Matrix Algebra
Gaussian Elimination
Homework 3

Show all steps and identify each row operation in working the following problems.

1. Use back-substitution to find all solutions of the following systems.

   (a)
$$\begin{array}{rcrcrcr} x & + & 3y & + & 5z & = & 3 \\ & & y & - & 2z & = & 2 \\ & & & & z & = & 1 \end{array}$$

   Result: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -14 \\ 4 \\ 1 \end{pmatrix}$

   (b)
$$\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 4 \\ & & y & + & 3z & = & 1 \\ & & & & z & = & 2 \end{array}$$

   Result: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -8 \\ -5 \\ 2 \end{pmatrix}$

   (c)
$$\begin{array}{rcrcrcr} x & + & y & + & 3z & = & 4 \\ & & & & z & = & 2 \end{array}$$

   Result: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-y \\ y \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
   where y can be anything (it is free).

   (d)
$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & x_3 & - & x_4 & = & 1 \\ & & x_2 & - & 4x_3 & + & 2x_4 & = & 2 \\ & & & & & & x_4 & = & 5 \end{array}$$

   Result: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 30-13x_3 \\ 4x_3-8 \\ x_3 \\ 5 \end{pmatrix} = \begin{pmatrix} 30 \\ -8 \\ 0 \\ 5 \end{pmatrix} + x_3 \begin{pmatrix} -13 \\ 4 \\ 1 \\ 0 \end{pmatrix}$
   where x_3 can be anything (it is free).

2. Put the following matrices in row-echelon form. (Use Gaussian elimination; do it one step at a time.)

   (a)
$$\begin{pmatrix} 2 & 4 & 6 & 2 \\ 1 & 3 & 8 & 1 \\ 2 & 2 & 3 & 4 \end{pmatrix}$$

   Result:
$$\begin{pmatrix} 2 & 4 & 6 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 7 & 2 \end{pmatrix}$$

   (b)
$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 3 & 8 & 1 \\ 2 & 7 & 19 & 7 \end{pmatrix}$$

   Result: One solution is
$$\begin{pmatrix} 1 & 3 & 8 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Use Gaussian elimination and back-substitution (if necessary) to find all solutions of the following systems.

   (a)
$$\begin{array}{rcrcrcr} 2x & + & 4y & + & 2z & = & 12 \\ x & + & y & - & 3z & = & 0 \\ 3x & - & 3y & + & 4z & = & 1 \end{array}$$

   Result: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

   (b)
$$\begin{array}{rcrcrcr} x & + & 3y & - & 2z & = & 5 \\ 2x & - & y & + & 2z & = & 7 \\ x & + & 10y & - & 8z & = & 3 \end{array}$$

   Result: No solution.

   (c)
$$\begin{array}{rcrcrcr} 2x & + & 2y & + & 4z & = & 6 \\ x & + & y & + & 3z & = & 2 \\ 4x & + & 4y & + & 10z & = & 10 \end{array}$$

Result: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5-y \\ y \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ where y can be anything (it is free).

4. The general equation for a circle in the plane is

$$Ax^2 + Ay^2 + Bx + Cy + D = 0.$$

   (a) If a circle passes through the three points $(3,2)$, $(-1,2)$ and $(1,0)$, then A, B and C will satisfy

$$\begin{array}{rcrcrcrcl} 13A & + & 3B & + & 2C & + & D & = & 0 \\ 5A & - & B & + & 2C & + & D & = & 0 \\ A & + & B & & & + & D & = & 0 \end{array}$$

   Find an equation for the circle. (D will be a free variable; you can find a specific solution by setting $D = 1$, for example.)

   Result: $x^2 + y^2 - 2x - 4y + 1 = 0$

   (b) Find an equation of the circle passing through the points $(4,5)$, $(6,1)$ and $(5,4)$

   Result: $x^2 + y^2 - 2x - 2y - 23 = 0$

5. Find all solutions of the following system of equations.

$$\begin{array}{rcrcrcr} \frac{2}{x} & + & \frac{3}{y} & + & \frac{4}{z} & = & 13 \\ \frac{1}{x} & + & \frac{2}{y} & + & \frac{1}{z} & = & 6 \\ \frac{3}{x} & & & + & \frac{1}{z} & = & 10 \end{array}$$

(Note: Since this system is not linear, you'll want to solve for something related to x, y and z, and then solve for x, y and z.)

Result: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1 \\ 1 \end{pmatrix}$