Section 04
Computing Derivatives

If we had to use the definition every time we needed to find a derivative, many of the advantages of the Calculus would be lost in the lengthy computations. Instead, we will develop formulas which we can use to compute derivatives relatively painlessly. To start, given a function f, instead of computing the derivative at a fixed number, we will compute it at some unspecified point x.

Example
Let $y = f(x) = 2x^2 - 2x$. Find $f'(x)$.

First, we'll find the average rate of change of y from x to $x + \Delta x$. Since

$$\begin{aligned} f(x + \Delta x) &= 2(x + \Delta x)^2 - 2(x + \Delta x) \\ &= 2(x^2 + 2x\Delta x + \Delta x^2) - 2x - 2\Delta x \\ &= 2x^2 + 4x\Delta x + 2\Delta x^2 - 2x - 2\Delta x \end{aligned}$$

we get

$$\begin{aligned} \Delta y &= f(x + \Delta x) - f(x) \\ &= 4x\Delta x + \Delta x^2 - 2\Delta x. \end{aligned}$$

So

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{4x\Delta x + \Delta x^2 - 2\Delta x}{\Delta x} \\ &= 4x + \Delta x - 2. \end{aligned}$$

For small values of Δx, this is basically $4x - 2$, so

$$\frac{dy}{dx} = f'(x) = 4x - 2$$

Given $f'(x)$, we can then easily compute $f'(a)$ for any number a. If $f(x) = 2x^2 - 2x$, we know $f'(x) = 4x - 2$, and so we have $f'(1) = 4 \cdot 1 - 2 = 2$, $f'(2) = 4 \cdot 2 - 2 = 6$, $f'(3) = 4 \cdot 3 - 2 = 10$, etc.

What we are getting is a new function f', called the ***derivative*** of f (since it is derived from the function f). Note that $f'(x)$ is *not* the same as $f'(a)$. While $f'(a)$ is a number, $f'(x)$ typically is not; it is an expression involving the variable x. In particular, $f'(x)$ *cannot* be interpreted as a velocity or a slope of a tangent line. Once we have evaluated $f'(x)$ at a point $x = a$ to get a number $f'(a)$, then this number, of course, has the usual interpretations as the slope of a tangent line, etc.

Example
Let $f(x) = 2x^2 - 2x$. Find the tangent line to the graph of f at $x = 2$.

The tangent line will be the line through $(2, f(2)) = (2, 4)$ with slope $m = f'(2)$. Since $f'(x) = 4x - 2$, we know $m = f'(2) = 4 \cdot 2 - 2 = 6$. So the tangent line is

$$y - 4 = 6(x - 2)$$

or, in slope-intercept form,

$$y = 6x - 8.$$

Warning:
Note that $f'(x) = 4x - 2$ is *not* the slope of the tangent line; the tangent line is *not* given by $y - 4 = (4x - 2)(x - 2)$.

The following proposition also follows from the definition of the derivative. (We'll omit the computations. You're welcome.)

Proposition
Let $f(x) = x^n$. *Then* $f'(x) = nx^{n-1}$.

Remark
The formulas

$$\begin{array}{ll} \text{if } f(x) = 1 & \text{then } f'(x) = 0 \\ \text{if } f(x) = x & \text{then } f'(x) = 1 \\ \text{if } f(x) = x^2 & \text{then } f'(x) = 2x \end{array}$$

are special cases of the above formula.

Example
Let $f(x) = x^5$. Then $f'(x) = 5x^{5-1} = 5x^4$.

The proposition holds for any number n, not just integer values of n.

Example
Let $f(x) = \sqrt{x} = x^{1/2}$. Then $f'(x) = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2}$.

Example
Let $f(x) = 1/x^2 = x^{-2}$. Then $f'(x) = -2x^{-2-1} = -2x^{-3}$.

