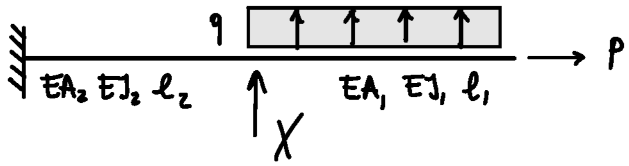
Exercise



Determine the value of the concentrated force X such that the vertical displacement at the corresponding point of application is zero.

Data

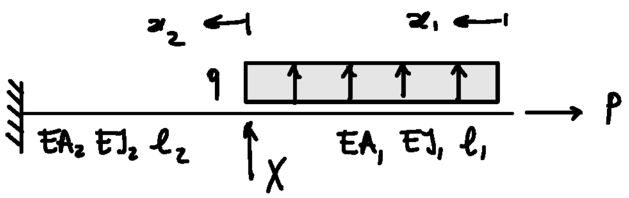
$\ell_1 = 1000\,(1 + B/10)$ mm
$\ell_2 = 1500$ mm

$q = 1$ N/mm
$P = 1000$ N

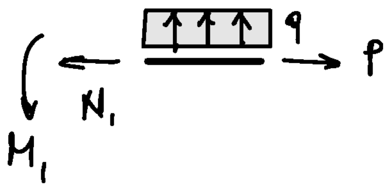
$EA_1 = 1 \cdot 10^6$ N
$EA_2 = 1 \cdot 10^7$ N
$EJ_1 = 1 \cdot 10^{12}$ Nmm^2
$EJ_2 = 1 \cdot 10^{12}$ Nmm^2

Solution

Real system

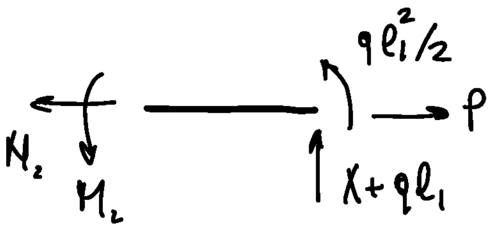


- **Beam 1**



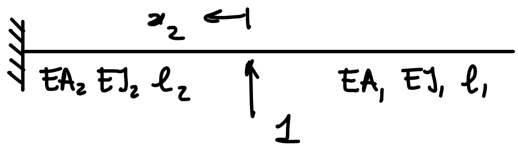
$$N_1 = P$$
$$M_1 = -\, q x_1^2 / 2$$

- **Beam 2**



$$N_2 = P$$
$$M_2 = -\, q l_1^2 / 2 - (X + q l_1)\, x_2$$

Dummy system



- **Beam 2**

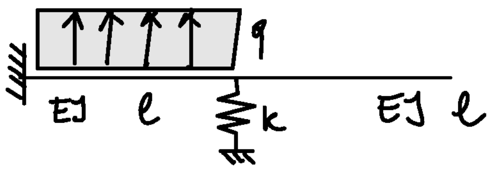
$\delta M = -x_2$

By application of the PCVW:

$$\int_0^{\ell_2} \frac{M\,\delta M}{EJ_2}\,dx_2 = 0$$, from which:

$$X = -\left(q\ell_1 + \tfrac{3}{4}\, q\, \ell_1^2/\ell_2\right) = -1500\ \mathrm{N}$$

Exercise 4



Determine the vertical displ. w in correspondence of the spring

Data

$l = 1000$ mm

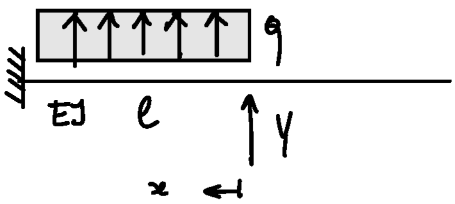
$EJ = 10^{12}$ N mm^2

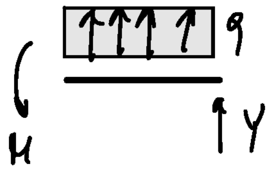
$k = 1500\,(1 + D/10)$ N/mm

$q = 15$ N/mm

Solution

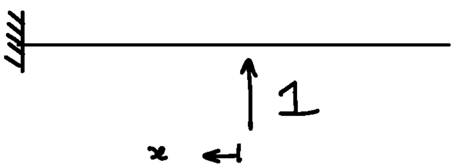
Real system





$$M = -Yx - qx^2/2$$

Dummy system



$$\delta M = -x$$

By application of the PCVW:

