Matrix Algebra
Projections
Homework 14

1. In each part of this problem, V is the space with orthogonal basis $\mathcal{B}$. Find the projection of $\boldsymbol{x}$ onto V.

   a) $\mathcal{B} = \left\{\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$, $\boldsymbol{x} = \begin{pmatrix}1\\2\\0\end{pmatrix}$

   Answer: The projection is $\boldsymbol{p} = \begin{pmatrix}3/2\\3/2\\0\end{pmatrix}$.

   b) $\mathcal{B} = \left\{\begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\2\end{pmatrix}\right\}$, $\boldsymbol{x} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$

   Answer: The projection is $\boldsymbol{p} = \begin{pmatrix}19/14\\33/14\\19/7\\27/14\end{pmatrix}$.

   c) $\mathcal{B} = \left\{\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\-3\\1)\end{pmatrix}\right\}$, $\boldsymbol{x} = \begin{pmatrix}2\\1\\2\\1\end{pmatrix}$.

   Answer: The projection is $\boldsymbol{p} = \begin{pmatrix}2/3\\-2/3\\2\\2/3\end{pmatrix}$.

2. In each part of this problem, V is the space with basis $\mathcal{B}$. Find the projection matrix P; i.e. the matrix P such that for each $\boldsymbol{x} \in \mathbb{R}^n$, $P\boldsymbol{x}$ is the projection of $\boldsymbol{x}$ onto V. Then find the projection of $\boldsymbol{b}$ onto V.

   a) $\mathcal{B} = \left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right\}$, $\boldsymbol{b} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

   Answer:

   $$P = \frac{1}{21}\begin{pmatrix}17 & -2 & 8\\-2 & 20 & 4\\8 & 4 & 5\end{pmatrix},$$

   $$P\boldsymbol{b} = \frac{1}{21}\begin{pmatrix}37\\50\\31\end{pmatrix}.$$

   b) $\mathcal{B} = \left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\1\end{pmatrix}\right\}$, $\boldsymbol{b} = \begin{pmatrix}1\\2\\2\\1\end{pmatrix}$

   Answer:

   $$P = \frac{1}{5}\begin{pmatrix}3 & 2 & 1 & 1\\2 & 3 & -1 & -1\\1 & -1 & 2 & 2\\1 & -1 & 2 & 2\end{pmatrix},$$

   $$P\boldsymbol{b} = \begin{pmatrix}2\\1\\1\\1\end{pmatrix}.$$

c) $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

Answer:

$$P = \frac{1}{4} \begin{pmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & 1 & 3 \end{pmatrix},$$

$$P\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

d) $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}$, $\boldsymbol{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix}$

Answer:

$$P = \frac{1}{5} \begin{pmatrix} 3 & 1 & -1 & 2 \\ 1 & 9 & 1 & 1 \\ -1 & 1 & 9 & -1 \\ -2 & 1 & -1 & 3 \end{pmatrix},$$

$$P\boldsymbol{b} = \frac{1}{10} \begin{pmatrix} 12 \\ 29 \\ 1 \\ 2 \end{pmatrix}.$$

3. In each part of this problem, find the line $y = mx + b$ which bests fits the data.

a)

x	1	2	3
y	2	2	1

Answer: $y = \frac{-1}{2}x + \frac{8}{3}$

b)

x	1	3	5
y	2	1	3

Answer: $y = \frac{1}{4}x + \frac{5}{4}$

c)

x	0	1	2	3
y	1	0	1	1

Answer: $y = \frac{1}{10}x + \frac{3}{5}$

4. In each part of this problem, find the parabola $y = ax^2 + bx + c$ which bests fits the data.

a)

x	0	1	2	3
y	0	1	1	9

Answer: $y = \frac{7}{4}x^2 + \frac{-51}{20}x + \frac{9}{20}$.

b)

x	0	1	2	3
y	0	1	0	1

Answer: $y = 0x^2 + \frac{1}{5}x + \frac{1}{5}$.