Small displacements approximation

- The formulation for the case of large displacements has been developed, with respect to the thermoelastic case, in the context of the Lagrangian approach.

  The solution of the governing PDE (or their variational counterpart) can be performed referring to numerical techniques (Finite elements, in most cases)

- The problem is now simplified, with the aim of facilitating the derivation of exact and approximate solutions without the need for solving the elastic problem numerically.

  More specifically assume:

  1. The displacements are so small that higher order terms can be neglected in the strain tensor expression
  2. Displacements and strains are so small that the equilibrium can now be written referring to the undeformed configuration

- It is important to understand these two assumptions as a special case of the more general nonlinear approach previously discussed.

  The linear approach should be conceptually understood as a specialization of the nonlinear case to a simpler case, provided the two assumptions reported above are acceptable.

  Whenever this is not the case, the assumptions of linearity should be rejected, and the problem solved in its intrinsic nonlinearity

- From assumption 1, it follows that:

$$\underline{\underline{E}} = \frac{1}{2}\left(\text{Grad}\,\underline{u} + \text{Grad}\,\underline{u}^T + \text{Grad}\,\underline{u}^T\,\text{Grad}\,\underline{u}\right)$$

$$\simeq \frac{1}{2}\left(\text{Grad}\,\underline{u} + \text{Grad}\,\underline{u}^T\right) = \frac{1}{2}\left(\text{grad}\,\underline{u} + \text{grad}\,\underline{u}^T\right) = \underline{\underline{\varepsilon}}$$

Note: no distinction exists between grad(•) and Grad(•).

- From assumption 2, it follows that:

$$\underline{\underline{\sigma}} = \underline{\underline{P}} = \underline{\underline{S}}$$

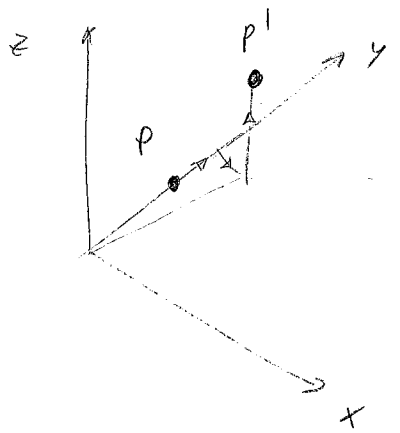
All the stress measures "collapse" into a single entity. Recalling the meaning of $\underline{\underline{P}}$ and $\underline{\underline{S}}$, it is clear that no distinctions exist when $\underline{n}\,da = \underline{N}\,dA$.

- Infinitesimal strain tensor

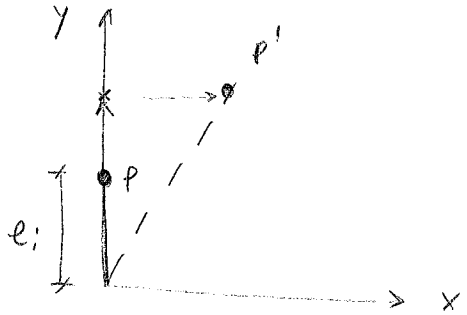
It is useful to recall the physical meaning of the components of the infinitesimal strain tensor $\underline{\underline{\varepsilon}}$.

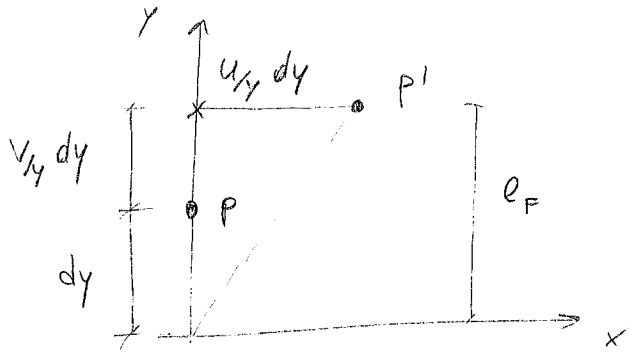
$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\text{grad}\,\underline{u} + \text{grad}\,\underline{u}^T\right) \quad \text{or} \quad \varepsilon_{ik} = \frac{1}{2}\left(u_{i/k} + u_{k/i}\right)$$

