Chapter 8

Determinants

Recall that the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible exactly when $ad - bc \neq 0$. The number $ad - bc$ also has a geometric interpretation, which we will discuss more later. Recall that two vectors in the plane determine a parallelogram; consider the parallelogram determined by the column vectors of A (see figure 8.1). The

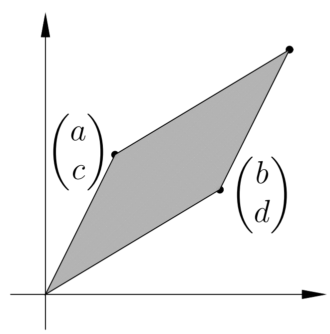

Figure 8.1: *The parallelogram determined by two vectors.*

area of this parallelogram is $|ad - bc|$. If $ad - bc > 0$, then the vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ are positively oriented, if $ad - bc < 0$ they are negatively oriented. The number $ad - bc$ is the ***determinant*** of the matrix A. Any square matrix has a determinant that tells us similar information about it, but before we can give a general definition of determinant, we need the following definition.

Definition. If A is a matrix, then a ***submatrix*** of A is a matrix obtained by deleting rows and/or columns of A. Important special cases are matrices of the form $A_{i,j}$, which is A with the ith row and jth column deleted. $\square$

Example. Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

Then

$$A_{2,1} = \begin{pmatrix} \cancel{1} & 2 & 3 & 4 \\ \cancel{5} & \cancel{6} & \cancel{7} & \cancel{8} \\ \cancel{9} & 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 10 & 11 & 12 \end{pmatrix}.$$

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Determinants are defined only for square matrices. We will define them recursively.

Definition. Let $A = (a)$ be a 1×1 matrix. Then the determinant of A, denoted $\det(A)$ or $|A|$, is

$$\det(A) = a.$$

$\square$

Example.

$$\det(4) = 4$$
$$\det(-3) = -3$$

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Note that even though determinants can be written using absolute value signs, they are not absolute values. A determinant can be positive, negative, or zero.

Now let's assume that we've defined the determinant of an arbitrary $(n-1) \times (n-1)$ matrix, and define the determinant of an $n \times n$ matrix.

Definition. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Then

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots \pm a_{1n}\det(A_{1n}),$$

where, as before, A_{ij} os the matrix A with row i and column j removed. $\square$

Let's find the determinants of some matrices.

1 × 1

$$\det(a) = a.$$

2 × 2

$$\begin{aligned} \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= a\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} - b\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= a\det(d) - b\det(c) = ad - bc \end{aligned}$$

Example.

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2.$$

$\not\square$

3 × 3

$$\begin{aligned} \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}. \end{aligned}$$

Fortunately, we have a mnemonic for the 3×3 case. To find the determinant of a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

place copies of the first two columns at the end:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}.$$

Then we multiply the numbers on the slants:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}.$$

To find the determinant, we add the numbers on the bottom and subtract the numbers on the top.

Example. ***Find the determinant of***

$$A = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \end{pmatrix}.$$

We write:

$$\begin{matrix} & & & \mathbf{18} & \mathbf{6} & \mathbf{32} \\ 1 & 2 & 3 & 1 & 2 & \\ 4 & 3 & 2 & 4 & 3 & \\ 2 & 3 & 4 & 2 & 3 & \\ & & & \mathbf{12} & \mathbf{8} & \mathbf{36} \end{matrix}.$$

So we get

$$\det(A) = (12 + 8 + 36) - (18 + 6 + 32) = 0. \qquad \boxtimes$$

This mnemonic only works for 3×3 matrices. For larger matrices, we could rely on the definition, but that quickly gets unwieldy. Instead, we start with the following proposition.

Proposition. *Let A be a triangular matrix (upper or lower triangular). Then the determinant of A is the product of the diagonal elements.* $\square$

Example.

$$\det \begin{pmatrix} 4 & 2 & 1 & 9 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 4 \cdot 2 \cdot 1 \cdot 2 = 16.$$

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Example.

$$\det \begin{pmatrix} 3 & 1 & 2 & 4 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 3 \cdot 2 \cdot 0 \cdot 2 = 0.$$

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This helps in general, since Gaussian elimination will turn any square matrix into a triangular matrix. But in order for this to be useful, we need to know how the elementary row operations affect determinants. To make the subsequent discussion simpler, we will often write matrices as

$$A = \begin{pmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \vdots \\ \boldsymbol{a_n} \end{pmatrix},$$

where $\boldsymbol{a_i}$ is the ith row of A.

