The University of New South Wales

Mid-Session Test

2014/09/01

COMP9020

Foundations of Computer Science

Time allowed: **1 hour**

Total number of questions: **5**

Maximum number of marks: **25**

Not all questions are worth the same.

Answer all questions.

Textbooks, lecture notes, etc. are not permitted.

Calculators may not be used.

Answers must be written in ink. Use a pencil or the back of the booklet for rough work. Your rough work will not be marked.

You can answer the questions in any order.

You may take this question paper out of the exam.

Write your answers into the answer booklet provided.

Question 1 (6 marks)

For each of the following formulae, analyse and prove whether it is valid or not.

$$A \vee \neg B \Leftrightarrow (B \Rightarrow (A \wedge B)) \tag{1}$$
$$(\neg(A \wedge B) \vee C) \Rightarrow (A \Rightarrow C) \tag{2}$$
$$(\neg(A \wedge B) \vee C) \Rightarrow \neg(A \Rightarrow C) \tag{3}$$

Answer:

A	B	$A \vee \neg B$	$(B \Rightarrow (A \wedge B))$
F	**F**	**T**	**T**
F	**T**	**F**	**F**
T	**F**	**T**	**T**
T	**T**	**T**	**T**

Figure 1: Truth table for question 1.1

$$A \vee \neg B \Leftrightarrow (B \Rightarrow (A \wedge B)) \qquad \text{is valid.}$$

We can prove this either using a truth table shown in Figure 1 where we note that the two columns on the right are equal, meaning that the two sub-formulae are indeed equivalent, or via a chain of equivalences:

$$
\begin{aligned}
A \vee \neg B &\Leftrightarrow \neg B \vee A & &\text{commutativity} \\
&\Leftrightarrow (\neg B \vee A) \wedge \mathbf{T} & &\text{identity} \\
&\Leftrightarrow (\neg B \vee A) \wedge (\neg B \vee B) & &\text{excluded middle} \\
&\Leftrightarrow (\neg B \vee (A \wedge B)) & &\text{distribution} \\
&\Leftrightarrow (B \Rightarrow (A \wedge B)) & &\text{implication} \\
(\neg(A \wedge B) \vee C) &\Rightarrow (A \Rightarrow C) & &\text{is not valid}
\end{aligned}
$$

Consider $A = \mathbf{T} \wedge C = \mathbf{F} \wedge B = \mathbf{F}$.

$$
\begin{aligned}
(\neg(\mathbf{T} \wedge \mathbf{F}) \vee \mathbf{F}) &\Leftrightarrow \mathbf{T} \\
&\nRightarrow \mathbf{F} \\
&\Leftrightarrow (\mathbf{T} \Rightarrow \mathbf{F}) \\
(\neg(A \wedge B) \vee C) &\Rightarrow \neg(A \Rightarrow C) \qquad\qquad \text{is not valid}
\end{aligned}
$$

Consider $A = \mathbf{T} \wedge C = \mathbf{T}$.

$$
\begin{aligned}
(\neg(\mathbf{T} \wedge B) \vee \mathbf{T}) &\Leftrightarrow \mathbf{T} \\
&\nRightarrow \mathbf{F} \\
&\Leftrightarrow \neg(\mathbf{T} \Rightarrow \mathbf{T})
\end{aligned}
$$

Question 2 (5 marks)

Prove that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

Answer: by induction on n. Base case $\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2^{0+1} - 1$. Inductive case

$$
\begin{aligned}
\sum_{i=0}^{n+1} 2^i &= 2^{n+1} + \sum_{i=0}^{n} 2^i \\
&= 2^{n+1} + 2^{n+1} - 1 \qquad\qquad \text{by the ind. hyp.} \\
&= 2^{n+2} - 1 \ .
\end{aligned}
$$

Question 3 (5 marks)

Prove or disprove that $S \setminus (T \cap U) = (S \setminus T) \cup (S \setminus U)$ for all sets S, T, and U.

Answer: $S \setminus (T \cap U) = (S \setminus T) \cup (S \setminus U)$ is valid.

Proof:

$$
\begin{aligned}
S \setminus (T \cap U) &= S \cap \overline{(T \cap U)} \\
&= S \cap (\overline{T} \cup \overline{U}) \\
&= (S \cap \overline{T}) \cup (S \cap \overline{U}) \\
&= (S \setminus T) \cup (S \setminus U)
\end{aligned}
$$

Question 4 (4 marks)

Consider the equivalence relation R defined on the set of natural numbers $\{1, 2, \ldots, 9, 10\}$ by xRy if x and y have exactly the same set of prime divisors. List its equivalence classes.

Answer: The equivalence classes are $\{1\}$, $\{2, 4, 8\}$, $\{3, 9\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{10\}$.

Question 5 (5 marks)

Consider the relation on the natural numbers defined by xRy iff x and y both have a divisor (not necessarily a common divisor) in the set $\{3, 5, 7, 13\}$.

Which of the basic properties (reflexive, anti-reflexive, symmetric, anti-symmetric, transitive) does this relation satisfy? (For each property explain briefly why it is satisfied or provide a counterexample if it is not satisfied.)

Answer:

(a) R is not reflexive: consider $2 \in \mathbb{N}$. It has no divisors in the set, hence $\neg 2R2$.

(b) R is not anti-reflexive: consider $3 \in \mathbb{N}$. We have $3R3$ since 3 has a divisor (3) in the set.

(c) R is symmetric: Any relation $R_p = \{ (x, y) : p(x) \wedge p(y) \}$ based on some property p of elements is an equivalence relation with at most two equivalence classes. As such it is in particular symmetric.

(d) R is not anti-symmetric: $3R5$ and $5R3$ but $3 \neq 5$.

(e) R is transitive: as for symmetry.

Reminders

For sets S and T, the *set difference* $S \setminus T$ is $\{\, s \in S : s \notin T \,\}$.

A binary relation $R \subseteq S \times S$ is

- *reflexive* iff $\forall x \in S\,((x, x) \in R)$;

- *anti-reflexive* $\forall x \in S\,((x, x) \notin R)$;

- *symmetric* iff $\forall x, y \in S\,((x, y) \in R \Rightarrow (y, x) \in R)$;

- *anti-symmetric* iff $\forall x, y \in S\,((x, y) \in R \wedge (y, x) \in R \Rightarrow x = y)$;

- *transitive* iff $\forall x, y, z \in S\,((x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R)$;

- an *equivalence* iff R is reflexive, transitive, and symmetric. An *equivalence class* is a set $[x] = \{\, y \in S : xRy \,\}$ for some $x \in S$.