COMP9020 Week 9
Term 3, 2020
Combinatorics

- [LLM] - Ch. 14
- [RW] - Ch. 5, 7
- [Rosen] - Ch. 6, 8

Counting Techniques

General idea: find methods, algorithms or precise formulae to count the number of elements in various sets or collections derived, in a structured way, from some basic sets.

Examples

Single base set $S = \{s_1, \ldots, s_n\}$, $|S| = n$; find the number of

- all subsets of S
- ordered selections of r different elements of S
- unordered selections of r different elements of S
- selections of r elements from S such that ...
- functions $S \longrightarrow S$ (onto, 1-1)
- partitions of S into k equivalence classes
- graphs/trees with elements of S as labelled vertices/leaves

Example

Example

A restaurant has the following menu:

Starter	Main Course	Dessert
Soup	Fish	Ice-cream
Bread	Beef	Fruit
	Pork	Cheese
	Chicken	

How many:

- 3 course meals (Starter-Main-Dessert) are possible?
- 3 course meals (Any item for each course) are possible?
- 3 course meals (Any item, no duplicates) are possible?
- Meals consisting of 3 items (order is unimportant)?

Applications of counting in CS

- Algorithmic analysis
- Data management
- Enumeration techniques
- Probability calculations

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Outline

- Basic counting rules
- Combinations and Permutations
- Alternative techniques
- Difficult counting problems

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Basic Counting Rules: Principles

Two simple rules:

- **Union rule** ("or"): If S and T are disjoint $|S \cup T| = |S| + |T|$
- **Product rule** ("followed by"): $|S \times T| = |S| \cdot |T|$

These cover many examples, though the rule application is not always obvious.

Common strategies:

- Direct application of the rule
- Relate unknown quantities to known quantities (e.g. $|S| + |T| = |S \cup T| + |S \cap T|$)
- Find a bijection to a set that can be counted

Basic Counting Rules (1)

Union rule — S and T *disjoint*

$$|S \cup T| = |S| + |T|$$

$S_1, S_2, \ldots, S_n$ pairwise disjoint ($S_i \cap S_j = \emptyset$ for $i \neq j$)

$$|S_1 \cup \ldots \cup S_n| = \sum |S_i|$$

Example

How many numbers in $A = [1, 2, \ldots, 999]$ are divisible by 31 or 41?

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Example

How many numbers in $A = [1, 2, \ldots, 999]$ are divisible by 31 or 41?

$\lfloor 999/31 \rfloor = 32$ divisible by 31
$\lfloor 999/41 \rfloor = 24$ divisible by 41
No number in A divisible by both
Hence, $32 + 24 = 56$ divisible by 31 or 41

Basic Counting Rules (1)

Union rule: Inferences

For arbitrary sets $S, T, \ldots$

$$
\begin{aligned}
|S \cup T| &= |S| + |T| - |S \cap T| \\
|T \setminus S| &= |T| - |S \cap T| \\
|S_1 \cup S_2 \cup S_3| &= |S_1| + |S_2| + |S_3| \\
&\quad - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| \\
&\quad + |S_1 \cap S_2 \cap S_3|
\end{aligned}
$$

Basic Counting Rules (2)

Product rule

$$|S_1 \times \ldots \times S_k| = |S_1| \cdot |S_2| \cdots |S_k| = \prod_{i=1}^{k} |S_i|$$

If all $S_i = S$ (the same set) and $|S| = m$ then $|S^k| = m^k$

NB

This counts the number of sequences where the first item is from S_1, the second is from S_2, and so on.

Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.
How many 5-letter words?

$$|\Sigma^5| = |\Sigma|^5 = 7^5 = 16,807$$

How many with no letter repeated?

Basic Counting Rules (2)

Product rule: Sequences of selections

Question

How can we count sequences when the underlying set changes?

Answer

- *Define an order on the whole underlying set*
- *Select from $[1, n]$, where n is the size of the "remaining" set, and a selection of i represents choosing the i-th element in that set*

Basic Counting Rules (2)

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Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.
How many 5-letter words with no letter repeated?

$$\prod^{4}(|\Sigma| - i) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2,520$$

Basic Counting Rules (2)

Product rule: Sequences with restrictions/duplications

Question

- *How can we count sequences when we have constraints in the underlying order?*
- *How can we count sequences when we have duplicates?*

Example

Let $\Sigma = \{a, b, c, d, e\}$.

- How many 5-letter words with no letter repeated and a before b before c?
- How many 5-letter words can be made from a, a, a, d, e?

Basic Counting Rules (2)

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NB

The answer will be the same.

Basic Counting Rules (2)

Product rule: Sequences with restrictions/duplications

- S_1 = sequences with constraints,
- S_2 = ways to define constraints,
- S = sequences without constraints

$$S = S_1 \times S_2,$$

so

$$|S_1| = |S|/|S_2|$$

Alternatively, $\frac{1}{|S_2|}$ of the $|S|$ unconstrained sequences meet the constraint.