Warning:
It is important to note that in order to use the formula in the above proposition, the function must be in *exactly* the form x^n. To find the derivative of $f(x) = 1/x^2$, we *cannot* use the proposition to get $f'(x) = 1/(2x)$; we must first rewrite $f(x) = x^{-2}$, and then apply the proposition to get $f'(x) = -2x^{-3}$.

The following rules also follow from the definition of the derivative.

- When taking the derivative of an expression with several terms, you can take the derivative of each term separately.

  Example
  Let $f(x) = x^6 + x^3 + 1$. Then $f'(x) = 6x^5 + 3x^2$.

- When taking the derivative of a term with a constant coefficient, leave the coefficient and take the derivative of the non-constant factor.

  Example
  Let $f(x) = 5x^3$. Then $f'(x) = 5 \cdot 3x^2 = 15x^2$

We now have the tools to take the derivative of any polynomial.

Example
Let $f(x) = 2x^3 - 5x^2 + 7x + 2$. Then $f'(x) = 2 \cdot 3x^2 - 5 \cdot 2x + 7 \cdot 1 + 0 = 6x^2 - 10x + 7$.

Example
Let $f(x) = 3x^2 + 5x$. Find the tangent line to the graph of f at $x = 2$.

The tangent line will be the line through $(2, f(2)) = (2, 22)$ with slope $m = f'(2)$. Since $f'(x) = 3 \cdot 2x + 5 \cdot 1 = 6x + 5$, we have $m = f'(2) = 6 \cdot 2 + 5 = 17$. The tangent line is then given by

$$y - 22 = 17(x - 2)$$

or, in slope-intercept form,

$$y = 17x - 12.$$

Example
Let $f(x) = 2x^4 - 4x^2 + 2x + 1$. Use the tangent approximation about $x = 1$ to approximate $f(1.05)$.

To use the tangent approximation about $x = 1$, we need to know $f(1)$ and $f'(1)$. We have

$$f(1) = 2 \cdot 1^4 - 4 \cdot 1^2 + 2 \cdot 1 + 1 = 1,$$

and since

$$f'(x) = 2 \cdot 4x^3 - 4 \cdot 2x + 2 \cdot 1 + 0 = 8x^3 - 8x + 2,$$

we also have

$$f'(1) = 8 \cdot 1^3 - 8 \cdot 1 + 2 = 2.$$

So the tangent approximation gives us

$$\begin{aligned} f(1.05) &\approx f(1) + f'(1)(1.05 - 1) \\ &= 1 + 2(.05) \\ &= 1.1 \end{aligned}$$

(Note that the actual value is $f(1.05) = 1.1210125$.)

Remark
Given $f(x)$, the derivative is denoted $f'(x)$. Using the "d" notation, we could also write it as $\frac{df(x)}{dx}$ or $\frac{d}{dx}f(x)$.[1]

If f is a reasonably nice function, then the derivative f' will be a new function. If f' is also a reasonably nice function, we can take the derivative of f' to get a new derived function, denoted f'' and called the ***second derivative*** of f.

Example
Let $f(x) = 2x^3 + 4x^2 + 9$. Then

$$\begin{aligned} f'(x) &= 2 \cdot 3x^2 + 4 \cdot 2x + 0 \\ &= 6x^2 + 8x. \end{aligned}$$

The second derivative, $f''(x)$, is the derivative of $f'(x) = 6x^2 + 8x$, so

$$\begin{aligned} f''(x) &= 6 \cdot 2x + 8 \cdot 1 \\ &= 12x + 8. \end{aligned}$$

This can be continued, of course, to get the third derivative f''', the fourth derivative, etc. (To avoid writing too many primes, the fourth derivative of f is usually denoted $f^{(4)}$, and similarly for higher order derivatives.) These higher order derivatives are important, but it is the first two that have simple interpretations, so those are the only derivatives we will consider.

[1] If we write $y = f(x)$, then $f'(x)$ is also denoted dy/dx, which we have also used to denote the derivative of f at a point. Whether dy/dx denotes the derivative at a point should be inferred from context.