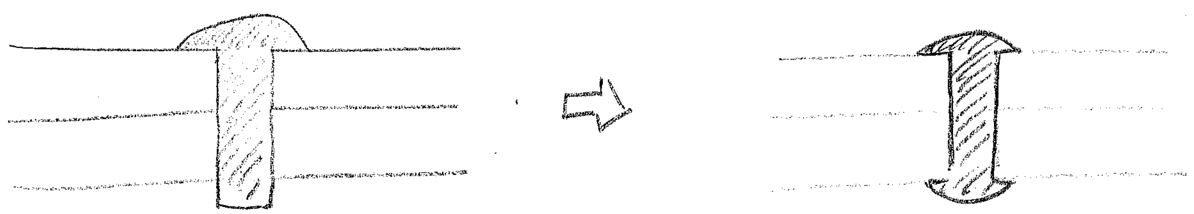
Junctions

The various components of a typical light-weight structure – panels, stringers, longerons, ribs and frames – are connected each other by means of junctions.
A common classification divides the junctions into:

1. continuous junctions (welding, bonding)
2. discontinuous junctions (bolts, rivets)

Discontinuous junctions

Are widely used both in metal and composite structures (although their latter are particularly affected by the interruption of the fibers).
In the aerospace field, the rivet is a widely used kind of discontinuous connection.

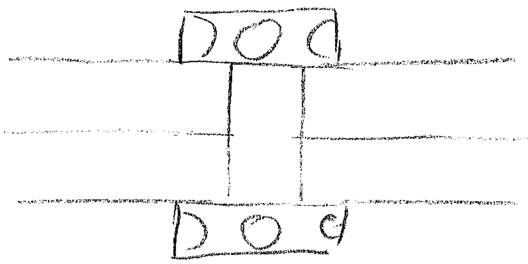


The rivet is inserted and then, by compression, it is created the second "head" of the rivet.

Fatigue problems (potential issues associated with repeated cycles of loads) may affect the region close to the hole, where the concentration of stresses can determine the onset and propagation of cracks

With this regard, a useful strategy consists in adopting a rivet with a diameter slightly larger in comparison to the hole's diameter. The result is a state of internal compression which mitigates the stress concentration around the hole and improve the fatigue life of the part.

Another common solution is constituted by bolts.

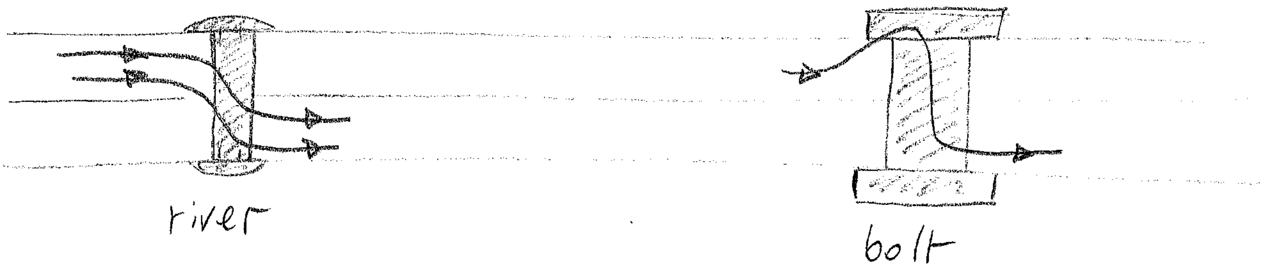


Bolts are generally used

1. when it is necessary to guarantee the possibility of disassemblying the part
2. in those areas characterized by high traction loads, where fatigue is a crucial issue (e.g. the bottom of a wing structure)

At the same time, it should be highlighted that they are a relatively heavy junction and, in general, do not offer good performance when loaded in shear

Overall, the underlying load transfer mechanisms in rivets and bolts is inherently different.



Indeed rivets work mainly in shear, whereas bolt are associate with a axial load transfer mechanism.
The two solutions can be compared by illustrating, in a simplified manner, the loads transmitted by the junction itself

<u>Rivet:</u> $\tau_y = \dfrac{\sigma_y}{\sqrt{3}} = \dfrac{T^{rivet}_{max}}{A} \Rightarrow T^{rivet}_{max} = \dfrac{\sigma_y A}{\sqrt{3}} \simeq 0.58\, \sigma_y A$

<u>Bolt</u> $T^{bolt}_{max} = \mu F_{max}$ with $F_{max} = \sigma_y A$

$T^{bolt}_{max} = \mu \sigma_y A \qquad \mu \simeq 0.1$

(The rivet transfers the shear load working in shear. The maximum stress τ can be taken as the yielding stress τ_y which, according to von Mises is $\sigma_y / \sqrt{3}$;
The bolt transfers load via friction, leading to a shear force T^{bolt}_{max} which is obtained by application of the Coulomb's law for friction, where μ is the coefficient for friction)

It is then clear that, for a given amount of area,

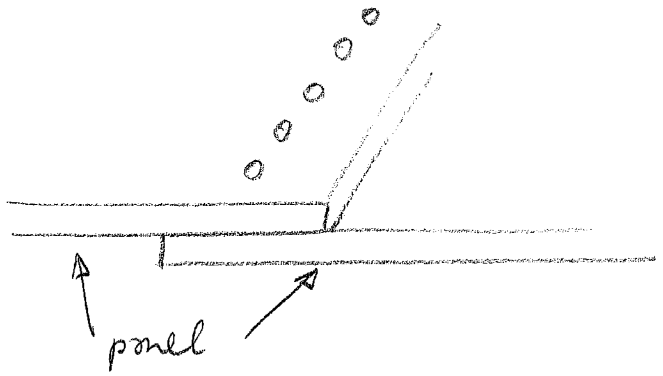
$$\tau_{max}^{rivet} = 0.58\,\sigma_y\,A \quad > \quad \tau_{max}^{bolt} = 0.10\,\sigma_y\,A$$

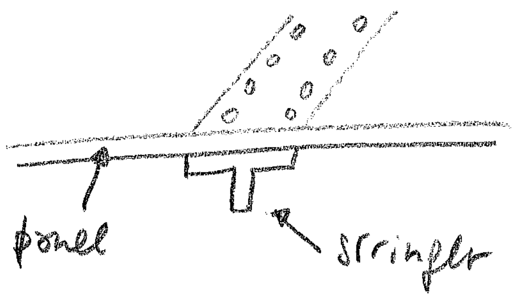
thus the rivets can transfer a higher shear force per unit of area. (and of weight)

Kind of junctions and their modeling

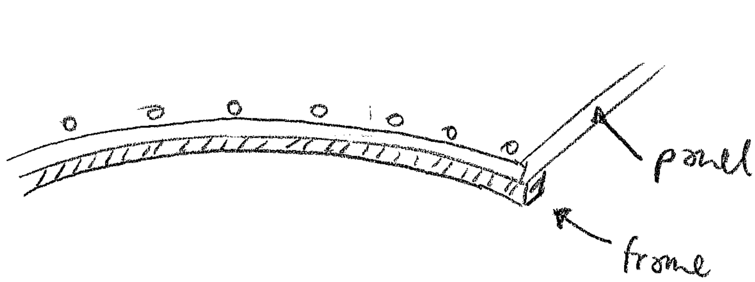
The junctions may regard the connection of

