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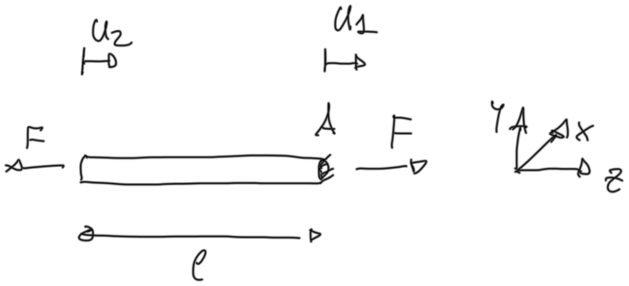
Principle of Virtual Work

Static equilibrium $\Leftrightarrow$ $\delta W_i = \delta W_e$ for any arbitrary virtual compatible displacement δu $\delta \varepsilon$

$\delta W = F \cdot \delta u$

We are looking for the only equilibrated solution among the infinite compatible ones.



TRUSS: it can carry only axial load

linear elastic C.L.

$$\sigma_{zz} = \frac{F}{A} = E \cdot \varepsilon_{zz}$$

$$\varepsilon_{zz} = u_{/z} = \frac{\Delta \ell}{\ell}$$

it holds for infinitesimal displacements

VIRTUAL

$$\delta W_i = \int_V \delta \varepsilon_{zz} \cdot \sigma_{zz} \, dV = \int_0^\ell \int_A \delta \varepsilon_{zz} \cdot \sigma_{zz} \, dA \cdot dz$$

$$= \int_0^\ell \delta \varepsilon_{zz} \cdot \int_A \sigma_{zz} \, dA \cdot dz = \int_0^\ell \delta \varepsilon_{zz} \cdot F \, dz = \underline{\underline{\delta \Delta \ell \cdot F}}$$

$$= \delta (u_1 - u_2) \cdot F$$

it's not the position, it's the displacements

we can describe our truss as

a spring, which kinematics is described by u_1 and u_2

$$F = K(u_1 - u_2) = K \cdot [1 \quad -1] \cdot \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$ ← curly brackets column vector

PVW $\quad \delta W_i = \delta W_e$

REAL

$W_i \qquad W_e$

$$(u_1 - u_2) \cdot F = \Delta\ell \cdot F$$

VIRTUAL

$\delta W_i \qquad \delta W_e$

$$\delta(u_1 - u_2) \cdot F = \delta\Delta\ell \cdot F$$

$$\Delta\ell = u_1 - u_2$$

$$\Delta\ell = [1 \quad -1] \cdot \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \underline{\delta\Delta\ell = [1 \quad -1] \cdot \delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}$$

$$\delta W_i = \int_V \delta \underline{\underline{\varepsilon}} \cdot \underline{\underline{\sigma}} \, dV = \underline{\delta\Delta\ell} \cdot \underline{F} =$$

$$= \underbrace{\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{VIRTUAL}} \cdot \underbrace{K \, [1 \quad -1]}_{\text{REAL}} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} =$$

OUTER SCALAR PRODUCT

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes [1 \quad -1] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \cdot \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \delta W_i$$

↑ stiffness matrix of our spring

$$\delta W_e = \underline{\delta\Delta\ell} \cdot F = \underline{\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}} \cdot F = \delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \cdot \begin{bmatrix} F \\ -F \end{bmatrix}$$

PVW $\delta W_i = \delta W_e$

$$\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \cdot \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \cdot \begin{Bmatrix} F \\ -F \end{Bmatrix}$$

TBN: you cannot eliminate $\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T$ from both sides of the equation:

- → It's mathematically WRONG
- → It's conceptually misleading

What we actually do:

