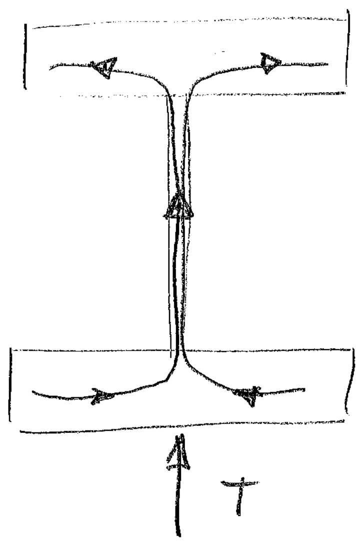
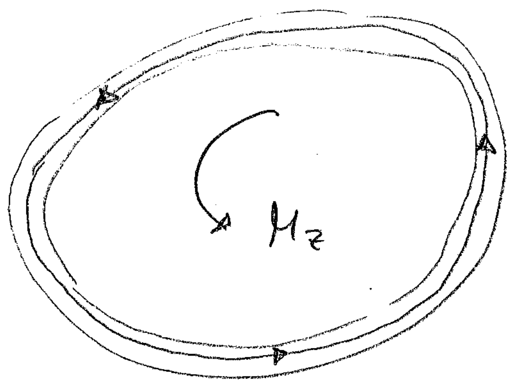
Semi-monocoque beam approximation

- The solution obtained for the direct stress component σ_{zz} is an exact and closed-form result that can be applied independently on the beam section shape.
- Contrarily, the evaluation of the shear stress components σ_{xz} and σ_{yz} (which are not null whenever the shear and/or torsion internal action is different from zero) is generally a more complex task to be accomplished. In this sense, the analogy with fluid motion allowed to gather insight into the <u>qualitative</u> description of the distribution of the shear flows.





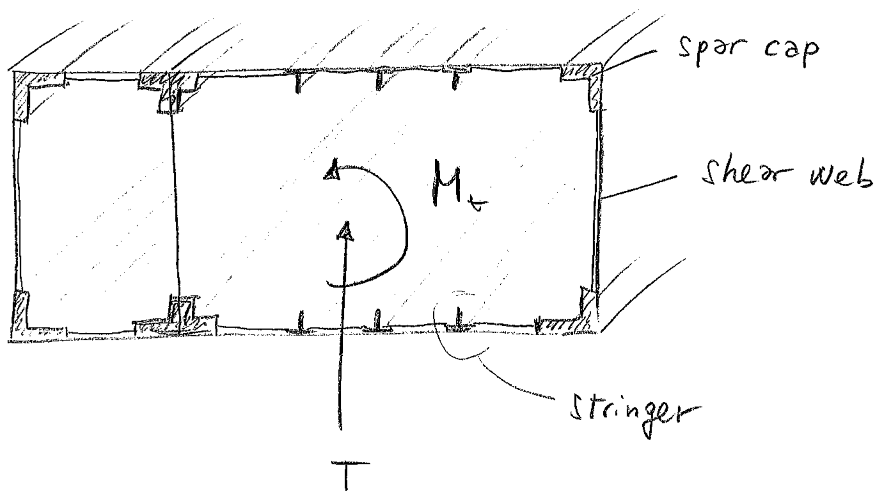
The qualitative description of the stress distribution was then used as a basis for developing <u>approximate</u> solutions for

a. Torsion of thin open profiles
b. Torsion of closed, single cell thin profiles

It is now necessary to extend the range of applications, with regard to typical aerospace thin-walled constructions, and derive approximate solutions for:

a. Torsion of thin walled single/multi cell section
b. Shear of thin walled single/multi cell section

The typical application can be imagined as a multi-cell stringer-stiffened beam section



How to determine the stress distribution in the beam section sketched above?

Again, recall that the stress σ_{zz} is obtained as:

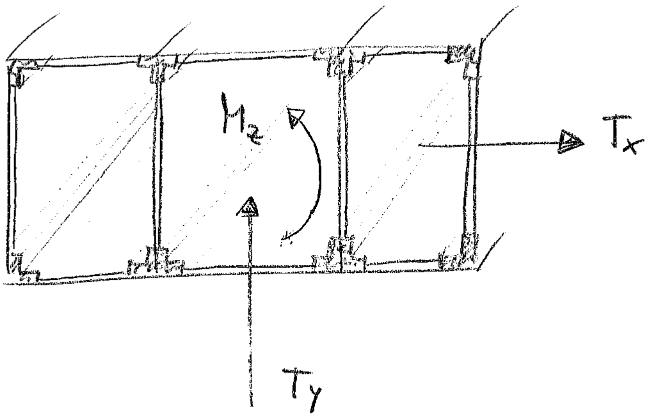
$$\sigma_{zz} = \frac{T_z}{A} + \frac{M_x}{J_{xx}} y - \frac{M_y}{J_{yy}} x$$

where M_x and M_y depend linearly on the applied shear forces.

The shear stress components will be evaluated in the context of the semi-monocoque approximation. It is then important to understand that with the term "semi-monocoque" it is here intended the approximating technique for evaluating the shear stresses of thin-walled beams

The shear flow equation

The evaluation of the shear flow was introduced in the context of the torsion of single-cell thin-walled beams. However, the effect of shear forces has not been assessed yet. To this aim, consider a generic thin-walled beam section, subjected to generic internal forces T_x and T_y (shearing) and torsional moment M_z.



A strategy for obtaining a solution for the shear stress distribution consists in considering the equilibrium equations

$$\operatorname{div} \underline{\underline{\sigma}} = \underline{0} \quad \Rightarrow \quad \begin{cases} \sigma_{xz/z} = 0 \\ \sigma_{yz/z} = 0 \\ \sigma_{xz/x} + \sigma_{yz/y} + \sigma_{zz/z} = 0 \end{cases}$$

together with the boundary conditions

$$\underline{t} \cdot \underline{n} = 0$$

and the equivalence with internal actions

$$T_y = \int_A \sigma_{yz}\, dA; \quad T_x = \int_A \sigma_{xz}\, dA; \quad M_z = \int_A (\sigma_{yz} x - \sigma_{xz} y)\, dA$$

Compatibility is not accounted for, this constituting a source of approximation in the solution that will be derived. More specifically, the solution will be respectful of the equilibrium requirements, but will not guarantee compatibility in the thickness-wise direction.

As already discussed in the context of DSV beam theory, the first two equations simply state that

$$\sigma_{xz} = \sigma_{xz}(x,y)$$
$$\sigma_{yz} = \sigma_{yz}(x,y)$$

The third equation is then the one to be solved:

$$\sigma_{xz/x} + \sigma_{yz/y} = - \sigma_{zz/z}$$

Recall the DSV solution for σ_{zz}:

$$\sigma_{zz} = \frac{T_z}{A} + \frac{M_x}{J_{xx}} y - \frac{M_y}{J_{yy}} x$$

with $M_x = \hat{M}_x + T_y z$ and $M_y = \hat{M}_y - T_x z$

It follows that:

$$\sigma_{zz/z} = \frac{M_{x/z}}{J_{xx}} y - \frac{M_{y/z}}{J_{yy}} x$$

$$= \frac{T_y}{J_{xx}} y + \frac{T_x}{J_{yy}} x \quad , \text{ and so:}$$

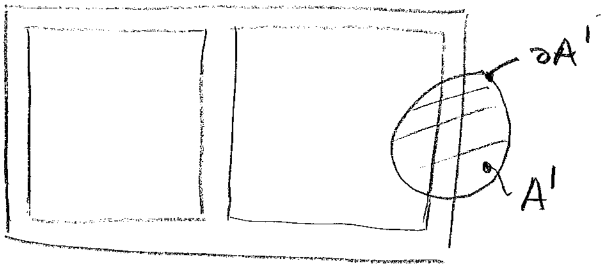
$$\sigma_{xz/x} + \sigma_{yz/y} = - T_y \frac{y}{J_{xx}} - T_x \frac{x}{J_{yy}}$$

More conveniently, the expression can be written as:

$$\operatorname{div} \underline{t} = - T_y \frac{y}{J_{xx}} - T_x \frac{x}{J_{yy}}$$

where, as usual, $\underline{t} = \sigma_{xz} \underline{e}_x + \sigma_{yz} \underline{e}_y$

Consider now an arbitrary region A' over the beam section A, e.g.



The equilibrium equation can be integrated over A':

$$\int_{A'} \operatorname{div} \underline{t} \, dA' = - \int_{A'} T_y \frac{y}{J_{xx}} dA' - \int_{A'} T_x \frac{x}{J_{yy}} dA'$$

Noting that T_x, T_y, J_{xx} and J_{yy} do not depend on A', they can be taken out of the integral. Applying the divergence theorem, it is obtained:

$$\int_{\partial A'} \underline{t} \cdot \underline{n} \, d\Gamma = - \frac{T_y}{J_{xx}} \int_{A'} y \, dA' - \frac{T_x}{J_{yy}} \int_{A'} x \, dA'$$

$$= - T_y \frac{S_x'}{J_{xx}} - T_x \frac{S_y'}{J_{yy}}$$

having denoted with

$$S_x' = \int_{A'} y \, dA' ; \quad S_y' = \int_{A'} x \, dA'$$

the first order moments of inertia of the area A'

Recalling that

$\phi = \int_{\partial A'} \underline{t} \cdot \underline{n} \, d\Gamma$, where ϕ denotes the net flow of shear stresses outflowing from A', the previous equation can be written as:

$$\boxed{\phi = - T_y \frac{S_x'}{J_{xx}} - T_x \frac{S_y'}{J_{yy}}}$$

shear flow equation

An alternative representation of the shear flow equation is obtained as:

$\sigma_{xz/x} + \sigma_{yz/y} = - \sigma_{zz/z}$ (equilibrium equation)

$\int_{A'} \operatorname{div} \underline{t} \, dA' = - \int_{A'} \sigma_{zz/z} \, dA'$

and, after applying the divergence theorem:

$$\boxed{\phi = - T_z'{}_{/z}}$$

where $T_z' = \int_{A'} \sigma_{zz} \, dA'$ is the axial force associated with the portion of area A'

Remarks

- The shear flow equation, in the form

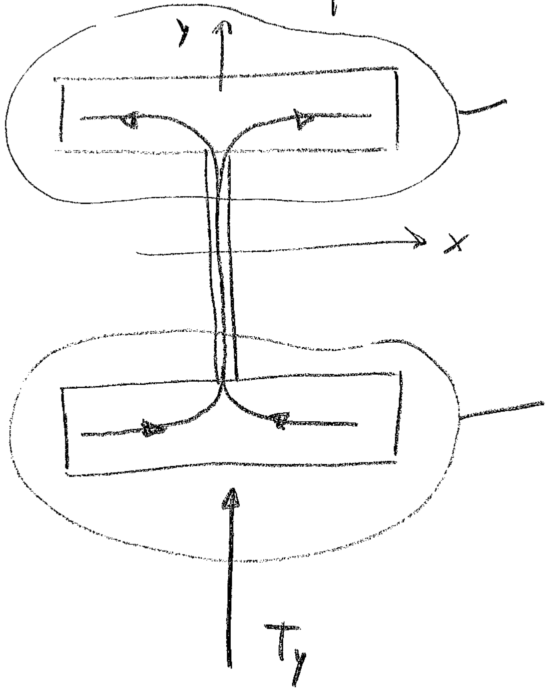
$$\phi = -T_x \frac{S_y'}{J_{yy}} - T_y \frac{S_x'}{J_{xx}}$$

provides a meaningful analogy with fluid motion

Recalling that

$$\phi = \oint_{\partial A'} \underline{t} \cdot \underline{n} \, d\Gamma \quad \begin{array}{l} \to > 0 \quad \text{positive outflow (\underline{source})} \\ \to < 0 \quad \text{negative outflow. (\underline{sink})} \end{array}$$

As an example consider



$S_x' > 0 \Rightarrow \phi < 0 \Rightarrow$ sink

$S_x' < 0 \Rightarrow \phi > 0 \Rightarrow$ source

- Whenever no shear forces are present, the shear flow equation reduces to

$\boxed{\phi = 0}$ which is the result already available for the torsion

$\phi = 0$ means <u>constant</u> shear flow (not null!)

- It is useful to illustrate that the shear flow equation implies the fulfillment of the in-plane equivalence conditions with the internal shear forces T_x, T_y. viz. if the shear flow equation is satisfied, the resultant of the shear stresses over the beam section A is equal to the internal shear forces T_x and T_y.

To this aim multiply both sides of the equation with x:

$$\int_A x \operatorname{div} \underline{t}\, dA = \int_A -x\,\sigma_{zz/z}\, dA$$

(Note: the integrals are now taken over A to obtain an expression where the internal forces appear)

a. left-hand term

$$\operatorname{div}(x\underline{t}) = (x\sigma_{xz})_{/x} + (x\sigma_{yz})_{/y}$$

$$= \sigma_{xz} + x\,\sigma_{xz/x} + x\,\sigma_{yz/y}$$

$$= \sigma_{xz} + x \operatorname{div} \underline{t}$$

So: $\quad x \operatorname{div}\underline{t} = \operatorname{div}(x\underline{t}) - \sigma_{xz}$

$$\int_A x \operatorname{div}\underline{t}\, dA = \int_A (\operatorname{div} x\underline{t} - \sigma_{xz})\, dA$$

$$= \int_{\partial A} x\,\underline{t}\cdot\underline{n}\, d\Gamma - \int_A \sigma_{xz}\, dA = -\int_A \sigma_{xz}\, dA$$

b. Right-hand term

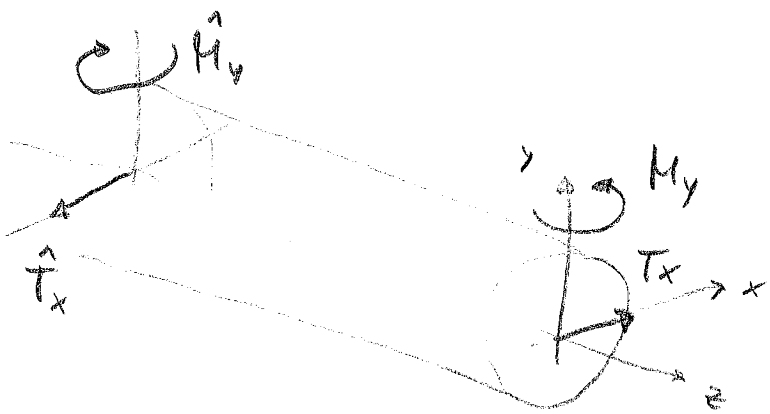
$$-\int_A x\,\sigma_{zz/z}\,dA = -\int_A (x\,\sigma_{zz})_{/z}\,dA$$

$$= +\,M_{y/z}$$

The initial equation is then re-written as:

$$\int_A \sigma_{xz}\,dA = -\,M_{y/z}$$

However the internal bending moment M_y is



$$M_y = \hat{M}_y - \hat{T}_x z$$
$$= \hat{M}_y - T_x z$$
$$M_{y/z} = -\,T_x$$

and so:

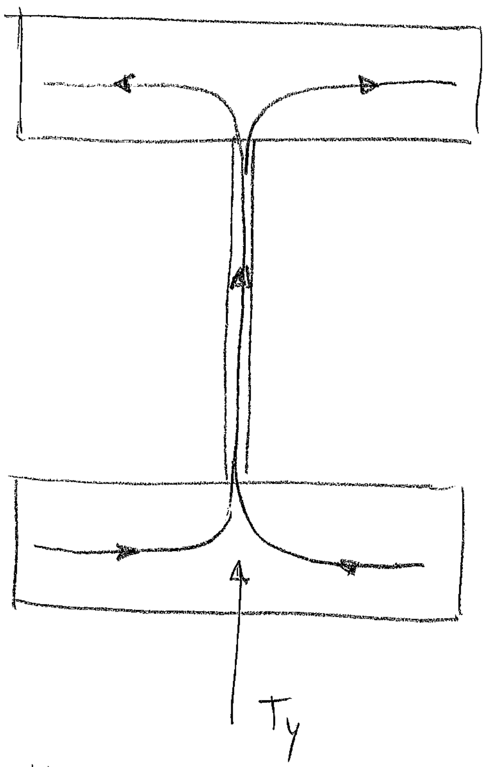
$\boxed{\int_A \sigma_{xz}\,dA = T_x}$, meaning that the integral of the shear stress component σ_{xz} over the section satisfies the equivalence condition with the internal action T_x.