1. Panel - panel
   panel - stringer } longitudinal



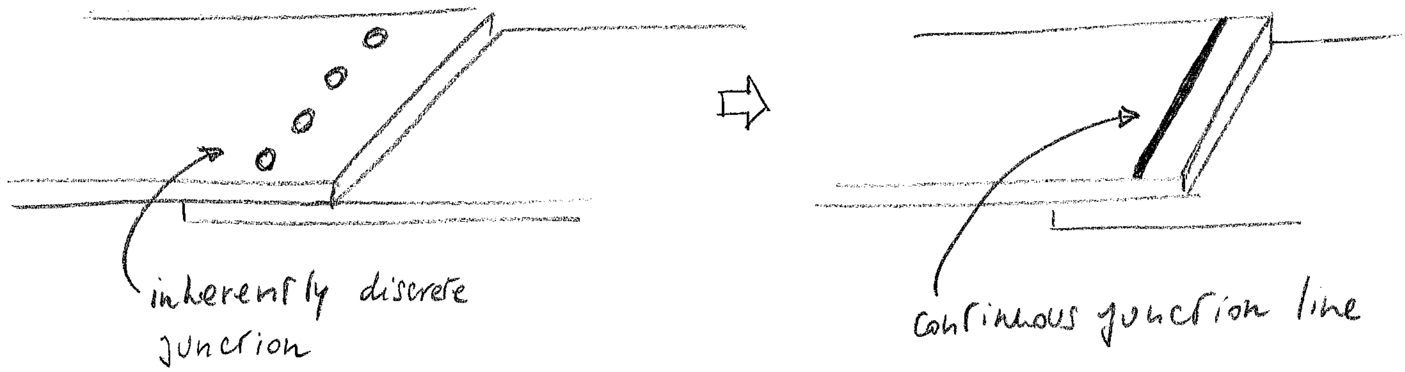


2. Panel - rib
   Panel - frame } transversal



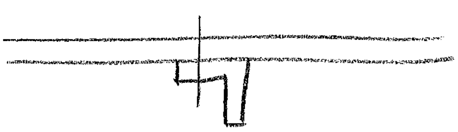
Within the modeling assumptions here introduced the junctions are represented as continuous lines responsible for the load transfer between the two connected parts

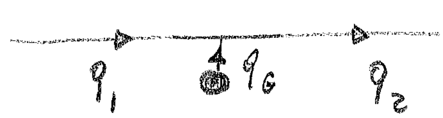
Example:



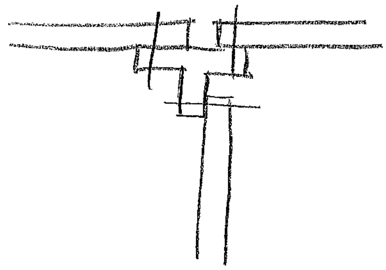
The conceptual representation of the junction can refer to the semi-monocoque description based on lumped areas, thus facilitating the understanding of how the shear flows run inside the structure and the junction.

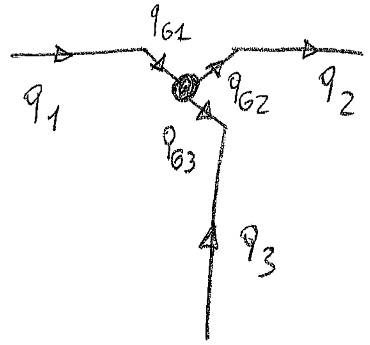
Example





$$q_2 = q_1 + q_G$$

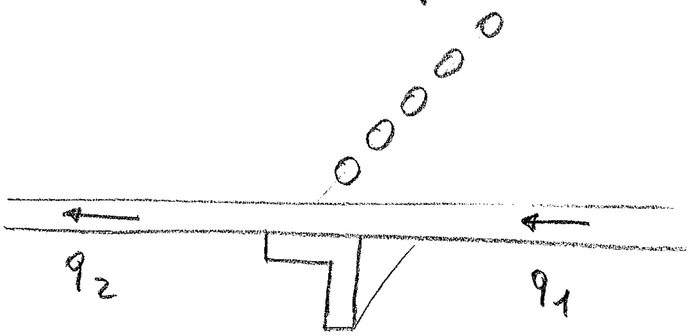


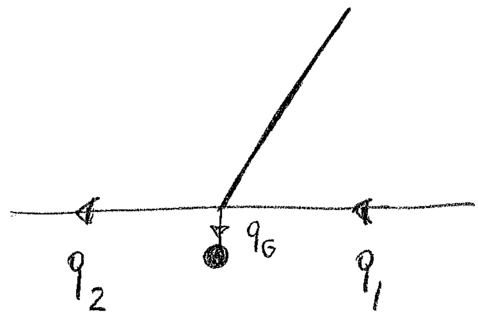


$$q_{G1} = q_1$$
$$q_{G2} = q_2$$
$$q_{G3} = q_3$$

Evaluation of the forces over the element of a junction

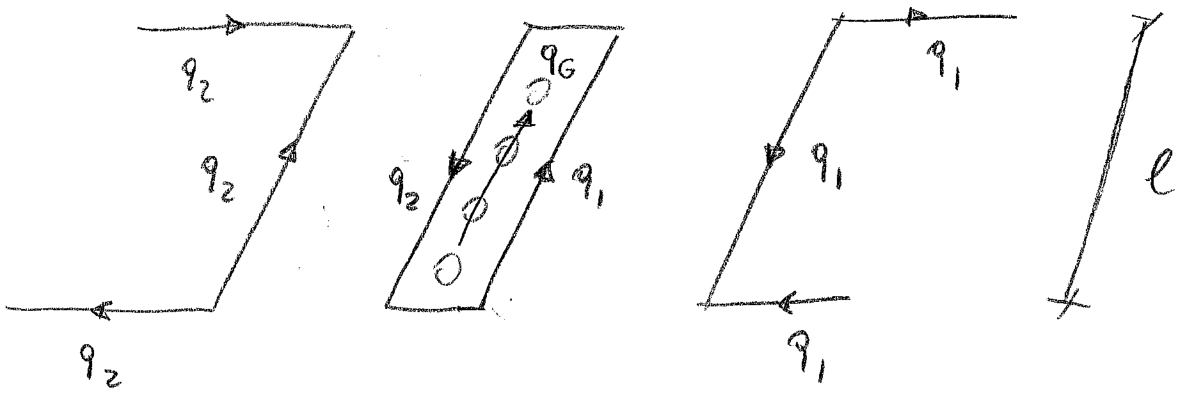
Consider the case of a stringer connected to a panel by means of one single row of rivets





(idealization)

By separating the two portions of panel and isolating the junction line, it is obtained

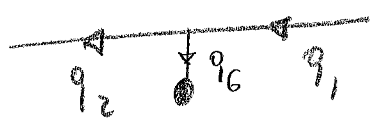


The flow along the junction q_G can then be seen as:

$$q_G = q_1 - q_2$$

(note, this is an equivalence condition)

which exactly what is intended by the idealized scheme

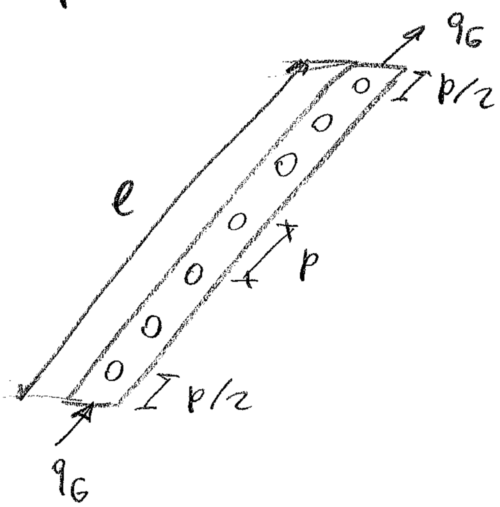


To evaluate the amount of shear force transmitted by each single rivet, define:

