- Stiffness and compliance matrices

Within the context of force-based approaches, the PCVW (and Menabrea's theorem) can be used for evaluating:

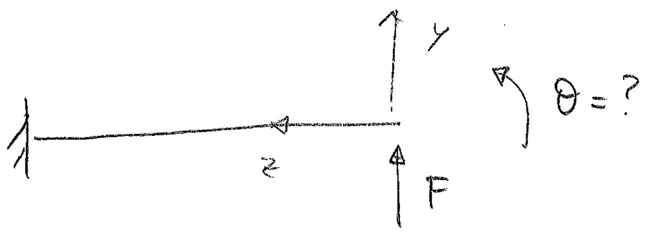
1. Unknown reaction forces (statically indeterminate systems)
2. Displacements in given points for given set of loads

Consider now this second case. The PCVW for the DSV beam was found to be

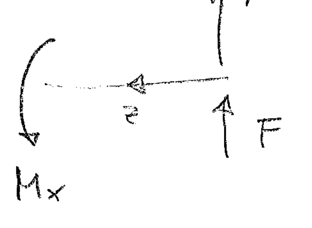
$$\boxed{\int_e \left( \delta T_z \frac{T_z}{EA} + \delta M_x \frac{M_x}{EJ_{xx}} + \delta M_y \frac{M_y}{EJ_{yy}} + \delta T_x \frac{T_x}{GA_x^*} + \delta T_y \frac{T_y}{GA_y^*} + \delta M_z \frac{M_z}{GJ} \right) dz = \delta F_i \hat{u}_i + \delta M_j \hat{\theta}_j + \delta \hat{F}_r u_r + \delta \hat{M}_s \theta_s}$$

↳ the components due to $T_x$, $T_y$, $M_z$ have not been discussed yet, and are reported here for generality purposes.

To understand how to evaluate displacement/rotations, consider this simple example




Which is the rotation $\theta$?

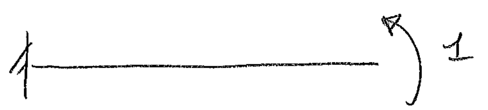



$M_x = -Fz$

The real generalized deformation is then $\frac{M_x}{EJ_{xx}} = -\frac{F z}{EJ_{xx}}$

Consider now a properly selected "dummy" system for evaluating a suitable set of equilibrated variation of forces.

The dummy system is chosen with regard to the displacement component to be evaluated, in such a manner that the unknow, in this case $\theta$, appears in the external work.



$\delta M_x = -1$

"dummy"

The application of the PCVW is:

$$\int_0^l \frac{F z}{EJ_{xx}} \, dz = \theta \quad \Longrightarrow \quad \boxed{\theta = \frac{F l^2}{2EJ_{xx}}}$$

- In this example no displacements were imposed, so

$$\delta W_e^* = \boxed{\delta F_i \hat{u}_i + \delta M_j \hat{\theta}_j} + \delta \hat{F}_r u_r + \delta \hat{M}_s \theta_s$$

The dummy system was taken by considering a variation of the applied moment $\delta \hat{M} = 1$, while $\delta \hat{F} = 0$

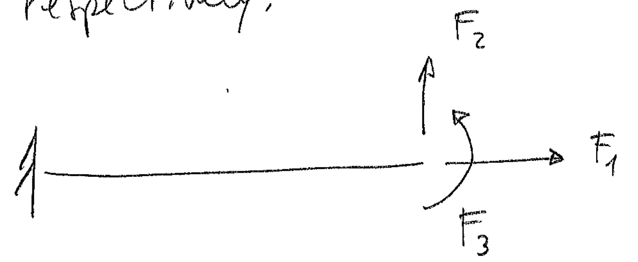
so

$$\delta W_e^* = \delta \hat{M} \theta = \theta$$

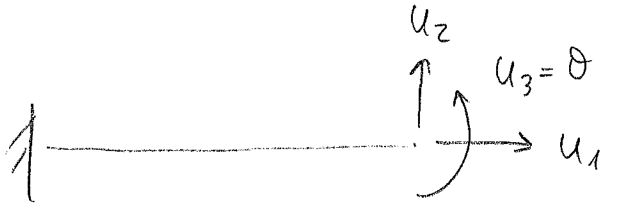
The quantity $\delta M_x$ in the expression of $\delta W_i^*$ is then the equilibrated variation of bending moment due to the application of $\delta \hat{M}$.

