Section 3
Gaussian Elimination

Photosynthesis is a process that takes molecules of carbon dioxide (CO_2) and water (H_2O) (and a dash of sunlight) and produces glucose ($C_6H_{12}O_6$) and oxygen (O_2). How many molecules of each compound are needed? Suppose that x carbon molecules and y water molecules are used to produce z glucose molecules and w oxygen molecules. By keeping track of the number of atoms in the process, we can determine how many molecules are required.

	INPUT		OUTPUT	
Molecules	Carbon dioxide	Water	Glucose	Oxygen
Carbon atoms	1	0	6	0
Oxygen atoms	2	1	6	2
Hydrogen atoms	0	2	12	0

If we look at the number of carbon atoms in the process, we get

$$1 \cdot x + 0 \cdot y = 6 \cdot z + 0 \cdot w.$$

The number of oxygen atoms gives us

$$2 \cdot x + 1 \cdot y = 6 \cdot z + 2 \cdot w,$$

and the number of hydrogen atoms gives us

$$0 \cdot x + 2 \cdot y = 12 \cdot z + 0 \cdot w.$$

Altogether, we get the simulaneous equations

$$\begin{array}{rcrcrcrcl} 1x & + & 0y & - & 6z & - & 0w & = & 0 \\ 2x & + & 1y & - & 6z & - & 2w & = & 0 \\ 0x & + & 2y & - & 12z & - & 0w & = & 0 \end{array}$$

We can solve this system (and get that for every glucose molecule that six oxygen molecules are also produced and it requires six carbon dioxide molecules and six carbon dioxide molecules), but this is a type of common and important system, and we should discuss how to methodically solve them.

Since we will be dealing with systems of equations with several variables, we need some consistent way of denoting these variables. For n variables, we will typically use $x_1, x_2, x_3, \ldots, x_n$. If we only have two or three variables, to avoid subscripts we will typically use x, y or x, y, z.

Definition
A ***linear equation*** in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Example

$$2x_1 + x_2 - 4x_3 + x_4 = 5.$$

Example

$$x + 2y - z = 7.$$

We will be interested in several simultaneous equations. A ***linear system of*** m ***equations in*** n ***unknowns*** is a system of the form

$$\begin{array}{rcrcccrcl}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & & & & & & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

Example
The system

$$\begin{array}{rcrcrcl}
x & - & 3y & + & 2z & = & 5 \\
x & + & y & + & z & = & 1 \\
-3x & & & + & z & = & 0 \\
 & & y & - & 2x & = & 5
\end{array}$$

is a system of 4 equations in 3 unknowns.

Definition
A linear system is ***homogeneous*** if all the constants are 0; i.e., if it is of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

Definition
A ***solution*** to a system of m equations in n unknowns is a set of n numbers

$$x_1 = c_1, \quad x_2 = c_2, \quad \cdots, \quad x_n = c_n$$

which simultaneously satisfy all m equations.

Example
The values $x = 1, y = 4, z = 2$ is a solution of

$$\begin{array}{ccccccc} 2x & + & y & - & 2z & = & 2 \\ 3x & + & y & + & 5z & = & 17 \end{array}$$

since

$$\begin{array}{ccccccc} 2 \cdot \mathbf{1} & + & \mathbf{4} & - & 2 \cdot \mathbf{2} & = & 2 \\ 3 \cdot \mathbf{1} & + & \mathbf{4} & + & 5 \cdot \mathbf{2} & = & 17 \end{array}$$

To ***solve*** a system of equations is to find the solution set; i.e., the set of all solutions. There are three possibilities; a linear system of equations will have either

- exactly 0 solutions
- exactly 1 solution
- infinitely many solutions.

(Note that a homogeneous system always has at least one solution; namely, $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$. So for a homogeneous system, this will be the only solution or there will be infinitely many solutions.)

Here are some examples of each possibility.

