

COMP9020 Week 2

Sets and Formal Languages

- [RW] - Ch. 1, Ch. 3
- [LLM] - Sections 4.1, 4.2

Applications of Sets and Formal Languages

- Sets are the building blocks of nearly all mathematical structures
- Data structures based around sets can be a space-efficient storage system
- Formal languages are essential for compilers and programming language design
- Set theory is a good introduction to formal reasoning (logic)
- Formal languages provide a good introduction to recursive structures (recursion and induction)

Outline

- Introduction to Sets
- Formal Languages
- Set Equality
- Laws of Set Operations

Sets

Definition

A **set** is a collection of objects (**elements**). If x is an element of A we write $x \in A$.

NB

- Elements are taken from a **universe**, \mathcal{U} , – but this can be quite complex. e.g. numbers, and sets of numbers, and sets of sets of numbers, etc.
- Not all “well-defined” universes are possible. e.g.
 - No “set of all sets” (Cantor’s paradox)
 - No “sets which do not contain themselves” (Russell’s paradox)

Sets

- A set is defined by the collection of its elements. Order and multiplicity of elements is not considered.
- We distinguish between an element and the set comprising this single element. Thus always $a \neq \{a\}$.
- Set $\emptyset = \{\}$ is empty (no elements);
- Set $\{\{\}\}$ is nonempty — it has one element.
- There is only one empty set; only one set consisting of a single a ; only one set of all natural numbers.

Subsets

Definition

For sets S and T , we say S is a **subset** of T , written $S \subseteq T$, if every element of S is an element of T .

NB

- $S \subseteq T$ includes the case of $S = T$
- $S \subset T$ — a **proper** subset: $S \subseteq T$ and $S \neq T$
- $\emptyset \subseteq S$ for all sets S
- $S \subseteq \mathcal{U}$ for all sets S
- $\mathbb{N}_{>0} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- An element of a set; and a subset of that set are two different concepts

$$a \in \{a, b\}, \quad a \not\subseteq \{a, b\}; \quad \{a\} \subseteq \{a, b\}, \quad \{a\} \notin \{a, b\}$$

Defining sets

Sets are typically described by:

(a) Explicit enumeration of their elements

$$\begin{aligned} S_1 &= \{a, b, c\} = \{a, a, b, b, b, c\} \\ &= \{b, c, a\} = \dots \quad \text{three elements} \end{aligned}$$

$$S_2 = \{a, \{a\}\} \quad \text{two elements}$$

$$S_3 = \{a, b, \{a, b\}\} \quad \text{three elements}$$

$$S_4 = \{\} \quad \text{zero elements}$$

$$S_5 = \{\{\{\}\}\} \quad \text{one element}$$

$$S_6 = \{\{\}, \{\{\}\}\} \quad \text{two elements}$$

Defining sets

(b) Defining a subset of the universal set \mathcal{U} . Including:

- Specifying the properties their elements must satisfy. A typical description involves a **logical** property $P(x)$. For example, with $\mathcal{U} = \mathbb{N}$ and $P(x) = \text{"x is even"}$:

$$\{x : x \in \mathbb{N} \text{ and } x \text{ is even}\} = \{0, 2, 4, \dots\}$$

- Derived sets of integers

$$2\mathbb{Z} = \{ 2x : x \in \mathbb{Z} \}$$

the even numbers

$$3\mathbb{Z} + 1 = \{ 3x + 1 : x \in \mathbb{Z} \}$$

- Using interval notation.

Intervals

Intervals of numbers (applies to any type)

$$[a, b] = \{x : a \leq x \leq b\}; \quad (a, b) = \{x : a < x < b\}$$

$$[a, b) = \{x : a \leq x < b\}; \quad (a, b] = \{x : a < x \leq b\}$$

$$(-\infty, b] = \{x : x \leq b\}; \quad (-\infty, b) = \{x : x < b\}$$

$$[a, \infty) = \{x : a \leq x\}; \quad (a, \infty) = \{x : a < x\}$$

NB

$(a, a) = (a, a] = [a, a] = \emptyset$; however $[a, a] = \{a\}$.

