 = A_1 = 0$

(3) : $A_2 + A_3 l = 0$

(4) : $B_0 = 0$

(5) : $B_3 = 0$

(6) : $B_1 + 2B_2 l = -\hat{\theta}$

(7) : $2A_2 + 6A_3 l = 2B_2$

(8) : $2A_2 l + 3A_3 l^2 = B_1$

So:

$$\begin{cases} A_2 + A_3 l = 0 & \longrightarrow \quad A_2 = -A_3 l \\ B_1 + 2B_2 l = -\hat{\theta} & \longrightarrow \quad B_1 = -\hat{\theta} - 2B_2 l \\ 2A_2 + 6A_3 l - 2B_2 = 0 & \longrightarrow \quad 2A_3 l = B_2 \\ 2A_2 l + 3A_3 l^2 - B_1 = 0 & \longrightarrow \quad A_3 = -\hat{\theta}/5l^2 \end{cases}$$

And then:

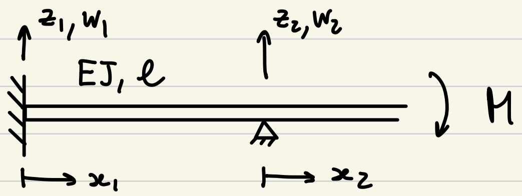
$A_0 = 0$ $\qquad$ $B_0 = 0$

$A_1 = 0$ $\qquad$ $B_1 = -\hat{\theta}/5$

$A_2 = \hat{\theta}/5l$ $\qquad$ $B_2 = -2\hat{\theta}/5l$

$A_3 = -\hat{\theta}/5l^2$ $\qquad$ $B_3 = 0$

The bending moment at $x_2 = \ell/2$ is

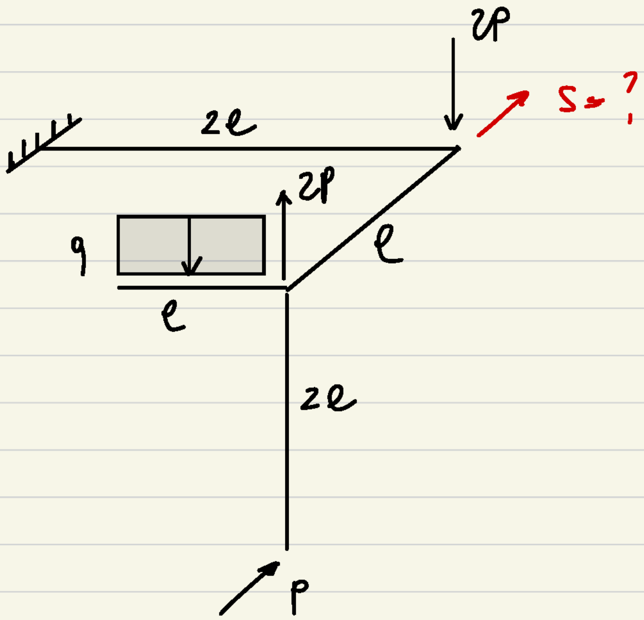


$$M = -EJ\, w_2''(\ell/2) =$$

$$= -EJ\left(2B_2 + 3B_3\ell\right) = EJ\, \frac{4}{5}\, \frac{\hat{\theta}}{\ell}$$

$$= 1.3963 \cdot 10^8 \text{ Nmm}$$

Force-based #6



Use a force-based approach to determine the displacement s. The contribution of shear deformability is negligible.

