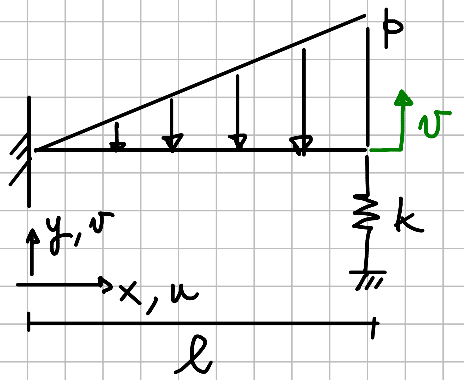
EXERCISE SESSION 12

Ex 1 (EXAM 25/01/2023)



$p = 14 \ N/mm$

$l = 2000 \ mm$

$EJ = 10^{10} \ N mm^2$

$k = 100 \ N/mm$

FIND v

USING 1 TERM POLY APPROX

SOL $\quad v(x) = a_0 + a_1 x + a_2 x^2$ $\quad$ BC $\quad v(0) = 0 \quad a_0 = 0$; $\quad v_{/x}(0) = 0 \quad a_1 = 0$

$$\longrightarrow v(x) = a x^2$$

$$\delta W_e = \int_0^l - \delta v \, q(x) \, dx - \delta v(l) \cdot k \, v(l)$$

$$\delta W_i = \int_0^l \delta v_{/xx} \, EJ \, v_{/xx} \, dx$$

WITH $\quad \delta v = \delta a \cdot x^2$

$$v_{/xx} = a \cdot 2 \qquad \delta v_{/xx} = \delta a \cdot 2$$

$$q(x) = p \cdot \frac{x}{l}$$

PVW $\quad \delta W_i = \delta W_e$

$$\delta a \int_0^l 2EJ \, 2a \, dx = -\delta a \left(\int_0^l x^2 \cdot p \cdot \frac{x}{l} dx + l^2 \cdot k \cdot a \cdot l^2 \right)$$

$$a \ 4EJ \cdot l = -\frac{1}{4} \frac{pl^4}{l} - a \cdot kl^4$$

$$a = -\frac{pl^2}{16EJ + 4kl^3}$$

$$v(l) = a \cdot l^2 = -66.67 \ mm$$

Contents


```
clear variables
close all
home
```

Data

```
f = 14; % [N/mm]
EJ = 1E+10; % [Nmm^2]
l = 2000;    % [mm]
k = 100; % [N/mm]
```

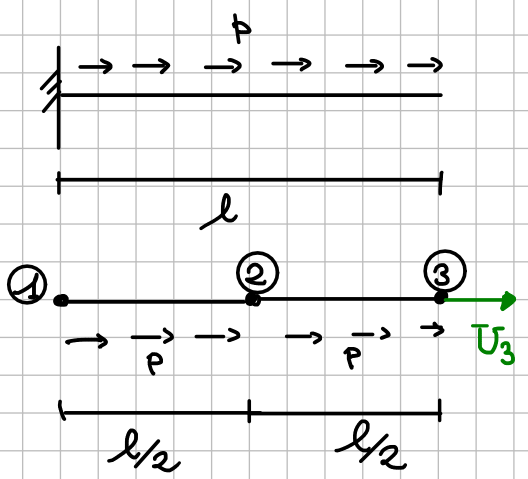
Solution

```
syms c x

q = f*x/l;

Phi = x^2;
w = c*Phi;

LHS = int(diff(Phi, x, 2)^2*EJ*c, x, 0, l);
RHS = int(-Phi*q, x, 0, l) - subs(Phi, x, l)^2*c*k;

c_d = double(solve(LHS == RHS, c));

w_a = double(subs(w, [c x], [c_d l]))
```

```
w_a =

 -66.666666666666671
```

Published with MATLAB® R2022b

Ex 2 (EXAM 05/07/2023)