In a similar manner it can be proved that

$$\boxed{\int_A \sigma_{yz}\,dA = T_y}$$

The assumption of constant shear flow for thin panels

Consider for simplicity, but without loss of generality, a beam characterized by an I section. The internal forces are in the form of a shear force T_y.



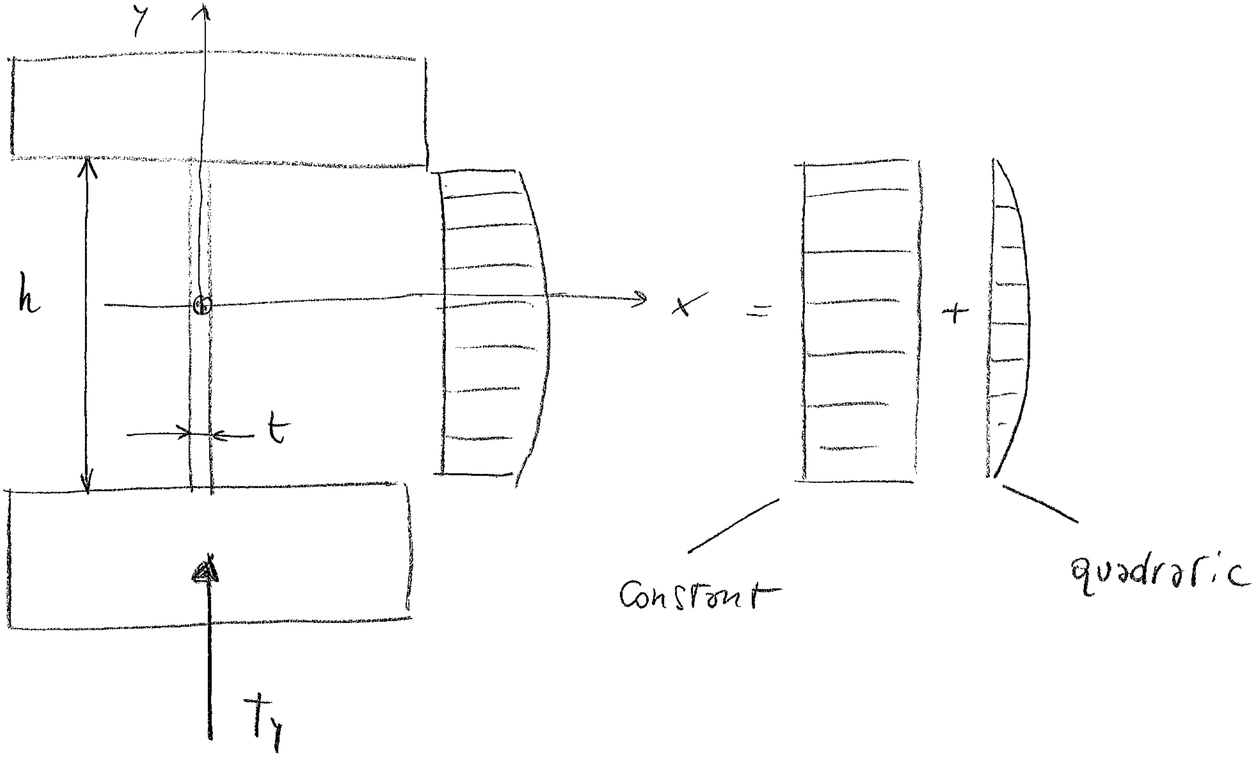
Assume that the web is thin. (in relation to the typical dimensions of the section)

The distribution of the internal shear flow can be traced by recalling the analogy with fluid motion

It is then clear that

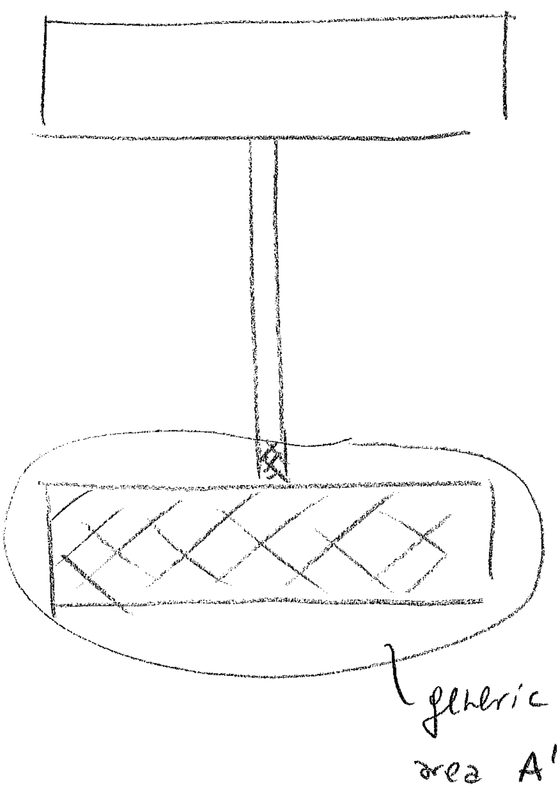
1. the shear stresses are mainly carried by the web
2. the two flanges do not contribute to the shear load carrying capabilities of the section, whilst they do contribute to the bending stiffness of the section.

The evaluation of the shear flow along the web of the section can be evaluated by making use of the shear flow equation.



Indeed, from the shear flow equation:

$$\phi = -T_y \frac{S_x'}{J_{xx}}$$



$$S_x' = \int_{A'} y \, dA' =$$

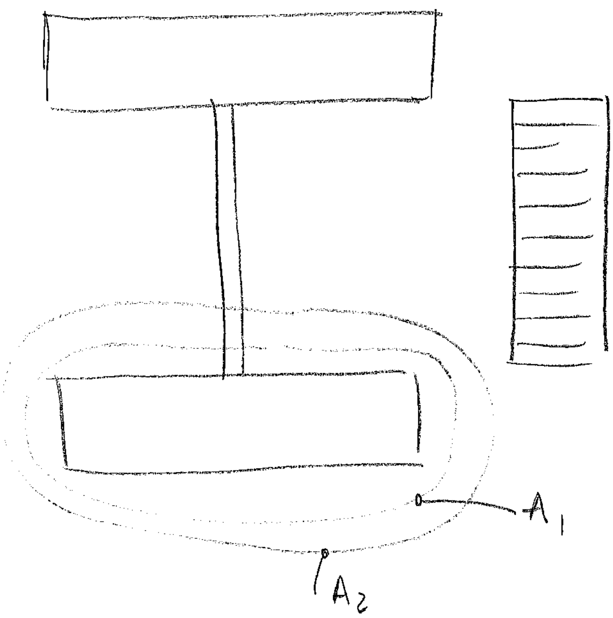
$$= S_{x\,\text{flange}} + \int_{-h/2}^{y} \xi \, t \, d\xi$$

$$= \underbrace{S_{x\,\text{flange}}}_{\text{constant}} + \underbrace{\frac{t}{2}\left(y^2 - \frac{h^2}{4}\right)}_{\text{quadratic}}$$

Whenever the web is <u>thin</u>, the contribution S_x' is mainly affected by the flange-related part of the expression, meaning that:

$$S_x' = S_{x_{flange}} + \frac{t}{2}\left(y^2 - h^2/4\right) \simeq S_{x_{flange}}$$

In other words, this means that the static moment S_x' does not vary as the position along the web is varied

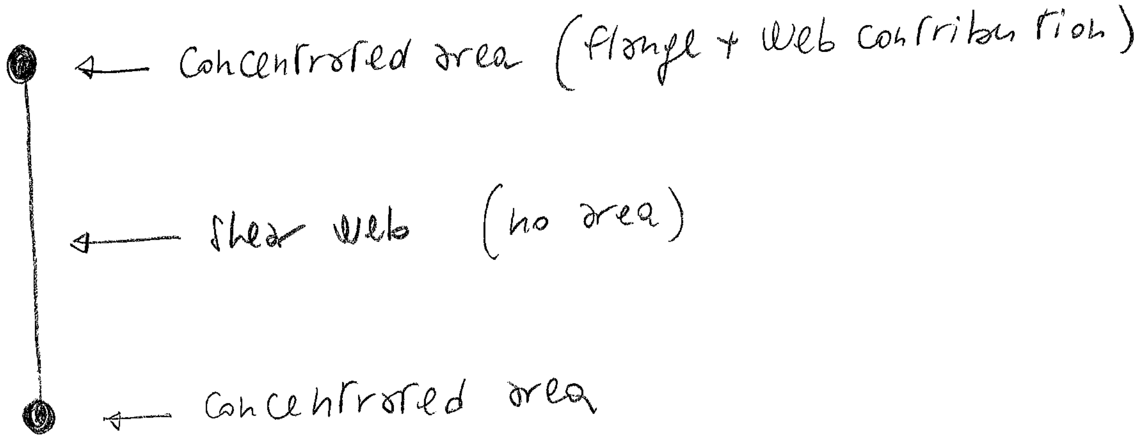


$$S_{x_1}' \simeq S_{x_2}'$$

and so the shear stress distribution can be <u>approximated</u> as <u>constant</u>

$$(\phi_1 = \phi_2)$$

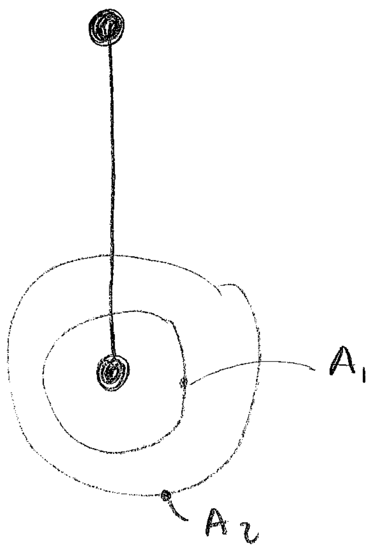
According to these observations, a natural way of representing the section would be the following



In particular:

1. the web is modelled as a line with no associated area, capable of transferring shear forces only
2. the flanges are modelled as lumped areas where the contribution of the shear web is introduced.

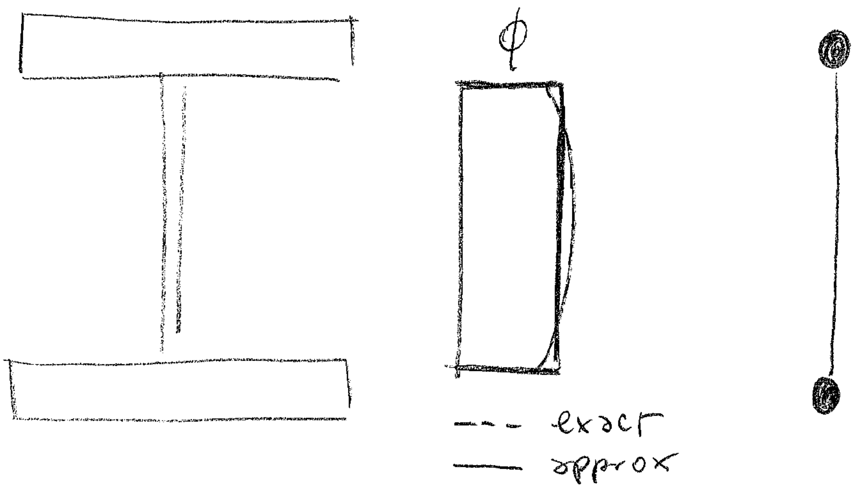
According to this model, it follows that:



$$\phi_1 = -T_y \frac{S'_{x_1}}{J_{xx}}$$

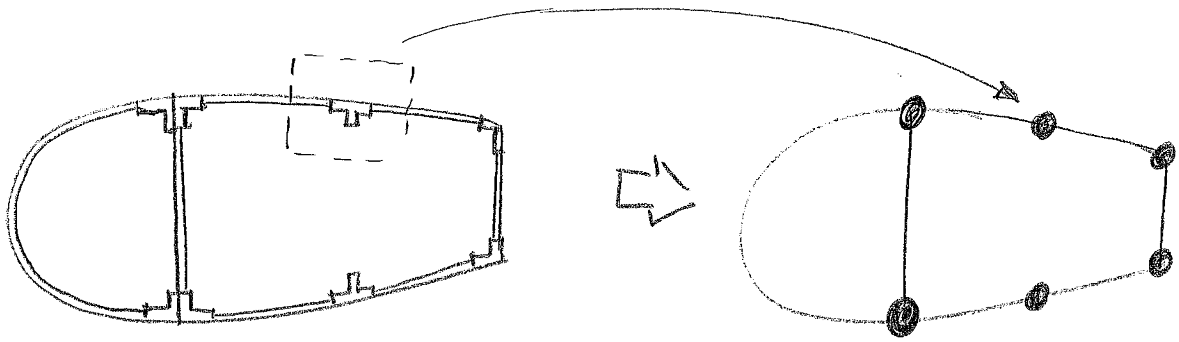
$$\phi_2 = -T_y \frac{S'_{x_2}}{J_{xx}}$$

but $S'_{x_1} = S'_{x_2}$ (the area is only in the lumped areas), and so $\boxed{\phi_1 = \phi_2}$ $\Rightarrow$ the shear flow over the web is constant.



This type of approximation is one of the key ideas of the semi-monocoque scheme.

For instance, using the same approach with the same motivations, the following scheme will be applied to realize a conceptual scheme of a two cell wing-box structure

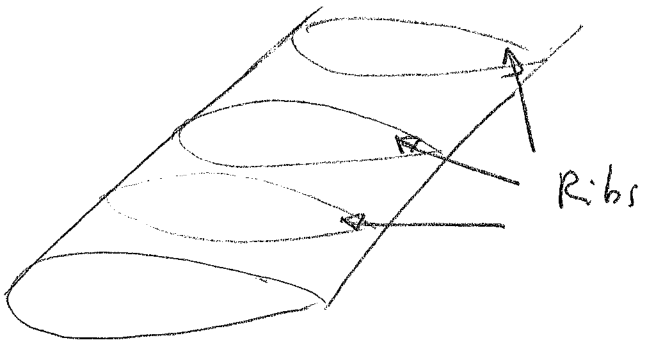


<u>Some remarks on the semi-monocoque scheme</u>

1. The panels are assumed to be <u>thin</u>, such that the shear flow can be assumed to be constant. Whenever the panels get thicker, it means that the assumption of constant shear flow becomes stronger.
   The model could still be used as a preliminary mean to perform calculations. However the evaluation of the appropriateness of the model is responsibility of the analyst.

2. The stringers should be characterized by a section which is <u>small in comparison to the dimensions of the section</u>.

3. Ribs / frames are assumed to be highly (infinitely) stiff in the in-plane directions, while they are considered highly compliant in the out-of plane direction.



The in-plane rigidity is due to the fact that a rib, when loaded in the in-plane direction, works as a membrane. This means that the stiffness is offered by its membrane behaviour, which is characterized by much higher values in comparison to the bending one (see previous discussions on the axial and bending stiffnesses of beams)

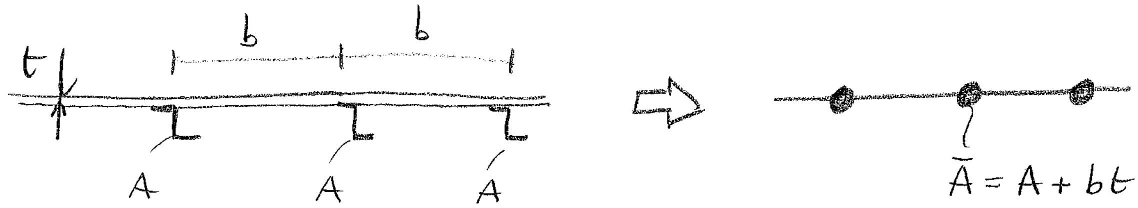
The out-of-plane flexibility is associated with the weaker bending properties of a thin plate with respect to the membrane ones.

The fact that a rib is flexible in terms of bending means that the beam sections are not prevented from warping in correspondence of the ribs.