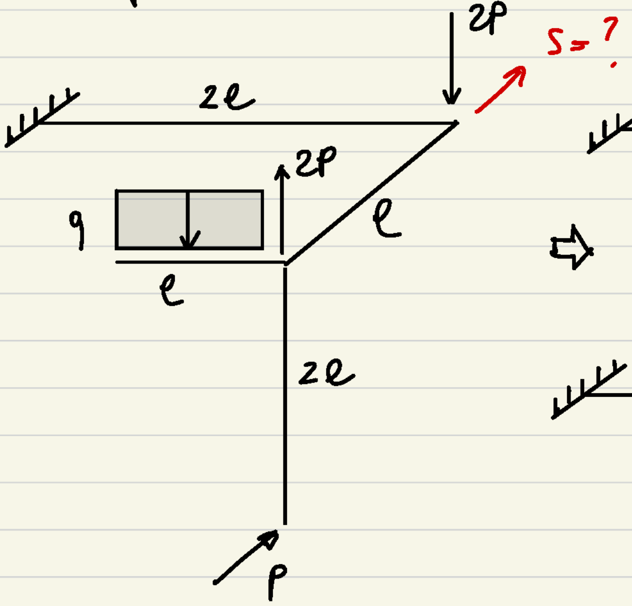
Data

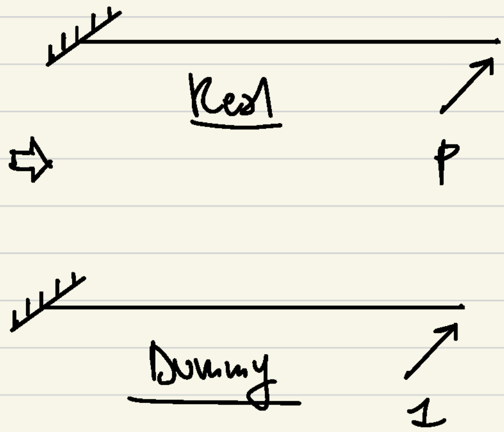
$l = 1200$ mm $q = 2$ N/mm

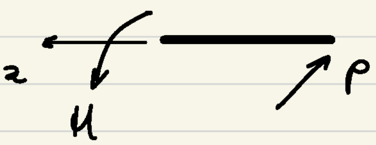
$EJ = 10^{12}$ Nmm² $P = 2000$ N

Solution

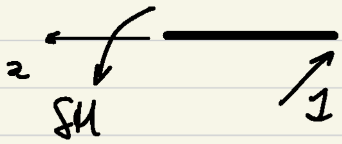
As in the previous exercise, only the real forces producing a not null energy contribution are reported.







$$M = -Px$$



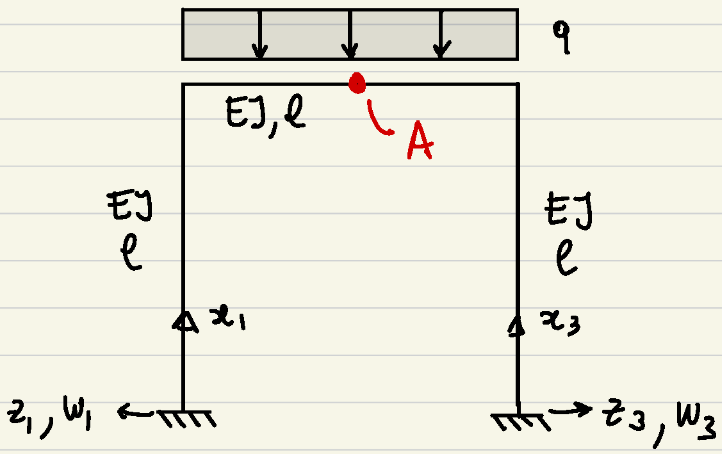
$$\delta M = -x$$

The PCVW reads

$$\int_0^{2\ell} \delta M \, M/EJ \, dx = \delta \quad , \quad \text{and so:}$$

$$\delta = \frac{8}{3} \frac{P\ell^3}{EJ} = 9.22 \text{ mm}$$

Ritz #4



Determine the downward deflection of the point A, at the middle of the horizontal beam.

For this purpose, use the Ritz method where the deflection for beams #1 and #3 is taken as

$w_1 = c\,\phi(x_1)$ and $w_3 = c\,\phi(x_3)$

Note: c is the same owing to the symmetry of the problem.

Formulate the problem using the only dof c and choose between $\phi(x) = x/\ell$ $\quad \phi(x) = \sin\frac{\pi x}{\ell}$

$\phi(x) = (x/\ell)^2$

Neglect the axial energy contributions.

Data

$l = 2000$ mm

$EJ = 10^{12}$ Nmm^2

$q = 50$ N/mm

Solution

The essential conditions of the problem require that:

