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UNSW COMP9020（week8）

时间：2020-9-27

时长：1.5h

地点：zoom

导师：Lindsay

logic II

syntax
语法

- well–formed formula
- parse tree
- CNF & DNF

sematics
语义

- Satisfiability
- Logical Equivalence
- Entailment&Vadility

logic II

The first step in the formal definition of logic is the separation of **syntax** and **semantics**

- Syntax is how things are written: what *defines* a formula
- Semantics is what things mean: what does it mean for a formula to be "true"?

Example

"Rabbit" and "Bunny" are syntactically different, but semantically the same.

A	$\neg A$	$\neg(\neg A)$
True	False	True
False	True	False

logic II

syntax

- well–formed formula
- parse tree
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syntax_well formed formula

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Let $\textsc{Prop} = \{p, q, r, \ldots\}$ be a set of propositional letters.
Consider the alphabet

$$\Sigma = \textsc{Prop} \cup \{\top, \bot, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, (,)\}.$$

The **well-formed formulas** (wffs) over $\textsc{Prop}$ is the smallest set of words over Σ such that:

- $\top$, $\bot$ and all elements of $\textsc{Prop}$ are wffs
- If φ is a wff then $\neg\varphi$ is a wff
- If φ and ψ are wffs then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$ are wffs.

syntax_well formed formula

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The following are well-formed formulas:

- $(p \wedge \neg \top)$
- $\neg(p \wedge \neg \top)$
- $\neg\neg(p \wedge \neg \top)$

The following are **not** well-formed formulas:

- $p \wedge \wedge$
- $p \wedge \neg \top$
- $(p \wedge q \wedge r)$
- $\neg(\neg p)$

syntax_well formed formula

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To aid readability some conventions and binding rules can and will be used.

- Parentheses omitted if there is no ambiguity (e.g. $p \wedge q$)
- $\neg$ binds more tightly than $\wedge$ and $\vee$, which bind more tightly than $\rightarrow$ and $\leftrightarrow$ (e.g. $p \wedge q \rightarrow r$ instead of $((p \wedge q) \rightarrow r)$

Other conventions (rarely used/assumed in this course):

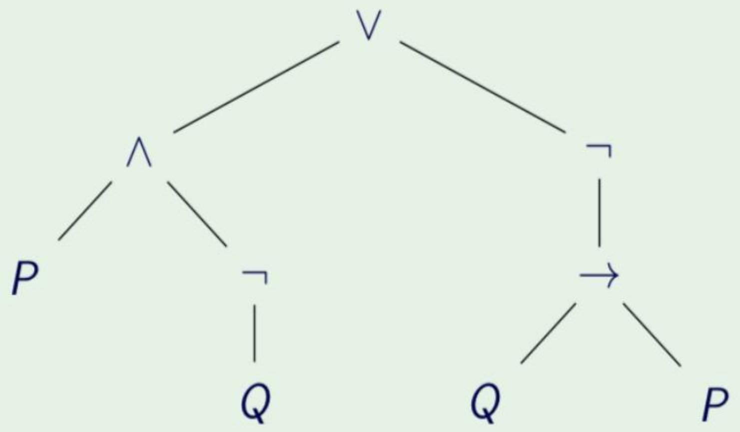
- $'$ or $\bar{\cdot}$ for $\neg$
- $+$ for $\vee$
- $\cdot$ or juxtaposition for $\wedge$
- $\wedge$ binds more tightly than $\vee$
- $\wedge$ and $\vee$ associate to the left: $p \vee q \vee r$ instead of $((p \vee q) \vee r)$
- $\rightarrow$ and $\leftrightarrow$ associate to the right: $p \rightarrow q \rightarrow r$ instead of $(p \rightarrow (q \rightarrow r))$

syntax_parse tree

The structure of well-formed formulas (and other grammar-defined syntaxes) can be shown with a **parse tree**.