N = number of rivets

p = pitch of rivets

F = force transmitted by single rivets



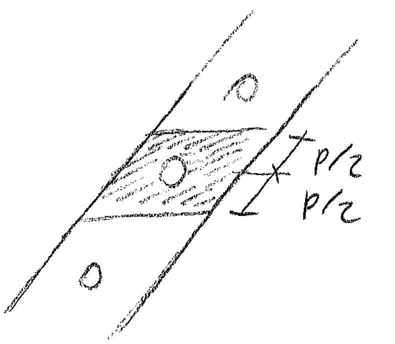
$$q_G l = NF$$

but $N = l/p$

so:

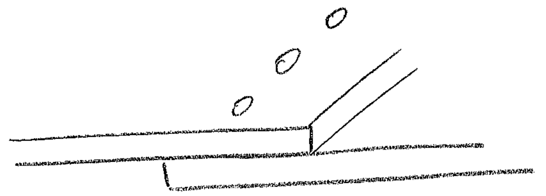
$$q_G l = \frac{l}{p} F \Rightarrow \boxed{F = p q_G}$$

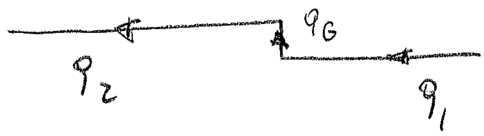
This result illustrates that each rivet is responsible for the transfer of the shear flow associated with the region highlighted below



$$\Rightarrow F = \left(\frac{p}{2} + \frac{p}{2}\right) q_G$$

Analysis of panel-panel connection with 1 row of rivets





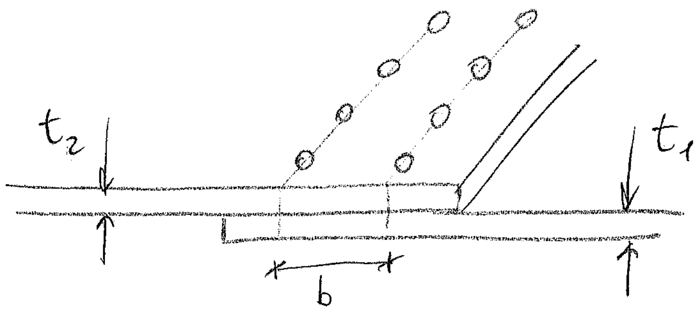
In this simple case $q_1 = q_2 = q_G = q$

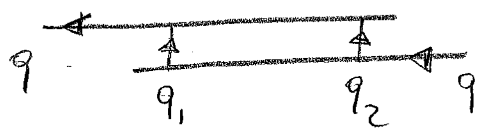
Known the shear flow q (which is equal to q_1 and q_2) the value of q_G is readily available as $q_G = q$.

The force transmitted by the rivet will be determined as

$F = qp$.

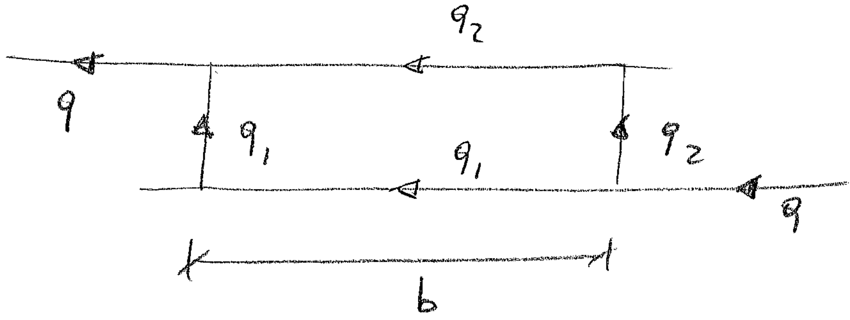
Analysis of panel-panel connection with 2 rows of rivets





- The solution of the problem is trivial if the two panels have equal thickness $t_1 = t_2$. Indeed by symmetry: $q_1 = q_2$ and, as far as $q_1 + q_2 = q$ it follows that $\boxed{q_1 = q_2 = q/2}$

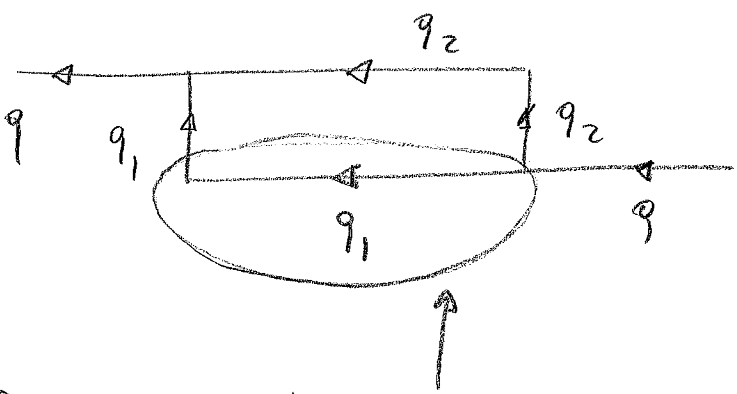
- The solution is not trivial if $t_1 \neq t_2$ and has to be calculated.



The problem is formulated within the framework of the force-based approach.
In this case, the equilibrium condition reads:

$$\boxed{q_1 + q_2 = q} \quad \text{equilibrium}$$

It follows that the problem is <u>statically indeterminate</u> as the evaluation of the two unknowns q_1 and q_2 requires an additional equation, which cannot be obtained from the equilibrium requirements.
As usual, the additional condition should represent the <u>compatibility</u> and, as such, can be obtained using the <u>PCVW</u> or <u>Menabrea's</u> theorem.



Consider the highlighted portion

$$U^* = \sum_{i=1}^{2} \left(U^*_{panel_i} + U^*_{rivet_i} \right)$$

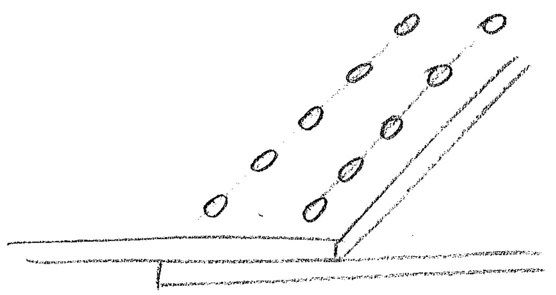
Contribution of panels

Contribution of rivets

The density of strain energy reads:

$$u^*_{panel} = \frac{1}{2}\tau_1\gamma_1 = \frac{1}{2}\frac{q_1}{t}\frac{q_1}{Gt} = \frac{1}{2}\frac{q_1^2}{Gt^2}$$

Recall now that Menabrea's theorem aims at minimizing the complementary energy of the structure:

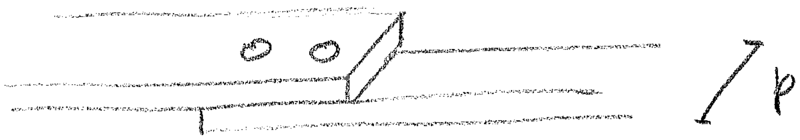


full structure

$$U^*_{panel} = \int_V u^*_{panel}\, dV = N U^*_{panel,unit}$$

Clearly the overall energy stored in the structure is a multiple of the strain energy stored in a repeating unit of the structure

