

**University of New South Wales**  
**School of Computer Science and Engineering**  
**Foundations of Computer Science (COMP9020)**  
**FINAL EXAM — Term 3, 2020**

Instructions:

- The exam consists of 10 questions for a total of **200 marks**
- Solutions should be submitted as a pdf, maximum size 20Mb.
- Solutions should be submitted via give/webCMS. Multiple submissions are allowed, your last submission before the due date will be the submission marked.
- Deadline for submissions is **10am, Tuesday 8 December 2020 (AEDST)**.
- Late submissions will be penalized at 25% (applied to raw mark) per 15 minutes or part thereof.
- Existing written materials (i.e. passive resources) may be used, but should be properly referenced. Obtaining, or trying to obtain solutions from active resources (e.g. social media, forums, other people) is a breach of the University's Plagiarism Policy and will be subject to disciplinary action as outlined in the Student Code of Conduct.
- Unless explicitly stated, you should prove/explain/motivate your answers. You are being assessed on your understanding and ability.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for your worst answer, as this indicates how well you understood the question.
- Number of pages in this exam paper: 8, including this cover page.

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**Question 1**

(20 marks)

- (a) Using only the functions  $\lfloor \cdot \rfloor$ ,  $\lceil \cdot \rceil$ ,  $\text{div}$ ,  $\%$ , basic arithmetic functions ( $+$ ,  $\times$ ,  $-$ , and  $\div$ ), and numeric constants, define the following functions:

(i)  $f : \mathbb{R} \setminus \{0\} \rightarrow \{-1, 1\}$  given by  $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$  (2 marks)

(ii)  $g : \mathbb{R} \rightarrow \{0, 1\}$  given by  $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \notin \mathbb{Z} \end{cases}$  (2 marks)

(iii)  $h : \mathbb{N} \rightarrow \{0, 1\}$  given by  $h(x) = \begin{cases} 1 & \text{if } 5|x \\ 0 & \text{if } 5 \nmid x \end{cases}$  (2 marks)

(iv)  $\min : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  given by  $\min(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases}$  (2 marks)

(v)  $q : \mathbb{R} \rightarrow \{0, 1\}$  given by  $q(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \text{ and } 5|x \\ 0 & \text{otherwise} \end{cases}$  (2 marks)

- (b) The following process leads to a rule for determining if a large number  $n$  is divisible by 17:

- Remove the last digit,  $b$ , of  $n$  leaving a smaller number  $a$ .
- Let  $n' = a - 5b$ .
- Repeat with  $n'$  in place of  $n$ .

So, for example, if  $n = 12345$ , then  $n' = 1234 - 5 \cdot 5 = 1209$ . Repeating would create  $120 - 5 \cdot 9 = 75$ ;  $7 - 5 \cdot 5 = -18$ ; and so on.

- (i) Prove that  $17|n$  if and only if  $17|n'$ . (6 marks)

This shows that we can repeat the process with  $n'$ , and so on, until we reach a “small enough” value that checking divisibility by 17 is straightforward.

- (ii) Give, with justification, an asymptotic upper bound in terms of  $n$ , for the number of times this process needs to be repeated before a negative  $n'$  is reached. (4 marks)

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**Question 2**

(20 marks)

- (a) Prove, or provide a counterexample to disprove, the following for all sets  $A, B$ :
- (i) If there is a set  $X$  such that  $A \cap X = B \cap X$  then  $A = B$ . (4 marks)
  - (ii) If there is a set  $X$  such that  $A \oplus X = B \oplus X$  then  $A = B$ . (4 marks)
  - (iii) If there is a set  $X$  such that  $A \cap X = B \cap X$  and  $A \cup X = B \cup X$  then  $A = B$ . (4 marks)
- (b) Let  $\Sigma$  be a finite set. Prove, or provide a counterexample to disprove, the following for all languages  $X, Y, Z \subseteq \Sigma^*$ :
- (i) If  $XY = YZ$  then  $X^*Y = YZ^*$ . (4 marks)
  - (ii) If  $X^*Y = YZ^*$  then  $XY = YZ$ . (4 marks)

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**Question 3**

(20 marks)

Let  $(A, \preceq)$  be a poset. Define  $f : A \rightarrow \text{Pow}(A)$  as follows:

$$f(a) = \{x : x \leq a\}.$$

- (a) Prove that  $a \preceq b$  if, and only if,  $f(a) \subseteq f(b)$ . (6 marks)
- (b) Prove that  $f$  is injective. (4 marks)
- (c) Prove that  $f$  is not surjective. (4 marks)
- (d) Prove that if  $c = \text{glb}(a, b)$  exists, then  $f(c) = f(a) \cap f(b)$ . (6 marks)