# Stiffness matrix for a cantilever beam

Consider a cantilever beam and denote with $F_i$ and $u_i$ the generalized forces and displacements, respectively.




forces




displacement

The convention for $F_i$ and $u_i$ is completely arbitrary. Obtaining the <u>compliance</u> and the <u>stiffness</u> matrices means obtaining a relation in the form:

| | | |
|---|---|---|
| $\underline{u} = \underline{\underline{C}}\,\underline{F}$ | compliance | |
| $\underline{F} = \underline{\underline{K}}\,\underline{u}$ | stiffness | with $\underline{\underline{K}} = \underline{\underline{C}}^{-1}$ |

## Remarks

1. The procedure relies upon the linearity assumption of the problem. It follows that the <u>principle of superposition</u> holds
   (the displacement $u_i$ due to the application of $F_A$ and $F_B$ is $u_i = u_A + u_B$ where $u_A$ is the displacement due to to $F_A$ and $u_B$ is the displacement due to $F_B$)

2. The compliance/stiffness matrix will be derived in the context of a force-based approach. The result will be the <u>compliance matrix</u> while the stiffness matrix will be derived indirectly, after inverting $\underline{\underline{C}}$.

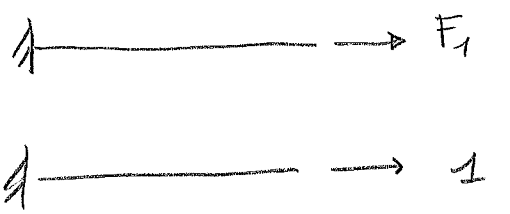
- Force-based approaches $\rightarrow$ $\underline{\underline{C}}$ $\left(\underline{\underline{K}} = \underline{\underline{C}}^{-1}\right)$
- Displacement-based approaches $\rightarrow$ $\underline{\underline{K}}$ $\left(\underline{\underline{C}} = \underline{\underline{K}}^{-1}\right)$

<u>Evaluation of the coefficients</u>

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ & & C_{33} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

<u>The generic term $C_{ij}$ denotes the i-th displacement due to the j-th unitary force</u>

① $C_{11}$



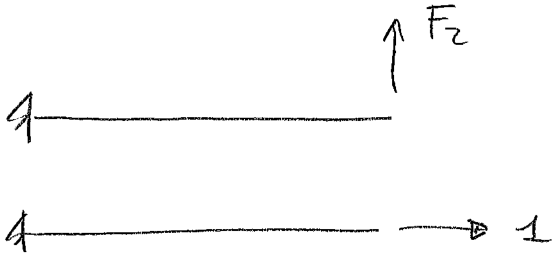

$T_z = F_1$

$\delta T_z = 1$

$$\int_0^{\ell} \delta T_z \frac{T_z}{EA} = 1 \cdot u_1$$

$$F_1 \frac{\ell}{EA} = u_1 \quad \Rightarrow \quad \boxed{C_{11} = \frac{\ell}{EA}}$$