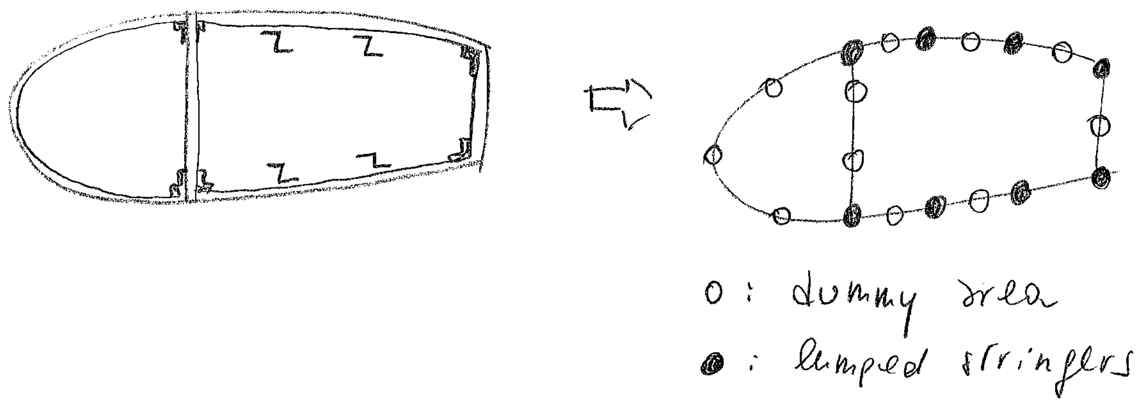
As seen in the problem of torsion, the beam section was assumed free to warp, which is exactly what happens thanks to the rib flexibility.

Whenever this is not the case, and the beam is prevented from warping, different modeling choices are needed (e.g. Vlasov beam model) to account for the stresses σ_{zz} arising from this kind of constraint (recall σ_{zz} is equal to zero in classical DSV torsion problem)

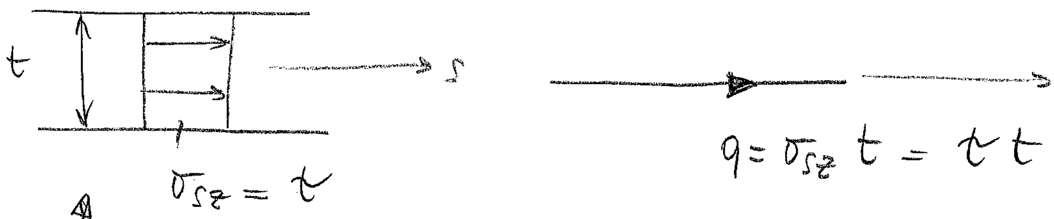
4. According to the semi-monocque model, the areas are lumped, thus any stringer is replaced with a concentrated area inclusive of the panel area



However, it is useful to highlight that the model can be refined by refining the lumping process and making use of "dummy" lumped areas (note: "dummy" has nothing to do with the dummy systems used in PCVW). As an example

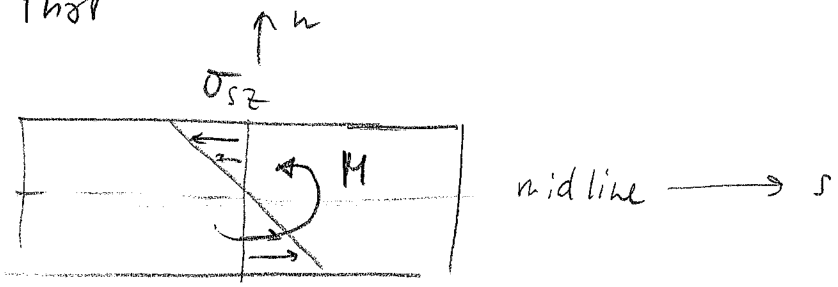


5. The panels are assumed to be thin, and the thickness is not sketched as the internal shear stress is referred to the panel midline

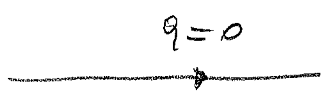


Assumption of constant stress in the thickness-wise direction (as the panel is thin)

It is then clear that, according to the semi-monocoque approximation, a thin panel cannot react a torsional load. Recall, from the solution relative to the torsion of thin-walled open profiles (narrow strips) that



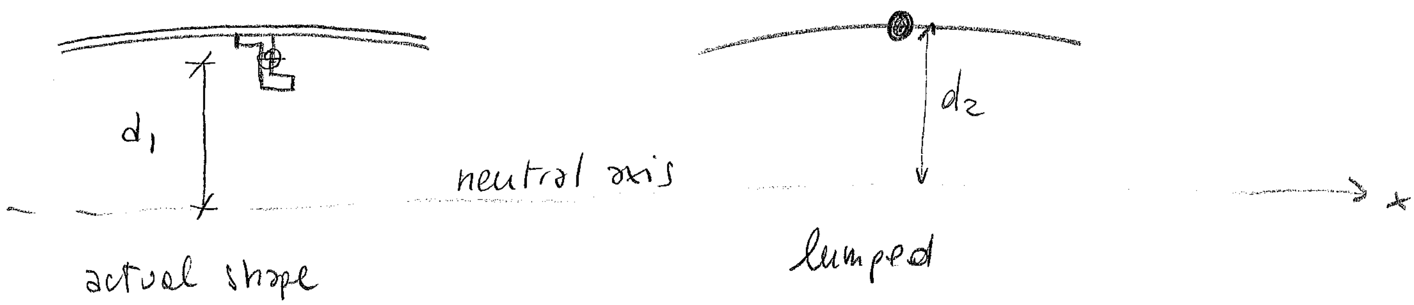
In the context of the semi-monocoque approximation the shear flow, referred to the midline, will be zero, as the integral of the linear distribution is null



$$q = \int_t \sigma_{sz}(n)\, dn = 0$$

The inability to react a torsional load stems from the mathematical model (semi-monocoque approximation) and is not an intrinsic property of the structure (which, as seen, is characterized by a small but not null torsional stiffness)

6. When the stringer area is lumped, it should be paid attention not to artificially increase the bending stiffness of the section.



The contribution of the stringer to the second moment of inertia is:

$J_{xx} = J_{xx}^{local} + A d_1^2$ (looking at the actual shape)

where J_{xx}^{local} is the moment of inertia referred to the stringer local axes.

When the stringer is lumped:

$J_{xx} = A d_2^2$ (looking at the lumped scheme)

It follows that:

a. the contribution J_{xx}^{local} is lost. (the lumped stringer has no shape)

This is always an acceptable approximation as $Ad_1^2 >> J_{xx}^{local}$

6. The process of lumping tends to raise the value of J_{xx} as $d_2 > d_1$, so

$Ad_1^2 > Ad_2^2$ (note that J_{xx} is quadratic in d_i, so the effect can be not completely negligible.

For this reason the lumped area of the stringer would be better approximated by imposing:

$$Ad_1^2 = A_{lumped}\, d_2^2$$

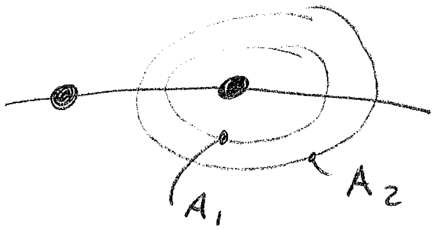
$$\Rightarrow \boxed{A_{lumped} = A\,\frac{d_1^2}{d_2}}$$

7. Having lumped the areas in the stringers, it follows that:

    a. the normal stress is carried by the lumped stringers

    b. the shear stress is carried by the panels

Variations of shear stresses are possible only when crossing lumped areas



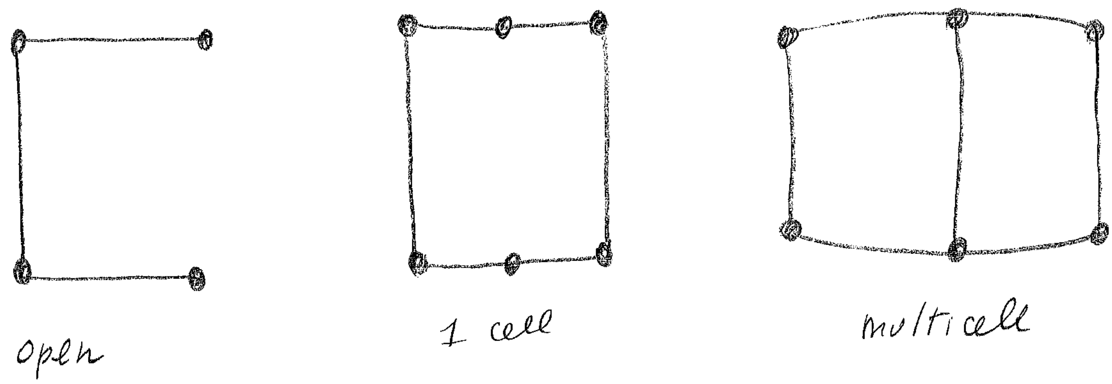
$\phi_1 = \phi_2 \Rightarrow \phi = \text{const}$

Different kinds of sections

The sections can be classified as:

1. Open sections
2. Closed sections (1 cell)
3. Closed sections (multicell)

Three examples are reported below:



The following nomenclature will be adopted:

a. n = number of stringers
b. m = number of panels
c. $N = m - n + 1$: number of cells

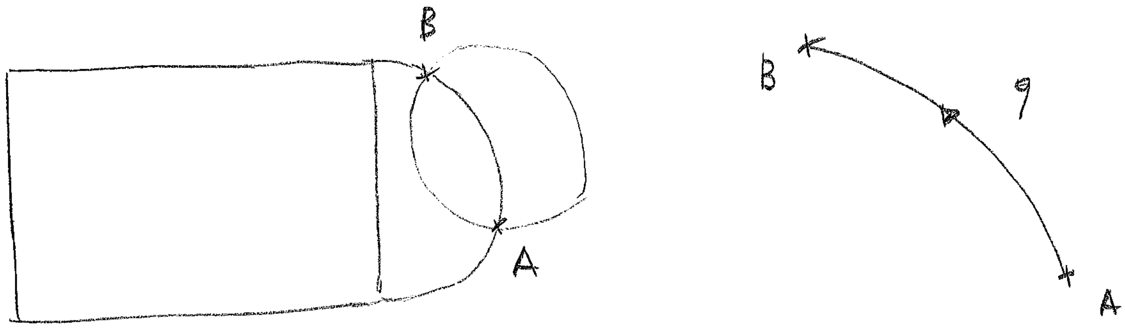
In the three previous examples it is:

$n = 4$	$n = 6$	$n = 6$
$m = 3$	$m = 6$	$m = 7$
$N = 0$	$N = 1$	$N = 2$

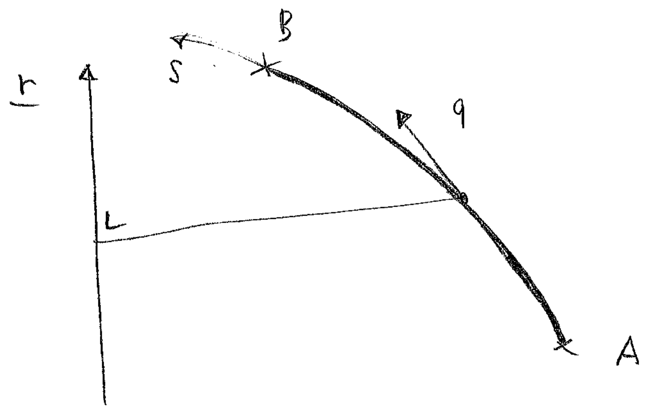
Resultant of the flow in a panel

It is important to understand how to evaluate the resultants associated with the shear flow acting in a generic panel.

To this aim consider a generic panel of a section

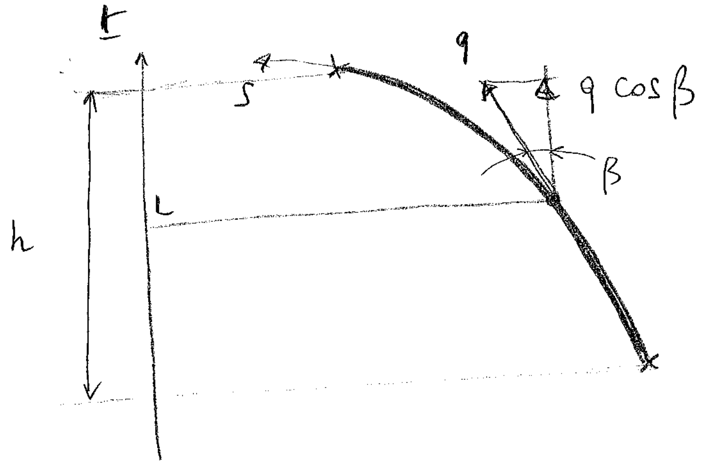


The evaluation of the resultant along a generic direction $\underline{r}$ can be performed as it follows.



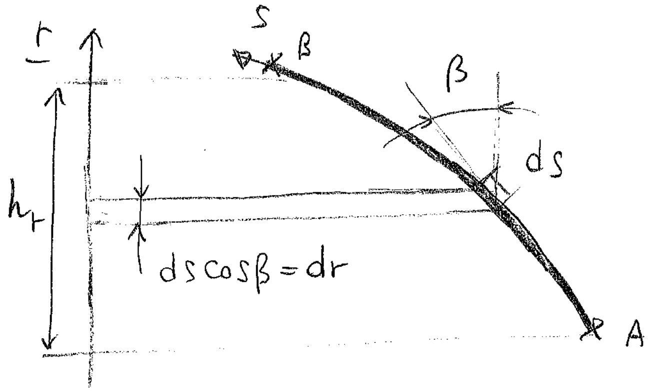
$\underline{r}$ arbitrary direction

<u>Note</u>: q is locally tangent to the panel midline



The contribution due to the infinitesimal portion ds is:

$$\boxed{dR = q\cos\beta \, ds}$$



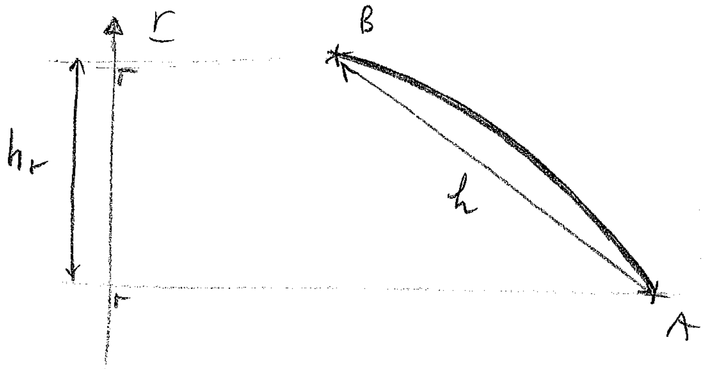
$$dr = ds\cos\beta$$

It follows that: $dR = q\cos\beta\, ds = q\, dr$

The resultant along $\underline{r}$ is denoted as $R_{\underline{r}}$ and is:

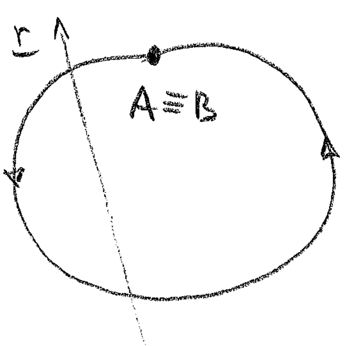
$$R_{\underline{r}} = \int_A^B dR = \int_A^B q\, dr = q\, h_r$$

It is worth noting that h_r is the projection of the length h over the direction $\underline{r}$ (see figure below).



