$\lfloor . \rfloor : R \longrightarrow Z$ — floor of x, the greatest integer ≤ x

$\lceil . \rceil : R \longrightarrow Z$ — ceiling of x, the least integer ≥ x

$\gcd(m,n) \cdot \mathrm{lcm}(m,n) = |m| \cdot |n|$

$\gcd(0, n) = |n|$

For $m, n \in \mathbb{Z}$, if $m > n$ then $\gcd(m,n) = \gcd(m-n, n)$

Let $k, m, n \in \mathbb{Z}$ such that $k > 0$ and $m \geq n$. The number of multiples of k in the interval $[n, m]$ is

$$\left\lfloor \frac{m}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor$$

Power set $\mathrm{Pow}(X) = \{ A : A \subseteq X \}$

$|\mathrm{Pow}(X)| = 2^{|X|}$

$|\varnothing| = 0$ $\mathrm{Pow}(\varnothing) = \{\varnothing\}$ $|\mathrm{Pow}(\varnothing)| = 1$ $\mathrm{Pow}(\mathrm{Pow}(\varnothing)) = \{\varnothing, \{\varnothing\}\}$ $|\mathrm{Pow}(\mathrm{Pow}(\varnothing))| = 2$

$|\{a\}|=1$ $\mathrm{Pow}(\{a\})=\{\varnothing,\{a\}\}$ $|\mathrm{Pow}(\{a\})|=2$

$A \oplus B = (A \setminus B) \cup (B \setminus A)$

Notation: Σ^k — set of all words of length k
We often identify $\Sigma^0 = \{\lambda\}$, $\Sigma^1 = \Sigma$
Σ^* — set of all words (of all lengths)
Σ^+ — set of all nonempty words (of any positive length)

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots; \quad \Sigma^{\leq n} = \bigcup_{i=0}^{n} \Sigma^i$$

$$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \ldots = \Sigma^* \setminus \{\lambda\}$$

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup A = A$$
$$A \cap A = A$$
$$A \cup \emptyset = A$$
$$A \cap \emptyset = \emptyset$$
$$(A^c)^c = A$$
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

$g \circ f : x \mapsto g(f(x))$, requiring $\mathrm{Im}(f) \subseteq \mathrm{Dom}($

every function maps its domain into its codomain, but only onto its image.

Σ^k — set of all words of length k

A unless B ➡ $\neg B \Rightarrow A$

$S \times T = \{ (s, t) : s \in S, t \in T \}$ where (s, t) is an ordered pair

S — domain of f , symbol: Dom(f)

T — codomain of f , symbol: Codom(f)

$\{f(x):x \in \mathrm{Dom}(f)\}$—image of f : $\mathrm{Im}(f) \subseteq \mathrm{Codom}(f)$

Pow(S) — subsets of S

join: $A \cup B$, *meet:* $A \cap B$, *complement:* $A^c = S \setminus A$

$$xy = xy \cdot 1 = xy \cdot (z + \overline{z}) = xyz + xy\overline{z}$$
$$\overline{z} = xy\overline{z} + x\overline{yz} + \overline{x}y\overline{z} + \overline{xyz}$$
$$xy + \overline{z} = \text{combine the 6 product terms above}$$
$$1 = \text{sum of all 8 possible product terms: } xyz + \overline{x}yz + \ldots + \overline{xyz}$$

Exercise 8 Let S be a finite set and let $n \in \mathbb{N}$. How many

1. functions,
2. onto functions,
3. binary relations, and
4. n-ary relations

are there on S? Explain your answers briefly.

1. $|S|^{|S|}$ — for every element a free choice between all eleme
2. $|S|!$ — onto functions on finite sets are 1–1 (permutation
3. $2^{(|S|^2)}$ — size of the powerset of the set of pairs
4. $2^{(|S|^n)}$ — size of the powerset of the set of n-tuples

1–1 (one-to-one) or injective : $f(x)=f(y) \Rightarrow x = y$

onto (or surjective): if every element of the codomain is mapped to by at least on x in the domain, i.e. $\mathrm{Im}(f) = T$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

A matrix M is called symmetric if $M^T = M$

Row : hang Column : lie rotating :xuanzhuan

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

(R) reflexive $(x,x) \in \mathcal{R}$ for all $x \in S$ $\begin{bmatrix} \bullet & \bullet & \circ \\ \circ & \bullet & \circ \\ \bullet & \circ & \bullet \end{bmatrix}$

(AR) antireflexive $(x,x) \notin \mathcal{R}$ $\begin{bmatrix} \circ & \bullet & \bullet \\ \circ & \circ & \circ \\ \bullet & \circ & \circ \end{bmatrix}$

(S) symmetric $(x,y) \in \mathcal{R} \Rightarrow (y,x) \in \mathcal{R}$ $\begin{bmatrix} \bullet & \circ & \bullet \\ \circ & \circ & \bullet \\ \bullet & \bullet & \circ \end{bmatrix}$

(AS) antisymmetric $(x,y), (y,x) \in \mathcal{R} \Rightarrow x = y$
$\begin{bmatrix} \bullet & \bullet & \circ \\ \circ & \circ & \bullet \\ \bullet & \circ & \circ \end{bmatrix}$

(T) transitive $(x,y), (y,z) \in \mathcal{R} \Rightarrow (x,z) \in \mathcal{R}$
$\begin{bmatrix} \circ & \circ & \bullet \\ \bullet & \bullet & \bullet \\ \circ & \circ & \circ \end{bmatrix}$

Most important kinds of relations on S

- total order $\begin{bmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet \\ \circ & \circ & \bullet \end{bmatrix}$
- partial order $\begin{bmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$, $\begin{bmatrix} \bullet & \bullet & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$
- equivalence $\begin{bmatrix} \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$
- identity $\begin{bmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{bmatrix}$

$\mathcal{R}(A) \stackrel{\text{def}}{=} \{t \in T | (s,t) \in \mathcal{R} \text{ for some } s \in A \subseteq S\}$

$\mathcal{R}^{\leftarrow}(B) \stackrel{\text{def}}{=} \{s \in S | (s,t) \in \mathcal{R} \text{ for some } t \in B \subseteq T$

$f^{\leftarrow}$ is a relation; when is it a function?

When f is 1-1 and onto.

Converse relation $\mathcal{R}^{\leftarrow}$

$$\mathcal{R}^{\leftarrow} = \{(t,s) \in T \times S | (s,t) \in \mathcal{R}\}$$

Note that $\mathcal{R}^{\leftarrow} \subseteq T \times S$.
Observe that $(\mathcal{R}^{\leftarrow})^{\leftarrow} = \mathcal{R}$.