Chapter 4

Gaussian Elimination

Recall:

Definition. A ***linear equation*** in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b. \qquad \square$$

Example.

$$2x_1 + x_2 - 4x_3 + x_4 = 5. \qquad \boxtimes$$

Example.

$$x + 2y - z = 7. \qquad \boxtimes$$

Also, a ***linear system of m equations in n unknowns*** is a system of the form

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = b_2 \\
\vdots & & & & & & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = b_m
\end{array}$$

Example. The system

$$\begin{array}{rcrcrl}
x & - & 3y & + & 2z & = 5 \\
x & + & y & + & z & = 1 \\
-3x & & & + & z & = 0 \\
 & & y & - & 2x & = 5
\end{array}$$

is a system of 4 equations in 3 unknowns. ☑

Definition. A linear system is ***homogeneous*** is all the constants are 0; i.e., if it is of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = 0 \\ \vdots & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = 0 \end{array}$$

□

Definition. A ***solution*** to a system of m equations in n unknowns is a set of n numbers

$$x_1 = c_1, \quad x_2 = c_2, \quad \cdots, \quad x_n = c_n$$

which simultaneously satisfy all m equations. □

Example. The values $x = 1, y = 4, z = 2$ is a solution of

$$\begin{array}{cccccc} 2x & + & y & - & 2z & = 2 \\ 3x & + & y & + & 5z & = 17 \end{array}$$

since

$$\begin{array}{cccccc} 2 \cdot \mathbf{1} & + & \mathbf{4} & - & 2 \cdot \mathbf{2} & = 2 \\ 3 \cdot \mathbf{1} & + & \mathbf{4} & + & 5 \cdot \mathbf{2} & = 17 \end{array}$$

☑

To ***solve*** a system of equations is to find the solution set; i.e., the set of all solutions. There are three possibilities; a linear system of equations will have either

- exactly 0 solutions
- exactly 1 solution
- infinitely many solutions.

(Note that a homogeneous system always has at least one solution; namely, $x_1 = 0$, $x_2 = 0$,…,$x_n = 0$. So for a homogeneous system, this will be the only solution or infinitely many solutions.)

Here are some examples of each possibility.

Example. ***Exactly* 0 *solutions.***

$$\begin{aligned} x + y + 2z &= 4 \\ 2y + z &= 3 \\ 0 &= 2 \end{aligned}$$

☑

Example. ***Exactly* 1 *solution.***

$$\begin{aligned} x + y - 2z &= 2 \\ y - z &= 3 \\ z &= 4 \end{aligned}$$

Finding z and working our way backwards (this is called ***back-substitution***), we get

$$\begin{aligned} &z = 4 \\ &y = z + 3 = 4 + 3 = 7 \\ &x = -y + 2z + 2 = -7 + 2 \cdot 4 + 2 = 3 \end{aligned}$$

☑

Example. ***Infinitely many solutions.***

$$\begin{aligned} x - y + 2z &= 3 \\ z &= 2 \end{aligned}$$

Finding z and working our way backwards, we see that $z = 2$. We would next find y, but we can't solve for y in terms of already determined values. The variable y is called a ***free variable***. Next, $x = y - 2z + 3 = y - 2 \cdot 2 + 3 = y - 1$. So the solutions look like

$$\begin{aligned} x &= y - 1 \\ y &= y \\ z &= 2 \end{aligned}$$

This describes the solution set. ☑

These examples were given to us in a particularly nice form; we would like every system to end up like one of these.

Definition. Two systems of equations are ***equivalent*** if they have the same solution set. □

Given a system of equations, we would like to find an equivalent system in a nice form. There are three basic ways to manipulate a system to get an equivalent system:

- Switch two equations.

  Example.

  $$\begin{aligned} x + y &= 4 \\ 2x - 3y &= 5 \end{aligned}$$

  is equivalent to

  $$\begin{aligned} 2x - 3y &= 5 \\ x + y &= 4 \end{aligned}$$ ☑

- Multiply an equation by a non-zero number.

  Example.

  $$\begin{aligned} x + y &= 3 \\ 2x - y &= 5 \end{aligned}$$

  is equivalent to (multiplying the top equation by 2)

  $$\begin{aligned} 2x + 2y &= 6 \\ 2x - y &= 5 \end{aligned}$$ ☑

- Add a multiple of one equation to another.

  Example.

  $$\begin{aligned} x + y &= 3 \\ 2x + 4y &= 1 \end{aligned}$$

  is equivalent to (adding twice the first equation to the second)

  $$\begin{aligned} x + y &= 3 \\ 4x + 6y &= 7 \end{aligned}$$ ☑

Recall that a system of equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = b_2 \\ \vdots & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = b_m \end{array}$$

can be written in matrix form as $A\boldsymbol{x} = \boldsymbol{b}$, where A is the ***matrix of coefficients***

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$. We sometimes want to combine all this information into a single matrix consisting of the coefficients and constants:

$$(A|\boldsymbol{b}) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix},$$

which is simply called the matrix of the system. Note that this second matrix is the coefficient matrix augmented with the column vector of the constants of the equations, and contains all the information of the system of equations. Each row of the matrix represents an equation; the numbers in the first column represent the x_1 coefficients, the numbers in the second column represent the x_2 coefficients, etc., and the numbers in the last column represent the constants to the right of the equal signs.