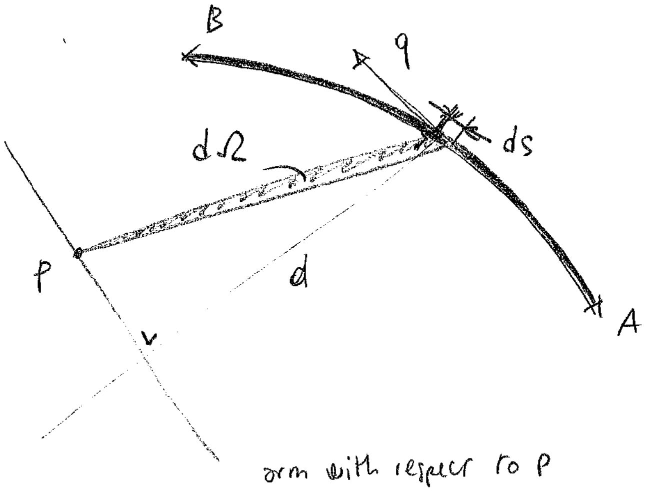
$$\boxed{R_{\underline{r}} = q\, h_r \quad \text{with } h_r \text{ projection of } h \text{ on } \underline{r}}$$

As a consequence of this result, it follows that $R_{\underline{r}} = 0$ whenever $A \equiv B$, e.g.



$$R_{\underline{r}} = 0$$

The evaluation of the moment with respect to a generic point P is:

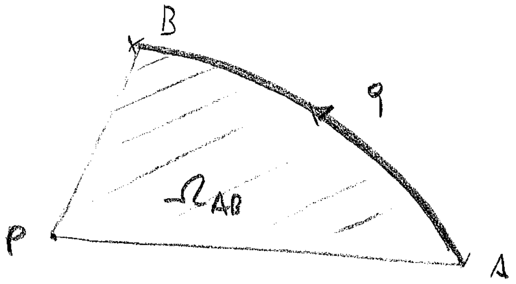


infinitesimal area of the triangle with basis ds and height d,

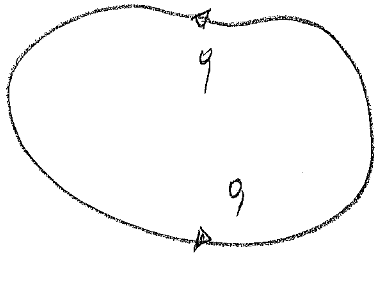
$$dM = \underbrace{q\,ds}_{\text{infinitesimal force acting on ds}} \cdot \overbrace{d}^{\text{arm with respect to P}} = q\,d \cdot ds = 2q\,d\Omega$$

$$M = \int_A^B dM = \int_A^B 2q\,d\Omega = 2q\int_A^B d\Omega = 2q\,\Omega_{AB}$$

$\boxed{M = 2q\,\Omega_{AB}}$ where Ω_{AB} is the area PAB



In the case of a closed section (see also Bredt's formula)



$$M = 2q\,\Omega_{cell}$$

Solution of the section

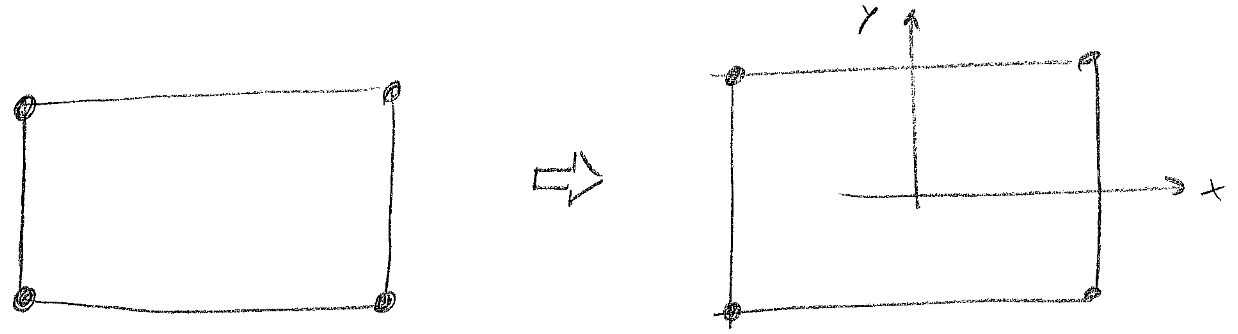
The solution strategy relies upon the discretization of the section into lumped areas, and the application of the shear flow equation. Recall that the solution

$$\phi = - T_y \frac{S_x'}{J_{xx}} - T_x \frac{S_y'}{J_{yy}}$$

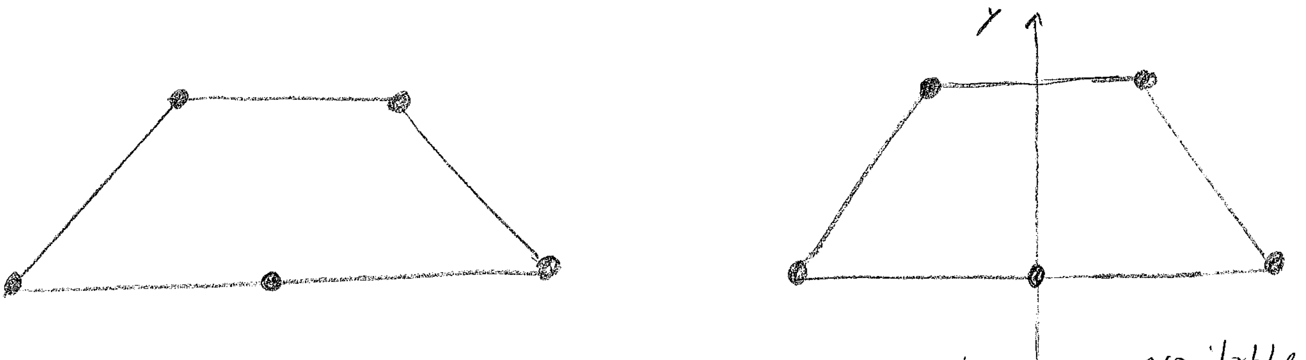
was derived after substituting the DSV solution for σ_{zz} into the equilibrium equation. In turn, the DSV solution was derived by considering principal axes.

Thus, principal axis will always be considered during the analysis of the section.

Often they are available from symmetry consideration

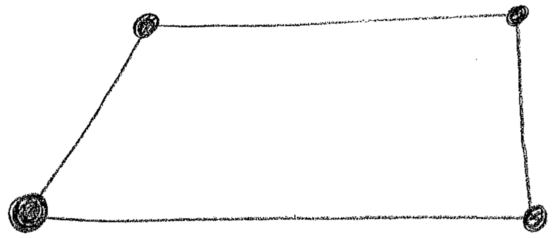


principal axes



one single axis available from symmetry

Whenever the section does not display any kind of symmetry, the principal axes can be determined as follows:



1. Consider a generic reference system $\bar{x}\bar{y}$
2. Evaluate the position of the centroid as:

$$\boxed{x_{CG} = \frac{\sum_i A_i \bar{x}_i}{\sum_i A_i} \qquad y_{CG} = \frac{\sum_i A_i \bar{y}_i}{\sum_i A_i}}$$

3. Determine the moments of inertia with respect to a set of axis $\tilde{x}, \tilde{y}$ with origin in (x_{CG}, y_{CG})

$$\begin{aligned} J_{\tilde{x}\tilde{x}} &= \sum_i A_i \tilde{y}_i^2 \\ J_{\tilde{y}\tilde{y}} &= \sum_i A_i \tilde{x}_i^2 \qquad \Rightarrow \qquad \alpha = \frac{1}{2} \operatorname{atan} \frac{2 J_{\tilde{x}\tilde{y}}}{J_{\tilde{y}\tilde{y}} - J_{\tilde{x}\tilde{x}}} \\ J_{\tilde{x}\tilde{y}} &= \sum_i A_i \tilde{x}_i \tilde{y}_i \end{aligned}$$

4. Rotate $\tilde{x}\tilde{y}$ by an angle α

$$\boxed{\begin{aligned} x &= \cos\alpha \, \tilde{x} + \sin\alpha \, \tilde{y} \\ y &= -\sin\alpha \, \tilde{x} + \cos\alpha \, \tilde{y} \end{aligned}}$$

(5. Evaluate now J_{xx} and J_{yy} in the novel reference system xy. Verify that $J_{xy} = 0$!)

Evaluation of the stresses

1 Normal stress σ_{zz}

Once the principal axes are available, the axial stress is obtained as:

$$\boxed{\sigma_{zz_i} = \frac{T_z}{A} + \frac{M_x}{J_{xx}} y_i - \frac{M_x}{J_{yy}} x_i}$$

i = i-th stringer

where σ_{zz_i} : axial stress on the i-th stringer

A : total section area $(A = \Sigma_i A_i)$

J_{xx}, J_{yy} : second moments of inertia of the section

2. Shear stresses

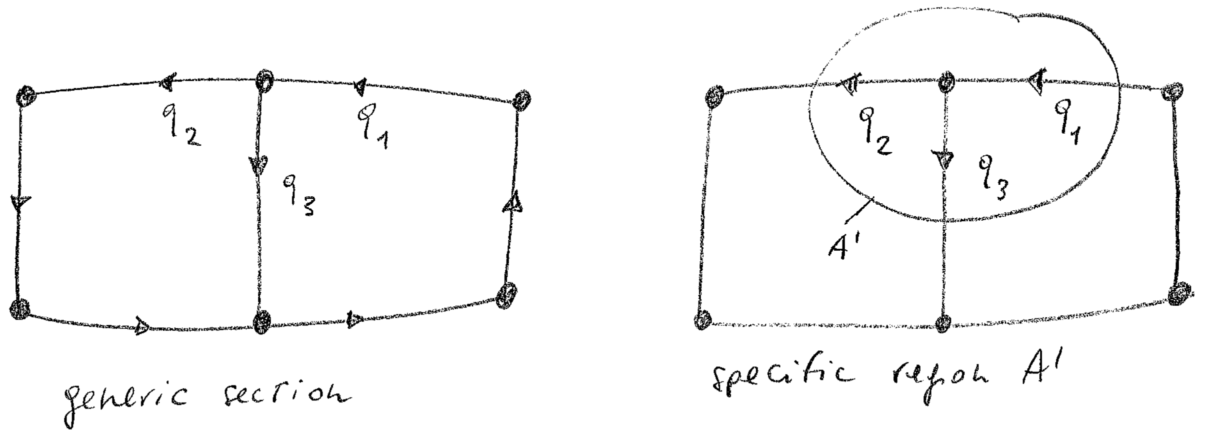
They are obtained by making use of the shear flow equation.

$$\phi = - T_y \frac{S_x'}{J_{xx}} - T_x \frac{S_y'}{J_{yy}} \qquad (a)$$

$$= - T_z'/z \qquad (b)$$

The equation (a) provides the "operative" formula. However, equation (b) is a useful alternative expression (and equal to (a)) which provides a clear insight into the actual meaning of the shear flow equation

Consider a generic section, as illustrated in the sketch:



Applying the shear flow equation with respect to the region A' leads to:

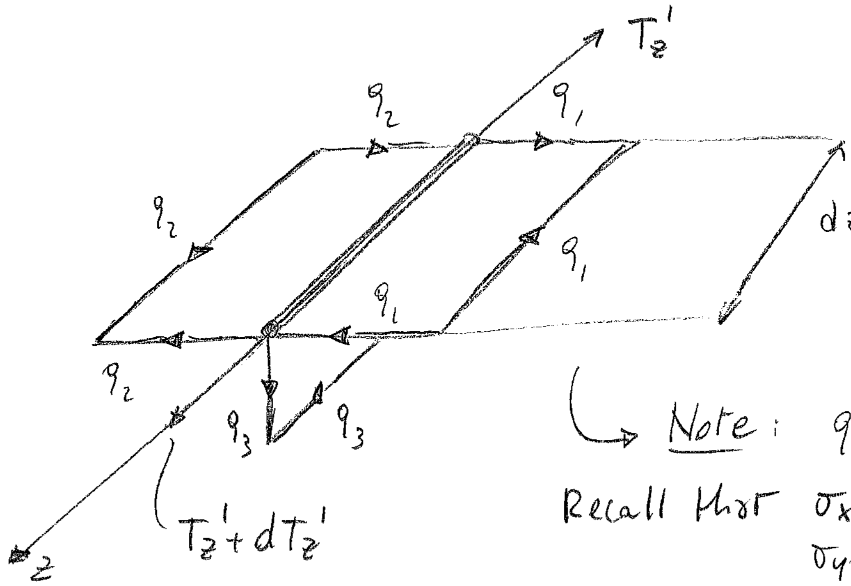
$$\phi = \int_{\partial A'} \underline{t} \cdot \underline{n} \, d\Gamma = q_2 + q_3 - q_1$$

and recalling that $\phi = -T_z'/_z$ it follows that the shear flow equation can be expressed as:

$$\boxed{q_2 + q_3 - q_1 = -T_z'/_z}$$

To clearly understand the equilibrium significance of the previous equation, it can be considered – as it is typically done when seeking differential equilibrium conditions – an infinitesimal portion of the beam.

It can now be highlighted that the equilibrium of an infinitesimal slice of beam leads to the same result.



↳ <u>Note</u>: q_i does not vary along z

Recall that $\sigma_{xz} = \sigma_{xz}(x,y)$ and
$\sigma_{yz} = \sigma_{yz}(x,y)$

The equilibrium along z is written as:

$$T_z' + dT_z' - T_z' + q_2\,dz - q_1\,dz + q_3\,dz = 0$$

$$-dT_z' = (q_2 - q_1 + q_3)\,dz \quad , \text{ and so:}$$

$$\boxed{q_2 + q_3 - q_1 = -T'_{z/z}}$$ as previously obtained from the shear flow equation

The shear flow equation, obtained after integrating over A' the third equilibrium equation, is then an equation expressing the equilibrium condition along the z-axis.

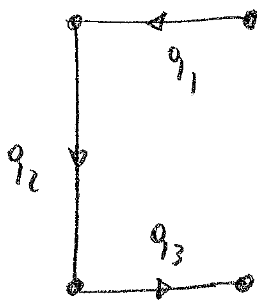
Number of equations

When solving a section, it is possible to write

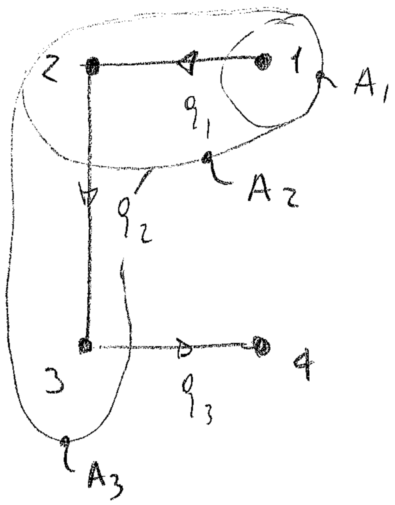
- $n-1$ shear flow equations, linearly independent each other.
- Considering that the number of unknown shear flows to be determined is m, it follows that the number of additional equations to satisfy the balance with the number of unknowns is:

$$\underbrace{m}_{\text{unknowns}} - \underbrace{(n-1)}_{\text{shear flow equations}} = m-n+1 = N \quad (\text{number of cells})$$

- As an example, consider an open section



$n=4 \Rightarrow$ 3 linearly independent shear flow equations



$$q_1 = -T'_{z_1}/z$$

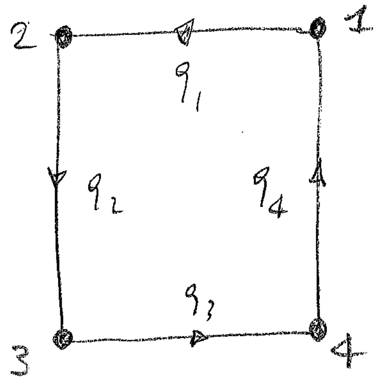
$$q_2 = -T'_{z_2}/z \quad \Rightarrow \text{the problem can be solved}$$

$$q_3 = -T'_{z_3}/z$$

where

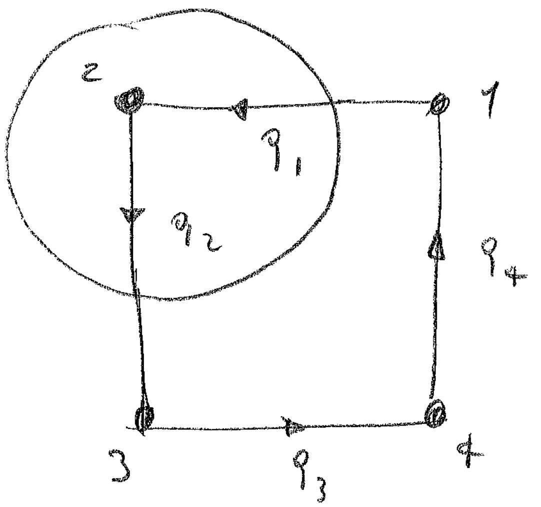
$$T'_{z_i} = \int_{A_i} \sigma_{zz} \, dA_i$$

• Consider now a closed section with one cell



$n = 4$ (stringers)
$m = 4$ (panels)
$N = 1$ (cell)

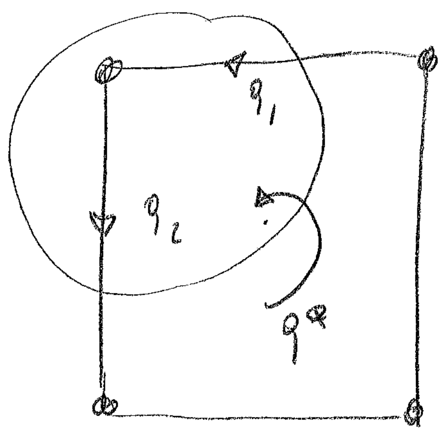
It is now possible to apply 3 shear flow equations, while 1 equation is missing.
To understand why there is a missing equation, it can be analyzed the closed region here below



The shear flow equation states that

$$q_2 - q_1 = \phi = -T'_{z2/z}$$

What happens if an additional shear flow q^*, flowing across all the panels is added?