Consider the displacement of a point P of the 3D space into another point P'



x-y plane view





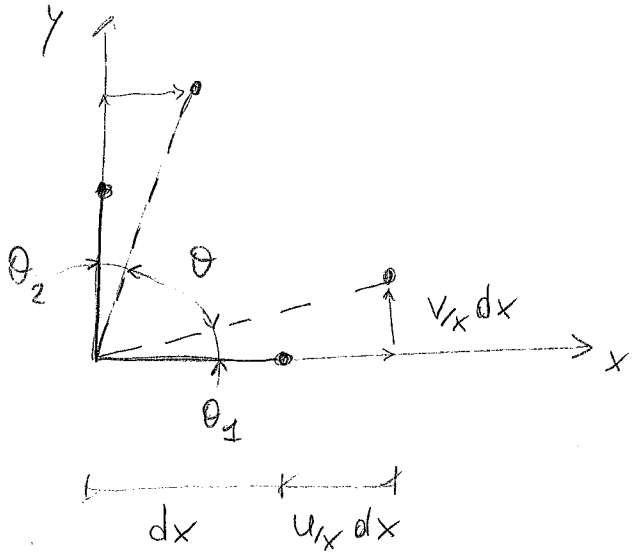
$$e_F = dy\,(1+v_{/y})$$

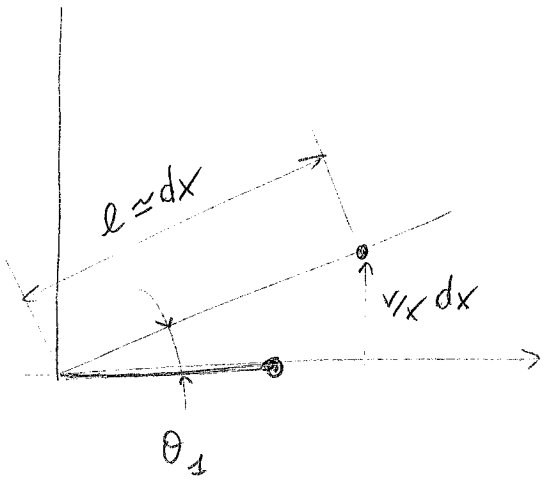
$$\varepsilon_{yy} = \frac{e_F - e_i}{e_i} = \frac{dy\,(1+v_{/y}) - dy}{dy} = v_{/y}$$

$$\Rightarrow \boxed{\varepsilon_{yy} = v_{/y}} \Rightarrow$$ ε_{yy} represents the variation of length of a unitary fiber, initially aligned along y

Clearly the same considerations hold for the components ε_{xx} and ε_{zz}.

It is useful to consider the case of mixed components $\varepsilon_{\alpha\beta}$ with $\alpha \neq \beta$. (shearing components)





For small displacements, $l \simeq dx$, so:

$$l \sin\theta_1 = v_{/x}\, dx$$

$$dx \sin\theta_1 = v_{/x}\, dx \Rightarrow v_{/x} = \sin\theta_1 \Rightarrow \boxed{v_{/x} \simeq \theta_1}$$

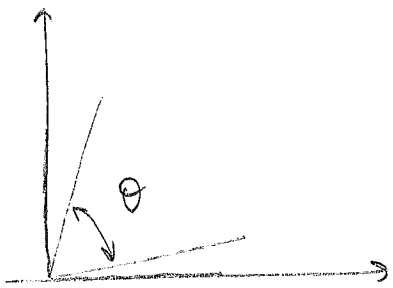
The same considerations hold for θ_2:

$$l \sin\theta_2 = u_{/y}\, dy$$

$$dy \sin\theta_2 = u_{/y}\, dy \Rightarrow u_{/y} = \sin\theta_2 \Rightarrow \boxed{u_{/y} \simeq \theta_2}$$

It follows that:

$$\boxed{v_{/x} + u_{/y} = \theta_1 + \theta_2 = \frac{\pi}{2} - \theta}$$



or:

$$\boxed{\gamma_{xy} = 2\varepsilon_{xy} = \frac{\pi}{2} - \theta}$$

Compatibility equations

- Although compatibility equations seem complex and somewhat obscure, they express a very simple idea: the displacement field is required to respect some conditions in order to be "compatible". A randomly defined function representing the displacement field can, in fact, be associated with a non admissible displacement field.
  The condition to be fulfilled is $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\text{grad}\,\underline{u} + \text{grad}\,\underline{u}^T\right)$

- Sometimes, to highlight that the previous one is an equation and not a definition, it is written as:
  $\underline{\underline{\varepsilon}} - \frac{1}{2}\left(\text{grad}\,\underline{u} + \text{grad}\,\underline{u}^T\right) = 0$

- Often the compatibility condition is implicitly imposed in a strong form manner, and does not need to be imposed. Whenever the problem is formulated by imposing, say, $\sigma = E u_{/x}$, the condition is already imposed.