Intervals of \mathbb{N}, \mathbb{Z} are finite: if $m \leq n$

$$[m, n] = \{m, m+1, \dots, n\}$$

Examples

Examples

- $[1, 5] = \{1, 2, 3, 4, 5\}$ (when $\mathcal{U} = \mathbb{Z}$)
- $[1, 5] = \{1, 1.1, 1.01, 1.001, \dots, 2, \dots, \pi, e, \dots\}$ (when $\mathcal{U} = \mathbb{R}$)
- Number of multiples of k between n and m (inclusive):

$$\lfloor \frac{m}{k} \rfloor - \lfloor \frac{n-1}{k} \rfloor$$

- $0 \leq (m \% n) < n$

Examples

Examples

- $[1, 5] = \{1, 2, 3, 4, 5\}$ (when $\mathcal{U} = \mathbb{Z}$)
- $[1, 5] = \{1, 1.1, 1.01, 1.001, \dots, 2, \dots, \pi, e, \dots\}$ (when $\mathcal{U} = \mathbb{R}$)
- Number of multiples of k in $[n, m]$:

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Examples

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- $[1, 5] = \{1, 2, 3, 4, 5\}$ (when $\mathcal{U} = \mathbb{Z}$)
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- Number of multiples of k in $[n, m]$:

$$\lfloor \frac{m}{k} \rfloor - \lfloor \frac{n-1}{k} \rfloor$$

- $(m \% n) \in [0, n)$

Defining sets

(c) Constructions from other, already defined, sets

- Union (\cup), intersection (\cap), complement (\cdot^c), set difference (\setminus), symmetric difference (\oplus)
- Power set $\text{Pow}(X) = \{ A : A \subseteq X \}$
- Cartesian product (\times)

Set Operations

Definition

$A \cup B$ – **union** (a or b):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

$A \cap B$ – **intersection** (a and b):

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A^c – **complement** (with respect to a universal set \mathcal{U}):

$$A^c = \{x : x \in \mathcal{U} \text{ and } x \notin A\}.$$

We say that A, B are **disjoint** if $A \cap B = \emptyset$

Set Operations

Other set operations

Definition

$A \setminus B$ – **set difference**, relative complement (a but not b):

$$A \setminus B = A \cap B^c$$

$A \oplus B$ – **symmetric difference** (a and not b or b and not a ; also known as a or b exclusively; a xor b):

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

Set Operations

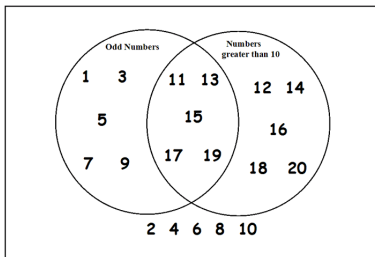
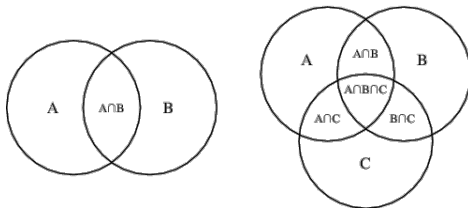
There is a correspondence between set operations and logical operators (to be discussed in Week 5).

NB

$$A \cup B = B \quad \text{iff} \quad A \cap B = A \quad \text{iff} \quad A \subseteq B$$

Venn Diagrams

A simple graphical approach to reason about the algebraic properties of set operations.