$$\int_0^\ell \delta M \, \frac{M}{EJ} \, dx + Y/k = 0$$

From which:

$$\frac{\ell^3}{24EJ}\left(8Y + 3q\ell\right) + Y/k = 0$$

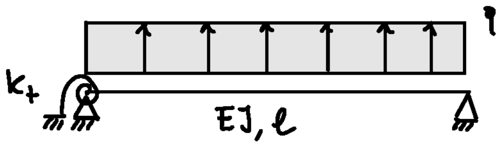
So:

$$Y = -\frac{3kq\ell^4}{8k\ell^3 + 24EJ} = -1875\ N$$

and

$$w = -Y/k = \frac{3q\ell^4}{8k\ell^3 + 24EJ} = 1.25\ mm$$

Exercise 5



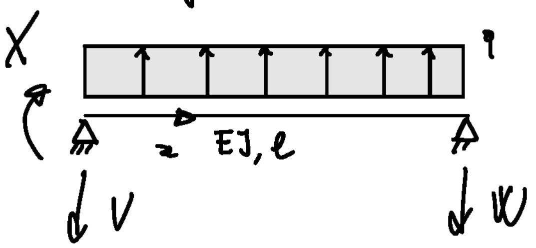
Determine the strain energy stored in the spring of torsional stiffness k_t.

Data

$\ell = 1000$ mm $EJ = 10^{12}$ Nmm^2 $q = 1$ N/mm

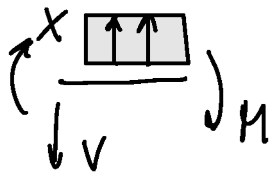
$k_t = 10^9$ Nmm

Real system



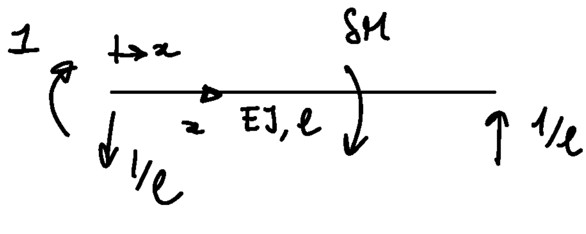
$$\begin{cases} V + W = ql & \Rightarrow V = ql - W \\ X = ql^2/2 - Wl & \Rightarrow W = ql/2 - X/l \end{cases}$$

so: $V = ql - ql/2 + X/l = ql/2 + X/l$



$$M = -X + Vx - qx^2/2$$

Dummy system



$$\delta M = -1 + \frac{1}{l}x$$

By application of the PCVW:

$$\int_0^l \delta M\, M/_{EJ}\, dx + X/_{k_t} = 0$$

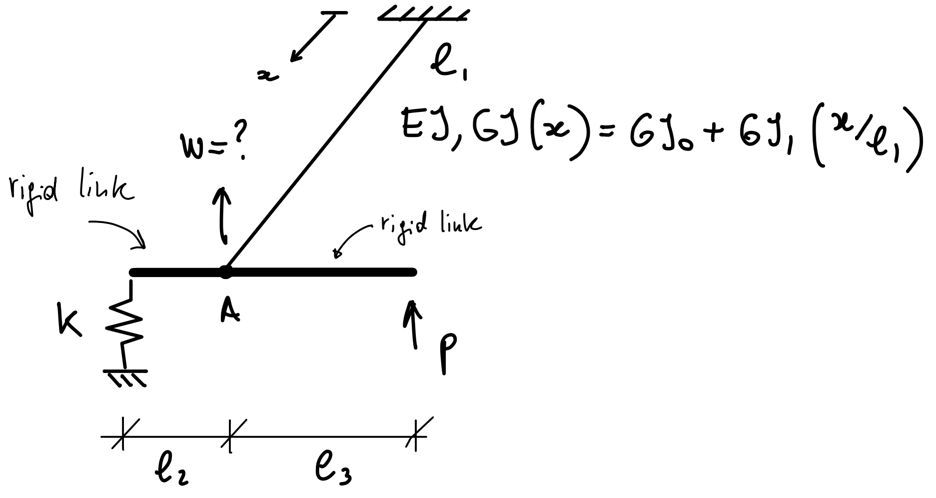
From which:

$$X = k_t \frac{q l^3}{24 EJ + 8 k_t l} = 31250 \text{ Nmm}$$

And so:

$$U = \frac{1}{2} X^2/_{k_t} = 0.4883 \text{ Nmm}$$

Exercise 9



Determine the vertical displacement at the point A using the simplest polynomial approximation and the Ritz method