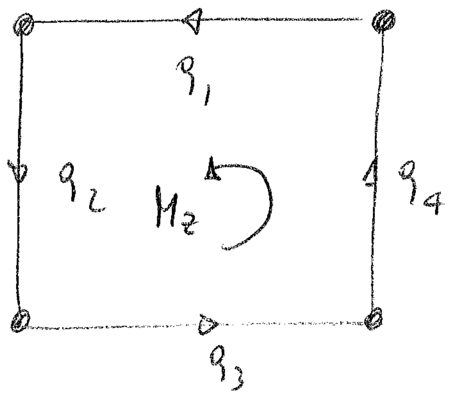
$$q_2 - q_1 + \cancel{q^*} - \cancel{q^*} = \phi = -T'_{z2/z}$$

$$q_2 - q_1 = \phi = -T'_{z2/z}$$

which is still the same result

This means that given two values of q_1 and q_2 which satisfy the shear flow equation, any other set of shear flows $q_1 + q^*$ and $q_2 + q^*$ will satisfy the shear flow equation as well. The solution is then indeterminate, i.e. one equation is missing.

The missing equation regards the equivalence with the internal torsional moment



The equation imposes that

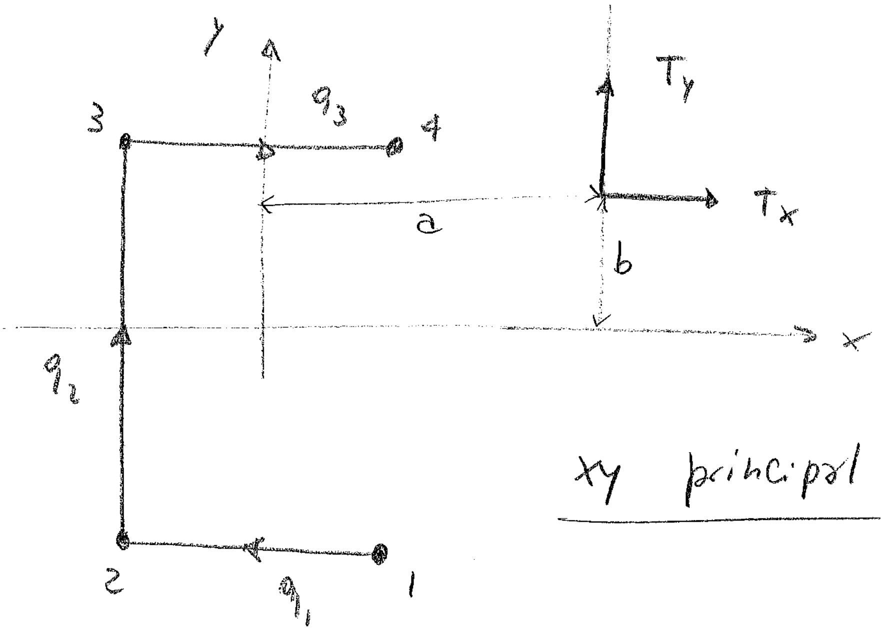
$$\boxed{M_z = 2 \sum_i \Omega_i q_i}$$

Remarks

1. The equation $M_z = 2 \sum_i \Omega_i q_i$ is an equivalence condition, not an equilibrium one.

Solution procedure for an open section

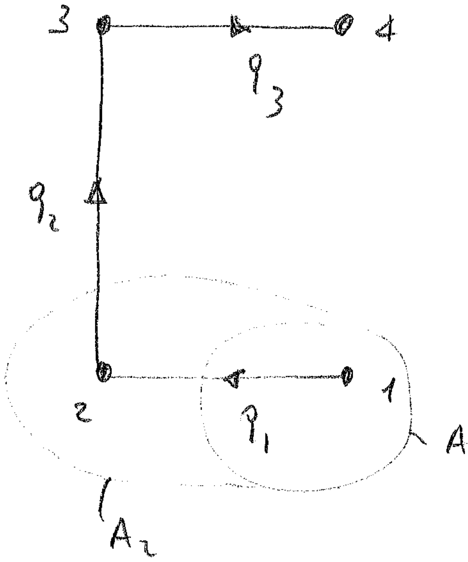
Consider, as an example, a C-section subjected to internal shear forces T_x and T_y



xy principal axes

The number of unknown shear flows is 3, and 3 independent shear flows equations are available.

Recalling that $\phi = -T_x \frac{S_y'}{J_{yy}} - T_y \frac{S_x'}{J_{xx}}$, the solution reads:



$$\boxed{q_1 = -T_x \frac{S_{y_1}}{J_{yy}} - T_y \frac{S_{x_1}}{J_{xx}}}$$

$$q_2 = -T_x \frac{S_{y_2}'}{J_{yy}} - T_y \frac{S_{x_2}'}{J_{xx}}$$

but $S_{y_2}' = S_{y_1} + S_{y_2}$

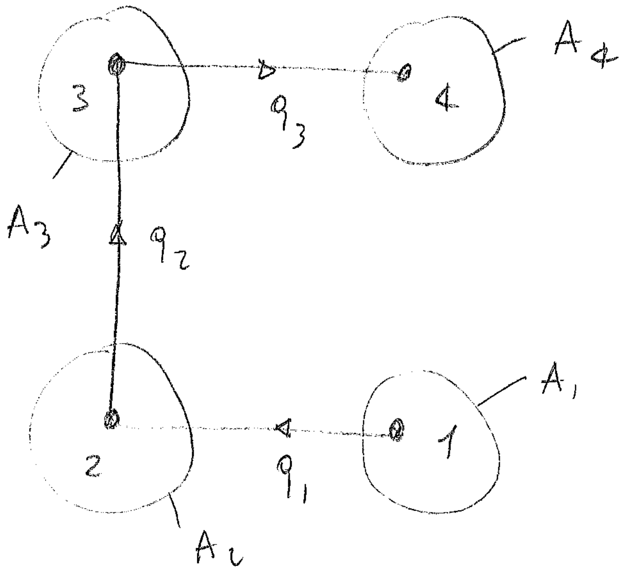
$S_{x_2}' = S_{x_1} + S_{x_2}$ so:

$$\boxed{q_2 = -T_x \frac{S_{y_1} + S_{y_2}}{J_{yy}} - T_y \frac{S_{x_1} + S_{x_2}}{J_{xx}}}$$

and similarly $$\boxed{q_3 = -T_x \frac{S_{y_1}+S_{y_2}+S_{y_3}}{J_{yy}} - T_y \frac{S_{x_1}+S_{x_2}+S_{x_3}}{J_{xx}}}$$

The shear flows q_1, q_2, q_3 are then obtained.

Note that the choice of paths for applying the shear flow equation are arbitrary. Another possible choice would be:



$$q_1 = -T_x \frac{S_{y_1}}{J_{yy}} - T_y \frac{S_{x_1}}{J_{xx}}$$

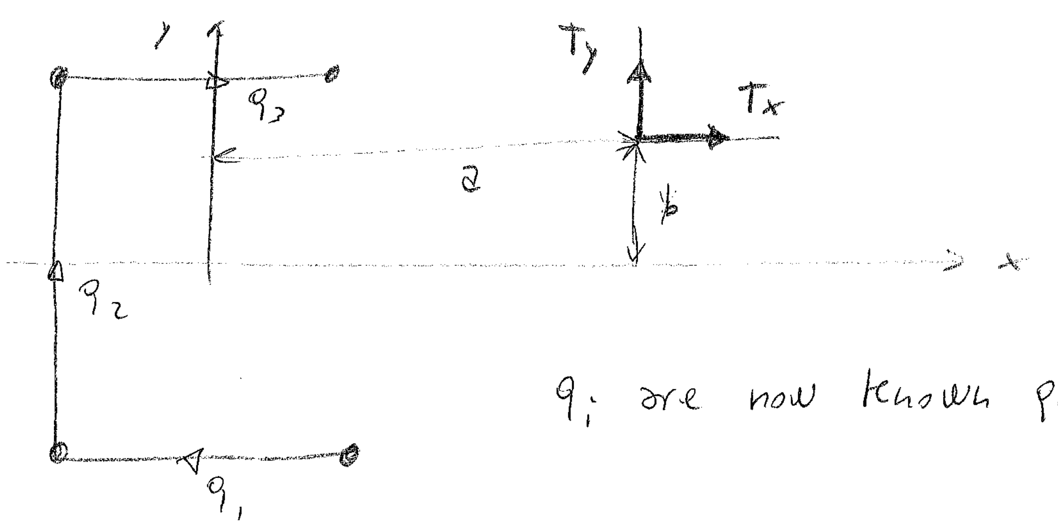
$$q_2 - q_1 = -T_x \frac{S_{y_2}}{J_{yy}} - T_y \frac{S_{x_2}}{J_{xx}}$$

$$q_3 - q_2 = -T_x \frac{S_{y_3}}{J_{yy}} - T_y \frac{S_{x_3}}{J_{xx}}$$

which obviously leads to the same results.

Position of the shear center

An important property of the section is the shear center position



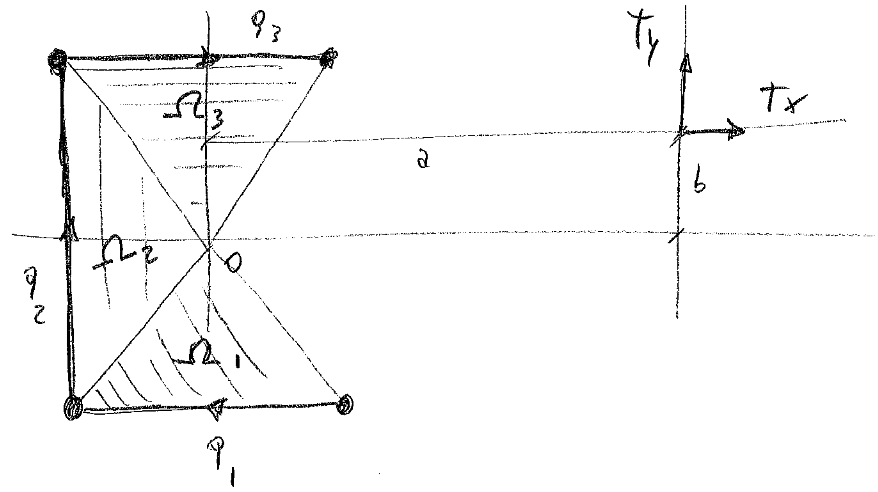
q_i are now known quantities

Write the equivalence with the internal moment:

(note: this is not an additional equation to solve the problem, as q_i were already determined; it is here applied for evaluating the shear center position)

$$\boxed{2q_1\Omega_1 + 2q_2\Omega_2 + 2q_3\Omega_3 = T_x b - T_y a}$$

(equivalence to torsional moment with respect to O



or, in compact form:

$$2\sum_{i=1}^{3} q_i \Omega_i = T_x b - T_y a$$

Recall now that the generic flow q_i is expressed as:

$$q_i = -T_x \frac{S'_{y_i}}{J_{yy}} - T_y \frac{S'_{x_i}}{J_{xx}} \qquad (i = 1,2,3)$$

where S'_{x_i} / S'_{y_i} are the static moment calculated with respect to the area A'. For instance:

$$S'_{y_2} = S_{y_1} + S_{y_2}$$

The equivalence to internal torsion is then:

$$2\sum_{i=1}^{3}\left(-T_x\frac{S'_{y_i}}{J_{yy}} - T_y\frac{S'_{x_i}}{J_{xx}}\right)\Omega_i = T_x b - T_y a$$

$$-2\frac{T_x}{J_{yy}}\Sigma_i S'_{y_i}\Omega_i - 2\frac{T_y}{J_{xx}}\Sigma_i S'_{x_i}\Omega_i = T_x b - T_y a$$

$$T_y\left(a - 2/_{J_{xx}}\Sigma_i S'_{x_i}\Omega_i\right) = T_x\left(b + 2/_{J_{yy}}\Sigma_i S'_{y_i}\Omega_i\right)$$

which can be written as

$$T_y(a - x_{sc}) = T_x(b - y_{sc})$$

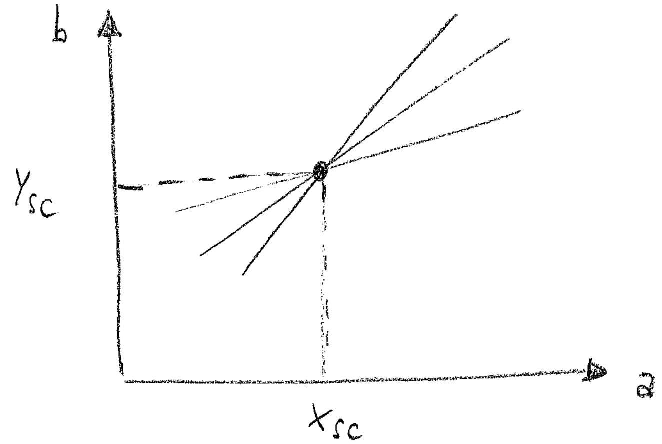
or:

$$\boxed{b - y_{sc} = T_y/_{T_x}(a - x_{sc})}$$

where

$$\left.\begin{aligned} x_{sc} &= 2/_{J_{xx}}\Sigma_i S'_{x_i}\Omega_i \\ y_{sc} &= -2/_{J_{yy}}\Sigma_i S'_{y_i}\Omega_i \end{aligned}\right\} \text{section properties}$$

The equation in the box can be graphically represented as:

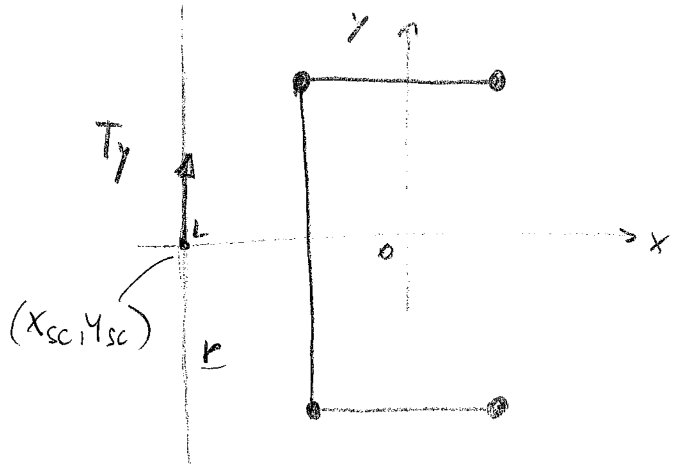


- All the curves pass through the point (x_{sc}, y_{sc}).
- The slope depends on T_y/T_x

Comments:

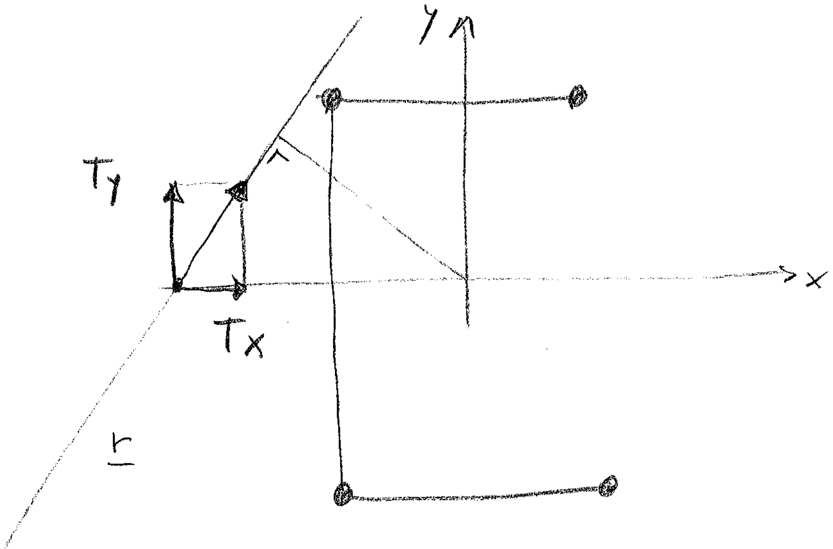
- For a given ratio T_y/T_x, there exist an <u>infinite</u> number of points (a,b) allowing for an equivalence between internal moments.