$l = 3000 \text{ mm}$

$EA = 7 \cdot 10^5 \text{ N}$

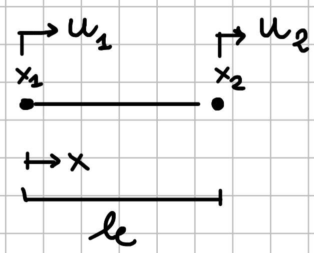
$p(x) = 10 \text{ N/mm}$

LINEAR FE ELEMENTS

FIND U_3

SOL ELEMENT LENGTH $l_e = \dfrac{l}{2}$

LOCAL REF SYST. FOR EACH ELEMENT



LOCAL

$$u_e(x) = N_1(x) \cdot u_1 + N_2(x) \cdot u_2 = \underline{N}_e \cdot \underline{q}_e = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$N_1(x) = \frac{x - x_2}{x_1 - x_2} = \frac{x_2 - x}{l_e}$$

$$N_2(x) = \frac{x - x_1}{x_2 - x_1} = \frac{x - x_1}{l_e}$$

$$\varepsilon_e(x) = u_{e/x}(x) = \sum_i N_{i/x} \cdot u_i$$

$$= -\frac{1}{l_e} \cdot u_1 + \frac{1}{l_e} u_2 = \begin{bmatrix} -1/l_e & 1/l_e \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \underline{B}_e \cdot \underline{q}_e$$

$$\delta W_{i_e} = \int_{x_1}^{x_2} \delta \varepsilon_e^T \cdot EA\, \varepsilon_e \, dx$$

$$= \int_{x_1}^{x_2} \delta \underset{1\times 2}{[q_e]^T} \underset{2\times 1}{[B_e]^T} EA \underset{1\times 2}{[B_e]} \underset{2\times 1}{[q_e]} \, dx$$

$$= \delta \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \cdot \underline{\int_{x_1}^{x_2} \begin{bmatrix} -1/l_e & 1/l_e \end{bmatrix}^T EA \cdot \begin{bmatrix} -1/l_e & 1/l_e \end{bmatrix} dx} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\frac{1}{l_e^2} \cdot EA \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot [x_2 - x_1]$$

$$\frac{EA}{l_e} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underline{\underline{K}}_e$$

$$= \delta \underset{1\times 2}{\underline{q}_e^T} \cdot \underset{2\times 2}{\underline{\underline{K}}_e} \cdot \underset{2\times 1}{\underline{q}_e}$$

$$\delta W_{e_e} = \int_{x_1}^{x_2} \delta \underline{u}_e^T \cdot p(x)\, dx$$

$$= \delta \underline{q}_e^T \cdot \int_{x_1}^{x_2} [N_1 \;\; N_2]^T \cdot p(x)\, dx$$

$$= \delta \underline{q}_e^T \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{p \cdot l_e}{2} = \delta \underline{q}_e^T \cdot \underline{f}_e$$

$$\int_{x_1}^{x_2} [N_1 \;\; N_2]^T \cdot p\, dx$$

$$\int_{x_1}^{x_2} \begin{bmatrix} \frac{x_2 - x}{x_2 - x_1} & \frac{x - x_1}{x_2 - x_1} \end{bmatrix}^T \cdot p \cdot dx$$

$$\begin{bmatrix} \frac{x_2 x - x^2 \cdot 1/2}{x_2 - x_1} & \frac{x^2 \cdot 1/2 - x_1 x}{x_2 - x_1} \end{bmatrix}_{x_1}^{x_2\,T} \cdot p$$

$$\begin{bmatrix} x_2^2 - \frac{1}{2} x_2^2 - x_1 x_2 + \frac{1}{2} x_1^2 & \frac{1}{2} x_2^2 - x_1 x_2 - \frac{1}{2} x_1^2 + x_1^2 \end{bmatrix}^T \cdot \frac{p}{l_e}$$

$$\begin{bmatrix} \frac{(x_2 - x_1)^{\cancel{2}}}{2} & \frac{(x_2 - x_1)^{\cancel{2}}}{2} \end{bmatrix}^T \frac{p}{\cancel{l_e}}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}^T \frac{p \cdot l_e}{2}$$

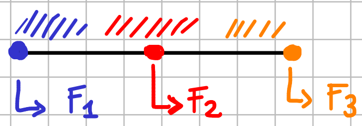
ASSEMBLE : FOR EL #1 $\rightsquigarrow$ $u_1 = U_1$ $u_2 = U_2$