Data

$\ell_1 = 1300$ mm

$\ell_2 = 500$ mm

$\ell_3 = 700$ mm

$EJ = 10^{12}$ Nmm^2

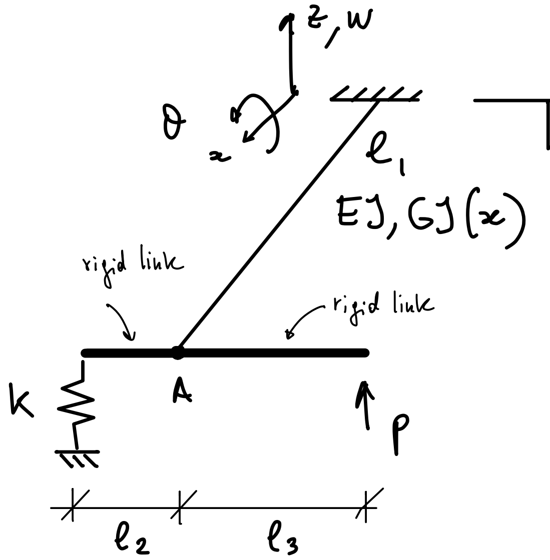
$GJ_0 = 2 \cdot 10^{11}$ Nmm^2

$GJ_1 = 5 \cdot 10^{11}$ Nmm^2

$k = 500 \, (1 + H/10)$ N/mm

$P = 7000$ N

Solution



$$\theta(0)=0$$
$$w(0)=0$$
$$w_{/x}(0)=0$$

Essential boundary conditions

$$GJ(x) = GJ_0 + GJ_1\, x/\ell_1$$

The simplest polynomial trial functions satisfying the essential boundary conditions are found as:

$$\theta(x) = c_1 \left(\frac{x}{\ell_1}\right) \quad \Rightarrow \theta' = c_1\, 1/\ell_1$$

$$w(x) = d_1 \left(\frac{x}{\ell_1}\right)^2 \quad \Rightarrow w'' = d_1\, 2/\ell_1^2$$

The Principle of Virtual Works reads:

$$\delta W_i = \int_0^{\ell_1} \left(\delta w'' \, EJ \, w'' + \delta\theta' \, GJ(x) \, \theta' \right) dx$$

$$+ \, \delta\left(w(\ell_1) - \theta(\ell_1)\ell_2 \right) k \left(w(\ell_1) - \theta(\ell_1)\ell_2 \right)$$

$$\delta W_e = \delta\left(w(\ell_1) + \theta(\ell_1)\ell_3 \right) P$$

From which:

$$\underline{\underline{K}} = \begin{bmatrix} k\ell_2^2 + \left(\dfrac{2\,GJ_0 + GJ_1}{2\ell_1} \right) & -k\ell_2 \\ -k\ell_2 & k + \dfrac{4EJ}{\ell_1^3} \end{bmatrix}$$

$$\underline{F} = \begin{Bmatrix} \ell_3 \\ 1 \end{Bmatrix} P \quad , \quad \text{so:}$$

$$\underline{\underline{K}} \begin{Bmatrix} c_1 \\ d_1 \end{Bmatrix} = \underline{F}$$

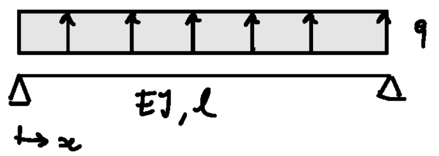
and $\quad c_1 = 0.0127 \quad rad$

$d_1 = 4.3875 \quad mm$

The vertical displacement at $x = \ell_1$ reads:

$w(\ell_1) = d_1 = 4.3875 \quad mm$

Exercise 14



Determine the vertical displacement at the middle using a Ritz approximation with 1 trigonometric trial function

Recall the integration rule

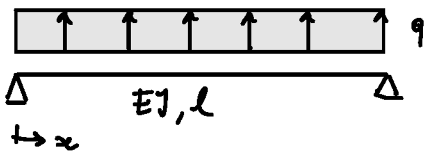
$$\int_0^a \left(\sin\frac{\pi x}{a}\right)^3 dx = \frac{4a}{3\pi}$$