Example

$$((P \wedge \neg Q) \vee \neg(Q \rightarrow P))$$



Formally, we can define a parse tree as follows:

A parse tree is either:

- (B) A node containing $\top$;
- (B) A node containing $\perp$;
- (B) A node containing a propositional variable;
- (R) A node containing $\neg$ with a single parse tree child;
- (R) A node containing $\wedge$ with two parse tree children;
- (R) A node containing $\vee$ with two parse tree children;
- (R) A node containing $\rightarrow$ with two parse tree children; or
- (R) A node containing $\leftrightarrow$ with two parse tree children.

syntax_ CNF DNF

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Definition

- A **literal** is an expression p or $\neg p$, where p is a propositional atom.
- A propositional formula is in CNF (conjunctive normal form) if it has the form
$$\bigwedge_i C_i$$
where each **clause** C_i is a disjunction of literals e.g. $p \vee q \vee \neg r$.
- A propositional formula is in DNF (disjunctive normal form) if it has the form
$$\bigvee_i C_i$$
where each clause C_i is a conjunction of literals e.g. $p \wedge q \wedge \neg r$.

NB

CNF and DNF are syntactic forms.

Theorem

For every Boolean expression φ, there exists an equivalent expression in conjunctive normal form and an equivalent expression in disjunctive normal form.

logic II

sematics

- Satisfiability
- Logical Equivalence
- Entailment&Vadility

semantics_truth valuation

A *truth assignment* is a function $v : Prop \to \mathbb{B}$.

We can extend a truth valuation, v, to all wffs of propositional logic as follows:

- $v(\top) = \texttt{true}$,
- $v(\bot) = \texttt{false}$,
- $v(\neg\varphi) = !v(\varphi)$,
- $v(\varphi \wedge \psi) = v(\varphi)\ \&\&\ v(\psi)$
- $v(\varphi \vee \psi) = v(\varphi) \parallel v(\psi)$
- $v(\varphi \to \psi) = v(\varphi) \rightsquigarrow v(\psi)$
- $v(\varphi \leftrightarrow \psi) = v(\varphi) \leftrightsquigarrow v(\psi)$

Recall the two-element Boolean Algebra $\mathbb{B} = \{\text{true}, \text{false}\} = \{T, F\} = \{1, 0\}$ together with the operations $!$, $\&\&$, $\parallel$.

Define $\rightsquigarrow$, $\leftrightsquigarrow$ as derived operations:

- $x \rightsquigarrow y = (!x) \parallel y$
- $x \leftrightsquigarrow y = (x \rightsquigarrow y)\ \&\&\ (y \rightsquigarrow x)$

semantics_truth table

Symbol	Default	Also known as
$\wedge$	and	but, ";"
$\vee$	or	"either .. or .."
$\neg$	not	not the case
$\rightarrow$	"if .. then .."	implies whenever is sufficient for
$\leftrightarrow$	".. if and only if .."	bi-implies necessary and sufficient exactly when just in case

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$A \rightarrow B$	$A \leftrightarrow B$
True	True	True	True	False	True	True
False	True	False	True	True	True	False
True	False	False	True	False	False	False
False	False	False	False	True	True	True

Exercises

LLM: Problem 3.2

$$
\begin{aligned}
p &= \text{“you get an HD on your final exam”} \\
q &= \text{“you do every exercise in the book”} \\
r &= \text{“you get an HD in the course”}
\end{aligned}
$$

Translate into logical notation:

(a) You get an HD in the course although you do not do every exercise in the book.

(c) To get an HD in the course, you must get an HD on the exam.

(d) You get an HD on your exam, but you don't do every exercise in this book; nevertheless, you get an HD in this course.

semantics_truth

Exercises

Evaluate the following formulae with the truth assignment $v(p) = v(q) = \mathtt{false}$

- $p \to q$
- $(p \to q) \to (p \to q)$
- $\neg\neg p$
- $\top \wedge \neg\bot \to p$

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$A \to B$	$A \leftrightarrow B$
True	True	True	True	False	True	True
False	True	False	True	True	True	False
True	False	False	True	False	False	False
False	False	False	False	True	True	True

semantics_satisfiability

A formula φ is

- **satisfiable** if $v(\varphi) = \texttt{true}$ for some truth assignment v (v **satisfies** φ)
- a **tautology** if $v(\varphi) = \texttt{true}$ for all truth assignments v
- **unsatisfiable** or a **contradiction** if $v(\varphi) = \texttt{false}$ for all truth assignments v

Example

- Contingency: It is raining
- Tautology: It is raining or it is not raining
- Contradiction: It is raining and it is not raining

semantics_satisfiability

Which of the following formulae are *always* true?