EL #2 $\rightsquigarrow$ $u_1 = U_2$ $u_2 = U_3$

$$\underline{q} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\underline{\underline{K}} = \underset{\text{EL \#1}}{\begin{bmatrix} \underline{\underline{K}}_e & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix}} + \underset{\text{EL \#2}}{\begin{bmatrix} 0 & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \underline{\underline{K}}_e \end{bmatrix}} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \frac{EA}{l_e}$$

$$\underline{f} = \begin{bmatrix} \underline{f}_e \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \underline{f}_e \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \frac{p \cdot l_e}{2}$$



SOLVE $\underline{\underline{K}} \cdot \underline{q} = \underline{f}$

$$\frac{EA}{\ell}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \frac{p \cdot \ell_e}{2}$$

FROM BC' $U_1 = 0$

$$\begin{cases} 2U_2 - U_3 = \dfrac{p\ell_e^2}{EA} \\ -U_2 + U_3 = \dfrac{p\ell_e^2}{2EA} \end{cases} \longrightarrow \begin{cases} U_2 = \dfrac{3}{2}\dfrac{p\ell_e^2}{EA} \\ U_3 = 2\dfrac{p\ell_e^2}{EA} = 64.286 \text{ mm} \end{cases}$$

Contents


```
close all
clear variables
home
```

Data

```
f = 10;     % N/mm
l = 3000;   % mm
EA = 7E+05; % N
```

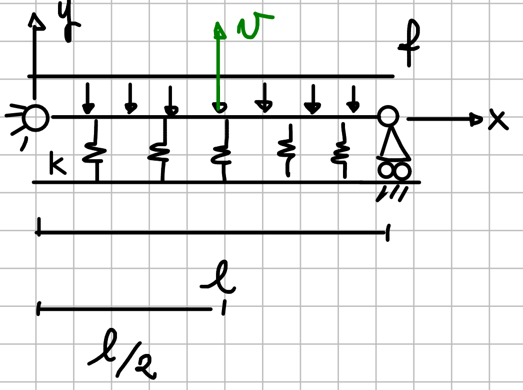
Sol

```
syms x_e u_1 u_2 u_3

l_e = l/2;

N_1 = (l_e-x_e)/l_e;
N_2 = (x_e)/l_e;

Ne_1 = [N_1 N_2 0];
Ne_2 = [0   N_1 N_2];

u = [u_1; u_2; u_3];

Be_1 = diff(Ne_1, x_e, 1);
Be_2 = diff(Ne_2, x_e, 1);

K = (int(Be_1.'*EA*Be_1, x_e, 0, l_e) + int(Be_2.'*EA*Be_2, x_e, 0, l_e));

dW_e = (int(Ne_1.', x_e, 0, l_e) + int(Ne_2.', x_e, 0, l_e))*f;

sol = solve(K(2:end, 2:end)*u(2:end) == dW_e(2:end), u(2:end));

double(sol.u_3)
```

```
ans =

  64.285714285714292
```

Published with MATLAB® R2022b

Ex 3 (EXAM 05/09/2023)



DATA

$\ell = 1000\ mm$

$EJ = 1E{+}12\ N\,mm^2$

$k = 8\ MPa \quad \left(\frac{N}{mm} \cdot \frac{1}{mm}\right)$

$f = 100\ N/mm$

FIND v USING TRIG. APPROX

SOL

$$v(x) = A \cdot \sin\left(\frac{\pi \cdot x}{\ell}\right) = A \cdot \phi(x) \quad \leadsto \text{SATISFY BC}$$

$$v_{/x}(x) = A\frac{\pi}{\ell} \cdot \cos\left(\frac{\pi x}{\ell}\right) = A\, \phi_{/x}(x)$$

$$v_{/xx}(x) = -A\frac{\pi^2}{\ell^2} \cdot \sin\left(\frac{\pi x}{\ell}\right) = A \cdot \phi_{/xx}(x)$$