Example
Exactly 0 solutions.

$$\begin{array}{rcrcrcl} x & + & y & + & 2z & = & 4 \\ & & 2y & + & z & = & 3 \\ & & & & 0 & = & 2 \end{array}$$

Example
Exactly 1 solution.

$$\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 2 \\ & & y & - & z & = & 3 \\ & & & & z & = & 4 \end{array}$$

Finding z and working our way backwards (this is called ***back-substitution***), we get

$$\begin{aligned} &z = 4 \\ &y = z + 3 = 4 + 3 = 7 \\ &x = -y + 2z + 2 = -7 + 2 \cdot 4 + 2 = 3 \end{aligned}$$

Example
Infinitely many solutions.

$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 3 \\ & & & & z & = & 2 \end{array}$$

Finding z and working our way backwards, we see that $z = 2$. We would next find y, but we can't solve for y in terms of already determined values. The variable y is called a ***free variable***. Next, $x = y - 2z + 3 = y - 2 \cdot 2 + 3 = y - 1$. So the solutions look like

$$\begin{aligned} x &= y - 1 \\ y &= y \\ z &= 2 \end{aligned}$$

This describes the solution set.

These examples were given to us in a particularly nice form; we would like every system to end up like one of these.

Definition
Two systems of equations are ***equivalent*** if they have the same solution set.

Given a system of equations, we would like to find an equivalent system in a nice form. There are three basic ways to manipulate a system to get an equivalent system:

- Switch two equations.

  Example

  $$\begin{array}{rcrcr} x & + & y & = & 4 \\ 2x & - & 3y & = & 5 \end{array}$$

  is equivalent to

  $$\begin{array}{rcrcr} 2x & - & 3y & = & 5 \\ x & + & y & = & 4 \end{array}$$

- Multiply an equation by a non-zero number.

  Example

$$\begin{array}{rcrcl} x & + & y & = & 3 \\ 2x & - & y & = & 5 \end{array}$$

  is equivalent to (multiplying the top equation by 2)

$$\begin{array}{rcrcl} 2x & + & 2y & = & 6 \\ 2x & - & y & = & 5 \end{array}$$

- Add a multiple of one equation to another.

  Example

$$\begin{array}{rcrcl} x & + & y & = & 3 \\ 2x & + & 4y & = & 1 \end{array}$$

  is equivalent to (adding twice the first equation to the second)

$$\begin{array}{rcrcl} x & + & y & = & 3 \\ 4x & + & 6y & = & 7 \end{array}$$

A system of equations

$$\begin{array}{rcrcccrcl} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

has two matrices connected with it. The matrix of coefficients:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

called the coefficient matrix, and matrix consisting of the coefficients and constants:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix},$$

which is simply called the matrix of the system. Note that this second matrix is the coefficient matrix augmented with the column vector of the constants of the equations, and contains all the information of the system of equations. Each row of the matrix represents an equation; the numbers in the first column represent the x_1 coefficients, the numbers in the second column represent the x_2 coefficients, etc., and the numbers in the last column represent the constants to the right of the equal signs.

Example

The system

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & - & x_4 & = & 1 \\ x_1 & & & - & 3x_3 & + & 9x_4 & = & 0 \\ x_1 & + & 9x_2 & & & - & x_4 & = & 3 \end{array}$$

has matrix

$$\begin{pmatrix} 1 & -2 & 3 & -1 & 1 \\ 1 & 0 & -3 & 9 & 0 \\ 1 & 9 & 0 & -1 & 3 \end{pmatrix}$$

and coefficient matrix

$$\begin{pmatrix} 1 & -2 & 3 & -1 \\ 1 & 0 & -3 & 9 \\ 1 & 9 & 0 & -1 \end{pmatrix}.$$

In terms of matrices, the three operations which transform a system into an equivalent system are called the ***elementary row operations***. If we want to keep track of these operations, we can draw an arrow from the original matrix to the transformed matrix with an indication of what was done.

- Switch any two rows.
  We can indicate that we are switching rows i and j with $\rho_i \leftrightarrow \rho_j$.

  Example
  (Switch rows 2 and 3.)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} < \rho_2 \leftrightarrow \rho_3 > \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix}$$

- Multiply a row by a non-zero number.
  We can indicate that we are multiplying row i by the number c with $c\rho_i$.

