Chapter 6

Inverses

So now we can add, subtract and multiply matrices. Next is division.

Definition. Let A be a matrix. A matrix B is a ***left-inverse*** for A if $B \cdot A = I$. A matrix C is a ***right-inverse*** for A if $A \cdot C = I$. $\square$

It is possible that a matrix have a left inverse and not a right, or a right inverse and not a left. However, if it has both a left and right inverse, they are the same matrix.

Definition. A matrix A is ***invertible*** (or ***non-singular***) if there is a matrix B with

$$AB = I$$
$$BA = I$$

$\square$

Example. Let

$$A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}.$$

For

$$B = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix},$$

we have

$$AB = BA = I.$$

So A is invertible. ☑

Remark. Only square matrices can be invertible. (It is possible for non-square matrices to have left or right inverses, but we'll save that for later.) What's more, if A and B are square matrices with $AB = I$, then $BA = I$ also. ⊠

Remark. If A is invertible, then the matrix B with $AB = BA = I$ is unique. It is denoted A^{-1} and called the ***inverse*** of A. ⊠

Example. If

$$A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$$

then

$$A^{-1} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}.$$

☑

Remark. Not all matrices (not even all square matrices) are invertible. Consider

$$A = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$$

(the $*$s indicate the value of the entry is unimportant). For $x = \begin{pmatrix} 0 & 1 \end{pmatrix}$, we get

$$xA = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

If there were a matrix B with $AB = I$, the we would have

$$xAB = x(AB) = xI = x = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

as well as

$$xAB = (xA)B = \begin{pmatrix} 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 \end{pmatrix},$$

which would mean

$$\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Since this is not the case, no such matrix B exists, and A cannot be invertible. ⊠

In fact, we can show the following:

Proposition. *Let A be a matrix with a row of all 0s. Then A cannot be invertible.* $\square$

In fact, as we shall later see, if elementary row operations can turn a matrix A into a matrix with a row of all 0s, then A cannot be invertible.

In general, how can we tell if a matrix is invertible, and how can we find the inverse? Given a (square) matrix A, suppose we can use elementary row operations to turn A into the identity matrix. Since performing a sequence of elementary row operations is equivalent to multiplying by some matrix E, this would mean that we can multiply E by A to get I; in other words, it would mean that A is invertible with $E = A^{-1}$. What's more, as we did in the previous section, we can keep track of the operations and find A^{-1} by starting off by augmenting A with the identity matrix.

Example. ***Let***

$$\boldsymbol{A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}.}$$

Try to find $\boldsymbol{A^{-1}}$.

We start with

$$(A|I) = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{pmatrix}.$$

Dividing row 1 by 3, we get

$$\begin{pmatrix} 1 & 2/3 & 1/3 & 0 \\ 4 & 3 & 0 & 1 \end{pmatrix} \tag{6.1}$$

Subtracting 4 times row 1 from row 2 gives us

$$\begin{pmatrix} 1 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & -4/3 & 1 \end{pmatrix} \tag{6.2}$$

Multiplying row 2 by 3 gets us

$$\begin{pmatrix} 1 & 2/3 & 1/3 & 0 \\ 0 & 1 & -4 & 3 \end{pmatrix} \tag{6.3}$$

Finally, subtract 2/3 times row 2 from row 1 gives us

$$\begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{pmatrix} \tag{6.4}$$

The first part of this augmented matrix is the identity, so the second part is A^{-1}:

$$A^{-1} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}.$$

☑

The previous example asked to "try" to find A^{-1}; there's no reason why the inverse has to exist. If we were not able to turn A into the identity matrix using Gauss-Jordan reduction (which is what we did), that would be because we came across a row of 0s at some point. In this case, the original matrix A would not be invertible.