② $C_{12}$

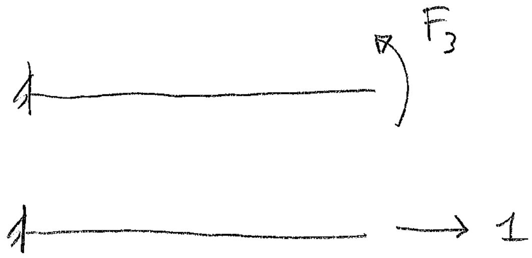



$M_x = 0 \qquad T_z = 0$

$\delta M_x = 0 \qquad \delta T_z = 1$

$$\boxed{C_{12} = 0}$$

③ $C_{13}$

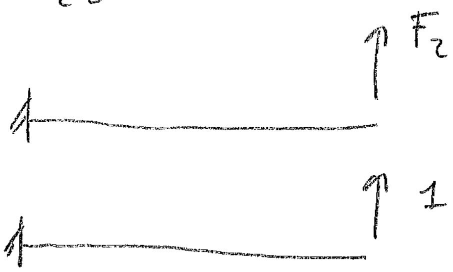



$M_x = -F_3 \qquad T_z = 0$

$\delta M_x = 0 \qquad \delta T_z = -1$

$$\boxed{C_{13} = 0}$$

④ $C_{22}$

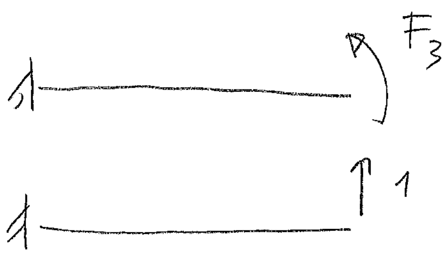



$M_x = -F_2 z$

$\delta M_x = -z$

$$u_2 = \int_0^\ell \frac{Fz^2}{EJ}\,dz = \frac{F\ell^3}{3EJ} \qquad \boxed{C_{22} = \frac{\ell^3}{3EJ}}$$

⑤ $C_{23}$

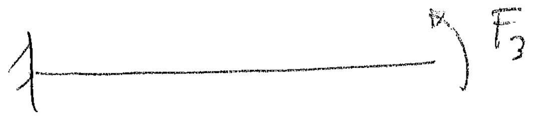



$M_x = -F_3$

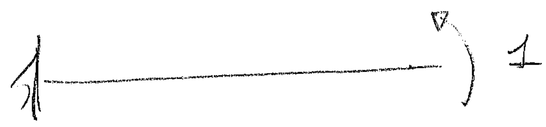
$\delta M_x = -z$

$$u_2 = \int_0^\ell \frac{Fz}{EJ}\,dz = \frac{F\ell^2}{2EJ} \qquad \boxed{C_{23} = \frac{\ell^2}{2EJ}}$$

⑥ $C_{33}$




$M_x = -F_3$




$\delta M_x = -1$

$$u_3 = \int_0^{\ell} \frac{F}{EJ}\, dz = \frac{F\ell}{EJ} \qquad \boxed{C_{33} = \ell/EJ}$$

The compliance matrix is then obtained as:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \ell/EA & 0 & 0 \\ 0 & \ell^3/3EJ & \ell^2/2EJ \\ 0 & \ell^2/2EJ & \ell/EJ \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad \underline{\text{Compliance}}$$

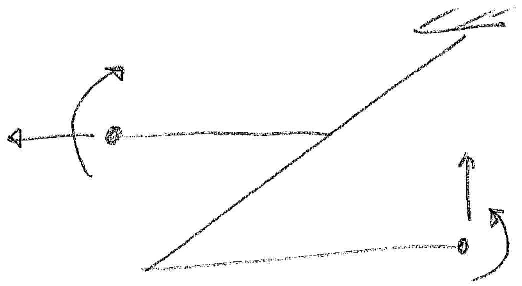
The stiffness matrix is available after inverting $\underline{\underline{C}}$:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} EA/\ell & & \\ & 12EJ/\ell^3 & -6EJ/\ell^2 \\ & -6EJ/\ell^2 & 4EJ/\ell \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \underline{\text{stiffness}}$$

# Extension to a generic beam system

The same approach can be adopted when dealing with a more complex system of beams

The derivation of the compliance matrix represents a sort of condensation of the structure's response to a few points of interest



After defining the forces and displacement of interest a relation can be found in the form $\underline{u} = \underline{\underline{C}}\,\underline{F}$

where the generic contribution $C_{ij}$ represents the i-th displacement due to a unitary force j.

