Logic

Problem 1
Let F be the set of well-formed formulas with propositional variables from PROP. Define a relation, $R \subseteq F \times F$ by $(\varphi, \psi) \in R$ if $\varphi \models \psi$. Prove or give a counter-example to disprove:

(a) R is a partial order.

(b) $R \cup R^{\leftarrow}$ is an equivalence relation.

(c) $R \cap R^{\leftarrow}$ is an equivalence relation.

Solution

1. R is **not** a partial order: it does not satisfy anti-symmetry. Take, for example $\varphi = p$ and $\psi = p \wedge p$. Then $(\varphi, \psi), (\psi, \varphi) \in R$, but $\varphi \neq \psi$.

2. $R \cup R^{\leftarrow}$ is **not** a partial order: it does not satisfy transitivity. Take, for example, $\varphi = p \wedge q$, $\psi = p$, and $\theta = p \wedge r$. Then
$$\varphi \models \psi \text{ and } \theta \models \psi,$$
so we have $(\varphi, \psi), (\theta, \psi) \in R$. However
$$\varphi \not\models \theta \text{ and } \theta \not\models \varphi$$
as there are truth assignments that make one formula true and the other false. So $(\varphi, \theta), (\theta, \varphi) \notin R$. Therefore, we have
$$(\varphi, \psi), (\psi, \theta) \in R \cup R^{\leftarrow}, \text{ but } (\varphi, \theta) \notin R \cup R^{\leftarrow}.$$

3. $R \cap R^{\leftarrow}$ **is** an equivalence relation. We show that $R \cap R^{\leftarrow}$ satisfies Reflexivity (R), Symmetry (S), and Transitivity (T) as follows:

   Reflexivity. For any formula $\varphi \in F$, we have $\varphi \models \varphi$, so $(\varphi, \varphi) \in R$ and (trivially) $(\varphi, \varphi) \in R^{\leftarrow}$. So $(\varphi, \varphi) \in R \cap R^{\leftarrow}$ and hence it is reflexive.

   Symmetry. Suppose $(\varphi, \psi) \in R \cap R^{\leftarrow}$. Then because (φ, ψ) is in R we have $(\psi, \varphi) \in R^{\leftarrow}$. Also, because (φ, ψ) is in $R^{\leftarrow}$ we have $(\psi, \varphi) \in R$. Therefore $(\psi, \varphi) \in R \cap R^{\leftarrow}$, and so $R \cap R^{\leftarrow}$ is symmetric.

   Transitivity. Supopse $(\varphi, \psi), (\psi, \theta) \in R \cap R^{\leftarrow}$. Then
$$\varphi \models \psi \quad \psi \models \theta \quad \psi \models \varphi \quad \theta \models \psi.$$
   That is, every valuation that makes φ true will also make ψ true and vice-versa. And every valuation that makes ψ true, will also make θ true and vice-versa. It follows that $\varphi \models \theta$ and $\theta \models \varphi$, so $(\varphi, \theta) \in R \cap R^{\leftarrow}$. So $R \cap R^{\leftarrow}$ is transitive.

   Alternatively, If $(\varphi, \psi) \in R \cap R^{\leftarrow}$, then $\varphi \models \psi$ and $\psi \models \varphi$. So φ and ψ are logically equivalent. Conversely, if φ and ψ are logically equivalent then $\varphi \models \psi$ and $\psi \models \varphi$ and so $(\varphi, \psi) \in R \cap R^{\leftarrow}$. Therefore $R \cap R^{\leftarrow}$ is the logical equivalence relation, which, from the lectures, is an equivalence relation.

Problem 2
Prove that $\neg N$ follows logically from $H \wedge \neg R$ and $(H \wedge N) \rightarrow R$.

Solution

We will show this using truth tables:

	H	R	N	$H \wedge N$	$(H \wedge N) \rightarrow R$	$H \wedge \neg R$	$\neg N$
v_1	T	T	T	T	T	F	F
v_2	T	T	F	F	T	F	T
v_3	T	F	T	T	F	T	F
v_4	T	F	F	F	T	T	T
v_5	F	T	T	F	T	F	F
v_6	F	T	F	F	T	F	T
v_7	F	F	T	F	T	F	F
v_8	F	F	F	F	T	F	T

From the above table, we see that there is exactly one valuation, v_4, that makes both $(H \wedge N) \rightarrow R$ and $H \wedge \neg R$ evaluate to true. That valuation makes $\neg N$ true, so

$$(H \wedge N) \rightarrow R,\ H \wedge \neg R \models \neg N$$

as required.

