University of New South Wales
School of Computer Science and Engineering
Foundations of Computer Science (COMP9020)
FINAL EXAM — Term 2, 2020

Instructions:

- The exam consists of 10 questions for a total of **200 marks**
- Solutions should be submitted as either a pdf or a zipped file of jpgs, maximum size 20Mb. It is your responsibility to ensure your submission is legible.
- Solutions should be submitted via give/webCMS. Multiple submissions are allowed, your last submission before the due date will be the submission marked.
- Deadline for submissions is **9am (Sydney time), Tuesday 18 August 2020**.
- Late submissions will be penalized at 25% (applied to raw mark) per 15 minutes or part thereof.
- Existing written materials may be used, but should be properly referenced.
- Unless explicitly stated, you should prove/explain/motivate your answers. You are being assessed on your understanding and ability.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for your worst answer, as this indicates how well you understood the question.
- Number of pages in this exam paper: 6, including this cover page.

Question 1 (20 marks)

(a) Prove, or give a counterexample to disprove:

(i) For all $x \in \mathbb{R}$:

$$|\lfloor x \rfloor| = \lfloor |x| \rfloor$$

(4 marks)

(ii) For all $x \in \mathbb{Z}$:

$$42 | x^7 - x$$

(4 marks)

(iii) For all $x, y, z \in \mathbb{Z}$, with $z > 1$ and $z \nmid y$:

$$(x \text{ div } y) = ((x \% z) \text{ div } (y \% z)) \pmod z$$

(4 marks)

(b) The following code snippet could be used to find the smallest i such that $i > n^2$ and $k|i$:

```
myFunction(n,k):
  i = 0
  while i ≤ n^2 :
    j = 0
    while j < k :
      i = i + 1
      j = j + 1
    end while
  end while
  return i
```

(i) Assuming k is a constant that is not dependent on n, provide an asymptotic upper bound, in terms of n, for the running time of this algorithm. Partial marks may be awarded for correct, but sub-optimal bounds. (4 marks)

(ii) Assuming the number-theoretic functions covered in this course ($|\cdot|$, $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$, div, %), and the standard arithmetic operations, can be computed in $O(1)$ time, find a constant-time replacement for the above code. (4 marks)

Question 2 (20 marks)

(a) Prove, or give a counterexample to disprove for all sets A, B, C, D:

(i) $(A \oplus B) = (B \oplus A)$ (4 marks)

(ii) $A \cup (B \oplus C) = (A \oplus B) \cup (A \oplus C)$ (4 marks)

(iii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ (4 marks)

(b) (i) The Laws of Set Operations only define equality between sets. How can they be used to show, say, $A \subseteq B$? (2 marks)

(ii) Use the Laws of Set Operations to show

$$A \oplus B \subseteq A \cup B.$$

Partial marks are available for a proof that does not use the Laws of Set Operations. (6 marks)

Question 3 (20 marks)

Let $(B, \preceq)$ be a partially ordered set, A be a set, and $f : A \to B$ a function from A to B. Define $R \subseteq A \times A$ as follows:

$$(a, b) \in R \quad \text{if and only if} \quad f(a) \preceq f(b)$$

(a) Give a counterexample to show that in general R is not a partial order. (4 marks)

(b) (i) State a restriction on f that will ensure R is a partial order, and (1 mark)

(ii) Prove that under that restriction R is a partial order. (9 marks)

(c) Prove that $R \cap R^{\leftarrow}$ is an equivalence relation. (6 marks)

Question 4 (20 marks)

Let $\Sigma = \{a, b\}$ and define $f : \Sigma^* \to \mathbb{R}$ recursively as follows:

- $f(\lambda) = 0$,
- $f(aw) = \frac{1}{2} + \frac{1}{2}f(w)$ for $w \in \Sigma^*$, and
- $f(bw) = -\frac{1}{2} + \frac{1}{2}f(w)$ for $w \in \Sigma^*$.