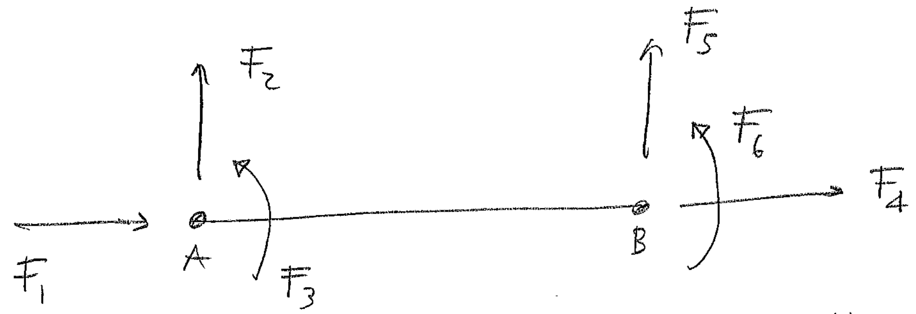
$\Rightarrow$ $C_{ij}$ is obtained by considering a real system where the force $F_j$ is applied, and a dummy system with unitary force $F_i$

| | |
|---|---|
| $C_{ij}$ is obtained with | i-th dummy system<br>j-th real system |

Free-Free beam

The most general case is given by a beam which is free at the two ends.

- From this case any set of boundary conditions can be derived as a special case.
- The resulting stiffness matrix will be useful for the computer implementation of a simple code for analyzing relatively complex systems of beams.




The goal is to derive a relation in the form

$$\begin{Bmatrix} \underline{F}_A \\ \underline{F}_B \end{Bmatrix} = \begin{bmatrix} \underline{\underline{k}}_{AA} & \underline{\underline{k}}_{AB} \\ \underline{\underline{k}}_{BA} & \underline{\underline{k}}_{BB} \end{bmatrix} \begin{Bmatrix} \underline{u}_A \\ \underline{u}_B \end{Bmatrix}$$

where

$$\underline{F}_A = \{F_1 \; F_2 \; F_3\}^T \qquad \underline{u}_A = \{u_1 \; u_2 \; u_3\}^T$$

$$\underline{F}_B = \{F_4 \; F_5 \; F_6\}^T \qquad \underline{u}_B = \{u_4 \; u_5 \; u_6\}^T$$

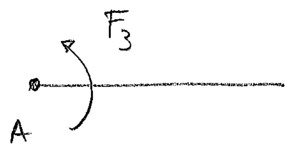
1. $\underline{\underline{k}}_{BB}$ is available from the previous results of the cantilever beam

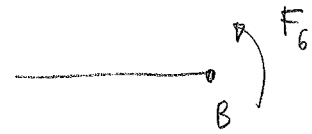
2. $\underline{\underline{k}}_{AA}$ can be obtained with a similar procedure (by repeating all the steps)

A smarter approach is to consider that:

1. The direct terms $C_{\alpha\alpha}$ do not change
   (the displacement $u_1$ due to $F_1$ is equal to the displacement $u_4$ due to $F_4$, and similarly for the other terms)

2. The mixed contributions $C_{\alpha\beta}$ may change the sign. ($C_{23}$ and $C_{32}$, as $C_{13} = C_{23} = 0$)



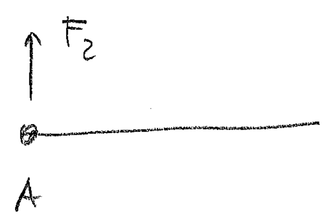



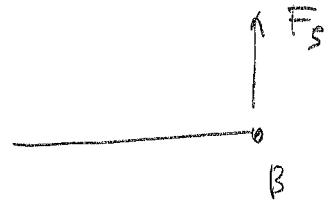

$F_3$ determines a displacement $u_2$ in the <u>downward</u> direction

$F_6$ determines a displacement $u_5$ in the <u>upward</u> direction

<u>$C_{23}$ changes sign!</u>

similarly




$F_2$ determines a rotation in the <u>clockwise</u> direction (negative in the conventions here used)

$F_5$ determines a rotation in the <u>counterclockwise</u> direction (positive in the conventions here used)