repeating unit

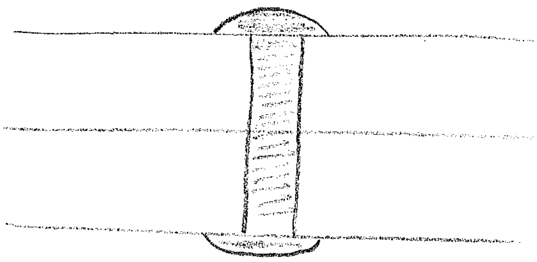


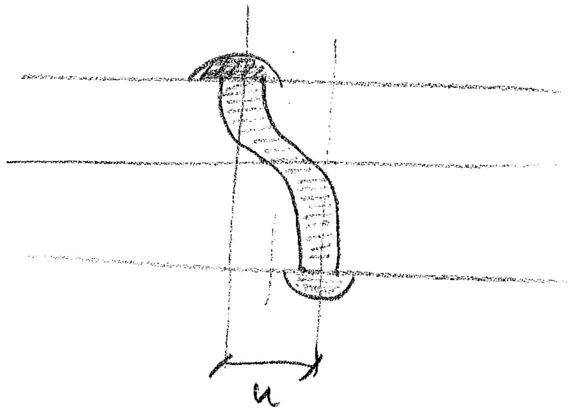
The strain energy of the generic panel of the repeating unit is:

$$U^*_{panel,unit} = \int_V u^*\, dV = \int_V \frac{1}{2}\frac{q^2}{Gt^2}\, dV = \frac{1}{2}\frac{q^2}{Gt^2}\, btp$$

$$\boxed{U^*_{panel,unit} = \frac{1}{2}\frac{q^2 bp}{Gt}}$$

The contribution associated with the rivets deformability can be determined as





$$U^*_{rivet} = \frac{1}{2} k u^2 \qquad \text{but} \quad u = F/k$$

$$= \frac{1}{2} \frac{F^2}{k}$$

F = shear force transmitted by the rivet.

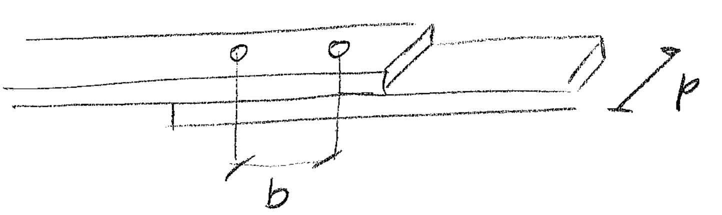
k = rivet stiffness

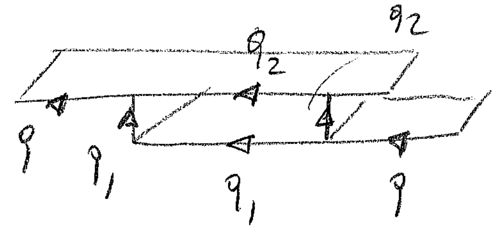
Recalling that the force transmitted by a rivet is

$F = q_G \, p$ it follows that

$$\boxed{U^*_{rivet} = \frac{1}{2} \frac{q_G^2 \, p^2}{k}}$$

At this point, it is possible to apply Menabrea's theorem to the problem under investigation





$$\boxed{U^* = \underbrace{\frac{1}{2}\frac{q_1^2 b p}{Gt} + \frac{1}{2}\frac{q_2^2 b p}{Gt}}_{\text{panels' contrib.}} + \underbrace{\frac{1}{2}\frac{q_1^2 p^2}{k} + \frac{1}{2}\frac{q_2^2 p^2}{k}}_{\text{rivets' contrib.}} + U^*(q)}$$

Note: the contribution $U^*(q)$ is due to the outer portions of the structure. This contribution does not depend up q_1 and q_2 thus is not contributing to the compatibility conditions.

Recalling the equilibrium condition $\boxed{q = q_2 + q_1}$

or $q_2 = q - q_1$

the energy U^* can then be expressed as function of q_1 only by replacing q_2 with $q - q_1$:

$$U^* = \frac{1}{2}\frac{q_1^2 bp}{Gt_1} + \frac{1}{2}\frac{(q_1^2 - 2qq_1 + q^2)}{Gt_2}bp + \frac{1}{2}\frac{q_1^2 p^2}{k} +$$

$$+ \frac{1}{2}\frac{(q_1^2 - 2qq_1 + q^2)p^2}{k} + U^*(q)$$

Again, the contributions in q do not contribute to the compatibility equations, so they can be neglected and collected in $U^*(q)$

$$U^* = \frac{1}{2}\frac{bp}{Gt_1}\left(q_1^2 + \frac{t_1}{t_2}\left(q_1^2 - 2qq_1\right)\right) +$$

$$+\frac{1}{2}\frac{bp}{Gt_1}\left(\frac{Gt_1 p}{kb}\left(2q_1^2 - 2qq_1\right)\right) + U^*(q)$$

Define now the nondimensional parameters

$$\boxed{\alpha = \frac{Gt_1 p}{kb} \qquad \beta = t_1/t_2}$$

α is a ratio between the stiffness of the panel and the stiffness of the rivet

and so:

$$U^* = \frac{1}{2}\frac{bp}{Gt_1}\left(q_1^2 + \beta\left(q_1^2 - 2qq_1\right)\right) +$$

$$\frac{1}{2}\frac{bp}{Gt_1}\left(\alpha\left(2q_1^2 - 2qq_1\right)\right) + U^*(q)$$

The stationarity condition can be imposed by setting:

$$\boxed{\frac{\partial U^*}{\partial q_1} = 0}$$

$$\frac{\partial U^*}{\partial q_1} = q_1 + \beta(q_1 - q) + 2q_1\alpha - q\alpha = 0$$

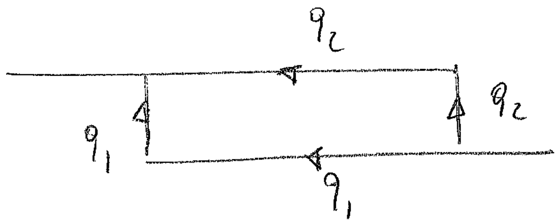
$$q_1(1+\beta+2\alpha) = q(\alpha+\beta)$$

$$\boxed{q_1 = \frac{\alpha+\beta}{1+\beta+2\alpha}\, q}$$

Clearly if $t_1 = t_2$, i.e. $\beta = 1$, the shear flow is evenly distributed between the two rivets.

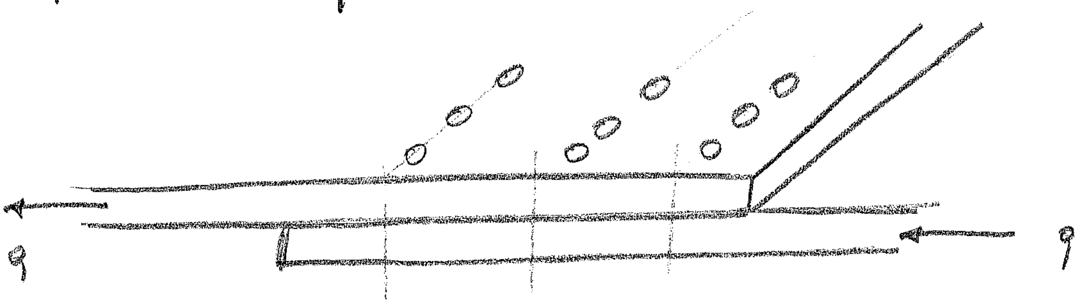
$$q_1 = \frac{\alpha + 1}{2\alpha + 2} q = \frac{1}{2} q$$

Whenever $t_1 \neq t_2$ the shear flow is unevenly distributed between the two rivets. In particular, higher values of β $(t_2 > t_1)$ will lead to $q_2 > q_1$

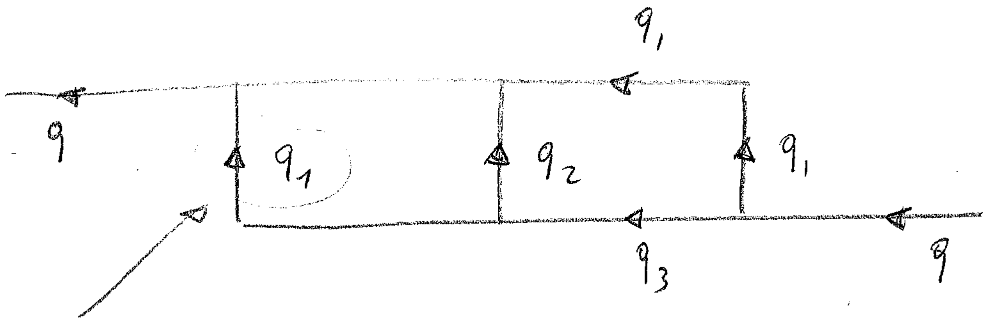


Analysis of panel-panel connections with 3 rows of rivets

The analysis can be easily extended to consider the case of a junction with 3 lines of rivets. For simplicity the analysis is here restricted to the case of panels of equal thickness.