(a) $(p \wedge (p \rightarrow q)) \rightarrow q$?

(b) $((p \vee q) \wedge \neg p) \rightarrow \neg q$?

(e) $((p \rightarrow q) \vee (q \rightarrow r)) \rightarrow (p \rightarrow r)$?

(f) $(p \wedge q) \rightarrow q$?

semantics_logic equivalence

Definition

Two formulas, φ and ψ, are **logically equivalent**, $\varphi \equiv \psi$, if $v(\varphi) = v(\psi)$ for all truth assignments v.

Fact

$\equiv$ *is an equivalence relation.*

表 1-12 $\neg(p \vee q)$和$\neg p \wedge \neg q$的真值表

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

semantics_logic equivalence

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Example

For all propositions P, Q, R:

$$
\begin{array}{lrcl}
\text{Commutativity:} & P \vee Q & \equiv & Q \vee P \\
 & P \wedge Q & \equiv & Q \wedge P \\
\text{Associativity:} & (P \vee Q) \vee R & \equiv & P \vee (Q \vee R) \\
 & (P \wedge Q) \wedge R & \equiv & P \wedge (Q \wedge R) \\
\text{Distributivity:} & P \vee (Q \wedge R) & \equiv & (P \vee Q) \wedge (P \vee R) \\
 & P \wedge (Q \vee R) & \equiv & (P \wedge Q) \vee (P \wedge R) \\
\text{Identity:} & P \vee \bot & \equiv & P \\
 & P \wedge \top & \equiv & P \\
\text{Complement:} & P \vee \neg P & \equiv & \top \\
 & P \wedge \neg P & \equiv & \bot
\end{array}
$$

- Implication: $p \to q \equiv \neg p \vee q$
- Double negation: $\neg\neg p \equiv p$
- Contrapositive: $(p \to q) \equiv (\neg q \to \neg p)$
- De Morgan's: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Fact

$\varphi \equiv \psi$ *if, and only if,* $(\varphi \leftrightarrow \psi)$ *is a tautology.*

Strategies for showing logical equivalence:

- Compare all rows of truth table.
- Show $(\varphi \leftrightarrow \psi)$ is a tautology.
- Use transitivity of $\equiv$.

Examples

$\boxed{2.2.18}$ Prove or disprove:

(a) $p \to (q \to r) \equiv (p \to q) \to (p \to r)$

(c) $(p \to q) \to r \equiv p \to (q \to r)$

semantics_logic equivalence

Examples

(a) $(p \to q) \to (p \to r)$

$\equiv \neg(p \to q) \vee (p \to r)$	[Implication]
$\equiv \neg(\neg p \vee q) \vee (\neg p \vee r)$	[Implication]
$\equiv (\neg\neg p \wedge \neg q) \vee (\neg p \vee r)$	[De Morgan's]
$\equiv (p \vee (\neg p \vee r)) \wedge (\neg q \vee (\neg p \vee r))$	[Distributivity]
$\equiv ((p \vee \neg p) \vee r) \wedge ((\neg q \vee \neg p) \vee r)$	[Associativity]
$\equiv \top \wedge ((\neg q \vee \neg p) \vee r))$	[Complement]
$\equiv (\neg q \vee \neg p) \vee r$	[Identity]
$\equiv (\neg p \vee \neg q) \vee r$	[Commutativity]
$\equiv \neg p \vee (\neg q \vee r)$	[Associativity]
$\equiv p \to (q \to r)$	[Implication]

(c) $(p \to q) \to r \not\equiv p \to (q \to r)$

Counterexample:

p	q	r	$(p \to q) \to r$	$p \to (q \to r)$
F	T	F	F	T

semantics_Entailment & Validity

An *argument* consists of a set of propositions called *premises* and a declarative sentence called the *conclusion*.

