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Combinatorics and Probability

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**Problem 1**

- (a) In how many ways can the letters  $a, b, c, d, e, f$  be arranged so that the letters  $a$  and  $b$  are next to each other?
- (b) In how many ways can the letters  $a, b, c, d, e, f$  be arranged so that the letters  $a$  and  $b$  are not next to each other?
- (c) In how many ways can the letters  $a, b, c, d, e, f$  be arranged so that the letters  $a$  and  $b$  are next to each other but  $a$  and  $c$  are not?

**Solution**

- (a) Assume  $a$  occurs before  $b$  and group  $a$  and  $b$  together as a single “letter”:  $(ab)$ . There are  $5!$  ways to arrange this new letter with the other 4. Similarly if  $b$  occurs before  $a$ , there are  $5!$  ways; giving a total of  $2 \cdot 5! = 240$  ways of arranging the letters so that  $a$  and  $b$  are next to each other.
- (b) There are  $6! = 720$  ways of arranging the letters, and from the previous question, 240 of them have  $a$  next to  $b$ . So  $720 - 240 = 480$  do not have  $a$  next to  $b$ .
- (c) Treat ‘ $cab$ ’ as one long symbol: there are  $4! = 24$  arrangements. Similarly 24 arrangements including ‘ $bac$ ’. Therefore  $240 - 24 - 24 = 192$  arrangements that include either ‘ $ab$ ’ or ‘ $ba$ ’ but do not include ‘ $cab$ ’ or ‘ $bac$ ’.

**Problem 2**

A 4-letter word is selected at random from  $\Sigma^4$ , where  $\Sigma = \{a, b, c, d, e\}$ .

- (a) What is the probability that the letters in the word are distinct?
- (b) What is the probability that there are no vowels in the word?
- (c) What is the probability that the word begins with a vowel?
- (d) What is the expected number of vowels in the word?
- (e) Let  $x$  be the answer to the previous question. What is the probability of the word having  $\lceil x \rceil$  or more vowels?

### Solution

- (a) Out of  $5^4 = 625$  words, there are  $5 \cdot 4 \cdot 3 \cdot 2 = 120$  words consisting of distinct letters  $\rightarrow$  the probability is  $\frac{120}{625} = 19.2\%$ .
- (b) There are  $3^4 = 81$  words consisting of only the letters  $b, c, d$  (not the vowels  $a, e$ ); the probability is  $\frac{81}{625} \approx 13\%$ .
- (c) Since there are two vowels, the probability that the first letter is a vowel is  $\frac{2}{5}$  (the subsequent letters are irrelevant).
- (d) Let  $X_i$  ( $1 \leq i \leq 4$ ) be a random variable that counts the number of vowels in the  $i$ -th position (so each  $X_i$  is either 0 [with probability  $\frac{3}{5}$ ] or 1 [with probability  $\frac{2}{5}$ ]). The expected number of vowels is then  $E(X_1 + X_2 + X_3 + X_4) = E(X_1) + E(X_2) + E(X_3) + E(X_4)$  by the linearity of expectation.  $E(X_i) = 0 \cdot \frac{3}{5} + 1 \cdot \frac{2}{5} = \frac{2}{5}$ ; so the expected number of vowels is  $4 \cdot \frac{2}{5} = \frac{8}{5}$ .
- (e)  $\lceil \frac{8}{5} \rceil = 2$ . From (b), the probability that the word contains no vowels is  $\frac{81}{625}$ ; and the probability that the word contains exactly one vowel is  $\frac{4 \cdot 2 \cdot 3^3}{5^4}$  (there are 4 choices for the location of the vowel; 2 choices for the vowel; and  $3^3$  choices for the remaining 3 consonants). So the probability of having 1 or fewer vowels is  $\frac{4 \cdot 2 \cdot 3^3 + 81}{625} = \frac{216 + 81}{625} = \frac{297}{625}$ . Therefore, the probability of having  $\lceil \frac{8}{5} \rceil$  or more vowels is  $1 - \frac{297}{625} = \frac{328}{625} \approx 52.5\%$ .

### Problem 3

A black die and a red die are tossed. What is the probability that

- (a) the sum of the values is even?
- (b) the number on the red die is bigger than the number on the black die?
- (c) the number on the red die is twice the number on the black die?

### Solution

- (a) Regardless of the outcome of the black die, there will be 3 outcomes of the red die (out of the possible 6) for which the sum of the values is even. Therefore the probability is  $\frac{1}{2}$ . (Alternatively, verify that the sum is even in 18 out of the 36 possible outcomes of both dice.)
- (b) There are 6 outcomes (among the 36 possible) where the numbers are equal, and of the remaining 30, half (that is, 15 outcomes) have a higher red number than black numbers. Therefore the probability is  $\frac{15}{36} = \frac{5}{12}$ .
- (c) The relevant outcomes are (1,2), (2,4), or (3,6); therefore the probability is  $\frac{3}{36} = \frac{1}{12}$ .

### Problem 4

Team  $\alpha$  faces team  $\beta$  in a 5-match series. Matches are either won or lost, i.e., there are no draws. It takes 3 wins to win the series. Team  $\alpha$  has probability  $p$  ( $0 < p < 1$ ) of winning a match. Consider each of the following situations and calculate the probability that they will lose the whole series.

- (a) They have lost the first match of the series already.
- (b) They have lost one of the first two matches of the series already.
- (c) They have lost the first two matches of the series already.
- (d) They have lost one of the first three matches of the series already.
- (e) They have lost two of the first three matches of the series already.

#### Solution

- (a) To lose at least two more matches out of the remaining 4, the probability is  $\binom{4}{2}p^2(1-p)^2 + \binom{4}{3}p(1-p)^3 + (1-p)^4$ .
- (b) To lose at least two more matches out of the remaining 3, the probability is  $\binom{3}{2}p(1-p)^2 + (1-p)^3$ .
- (c) To lose at least one more match out of the remaining 3, the probability is  $\binom{3}{1}p^2(1-p) + \binom{3}{2}p(1-p)^2 + (1-p)^3$  (alternatively,  $1 - p^3$ ).
- (d) To lose both of the remaining two matches, the probability is  $(1-p)^2$ .
- (e) To lose at least one of the remaining two matches, the probability is  $1 - p^2$ .

#### Problem 5

Let  $E_1, E_2$  be two events. Prove that  $P(E_1 \setminus E_2) = P(E_1) - P(E_2)$  implies  $P(E_2 \setminus E_1) = 0$ .

#### Solution

Since  $E_1 \setminus E_2$  and  $E_2$  are disjoint and  $E_2 \setminus E_1$  and  $E_1$  are disjoint, we have:

$$P(E_2 \setminus E_1) + P(E_1) = P(E_2 \cup E_1) = P(E_1 \setminus E_2) + P(E_2).$$

So  $P(E_2 \setminus E_1) = P(E_1 \setminus E_2) + P(E_2) - P(E_1) = 0$  if  $P(E_1 \setminus E_2) = P(E_1) - P(E_2)$ .