- In many other cases - the De Saint Venant (DSV) problem is an example - the strains are part of the problem unknowns.
  In such cases, the deformation field must satisfy the compatibility conditions in order to make it sure that the solution, both in terms of displacements and strains, is <u>compatible</u>

- Compatibility equations = relations between the various components of ε_{ij} to guarantee the compatibility of the resulting strain field.
  ε_{ij} are <u>not independent each other!</u>

Derivation of the compatibility equations

1. Begin with the deformation components associated with the xy plane

$$\varepsilon_{xx} = u_{/x}$$
$$\varepsilon_{yy} = v_{/y}$$
$$\gamma_{xy} = u_{/y} + v_{/x}$$

How can they be related each other?

$$\boxed{\varepsilon_{xx/yy} = u_{/xyy}}$$
$$\varepsilon_{yy} = v_{/y}$$
$$\gamma_{xy} = u_{/y} + v_{/x}$$

$$\Rightarrow$$

$$\varepsilon_{xx/yy} = u_{/xyy}$$
$$\varepsilon_{yy} = (v_{/y}) \leftrightarrow (+ v_{/xxy})$$
$$\gamma_{xy/xy} = u_{/xyy} (+ v_{/xxy})$$

$$\Rightarrow$$

$$\varepsilon_{xx/yy} = u_{/xyy}$$
$$\varepsilon_{yy/xx} = v_{/xxy}$$
$$\gamma_{xy/xy} = u_{/xyy} + v_{/xxy}$$

$$\Rightarrow \boxed{\varepsilon_{xx/yy} + \varepsilon_{yy/xx} = \gamma_{xy/xy}}$$

2. Consider now the three shearing deformations:

$$\gamma_{xy} = v_{/x} + u_{/y}$$
$$\gamma_{xz} = w_{/x} + u_{/z}$$
$$\gamma_{yz} = w_{/y} + v_{/z}$$
$$\varepsilon_{xx} = u_{/x}$$

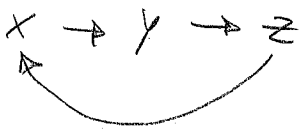
$$\Rightarrow$$

$$\gamma_{xy/xz} = v_{/xxz} + u_{/xyz}$$
$$\gamma_{xz/xy} = w_{/xxy} + u_{/xyz}$$
$$\gamma_{yz} = w_{/y} + v_{/z}$$
$$\varepsilon_{xx/yz} = u_{/xyz}$$

$$\gamma_{xy/xz} = v_{/xxz} + u_{/xyz}$$
$$\gamma_{xz/xy} = w_{/xxy} + u_{/xyz}$$
$$\gamma_{yz/xx} = w_{/xxy} + v_{/xxz}$$
$$\varepsilon_{xx/yz} = u_{/xyz}$$

$$\Rightarrow \boxed{2\,\varepsilon_{xx/yz} = \gamma_{xy/xz} + \gamma_{xz/xy} - \gamma_{yz/xx}}$$

Starting from the two equations here derived, the complete set of compatibility equations is readily available by cycling the indexes



So:

$$\varepsilon_{xx/yy} + \varepsilon_{yy/xx} = \gamma_{xy/xy}$$

$$\varepsilon_{yy/zz} + \varepsilon_{zz/yy} = \gamma_{yz/yz}$$

$$\varepsilon_{zz/xx} + \varepsilon_{xx/zz} = \gamma_{zx/zx}$$

$$2\,\varepsilon_{xx/yz} = \gamma_{xy/xz} + \gamma_{xz/xy} - \gamma_{yz/xx}$$

$$2\,\varepsilon_{yy/zx} = \gamma_{yz/yx} + \gamma_{yx/yz} - \gamma_{zx/yy}$$

$$2\,\varepsilon_{zz/xy} = \gamma_{zx/zy} + \gamma_{zy/zx} - \gamma_{xy/zz}$$

Compatibility equations

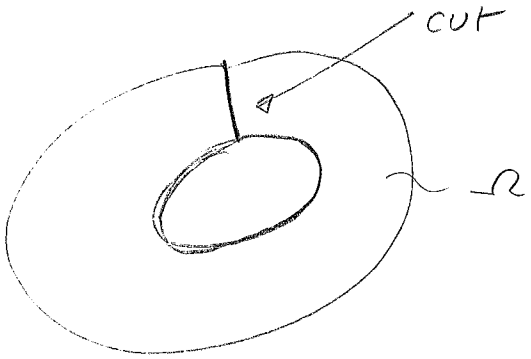
In elasticity it is sometimes useful to solve the problems by using the compatibility equation (which are a set of relations among ε_{ik}) instead of imposing $\varepsilon_{ik} = \frac{1}{2}(u_{i/k} + u_{k/i})$ (which involves, contemporarily the strain components and the displacement field components)