Consider, for instance:



If T_y is shifted along $\underline{r}$, the resulting moment with respect to o is unchanged. Thus all the points belonging to $\underline{r}$ satisfy the equivalence of internal moments.

Consider the case of $T_x \neq 0$ and $T_y \neq 0$



- Any point along $\underline{r}$ satisfies the equivalence.

- The slope of $\underline{r}$ is then function of T_y/T_x

$$\boxed{\text{The one and only point where the equivalence condition can be satisfied independently on the ratio } T_y/T_x \text{ is the shear center}}$$

- It follows that an open section, when modelled according to the semi-monocoque scheme, cannot react torsion loads.

  Shear forces can be applied <u>only</u> in a specific point, the shear center, so that the load is purely of shearing type (no torsion with respect to the shear center).

- The balance between shear flows and linearly independent shear flow equations is such that the internal moment with respect to a generic point is automatically determined once q_i are determined.

- The shear center position is

$$\begin{cases} x_{sc} = 2/J_{xx} \sum_i S'_{x_i} \Omega_i \\ y_{sc} = -2/J_{yy} \sum_i S'_{y_i} \Omega_i \end{cases}$$

  and is fully determined by the geometric properties of the section. In this sense it is a <u>section property</u>.

Solution procedure for single-cell closed sections

As previously discussed, a closed section with one single cell is characterized by:

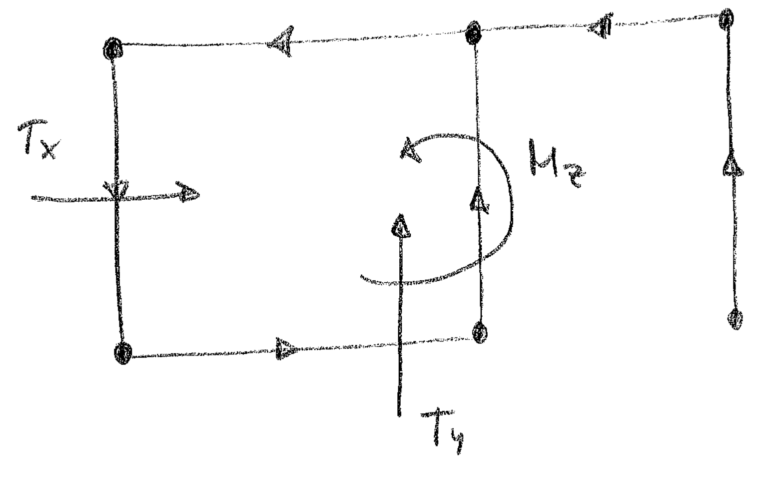
- n unknown shear flows

while the available equations are:

- $n-1$ shear flow equations
- 1 equivalence condition with internal moment

In this sense, the single-cell sections are a <u>statically determined</u> scheme. Indeed, the solution can be found by considering equilibrium conditions only (recall that the shear flow equations express the equilibrium along the beam axis).
Compatibility does not need to be enforced. This is in analogy with the case of beam problems which are statically determined.

To outline the procedure consider a generic section with one single closed loop



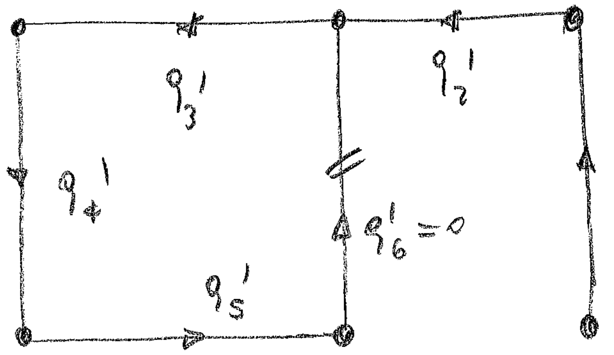
In general the internal actions are characterized by T_x, T_y, M_z

$n = 6$

$m = 6$

$N = 1$

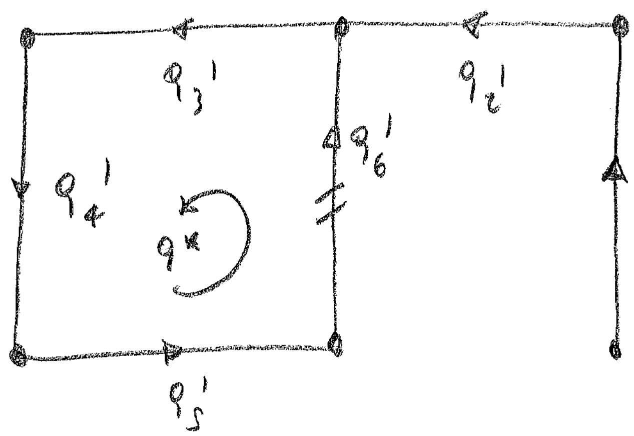
The $n-1$ shear flow equations are applied after "cutting" one panel (the choice is arbitrary)



The flow along the cut is zero.
The other flows are denoted as q_i'.

$$\boxed{q_i' = -T_x \frac{S_y'}{J_{yy}} - T_y \frac{S_x'}{J_{xx}} \qquad i = 1, \dots 5}$$

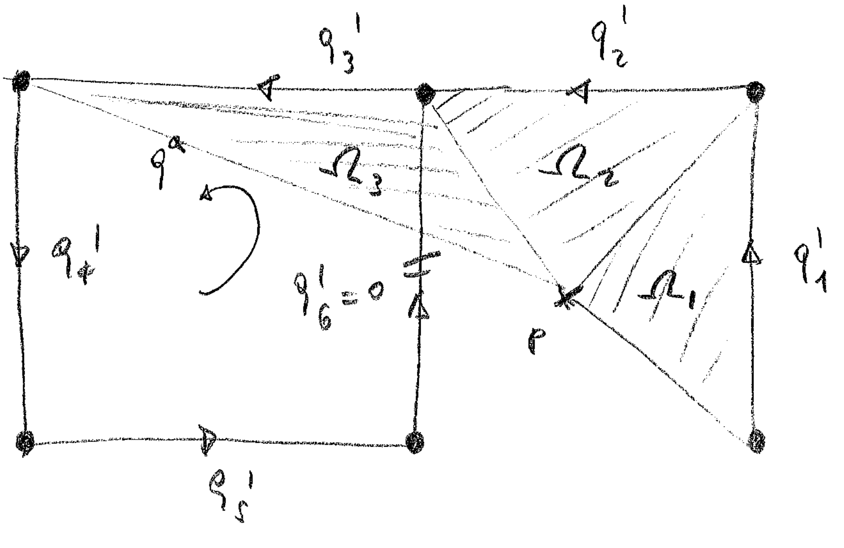
A circulating flow q^* is then introduced in the closed cell. The flow circulates through all the panels belonging to the cell, thus the shear flow equations are still satisfied.



$q_i = q_i' + q^*$ if panel $i \in$ cell

$q_i = q_i'$ if panel $i \notin$ cell

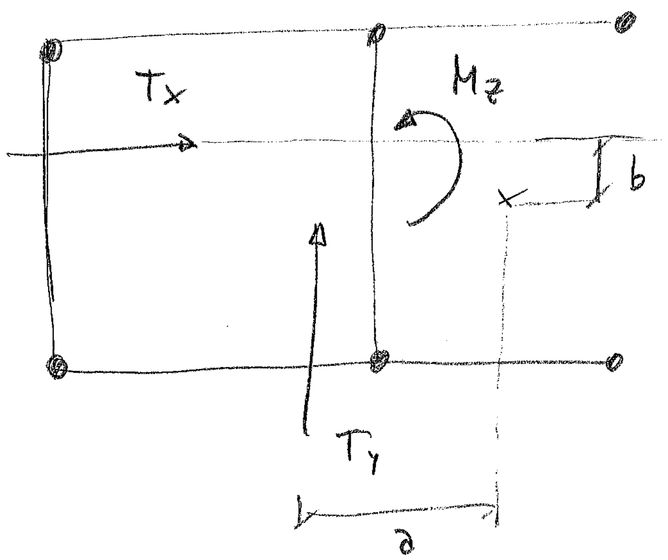
The value of q^* is finally obtained after imposing the equivalence to internal moment with respect to an arbitrary point P



The equivalence condition reads:

$$2 \sum_i q_i' \Omega_i + 2 q^* \Omega_{cell} = M_P$$

where M_P is the moment due to the internal actions with respect to P.



In this case

$$M_P = M_z - T_x b - T_y a$$

The value of q^* is then obtained as:

$$\boxed{q^* = \frac{M_P - 2 \sum_i q_i' \Omega_i}{2 \Omega_{cell}}}$$

Remarks

+ The sign of the moment contribution due to the generic panel i, $M_i = 2\Omega_i q_i$, depends on the direction of q_i.

  In the previous example, q_i' (with $i = 1,..5$) determined a moment contribution with respect to P in the counterclockwise direction.

  Similarly the contribution due to q^* was directed in the counterclockwise direction.

+ Note, again, that the condition imposed determines the equivalence (not the equilibrium!) between internal forces.

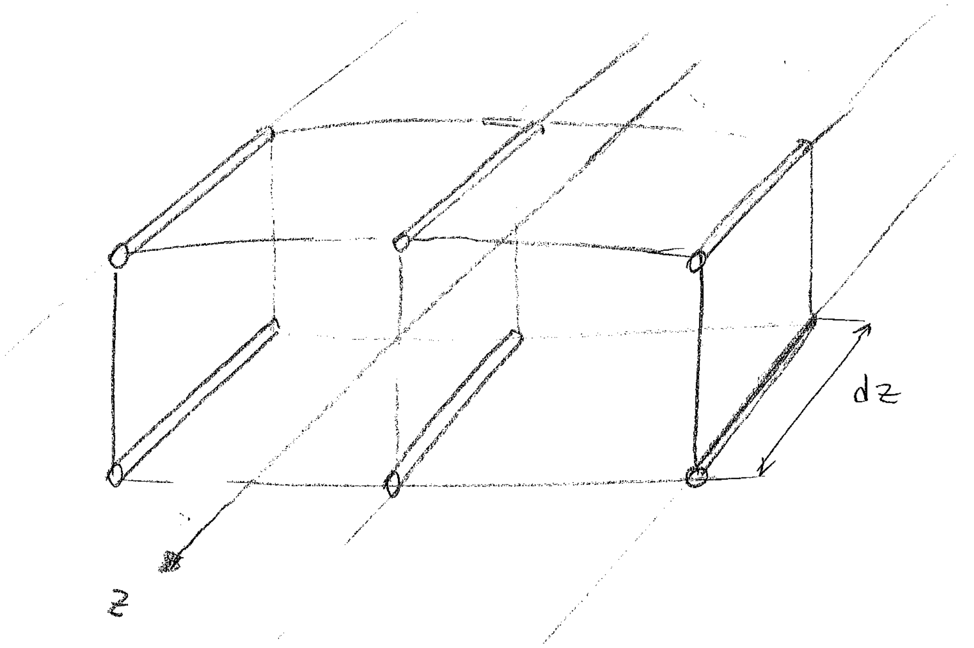
$$\underbrace{2\Sigma_i q_i'\Omega_i + 2q^*\Omega_{cell}}_{>0 \text{ in counterclockwise direction}} = \underbrace{M_P}_{>0 \text{ in counterclockwise direction}}$$

Evaluation of torsion

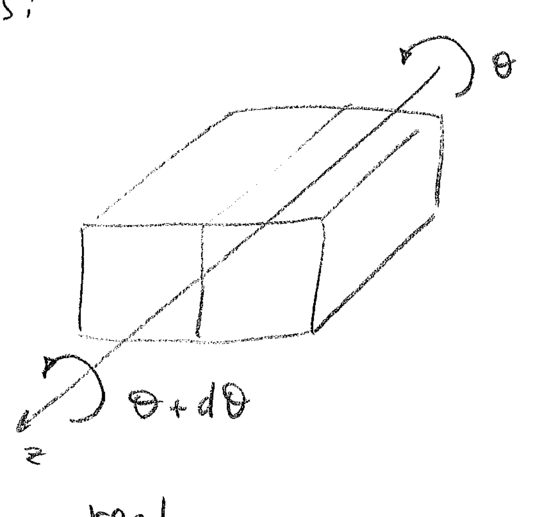
The evaluation of the torsion θ' is necessary for:

1. solving sections with more than one-cell.
2. evaluating the torsional stiffness of generic sections

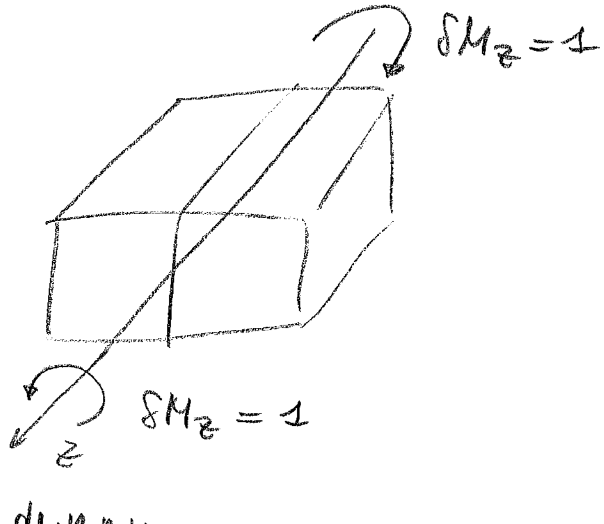
Consider, for generality purposes, a section with two cells and extract an infinitesimal portion dz.



The evaluation of the torsion θ' is conducted by applying the PCVW. The shear flows are assumed to be known. The real and the dummy systems are then obtained as:



real



dummy

In the real configuration the rotation angle θ will be different when looking at the two faces of the beam infinitesimal portions.

The dummy system, as usual, is supposed to satisfy the equilibrium conditions, and so the infinitesimal variation of torsional moment M_z is δM_z on both the faces.

External work

$$\delta W_e^* = \delta M_z (\theta + d\theta) - \delta M_z \theta$$

$$= \delta M_z \, d\theta = d\theta$$

Internal work

$$\delta W_i^* = \int_V \delta \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} \, dV = \int_V \delta \sigma_{ik} \, \varepsilon_{ik} \, dV$$

where $\delta \sigma_{ik}$ identifies the variation, subjected to the requirement of equilibrium, due to δM_z.

The terms ε_{ik} denote the real deformations.