Exercises

Exercises

RW: 1.4.4 (d) All subsets of $\{a, b\}$:

RW: 1.4.7 (a) $A \oplus A =$

RW: 1.4.7 (b) $A \oplus \emptyset =$

Exercises

Exercises

RW: 1.4.4 (d) All subsets of $\{a, b\} : ?$

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Exercises

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RW: 1.4.4 (d) All subsets of $\{a, b\} : ?$

RW: 1.4.7 (a) $A \oplus A = ?$

RW: 1.4.7 (b) $A \oplus \emptyset = ?$

Cardinality

Number of elements in a set X (various notations):

$$|X| = \#(X) = \text{card}(X)$$

Fact

Always $|\text{Pow}(X)| = 2^{|X|}$

Exercises

Exercises

- $|\emptyset| =$
- $\text{Pow}(\emptyset) =$
- $|\text{Pow}(\emptyset)| =$
- $\text{Pow}(\text{Pow}(\emptyset)) =$
- $|\text{Pow}(\text{Pow}(\emptyset))| =$
- $|\{a\}| =$
- $\text{Pow}(\{a\}) =$
- $|\text{Pow}(\{a\})| =$
- $|[m, n]| =$

Exercises

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- $|\text{Pow}(\{a\})| = ?$
- $|[m, n]| = ?$

Exercises

RW: 1.3.2 Find the cardinalities of sets

① $|\{ \frac{1}{n} : n \in [1, 4] \}| =$

② $|\{ n^2 - n : n \in [0, 4] \}| =$

③ $|\{ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11 \}| =$

④ $|\{ 2 + (-1)^n : n \in \mathbb{N} \}| =$

Exercises

RW: 1.3.2 Find the cardinalities of sets

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Exercises

RW: 1.3.2 Find the cardinalities of sets

- 1 $|\{ \frac{1}{n} : n \in [1, 4] \}| = ?$
- 2 $|\{ n^2 - n : n \in [0, 4] \}| = ?$
- 3 $|\{ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11 \}| =$
- 4 $|\{ 2 + (-1)^n : n \in \mathbb{N} \}| =$

Exercises

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Exercises

RW: 1.3.2 Find the cardinalities of sets

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- ② $|\{ n^2 - n : n \in [0, 4] \}| = ?$
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- ④ $|\{ 2 + (-1)^n : n \in \mathbb{N} \}| = ?$

Exercises

RW: 1.4.8 Relate the cardinalities to $|A \cap B|$, $|A|$, $|B|$

- $|A \cup B| =$

- $|A \setminus B| =$

- $|A \oplus B| =$

Exercises

RW: 1.4.8 Relate the cardinalities to $|A \cap B|$, $|A|$, $|B|$

- $|A \cup B| = ?$
?
- $|A \setminus B| = ?$
- $|A \oplus B| = ?$

Cartesian Product

$S \times T \stackrel{\text{def}}{=} \{ (s, t) : s \in S, t \in T \}$ where (s, t) is an **ordered** pair

$\times_{i=1}^n S_i \stackrel{\text{def}}{=} \{ (s_1, \dots, s_n) : s_k \in S_k, \text{ for } 1 \leq k \leq n \}$

$S^2 = S \times S, \quad S^3 = S \times S \times S, \dots, \quad S^n = \times_{i=1}^n S, \dots$

$\emptyset \times S = \emptyset$, for every S

$|S \times T| = |S| \cdot |T|, \quad |\times_{i=1}^n S_i| = \prod_{i=1}^n |S_i|$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\ &= \{(0, a), (1, a), (0, b), (1, b)\} \end{aligned}$$

$$B \times A =$$

$$A^2 =$$

$$A^3 =$$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\ &= \{(0, a), (1, a), (0, b), (1, b)\} \end{aligned}$$

$$B \times A = \{(a, 0), (b, 0), (a, 1), (b, 1)\} \neq A \times B$$

$$A^2 =$$

$$A^3 =$$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

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$$A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

$$A^3 =$$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\ &= \{(0, a), (1, a), (0, b), (1, b)\} \end{aligned}$$

$$B \times A = \{(a, 0), (b, 0), (a, 1), (b, 1)\} \neq A \times B$$

$$A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

$$\begin{aligned} A^3 &= \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), \\ &\quad (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}. \end{aligned}$$

Exercise

Exercise

Let A, B, C be sets.

Is $A \times (B \times C) = (A \times B) \times C$?