Remarks

1. For simply connected domains, the compatibility equations are equivalent to ε_{ik}, and represent a sufficient condition for a compatible solution

2. For multiconnected domains, the compatibility conditions are NOT sufficient

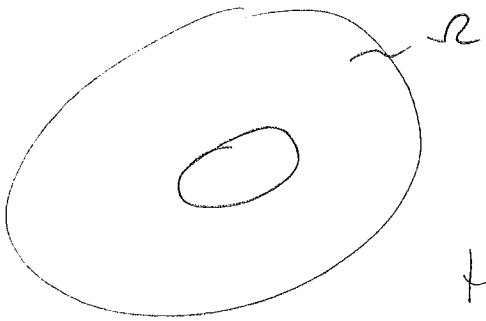
This can be easily understood by considering a domain as reported in the figure



Due to the presence of the cut the domain is simply connected $\Rightarrow$ compatibility equations are a sufficient condition. Assume they are satisfied

In correspondence of the cut the displacements will <u>not</u>, in general be continuous.

Remove now the cut



The previous displacement field still satisfies the compatibility equations.

However the discontinuities of displacement in correspondence of the previously cutted region are now not acceptable if compatibility has to be guaranteed.

Constitutive laws

We consider now different material models that will be adopted throughout the course.

1. Elastic material

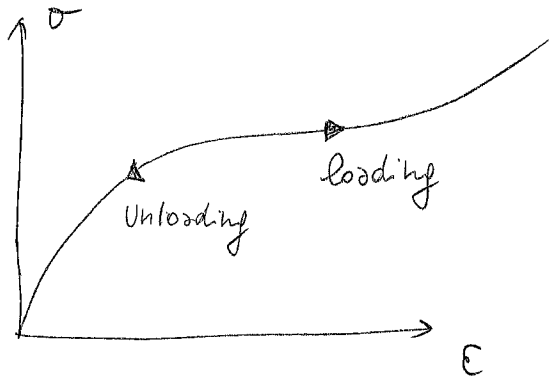
A first model of interest is given by the class of elastic materials.

The assumption of elasticity means that the response is reversible although not necessarily linear

The constitutive law will be generically expressed as:

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}) \quad \text{or} \quad \sigma_{ik} = \sigma_{ik}(\varepsilon_{ik})$$

For instance, an elastic response could be the one reported below



The loading and unloading phases are directed along the same path

For a space application, this response could be the one associated with a foam core of a sandwich panel

2. Linear elastic material

With respect to the previous case it adds the assumption of linearity.

This is the case for the generalized Hooke's law.

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{\varepsilon}} \qquad \text{or} \qquad \sigma_{ik} = C_{ikrs}\,\varepsilon_{rs}$$

The material response is thus reversible and linear.

Note: in this case the only conditions of symmetry are dictated by the symmetry of the tensors

$\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$

This means that $\sigma_{ik} = \sigma_{ki}$

$\varepsilon_{rs} = \varepsilon_{sr}$

$$\Rightarrow C_{ikrs} = C_{kirs} = C_{iksr} = C_{kisr}$$

3. Hyperelastic material

An hyperelastic material is based on the assumption of existence of a strain energy function or, to a more general extent, of a potential function.

It was obtained that

$$\underline{\underline{S}} = \frac{1}{\rho_0}\,\frac{\partial \Psi}{\partial \underline{\underline{\varepsilon}}} \qquad \text{with} \quad \Psi = E - TS$$

Ψ = Helmholtz free energy

For an elastic solid the internal energy is elastic energy only, which is generally denoted as u, so:

$$u = \rho E = \rho e \qquad \Rightarrow \qquad \psi = E - TS$$

↕ the distinction between Lagrangian and Eulerian is now removed

$$\rho\psi = \rho E - \rho TS$$

$$\rho\psi = u - \rho TS$$

Consider the elastic response and neglect the thermal contribution:

$$\rho\psi = u \quad \text{or} \quad \psi = \frac{1}{\rho} u$$

Recalling the thermodynamic definition of the stresses:

$$\underline{\underline{S}} = \rho_0 \, \psi_{/\underline{\underline{\epsilon}}} \qquad \rightsquigarrow \qquad \underline{\underline{\sigma}} = \rho \, \psi_{/\underline{\underline{\varepsilon}}}$$

It is:

$$\underline{\underline{\sigma}} = \frac{1}{\rho} \frac{\partial}{\partial \underline{\underline{\varepsilon}}} (\rho u) = u_{/\underline{\underline{\varepsilon}}} \qquad \Rightarrow \qquad \boxed{\underline{\underline{\sigma}} = u_{/\underline{\underline{\varepsilon}}}}$$

- Let's try to understand the actual meaning of this result.

