Student Name:	
Student Number:	
Signature:	

University of New South Wales
School of Computer Science and Engineering
Foundations of Computer Science (COMP9020)
FINAL EXAM — Session 1, 2017

This paper must be submitted and cannot be retained by the student

Instructions:

- **Ensure you enter your correct name and student number above!**
- This exam paper contains 10 multiple-choice questions (pages 1-3) plus 5 open questions (pages 4-8).
  Each multiple-choice question is worth 4 marks ($10 \times 4 = 40$).
  Each open question is worth 12 marks ($5 \times 12 = 60$).
  Total exam marks = 100.
- Only use a blue or black pen. **All answers must be recorded in this paper.**
- For the multiple-choice questions, tick **one** box for your answer directly (each multiple-choice question has only one correct answer).
  To make a correction, tick *all* boxes, then *circle* one box for your answer.
- For the open questions, write your answer in the space provided (if you need more space, you can write on the back of the sheet).
- A separate white booklet is provided for scratch work only. **Do not write your answers in the Examination Answer Book, it will not be marked.**
- Time allowed – 120 minutes + 10 minutes reading time.
- The exam is *closed book.* Reference materials are not allowed, apart from one A4-sized sheet (double-sided is ok) of your own notes.
- Number of pages in this exam paper: 8 (in addition to this cover sheet).

1. How many integers in the interval $[-100, 100]$ are divisible by 5 **or** 7 (or both)?

   ☐ 64

   ☒ 65
   $N = 2 \cdot (\lfloor 100/5 \rfloor + \lfloor 100/7 \rfloor - \lfloor 100/35 \rfloor) + 1 = 2 \cdot (20 + 14 - 2) + 1 = 65$

   ☐ 67

   ☐ 68

2. Consider the alphabets $\Sigma = \{s, e, a\}$ and $\Psi = \{a, r, t\}$. How many words are in the set $\{\omega \in (\Sigma \setminus \Psi)^* : \text{ length}(\omega) \leq 2\}$?

   ☐ 2

   ☐ 4

   ☐ 6

   ☒ 7

3. Which of the following is **not** a correct equivalence?

   ☒ $\neg A \vee B \;\equiv\; \neg(B \wedge \neg A)$

   ☐ $A \wedge \neg B \;\equiv\; \neg(B \vee \neg A)$

   ☐ $A \Rightarrow \neg B \;\equiv\; B \Rightarrow \neg A$

   ☐ $\neg(A \Rightarrow B) \;\equiv\; \neg B \wedge A$

4. Consider the functions $f : \mathbb{N} \longrightarrow \{0, 1, 2\}$ and $g : \{0, 1, 2\} \longrightarrow \{0, 1, 2\}$ defined by

$$f(x) = x \bmod 3$$
$$g(x) = |x - 2|$$

   Which of the following statements is true?

   ☐ $f \circ f \neq f$

   ☒ $g \circ g = \mathrm{Id}_{\{0, 1, 2\}}$

   ☐ $f \circ g$ is **not** onto

   ☐ $g \circ f$ is **not** onto

5. Consider the partial order $\preceq$ on $S = \{1, 2, 3, 4, 6, 12\}$ defined by

$$x \preceq y \text{ if and only if } x \mid y \qquad \text{(i.e., } x \text{ is a divisor of } y\text{)}$$

Which of the following is **not** true?

- ☐ $\text{lub}(\{1, 4, 6\}) = 12$
- ☒ $\text{glb}(\{4, 6, 12\}) = 1$
  correct is $\text{glb}(\{4, 6, 12\}) = 2$
- ☐ $(S, \preceq)$ is a lattice
- ☐ $1 < 3 < 2 < 6 < 4 < 12$ is a topological sort of $(S, \preceq)$

6. All connected graphs with n vertices and k edges satisfy

- ☐ $n \geq k + 1$
- ☐ $n \geq k$
- ☐ $n \leq k$
- ☒ $n \leq k + 1$
  a tree has $k + 1$ vertices