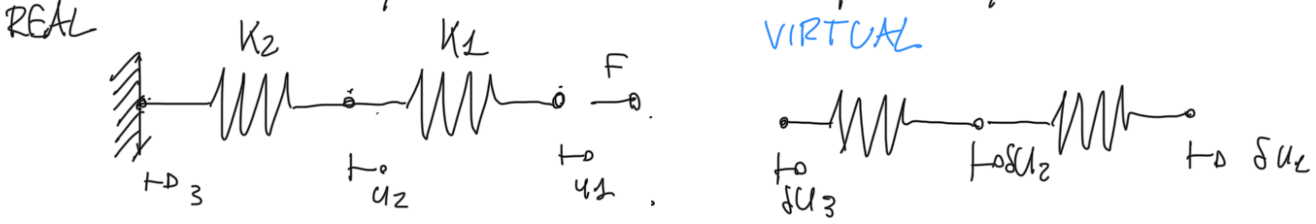
$$\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \left(\begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} F \\ -F \end{Bmatrix} \right) = \emptyset$$

This must be $\emptyset$ for any value of $\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T$

$$\begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} F \\ -F \end{Bmatrix} = 0$$

SPRING CONSTITUTIVE EQUATION

Let's try with a more complex system



PVW $\delta W_i = \delta W_e$

for the first spring $\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

for the second $\begin{Bmatrix} f_2 \\ \ldots \end{Bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ \ldots & \ldots \end{bmatrix} \cdot \begin{Bmatrix} u_2 \\ \ldots \end{Bmatrix}$

spring $\quad \begin{Bmatrix} f_3 \end{Bmatrix} \quad \begin{bmatrix} -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_3 \end{Bmatrix}$

real force

$$\delta W_i = \delta W_{i1} + \delta W_{i2} = \delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \overbrace{\begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix}} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \delta \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}^T \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\delta W_e = \delta \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^T \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

PVW $$\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \delta \begin{Bmatrix} u_2 \\ \cancel{u_3}\,\emptyset \end{Bmatrix}^T \begin{bmatrix} K_2 & \cancel{-K_2} \\ \cancel{-K_2} & \cancel{K_2} \end{bmatrix} \begin{Bmatrix} u_2 \\ \cancel{u_3}\,\emptyset \end{Bmatrix} = \delta \begin{Bmatrix} u_1 \\ \cancel{u_2} \\ \cancel{u_3}\,\emptyset \end{Bmatrix}^T \begin{Bmatrix} f_1 \\ \cancel{f_2}\,\emptyset \\ \cancel{f_3} \end{Bmatrix}$$

let's add the BC

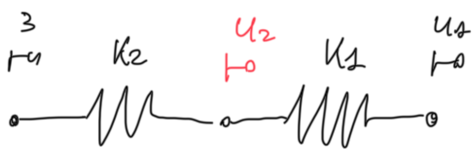
$$u_3 = \emptyset \qquad \delta u_3 = \emptyset$$

$$\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \delta u_2 K_2 u_2 = \delta u_1 \cdot f_1$$

$$\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \overbrace{\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}} - \overbrace{\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \cdot \begin{Bmatrix} f_1 \\ \emptyset \end{Bmatrix}} = 0$$

$$\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \left(\begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} f_1 \\ \emptyset \end{Bmatrix} \right) = \emptyset$$

$$\begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 + K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} f_1 \\ \emptyset \end{Bmatrix} = 0$$



Let's do it in a different way

ASSEMBLY $$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & \emptyset \\ -K_1 & K_1 + K_2 & -K_2 \\ \emptyset & -K_2 & K_2 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

PVW $$\delta \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^T \begin{bmatrix} K_1 & -K_1 & \emptyset \\ -K_1 & K_1 + K_2 & -K_2 \\ \emptyset & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \delta \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}^T \cdot \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

B.C. $\quad \begin{Bmatrix} u_1 \end{Bmatrix}^T \begin{bmatrix} K_1 & -K_1 & \cancel{\emptyset} \end{bmatrix} \begin{Bmatrix} u_1 \end{Bmatrix} \qquad \begin{Bmatrix} u_1 \end{Bmatrix}^T \begin{Bmatrix} F \end{Bmatrix}$