Problem 3

Consider the formulae $\phi_1 = (r \rightarrow p)$ and $\phi_2 = (p \rightarrow (q \vee \neg r))$. Transform the formula $\phi = (\neg q \rightarrow (\phi_1 \wedge \phi_2))$ into

(a) **DNF**, and

(b) **CNF**.

Simplify the result as much as possible.

Solution

Let us first consider the truth table of ϕ.

p	q	r	ϕ_1	$q \vee \neg r$	ϕ_2	ϕ
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	F	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	F	T	F	F	T	F
F	F	F	T	T	T	T

So the canonical DNF for ϕ is

$$pqr + pq\bar{r} + p\bar{q}\bar{r} + \bar{p}qr + \bar{p}q\bar{r} + \bar{p}\bar{q}\bar{r}.$$

Examining the Karnaugh map:

	pq	$p\bar{q}$	$\bar{p}\bar{q}$	$\bar{p}q$
r	+			+
$\bar{r}$	+	+	+	+

We observe that the +'s can be covered by a 2×2 rectangle (blue) and a 1×4 rectangle (orange). So the minimal DNF for ϕ is:

$$\phi = q \vee \neg r.$$

We note that this is also in CNF; and it is straightforward to check that the CNF obtained by finding a minimal DNF for $\neg\phi$ is identical.

Problem 4

Let $(T, \wedge, \vee, ', 0, 1)$ be a Boolean Algebra. Define $\oplus : T \times T \to T$ as follows:

$$x \oplus y = (x \wedge y') \vee (x' \wedge y)$$

(a) Prove using the laws of Boolean Algebra that for all $x \in T$, $x \oplus 1 = x'$.

(b) Prove using the laws of Boolean Algebra that $x \wedge (y \oplus z) = (x \wedge y) \oplus (x \wedge z)$.

(c) Find a Boolean Algebra (and x, y, z) which demonstrates that $x \oplus (y \wedge z) \neq (x \oplus y) \wedge (x \oplus z)$

Solution

Outside of the lecture material, we need the law of idempotence:

$$\begin{aligned}
x &= x \wedge 1 && \text{(Identity)} \\
&= x \wedge (x \vee x') && \text{(Complement)} \\
&= (x \wedge x) \vee (x \wedge x') && \text{(Distributivity)} \\
&= (x \wedge x) \vee 0 && \text{(Complement)} \\
&= x \wedge x && \text{(Identity);}
\end{aligned}$$

the law of annihilation:

$$\begin{aligned}
x \wedge 0 &= x \wedge (x \wedge x') && \text{(Complement)} \\
&= (x \wedge x) \wedge x' && \text{(Associativity)} \\
&= x \wedge x' && \text{(Idempotence)} \\
&= 0 && \text{(Identity);}
\end{aligned}$$

and their duals (which follow from the Principle of Duality). We also observe that $1' = 0$ which follows directly from the uniqueness of complement (as $1 \wedge 0 = 0$ and $1 \vee 0 = 1$). For simplicity we will make extensive use of associativity and commutativity to minimize parentheses and manipulate terms.

(a)

$$\begin{aligned}
x \oplus 1 &= (x \wedge 1') \vee (x' \wedge 1) \\
&= (x \wedge 0) \vee x' && \text{($1' = 0$ and Identity)} \\
&= 0 \vee x' && \text{(Annihilation)} \\
&= x' && \text{(Identity).}
\end{aligned}$$