Outline

- Introduction to Sets
- **Formal Languages**
- Set Equality
- Laws of Set Operations

Formal Languages: Symbols

Σ — **alphabet**, a finite, nonempty set

Examples (of various alphabets and their intended uses)

$\Sigma = \{a, b, \dots, z\}$ for single words (in lower case)

$\Sigma = \{\text{true}, -, a, b, \dots, z\}$ for composite terms

$\Sigma = \{0, 1\}$ for binary integers

$\Sigma = \{0, 1, \dots, 9\}$ for decimal integers

The above cases all have a natural ordering; this is not required in general, thus the set of all Chinese characters forms a (formal) alphabet.

Formal Languages: Words

Definition

word — any finite string of symbols from Σ

empty word — λ

Example

$w = aba$, $w = 01101 \dots 1$, etc.

$\text{length}(w)$ — # of symbols in w

$\text{length}(aaa) = 3$, $\text{length}(\lambda) = 0$

The only operation on words (discussed here) is **concatenation**, written as juxtaposition vw , ww , abw , wbv , \dots

NB

$\lambda w = w = w\lambda$

$\text{length}(vw) = \text{length}(v) + \text{length}(w)$

Examples

Examples

Let $w = abb$, $v = ab$, $u = ba$

- $vw = ababb$
- $ww = abbabb = vubb$
- $w\lambda v = abbab$
- $\text{length}(vw) = \text{length}(ababb) = 5$

Formal Languages: Languages

Notation: Σ^k — set of all words of length k

We often identify $\Sigma^0 = \{\lambda\}$, $\Sigma^1 = \Sigma$

Σ^* — set of all words (of all lengths)

Σ^+ — set of all nonempty words (of any positive length)

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \dots; \quad \Sigma^{\leq n} = \bigcup_{i=0}^n \Sigma^i$$

$$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \dots = \Sigma^* \setminus \{\lambda\}$$

Definition

A **language** is a subset of Σ^* .

Typically, only the subsets that can be formed (or described) according to certain rules are of interest. Such a collection of 'descriptive/formative' rules is called a **grammar**.

Example (Decimal numbers)

The “language” of all numbers written in decimal to at most two decimal places can be described as follows:

- $\Sigma = \{-, ., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Consider all words $w \in \Sigma^*$ which satisfy the following:
 - w contains at most one instance of $-$, and if it contains an instance then it is the first symbol.
 - w contains at most one instance of $.$, and if it contains an instance then it is preceded by a symbol in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and followed by either one or two symbols in that set.
 - w contains at least one symbol from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

NB

According to these rules 123, 123.0 and 123.00 are all (distinct) words in this language.

Example (HTML documents)

Take $\Sigma = \{ \text{“<html>”, “</html>”, “<head>”, “</head>”, “<body>”, ...} \}$.

The (language of) **valid HTML documents** is loosely described as follows:

- Starts with “<html>”
- Next symbol is “<head>”
- Followed by zero or more symbols from the set of HeadItems (defined elsewhere)
- Followed by “</head>”
- Followed by “<body>”
- Followed by zero or more symbols from the set of BodyItems (defined elsewhere)
- Followed by “</body>”
- Followed by “</html>”

Exercises

RW: 1.3.10 Number of elements in the sets (cont'd)

(e) Σ^* where $\Sigma = \{a, b, c\}$ —

(f) $\{ w \in \Sigma^* : \text{length}(w) \leq 4 \}$ where $\Sigma = \{a, b, c\}$
 $|\Sigma^{\leq 4}| =$

Exercises

RW: 1.3.10 Number of elements in the sets (cont'd)

(e) Σ^* where $\Sigma = \{a, b, c\}$ — ?

(f) $\{ w \in \Sigma^* : \text{length}(w) \leq 4 \}$ where $\Sigma = \{a, b, c\}$
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Exercises

RW: 1.3.10 Number of elements in the sets (cont'd)

(e) Σ^* where $\Sigma = \{a, b, c\}$ — ?