$w_1(0) = 0$ $\quad\quad$ $w_3(0) = 0$

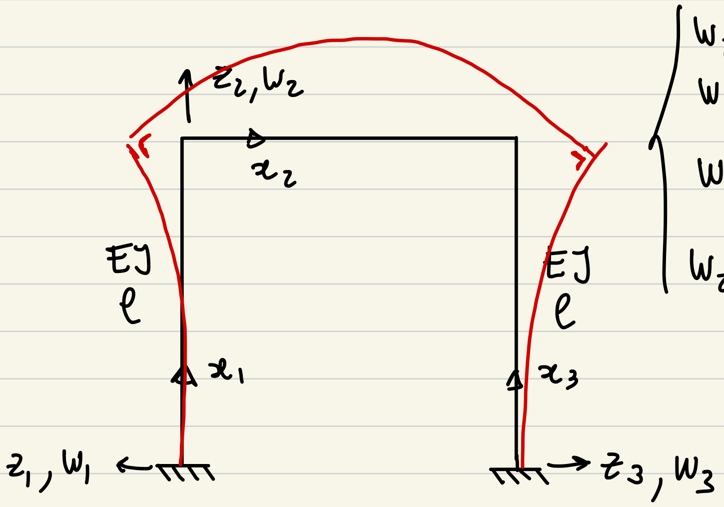
$w_1'(0) = 0$ $\quad\quad$ $w_3'(0) = 0$

It follows that $\phi(x) = \left(x/\ell\right)^2$

In addition, the approximation of the bending displacement in the second beam is taken as:

$$w_2 = a_0 + a_1\, x/\ell + a_2 \left(x/\ell\right)^2$$

Under the additional essential conditions:



$$\begin{cases} w_2(0) = 0 \\ w_2(\ell) = 0 \\ w_2'(0) = w_1'(\ell) \\ w_2'(\ell) = -w_3'(\ell) \end{cases}$$

From which:

$$c\,x^2/\ell^2 \rightarrow c\,\frac{2x}{\ell^2}$$

$$\begin{cases} a_0 = 0 \\ a_1 + a_2 = 0 \\ 1/\ell\, a_1 = 2/\ell\, c \\ 1/\ell\, a_1 + 2/\ell\, a_2 = -2/\ell\, c \end{cases} \Rightarrow \begin{aligned} a_1 &= 2c \\ a_2 &= -2c \end{aligned}$$

(Note, one equation is redundant due to the assumed symmetry of the solution)

The displacements are then approximated as:

$$w_1 = c\left(x_1/\ell\right)^2$$

$$w_3 = c\left(x_3/\ell\right)^2$$

$$w_2 = c\,2\left[x_2/\ell - \left(x_2/\ell\right)^2\right]$$

The PVW reads: $\delta W_i = \delta W_e$ with:

$$\delta W_i = \int_0^\ell \delta w_1'' EJ\, w_1''\, dx_1 + \int_0^\ell \delta w_2'' EJ\, w_2''\, dx_2 + $$

$$+ \int_0^\ell \delta w_3'' EJ\, w_3''\, dx_3$$

$$\delta W_e = -\int_0^{\ell} \delta w_2 \, q \, dx_2$$

Upon substitution of the approximations above, one obtains:

$$\frac{24 EJ}{\ell^3} c = - \frac{q\ell}{3}$$

And so:

$$c = - \frac{q\ell^4}{72 EJ}$$

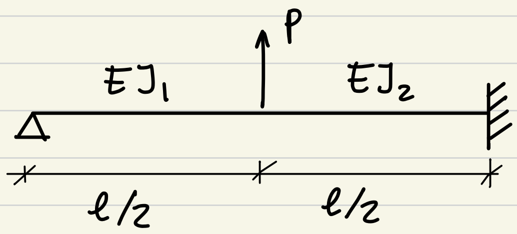
The downward deflection in the point A is then:

$$w_A = -w_2(\ell/2) = -c\,2\left[x_2/\ell - \left(x_2/\ell\right)^2 \right]_{x_2 = \ell/2}$$

$$= -c/2$$

$$= 5.56 \text{ mm}$$

Ritz #5



Determine the vertical displacement in the middle of the beam using the Ritz method. Approximate the displacement field using the simplest polynomial approximation.

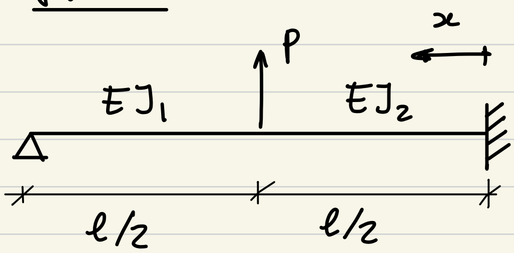
Data

$\ell = 1500\,mm$ $\qquad$ $P = 3000\,N$

$EJ_1 = 10^{10}\,Nmm^2$

$EJ_2 = 3 \cdot 10^{10}\,Nmm^2$

Solution



The essential boundary conditions are:

$$\begin{cases} w(0)=0 \\ w'(0)=0 \\ w(\ell)=0 \end{cases}$$

The simplest polynomial approximation reads:

$$w = a_0 + a_1\left(\frac{x}{\ell}\right) + a_2\left(\frac{x}{\ell}\right)^2 + a_3\left(\frac{x}{\ell}\right)^3$$