PVW

$$\delta W_i = \int_0^\ell \delta v_{/xx}\, EJ\, v_{/xx}\, dx$$

$$\delta W_e = -\int_0^\ell \delta v \cdot f(x)\, dx - \int_0^\ell \delta v\, k \cdot v\, dx$$

$$\delta W_i = \delta A \cdot \int_0^\ell -\frac{\pi^2}{\ell^2} \sin\left(\frac{\pi x}{\ell}\right) EJ \cdot -\frac{\pi^2}{\ell^2} \cdot A \cdot \sin\left(\frac{\pi x}{\ell}\right) dx$$

KNOWN $\quad \int_0^\ell \sin^2\left(\frac{\pi x}{\ell}\right) dx = \frac{\ell}{2}$

$$\delta W_e = \delta A \cdot \left(-\int_0^\ell \sin\left(\frac{\pi x}{\ell}\right) \cdot f\, dx - \int_0^\ell \sin^2\left(\frac{\pi x}{\ell}\right) \cdot k \cdot A\, dx\right)$$

KNOWN $\quad \int_0^\ell \sin\left(\frac{\pi x}{\ell}\right) dx = \frac{2\ell}{\pi}$

PVW $\leadsto \quad \delta W_i = \delta W_e \quad \Rightarrow \quad EJ \cdot A \cdot \frac{\pi^4}{\ell^{\cancel{4}3}} \cdot \frac{\ell}{2} = -f \cdot \frac{2\ell}{\pi} - kA \cdot \frac{\ell}{2}$

$$v\left(\frac{\ell}{2}\right) = A \cdot \sin\left(\frac{\pi \cdot \ell/2}{\ell}\right) = A = -\frac{f\, 2\ell/\pi}{\frac{1}{2} EJ\, \pi^4/\ell^3 + k\ell/2} = -1.208\ mm$$

Contents


```
close all
clear variables
home
```

Data

```
l = 1000;   % mm
EJ = 1E+12; % Nmm^2
k = 8;      % MPa
f = 100;    % N/mm
```

Sol

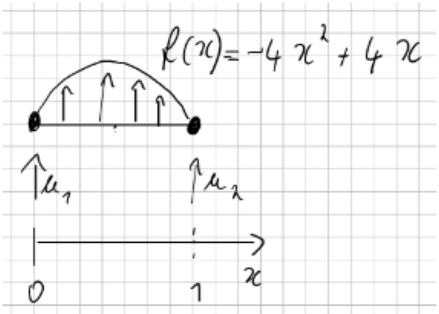
```
syms z A

phi = sin(pi*z/l);

LHS = int(diff(phi, 2, z)^2*EJ*A, z, 0, l) + ...
        int(phi*f, z, 0, l) + ...
        int(phi*k*A*phi, z, 0, l);

sol = solve(LHS == 0, A);

v = double(sol*subs(phi, z, l/2))
```

```
v =

  -1.207902973306587
```

Published with MATLAB® R2022b

Ex 4 (EXAM 13/06/2023)

The two-node finite element sketched in the figure is loaded by the distributed force $f(x)$. Compute the virtual work of the force for the virtual displacement of node #2. The overall lenght of the finite element is equal to 1 mm
(Unit for result: N mm)



Data
$f(x) = -4x^2 + 4x$ N/mm

FIND

$$\delta W_e = \delta u_1 \cdot [\ldots] + \delta u_2 \cdot \overbrace{[\ \ldots\ ]}^{\text{THIS}}$$

SOL

$$N_1(x) = (1-x)$$
$$N_2(x) = x$$

$$u(x) = N_1 \cdot u_1 + N_2 \cdot u_2$$

$$\delta W_e = \int_0^\ell \delta u \cdot f(x)\,dx$$
$$= \delta u_1 \cdot \int_0^\ell N_1 \cdot f(x)\,dx + \delta u_2 \int_0^\ell N_2 \cdot f(x)\,dx$$

- $$\int_0^\ell N_2 \cdot f(x)\,dx = \int_0^\ell x\,(-4x^2+4x)\,dx$$
$$= \int_0^\ell -4x^3+4x^2\,dx$$
$$= \left[-x^4 + \frac{4}{3}x^3\right]_0^\ell = -1 + \frac{4}{3} = \frac{1}{3}$$