  Example
  (Multiply row 2 by 5.)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} < 5\rho_2 > \begin{pmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{pmatrix}$$

- Add a multiple of one row to another.
  We can indicate that we are adding c times row i to row j with $c\rho_i + \rho_j$.
  (Note that the matrix being changed is at the end.)

  Example
  (Add 4 times row 2 to row 1.)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} < 4\rho_2 + \rho 1 > \begin{pmatrix} 17 & 22 & 27 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We want to use the elementary row operations to reduce a system of equations to an equivalent system which is easier to solve. The idea is this: it is easier to solve a system of equations if there are fewer variables and equations. So given a system of m equations and n variables, we want to get rid of one of the variables from all but one of the equations. The equations which don't involve this variable will then be a system of $m-1$ equations with $n-1$ variables.

The procedure we use is called ***Gaussian Elimination***. While there are different variations of Gaussian elimination that put a matrix in a convenient form, the specific method given here will be the version that we mean when we talk about Gaussian elimination.

Given the matrix of a linear system of equations, start with the top row.

1. Look at the leftmost non-zero element in the row. If this is also the leftmost non-zero element including any lower rows, don't do anything for this step. Otherwise, switch the current row with a lower row so that the leftmost non-zero element in the current row is as far left as possible. The position of this non-zero element is called a ***pivot***.

2. Add multiples of the current row the lower rows it to get 0s below the pivot.

3. If the current row is not the bottom row, go to the next row and repeat these steps. If the current row is the bottom row, this procedure is finished.

Example

Starting with the system of equations

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & 4x_2 & + & 2x_3 & + & 6x_4 & + & 2x_5 & = & 6 \\ x_1 & + & 2x_2 & + & x_3 & + & 3x_4 & + & 2x_5 & = & 5 \\ 2x_1 & + & 4x_2 & + & x_3 & + & 4x_4 & + & x_5 & = & 5 \end{array}$$

we get the (augmented) matrix

$$\begin{pmatrix} \mathbf{2} & 4 & 2 & 6 & 2 & 6 \\ 1 & 2 & 1 & 3 & 2 & 5 \\ 2 & 4 & 1 & 4 & 1 & 5 \end{pmatrix}$$

There is a 2 on the far left of the first row; this is the first pivot. To get zeros below this pivot, add $-1/2$ times the first row to the second row, and then add -1 times the first row to the third row.

$$\begin{pmatrix} \mathbf{2} & 4 & 2 & 6 & 2 & 6 \\ 1 & 2 & 1 & 3 & 2 & 5 \\ 2 & 4 & 1 & 4 & 1 & 5 \end{pmatrix} < \substack{(-1/2)\rho_1+\rho_2 \\ (-1)\rho_1+\rho 3} > \begin{pmatrix} \mathbf{2} & 4 & 2 & 6 & 2 & 6 \\ \mathbf{0} & 0 & 0 & 0 & 1 & 2 \\ \mathbf{0} & 0 & -1 & -2 & -1 & -1 \end{pmatrix}$$

There are now 0s below the first pivot. Moving to the second row, we would like a non-zero term as far to the left as possible. There is a zero in the second column, but there are also 0s in the second column of all lower rows. There is a zero in the third column and a non-zero term in the third column of the third row. To get a non-zero term as far to the left as possible in the second row, we will switch the second and third rows.

$$\begin{pmatrix} \mathbf{2} & 4 & 2 & 6 & 2 & 6 \\ \mathbf{0} & 0 & 0 & 0 & 1 & 2 \\ \mathbf{0} & 0 & -1 & -2 & -1 & -1 \end{pmatrix} < \rho_2 \leftrightarrow \rho_3 > \begin{pmatrix} 2 & 4 & 2 & 6 & 2 & 6 \\ 0 & 0 & \mathbf{-1} & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

The $\mathbf{-1}$ in the second row is now the second pivot. There is only a zero below it, so that's taken care of. There is nothing to do with the third row, the **1** in the fifth column of the third row is the third pivot. We are done with Gaussian elimination.