Compute the elastic work to move from ε_1 to ε_2:

$$W_i = \int_{\varepsilon_1}^{\varepsilon_2} \sigma(\varepsilon) \, d\varepsilon$$

If a potential function exists, then

$$W_i = \int_{\varepsilon_1}^{\varepsilon_2} \sigma(\varepsilon)\, d\varepsilon = \int_{\varepsilon_1}^{\varepsilon_2} U_{/\varepsilon}\, d\varepsilon = U(\varepsilon_2) - U(\varepsilon_1)$$

$\Rightarrow$ The elastic work, thus the energy of deformation, depends on the initial e final values of the deformation. It is independent on the path from ε_1 to ε_2

Clarly the <u>reversibility</u> of the process, i.e. the fact that the material is elastic, is a consequence of this. Indeed:

$$\text{from } \varepsilon_1 \text{ to } \varepsilon_2 \quad \Rightarrow \quad W_i = U(\varepsilon_2) - U(\varepsilon_1)$$

$$\text{from } \varepsilon_2 \text{ to } \varepsilon_1 \quad \Rightarrow \quad U(\varepsilon_1) - U(\varepsilon_2) = 0$$

4. Hyperelastic linear material

This is the kind of material that we will generally assume. This is a good approximation for several materials of aerospace interest.

As implicit in the definition, this material model is

1. Hyperelastic
2. Characterized by a linear constitutive law

It follows that

$$\underline{\underline{\sigma}} = U_{/\underline{\underline{\varepsilon}}} = \underline{\underline{\underline{\underline{C}}}}\ \underline{\underline{\varepsilon}} \qquad \text{or} \qquad \sigma_{ik} = U_{/\varepsilon_{ik}} = C_{ikrs}\, \varepsilon_{rs}$$

Note: this is different from the case of linear elastic materials, as the existence of a potential function is now implied.

The elastic tensor $\underline{\underline{\underline{\underline{C}}}}$, which is a 4th order tensor, is characterized by a number of symmetries, i.e. not all the 3^4 components are independent each other.

a. The number of components is 81

C_{ikrs} with $i,k,r,s = 1,2,3$

b. Consider the symmetry of $\underline{\underline{\sigma}}$

$$\sigma_{ik} = C_{ikrs}\,\varepsilon_{rs}$$
$$\sigma_{ki} = C_{kirs}\,\varepsilon_{rs} \quad \Rightarrow \quad \boxed{C_{ikrs} = C_{kirs}}$$

6 independent couples of ik

The number of independent terms is then $6 \times 3 \times 3 = 54$

c. Consider the symmetry of $\underline{\underline{\varepsilon}}$

$$\sigma_{ik} = C_{ikrs}\,\varepsilon_{rs} = C_{iksr}\,\varepsilon_{sr} \quad \Rightarrow \quad \boxed{C_{ikrs} = C_{iksr}}$$

6 couples

6 couples

The number of independent terms is $6 \times 6 = 36$

d. Consider now the assumption of hyperelasticity

$$\sigma_{ik} = \frac{\partial U}{\partial \varepsilon_{ik}} \quad \text{but} \quad \sigma_{ik} = C_{ikrs}\,\varepsilon_{rs}$$

so $$\frac{\partial \sigma_{ik}}{\partial \varepsilon_{rs}} = C_{ikrs} \Rightarrow \boxed{C_{ikrs} = \frac{\partial^2 U}{\partial \varepsilon_{ik}\,\partial \varepsilon_{rs}}}$$

As far as the order of derivation does not affect the result,

$$C_{ikrs} = \frac{\partial^2 U}{\partial \varepsilon_{ik}\,\partial \varepsilon_{rs}} = \frac{\partial^2 U}{\partial \varepsilon_{rs}\,\partial \varepsilon_{ik}}$$

$$C_{rsik} = \frac{\partial^2 U}{\partial \varepsilon_{rs}\,\partial \varepsilon_{ik}} = \frac{\partial^2 U}{\partial \varepsilon_{ik}\,\partial \varepsilon_{rs}}$$

$$\Rightarrow \boxed{C_{ikrs} = C_{rsik}}$$

From the 36 independent components, the number reduces to 21 for an hyperelastic material

ik \ rs	11	12	13	22	23	33
11	x	x	x	x	x	x
12	x	x	x	x	x	x
13	x	x	x	x	x	x
22	x	x	x	x	x	x
23	x	x	x	x	x	x
33	x	x	x	x	x	x

non hyperelastic (36 components)