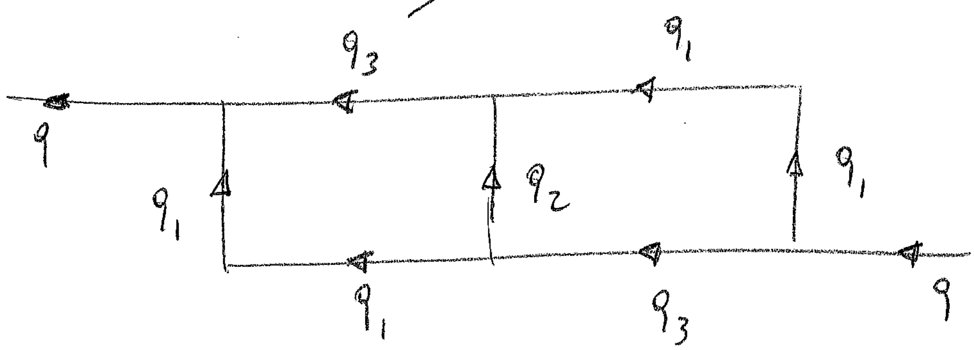


Or, schematically:

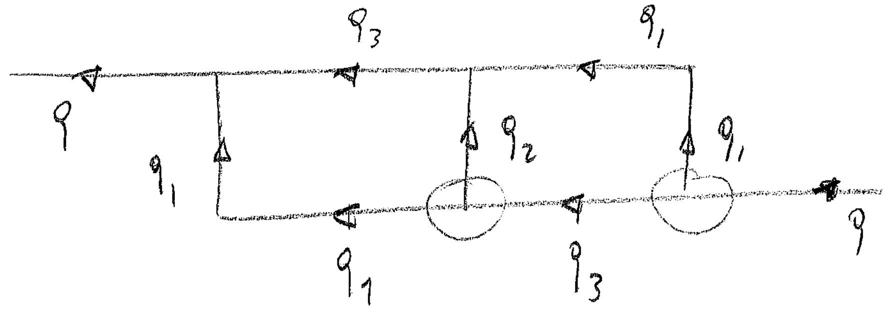


from the symmetry

and so: having recalled that $q_3 = q - q_1$



Write now the equilibrium conditions:



Equilibrium conditions

$$\begin{cases} q_1 + q_3 = q \\ q_2 + q_1 = q_3 \end{cases}$$

3 unknowns (q_i $i = 1,2,3$)
2 equations

The third equation is available by invoking the compatibility of the solution.

$$U^* = \underbrace{\frac{1}{2}\frac{(2q_1^2 + 2q_3^2)\,bp}{Gt}}_{\text{contribution of the four panels}} + \underbrace{\frac{1}{2}\frac{2q_1^2 p^2}{k}}_{\text{contrib. of two outer rivets}} + \underbrace{\frac{1}{2}\frac{q_2^2 p^2}{k}}_{\text{contrib. of middle rivet}} + U^*(q)$$

The functional U^* can be expressed as function of one single unknown (arbitrary), by using the two equilibrium conditions. Consider q_1:

$$\boxed{\begin{aligned} q_3 &= q - q_1 \\ q_2 &= q_3 - q_1 = q - 2q_1 \end{aligned}}$$

The expression of U^*, after a few algebraic manipulations, is:

$$U^* = \frac{bp}{Gt}\left[\left(2+3\alpha\right)q_1^2 - \left(2q+2\alpha q\right)q_1\right] + U^*(q)$$

And imposing:

$$\frac{\partial U^*}{\partial q_1} = 0$$ it is obtained:

$$\frac{\partial U^*}{\partial q_1} = 2\left(2+3\alpha\right)q_1 - 2q\left(1+\alpha\right) = 0$$

and so: $$\boxed{q_1 = \frac{1+\alpha}{2+3\alpha}\, q}$$

Comments to the result

α is defined as $\alpha = \frac{Gt_1 p}{kb}$

$\alpha \to +\infty$ means that the panel is much stiffer than the rivet

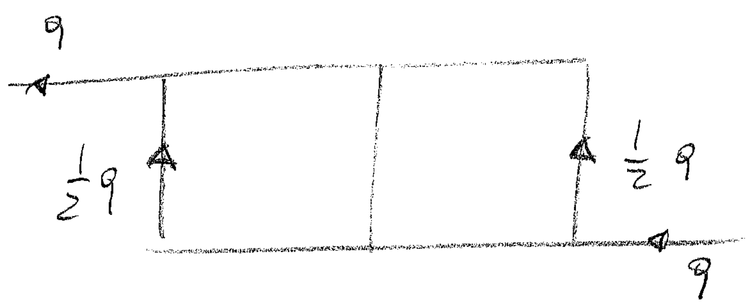
$\alpha \to 0$ means that the rivet is much stiffer than the panel. (situation closer to typical values)

If $\alpha \to 0$ then:

$q_1 = \frac{1}{2} q$

$q_2 = 0$

$q_3 = \frac{1}{2} q$



Results for $\alpha = 0$

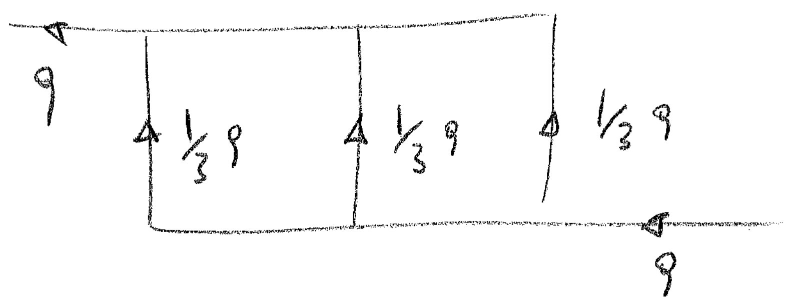
Whenever the rivet is much stiffer than the panel, the central row of rivets is unloaded and so unuseful.
This situation is typically close to common aerospace values and explains why using a triple row of rivets is not a good practice.

