Fundamentals (Numbers, Sets, Words, Functions, and Relations)

Problem 1
How many numbers are there between 100 and 1000 that are

(a) divisible by 3?

(b) divisible by 5?

(c) divisible by 15?

Solution

Using the formula $\left\lfloor \frac{m}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor$:

- $\left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{99}{3} \right\rfloor = 300$ numbers divisible by 3 $(102, 105, \ldots, 999)$;
- $\left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{99}{5} \right\rfloor = 181$ numbers divisible by 5 $(100, 105, \ldots, 1000)$;
- $\left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{99}{15} \right\rfloor = 60$ numbers divisible by 15 $(105, 120, \ldots, 990)$.

Problem 2
Let $\Sigma = \{a, b, c\}$ and $\Phi = \{a, c, e\}$.

(a) How many words are in the set Σ^2?

(b) What are the elements of $\Sigma^2 \setminus \Phi^*$?

(c) Is it true that $\Sigma^* \setminus \Phi^* = (\Sigma \setminus \Phi)^*$? Why?

Solution

(a) $\Sigma^2 = \{aa, ab, ac, ba, \ldots, cc\}$, hence $|\Sigma^2| = 3 \cdot 3 = 9$.
(b) $\Sigma^2 \setminus \Phi^* = \{ab, ba, bb, bc, cb\}$, that is, all words in Σ^2 with the letter b.
(c) No; for example, $ab \in \Sigma^*$ and $ab \notin \Phi^*$, hence $ab \in \Sigma^* \setminus \Phi^*$; but $\Sigma \setminus \Phi = \{b\}$, hence $ab \notin (\Sigma \setminus \Phi)^*$.

Problem 3
Prove that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

Solution

Using the laws of set operations (and the derived DeMorgan's law) we have:

$$\begin{aligned}
(A \setminus B) \cup (B \setminus A) &= (A \cap B^c) \cup (B \cap A^c) && \text{(Definition)}\\
&= ((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup A^c) && \text{(Distrib.)}\\
&= (B \cup (A \cap B^c)) \cap (A^c \cup (A \cap B^c)) && \text{(Commut.)}\\
&= ((B \cup A) \cap (B \cup B^c)) \cap ((A^c \cup A) \cap (A^c \cup B^c)) && \text{(Distrib.)}\\
&= ((A \cup B) \cap (B \cup B^c)) \cap ((A \cup A^c) \cap (A^c \cup B^c)) && \text{(Comm.)}\\
&= ((A \cup B) \cap \mathcal{U}) \cap (\mathcal{U} \cap (A^c \cup B^c)) && \text{(Complement)}\\
&= ((A \cup B) \cap \mathcal{U}) \cap ((A^c \cup B^c) \cap \mathcal{U}) && \text{(Comm.)}\\
&= (A \cup B) \cap (A^c \cup B^c) && \text{(Identity)}\\
&= (A \cup B) \cap (A \cap B)^c && \text{(De Morgan's)}\\
&= (A \cup B) \setminus (A \cap B) && \text{(Definition)}
\end{aligned}$$

Solution

Alternative proof: We show in both directions that if an element belongs to $(A \setminus B) \cup (B \setminus A)$ then it also belongs to $(A \cup B) \setminus (A \cap B)$ and vice versa:

- Suppose that an element $x \in (A \setminus B) \cup (B \setminus A)$. Therefore, either $x \in A \setminus B$ or $x \in B \setminus A$. In either case, we conclude that $x \in A \cup B$ and (by the definition of set difference) $x \notin A \cap B$. Therefore, $x \in (A \cup B) \setminus (A \cap B)$.
- Suppose than $x \in (A \cup B) \setminus (A \cap B)$. This means that $x \in A \cup B$ (and, therefore, either $x \in A$ or $x \in B$), but $x \notin A \cap B$. If $x \in A$ and $x \notin A \cap B$, then $x \in A \setminus B$; alternatively, if $x \in B$ and $x \notin A \cap B$, then $x \in B \setminus A$. In either case, we conclude that $x \in (A \setminus B) \cup (B \setminus A)$.

Problem 4

Consider the relation $R \subseteq \mathbb{R} \times \mathbb{R}$ defined by aRb if, and only if, $b + 0.5 \geq a \geq b - 0.5$. Is R

(a) reflexive?

(b) antireflexive?

(c) symmetric?

(d) antisymmetric?

(e) transitive?

Solution

(a) Yes, since $a + 0.5 \geq a \geq a - 0.5$ for all $a \in \mathbb{R}$

(b) No; see (a)

(c) Yes, since $(b + 0.5 \geq a) \wedge (a \geq b - 0.5)$ implies $(b \geq a - 0.5) \wedge (a + 0.5 \geq b)$.

(d) No; e.g. $(0, 0.1) \in R$ and $(0.1, 0) \in R$.

(e) No; e.g. $(1.1, 1.5) \in R$ and $(1.5, 1.9) \in R$ but $(1.1, 1.9) \notin R$ since $1.9 - 0.5 > 1.1$

Problem 5
For each of the following statements, provide a valid proof if it is true for all sets S and all relations $R_1 \subseteq S \times S$ and $R_2 \subseteq S \times S$. If the statement is not always true, provide a counterexample.

(a) If R_1 and R_2 are symmetric, then $R_1 \cap R_2$ is symmetric.

(b) If R_1 and R_2 are antisymmetric, then $R_1 \cup R_2$ is antisymmetric.

Solution

(a) We will show that if R_1 and R_2 are symmetric, then $R_1 \cap R_2$ is symmetric. Suppose $(a, b) \in R_1 \cap R_2$. Then $(a, b) \in R_1$ and $(a, b) \in R_2$. Because R_1 is symmetric, $(b, a) \in R_1$. Because R_2 is symmetric, $(b, a) \in R_2$. Therefore $(b, a) \in R_1 \cap R_2$. Therefore $R_1 \cap R_2$ is symmetric.

(b) This is not the case. Consider relations on $\mathbb{N}$: $R_1 = \leq$ and $R_2 = \geq$. We have $1 \leq 2$, so $(1, 2) \in R_1$ and $(2, 1) \in R_2$. Therefore, $(1, 2), (2, 1) \in R_1 \cup R_2$ but $1 \neq 2$.