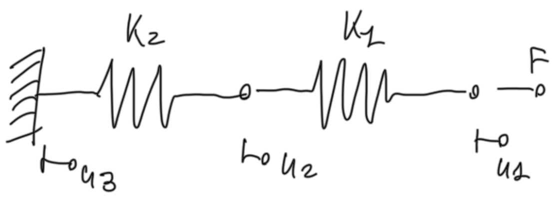
$$\delta \begin{Bmatrix} u_2 \\ \cancel{\emptyset} \end{Bmatrix} \begin{bmatrix} -k_1 & k_1+k_2 & \cancel{-k_2} \\ \cancel{\emptyset} & \cancel{-k_2} & \cancel{k_2} \end{bmatrix} \begin{Bmatrix} u_2 \\ \cancel{\emptyset} \end{Bmatrix} = \delta \begin{Bmatrix} u_2 \\ \cancel{\emptyset} \end{Bmatrix} \cdot \begin{Bmatrix} \cancel{\emptyset} \\ \cancel{f_3} \end{Bmatrix}$$

$$\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \begin{bmatrix} k_1 & -k_1 \\ -k_2 & k_2+k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^T \cdot \begin{Bmatrix} F \\ \emptyset \end{Bmatrix} = \emptyset$$

$$\delta \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \left(\begin{bmatrix} k_1 & -k_1 \\ -k_2 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} F \\ \emptyset \end{Bmatrix} \right) = \emptyset$$

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} F \\ \emptyset \end{Bmatrix} = \emptyset \qquad u_1 = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right)$$

	PVW	PCVW
problem as a function of →	displacements strains	~~forces~~ stresses
virtual →	displacements	forces
enforce A PRIORI →	compatibility Dirichlet/Essential BC	equilibrium Neumann/Natural BC
↓	↓	↓
find a solution →	equilibrated	compatible

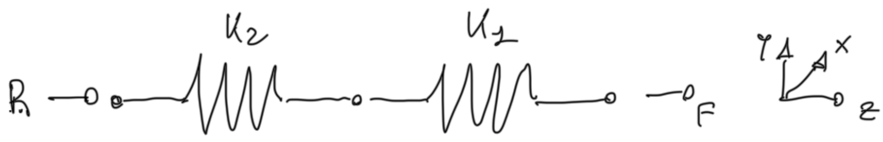


SERIES

the Internal force is constant along the structure

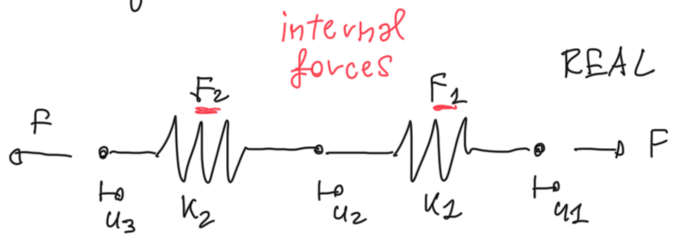
- Is this statically determined? Is this ISOSTATIC?
  YES

If our structure is isostatic, we can solve our elastic problem simply IMPOSING THE EQUILIBRIUM

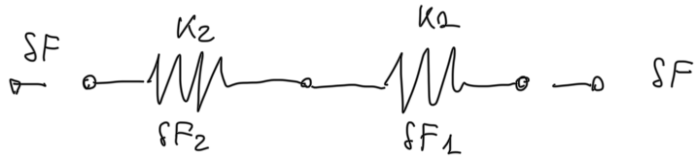


Equilibrium along z $\quad F + R = 0 \quad R = -F$

PCVW



$F = K \cdot \Delta l$



$$\delta W_i = \delta W_{i1} + \delta W_{i2} = \delta F_1 \cdot \Delta l_1 + \delta F_2 \cdot \Delta l_2 =$$

$$= \delta F_1 \cdot \frac{F_1}{K_1} + \delta F_2 \frac{F_2}{K_2} =$$

SERIES

$$\boxed{\begin{matrix} \delta F = \delta F_1 = \delta F_2 \\ F = F_1 = F \end{matrix}}$$

$$= \delta F \left(\frac{F}{K_1} + \frac{F}{K_2} \right)$$

$$\delta W_e = \delta F \cdot u_1 + \underline{\delta F \cdot \cancel{u_3}}^{\,\emptyset}$$