The other extreme situation is given by $\alpha \to +\infty$.
In this case

$q_1 = \frac{1}{3} q$

$q_2 = \frac{1}{3} q$

$q_3 = \frac{1}{3} q$

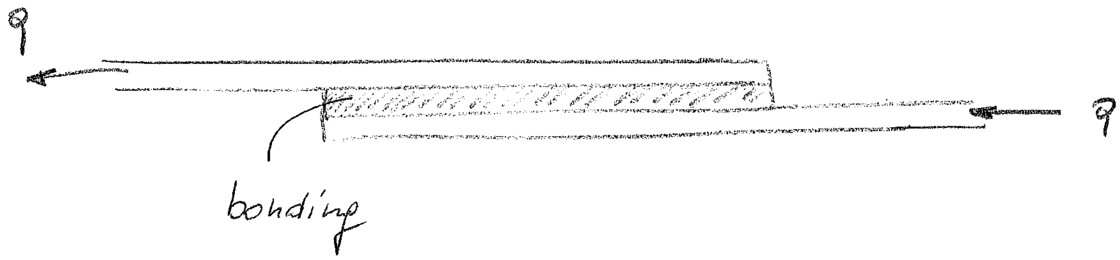


All the intermediate cases will be

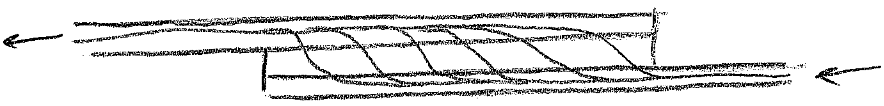
$$\boxed{\begin{array}{c} \frac{1}{3}q \le q_1 \le \frac{1}{2}q \\ 0 \le q_2 \le \frac{1}{3}q \\ \frac{1}{3}q \le q_3 \le \frac{1}{2}q \end{array}}$$

Bonding

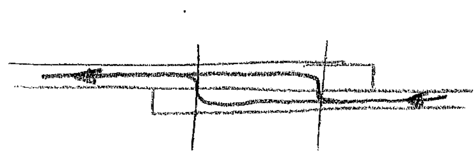
A continuous junction technique, which is commonly used in the aerospace field, is the bonding.
Often bonding makes use of thin films of pre-polymerized material, reinforced with glass fiber.
It is a cheap technology, offering the additional advantage of guaranteeing isolation with outside.
For metallic materials, the typical thickness of the bonding layers is of the order of $1/10$ mm.



In the presence of a bonding line the load transfer is continuous, as opposed to the case of discrete junctions, where the load transfer happens in discrete locations.

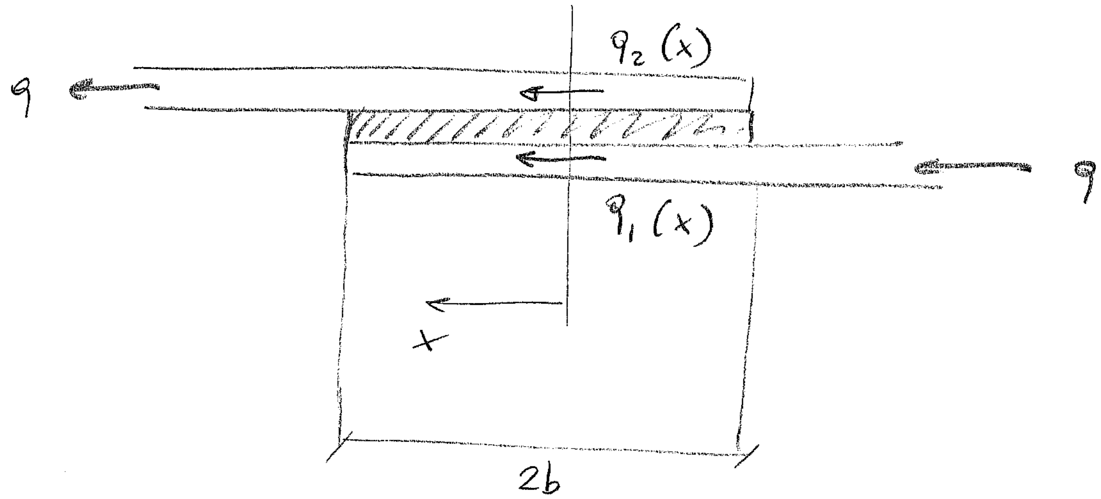


load transfer in continuous junction



load transfer in discrete junction

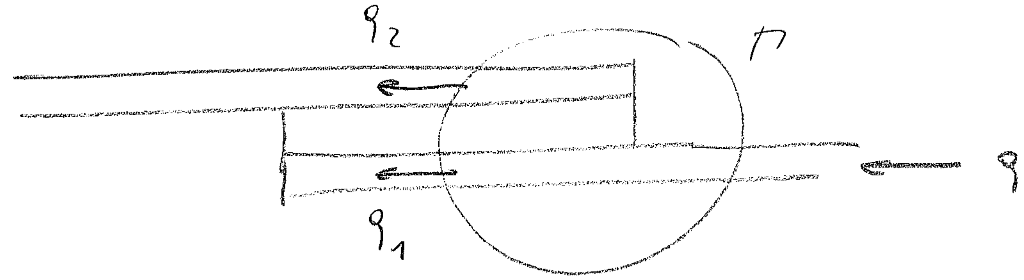
It follows that the problem of evaluating the shear flow transfer is a <u>differential</u> problem and not an algebraic one as in the case of discrete junctions.



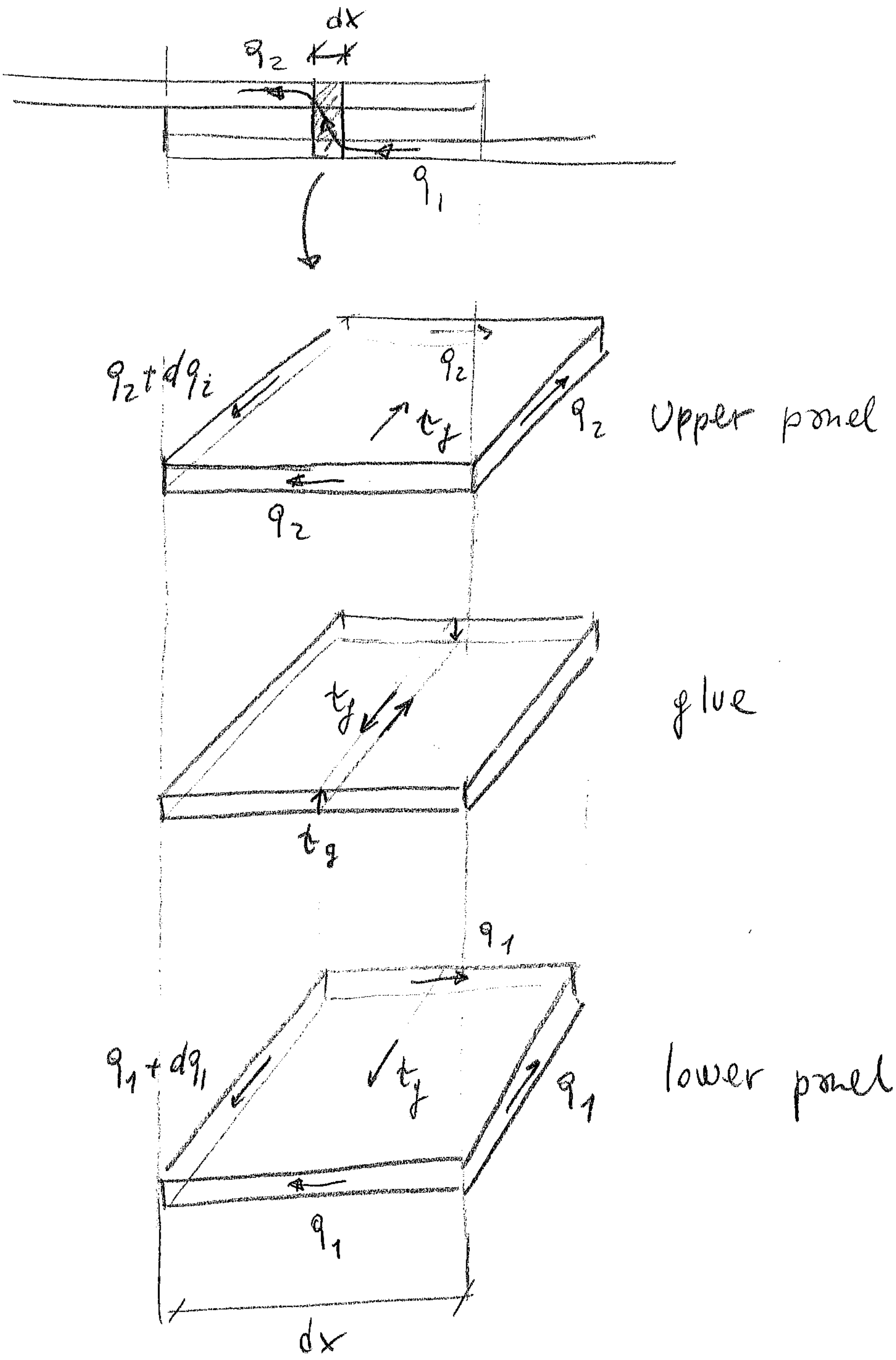
The shear flows q_1 and q_2 are now functions of the position x, meaning that the unknowns are no more scalars but functions.