The specific operations used in the previous example weren't the only ones we could have used for Gaussian elimination. For example, to avoid beginning by dividing the first row by 2, we could have started by switching the first two rows to get our leading 1 in the first row.

After Gaussian elimination, a matrix will be in a specific form, called row-echelon form.

Definition
A matrix is in ***row-echelon form*** if:

1. The leftmost non-zero element in any row is further to the right than the leftmost non-zero element of any higher row.
   The position of this leftmost non-zero element is called a pivot.

2. Any rows with all 0s are at the bottom.

If a matrix A can be reduced to the row-echelon matrix E using elementary row operations, we write $A \rightsquigarrow E$. Note that the row-echelon form of a matrix is not unique: it is possible to have $A \rightsquigarrow E_1$ and $A \rightsquigarrow E_2$ with $E_1 \neq E_2$.

Once a matrix is in row-echelon form, the system can be solved with ***back-substitution***:

- Start with the last equation and find the value of the last variable x_n.
- Go up to the next equation and solve for x_{n-1}, using our value for x_n. Continue.
- If the column corresponding to a variable doesn't have a leading 1, the equations don't determine a value for that variable. That variable can take on any value, and is called a ***free variable***.

Example

To finish our example, our matrix after Gaussian elimination is

$$\begin{pmatrix} 2 & 4 & 2 & 6 & 2 & 6 \\ 0 & 0 & -1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

The corresponding equations are

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 + 6x_4 + 2x_5 &= 6 \\ -x_3 - 2x_4 - x_5 &= -1 \\ x_5 &= 2 \end{aligned}$$

From the last equation, we get $x_5 = 2$. The variable x_4 is a free variable. From the second equation from the bottom, $x_3 = 1 - 2x_4 - x_5 = 1 - 2x_4 - 2 = -1 - 2x_4$. The variable x_2 is a free variable. From the top equation, we get $x_1 = 3 - 2x_2 - x_3 - 3x_4 - x_5 = 3 - 2x_2 - (-1 - 2x_4) - 3x_4 - 2 = 2 - 2x_2 - x_4$.

It is a good idea to write the solution(s) as a vector, so we will do that. The solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 - 2x_2 - x_4 \\ x_2 \\ -1 - 2x_4 \\ x_4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

Remark
The back-substitution can be done as part of the elimination process, while in matrix form, by multiplying the rows by appropriate numbers to put a 1 in each pivot and getting 0s above as well as below the pivots. This is called ***Gauss-Jordan Elimination***.

Example
Perform back-substitution for the matrix in the previous example.

The matrix in row-echelon form is:

$$\begin{pmatrix} 2 & 4 & 2 & 6 & 2 & 6 \\ 0 & 0 & -1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Multiplying the first row by $1/2$ and the second row by -1,

$$\begin{pmatrix} 2 & 4 & 2 & 6 & 2 & 6 \\ 0 & 0 & -1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} < (1/2)\rho_1 >$$
$$\begin{pmatrix} 1 & 2 & 1 & 3 & 1 & 3 \\ 0 & 0 & -1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} < (-1)\rho_2 >$$
$$\begin{pmatrix} 1 & 2 & 1 & 3 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

gives us 1s in the pivot positions.

There are no rows above the first pivot, so we'll look at the pivot in the second row. We want a 0 above that, so we'll subtract row 2 from row 1 (i.e., add -1 times row 2 to row 1):

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} < (-1)\rho_2 + \rho_1 > \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Next, we want zeros above the leading 1 in the third row. We only have to take care of the second row for this example, so we'll subtract row 3 from row 2:

$$\begin{pmatrix}1&2&0&1&0&2\\0&0&1&2&1&1\\0&0&0&0&1&2\end{pmatrix} < (-1)\rho_3+\rho_2 > \begin{pmatrix}1&2&0&1&0&2\\0&0&1&2&0&-1\\0&0&0&0&1&2\end{pmatrix}$$