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**Question 4**

(20 marks)

- (a) Prove that for all sets  $A, B, C$  there is a bijection between  $(A \times B) \times C$  and  $A \times (B \times C)$ . (4 marks)
- (b) Prove that  $f : S \rightarrow T$  is injective if, and only if, for all functions  $g, h : U \rightarrow S$ :

$$f \circ g = f \circ h \quad \text{implies} \quad g = h$$

(8 marks)

(c) Order the following functions in increasing order of asymptotic complexity:

- (I)  $n^3\sqrt{n}$
- (II)  $n^3 \log(n)$
- (III)  $T(n)$  where  $T(0) = 1$  and  $T(n) = T(n-3) + n^3$
- (IV)  $T(n)$  where  $T(0) = 1$  and  $T(n) = 6T(n/2) + n^3$
- (V)  $(3^{\log n})^2$
- (VI)  $3^{(\log n)^2}$

Justify your ordering.

(8 marks)

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### Question 5

(20 marks)

Consider the following code snippet, which takes an acyclic directed graph (which does not contain loops) and returns a topological ordering of the vertices (i.e. vertex order respects edge direction):

```
topSort(G):
  If G has no vertices :
    return an empty list
  endif
  Let  $v$  be any vertex of  $G$ 
  while  $v$  has in-degree  $> 0$ :
    Let  $v'$  be a vertex such that  $(v', v) \in E$ 
    Let  $v = v'$ 
  end while
  Let  $G'$  be the graph that results from removing  $v$  and all incident edges
  Let  $L = \text{topSort}(G')$ 
  prepend  $L$  with  $v$ 
  return  $L$ 
```

For this question you may assume that removing a vertex and all incident edges; or prepending a vertex to a list, can be executed in  $O(1)$  time. Assume that accessing a single matrix entry can be executed in  $O(1)$  time, but searching a row (column) of an  $n \times m$  matrix will take  $O(m)$  time ( $O(n)$  time, respectively).

Calculate the asymptotic running time of the above code snippet in terms of  $n$ : the number of vertices of  $G$ , and  $m$ : the number of edges of  $G$ :

- (a) When  $G$  is given as an  $n \times n$  adjacency matrix. (6 marks)
- (b) When  $G$  is given as an adjacency list. (6 marks)
- (c) When  $G$  is given as an  $n \times m$  incidence matrix. (8 marks)

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**Question 6**

(20 marks)

Recall from Assignment 2 the recursive definition of a binary tree:

- (B): An empty tree  $\tau$
- (R): An ordered pair of binary trees  $(T_{\text{left}}, T_{\text{right}})$

The *height* of a binary tree is defined to be the length of the longest path from the root node (node with no predecessors) to a leaf. For convenience, we define the height of  $\tau$  to be  $-1$ . The example tree in Assignment 2 has height 2.

- (a) Based on the recursive definition above, recursively define a function  $\text{height}(T)$  that computes the height of a binary tree  $T$ . (4 marks)

A binary tree is *balanced* if it is the empty tree, or if the heights of its two subtrees  $T_{\text{left}}$  and  $T_{\text{right}}$  differ by at most 1. A binary tree is *fully-balanced* if every subtree is balanced. The example tree in Assignment 2 is fully-balanced.

- (b) Based on part (a) and the recursive definition above, recursively define a function  $\text{balanced}(T)$  that returns an element of  $\mathbb{B}$  indicating whether or not the binary tree  $T$  is fully-balanced. (4 marks)

Recall from Assignment 2 the count function that counts the number of nodes in a binary tree:

- $\text{count}(\tau) = 0$
- $\text{count}((T_{\text{left}}, T_{\text{right}})) = 1 + \text{count}(T_{\text{left}}) + \text{count}(T_{\text{right}})$

- (c) Let  $P(T)$  be the proposition that:

$$\text{count}(T) \leq 2^{\text{height}(T)+1} - 1.$$

Prove that  $P(T)$  holds for all binary trees  $T$ . (6 marks)

- (d) Let  $Q(T)$  be the proposition that:

$$\text{If } \text{balanced}(T) \text{ then } \text{count}(T) \geq 2^{\frac{\text{height}(T)}{2}} - 1.$$