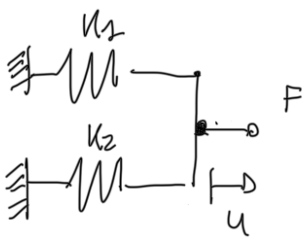
the RF don't produce work

$$= \delta F \cdot u_1$$

PCVW $\quad \delta W_i = \delta W_e \quad \delta F \cdot F \left(\frac{1}{K_1} + \frac{1}{K_2} \right) - u_1 \delta F = 0$

$$\delta F \left(\underline{F \left(\frac{1}{K_1} + \frac{1}{K_2} \right) - u_1} \right) = 0$$

$$\underbrace{u_1 = F \left(\frac{1}{K_1} + \frac{1}{K_2} \right)}$$



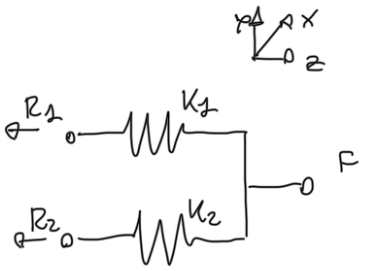
TBN: we have only axial behavior.

PARALLEL $\quad \Delta l = \Delta l_1 = \Delta l_2$

given F, let's find u

- statically determined ? 3 rigid DOF

NO! the system is hyperstatic



Impose the equilibrium along z

$$F - R_1 - R_2 = \emptyset$$

unknowns

First method: PCVW

I

VIRTUAL

$\delta F = 1$

$\mapsto u$

$$\delta F = \cancel{\delta F_1} + \delta F_2$$

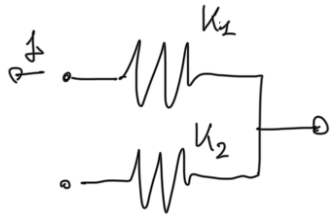
$$\delta W_i = \delta W_{i1} + \delta W_{i2} = \cancel{\delta F_1}^{\,\emptyset} \cdot \frac{F_1}{K_1} + \delta F_2 \cdot \frac{F_2}{K_2}$$

$$\delta W_e = \overset{1}{\delta F} \cdot u = u \quad [N \cdot mm]$$

$$\overset{1}{\delta F_2} \cdot \frac{F_2}{K_2} = u \qquad \frac{F_2}{K_2} = u$$

$\delta F_2 = \delta F$

unknowns



$\delta F = 1$

$$\frac{F_1}{K_1} = u \qquad F_1 = F - F_2$$

$$u = \frac{F_1}{K_1} = \frac{F - F_2}{K_1} = \frac{F_2}{K_2}$$

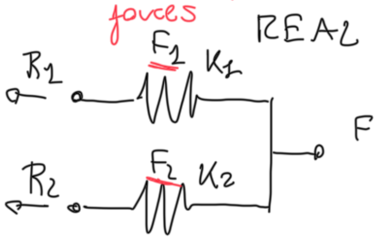
$$K_2 F - K_2 F_2 - F_2 K_1 = \emptyset \qquad F_2 = F \cdot \frac{K_2}{K_1+K_2}$$

$$u = \frac{F_2}{K_2} = \frac{F}{K_1+K_2}$$

II

internal forces

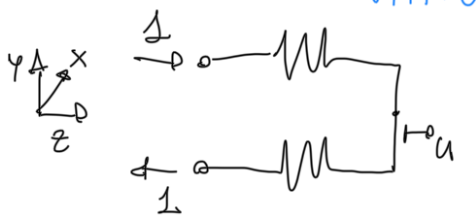
REAL



$$F_1 = F - F_2$$

we can chose a smarter system of EQUILIBRATED FORCES

VIRTUAL



$$\delta W_i = \delta F_2 \cdot \frac{F_2}{K_2} + \delta F_1 \frac{F_1}{K_1}$$

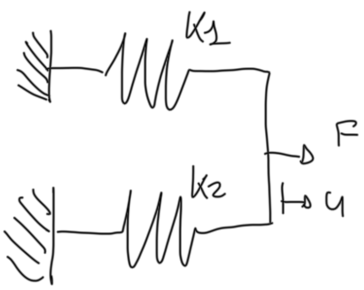
$$= -1 \cdot \frac{F_2}{K_2} + 1 \, \frac{F - F_2}{K_1}$$

$$\delta W_e = u \cdot \emptyset = \emptyset$$

$$-K_1 F_2 + K_2 F - K_2 F_2 = \emptyset \qquad -F_2(K_1+K_2) = -K_2 F$$

$$F_2 = F \cdot \frac{K_2}{K_1+K_2} \qquad u = \frac{F_2}{K_2} = \frac{F}{K_1+K_2}$$