Basic Counting Rules (2)

Example

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Let $\Sigma' = \{a, b, c\}$.

$$\begin{aligned} S &= \prod_{i=0}^{4}(|\Sigma| - i) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\ S_2 &= \prod_{i=0}^{2}(|\Sigma'| - i) = 3 \cdot 2 \cdot 1 = 6 \end{aligned}$$

$$\text{So } S_1 = 120/6 = 20$$

Exercises

Exercises

S, T finite. How many functions $S \longrightarrow T$ are there?

$\boxed{\text{RW: 5.1.19}}$ Consider a *complete* graph on n vertices.

(a) No. of paths of length 3

(b) paths of length 3 with all vertices distinct

(c) paths of length 3 with all edges distinct

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Exercises

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$\boxed{\text{RW: 5.3.1}}$ 200 people. 150 swim or jog, 85 swim and 60 do both. How many jog?

$\boxed{\text{RW: 5.6.38}}$ (Supp) There are 100 problems, 75 of which are 'easy' and 40 'important'.
What's the smallest number of easy *and* important problems?

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Exercise

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$\boxed{\text{RW: 5.3.2}}$ $S = [100 \ldots 999]$, thus $|S| = 900$.

(b) How many numbers have a 3 *and* a 7?

(a) How many numbers have at least one digit that is a 3 or 7?

?

Exercise

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Corollaries

- If $|S \cup T| = |S| + |T|$ then S and T are disjoint
- If $|\bigcup_{i=1}^{n} S_i| = \sum_{i=1}^{n} |S_i|$ then S_i are pairwise disjoint
- If $|T \setminus S| = |T| - |S|$ then $S \subseteq T$

These properties can serve to identify cases when sets are disjoint (resp. one is contained in the other).

Proof.

$|S| + |T| = |S \cup T|$ means $|S \cap T| = |S| + |T| - |S \cup T| = 0$

$|T \setminus S| = |T| - |S|$ means $|S \cap T| = |S|$ means $S \subseteq T$ $\square$

Outline

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- Combinations and Permutations
- Alternative techniques
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Combinatorial Objects: How Many?

permutations

Ordering of all objects from a set S; equivalently: Selecting all objects while *recognising* the order of selection.

The number of permutations of n elements is

$$n! = n \cdot (n-1) \cdots 1, \quad 0! = 1! = 1$$

r-permutations (sequences without repetition)

Selecting any r objects from a set S of size n without repetition while *recognising* the order of selection.

Their number is

$$\Pi(n, r) = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with duplicates

Example

How many anagrams of ASSESS?

Permutations with duplicates

Example

How many anagrams of ASSESS?

Label S's: $\mathrm{AS_1S_2ES_3S_4}$: $6!$

In each anagram we can label the S's in $4!$ ways.

Suppose there are m anagrams. So $m.4! = 6!$, i.e. $m = \frac{6!}{4!}$

Permutations with duplicates

Example

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Example

Number of anagrams of MISSISSIPPI?

Permutations with duplicates

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Example

Number of anagrams of MISSISSIPPI? $\frac{11!}{3!4!2!}$

r-**selections** (or: r-**combinations**)
Collecting any r distinct objects without repetition;
equivalently: selecting r objects from a set S of size n and *not* recognising the order of selection.
Their number is

$$\binom{n}{r} = \frac{\Pi(n,r)}{r!} = \frac{n!}{(n-r)!r!} = \frac{n\cdot(n-1)\cdots(n-r+1)}{1\cdot 2\cdots r}$$

NB

These numbers are usually called binomial coefficients due to

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$$

Also defined for any $\alpha \in \mathbb{R}$ *as* $\displaystyle\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$

Simple Counting Problems

Example

$\boxed{\text{RW: 5.1.2}}$ Give an example of a counting problem whose answer is

(a) $\Pi(26, 10)$

(b) $\binom{26}{10}$

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Draw 10 cards from a half deck (eg. black cards only)
(a) the cards are recorded in the order of appearance
(b) only the complete draw is recorded

Examples

- Number of edges in a complete graph K_n
- Number of diagonals in a convex polygon
- Number of poker hands
- Decisions in games, lotteries etc.

Exercises

Exercises

$\boxed{\text{RW: 5.1.6}}$ From a group of 12 men and 16 women, how many committees can be chosen consisting of

(a) 7 members?

(b) 3 men and 4 women?

(c) 7 women or 7 men?

$\boxed{\text{RW: 5.1.7}}$ As above, but any 4 people (male or female) out of 9 and two, Alice and Bob, unwilling to serve on the same committee.