Example. ***Let***

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{6} & \mathbf{4} \\ \mathbf{9} & \mathbf{6} \end{pmatrix}.$$

Try to find A^{-1}.
We start with

$$(A|I) = \begin{pmatrix} 6 & 4 & 1 & 0 \\ 9 & 6 & 0 & 1 \end{pmatrix}.$$

Dividing row 1 by 6, we get

$$\begin{pmatrix} 1 & 2/3 & 1/6 & 0 \\ 9 & 6 & 0 & 1 \end{pmatrix} \tag{6.5}$$

Subtracting 9 times row 1 from row 2 gives us

$$\begin{pmatrix} 1 & 2/3 & 1/6 & 0 \\ 0 & 0 & -3/2 & 1 \end{pmatrix} \tag{6.6}$$

Now row 2 of the "A" portion of this augmented matrix is all 0s. This means the original matrix A is not inverible. ☑

For 2×2 matrices, there is a nice formula for determining when a matrix is invertable and finding the inverse.

Proposition. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then A is invertible exactly when $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

□

Example. Let
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$
Since $1 \cdot 4 - 3 \cdot 2 = -2$ is not zero, A is invertible and
$$A^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}.$$
☑

For larger matrices, Gaussian elimination should still be used.

Example. ***Let***
$$\boldsymbol{A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & -2 & 0 \\ 2 & -1 & 1 \end{pmatrix}.}$$
Find $\boldsymbol{A^{-1}}$, if it exists.
Starting with the augmented matrix $(A|I)$:
$$\begin{pmatrix} 0 & 4 & 2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{pmatrix},$$
Gauss-Jordan elimination will reduce this to
$$\begin{pmatrix} 1 & 0 & 0 & -1 & -3 & 2 \\ 0 & 1 & 0 & -1/2 & -2 & 1 \\ 0 & 0 & 1 & 3/2 & 4 & -2 \end{pmatrix}.$$
Since the first part of this matrix is the identity matrix, the second part is A^{-1}:
$$A^{-1} = \begin{pmatrix} -1 & -3 & 2 \\ -1/2 & -2 & 1 \\ 3/2 & 4 & -2 \end{pmatrix}.$$
☑

The following properties follow from the definition of the inverse.

Proposition. *Let A and B be invertible matrices of the same size.*
- $(A^{-1})^{-1} = A$
- $(A^k)^{-1} = (A^{-1})^k$
- $(AB)^{-1} = B^{-1}A^{-1}$ □

A demonstration of (3) will show why the order of the matrices must be reversed. Since

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I \tag{*}$$

and

$$(B^{-1}A^{-1})(AB) = B(AA^{-1})B^{-1} = BIB^{-1} = BB^{-1} = I, \tag{**}$$

it follows immediately from the definition of the inverse that the inverse of AB is $B^{-1}A^{-1}$. Note that since matrix multiplication is not commutative, the simplifications done in (*) and (**) would not have been possible if $B^{-1}A^{-1}$ were replaced by $A^{-1}B^{-1}$.

In some cases, matrix inverses can be used to solve systems of equations. Recall, the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = b_2 \\ \vdots & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = b_m \end{array}$$

can be written

$$A\boldsymbol{x} = \boldsymbol{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

If A is invertible, then

$$A\boldsymbol{x} = \boldsymbol{b}$$

can be rewritten

$$A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

or simply

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}.$$

Example. The system of equations

$$\begin{aligned} x &+ 2y &= 2 \\ 2x &+ 3y &= 1 \end{aligned}$$

can be written

$$A\boldsymbol{x} = \boldsymbol{b}.$$

where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Since

$$A^{-1} = \frac{1}{3-4}\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix},$$

the solution is

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}.$$

So $x = -4$, $y = 3$. ☑

Example. ***Solve the system***

$$\begin{aligned} & & 4y &+ 2z &= 1 \\ x &- & 2y & &= 2 \\ 2x &- & y &+ z &= 3 \end{aligned}$$

This system is simply

$$A\boldsymbol{x} = \boldsymbol{b}.$$

where

$$A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & -2 & 0 \\ 2 & -1 & 1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

We know

$$A^{-1} = \begin{pmatrix} -1 & -3 & 2 \\ -1/2 & -2 & 1 \\ 3/2 & 4 & -2 \end{pmatrix},$$

so the solution is

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{pmatrix} -1 & -3 & 2 \\ -1/2 & -2 & 1 \\ 3/2 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -3/2 \\ 7/2 \end{pmatrix}.$$

☑

So $x = -1$, $y = -3/2$ and $z = 7/2$.