Example. If

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

then $\boldsymbol{a_1} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, $\boldsymbol{a_2} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$, $\boldsymbol{a_3} = \begin{pmatrix} 7 & 8 & 9 \end{pmatrix}$. ☑

For the rest of this section and the next, we will assume A is a square matrix.

The first row operation is switching rows.

Proposition. *If two rows of A are interchanged, then the determinant of the resulting matrix is* $-\det(A)$. $\square$

Example. We know

$$\det\begin{pmatrix}1 & 1 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} = 6,$$

so

$$\det\begin{pmatrix}1 & 1 & 3\\ 7 & 8 & 9\\ 4 & 5 & 6\end{pmatrix} = -6.$$

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Now suppose A has two identical rows. Switching these two rows will leave A unchanged, so $\det(A) = -\det(A)$. We get

Corollary. *If two rows of A are identical, then* $\det(A) = 0$. $\square$

The next operation is multiplying a row by a non-zero number.

Proposition. *If a row of A is multiplied by a number c, the determinant of the resulting matrix is* $c\det(A)$. *Put another way,*

$$\det\begin{pmatrix}\boldsymbol{a_1}\\ \vdots\\ c\boldsymbol{a_i}\\ \vdots\\ \boldsymbol{a_n}\end{pmatrix} = c\det\begin{pmatrix}\boldsymbol{a_1}\\ \vdots\\ \boldsymbol{a_i}\\ \vdots\\ \boldsymbol{a_n}\end{pmatrix}.$$

$\square$

Example. Since

$$\det\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 4 & 5 & 4\end{pmatrix} = -2,$$

we know

$$\det\begin{pmatrix}1&1&1\\3&6&9\\4&5&4\end{pmatrix} = \det\begin{pmatrix}1&1&1\\3\cdot 1&3\cdot 2&3\cdot 3\\4&5&4\end{pmatrix}$$
$$= 3\cdot\det\begin{pmatrix}1&1&1\\1&2&3\\4&5&4\end{pmatrix}$$
$$= 3(-2)$$
$$= -6. \qquad \boxtimes$$

As a corollary, we get

Corollary. *If A has a row consisting of all* 0*s, then* $\det(A) = 0$. □

The last row operation is adding a multiple of one row to another.

Corollary. *If a multiple of one row of A is added to a different row, the determinant of the resulting matrix is the same as the determinant of A:*

$$\det\begin{pmatrix}\boldsymbol{a}_1\\\vdots\\\boldsymbol{a}_i + c\boldsymbol{a}_j\\\vdots\\\boldsymbol{a}_n\end{pmatrix} = \det\begin{pmatrix}\boldsymbol{a}_1\\\vdots\\\boldsymbol{a}_i\\\vdots\\\boldsymbol{a}_n\end{pmatrix}.$$

□

Example. If we start with the matrix

$$\begin{pmatrix}1&1&3\\2&3&0\\7&8&9\end{pmatrix}$$

and add 2 times the first row to the second, we get

$$\begin{pmatrix}1&1&3\\4&5&6\\7&8&9\end{pmatrix}.$$

So

$$\det\begin{pmatrix}1 & 1 & 3\\ 2 & 3 & 0\\ 7 & 8 & 9\end{pmatrix} = \det\begin{pmatrix}1 & 1 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}.$$

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So if we use Gaussian elimination to reduce a matrix to a triangular matrix, we know how the row operations affect the determinant:

1. **Switch two rows.**
   Multiply the determinant by -1.
2. **Multiply a row by a number.**

$$\det\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ c\mathbf{a}_i\\ \vdots\\ \mathbf{a}_n\end{pmatrix} = c\det\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_i\\ \vdots\\ \mathbf{a}_n\end{pmatrix}.$$

3. **Add a multiple of one row to another.**
   The determinant remains unchanged.

Example.

$$
\begin{aligned}
\det\begin{pmatrix} 0 & 2 & 8 & 0 \\ 1 & 3 & 5 & 2 \\ 2 & 1 & 0 & -3 \\ 2 & -5 & -2 & -7 \end{pmatrix} &= -\det\begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & 2 & 8 & 0 \\ 2 & 1 & 0 & -3 \\ 2 & -5 & -2 & -7 \end{pmatrix} \\
&= -\det\begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & 2 & 8 & 0 \\ 0 & -5 & -10 & -7 \\ 0 & -11 & -12 & -11 \end{pmatrix} \\
&= -2\det\begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & -5 & -10 & -7 \\ 0 & -11 & -12 & -11 \end{pmatrix} \\
&= -2\det\begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 10 & -7 \\ 0 & 0 & 32 & -11 \end{pmatrix} \\
&= -20\det\begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & -7/10 \\ 0 & 0 & 32 & -11 \end{pmatrix} \\
&= -20\det\begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & -7/10 \\ 0 & 0 & 0 & 71/5 \end{pmatrix} \\
&= -20 \cdot 1 \cdot 1 \cdot 1 \cdot \frac{71}{5} \\
&= -284.
\end{aligned}
$$