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Counting Poker Hands

Exercises

$\boxed{\text{RW: 5.1.15}}$ A poker hand consists of 5 cards drawn without replacement from a standard deck of 52 cards

$$\{A, 2\text{-}10, J, Q, K\} \times \{\text{club, spade, heart, diamond}\}$$

(a) Number of "4 of a kind" hands (e.g. 4 Jacks)

(b) Number of non-straight flushes, i.e. all cards of same suit but *not* consecutive (e.g. 8,9,10,J,K)

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Selecting items summary

Selecting k items from a set of n items:

With replacement	Order matters	Examples	Formula
Yes	Yes	Words of length k (sequences of length k)	n^k
No	Yes	k-permutations	$\Pi(n, k)$
No	No	Subsets of size k	$\binom{n}{k}$
Yes	No		

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Yes	No	Multisets of size k	$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$

"Balls in boxes"

Have n "distinguishable" boxes.
Have k balls which are either:

1. Indistinguishable
2. Distinguishable

How many ways to place balls in boxes with

A. At most one
B. Any number of

balls per box?

NB

Suppose K is a set with $|K| = k$ and N is a set with $|N| = n$:

- *2A counts the number of injective functions from K to N*
- *2B counts the number of functions from K to N*

“Balls in boxes”

Case	Balls	Balls per box	Number
1A	Indist.	At most 1	
1B	Indist.	Any number	
2A	Dist.	At most 1	
2B	Dist.	Any number	

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2B	Dist.	Any number	n^k

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Alternative techniques

What if the current techniques are unwieldy?
Other techniques for obtaining an exact count:

- Find a different approach for counting
- Make use of symmetries
- Make use of recursion
- Write a program (running time?)

Example

Example

How many sequences of 15 coin flips have an even number of heads?

Example

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- Using "balls in boxes": $\binom{15}{0} + \binom{15}{2} + \ldots + \binom{15}{14}$

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Example

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How many sequences of 15 coin flips have an even number of heads?

- Using "balls in boxes": $\binom{15}{0} + \binom{15}{2} + \ldots + \binom{15}{14}$
- Use symmetry: $\frac{1}{2} \times 2^{15}$
- Use recursion: $\mathrm{Even}(n) = \mathrm{Odd}(n-1) + \mathrm{Even}(n-1)$; $\mathrm{Odd}(n) = \mathrm{Even}(n-1) + \mathrm{Odd}(n-1)$

Example

Example

How many sequences of n coin flips contain HH?

Example

Example

How many sequences of n coin flips contain HH?

$$
\begin{aligned}
C(0) &= 0 \\
C(1) &= 0 \\
C(n) &= C(n-1) + C(n-2) + 2^{n-2}
\end{aligned}
$$

Example

Example

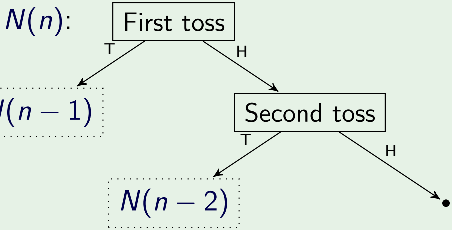
How many sequences of n coin flips do not contain HH?

$$\begin{aligned} N(0) &= 1 \\ N(1) &= 2 \\ N(2) &= 3 \\ N(n) &= N(n-1) + N(n-2) \end{aligned}$$

Example

Example

How many sequences of n coin flips do not contain HH?
We can summarise all possible outcomes in a **recursive tree**



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Difficult Counting Problems

Example (Ramsay numbers)

An example of a *Ramsay number* is $R(3,3) = 6$, meaning that

> *"K_6 is the smallest complete graph such that if all edges are painted using two colours, then there must be at least one monochromatic triangle"*

This serves as the basis of a game called S-I-M (invented by Simmons), where two adversaries connect six dots, respectively using blue and red lines. The objective is to *avoid* closing a triangle of one's own colour. The second player has a winning strategy, but the full analysis requires a computer program.

Using Programs to Count

Two dice, a red die and a black die, are rolled.
(Note: one *die*, two or more *dice*)

Write a program to list all the pairs $\{(R, B) : R > B\}$

Similarly, for three dice, list all triples $R > B > G$

Generally, for n dice, all of which are m-sided ($n \leq m$), list all *decreasing* n-tuples

NB

In order to just find the number of such n-tuples, it is not necessary to list them all. One can write a recurrence relation for these numbers and compute (or try to solve) it.

Approximate Counting

NB

A Count may be a precise value or an **estimate**.

The latter should be asymptotically correct or at least give a good asymptotic bound, whether upper or lower. If S *is the base set,* $|S| = n$ *its size, and we denote by* $c(S)$ *some collection of objects from* S *we are interested in, then we seek constants* a, b *such that*

$$a \leq \lim_{n \to \infty} \frac{est(|c(S)|)}{|c(S)|} \leq b$$

In other words $est(|c(S)|) \in \Theta(|c(S)|)$.