Section 03
Tangent Approximation

If we know an object's velocity over a given interval, we can determine how far the object travels.

Example
For the next $\frac{1}{2}$ hour, a car is traveling at an average velocity of 30 miles/hour. How far does the car travel?

The distance traveled will be velocity $\times$ time, which in this case is

$$30 \times \frac{1}{2} = 15 \text{ miles.}$$

If we only know the approximate velocity over an interval, we can only approximate the distance traveled, but often that is sufficient.

If we know the instantaneous velocity of an object, we can approximate its velocity over small intervals.[1] If we know a car currently has an instantaneous velocity of 30 miles per hour, then (since 1 minute is a fairly short time interval) the car's velocity for the next minute will be about 30 miles/hour.

Example
A car has an instantaneous velocity of 30 miles/hour. Approximately how far does it travel in the next minute?

Over the next minute ($\frac{1}{60}$ hour), we can approximate the car's velocity by 30 miles/hour, so the car will travel about

$$30 \times \frac{1}{60} = \frac{1}{2} \text{ mile.}$$

Example
A man is walking away. In 5 minutes, he will be 100 yards away and traveling at 50 yards/minute. How far away will he be in $5\frac{1}{2}$ minutes?

[1] We won't concern ourselves with what exactly we mean by "small" or how close of an approximation we get. While these are important questions, they are beyond the scope of this class.

In the $\frac{1}{2}$ minute from 5 to $5\frac{1}{2}$ minutes, we can use 50 yards/minute to approximate his velocity. So he will travel about

$$50 \times \frac{1}{2} = 25 \text{ yards}$$

in this time. Adding this to the distance away he already is, he will be

$$100 + 25 = 125 \text{ yards away.}$$

Using this same idea, if we know the value of a function at a point and the instantaneous rate of change (derivative) of the function at the same point, we can approximate the function at nearby points.

Example

Suppose $f(10) = 12$ and $f'(10) = 5$. Approximate $f(11)$.

We want to see what happens to $f(x)$ as x increases from $x = 10$ to $x = 11$. Since $\Delta x = 1$ and $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} = f'(10) = 5$, we get $\Delta y \approx 5\Delta x = 5$. Since $\Delta y = f(11) - f(10)$, we get $f(11) = f(10) + \Delta y \approx 12 + 5 = 17$.

All we are doing here is using the approximation $\Delta y/\Delta x \approx dy/dx$, from which we get $\Delta y \approx (dy/dx)\Delta x$. This is called the tangent approximation, and can be written as a simple formula.

Proposition (Tangent Approximation)

For x close to a,

$$f(x) \approx f(a) + f'(a)\Delta x$$

Since $\Delta x = x - a$, the tangent approximation formula can also be written

$$f(x) \approx f(a) + f'(a)(x - a).$$

Example

Suppose $f(4) = 10$ and $f'(4) = \frac{1}{2}$. Approximate $f(4.3)$.

According to the above formula,

$$\begin{aligned} f(4.3) &\approx f(4) + f'(4)(4.3 - 4) \\ &= 10 + \frac{1}{2}(.3) \\ &= 10 + .15 = 10.15 \end{aligned}$$

Note that the tangent line to the graph

$$y = f(x)$$

about $x = a$ is

$$y = f(a) + f'(a)(x - a).$$

The tangent approximation, $f(x) \approx f(a) + f'(a)(x - a)$, just reminds us that the tangent line is close to the graph near the point in question.

Example
Suppose $f(10) = 40$ and $f'(10) = 2$. Approximate $f(9.5)$.

According to the above formula,

$$\begin{aligned} f(9.5) &\approx f(10) + f'(10)(9.5 - 10) \\ &= 40 + 2 \cdot (-0.5) \\ &= 40 + (-1) = 39 \end{aligned}$$