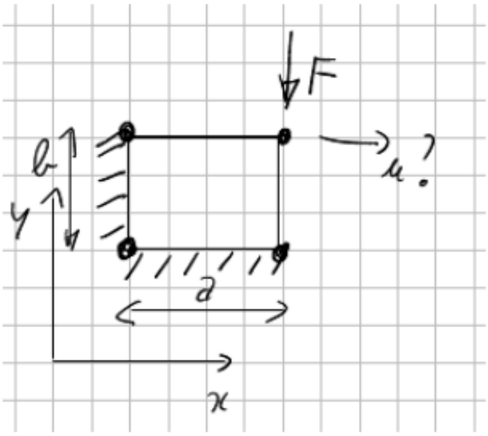
Ex 5 (EXAM 09/09/2024)

The single bilinear finite element sketched in the figure has the displacement of the top left, bottom left and bottom right nodes completely constrained. The element has unit thickness, and the material works in a state of plane stress, so that

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{1}$$

The top right node is loaded by the vertical force F, as sketched. Compute the horizontal displacement u of the top right node.

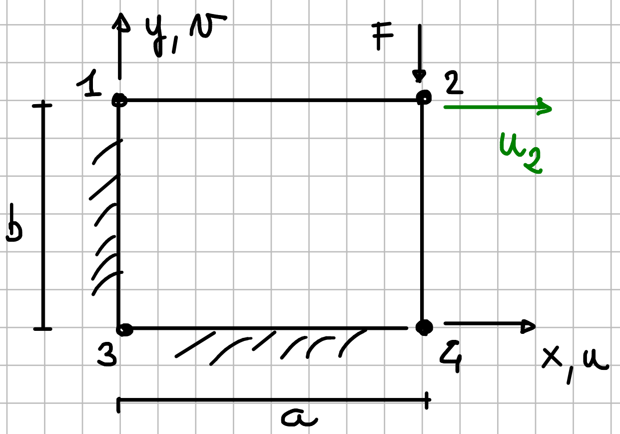
(Unit for result: mm)



Data
$t = 1$ mm
$a = 4$ mm
$b = 3$ mm
$E = 72000$ MPa
$\nu = 0.3$
$F = 100$ N

BILINEAR MEANS THAT $u(x)$ & $v(x)$ ARE LINEAR

SOL



FROM BC $\begin{cases} u_1, u_3, u_4 = 0 \\ v_1, v_3, v_4 = 0 \end{cases}$

PVW
$$\int_V \delta\underline{\varepsilon}^T \cdot \underline{\sigma} \, dV = \delta\underline{u}^T(a,b) \cdot \underline{F}$$

KNOW FROM PLANE STRESS EVERYTHING IS CONSTANT IN z

$$\int_V \delta\underline{\varepsilon}^T \cdot \underline{\sigma} \, dV = \int_A \delta\underline{\varepsilon}^T \cdot \underline{\sigma} \, dA \cdot t \quad \text{WHERE} \quad t = 1$$

KNOW $\underline{u} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$ & $\underline{F} = \begin{Bmatrix} 0 \\ -F \end{Bmatrix}$

IF FULLY UNCONSTRAINED

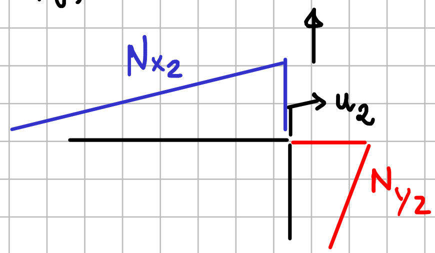
$$\underline{u}(x,y) = \underline{\underline{N}} \cdot \underline{q}$$

$$[\underline{\underline{N}}] \cdot \{\underline{q}\}$$
$$2\times8 \qquad 8\times1$$

$$\{\underline{q}\} = \left\{ \begin{matrix} \{u_i\} \\ \{v_i\} \end{matrix} \right\}$$

SINCE 1, 3, 4 ARE CONSTRAINED $\quad q$ IS $2\times1 \quad N$ IS 2×2

$$\underline{u}(x,y) = \left\{ \begin{matrix} u(x,y) \\ v(x,y) \end{matrix} \right\} = \begin{bmatrix} xy/ab & 0 \\ 0 & xy/ab \end{bmatrix} \left\{ \begin{matrix} u_2 \\ v_2 \end{matrix} \right\} = \underline{\underline{N}} \cdot \underline{q}$$



$$\underline{u} = \left[\left\{ \begin{matrix} \overline{(u_2 \cdot x/a)} \cdot \overline{y/b} \\ \overline{(v_2 \cdot y/b)} \cdot \overline{x/a} \end{matrix} \right\} \right]$$

HOW TO INTERPOLATE DISPLACEMENT:

FROM u_2 FIX $y = b$ GET $u(x)$

USE INTERPOLATION $N_{x_2} = x/a$

$u(x) = u_2 \cdot x/a$, THEN INTERP.