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 $|\Sigma^{\leq 4}| = ?$

Set Operations for Languages

Languages are sets, so the standard set operations (\cap , \cup , \setminus , \oplus , etc) can be used to build new languages.

Two set operations that apply to languages uniquely:

- Concatenation (written as juxtaposition):
 $XY = \{xy : x \in X \text{ and } y \in Y\}$
- Kleene star: X^* is the set of words that are made up by concatenating 0 or more words in X
 - $X^0 = \{\lambda\}$; $X^{i+1} = XX^i$
 - $X^* = X^0 \cup X^1 \cup X^2 \cup \dots$

NB

The set of all finite words over Σ is the Kleene star of Σ (hence notation).

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$
- $A^* = \{\lambda, aa, bb, aaaa, aabb, bbaa, bbbb, aaaaaa, \dots\}$
- $B^* = \{\lambda, c, cc, ccc, cccc, \dots\}$

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
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- $\{\lambda\}^* =$

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

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- $B^* = \{\lambda, c, cc, ccc, cccc, \dots\}$
- $\{\lambda\}^* = \{\lambda\}$

Set Operations for Languages

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Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

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- $B^* = \{\lambda, c, cc, ccc, cccc, \dots\}$
- $\{\lambda\}^* = \{\lambda\}$
- $\emptyset^* =$

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
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- $B^* = \{\lambda, c, cc, ccc, cccc, \dots\}$
- $\{\lambda\}^* = \{\lambda\}$
- $\emptyset^* = \{\lambda\}$

Outline

- Introduction to Sets
- Formal Languages
- Set Equality
- Laws of Set Operations

Set Equality

Two sets are **equal** ($A = B$) if they contain the same elements

To show equality:

- Examine all the elements
- Show $A \subseteq B$ and $B \subseteq A$
- Use the Laws of Set Operations

NB

Venn diagrams can help visualize, but are not rigorous.

Examples

Example

Show $\{3, 2, 1\} = (0, 4)$.

Examples

Example

Show $\{3, 2, 1\} = (0, 4)$.

$(0, 4) = \{1, 2, 3\} = \{3, 2, 1\}$.

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$

$$\begin{aligned}\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} &= \{-2, -1, 0, 1, 2\} \\ &= \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}\end{aligned}$$

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 > 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| > 2\}$

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Show $\{n : n \in \mathbb{Z} \text{ and } n^2 > 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| > 2\}$

Show:

- For all $n \in \mathbb{Z}$, if $n^2 > 5$ then $|n| > 2$; and
- For all $n \in \mathbb{Z}$, if $|n| > 2$ then $n^2 > 5$.

That is, show:

For all $n \in \mathbb{Z}$: $n^2 > 5$ if, and only if $|n| > 2$

Outline

- Introduction to Sets
- Formal Languages
- Set Equality
- Laws of Set Operations

Laws of Set Operations

For all sets A , B , C :

Commutativity

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associativity

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distribution

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Identity

$$A \cup \emptyset = A$$

$$A \cap \mathcal{U} = A$$

Complementation

$$A \cup (A^c) = \mathcal{U}$$

$$A \cap (A^c) = \emptyset$$

Substitution

Because the laws hold for all sets, we can substitute complex expressions for each set symbol.

Example

Commutativity

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Substitution

Because the laws hold for all sets, we can substitute complex expressions for each set symbol.

Example

Commutativity

$$A \cup B = B \cup A$$

Therefore: $(C \cap D) \cup (D \oplus E) = (D \oplus E) \cup (C \cap D)$

Example

Example

Show that for all sets $A \cap (B \cap C) = C \cap (B \cap A)$:

Example

Example

Show that for all sets $A \cap (B \cap C) = C \cap (B \cap A)$:

$$\begin{aligned} A \cap (B \cap C) &= (A \cap B) \cap C && [\text{Associativity}] \\ &= C \cap (A \cap B) && [\text{Commutativity}] \\ &= C \cap (B \cap A) && [\text{Commutativity}] \end{aligned}$$

Important!

