Section 05
Some Transcendental Functions

While we can now take the derivative of any polynomial, not all functions are polynomials or even built up from polynomials. Functions which are not based on polynomials are called ***transcendental*** functions. Examples of transcendental functions (in fact the only examples that we'll worry about) are the trigonometric functions, exponential functions, and logarithmic functions.

Let's start with the trigonometric functions. When you first learned about the trigonometric functions, you were probably told that radians were preferable to degrees to measure angles with. Why is that? If angles are measured in degrees, then for small values of x, $\sin(x)$ will take on the following values:

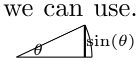
$$\begin{aligned}
\sin(0.1^\circ) &= 0.0174532\ldots \\
\sin(0.01^\circ) &= 0.0017453\ldots \\
\sin(0.001^\circ) &= 0.0001745\ldots.
\end{aligned}$$

With angles measured in radians, we get the following:

$$\begin{aligned}
\sin(0.1) &= 0.0998334\ldots \\
\sin(0.01) &= 0.0099998\ldots \\
\sin(0.001) &= 0.0009999\ldots
\end{aligned}$$

We note that if angles are measured in radians, then for small values of x we have $\sin(x) \approx x$, and so $\sin(x)/x \approx 1$.[1] We'll see that this is important, and so we'll measure all of our angles in radians from now on.

Let $f(x) = \sin(x)$. Then $\frac{f(0+\Delta x)-f(0)}{\Delta x} = \frac{\sin(\Delta x)}{\Delta x} \approx 1$ for small Δx, and so $f'(0) = 1$.[2] Looking at the graph of the sin function (see figure 1), we can see that we also have $f'(\pi/2) = 0$, $f'(\pi) = -1$, $f'(3\pi/2) = 0$ and $f'(2\pi) = 1$. Plotting these points and connecting them by a smooth curve, we should get a familiar curve. (See figure 2.) It is the graph of the cosine function.[3]

[1] While we can't definitely conclude this using a few values, there is a geometric argument that we can use. Consider a small angle with measure θ cutting off an arc from a unit circle: 

. If the angle is measured in radians, then length of the arc is θ, while the length of the vertical line segment is $\sin(\theta)$. For small angles, these lengths will be close.

[2] If we measure angles in degrees, then $f'(0) = \pi/180$, which is not as convenient.

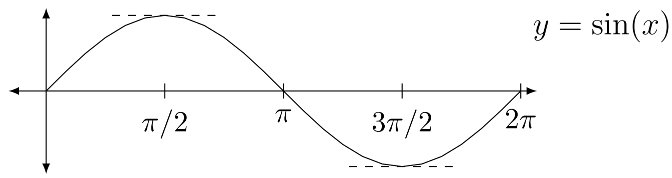


Figure 1: The graph $y = \sin(x)$.

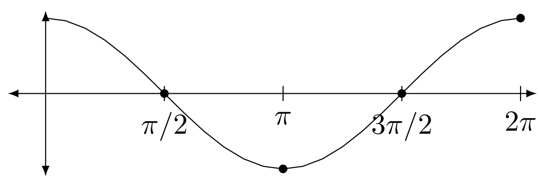


Figure 2: The graph $y = \sin(x)$.

Proposition
Let $f(x) = \sin(x)$. Then $f'(x) = \cos(x)$.

Similarly, we have the following.

Proposition
Let $f(x) = \cos(x)$. Then $f'(x) = -\sin(x)$.

(Note the minus sign in this last proposition.)

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[3]We could have arrived at this more formally by noting that for $f(x) = \sin(x)$, the sum formula for sines gives us

$$f(x + \Delta x) = \sin(x + \Delta x) = \sin(x)\cos(\Delta x) + \sin(\Delta x)\cos(x),$$

and so

$$\begin{aligned} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{\sin(x)\cos(\Delta x) + \sin(\Delta x)\cos(x) - \sin(x)}{\Delta x} \\ &= \sin(x)\frac{\cos(\Delta x) - 1}{\Delta x} + \cos(x)\frac{\sin(\Delta x)}{\Delta x}. \end{aligned}$$

For small Δx, we know $\frac{\sin(\Delta x)}{\Delta x} \approx 1$, and we can show that $\frac{\cos(\Delta x)-1}{\Delta x} \approx 0$. So

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} \approx \cos(x),$$

and so $f'(x) = \cos(x)$.

Example
Let $f(x) = 2\sin(x) + 4\cos(x)$. Then $f'(x) = 2\cos(x) + 4(-\sin(x)) = 2\cos(x) - 4\sin(x)$.

Now let's look at exponential functions. Let $f(x) = b^x$ for some $b > 0$. Then $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{b^{x+\Delta x}-b^x}{\Delta x} = \frac{b^x b^{\Delta x}-b^x}{\Delta x} = b^x \frac{b^{\Delta x}-1}{\Delta x}$. For small values of Δx, the value of $\frac{b^{\Delta x}-1}{\Delta x}$ will be close to a number which depends on b.[4] If b is the particularly nice number $e = 2.71828182846\ldots$, then $\frac{b^{\Delta x}-1}{\Delta x}$ will be close to 1, and so $\frac{e^{x+\Delta x}-e^x}{\Delta x}$ will be close to e^x. What could be nicer than that?

Proposition
Let $f(x) = e^x$. Then $f'(x) = e^x$.

Not surprisingly, the logarithm base e, written $\log_e$ or ln, has a particularly nice derivative, which we will state without proof.

Proposition
Let $f(x) = \ln(x)$. Then $f'(x) = 1/x$.

Example
Let $f(x) = 4e^x + 5\ln(x) + x^2$. Then $f'(x) = 4e^x + 5\frac{1}{x} + 2x$.

[4] Actually, it will be close to $\ln(b)$.