<u>$C_{32}$ changes sign</u>

Thus:

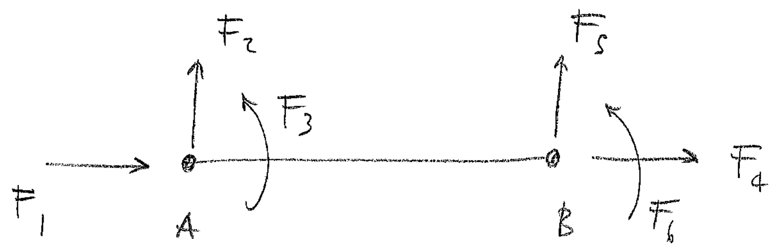
$$\underline{\underline{C}}_{AA} = \begin{bmatrix} C_{11} & 0 & 0 \\ 0 & C_{22} & -C_{23} \\ 0 & C_{-23} & C_{33} \end{bmatrix}$$

and so:

$$\underline{\underline{K}}_{AA} = \begin{bmatrix} EA/\ell & & \\ & 12EJ/\ell^3 & 6EJ/\ell^2 \\ & 6EJ/\ell^2 & 4EJ/\ell \end{bmatrix}$$

3 Evaluation of $\underline{\underline{K}}_{AB}$

Even in this case, it is possible to obtain $\underline{\underline{K}}_{AB}$ term by term.
A smarter strategy is to proceed as follows.




From equilibrium:

$$F_1 = -F_4$$
$$F_2 = -F_5$$
$$F_3 = -F_6 - F_5\,\ell$$

or $\underline{F}_A = \underline{\underline{T}}\,\underline{F}_B$ with $\underline{\underline{T}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -\ell & -1 \end{bmatrix}$

Recall now that:

$$\begin{Bmatrix} \underline{F}_A \\ \underline{F}_B \end{Bmatrix} = \begin{bmatrix} \underline{\underline{K}}_{AA} & \underline{\underline{K}}_{AB} \\ \underline{\underline{K}}_{AB}^T & \underline{\underline{K}}_{BB} \end{bmatrix} \begin{Bmatrix} \underline{u}_A \\ \underline{u}_B \end{Bmatrix}$$

- Assume the set $\underline{u}_A = \underline{0}$, then:

$$\underline{F}_A = \underline{\underline{K}}_{AB} \underline{u}_B$$

$$\underline{F}_B = \underline{\underline{K}}_{BB} \underline{u}_B$$

but $\underline{F}_A = \underline{\underline{T}}\, \underline{F}_B$ so: $\boxed{\underline{\underline{K}}_{AB} = \underline{\underline{T}}\, \underline{\underline{K}}_{BB}}$

It is then obtained that:

$$\underline{\underline{K}}_{AB} = \begin{bmatrix} -EA/\ell & & \\ & -12EJ/\ell^3 & 6EJ/\ell^2 \\ & -6EJ/\ell^2 & 2EJ/\ell \end{bmatrix}$$

- With a similar approach it is found $\underline{\underline{K}}_{BA} = \underline{\underline{K}}_{AB}^T$

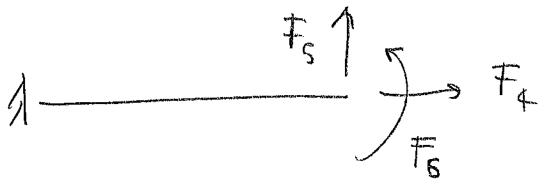
Summarizing, the stiffness matrix reads

$$\underline{\underline{K}} = \left[\begin{array}{ccc|ccc} EA/\ell & & & -EA/\ell & & \\ & 12EJ/\ell^2 & 6EJ/\ell^2 & & -12EJ/\ell^3 & 6EJ/\ell^2 \\ & 6EJ/\ell^2 & 4EJ/\ell & & -6EJ/\ell^2 & 2EJ/\ell \\ \hline & & & EA/\ell & & \\ & SYM & & & 12EJ/\ell & -6EJ/\ell^2 \\ & & & & -6EJ/\ell^2 & 4EJ/\ell \end{array}\right]$$

# Axial and bending stiffness

Evaluate now the role played by the axial and the bending stiffness and the corresponding load paths.