(b)

$$\begin{aligned}
x \wedge (y \oplus z) &= x \wedge ((y \wedge z') \vee (y' \wedge z)) \\
&= (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) && \text{(Distributivity)} \\
&= (0 \vee (x \wedge y \wedge z')) \vee (0 \vee (x \wedge y' \wedge z)) && \text{(Identity)} \\
&= ((x \wedge y \wedge x') \vee (x \wedge y \wedge z')) \vee ((x' \wedge x \wedge z) \vee (y' \wedge x \wedge z)) && \text{(Complement, Commutativity)} \\
&= ((x \wedge y) \wedge (x' \vee z')) \vee ((x' \vee y') \wedge (x \wedge z)) && \text{Distributivity} \\
&= ((x \wedge y) \wedge (x \wedge z)') \vee ((x \wedge y)' \wedge (x \wedge z)) && \text{De Morgan's laws} \\
&= (x \wedge y) \oplus (x \wedge z).
\end{aligned}$$

(c) Consider $\mathbb{B}$ with $x = z = 1 and y = 0$. We have:

$$\begin{aligned}
x \oplus (y \wedge z) &= 1 \oplus (0 \wedge 1) \\
&= 1 \oplus 0 \quad \text{(Identity)} \\
&= 0' \quad \text{(from (a))} \\
&= 1.
\end{aligned}$$

On the other hand we have:

$$\begin{aligned}
(x \oplus y) \wedge (x \oplus z) &= (1 \oplus 0) \wedge (1 \oplus 1) \\
&= 0' \wedge 1' \quad \text{(from (a))} \\
&= 1 \wedge 0 \\
&= 0 \quad \text{(Identity).}
\end{aligned}$$

Problem 5

(a) How many well-formed formulas can be constructed from one $\vee$; one $\wedge$; two parenthesis pairs (,); and the three literals p, $\neg p$, and q?

(b) Under the equivalence relation defined by **logical equivalence**, how many equivalence classes do the formulas in part (a) form?

Solution

(a) We will count the number of well-formed formulas that use all symbols exactly once. We note that the parentheses are tied to the operations $\wedge$ and $\vee$ and there are two "shapes" of formula: $(l_1 op_1 (l_2 op_2 l_3))$ and $((l_2 op_2 l_3) op_1 l_1)$. There are $2 \times 1 = 2$ choices for op_1, op_2. There are $3 \times 2 \times 1 = 6$ choices for l_1, l_2, l_3. Therefore, there are $2.2.6 = 24$ formulas in total.

(b) We note that since $(\varphi \vee \psi)$ is logically equivalent to $(\psi \vee \varphi)$ and $(\varphi \wedge \psi)$ is logically equivalent to $(\psi \wedge \varphi)$ we can reduce the 24 formulas from above to the following six (possibly not distinct) classes:

$I.$	$(p \vee (\neg p \wedge q))$	$II.$	$(\neg p \vee (p \wedge q))$	$III.$	$(q \vee (p \wedge \neg p))$
$IV.$	$(p \wedge (\neg p \vee q))$	$V.$	$(\neg p \wedge (p \vee q))$	$VI.$	$(q \wedge (p \vee \neg p))$

Since

$$(q \vee (p \wedge \neg p)) \equiv (q \vee \bot) \equiv q \equiv (q \wedge \top) \equiv (q \wedge (p \vee \neg p))$$

we see that III and VI are the same class.

For the other cases we have:

$$\begin{aligned}
\text{I} \quad & (p \vee (\neg p \wedge q)) \equiv ((p \vee \neg p) \wedge (p \vee q)) \equiv (\top \wedge (p \vee q)) \equiv (p \vee q) \\
\text{II} \quad & (\neg p \vee (p \wedge q)) \equiv ((\neg p \vee p) \wedge (\neg p \vee q)) \equiv (\top \wedge (\neg p \vee q)) \equiv (\neg p \vee q) \\
\text{III} \quad & (p \wedge (\neg p \vee q)) \equiv ((p \wedge \neg p) \vee (p \wedge q)) \equiv (\bot \vee (p \wedge q)) \equiv (p \wedge q) \\
\text{IV} \quad & (\neg p \wedge (p \vee q)) \equiv ((\neg p \wedge p) \vee (\neg p \wedge q)) \equiv (\bot \vee (\neg p \wedge q)) \equiv (\neg p \wedge q)
\end{aligned}$$

Each of these classes are distinct, as can be seen from the truth table:

p	q	$\neg p$	I	II	III	IV	V
T	T	F	T	T	T	T	F
T	F	F	T	F	F	F	F
F	T	T	T	T	T	F	T
F	F	T	F	T	F	F	F

So there are five equivalence classes.