Prove that  $Q(T)$  holds for all binary trees  $T$ . (6 marks)

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**Question 7**

(20 marks)

- (a) You are on a spaceship filled with crewmates and imposters. Imposters will always lie, and crewmates will always tell the truth. The following statements are made:
- Red says: Either myself and Blue are both crew, or we are both imposters.
  - Orange says: At least one of Red and Green is an imposter.
  - Green says: Red is an imposter.
  - Blue says: Red is an imposter.
- (i) Formulate this as a problem in propositional logic, defining the propositional variables and what they represent; any formulas that are relevant; and the logical problem you are solving. (6 marks)
- (ii) Use your answer from (i) to determine what possible combinations of crewmate/imposters there are that are consistent with this system. (6 marks)
- (b) Prove, or provide a counterexample to disprove the following hold in all Boolean Algebras:
- (i) There is no  $x$  such that  $x = x'$ . (4 marks)
- (ii) For all  $x, y, z$ :  $((x \wedge y) \vee (x' \wedge z)) \vee (y \wedge z) = (x \wedge y) \vee (x' \wedge z)$ . (4 marks)

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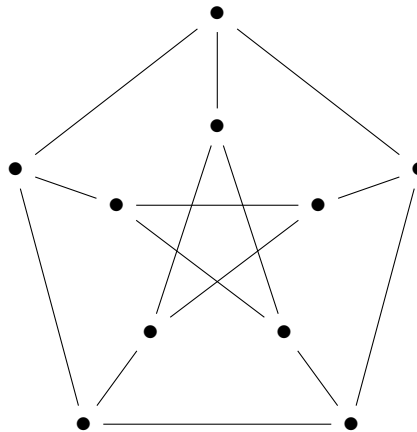
**Question 8**

(20 marks)

Prove or disprove:

- (a) The degree sequence of a graph provides enough information to determine if it has an Eulerian cycle. (4 marks)
- (b) If a graph  $G$  has an even number of vertices and an Eulerian cycle, then it has an even number of edges. (4 marks)
- (c) If a graph  $G$  has an even number of vertices and an Eulerian cycle, then it has chromatic number at most 2. (4 marks)

(d) The following graph:



contains a subdivision of  $K_4$ .

(4 marks)

(e)  $K_{2,16}$  contains a subdivision of  $K_{4,4}$ .

(4 marks)

### Question 9

(20 marks)

For  $n \in \mathbb{N}$  with  $n > 0$  let  $A_n = \{1, 2, \dots, n\}$ . Let  $E(n)$  denote the number of equivalence relations there are on  $A_n$ , and let  $F(n, k)$  denote the number of surjective functions there are from  $A_n$  to  $A_k$ .

- With justification, give a simple formula (using combinatorial functions defined in lectures) for  $F(n, n)$ . (4 marks)
- With justification, give a simple formula (using combinatorial functions defined in lectures) for  $F(n, 2)$ . (4 marks)
- With justification, give a recurrence equation for  $E(n)$  involving (one or more)  $E(n')$  where  $n' < n$ . Remember to include a base case. (4 marks)  
*Hint: How many equivalence relations are there on  $A_n$  where the equivalence class of  $n$  has exactly  $i$  elements?*
- With justification, give a recurrence equation for  $F(n, k)$  involving (one or more)  $F(n', k')$  where  $n' < n$  and  $k' \leq k$ . Remember to include a base case. (4 marks)  
*Hint: Each  $f \in A_k^{A_n}$  can be viewed as  $f = g \cup \{(n, y)\}$  where  $y \in A_k$  and  $g$  is a function with domain  $A_{n-1}$ .*
- Let  $E(n, k)$  denote the number of equivalence relations on  $A_n$  that have exactly  $k$  equivalence classes. With justification, define  $E(n, k)$  in terms of  $F(n, k)$ . (4 marks)

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**Question 10**

(20 marks)

Prove, or provide a counterexample to disprove the following for all discrete probability distributions  $P$ , all events  $A$  and  $B$ , and all random variables  $X$  and  $Y$ :

(a)  $P(A \setminus B) \geq P(A) - P(B)$  (6 marks)

(b) If  $0 < P(A) \leq P(B)$  then  $P(A|B) \leq P(B|A)$  (6 marks)

(c)  $P(X > E(X)) \geq \frac{1}{2}$  (4 marks)

(d) If  $E(X) > E(Y)$  then  $P(X > Y) > 0$  (4 marks)