This finishes the Gauss-Jordan elimination. This matrix corresponds to the equations:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_4 & = & 2\\ & & x_3 & + & 2x_4 & = & -1\\ & & & & x_5 & = & 2\end{array}$$

Without any more substitution, these give us

$$\begin{aligned} x_1 &= 2-2x_2-x_4\\ x_3 &= -1-2x_4\\ x_5 &= 2\end{aligned}$$

which are in terms of the free variables x_2 and x_4. Again, we can write this as

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}2-2x_2-x_4\\x_2\\-1-2x_4\\x_4\\2\end{pmatrix} = \begin{pmatrix}2\\0\\-1\\0\\2\end{pmatrix} + x_2\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix} + x_4\begin{pmatrix}-1\\0\\-2\\1\\0\end{pmatrix}$$

where x_2 and x_4 can be anything.

Remark

So far, all of our calculations have been exact. When approximate calculations are used, errors can accumulate the computed answer can be far from the actual answer.

Example

Solve the system

$$\begin{array}{rcrcr} .4x & + & 561.6y & = & 562\\ 73.03x & - & 43.03y & = & 30\end{array}$$

where the result of each calculation is rounded to 4 significant digits.

We can tell by inspection that $x = 1$, $y = 1$ is a solution. (It is, in fact, the only solution.) However, let's solve this with Gaussian elimination. Our matrix is

$$\begin{pmatrix} .4000 & 561.6 & 562.0 \\ 73.03 & -43.03 & 30.00 \end{pmatrix}$$

Dividing row 1 by .4000, we get

$$\begin{pmatrix} 1.000 & 1404 & 1405 \\ 73.03 & -43.03 & 30.00 \end{pmatrix}$$

Subtracting 73.03 times row 1 from row 2, we get

$$\begin{pmatrix} 1.000 & 1404 & 1405 \\ 0.000 & -43.03 - 102500 & 30.00 - 102600 \end{pmatrix}$$

or

$$\begin{pmatrix} 1.000 & 1404 & 1405 \\ 0.000 & -102500 & -102600 \end{pmatrix}$$

Dividing row 2 by -102500, we get

$$\begin{pmatrix} 1.000 & 1404 & 1405 \\ 0.000 & 1.000 & 1.001 \end{pmatrix}$$

This corresponds to the equations

$$\begin{aligned} 1.000x \; + \; 1404y \; &= \; 1405 \\ 1.000y \; &= \; 1.001 \end{aligned}$$

So our computed solution would be

$$\begin{aligned} &y = 1.001 \\ &x = 1405 - 1404y = 1405 - 1404 \cdot 1.001 = 1405 - 1405 = 0 \end{aligned}$$

The computed y value is close to the y value of the true solution, but the computer x value is not close.

One way to reduce this type of error in Gaussian elimination is with ***pivotal condensation***, where each leading 1 is obtained from the largest term in the corresponding column.

Example

The original matrix from the previous example was

$$\begin{pmatrix} .4000 & 561.6 & 562.0 \\ 73.03 & -43.03 & 30.00 \end{pmatrix}$$

We start by getting a leading 1 for the first row. This 1 will be in the first column, so instead of automatically dividing the first row by .4000, we look for the largest number in the first column. This is the 73.03 in the second row, so we switch the second row with the first to get

$$\begin{pmatrix} 73.03 & -43.03 & 30.00 \\ .4000 & 561.6 & 562.0 \end{pmatrix}$$

We now get the leading 1 for the first row by dividing by a larger, rather than smaller, number. Completing the problem as before, we get the solution $x = 1.001$, $y = 1.001$. This is a much better approximate solution than before.

While this is an important practical concern, in the future we will pretend that all calculations are exact.

Let's do some more examples of Gaussian elimination. We will solve a system with exactly one solution, and one with no solutions.