$u(x)$ IN y USING $N_{y_2} = y/b$

$\rightarrow u(x,y) = (u_2 \cdot x/a) \cdot y/b$

COMPUTE THE STRAIN

$$\underline{\varepsilon} = \left\{ \begin{matrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{matrix} \right\} = \underbrace{\begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix}}_{3\times2} \underbrace{\underline{u}}_{2\times1}$$

$$= \underbrace{\underbrace{\begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix}}_{3\times2} \underbrace{\begin{bmatrix} xy/ab & 0 \\ 0 & xy/ab \end{bmatrix}}_{2\times2}}_{\underline{\underline{B}}} \cdot \underbrace{\underbrace{\left\{ \begin{matrix} u_2 \\ v_2 \end{matrix} \right\}}_{2\times1}}_{\underline{q}}$$

$$\underline{\varepsilon} = \begin{bmatrix} y/ab & 0 \\ 0 & x/ab \\ x/ab & y/ab \end{bmatrix} \cdot \left\{ \begin{matrix} u_2 \\ v_2 \end{matrix} \right\} = \underline{\underline{B}} \cdot \underline{q}$$

STRESS AS FUNCTION OF STRAIN $\quad \underbrace{\underline{\sigma}}_{3\times1} = \underbrace{\underline{\underline{D}}}_{3\times3} \cdot \underbrace{\underline{\varepsilon}}_{3\times1}$

$$\delta W_i = \int_A \underset{1\times 2}{\delta \underline{q}^T} \cdot \underset{2\times 3}{\underline{\underline{B}}^T} \cdot \underset{3\times 3}{\underline{\underline{D}}} \cdot \underset{3\times 2}{\underline{\underline{B}}} \cdot \underset{2\times 1}{\underline{q}} \, dA$$

$$\delta W_e = \delta \underline{q}^T \cdot \underbrace{\underline{\underline{N}}^T(a,b)}_{\underline{\underline{I}}} \cdot \underline{F} = \underset{1\times 2}{\delta \underline{q}^T} \cdot \underset{2\times 1}{\underline{F}} = -\delta v_2 \cdot F$$

PVW $$\delta \underline{q}^T \cdot \underbrace{\int_A \underline{\underline{B}}^T \underline{\underline{D}}\, \underline{\underline{B}} \, dA}_{\underline{\underline{K}}} \cdot \underbrace{\underline{q}}_{\underline{q}} = \delta \underline{q}^T \cdot \underbrace{\underline{F}}_{\underline{f}}$$

SOLVE $\underline{\underline{K}}\,\underline{q} = \underline{f}$ FOR $\underline{q} = \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$ $\longrightarrow u_2 = 1.08 \cdot 10^{-3}$ mm

Contents


```
close all
clear variables
home
```

Sol

```
syms x y z u_2 v_2 t a b E nu F

q = [u_2; v_2];
B = [
    y/a/b 0;
    0 x/a/b;
    x/a/b y/a/b
];
D = E/(1-nu^2)*[
    1 nu 0;
    nu 1 0;
    0 0 (1-nu)/2
];
F_v = [
    0;
    -F
];

RHS = F_v;
K = int(...
    int( ...
    int( ...
    transpose(B)*D*B, x, 0, a ...
    ), y, 0, b ...
    ), z, 0, t);

LHS = K*q;
```

Data

```
t = 1;
a = 4;
b = 3;
E = 72000;
nu = .3;
F = 100;

sol = solve(subs(LHS) == subs(RHS), q);

u_2_d = double(sol.u_2)
```

```
u_2_d =

   0.001084801369375
```