Given the real system here considered, the only not null components of the strain tensor are γ_{xz} and γ_{yz}

or, considering the system snz, $\gamma_{sz} \neq 0$

(s: curvilinear coordinate along panel midline
n: normal to s)

For brevity γ_{sz} will be denoted as γ; similarly σ_{sz} will be denoted as τ

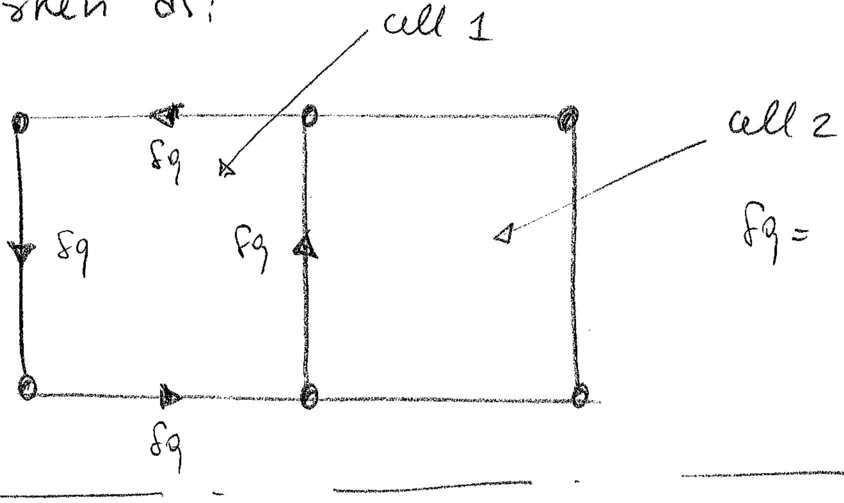
$$\delta W_i^* = \int_V \delta\sigma_{sz}\, \gamma_{sz}\, dV = \int_V \delta\tau\, \gamma\, dV$$

$$= \sum_m \int_{V_m} \delta\tau_m\, \gamma_m\, dV_m \qquad \text{(summatory over the panels composing the section)}$$

or, recalling the definition of shear flow $q = t\tau$:

$$\delta W_i^* = \sum_m \int_{V_m} \frac{\delta q_m}{t}\, \gamma_m\, dV_m$$

Recall now that the choice of $\delta\sigma_{in}$ (in this case δq) is arbitrary, the equilibrium representing the only requirement. It is then possible to properly select a set of δq_m which guarantees equilibrium and is particularly suitable for evaluating θ'. In particular δq_m are taken as:



$$\delta q = \frac{\delta M_z}{2\Omega_{cell_1}} = \frac{1}{2\Omega_{cell_1}}$$

<u>Important remark</u>: the set of δq_m illustrated above guarantees the equilibrium as far as they are taken equal to $\delta M / 2\Omega_{cell}$ (see Bredt formula).
However compatibility is not fulfilled (as the virtual

variations are not required to satisfy compatibility)
Having flows in one single cell is a useful choice for simplyfing the expression of δW_i^*!

Then:

$$\delta W_i^* = \sum_{①} \int_{V_m} \frac{1}{2\Omega_{cell_1} t_m} \gamma_m \, dV_m$$

where:
- Ω_{cell_k} is the area of the generic cell k. (In this case $k = 1$)
- $\sum_{ⓚ}$ denotes the summatory over the panels m belonging to the cell k.

Introducing now the constitutive equation:

$$\gamma_m = \frac{\tau_m}{G} = \frac{q_m}{G t_m}$$

$$\delta W_i^* = \sum_{①} \int_{V_m} \frac{1}{2\Omega_{cell_1} t_m^2} \frac{q_m}{G} \, dV_m$$

$$= \sum_{①} \frac{1}{2\Omega_{cell_1} t_m^2 G} \int_{V_m} dV_m$$

observe now that:

$$\int_{V_m} dV_m = \int_{S_m} dS_m \, dz = l_m t_m \, dz$$

l_m: length of m-th panel

$$\delta W_i^* = \sum_{①} \frac{1}{2\Omega_{cell_1} t_m} \frac{q_m l_m}{G} dz$$

$$= \frac{1}{2\Omega_{cell_1} G} \sum_{①} \frac{q_m l_m}{t_m} dz$$

Imposing now $\delta W_i^* = \delta W_e^*$ it is obtained:

$$\boxed{\frac{d\theta}{dz} = \theta' = \frac{1}{2\Omega_{cell_1} G} \sum_{①} \frac{q_m l_m}{t_m}}$$

Remarks

In the derivation of the previous expression the choice of the flows δq_m was arbitrary. All the steps could be repeated by considering the cell 2, leading to:

$$\theta' = \frac{1}{2\Omega_{cell_2} G} \sum_{②} \frac{q_m l_m}{t_m}$$

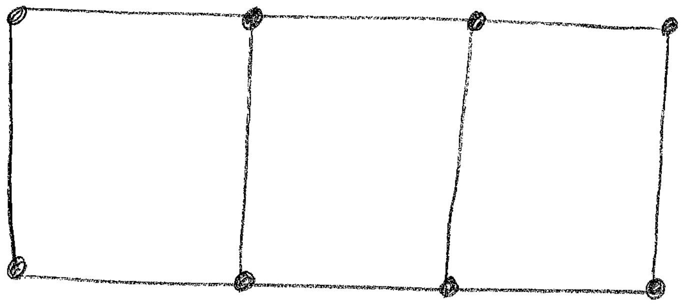
(as far as θ' is referred to the overall section and not to a specific cell)

It follows that, in general, the torsion θ' can be expressed as

$$\boxed{\theta' = \frac{1}{2\Omega_{cell_k} G} \sum_{(k)} \frac{q_m l_m}{t_m}}$$

Solution procedure for multi-cells closed sections

Consider the case of a section characterized by N cells with $N > 1$; for instance:



$n = 8$

$m = 10$

$N = 10 - 8 + 1 = 3$

This is an example of statically indeterminate problem (as it was for some beam problems): the equilibrium conditions (shear flow equations + equivalence with internal moment) do not suffice for satisfying the balance between equations and unknowns.

Overview of the equations:

- $n - 1$ shear flow equations
- 1 equivalence with internal moment

Remaining equations to be imposed:

$$\underbrace{m}_{\text{number of shear flows}} - \underbrace{(n - 1 + 1)}_{\text{equations}} = m - n$$

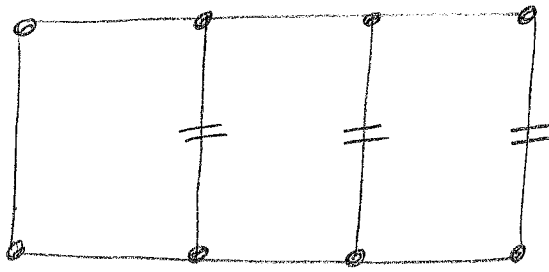
(and recalling that $N = n - m + 1$)

$$m - n = N - 1$$

- $N - 1$ compatibility conditions

The procedure is then summarized as:

1. Cut a number of panels equal to N, e.g.



2. Evaluate the flows q_i' as

$$\boxed{\begin{array}{ll} q_i' = -T_x \dfrac{S_y'}{J_{yy}} - T_y \dfrac{S_x'}{J_{xx}} & \text{non-cut panels} \\ q_i' = 0 & \text{cut panels} \end{array}}$$

3. Impose the equivalence with internal moment

$$\boxed{\sum_i 2\Omega_i q_i' + \sum_{k=1}^{N} 2\Omega_{cell_k} q^* = M_z}$$

4. Impose $N-1$ compatibility conditions in the form

$$\theta_1' = \theta_2' = \dots = \theta_N'$$

or

$$\boxed{\begin{array}{l} \dfrac{1}{2\Omega_{cell_1} G} \displaystyle\sum_{1} \dfrac{q_m l_m}{t_m} = \dfrac{1}{2\Omega_{cell_2} G} \displaystyle\sum_{2} \dfrac{q_m l_m}{t_m} = \dots \\ = \dfrac{1}{2\Omega_{cell_N} G} \displaystyle\sum_{N} \dfrac{q_m l_m}{t_m} \end{array}}$$

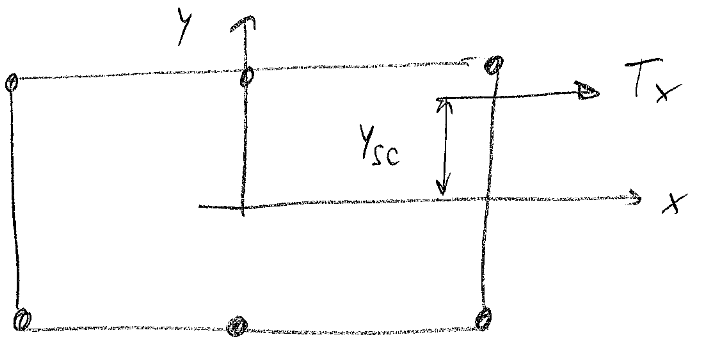
Remark

The compatibility condition $\theta_1' = \theta_2' = \dots \theta_N'$ is sometimes justified by observing that the torsion of the various cells should be equal as far as the section shape is preserved by the ribs (which are infinitely stiff along the in-plane direction).

However it is observed that the condition $\theta_1' = \theta_2' = \dots \theta_N'$ is a requirement of internal compatibility which holds irrespectively on the presence of the ribs.

Indeed it was obtained by application of the PCVW without any reference to the presence of ribs.

5. Repeat all the steps from 2 to 4 by considering T_x and obtain y_{sc}.



Multi-cell section

The procedure is analogous to the case of one-cell sections.

1. Determine principal axes
2. Evaluate shear flows as:

$$q_i' = - T_y \frac{S_x'}{J_{xx}} \quad \text{non-cut panel} \qquad n-1 \text{ equations}$$

$$q_i' = 0 \quad \text{cut panel}$$

$$\Sigma\, 2\Omega_i\, q_i' + \sum_{k=1}^{N} 2\Omega_{cell_k}\, q^* = M_z \qquad 1 \text{ equation}$$

3. Impose compatibility

$$\theta_1' = \theta_2' = \ldots \theta_N' \qquad N-1 \text{ equations}$$

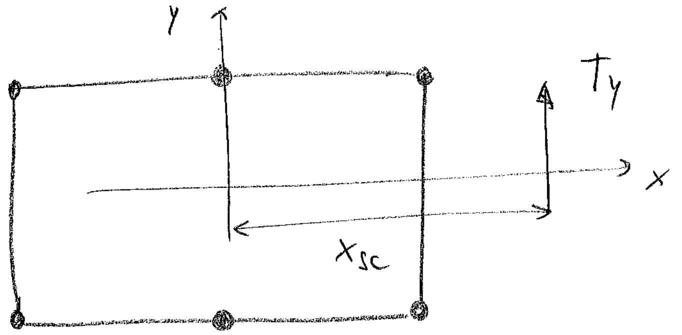
4. Impose $\theta_i' = 0$ 1 equation
5. Solve the governing equations and obtain q_i $(i = 1, \ldots m)$ and x_{sc}

Repeat all the steps from 2 to 5 to evaluate the position y_{sc} of the shear center.

Evaluation of the shear center for closed sections

The shear center is defined as the point where the application of the shear forces T_x and T_y determines a torsion $\theta' = 0$.

Single-cell section



1. Determine principal axes
2. Evaluate shear flows as:

$$q_i' = - T_y \frac{S_x'}{J_{xx}} \qquad \text{non-cut panel} \qquad n-1 \text{ equations}$$

$$q_i' = 0 \qquad \text{cut panel}$$

$$\sum 2\Omega_i q_i' + 2\Omega_{cell} q^* = T_y x_{sc} \qquad 1 \text{ equations}$$

3. Impose $\theta' = 0$

$$\theta' = \frac{1}{2\Omega_1 G t} \sum_{①} \frac{q_m l_m}{t_m} = 0 \qquad 1 \text{ equations}$$

4. Solve the set of equations and obtain q_i $(i = 1, \ldots m)$ and x_{sc}

Remarks

- The shear section is a property of the section (as already observed for open sections).
  To perform calculations, the values of T_x and T_y can be arbitrarily chosen (sometimes they are taken greater than 1, as it would seem natural, not to deal with very small values of q_i)
- The position of the shear center identifies the center of relative rotation between two infinitesimally close sections. when the internal action is purely torsional
- In general, each section can be characterized by a different position of the shear center. For this reason the center of the absolute rotations depends on the overall distribution of the shear centers.
- Whenever the shear centers are aligned along the same line, the center of relative rotation is also the center of the absolute rotation.
  This line is referred to as <u>elastic axis</u>

Evaluation of section properties, a few remarks

The internal complementary work for a beam, previously restricted to case of bending and axial behaviour, is now generalized to account also for torsion and shear.

The expression can be derived as:

$$\delta W_i^* = \int_V \delta \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} \, dV \qquad \text{(general expression)}$$

$$= \int_V \left(\delta\sigma_{zz} \varepsilon_{zz} + \delta\sigma_{xz} \gamma_{xz} + \delta\sigma_{yz} \gamma_{yz} \right) dV$$

having observed that $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$ within the context of the DSV solution for isotropic homogeneous beams.

After introducing the DSV solution

$$\sigma_{zz} = \frac{T_z}{A} + \frac{M_x}{J_{xx}} y - \frac{M_y}{J_{yy}} x$$

(which is obtained by considering principal axes)

into the expression of δW_i^* it is obtained:

$$\delta W_i^* = \int_\ell \left(\delta T_z \frac{T_z}{EA} + \delta M_x \frac{M_x}{EJ_{xx}} + \delta M_y \frac{M_y}{EJ_{yy}} \right) dz$$

$$+ \int_V \left(\delta\sigma_{xz} \gamma_{xz} + \delta\sigma_{yz} \gamma_{yz} \right) dV$$

The second contribution (the volume integral) is the part of internal work due to the shear stresses and the corresponding deformations.

In particular, the shear stresses σ_{xz} and σ_{yz} are associated with

- internal shear forces T_x, T_y
- internal torsional moment M_z

In order to separate the contribution due to shear (T_x, T_y) from the contribution due to torsion it is necessary to consider a reference system with axis z passing through the <u>shear center</u>.

Note that this reference system may not coincide with the principal axes.

Considering then a system with z passing through the shear center:

$$\int_V \left(\delta\sigma_{xz}\gamma_{xz} + \delta\sigma_{yz}\gamma_{yz}\right) dV$$

$$= \underbrace{\int_\ell \delta M_z \frac{M_z}{GJ}\, dz}_{\text{see notes on torsion}} + \underbrace{\int_V \left(\delta\sigma_{xz}^{\text{shear}}\gamma_{xz}^{\text{shear}} + \delta\sigma_{yz}^{\text{shear}}\gamma_{yz}^{\text{shear}}\right) dV}_{\text{contribution due to } T_x, T_y}$$

The internal work due to shear can be expressed, following the results obtained for the other components, as:

$$\int_V \delta\sigma_{xz}^{\text{shear}}\gamma_{xz}^{\text{shear}}\, dV = \int_\ell \delta T_x \frac{T_x}{GA_x^*}\, dz$$

meaning that the contribution to the internal work is still represented as the product between

- δT_x → virtual variation of the internal force
- $\dfrac{T_x}{GA^*_x}$ → generalized deformation where the term at the denominator GA^*_x is the shear stiffness (exactly as for bending the generalized deformation is M_x / EJ_{xx})

  The stiffness is represented as GA^*_x because it is <u>not</u> equal to the product between G and A, even in the case of homogeneous isotropic beams.