At any section of the structure the equilibrium is satisfied, meaning that

$$\boxed{q_1(x) + q_2(x) = q} \quad \text{equilibrium}$$



This can be easily seen by considering an arbitrary path Γ. The shear flow equation states that: $q_2 + q_1 = q$



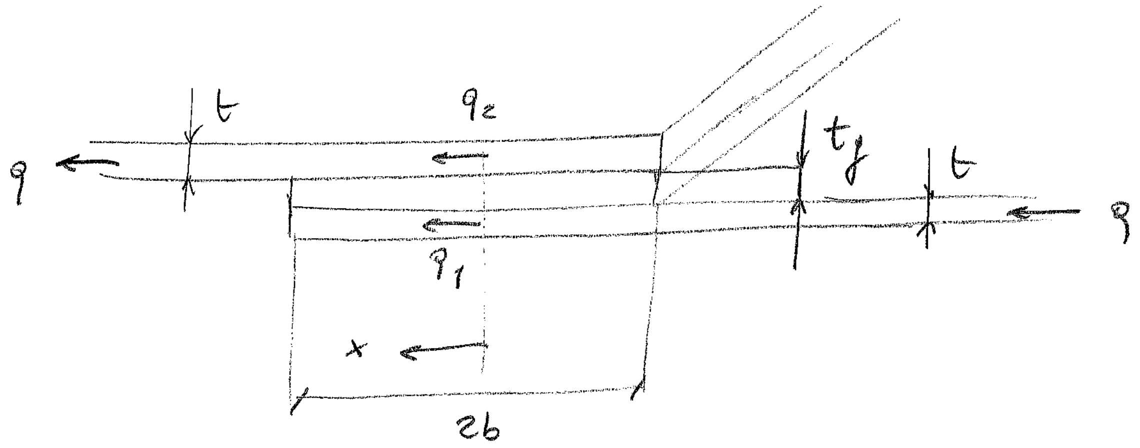
Equilibrium of the upper panel : $(q_2 + dq_2)\,dz - q_2\,dz - \tau_g\,dx\,dz = 0$

$$dq_2 = \tau_g\,dx \Rightarrow \boxed{q_{2/x} = \tau_g}$$

Equilibrium of the lower panel : $(q_1 + d_1)\,dz - q_1\,dz + \tau_g\,dx\,dz = 0$

$$\boxed{q_{1/x} = -\tau_g}$$

The problem is formulated, as done for the rivets, by referring to Menabrea's theorem for imposing the compatibility.



$$U^* = U^*_{panels} + U^*_{glue}$$

$$U^*_{panels} = \frac{1}{2}\int_V \frac{q^2}{Gt^2}\,dV = \frac{1}{2}\int_\ell \int_{-b}^{b} \frac{q^2}{Gt}\,dx\,dz$$

$$U^*_{glue} = \frac{1}{2}\int_V \tau_g \gamma_g\,dV = \frac{1}{2}\int_V \frac{\tau_g^2}{G_g}\,dV = \frac{1}{2}\int_\ell\int_{-b}^{b} \frac{\tau_g^2}{G_g}\,t_g\,dx\,dz$$

where G_g is the glue's shear modulus

So:

$$U^* = \frac{1}{2}\int_\ell\int_{-b}^{b}\left(\frac{q_1^2+q_2^2}{Gt} + \frac{\tau_g^2\,t_g}{G_g}\right)dx\,dz + U^*(q)$$

The problem does not depend on z, so it can be taken $dz = 1$, so:

$$\boxed{U^* = \frac{1}{2}\int_{-b}^{b}\left(\frac{q_1^2+q_2^2}{Gt} + \frac{\tau_g^2\,t_g}{G_g}\right)dx} + U^*(q)$$

Introduce now the equilibrium conditions and express U^* as function of q_2 only

$$q_1 + q_2 = q \qquad \Rightarrow \qquad \boxed{\begin{array}{l} q_1 = q - q_2 \\ q_{2/x} = t_f \end{array}} \qquad \text{equilibrium conditions}$$
$$q_{2/x} = t_f$$

The strain energy is then:

$$U^* = \frac{1}{2}\int_{-b}^{b}\left(\frac{q^2 + q_2^2 - 2qq_2 + q_2^2}{Gt} + \frac{q_{2/x}^2\, t_f}{G_f}\right)dx + U^*(q)$$

$$\boxed{U^* = \frac{1}{Gt}\int_{-b}^{b}\left(q_2^2 - qq_2\right)dx + \frac{1}{2}\int_{-b}^{b} q_{2/x}^2 \frac{t_f}{G_f}\,dx + U^*(q)}$$

The functional U^* depends on q_2 and $q_{2/x}$, or

$$U^* = U^*(q_2, q_{2/x})$$

The stationarity condition $\delta U^* = 0$ can be imposed by applying the <u>Euler-Lagrange equations</u>:

$$\boxed{\frac{\partial u^*}{\partial q_2} - \frac{d}{dx}\frac{\partial u^*}{\partial q_{2/x}} = 0}$$

where $U^* = \int_{-b}^{b} u^*\, dx$

which lead to:

$$\frac{\partial u^*}{\partial q_2} = \frac{1}{Gt}(2q_2 - q)$$

$$\frac{d}{dx}\frac{\partial u^*}{\partial q_{2/x}} = \frac{t_f}{G_f} q_{2/xx}$$

The Euler-Lagrange equation is then:

$$\frac{1}{Gt}(2q_2 - q) - \frac{t_f}{G_f} q_{2/xx} = 0$$

or:

$$-\frac{Gt\,t_f}{G_f} q_{2/xx} + 2q_2 = q$$

$$-\frac{Gt\,t_f}{2G_f} q_{2/xx} + q_2 = q/2$$

Define now the parameter (dimensional) $\boxed{\alpha^2 = \frac{2G_f}{Gt\,t_f}}$

and the equation becomes:

$$\boxed{-\frac{1}{\alpha^2} q_{2/xx} + q_2 = q/2}$$

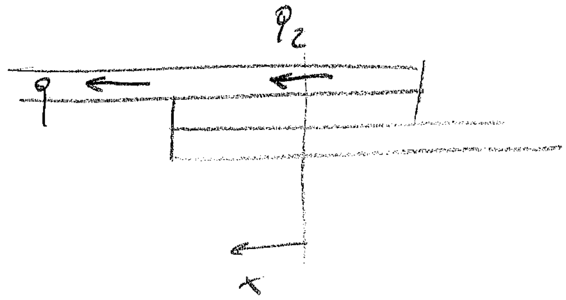
This is a non-homogeneous equation describing the compatibility requirements.