⇨

x	x	x	x	x	x
	x	x	x	x	x
		x	x	x	x
			x	x	x
				x	x
					x

hyperelastic (21 components)

In the special case of isotropic hyperelastic material, the number of independent constants reduces to 2.
(They can be taken as the engineering constants E, ν or the Lamé' parameters λ, μ)

The assumption of isotropy implies that the components of the constitutive law are independent on the reference system which is used for expressing the components of $\underline{\underline{\varepsilon}}$ and $\underline{\underline{\sigma}}$.

It follows that the strain energy should be expressed as function of quantities which are independent on the reference system.
The 3 invariants of the deformation are thus suitable candidates for modeling an isotropic material.
The 3 invariants are

- linear in ε_{ik}
- quadratic in ε_{ik}
- cubic in ε_{ik}

For a linear material the strain energy is quadratic thus the first two invariants are used $\Rightarrow$ the number of parameters is two.

Kelvin-Voigt notation

From the previous discussion, it is clear that a matrix representation of the constitutive law can be an effective way for "organizing" the coefficients of the elastic law.

$\underline{\underline{\underline{\underline{C}}}}$ 4th order tensor $\longrightarrow$ $[C]$ matrix

$\underline{\underline{\sigma}}$ 2nd order tensor $\longrightarrow$ $\{\sigma\}$ vector

$\underline{\underline{\varepsilon}}$ 2nd order tensor $\longrightarrow$ $\{\varepsilon\}$ vector

Note: This is a way for organizing the components of $\underline{\underline{\underline{\underline{C}}}}$, $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$. It is important to remark that matrix/vectors are not tensors!

According to the so-called Kelvin-Voigt notation, the constitutive law is written as

$$\{\sigma\} = [C]\{\varepsilon\}$$

where, in general, the components are ordered as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & \text{sym} & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} = 2\varepsilon_{yz} \\ \gamma_{xz} = 2\varepsilon_{xz} \\ \gamma_{xy} = 2\varepsilon_{xy} \end{Bmatrix}$$

Remarks

According to the Kelvin-Voigt notation, $\{\sigma\}$ and $\{\varepsilon\}$ are vectors, while $[C]$ is a matrix. However, the tensorial nature of these entities should be taken into account whenever, for instance, they need to be rotated from one reference system to another

1. $\{\sigma\}$ <u>does not</u> rotate as a vector
   The rotation of $\underline{\underline{\sigma}}$ implies the rotation of a 2^{nd} order tensor

2. $[C]$ does not rotate as a second order tensor.
   The rotation of $[C]$ implies the rotation of a 4^{th} order tensor

According to the Kelvin-Voigt notation, the strain energy reads:

$$u = \frac{1}{2}\,\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} = \frac{1}{2}\{\sigma\}^T\{\varepsilon\} = \frac{1}{2}\{\varepsilon\}^T\{\sigma\}$$

$$= \frac{1}{2}\{\varepsilon\}^T[C]\{\varepsilon\}$$

$$\boxed{u = \frac{1}{2}\,\underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}} = \frac{1}{2}\{\varepsilon\}^T[C]\{\varepsilon\}}$$

Everything can be extended to the case of a thermoelastic constitutive law.

In that case:

$\psi = \psi(\underline{\underline{\varepsilon}}, T)$ and

$$\rho\psi = \frac{1}{2}\,\underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}} - \underline{\underline{\beta}} : \underline{\underline{\varepsilon}}\,\Delta T - \frac{\rho c_v}{2T_0}\Delta T^2$$

from which:

$$\boxed{\underline{\underline{\sigma}} = \rho\,\frac{\partial \psi}{\partial \underline{\underline{\varepsilon}}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}} - \underline{\underline{\beta}}\,\Delta T}$$