(Aim to) limit each step to a single application of a single rule

Other useful set laws

The following are all derivable from the previous 10 laws.

Idempotence

$$A \cap A = A$$

$$A \cup A = A$$

Double complementation

$$(A^c)^c = A$$

Annihilation

$$A \cap \emptyset = \emptyset$$

$$A \cup \mathcal{U} = \mathcal{U}$$

de Morgan's Laws

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$

Example (Idempotence of \cup)

$$A = A \cup \emptyset$$

(Identity)

Example (Idempotence of \cup)

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \end{aligned}$$

Example (Idempotence of \cup)

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \end{aligned}$$

Example (Idempotence of \cup)

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\ &= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \end{aligned}$$

Example (Idempotence of \cup)

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Two useful results

Definition

If A is a set defined using \cap , \cup , \emptyset and \mathcal{U} , then $\text{dual}(A)$ is the expression obtained by replacing \cap with \cup (and vice-versa) and \emptyset with \mathcal{U} (and vice-versa).

Theorem (Principle of Duality)

If you can prove $A_1 = A_2$ using the Laws of Set Operations then you can prove $\text{dual}(A_1) = \text{dual}(A_2)$

Example

Absorption law: $A \cup (A \cap B) = A$

Dual: $A \cap (A \cup B) = A$

Application (Idempotence of \cap)

Recall Idempotence of \cup :

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\ &= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \\ &= (A \cup A) && \text{(Identity)} \end{aligned}$$

Application (Idempotence of \cap)

Invoke the dual laws!

$$\begin{aligned} A &= A \cap \mathcal{U} && \text{(Identity)} \\ &= A \cap (A \cup A^c) && \text{(Complementation)} \\ &= (A \cap A) \cup (A \cap A^c) && \text{(Distributivity)} \\ &= (A \cap A) \cup \emptyset && \text{(Complementation)} \\ &= (A \cap A) && \text{(Identity)} \end{aligned}$$

Two useful results

Theorem (Uniqueness of complement)

$A \cap B = \emptyset$ and $A \cup B = \mathcal{U}$ if, and only if, $B = A^c$.

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Theorem (Uniqueness of complement)

$A \cap B = \emptyset$ and $A \cup B = \mathcal{U}$ if, and only if, $B = A^c$.

Proof (Only if).

$$\begin{aligned} B &= B \cap \mathcal{U} && \text{(Identity)} \\ &= B \cap (A \cup A^c) && \text{(Complement)} \\ &= (B \cap A) \cup (B \cap A^c) && \text{(Distributivity)} \\ &= (A \cap B) \cup (A^c \cap B) && \text{(Commutativity)} \\ &= \emptyset \cup (A^c \cap B) && \text{(Given)} \\ &= (A \cap A^c) \cup (A^c \cap B) && \text{(Complement)} \\ &= (A^c \cap A) \cup (A^c \cap B) && \text{(Commutativity)} \\ &= A^c \cap (A \cup B) && \text{(Distributivity)} \\ &= A^c \cap \mathcal{U} && \text{(Given)} \\ &= A^c && \text{(Identity)} \end{aligned}$$



Application (Double complement)

Take $A = X^c$ and $B = X$:

$$\begin{aligned} X^c \cap X &= X \cap X^c && \text{(Commutativity)} \\ &= \emptyset && \text{(Identity)} \end{aligned}$$

$$X^c \cup X = \mathcal{U} \quad \text{(Principle of duality)}$$

By the uniqueness of complement, $(X^c)^c = X$.

Exercises

Exercises

Show the following for all sets A , B , C :

- $B \cup (A \cap \emptyset) = B$
- $(C \cup A) \cap (B \cup A) = A \cup (B \cap C)$
- $(A \cap B) \cup (A \cup B^c)^c = B$

Exercises

Give counterexamples to show the following do not hold for all sets:

- $A \setminus (B \setminus C) = (A \setminus B) \setminus C$
- $(A \cup B) \setminus C = A \cup (B \setminus C)$
- $(A \setminus B) \cup B = A$