Example

Premises:	Frank took the Ford or the Toyota.
	If Frank took the Ford he will be late.
	Frank is not late.
Conclusion:	Frank took the Toyota

An argument is *valid* if the conclusions are true *whenever* all the premises are true. Thus: if we believe the premises, we should also believe the conclusion.
(Note: we don't care what happens when one of the premises is false.)
Other ways of saying the same thing:

- The conclusion *logically follows* from the premises.
- The conclusion is a *logical consequence* of the premises.
- The premises **entail** the conclusion.

semantics_Entailment & Validity

A set of formulas is a **theory**

A truth assignment v *satisfies* a theory T if $v(\varphi) = \texttt{true}$ for all $\varphi \in T$

A theory T **entails** a formula φ, $T \models \varphi$, if $v(\varphi) = \texttt{true}$ for all truth assignments v which satisfy T

NB

Other notation (when $T = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$*)*

- $\varphi_1, \varphi_2, \ldots, \varphi_n \models \varphi$
- $\varphi_1, \varphi_2, \ldots, \varphi_n, \quad \therefore \varphi$
- $\varphi_1, \varphi_2, \ldots, \varphi_n \Longrightarrow \varphi$

semantics_Entailment & Validity

Example

We mark only true locations (blank = F)

Frd	$Tyta$	$Late$	$Frd \vee Tyta$	$Frd \rightarrow Late$	$\neg Late$	$Tyta$
F	F	F		T	T	
F	F	T		T		
F	T	F	T	T	T	T
F	T	T	T	T		T
T	F	F	T		T	
T	F	T	T	T		
T	T	F	T		T	T
T	T	T	T	T		T

This shows $Frd \vee Tyta,\ Frd \rightarrow Late,\ \neg Late \models Tyta$

semantics_Entailment & Validity

Theorem

The following are equivalent:

- $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$
- $\emptyset \models ((\varphi_1 \land \varphi_2) \land \ldots \varphi_n) \to \psi$
- $((\varphi_1 \land \varphi_2) \land \ldots \varphi_n) \to \psi$ *is a tautology*
- $\emptyset \models \varphi_1 \to (\varphi_2 \to (\ldots \to \varphi_n) \to \psi))\ldots)$
- $\varphi_1 \models \varphi_2 \to (\ldots \to \varphi_n) \to \psi))\ldots)$

Strategies for showing $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$:

- Draw a truth table with columns for $\varphi_1, \ldots, \varphi_n$ and φ. Check φ is true in rows where **all** the φ_i are true.
- Show $((\varphi_1 \land \varphi_2) \land \ldots \varphi_n) \to \psi$ is a tautology.
- Show $\varphi_1 \to (\varphi_2 \to (\ldots \to \varphi_n) \to \psi))\ldots)$ is a tautology.
- Show $\varphi_1 \models \varphi_2 \to (\ldots \to \varphi_n) \to \psi))\ldots)$
- Syntactic techniques: Natural deduction, Resolution, etc (not covered here)

semantics_Entailment & Validity

Example

You are planning a party, but your friends are a bit touchy about who will be there.

1. If John comes, he will get very hostile if Sarah is there.
2. Sarah will only come if Kim will be there also.
3. Kim says she will not come unless John does.

Who can you invite without making someone unhappy?

semantics_Entailment & Validity

Translation to logic: let J, S, K represent "John (Sarah, Kim) comes to the party". Then the constraints are:

1. $J \to \neg S$
2. $S \to K$
3. $K \to J$

Thus, for a successful party to be possible, we want the formula $\phi = (J \to \neg S) \land (S \to K) \land (K \to J)$ to be satisfiable.
Truth values for J, S, K making this true are called *satisfying assignments*, or *models*.

semantics_Entailment & Validity

We figure out where the conjuncts are false, below. (so blank = T)

J	K	S	$J \rightarrow \neg S$	$S \rightarrow K$	$K \rightarrow J$	ϕ
F	F	F				
F	F	T		F		F
F	T	F			F	F
F	T	T			F	F
T	F	F				
T	F	T	F	F		F
T	T	F				
T	T	T	F			F

Conclusion: a party satisfying the constraints can be held. Invite nobody, or invite John only, or invite Kim and John.

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Thanks !