(a) What is $f(abba)$? (2 marks)

(b) Prove that $f(w) \in (-1, 1)$ for all $w \in \Sigma^*$ (6 marks)

(c) Prove, or give a counterexample to disprove:

  (i) f is injective (6 marks)

  (ii) $Im(f) = (-1, 1)$ (6 marks)

Question 5 (20 marks)

Consider the following code snippet that defines a function that works on a set A:

```
myFunction(A):
  if |A| ≤ 1:
    print('*')
  else:
    (A1, A2) = split(A)
    myFunction(A1)
    myFunction(A2)
  end if
```

Here split(X) is a program that runs in $O(|X|)$ time and partitions the set X into two non-empty sets X_1 and X_2.

Give, with justification, asymptotic upper bounds, in terms of $n = |A|$, for the running time of myFunction(A) when:

(a) split(X) always partitions X into equal sized sets.[1] (6 marks)

(b) split(X) can partition X into non-empty sets of any size. (6 marks)

(c) split(X) always partitions X into non-empty sets of size at least $\frac{|X|}{3}$. (8 marks)

Partial marks may be awarded for correct, but sub-optimal bounds.

[1] You may assume that n is a power of 2 so this is always possible.

Question 6 (20 marks)

Prove, or give a counterexample to disprove, for all propositional formulas φ, ψ, θ:

(a) $((\varphi \wedge \psi) \to \theta) \equiv (\varphi \to (\psi \to \theta))$ (4 marks)

(b) $((\varphi \leftrightarrow \psi) \wedge (\psi \leftrightarrow \theta)) \equiv (\varphi \leftrightarrow \theta)$ (4 marks)

(c) $(\varphi \wedge \psi) \models (\varphi \leftrightarrow \psi)$ (4 marks)

(d) $(\varphi \to \psi), (\neg\varphi \to \theta) \models (\psi \vee \theta)$ (4 marks)

(e) $(\varphi \to (\psi \to \theta)) \models ((\varphi \to \psi) \to \theta)$ (4 marks)

Question 7 (20 marks)

(a) Prove that a graph with 6 vertices and at least 13 edges must have at least two vertices of degree 5. (4 marks)

(b) Prove that a graph with 6 vertices and at least 13 edges has clique number at least 4. (4 marks)

(c) Prove that a graph with 6 vertices and at least 13 edges is non-planar. (4 marks)

(d) Draw a planar graph with 6 vertices and 12 edges that has clique number 3. Justify each property. (8 marks)

Question 8 (20 marks)

You are taking an exam that has 6 easy questions and 4 difficult questions. Assuming all questions are distinguishable, how many ways are there of ordering the questions so that:

(a) All the easy questions come first. (4 marks)

(b) Each pair of difficult questions is separated by at least 2 easy questions. (4 marks)

(c) Each pair of difficult questions is separated by at least 1 easy question. (4 marks)

(d) Each pair of difficult questions is separated by at most 1 easy question. (4 marks)

(e) Each pair of difficult questions is separated by exactly 1 easy question. (4 marks)

Question 9 (20 marks)

Suppose two players, A and B, are playing the following game:

- A starts.
- The players take turns rolling a 6-sided die.
- Whoever rolls the first 6 wins the game.

(a) What is the probability that A wins? (6 marks)

(b) What is the expected number of die rolls before a winner is determined? (2 marks)

Now suppose we consider the following addition to the rules:

- If a player rolls a number that has already been seen then they roll again until an unseen number is rolled.

(c) What is the probability that A wins this game? (4 marks)

(d) If we say a turn ends when an unseen number is rolled, what is the expected number of turns before a winner is determined? (4 marks)

(e) At the start of B's second turn, what is the expected number of die rolls before a winner is determined? (4 marks)

Question 10 (20 marks)

(a) Let A be a set with n elements. If we choose a binary relation R uniformly at random from the set of all binary relations on A, what is the probability that:

(i) R is reflexive? (4 marks)

(ii) R is symmetric? (4 marks)

(b) Let F_n denote the set of well-formed formulas (of Propositional Logic) that can be formed using at most n different propositional variables (multiple occurrences of the same variable is acceptable). How many equivalence classes does the logical equivalence relation $\equiv$ define on F_n? (6 marks)

(c) Prove that

$$R = \{(G, H) : G, H \text{ are graphs and } G \text{ contains a subdivision of } H\}$$

is a partial order. (6 marks)