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Before we discuss some of the fun things that we can do with determinants, let's look more at the original formula we gave for determinants:

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots \pm a_{1n}\det(A_{1n}).$$

Definition. The number $\tilde{A}_{ij} = (-1)^{i+j}\det(A_{ij})$ is called the ***cofactor*** of a_{ij}. □

So we have

$$\begin{aligned}\det(A) &= a_{11}\tilde{A}_{11} + a_{12}\tilde{A}_{12} + \cdots + a_{1n}\tilde{A}_{1n} \\ &= \sum_{j=1}^{n} a_{1j}\tilde{A}_{1j}.\end{aligned}$$

This is known as the ***cofactor expansion*** in the first row. We have similar formulas for any row or column.

Proposition. *For any i from 1 to n:*

$$\begin{aligned}\det(A) &= a_{i1}\tilde{A}_{i1} + a_{i2}\tilde{A}_{i2} + \cdots + a_{in}\tilde{A}_{in} \\ &= \sum_{j=1}^{n} a_{ij}\tilde{A}_{ij}, \\ \det(A) &= a_{1i}\tilde{A}_{1i} + a_{2i}\tilde{A}_{2i} + \cdots + a_{ni}\tilde{A}_{ni} \\ &= \sum_{j=1}^{n} a_{ji}\tilde{A}_{ji}.\end{aligned}$$

$\square$

Example. Let's compute a determinant using the cofactor expansion in the second column:

$$\begin{aligned}\det\begin{pmatrix}1 & 1 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} &= -1\cdot\det\begin{pmatrix}1 & 1 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} + 5\cdot\det\begin{pmatrix}1 & 1 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} - 8\cdot\det\begin{pmatrix}1 & 1 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} \\ &= -1\cdot(-6) + 5\cdot(-12) - 8\cdot(-6) \\ &= -6\end{aligned}$$

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Our formula

$$\det(A) = a_{i1}\tilde{A}_{i1} + a_{i2}\tilde{A}_{i2} + \cdots + a_{in}\tilde{A}_{in}$$

can be rewritten

$$\det(A) = \begin{pmatrix}a_{i1} & a_{i2} & \ldots & a_{in}\end{pmatrix}\begin{pmatrix}\tilde{A}_{i1}\\ \tilde{A}_{i2}\\ \vdots\\ \tilde{A}_{in}\end{pmatrix}.$$

Notice that if the row vector $\begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{pmatrix}$ is replaced by $\begin{pmatrix} a_{j1} & a_{j2} & \dots & a_{jn} \end{pmatrix}$ for $j \neq i$, we get

$$\det(\hat{A}) = \begin{pmatrix} a_{j1} & a_{j2} & \dots & a_{jn} \end{pmatrix} \begin{pmatrix} \tilde{A}_{i1} \\ \tilde{A}_{i2} \\ \vdots \\ \tilde{A}_{in} \end{pmatrix}.$$

where $\hat{A}$ is the matrix A with row i replaced by row j. But this means in $\hat{A}$, rows i and j will be the same, and so $\det(\hat{A}) = 0$. We get

$$\begin{pmatrix} a_{j1} & a_{j2} & \dots & a_{jn} \end{pmatrix} \begin{pmatrix} \tilde{A}_{i1} \\ \tilde{A}_{i2} \\ \vdots \\ \tilde{A}_{in} \end{pmatrix} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Altogether, this gives us

$$\begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} \begin{pmatrix} \tilde{A}_{11} & \dots & \tilde{A}_{n1} \\ \vdots & & \vdots \\ \tilde{A}_{1n} & \dots & \tilde{A}_{nn} \end{pmatrix} = \begin{pmatrix} \det(A) & 0 & \dots & 0 \\ 0 & \det(A) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \det(A) \end{pmatrix} \tag{8.0}$$

$$= \det(A) \cdot I. \tag{8.0}$$

Notice that in the second matrix above, the indices are reverse from what they usually are. We'll fix this with a definition.