The solution of the problem involves

1. The definition of the boundary conditions
2. The solution of the homogeneous part and the particular one

1. Differential problem:

$$\boxed{\begin{cases} -\dfrac{1}{\alpha^2}\, q_{2/xx} + q_2 = q/2 \\ q_2(b) = q \\ q_2(-b) = 0 \end{cases}}$$



2a. Solution (homogeneous)

$$-\frac{1}{\alpha^2}\, q_{2/xx} + q_2 = 0$$

$$-\frac{\lambda^2}{\alpha^2} + 1 = 0 \quad \Rightarrow \quad \lambda^2 = \alpha^2 \quad \Rightarrow \quad \boxed{\lambda = \pm\alpha}$$

and so:

$$\boxed{q_2^H = A e^{\alpha x} + B e^{-\alpha x}}$$

A, B integration constants to evaluate by imposing BCs

26. Solution (particular)

$$\boxed{q_2^p = q/2}$$

The solution of the differential problem is then:

$$\boxed{q_2 = Ae^{\alpha x} + Be^{-\alpha x} + q/2}$$

The constants A, B are obtained by imposing the boundary conditions:

$$\begin{cases} q_2(b) = Ae^{\alpha b} + Be^{-\alpha b} + q/2 = q & (1) \\ q_2(-b) = Ae^{-\alpha b} + Be^{\alpha b} + q/2 = 0 & (2) \end{cases}$$

$$A(e^{\alpha b} + e^{-\alpha b}) + B(e^{\alpha b} + e^{-\alpha b}) = 0$$

$$\Rightarrow \quad B = -A$$

From (1): $\quad A(e^{\alpha b} - e^{-\alpha b}) = q/2$

Recalling that:

$$\sinh \alpha = \frac{e^{\alpha} - e^{-\alpha}}{2}$$

$$\cosh \alpha = \frac{e^{\alpha} + e^{-\alpha}}{2}$$

$$A = \frac{q}{2(e^{\alpha b} - e^{-\alpha b})} \Rightarrow A = \frac{q}{4 \sinh(\alpha b)}$$

Summarizing the solution is obtained as:

- $q_2 = A\left(e^{-\alpha x} - e^{\alpha x}\right) + q/2$

$$= 2A \sinh(\alpha x) + q/2$$

$$\boxed{q_2 = \frac{q}{2}\frac{\sinh(\alpha x)}{\sinh(\alpha b)} + q/2}$$

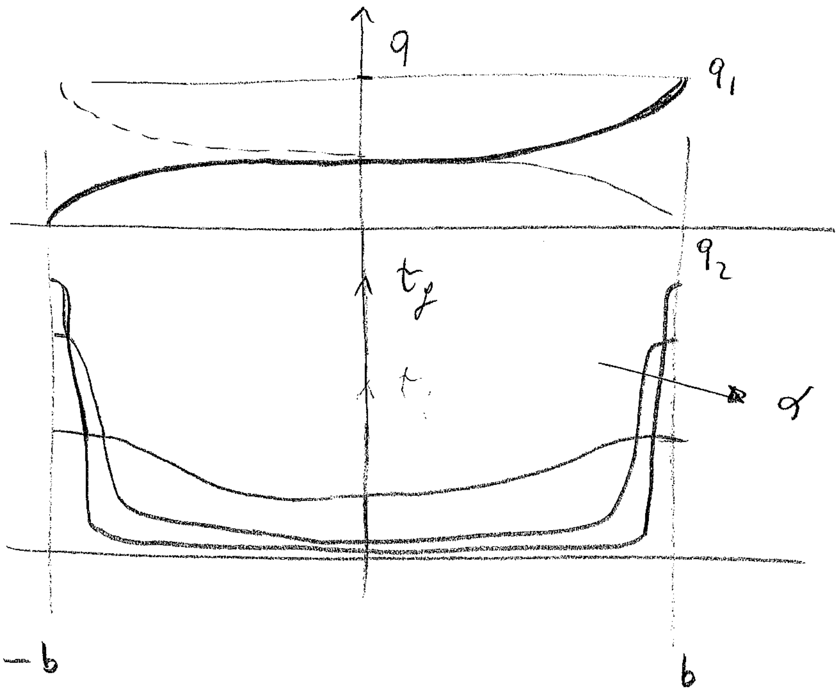
- $q_1 = q - q_2$

$$\boxed{q_1 = -\frac{q}{2}\frac{\sinh(\alpha x)}{\sinh(\alpha b)} + q/2}$$

- $\tau_g = q_2/x$

$$\boxed{\tau_g = \frac{q}{2}\alpha\frac{\cosh(\alpha x)}{\sinh(\alpha b)}}$$

The solutions can be reported graphically



Remarks

1. The behaviour of t_g depends on the parameter α

(Recall $\alpha^2 = \dfrac{2G_g}{G t t_g}$)

- high α : - high G_c
  - low t_g
- low α : - low G_c
  - high t_g

2. The bonding works fine for high values of α. In these cases (α high) the central region is substantially unloaded, which is beneficial from a viscoelastic phenomena (creep) point of view

3. The peaks of stresses on the outer regions will be, in any case, reduced by yielding phenomena

A remark on the application of Euler-Lagrange equations

Euler-Lagrange equations are used in variational calculus to determine the stationarity conditions for a functional.

Given a functional

$$\boxed{I = \int_a^b F(x, u, u_{/x}, u_{/xx})\,dx} \qquad I : \underset{\substack{\text{generic}\\\text{space}}}{U} \longrightarrow \mathbb{R}$$

the stationarity condition (the maximum or the minimum) is obtained by applying the so-called Euler-Lagrange equations:

$$\boxed{\frac{\partial F}{\partial u} - \left(\frac{\partial F}{\partial u_{/x}}\right)_{/x} + \left(\frac{\partial F}{\partial u_{/xx}}\right)_{/xx} = 0}$$

meaning that the function u which satisfies the Euler-Lagrange equations will determine an extremal value (maximum or minimum) for I.

Instead of applying the Euler-Lagrange equations the same result can be obtained by setting

$\delta I = 0$ and integrating by parts

Consider now this approach with reference to the problem of minimizing U^* for continuous junctions; it was found that:

$$U^* = \frac{1}{Gt}\int_{-b}^{b} \left(q_2^2 - q q_2\right) dx + \frac{1}{2}\int_{-b}^{b} q_{2/x}^2 \frac{t_f}{G_f}\, dx$$

Impose now $\delta U^* = 0$:

$$\delta U^* = \frac{1}{Gt}\int_{-b}^{b} \left(2 q_2\, \delta q_2 - q\, \delta q_2\right) dx + \int_{-b}^{b} q_{2/x}\, \delta q_{2/x} \frac{t_f}{G_f}\, dx$$

Integrate now the second contribution by parts:

$$= \frac{1}{Gt}\int_{-b}^{b} \delta q_2 \left(2 q_2 - q\right) dx - \int_{-b}^{b} \delta q_2\, q_{2/xx} \frac{t_f}{G_f}\, dx +$$

$$+ \delta q_2\, q_{2/x} \frac{t_f}{G_f}\Big|_{-b}^{b} = 0$$

Observe now that:

$$q_2(b) = q \quad \Rightarrow \quad \delta q_2(b) = 0$$

$$q_2(-b) = 0 \quad \Rightarrow \quad \delta q_2(-b) = 0$$

So:

$$= \int_{-b}^{b} \delta q_2 \left[\frac{1}{Gt}\left(2 q_2 - q\right) - \frac{t_f}{G_f}\, q_{2/xx}\right] dx = 0 \qquad \forall\, \delta q_2$$

As far as the previous equation should be verified for $\forall\, \delta q_2$, it follows that:

$$\boxed{\frac{1}{Gt}\left(2q_2 - q\right) - \frac{t_f}{G_f}\, q_{2/xx} = 0}$$

which is the same stationarity condition obtained using the Euler-Lagrange equations.