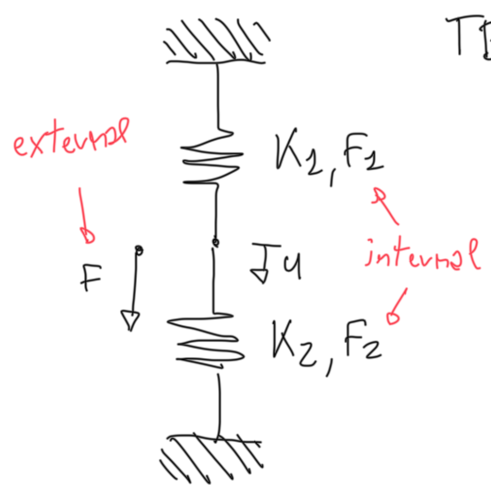
Second Method: PVW



$$\delta u \cdot K_1 u + \delta u K_2 u = \delta u F$$

$$F = (K_1+K_2)u$$

$$u = \frac{F}{K_1+K_2}$$



TBN: this is not a SERIES

$\Rightarrow$ if $K_1 \neq K_2$, then $F_1 \neq F_2$

$F = F_1 + F_2$

I) PVW

$$\delta\begin{Bmatrix}\emptyset\\u\end{Bmatrix}^T\begin{bmatrix}K_1 & -K_1\\-K_1 & K_1\end{bmatrix}\begin{Bmatrix}\emptyset\\u\end{Bmatrix}+\delta\begin{Bmatrix}u\\\emptyset\end{Bmatrix}^T\begin{bmatrix}K_2 & -K_2\\-K_2 & K_2\end{bmatrix}\begin{Bmatrix}u\\\emptyset\end{Bmatrix}=\delta u\cdot F$$

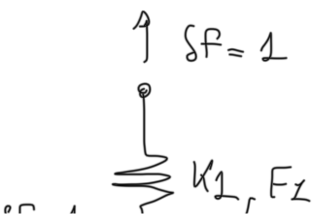
$$\delta u\, K_1 u + \delta u\, K_2 u - \delta u\, F = \emptyset$$

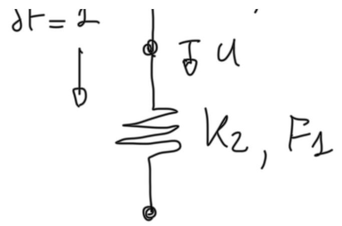
$$(K_1 + K_2)\cdot u = F$$

II PCVW

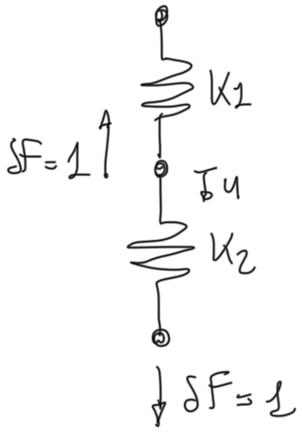
VIRTUAL 1

$$\delta F\cdot \frac{F_1}{K_1} = \delta F\cdot u \qquad u = \frac{F_1}{K_1}$$





VIRTUAL 2



they are both negative

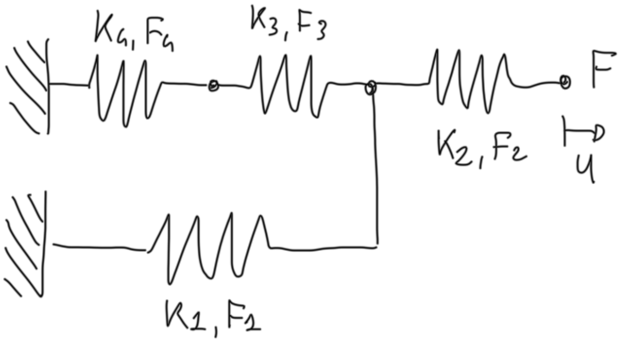
$$\delta F \cdot \frac{F_2}{K_2} = \delta F\, u \qquad u = \frac{F_2}{K_2}$$