Definition. The ***adjoint*** of A, denoted $\operatorname{adj}(A)$, is

$$\operatorname{adj}(A) = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \dots & \tilde{A}_{1n} \\ \tilde{A}_{21} & \tilde{A}_{22} & \dots & \tilde{A}_{2n} \\ \vdots & \vdots & & \vdots \\ \tilde{A}_{n1} & \tilde{A}_{n2} & \dots & \tilde{A}_{nn} \end{pmatrix}^T.$$

□

And now (8) becomes:

Proposition.

$$A \cdot adj(A) = \det(A) \cdot I.$$

□

Corollary. *If* $\det(A) \neq 0$*, then A is invertible, and*

$$A^{-1} = \frac{1}{\det(A)} adj(A).$$

□

This is a generalization of the formula for the inverse of a 2×2 matrix. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here, $\tilde{A}_{11} = d$, $\tilde{A}_{12} = -c$, $\tilde{A}_{21} = -b$ and $\tilde{A}_{22} = a$, so

$$\operatorname{adj}(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T \tag{8.0}$$

$$= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{8.0}$$

If $\det(A) = ad - bc$ is not 0, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) \tag{8.0}$$

$$= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{8.0}$$

The previous corollary gives us a simply stated condition for a matrix to have an inverse. With the aid of the following proposition, that allows us to easily prove many things about inverses.

Proposition. *Let A and B be* $n \times n$ *matrices. Then*

$$\det(AB) = \det(A)\det(B).$$

□

We can now prove the converse of the previous proposition.

Proposition. *A square matrix A is invertible if and only if* $\det(A) \neq 0$. □

PROOF. *We have already shown one direction (the "if" part). For the other direction, we need to show that if A is invertible, then* $\det(A) \neq 0$*. If A is invertible, then* $AB = I$ *for some matrix B (namely* $B = A^{-1}$*, and so* $\det(AB) = \det(I) = 1$*. Since* $\det(AB) = \det(A)\det(B)$*, this means* $\det(A)\det(B) = 1$*. If a product is not zero, then no factor can be zero, so* $\det(A) \neq 0$*.* □

Definition. The matrix A is ***singular*** if $\det(A) = 0$. If A is not singular, it is ***non-singular***. □

So a matrix is non-singular exactly when it is invertible.

As an example of another (albeit already known) result concerning inverses that we can easily prove, we have the following:

Proposition. *Let A and B be $n \times n$ matrices. If AB is invertible, then A and B are both invertible.* □

PROOF. *If AB is invertible, then $\det(AB) \neq 0$. Since $\det(AB) = \det(A)\det(B)$, that means $\det(A)\det(B) \neq 0$. This implies that both $\det(A)$ and $\det(B)$ are both non-zero, which means that A and B are both invertible.* □

We can state the condition on when a system of equations has a unique solution in terms of determinants.

Proposition. *The system*

$$A\boldsymbol{x} = \boldsymbol{b}$$

has a unique solution exactly when $\det(A) \neq 0$. □

Determinants can also be used to find the solution.

Proposition (Cramer's Rule). *If $\det(A) \neq 0$, then the unique solution of $A\boldsymbol{x} = \boldsymbol{b}$ is*

$$\begin{aligned} x_1 &= \det(B_1)/\det(A), && (8.0)\\ x_2 &= \det(B_2)/\det(A) && (8.0)\\ &\vdots && (8.0)\\ x_n &= \det(B_n)/\det(A), && (8.0) \end{aligned}$$

where B_i is the matrix A with the ith column replaced by the column vector $\boldsymbol{b}$. □

Example. Consider the system

$$\begin{aligned} 3x &- 2y &= 5\\ 4x &- 5y &= 2. \end{aligned}$$

☑

Here,

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -5 \end{pmatrix}, \tag{8.0}$$

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{8.0}$$

$$\boldsymbol{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}. \tag{8.0}$$

Since $\det(A) = 3 \cdot (-5) - (-2) \cdot 4 = -7$, there is a unique solution. We have

$$A = \begin{pmatrix} 5 & -2 \\ 2 & -5 \end{pmatrix}, \tag{8.0}$$

$$A = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}, \tag{8.0}$$

so $\det(B_1) = -21$ and $\det(B_2) = -14$, we get

$$x = \det(B_1)/\det(A) = (-21)/(-7) = 3 \tag{8.0}$$

$$y = \det(B_2)/\det(A) = (-14)/(-7) = 2. \tag{8.0}$$

While it's nice to have a formula for the solution of a system of equations, such as Cramer's rule provides, note that this rule isn't very efficient; Gaussian elimination is typically the best way to solve a system of linear equations.