Referring to the Kelvin–Voigt notation:

$$\rho\psi = \frac{1}{2}\{\varepsilon\}^T [C] \{\varepsilon\} - \{\varepsilon\}^T \{\beta\}\,\Delta T - \frac{\rho c_v}{2T_0}\Delta T^2$$

$$\Rightarrow \boxed{\{\sigma\} = [C]\{\varepsilon\} - \{\beta\}\,\Delta T}$$

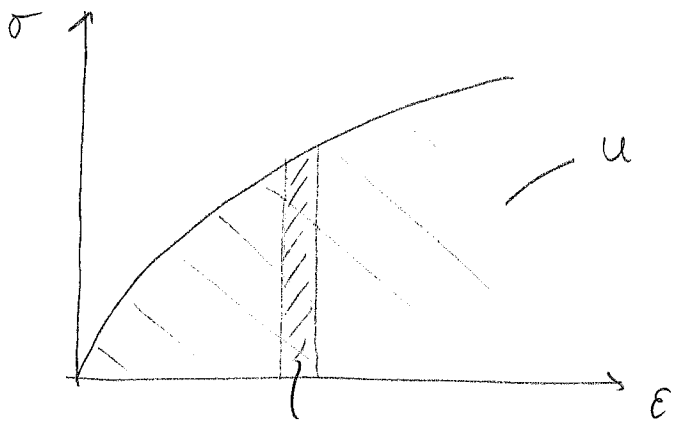
Complementary energy

The strain energy has been introduced as $u = \frac{1}{2}\,\underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}$

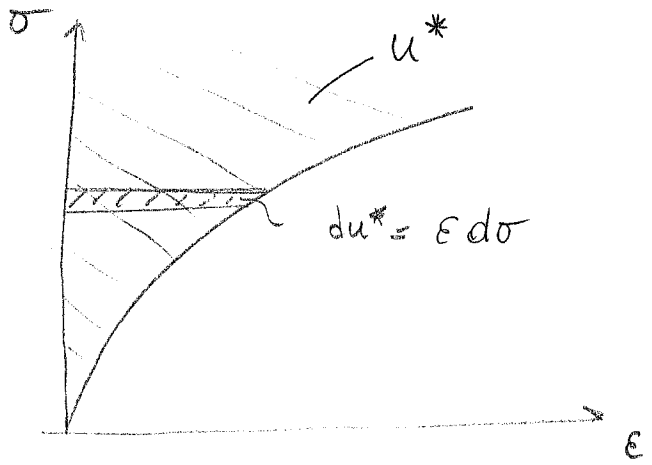
meaning that $u = u(\underline{\underline{\varepsilon}})$ and $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{\underline{\varepsilon}})$

Similarly, the complementary strain energy can be introduced as $u^* = u^*(\underline{\underline{\sigma}})$ with $\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}(\underline{\underline{\sigma}})$

A simple graphical representation is given for the 1-D case



Strain energy



Complementary strain energy

Clearly in the linear case $u = u^*$. However, they are conceptually different.

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{\underline{\varepsilon}})$

$u = u(\underline{\underline{\varepsilon}})$

$u = \int_0^{\varepsilon_{ik}} \sigma_{ik}\, d\varepsilon_{ik}$

$\underline{\underline{\sigma}} = u/\underline{\underline{\varepsilon}}$

$\underline{\underline{\underline{\underline{C}}}} = u/\underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}$

$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}(\underline{\underline{\sigma}})$

$u^* = u^*(\underline{\underline{\sigma}})$

$u^* = \int_0^{\sigma_{ik}} \varepsilon_{ik}\, d\sigma_{ik}$

$\underline{\underline{\varepsilon}} = u^*/\underline{\underline{\sigma}}$

$\underline{\underline{\underline{\underline{S}}}} = \underline{\underline{\underline{\underline{C}}}}^{-1} = u^*/\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}$

$$u = \frac{1}{2}\,\underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}$$

$$= \frac{1}{2}\{\varepsilon\}^T [C] \{\varepsilon\}$$

$$u^* = \frac{1}{2}\,\underline{\underline{\sigma}} : \underline{\underline{\underline{\underline{S}}}} : \underline{\underline{\sigma}}$$

$$= \frac{1}{2}\{\sigma\}^T [S] \{\sigma\}$$

From a mathematical point of view, u^* is the Legendre transform of u

A few words on the Legendre transform

Consider a function f:

$f = f(x, \varepsilon)$ st $f_{/\varepsilon} = \sigma$

Goal: build-up a function $g = g(x, \sigma)$ st $g_{/\sigma} = \varepsilon$

The variables ε and σ are said to be conjugate.