Consider the cantilever beam here below




The stiffness matrix was found to be:

$$\underline{\underline{K}}_{BB} = \begin{bmatrix} EA/\ell & & \\ & 12EJ/\ell^3 & -6EJ/\ell^2 \\ & -6EJ/\ell^2 & 4EJ/\ell \end{bmatrix}$$

For comparing the role played by the various contributions it is useful to highlight some nondimensional parameters.

1. EA is, dimensionally, a force so
   - Forces can be normalized with respect to EA
   - Moments " to $EA \cdot \ell$
2. The displacements can be normalized with respect to $\ell$

$$\bar{F} = \frac{F}{EA} \qquad \bar{M} = \frac{M}{EA\ell} \qquad \bar{u} = \frac{u}{\ell}$$

$$\begin{cases} F_4 = \dfrac{EA}{\ell} u_4 & \text{Multiply by } \dfrac{1}{EA} \\[2ex] F_5 = \dfrac{12EJ}{\ell^3} u_5 - \dfrac{6EJ}{\ell^2} u_6 & \text{Multiply by } \dfrac{1}{EA} \\[2ex] F_6 = -\dfrac{6EJ}{\ell^2} u_5 + \dfrac{4EJ}{\ell} u_6 & \text{Multiply by } \dfrac{1}{EA\ell} \end{cases}$$

$$\begin{cases} \dfrac{F_4}{EA} = \dfrac{u_4}{\ell} \\[2ex] \dfrac{F_5}{EA} = \dfrac{12EJ}{EA\ell^2} \dfrac{u_5}{\ell} - \dfrac{6EJ}{EA\ell^2} u_6 \\[2ex] \dfrac{F_6}{EA\ell} = -\dfrac{6EJ}{EA\ell^2} \dfrac{u_5}{\ell} + \dfrac{4EJ}{EA\ell^2} \end{cases}$$

Defining now $\boxed{\lambda^2 = \dfrac{EA\ell^2}{EJ} \quad \underline{\text{slenderness ratio}}}$, it

is possible to write the relation between nondimensional forces and non dimensional displacements

$$\boxed{\begin{Bmatrix} \bar{F}_4 \\ \bar{F}_5 \\ \bar{F}_6 \end{Bmatrix} = \begin{bmatrix} 1 & & \\ & 12/\lambda^2 & -6/\lambda^2 \\ & -6/\lambda^2 & 4/\lambda^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_4 \\ \bar{u}_5 \\ \bar{u}_6 \end{Bmatrix}}$$

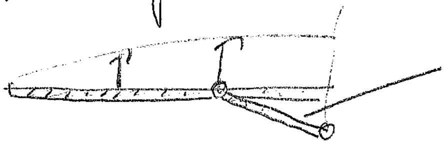
Remarks

- When $\lambda^2$ is high, the bending-related contributions tend to zero
- $\lambda^2$ is high when $EA\ell^2 >> EJ$. This condition is verified whenever:
  1. $EA$ is high
  2. $\ell$ is high (for fixed sectional properties $EA$ and $EJ$, the value of $\lambda$ increases as $\ell$ is increased)

so: the major contributions to the load carrying capability are due to the axial behavior whenever $\lambda^2$ is high.
This is true for beams characterized high values of axial stiffness $EA$ (high has to be intended with respect to $EJ$) and/or slender beams with high $\ell$.

- A structure which is designed for sustaining the loads via axial behaviour is an <u>extremely</u> <u>efficient</u> structure.
On the contrary, the efficiency reduces when the loads are carried via bending response.
- Systems of <u>trusses</u> are <u>highly efficient</u> structures and for this reason they are widely employed in the space field.