Example

Solve

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 9 \\ 3x & + & y & + & 2z & = & 11 \\ & & 2y & + & z & = & 7 \end{array}$$

The matrix for this system is

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 3 & 1 & 2 & 11 \\ 0 & 2 & 1 & 7 \end{pmatrix}$$

The Gaussian elimination will proceed as follows:

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & -2 & -4 & -16 \\ 0 & 2 & 1 & 7 \end{pmatrix}$$

$$< (-1/2)\rho_2 >$$

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 1 & 7 \end{pmatrix} < -2\rho_2 + \rho_3 >$$

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & -3 & -9 \end{pmatrix}$$

This is now in row-echelon form, and corresponds to the equations

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 9 \\ & & y & + & 2z & = & 8 \\ & & & & -3z & = & -9 \end{array}$$

Back substitution gives us:

$$\begin{aligned} & z = 3 \\ & y = 8 - 2z = 8 - 2 \cdot 3 = 2 \\ & x = 9 - y - 2z = 9 - 2 - 2 \cdot 3 = 1 \end{aligned}$$

So the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Example
Solve

$$\begin{array}{rcrcrcr} 2x & + & 2y & + & 4z & = & 12 \\ x & + & y & + & z & = & 8 \\ 3x & + & 3y & + & 2z & = & 19 \end{array}$$

Our matrix is

$$\begin{pmatrix} 2 & 2 & 4 & 12 \\ 1 & 1 & 1 & 8 \\ 3 & 3 & 2 & 19 \end{pmatrix} < \substack{-(1/2)\rho_1+\rho 2 \\ (-3/2)\rho_1+\rho_3} >$$
$$\begin{pmatrix} 2 & 2 & 4 & 12 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -4 & 1 \end{pmatrix} < -4\rho_2 + \rho_3 >$$
$$\begin{pmatrix} 2 & 2 & 4 & 12 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

We can see that the last row corresponds to the equation $0 = -7$, which is never true. So this system has no solutions.

Xcas

`Xcas` can solve systems of equations with the `solve` command. The equations and variables to solve for need to be given as lists. So to solve the system

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + 4z &= 7 \\ x + y - z &= 2 \end{aligned}$$

you can use the command

```
solve([x+y+z=6, 2*x-y+4*z=7, x+y-z=2],[x,y,z])
```

$$[[1, 3, 2]]$$

where the result is a list with the values of x, y and z. To solve the system

$$\begin{aligned} x + y + z &= 1 \\ x - y - z &= 2 \end{aligned}$$

you can use the command

```
solve([x + y + z = 1, x - y - z = 2],[x,y,z])
```

$$\left[\left[\frac{3}{2}, -z - \frac{1}{2}, z\right]\right]$$

where the values of x, y and z are given in terms of the free variable z.

We are interested in more than solving equations, and `Xcas` has commands for each elementary row operation.

The `rowSwap` can be used to switch two rows of a matrix. It takes as input a matrix and the indices (beginning at 0) of the two rows to switch. For example, for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

```
A := [[1,2,3],[4,5,6],[7,8,9]]
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

you can switch rows 1 and 2 (which have indices 0 and 1) with the command

```
rowSwap(A,0,1)
```

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Note that these commands will return the transformed matrix, but will not change the value of the given matrix; A will be the same matrix as before.

```
A
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

To change the matrix, you will need to assign it to the new matrix.

```
A := rowSwap(A,0,1)
```

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

```
A
```

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

The `mRow` command will multiply a row by a given number. It takes as input a number (the multiplier), a matrix, and the index of the row to be multiplied. Resetting A,

```
A := [[1,2,3],[4,5,6],[7,8,9]]
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

to multiply row 2 (index 1) by 7, you can use the command

```
mRow(7,A,1)
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 28 & 35 & 42 \\ 7 & 8 & 9 \end{bmatrix}$$

The `mRowAdd` command will add a multiple of one row to another. It takes as input a number (the multiplier), a matrix, the index of the row to be multiplied, and the index of the row the multiple is added to. With A as above, to add 5 times row 2 (index 1) to row 3 (index 2), you can use the command

```
mRowAdd(5,A,1,2)
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 27 & 33 & 39 \end{bmatrix}$$

`Xcas` can of course simply put a matrix in reduced echelon form for you, with the command `rref`.

```
A := [[1,2,3],[4,5,6],[7,8,9]]
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

```
rref(A)
```

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$