And by application of the boundary conditions:

$$\begin{cases} a_0 = 0 \\ a_1 = 0 \\ a_2 + a_3 = 0 \end{cases}$$

And so:

$$w = a_2\left[\left(\frac{x}{\ell}\right)^2 - \left(\frac{x}{\ell}\right)^3\right]$$

The PVW is $\delta W_i = \delta W_e$, with:

$$\delta W_i = \int_0^{\ell/2} \delta w'' EJ_2 w'' dx + \int_{\ell/2}^{\ell} \delta w'' EJ_1 w'' dx$$

$$\delta W_e = \delta w(\ell/2) P$$

From which:

$$\delta a_2 \frac{7EJ_1 + EJ_2}{2\ell^3} = \delta a_2 P/8 \qquad \forall \delta a_2$$

And then:

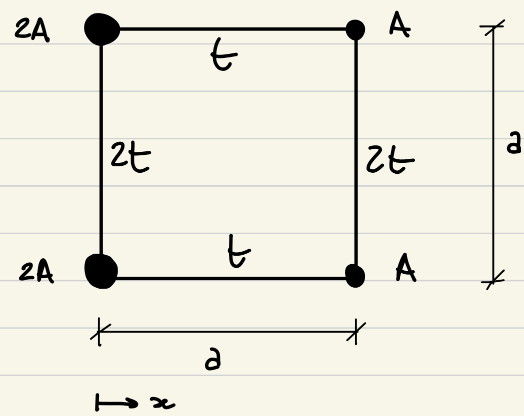
$$\frac{7EJ_1 + EJ_2}{2\ell^3} a_2 = P/8 \Rightarrow a_2 = 25.31 \text{ mm}$$

The displacement in the middle is then:

$$w = a_2 \left[\left(x/\ell\right)^2 - \left(x/\ell\right)^3 \right]\Big|_{x=\ell/2}$$

$$= a_2/8 = 3.16 \text{ mm}$$

Semi # 5

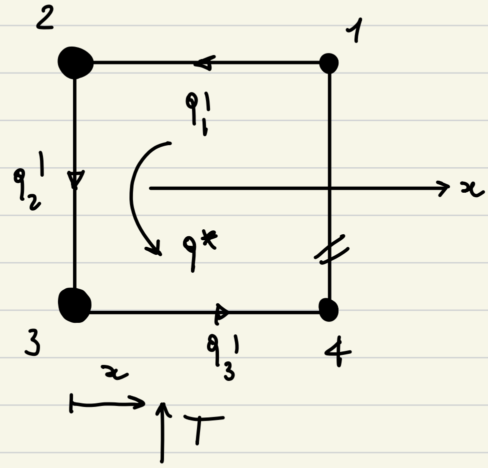


Determine the horizontal position of the shear center (report the coordinate x)

Data

$a = 400$ mm
$t = 1.3$ mm
$A = 800$ mm^2

Solution



The position of the x axis (principal system) is available from the symmetry of the section.

The section properties are:

$$J_{xx} = 6A\left(\frac{a}{2}\right)^2 = \frac{3}{2} A a^2$$

$$S_{x_1}' = \frac{1}{2} A a$$

$$S_{x_2}' = \frac{3}{2} A a$$

$$S_{x_3}' = \frac{1}{2} A a$$

- Shear flow equations

$$q_1' = -T \frac{S_{x_1}'}{J_{xx}} = -\frac{1}{3} T/a$$

$$q_2' = -T \frac{S_{x_2}'}{J_{xx}} = -T/a$$

$$q_3' = -T\frac{S_{x3}'}{J_{xx}} = -\frac{1}{3}T/a$$

- Equivalence with internal moment (refer to stringer 3)

$$2q_1'\Omega_1 + 2q^*\Omega_c = Tx$$

with:

$\Omega_1 = a^2/2$; $\Omega_c = a^2$

So:

$$\boxed{-\frac{1}{3}Ta + 2q^*a^2 = Tx}$$

- Compatibility $(\theta' = 0)$

$$q_1'a + q_2'\frac{a}{2} + q_3'a + q^*(2a + a) = 0$$

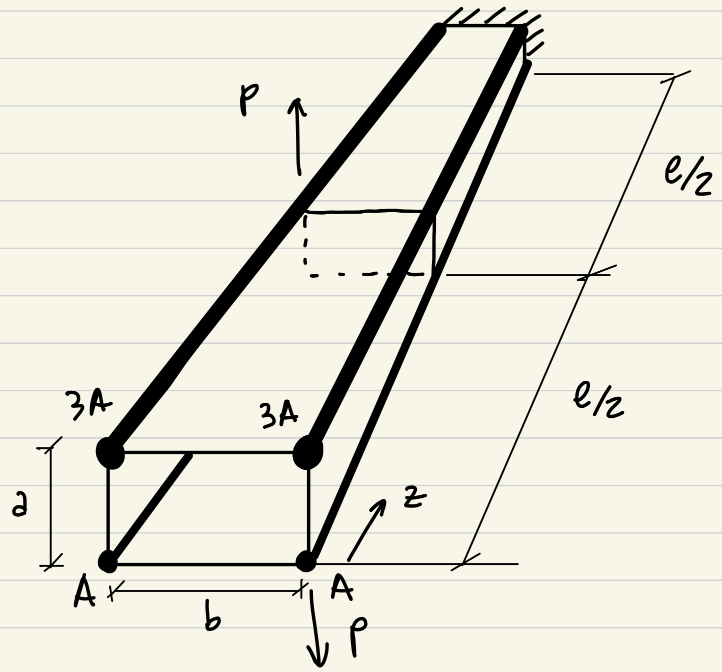
$$-\frac{1}{3}T - \frac{1}{2}T - \frac{1}{3}T + 3a\,q^* = 0$$

$$3a\,q^* = 7/6\,T \Rightarrow q^* = 7/18\,T/a$$

And upon substitution in the equivalence to moment

$$x = 4/9\,a = 177.78 \text{ mm}$$

Semi # 6



Evaluate the axial force carried by the top-right stringer at $z = l/2$

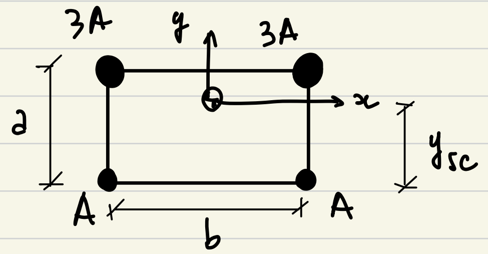
Data

$a = 250$ mm $\quad$ $A = 400$ mm^2

$b = 550$ mm

$l = 3000$ mm $\quad$ $P = 10$ kN

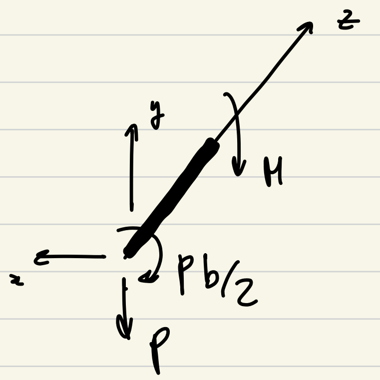
Solution



$$y_{sc} = \frac{6Aa}{8A} = \frac{3}{4}a$$

$$J_{xx} = 6A\left(\frac{1}{4}a\right)^2 + 2A\left(\frac{3}{4}a\right)^2 = \frac{3}{2}Aa^2$$

The internal actions are obtained as:



$M(z) = Pz$

and so:

$M(l/2) = Pl/2$

The stress on stringer 3 is then:

$$\sigma_{zz}(l/2) = \frac{M(l/2)}{J_{xx}}\frac{1}{4}a = \frac{Pl}{12Aa} = 25 \text{ MPa}$$

The axial force is: $N = \sigma_{zz}3A = \dfrac{Pl}{12a} = 70$ kN

- The axial stress of a bent beam is function of the its material elastic modulus
  - False
- When using a displacement-based method the Natural (Newmann) boundary conditions may not be satisfied exactly
  - True
- The cross-sections of a beam subject to a torsional moment do always rotate around the area center
  - False
- The PCVW allows to
  - find the compatible solution among the equilibrated ones
  - find the equilibrated solution among the compatible ones
  - find the compatible and equilibrated solutions among all the possible independent stress and displacement fields
  - none of the above
- Shear deformability needs to be accounted for
  - never
  - always
  - it depends on the beam at hand
- When a torsional moment is applied to a thin-walled beam, without any other load
  - the shear flows are null
  - the torsion is null
  - the torsion is different from zero, but only if the cross-section is free to warp
  - the torsion is different from zero
  - the torsion is different from zero only if the transverse shear deformability is not negligible