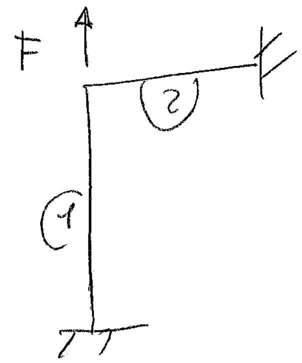
- Even the early aeronautical structures were based on the axial load carrying capabilities of the wings



truss

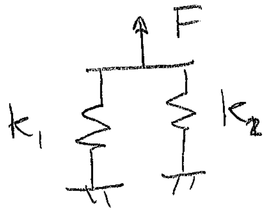
This kind of configuration was changed for aerodynamic reasons. The structure is efficient!

- According to what has been discussed, a statically indeterminate structure as below




will be characterized by a load path such that most of the load is carried via axial response by beam 1.

In fact this structure can be seen as

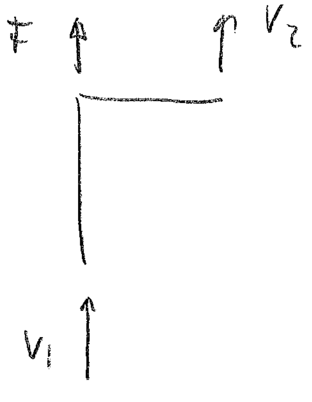



where $k_1 = \dfrac{EA}{\ell}$

$k_2 = \dfrac{3EJ}{\ell^3}$

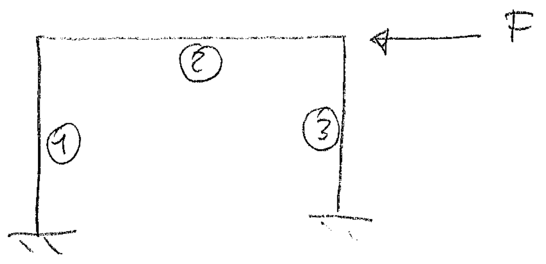
Thus $F_1 = k_1 u$, $F_2 = k_2 u$ $\Rightarrow \dfrac{F_1}{F_2} = \dfrac{EA}{\ell}\dfrac{\ell^3}{3EJ} = \dfrac{EA\ell^2}{3EJ} = \dfrac{\lambda^2}{3}$

Clearly the reaction forces behave accordingly



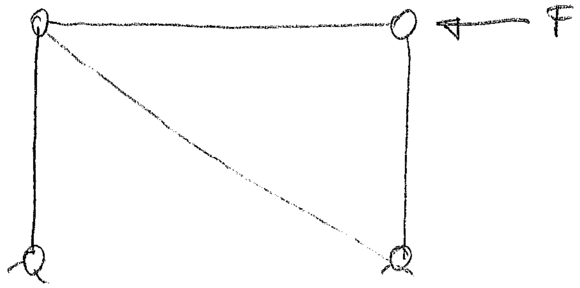

$V_1 >> V_2$

- For the same reasons a beam system as in the figure



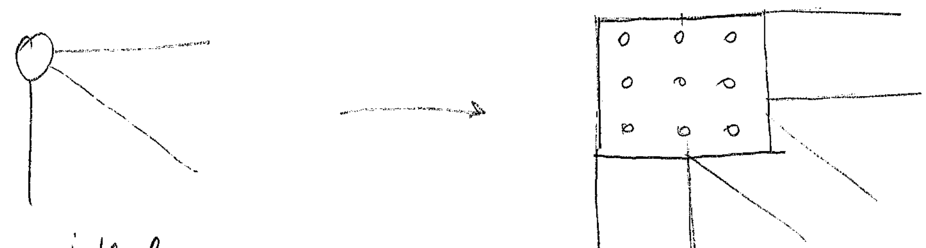

is characterized by a bending response for beams 1 and 3, while beam 2 carry the load via axial behaviour.