7. We would like to prove that $P(n)$ for all $n \geq 0$.
Which of the following conditions imply this conclusion?

- ☐ $P(0)$ and $\forall n \geq 1\,(P(n) \Rightarrow P(n + 1))$
- ☐ $P(0)$ and $P(1)$ and $\forall n \geq 1\,(P(n) \wedge P(n + 1) \Rightarrow P(n + 2))$
- ☒ $P(0)$ and $P(1)$ and $\forall n \geq 0\,(P(n) \wedge P(n + 1) \Rightarrow P(n + 2))$
  True
- ☐ $P(0)$ and $P(1)$ and $\forall n \geq 1\,(P(n) \Rightarrow P(n + 2))$

8. Consider the recurrence given by $T(1) = 1$ and $T(n) = 4 \cdot T(\frac{n}{2}) + n$.
   This has order of magnitude

   - ☐ $O(n)$
   - ☐ $O(n \cdot \log n)$
   - ☒ $O(n^2)$
     master theorem
   - ☐ $O(2^n)$

9. Let $S = \{1, 2, 3\}$ and $\mathbb{B} = \{0, 1\}$.
   How many different *onto* functions $f : S \longrightarrow \mathbb{B}$ are there?

   - ☐ 0
   - ☒ 6
     $2^3 - 2 = 6$ since there are $|\mathbb{B}|^{|S|} = 2^3$ functions in total, and two of them are not onto: $f_1 : s \mapsto 0$ and $f_2 : s \mapsto 1$
   - ☐ 8
   - ☐ 9

10. Which of the following is true for all A, B?

    - ☒ $P(A \cap B|B) = P(A|B)$
    - ☐ $P(A \cap B) = P(B) \cdot P(B|A)$
    - ☐ $P(A \cup B) \geq P(A) + P(B)$
    - ☐ $P(A|B) + P(A|\bar{B}) = 1$

11. Consider the following two formulae:

$$\begin{aligned}\phi &= \neg(A \Rightarrow (B \wedge C)) \\ \psi &= \neg A \vee C\end{aligned}$$

(a) Transform ϕ into ***disjunctive*** normal form (DNF).
(b) Prove that $\phi, \psi \models \neg B$ (i.e., $\neg B$ is a logical consequence of ϕ and ψ).
(c) Is $\phi \vee \psi$ a tautology (i.e., always true)? **Explain your answer.**

(a) $\overline{\overline{A} + BC} = \overline{\overline{A}} \cdot \overline{BC} = A \cdot (\overline{B} + \overline{C}) = A\overline{B} + A\overline{C}$

(b) From ψ it follows that $\neg(A \wedge \neg C)$.
From (a) it then follows that $A \wedge \neg B$, which implies $\neg B$.

Alternative solution using a truth table:

A	B	C	ϕ	ψ	$\neg B$
F	F	F	F	T	T
F	F	T	F	T	T
F	T	F	F	T	F
F	T	T	F	T	F
T	F	F	T	F	T
T	F	T	T	T	T
T	T	F	T	F	F
T	T	T	F	T	F

(c) $\phi \vee \psi$ is always true:
Case 1: A is false or C is true. Then ψ is true.
Case 2: Case 1 is false, then $A \wedge \neg C$, hence ϕ is true according to (a).

Alternative solution extends the truth table from above by $\phi \vee \psi$.

12. Prove that for all binary relations $\mathcal{R}_1 \subseteq S \times S$ and $\mathcal{R}_2 \subseteq S \times S$ the following holds:

If $\mathcal{R}_1$ and $\mathcal{R}_2$ are symmetric, then $\mathcal{R}_1 \setminus \mathcal{R}_2$ is symmetric.

If $(x, y) \in \mathcal{R}_1 \setminus \mathcal{R}_2$ then $(x, y) \in \mathcal{R}_1$ and $(x, y) \notin \mathcal{R}_2$.

By symmetry of $\mathcal{R}_1$ and $\mathcal{R}_2$ it follows that $(y, x) \in \mathcal{R}_1$ and $(y, x) \notin \mathcal{R}_2$.

Hence, $(y, x) \in \mathcal{R}_1 \setminus \mathcal{R}_2$.