  However GA^*_x illustrates that the stiffness depends on the shear modulus and on "area-related" quantity

  The dimensional analysis reveals that GA^*_x is indeed a force.

$$\int_\ell \delta T_x \frac{T_x}{GA^*_x}\, dz \qquad [F] \cdot \frac{[F]}{[F]} \cdot [\ell] = \underbrace{[F][\ell]}_{\text{work}}$$

The contribution due to T_y is then

$$\int_V \delta \sigma_{yz}\, \gamma_{yz}\, dV = \int_\ell \delta T_y \frac{T_y}{GA^*_y}\, dz$$

It is then concluded that the internal virtual work can be written as:

$$\boxed{\delta W_i^* = \int_0^l \left(\delta T_z \frac{T_z}{EA} + \delta M_x \frac{M_x}{EJ_{xx}} + \delta M_y \frac{M_y}{EJ_{yy}} + \delta T_x \frac{T_x}{GA_x^*} + \delta T_y \frac{T_y}{GA_y^*} + \delta M_z \frac{M_z}{GJ} \right) dz}$$

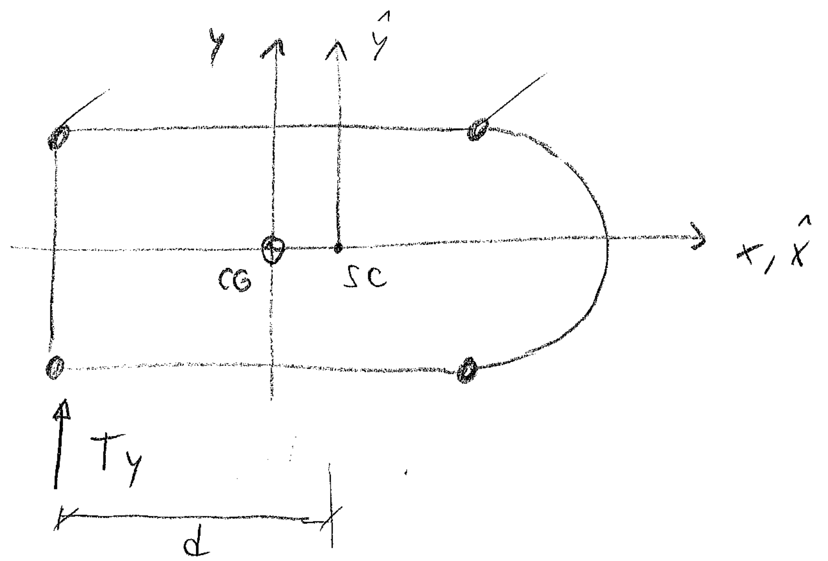
where:

1. the first two contributions are referred to the <u>principal axes</u>

2. the last three contributions are referred to the system passing through the <u>shear center</u>

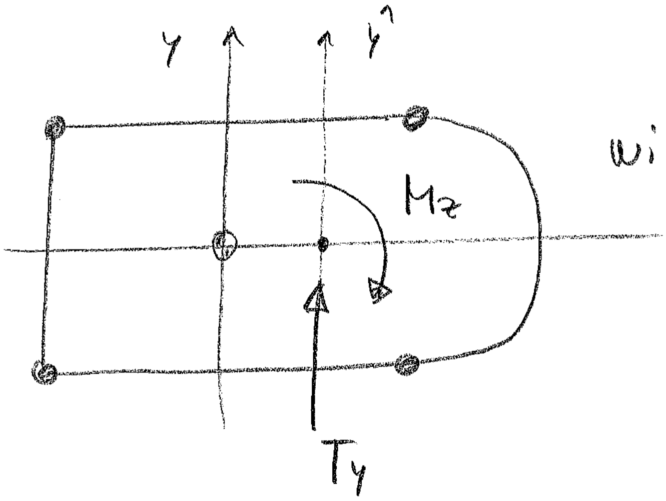
Example

Assume the following section is considered



The shear center is not coincident with the centroid

In order to decouple the internal work due to shear and torsion, T_y and M_z should be referred to the system $\hat{x}\hat{y}$



with $M_z = T_y d$

Assume now that the section properties of the beam, modelled according to the semi-monocoque scheme, need to be evaluated (for instance because they are required as input in a finite element beam model)

- Axial stiffness

EA is readily available from: $\boxed{EA = E \sum_{i=1}^{n} A_i}$

- Bending stiffnesses

EJ_{xx} and EJ_{yy} are obtained as:

$$\boxed{\begin{aligned} EJ_{xx} &= E \sum_i A_i y_i^2 \\ EJ_{yy} &= E \sum_i A_i x_i^2 \end{aligned}}$$

where x_i, y_i are the distances from the principal axes

- Torsional stiffness

a. Open sections modelled using the semi-monocoque scheme are characterized by $GJ = 0$

b. Single-cell closed sections

The solution for the torsional constant J is available as:

$$J = \frac{4\Omega^2}{\oint_{\Gamma} \frac{1}{t(s)}\, ds}$$

$$\oint_{\Gamma} \frac{1}{t(s)}\, ds = \sum_{i=1}^{m} \frac{l_i}{t_i}$$ so:

$$\boxed{J = \frac{4\Omega^2}{\sum_{i=1}^{m} l_i/t_i}}$$

c. Multi-cell sections

In this case the evaluation of GJ is obtained by recalling that: $M_z = GJ\theta'$ and

$$\theta' = \frac{1}{2\Omega_{cell} G} \sum_{(k)} \frac{q_i l_i}{t_i}$$ and so:

$$\boxed{GJ = \frac{M_z}{\theta'}}$$ where M_z is an arbitrary torsional moment

Clearly the torsional stiffness GJ can be evaluated as $GJ = \frac{M_z}{\theta'}$ even for single-cell section

(the result is equal to $G \frac{4\Omega^2}{\sum_{i=1}^{m} l_i/t_i}$)

- $\underline{\text{Shear stiffnesses } GA_x^* \text{ and } GA_y^*}$

The evaluation of GA_x^* / GA_y^* requires the ability to solve the section when an internal shear force T_x / T_y is applied on the shear center.

The shear stiffnesses are then obtained with an <u>energy equivalence</u> as outline here below. Consider an <u>arbitrary T_x</u>

$$\delta W_i^* = \int_V \left(\delta\sigma_{xz}^{shear} \gamma_{xz}^{shear} + \delta\sigma_{yz}^{shear} \gamma_{yz}^{shear} \right) dV \quad \text{or}$$

$$= \int_V \delta\tau\, \gamma\, dV \qquad \text{where } \tau = \sigma_{sz} \quad \gamma = \gamma_{sz}$$

$$= \sum_{i=1}^{m} \int_{V_m} \delta\tau_m \frac{\tau_m}{G} dV_m = \sum_{i=1}^{m} \int_{V_m} \delta q_m \frac{q_m}{G t_m^2} dV_m$$

$$= \int_e \sum_{i=1}^{m} \delta q_m \frac{q_m}{G t_m^2} t_m l_m \, dz$$

$$= \int_e \sum_{i=1}^{m} \delta q_m \frac{l_m}{G t_m} q_m \, dz$$

$q_m / \delta q_m$ obtained considering an arbitrary T_x

so: $$\boxed{\delta W_i^* = \int_e \sum_{i=1}^{m} \delta q_m \frac{l_m}{G t_m} q_m \, dz}$$

internal virtual work due to shear as obtained from the semi-monocoque scheme

Recall that the internal virtual work, expressed in terms of internal actions, was:

$$\delta W_i^* = \int_l \delta T_x \frac{T_x}{GA_x^*} dz$$

GA_x^* can so be obtained by equating the two expressions obtained for δW_i^*:

$$\int_l \delta T_x \frac{T_x}{GA_x^*} dz = \int_l \sum_{i=1}^{m} \delta q_m \frac{l_m}{G t_m} q_m \, dz$$

or:

$$\boxed{GA_x^* = \frac{\delta T_x \, T_x}{\sum_{i=1}^{m} \delta q_m \frac{l_m}{G t_m} q_m}}$$

Sometimes it is introduced the shear factor χ_x defined as:

$$\boxed{\chi_x = \frac{A_x^*}{A_{TOT}}}$$

which provides a measure of how much area is collaborating to resisting the shear force T_x

According to this definition:

$$\boxed{GA_x^* = \chi_x \, GA_{TOT}}$$

The evaluation of GA_y^* follows the same approach, thus:

$$\boxed{GA_y^* = \frac{\delta T_y \, T_y}{\sum_{i=1}^{m} \delta q_m \frac{l_m}{G t_m} q_m}}$$

where δq_m and q_m are the shear flows obtained by considering an arbitrary shear force δT_y / T_y

Remark

The choice of referring M_x, M_y and T_z to the principal axes is consistent with the assumption introduced when deriving the DSV solution.
T_x, T_y and M_z were then assumed to be referred to the system passing through the shear center of the section in order to separate the contribution of torsion and shear.
According to this approach, the internal work is written as:

$$\delta W_i^* = \int_0^l \left(\delta T_z \frac{T_z}{EA} + \delta M_x \frac{M_x}{EJ_{xx}} + \delta M_y \frac{M_y}{EJ_{yy}} + \delta T_x \frac{T_x}{GA_x^*} + \delta T_y \frac{T_y}{GA_y^*} + \delta M_z \frac{M_z}{GJ} \right) dz$$

or, in matrix notation:

$$\delta W_i^* = \int_0^l \delta \underline{T}^T \underline{\underline{C}}\, \underline{T}\, dz \quad \text{with:}$$

$$\underline{T} = \{ T_x \; T_y \; T_z \; M_x \; M_y \; M_z \}^T \quad \text{and}$$

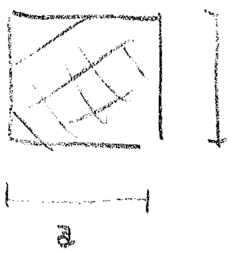
$$\underline{\underline{C}} = \begin{bmatrix} \frac{1}{GA_x^*} & & & & & \\ & \frac{1}{GA_y^*} & & & & \\ & & \frac{1}{EA} & & & \\ & & & \frac{1}{EJ_{xx}} & & \\ & & & & \frac{1}{EJ_{yy}} & \\ & & & & & \frac{1}{GJ} \end{bmatrix}$$

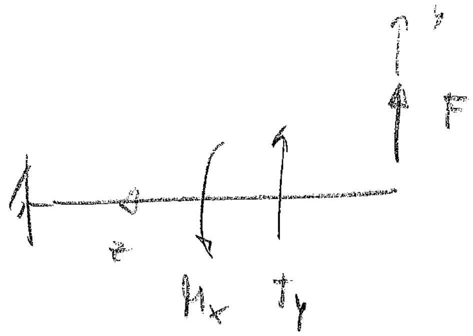
Compliance matrix of the section

Should a different set of axes be considered, the compliance matrix $\underline{\underline{C}}$ would be a fully populated. The proposed choice of principal axes and axes passing through the shear center is then a suitable choice for obtaining a simple expression of δW_i^*, where the section compliance matrix $\underline{\underline{C}}$ is <u>diagonal</u>.

Example

For quantifying the effect of shear deformability in thin walled beams, it can be useful to compare the strain energies in two sections: a compact square section and a thin walled one





$$J = \frac{1}{12} a^4 \qquad M_x = -Fz$$

$$A_y^* = \frac{5}{6} a^2 \qquad T_y = -F$$

$$U_{bending} = \frac{1}{2} \int_0^\ell \frac{M_x^2}{EJ} \, dz = \frac{1}{2} \frac{F^2 \ell^3}{3EJ}$$

$$U_{shear} = \frac{1}{2} \int_0^\ell \frac{T_y^2}{GA_y^*} \, dz = \frac{1}{2} \frac{F^2 \ell}{GA_y^*}$$

Thus,

$$\frac{U_{shear}}{U_{bending}} = \frac{\ell}{GA_y^*} \frac{3EJ}{\ell^3} = 3 \frac{E}{G} \frac{1}{12} a^4 \frac{6}{5a^2} \frac{1}{\ell^2} \simeq \frac{3}{5} \frac{a^2}{\ell^2}$$

thus, for the compact square section:

$$\boxed{\frac{U_{shear}}{U_{bending}} \simeq \frac{3}{5} \frac{a^2}{\ell^2}} \qquad \left(\text{having assumed that } \frac{E}{G} \simeq 2\right)$$

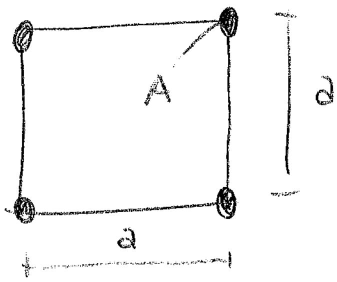
This means that, for a section with

$a = 100$ and $l = 1000$ it is:

$$\boxed{\frac{U_{shear}}{U_{bending}} \simeq \frac{3}{5}\, 10^{-2} = \frac{3}{500} < 1\%}$$

The energy associated with shear deformability is thus less than 1%, with respect to the bending energy

Consider now a thin-walled section



$$A = \frac{1}{12} a^2$$

$$t = \frac{1}{120} a \quad \left(\text{so that } A_{web} = \frac{1}{10} A\right)$$

The section is then characterized by the same bending stiffness of the compact section.

$$J = 4A\left(\frac{a}{2}\right)^2 = A a^2 = \frac{1}{12} a^4$$

However the shear area A_y^* is now:

$$A_y^* = 2at$$

The ratio between energies is thus

$$\frac{U_{shear}}{U_{bending}} = \frac{l}{G A_y^*} \frac{3EJ}{l^3} = \frac{3E}{G l^2} \frac{J}{A_y^*} \simeq \frac{6}{l^2} \frac{1}{12} a^4 \frac{1}{2at}$$

$$= \frac{1}{4} \frac{a^3}{l^2 t} = \frac{1}{4} \frac{a^3}{l^2} \frac{120}{a} = 30 \frac{a^2}{l^2}$$

$$\boxed{\frac{U_{shear}}{U_{bending}} \simeq 30 \frac{a^2}{\ell^2}}$$

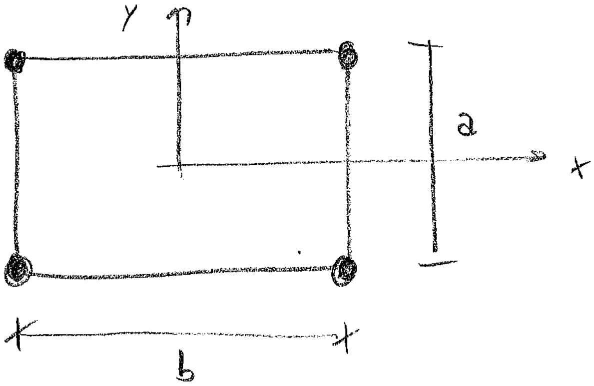
Considering again the case of $a = 100$ and $\ell = 1000$ it follows that:

$$\boxed{\frac{U_{shear}}{U_{bending}} \simeq 30 \cdot 10^{-2} = \frac{30}{100} = 30\,\%}$$

Thus the energy associated with the shear deformability is 30% of the bending energy. It is clear that the contribution is not negligible!

Example

Consider the following beam section:

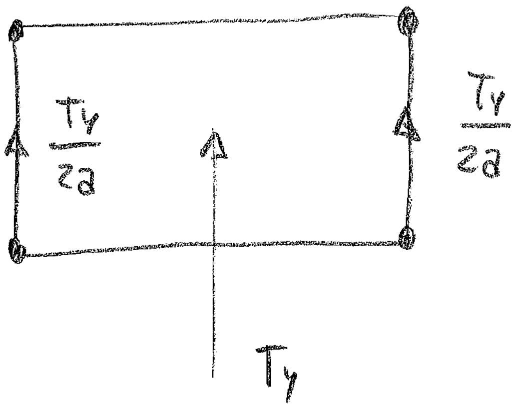


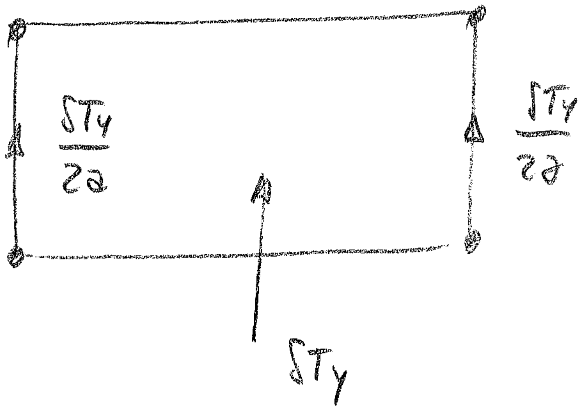
$t = \text{const}$

The torsional stiffness is obtained by calculating the torsional constant J as:

$$J = \frac{4\Omega^2}{\oint \frac{1}{t(s)}\, ds} = \frac{4a^2b^2t}{2(a+b)} = \frac{2a^2b^2t}{(a+b)} \Rightarrow GJ = G\frac{2a^2b^2t}{(a+b)}$$

The shear stiffness requires the evaluation of the shear flows for an arbitrary T_y applied on the shear center. From the symmetry of the section the shear center coincides with the centroid. The shear flows are readily obtained by exploiting the symmetry as:

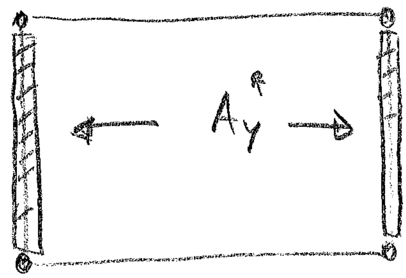




$$A_x^* = \frac{\delta T_y T_y}{\sum_{i=1}^{m} \delta q_m \frac{l_m}{t_m} q_m} = \frac{1}{\frac{1}{2a^2 t} a} = 2at$$

and so $GA_x^* = G \cdot 2at$

- Note that the area A_x^* is equal to the area of the shear webs, which is the fraction of the total area which is contributing to carry the shear load reacted by T_y.



$A_y^* = A_{web}$ ($q =$ const)

- The coincidence between A_y^* and the area of the shear webs is not, in general, a rule. It happens whenever the shear flow is constant along the webs. If the shear flow along the web is not constant

$A_y^* < A_{web}$