A much more <u>efficient</u> design would be the following:




where all of the beams are working in axial mode (no bending energy is involved)

- Note that the diagonal element is loaded in traction. If the diagonal were placed in the other direction, a buckling requirement would enter the design.
- In the previous sketch the elements were connected with hinges, thus no bending moments could be introduced. This is clearly an idealization as, in a real structure, the constraint will not be a perfect hinge and the introduction of some sort of bending moments would be permitted. Consider for instance this case

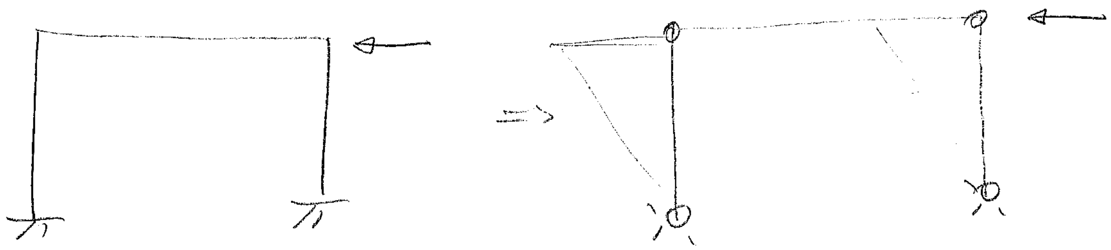


ideal real

If the real structure, after idealizing all the junctions as hinges, is statically determined (thus no kinematisms are possible) then the behaviour will resemble the ideal one, and the loads will be carried via axial behaviour.

In other words, if a purely axial load path is possible, then the structure will use it for sustaining loads. (it is stiffer with respect to bending).

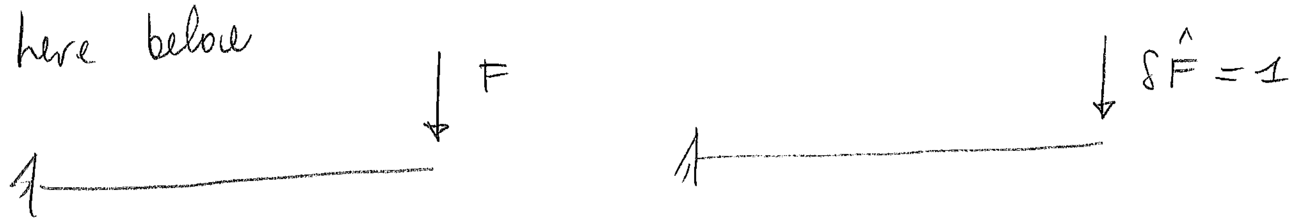
Clearly, if the axial path is not possible due to onset of a mechanism, the response will require the contribution of bending.



(not possible to have a purely axial load path)

- <u>Contribution of shear to the internal complementary virtual work</u>

It was mentioned that in many cases (more specifically, for slender beams) the energy contribution associated with the shear can be neglected.
To justify this assumption consider the example here below




$$\delta W^*_{i,\,bending} = \int_0^\ell \frac{F z^2}{EJ}\,dz = \frac{F\ell^3}{3EJ}$$

$$\delta W^*_{i,\,shear} = \int_0^\ell \delta T \frac{T}{GA^*}\,dz = \frac{F\ell}{GA^*}$$

The ratio between the two contribution is then:

$$\frac{\delta W^*_{i,\,bend}}{\delta W^*_{i,\,shear}} = \frac{F\ell^3}{3EJ}\,\frac{GA^*}{F\ell} = \frac{GA^*\,\ell^2}{3EJ} \propto \frac{A\ell^2}{J}$$

($A^*$ is of the order of $A$; $G$ is of the order of $E$)

Thus:

$$\boxed{\frac{\delta W^*_{i,\,bend}}{\delta W^*_{i,\,shear}} \propto \lambda^2}$$

with $\lambda$ slenderness ratio

For slender beams the strain energy due to bending is much higher in comparison to the shear contribution.

<u>For thin-walled beams</u> <u>this will not be, in general, true</u>

As a matter of fact

$$\frac{GA^* l^2}{3EJ} \propto \frac{A l^2}{J} \qquad \text{iif} \quad A^* \simeq A$$

This is the case of a rectangular beam, where

$A^*/A = 5/6 = \chi$ (shear factor)

For thin walled beams $\chi << 1$ then, in many cases, the shear deformability will not be negligible.