Alternative proof by contradiction:

If $\mathcal{R}_1 \setminus \mathcal{R}_2$ is not symmetric, then there exist $x, y \in S$ such that $(x, y) \in \mathcal{R}_1$ and $(x, y) \notin \mathcal{R}_2$ but $(y, x) \notin \mathcal{R}_1 \setminus \mathcal{R}_2$.

From $(y, x) \notin \mathcal{R}_1 \setminus \mathcal{R}_2$ it follows that $(y, x) \notin \mathcal{R}_1$ or $(y, x) \in \mathcal{R}_2$.

But $(y, x) \notin \mathcal{R}_1$ contradicts $(x, y) \in \mathcal{R}_1$ given that $\mathcal{R}_1$ is symmetric,
and $(y, x) \in \mathcal{R}_2$ contradicts $(x, y) \notin \mathcal{R}_2$ given that $\mathcal{R}_2$ is symmetric.

13. The *Fibonacci numbers* are defined as follows:

$$F_1 = 1; \quad F_2 = 1; \quad F_i = F_{i-1} + F_{i-2} \text{ for } i \geq 3$$

Write a proof by induction for the statement that every *third* Fibonacci number (that is, $F_3,\ F_6,\ F_9,\ \ldots$) is even (i.e., divisible by 2).

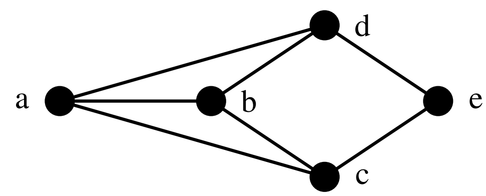
Base case $n = 3$:
$F_1 = 1;\ F_2 = 1;\ F_3 = 2$. Hence, $2 \mid F_3$.

Inductive step $n \longrightarrow n + 3$: By definition,

$$\begin{aligned} F_{n+3} &= F_{n+2} + F_{n+1} \\ &= (F_{n+1} + F_n) + F_{n+1} \\ &= 2 \cdot F_{n+1} + F_n \end{aligned}$$

From the induction hypothesis $2 \mid F_n$ it follows that $2 \mid (2F_{n+1} + F_n)$.

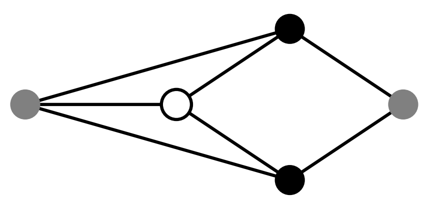
14. Consider the following graph G:



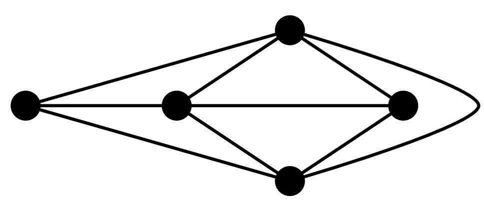
(a) Give all 3-cliques of G.
(b) What is the chromatic number $\chi(G)$ of G? **Explain your answer.**
(c) What is the maximal number of edges that can be added to G such that G remains planar? **Explain your answer.**

(a) {a,b,c}, {a,b,d}

(b) $\chi(G) = 3$.
3 colours are necessary because G contains a 3-clique.
3 colours are also sufficient:



(c) A maximum of 2 edges can be added, for example:



3 edges cannot be added since this would result in K_5, which is not planar.

15. Consider a deck of six cards containing 2 jacks and 4 aces. One card is randomly drawn from the deck at a time. Calculate the expected number of drawing attempts until an ace is drawn:

    (a) if the cards are put back into the deck after each drawing;
    (b) if the cards are **not** put back into the deck after each drawing.

    Briefly explain your answers.

(a) Each drawing event has the probability $p = \frac{4}{6} = \frac{2}{3}$.
Hence, the expected number of drawing attempts is $\frac{1}{p} = 1.5$

(b) $1 \cdot \frac{4}{6} + 2 \cdot \frac{2}{6} \cdot \frac{4}{5} + 3 \cdot \frac{2}{6} \cdot \frac{1}{5} \cdot 1 = \frac{2}{3} + \frac{8}{15} + \frac{1}{5} = \frac{21}{15} = \frac{7}{5} = 1.4$

— END OF EXAM —