Data

$l = 1700\,(1 + C/10)$ mm $\qquad q = 15$ N/mm

$EJ_0 = 10^{12}$ N mm^2

Solution



The essential conditions are:

$w(0)=0$
$w(l)=0$

And the natural ones:

$w''(0)=0$
$w''(l)=0$

A sine-type expansion satisfies both essential and natural conditions. Assuming a single-dof approximation:

$$w(x) = a \sin \frac{\pi x}{l}$$

And, by application of the PVW:

$$\int_0^l \delta w'' \, EJ(x) \, w'' \, dx = \int_0^l \delta w \, q \, dx$$

From which:

$ka = f$ where:

$k = \frac{4}{3} EJ_0 \left(\pi/\ell\right)^3$

$f = q\, 2\ell/\pi$

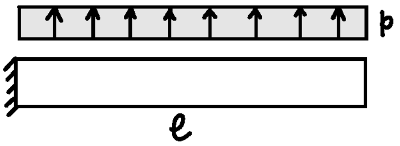
So:

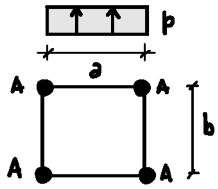
$a = f/k = 1.9292$ mm

And so:

$$w_{mid} = a \sin \frac{\pi x}{\ell} \Big|_{x=\ell/2} = 1.9292 \text{ mm}$$

Exercise 22





Determine A such that the maximum displacement is $u = 3$ mm

Data

$\ell = 3000\,(1 + A/10)$ mm

$a = 200$ mm

$b = 150$ mm

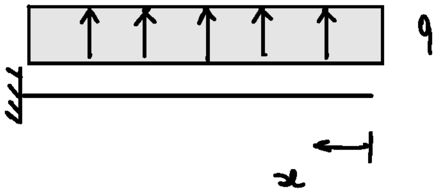
$t = 1.2$ mm

$p = 1.6 \cdot 10^{-3}$ N/mm^2

$E = 72\,000$ MPa

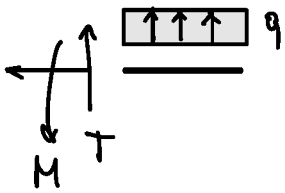
$G = 27000$ MPa

Solution



$q = pa$

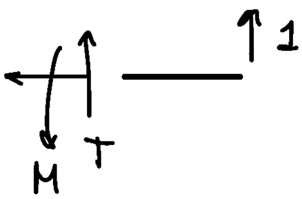
Real system



$T = -\,qx$

$M = -\,qx^2/2$

Dummy system



$\delta T = -1$

$\delta M = -x$

By application of the PCVW:

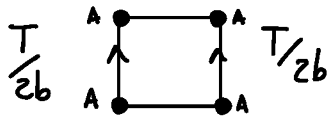
$$\int_0^{l} \left(\delta M \, M/EJ + \delta T \, T/GA^* \right) dx = u \quad , \text{ so:}$$

$$u = \frac{q l^4}{8} \frac{1}{EJ} + \frac{q l^2}{2} \frac{1}{GA^*}$$

But

$$EJ = EA\left(\frac{b}{2}\right)^2 4 = EAb^2$$

$$GA^* = G\,\frac{T^2}{\Sigma_i\, q_i^2\, l_i/t_i}$$



$$= G\;2bt$$ (Note, $2bt$ is the total area of the vertical webs)

So:

$$u = \frac{ql^4}{8}\frac{1}{EJ} + \frac{ql^2}{2}\frac{1}{GA^*}$$

$$\frac{ql^4}{8}\frac{1}{EAb^2} = S - \frac{ql^2}{4Gbt} \quad , \text{ So:}$$

$$A = \frac{ql^4}{8Eb^2}\frac{1}{\left(u - \frac{ql^2}{4Gbt}\right)} = 701.3 \text{ mm}^2$$

- Consider a cantilever beam modeled according to Euler-Bernoulli and loaded with a uniformly distributed load. The exact solution is
  - polynomial (quartic)
- A plane-strain constitutive law:
  - has null axial strain
- A two-cell section modeled according to the semi-monocoque scheme can be solved by using:
  - shear flow equations, equivalence to internal moment and the compatibility equation
- According to the Kirchhoff plate model, deformed sections remain normal to the reference surface
  - True
- The exact solution of the elasticity problem satisfies both natural and essential boundary conditions
  - True
- According to the semi-monocoque scheme, an open-section profile has no torsional stiffness
  - True