- Constructing g

The function g is the Legendre transform of f.

$$f = f(x, \varepsilon) \Rightarrow df = f_{/x}\, dx + f_{/\varepsilon}\, d\varepsilon$$
$$= p\, dx + \sigma\, d\varepsilon$$

with $p = f_{/x}$ (this is just a name to identify $f_{/x}$)

Assume now that g can be constructed according to this definition:

$$g = \sigma\varepsilon - f$$

(function to be transformed)

(conjugate variables)

It follows that

$$dg = d(\sigma\varepsilon - f) = \sigma\, d\varepsilon + \varepsilon\, d\sigma - df$$
$$= \sigma\, d\varepsilon + \varepsilon\, d\sigma - p\, dx - \sigma\, d\varepsilon$$
$$= \varepsilon\, d\sigma - p\, dx$$

$dg = \varepsilon\, d\sigma - p\, dx$, so:

1. The function g is function of σ and x

$$g = g(\sigma, x)$$

2. $\dfrac{dg}{d\sigma} = \varepsilon$ or $g_{/\sigma} = \varepsilon$ as initially required

- Complementary energy as the Legendre transform of the strain energy

$u = u(\underline{\underline{\varepsilon}})$ and $u_{/\underline{\underline{\varepsilon}}} = \underline{\underline{\sigma}}$

The complementary strain energy is:

$u^* = u^*(\underline{\underline{\sigma}})$ and $u^*_{/\underline{\underline{\sigma}}} = \underline{\underline{\varepsilon}}$

Apply the legendre transform to u

$$u^* = \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} - u$$

where $du = u_{/\underline{\underline{\varepsilon}}} : d\underline{\underline{\varepsilon}} = \underline{\underline{\sigma}} : d\underline{\underline{\varepsilon}}$

$$du^* = d(\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}) - du$$

$$= d\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} : d\underline{\underline{\varepsilon}} - du$$

$$= d\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} : d\underline{\underline{\varepsilon}} - \underline{\underline{\sigma}} : d\underline{\underline{\varepsilon}}$$

$$= \underline{\underline{\varepsilon}} : d\underline{\underline{\sigma}} \quad \Rightarrow u^* = u^*(\underline{\underline{\sigma}})$$

Furthermore it can be verified that:

$\underline{\underline{\sigma}} = \dfrac{\partial u^*}{\partial \underline{\underline{\varepsilon}}}$, indeed:

$$\frac{\partial u^*}{\partial \underline{\underline{\sigma}}} = \frac{\partial \left(\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} - u \right)}{\partial \underline{\underline{\sigma}}}$$

$$= \underline{\underline{\underline{\underline{\mathbb{I}}}}} : \underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} : \frac{\partial \underline{\underline{\varepsilon}}}{\partial \underline{\underline{\sigma}}} - \frac{\partial u}{\partial \underline{\underline{\sigma}}}$$

$$= \underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} : \frac{\partial \underline{\underline{\varepsilon}}}{\partial \underline{\underline{\sigma}}} - \frac{\partial u}{\partial \underline{\underline{\varepsilon}}} : \frac{\partial \underline{\underline{\varepsilon}}}{\partial \underline{\underline{\sigma}}}$$

$$= \underline{\underline{\varepsilon}} + \underline{\underline{\sigma}} : \frac{\partial \underline{\underline{\varepsilon}}}{\partial \underline{\underline{\sigma}}} - \underline{\underline{\sigma}} : \frac{\partial \underline{\underline{\varepsilon}}}{\partial \underline{\underline{\sigma}}}$$

$$= \underline{\underline{\varepsilon}} \qquad \Rightarrow \boxed{\frac{\partial u^*}{\partial \underline{\underline{\sigma}}} = \underline{\underline{\varepsilon}}}$$

Note: $\underline{\underline{\underline{\underline{\mathbb{I}}}}}$ is the fourth order identity defined as:

$$\underline{\underline{A}} : \underline{\underline{\underline{\underline{\mathbb{I}}}}} = \underline{\underline{\underline{\underline{\mathbb{I}}}}} : \underline{\underline{A}} = \underline{\underline{A}}$$

If the expression of u^* is sought, it is easy to obtain it as:

$$u^* = \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} - u$$

$$= \underline{\underline{\sigma}} : \underline{\underline{\underline{\underline{S}}}} : \underline{\underline{\sigma}} - \frac{1}{2} \underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}$$

$$= \underline{\underline{\sigma}} : \underline{\underline{\underline{\underline{S}}}} : \underline{\underline{\sigma}} - \frac{1}{2} \underline{\underline{\sigma}} : \underline{\underline{\underline{\underline{S}}}} : \underline{\underline{\sigma}}$$

$$= \frac{1}{2} \underline{\underline{\sigma}} : \underline{\underline{\underline{\underline{S}}}} : \underline{\underline{\sigma}}$$