$$u = \frac{F_1}{K_1} = \frac{F_2}{K_2} = \frac{F - F_2}{K_1}$$

$$\frac{F_2}{K_2} = \frac{F - F_2}{K_1} \qquad K_1 F_2 = (F - F_2) K_2$$

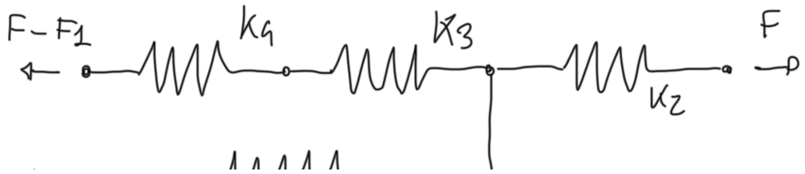
$$F = \frac{(K_1 + K_2)}{K_2} F_2 = \frac{K_1 + K_2}{\cancel{K_2}} \cdot \cancel{K_2} \cdot u$$

$$u (K_1 + K_2) = F$$

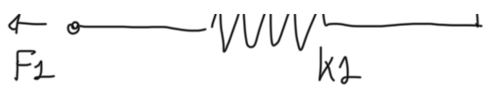


given F, find u

- Is it isostatic ? No, thus it's not statically determined
- Reaction forces

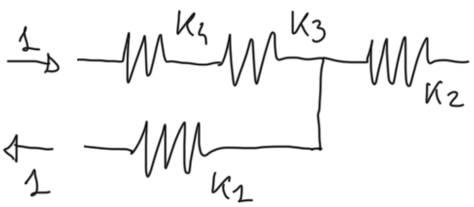


$F_2 = F$
$F_3 = F_4 = F - F_1$



- Let's use PCVW <u>two times</u>:
  - one to find the reaction forces
  - one to find u.

VIRTUAL 1 — as we did before, let's chose a smart system of equilibrated forces

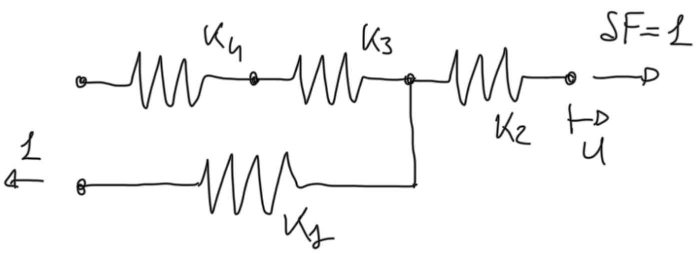


$$\underbrace{1\cdot\frac{F_1}{K_1}}_{\substack{\text{positive:}\\ \text{TENSED}}} - \underbrace{1\cdot\frac{(F-F_1)}{K_3}}_{\substack{\text{negative:}\\ \text{COMPRESSED}}} - \underbrace{1\cdot\frac{(F-F_1)}{K_4}}_{\substack{\text{negative:}\\ \text{COMPRESSED}}} = 1\cdot\varnothing$$

$$K_3K_4\cdot F_1 - K_1K_4(F-F_1) - K_1K_3(F-F_1) = \varnothing$$

$$F_1 = \frac{K_1K_4 + K_1K_3}{K_3K_4 + K_1K_4 + K_1K_3}\cdot F$$ then F_1, F_2, F_3, F_4 are known

VIRTUAL 2 — If we want to find u, let's put $\delta F_1 = 1$ there.



$$1\cdot\frac{F_1}{\cdot} + 1\cdot\frac{F}{\cdot} = 1\cdot u$$

k_1 k_2

$$u = \frac{F_1}{k_1} + \frac{F}{k_2}$$