Example. The system

$$\begin{array}{ccccccccc} x_1 & - & 2x_2 & + & 3x_3 & - & x_4 & = 1 \\ x_1 & & & - & 3x_3 & + & 9x_4 & = 0 \\ x_1 & + & 9x_2 & & & - & x_4 & = 3 \end{array}$$

has matrix

$$\begin{pmatrix} 1 & -2 & 3 & -1 & 1 \\ 1 & 0 & -3 & 9 & 0 \\ 1 & 9 & 0 & -1 & 3 \end{pmatrix}$$

and coefficient matrix

$$\begin{pmatrix} 1 & -2 & 3 & -1 \\ 1 & 0 & -3 & 9 \\ 1 & 9 & 0 & -1 \end{pmatrix}.$$

☑

In terms of matrices, the three operations which transform a system into an equivalent system are called the ***elementary row operations***:

- Switch any two rows.

  Example.

  $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

  Switching rows 2 and 3, we get

  $$\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix}$$

  ☑

- Multiply a row by a non-zero number.

  Example.

  $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

  Multiplying row 2 by 5, we get

  $$\begin{pmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{pmatrix}$$

  ☑

- Add a multiple of one row to another.

Example.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Adding 4 times row 2 to row 1, we get

$$\begin{pmatrix} 17 & 22 & 27 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

☑

We want to use the elementary row operations to reduce a system of equations to an equivalent system which is easier to solve. The idea is this: it is easier to solve a system of equations if there are fewer variables and equations. So given a system of m equations and n variables, we want to get rid of one of the variables from all but one of the equations. The equations which don't involve this variable will then be a system of $m-1$ equations with $n-1$ variables.

The procedure we use is called ***Gaussian Elimination***.

Given the matrix of a linear system of equations:

1. Starting with the top row, get a 1 on the left side. This can be done by getting a non-zero number there (switching rows if necessary) and dividing the row by that number. (If the first column is all 0s, go to the next column.) This 1 is called a ***leading 1***.
2. Add multiples of the row with the leading 1 to the rows below it to get 0s below the leading 1.
3. Go to the next row, repeat steps 1 and 2. Keep repeating as long as possible.

Example. Starting with the system of equations

$$\begin{array}{rcrcrcrcrl} 2x_1 & + & 4x_2 & + & 2x_3 & + & 6x_4 & + & 2x_5 & = 6 \\ x_1 & + & 2x_2 & + & x_3 & + & 3x_4 & + & 2x_5 & = 5 \\ 2x_1 & + & 4x_2 & + & x_3 & + & 4x_4 & + & x_5 & = 5 \end{array}$$

we get the (augmented) matrix

$$\begin{pmatrix} \mathbf{2} & 4 & 2 & 6 & 2 & 6 \\ 1 & 2 & 1 & 3 & 2 & 5 \\ 2 & 4 & 1 & 4 & 1 & 5 \end{pmatrix}$$

There is a 2 on the left of the first row; we will divide the first row by 2 to get

$$\begin{pmatrix} \mathbf{1} & 2 & 1 & 3 & 1 & 3 \\ 1 & 2 & 1 & 3 & 2 & 5 \\ 2 & 4 & 1 & 4 & 1 & 5 \end{pmatrix}$$

The first row now had a leading 1. Subtract 1 times row 1 from row 2 and 2 times row 1 from row 3 to get 0s below this leading 1.

$$\begin{pmatrix} \mathbf{1} & 2 & 1 & 3 & 1 & 3 \\ \mathbf{0} & 0 & 0 & 0 & 1 & 2 \\ \mathbf{0} & 0 & -1 & -2 & -1 & -1 \end{pmatrix}$$

There are now 0s below the leading 1 in the first row. Moving to the second row, we would like a non-zero term as far to the left as possible. There is a zero in the second column, but there are also 0s in the second column of all lower rows. There is a zero in the third column, but a non-zero term in the third column of the third row. To get a non-zero term as far to the left as possible in the second row, we will switch the second and third rows.

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 1 & 3 \\ 0 & 0 & \mathbf{-1} & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

We will now divide row 2 by -1 to get a leading 1 in that row.

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 1 & 3 \\ 0 & 0 & \mathbf{1} & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

There are 0s below this leading 1, so that's taken care of. Finally there is a leading 1 already in the third row, so we are done with Gaussian elimination. ☑

The specific operations used in the previous example weren't the only ones we could have used for Gaussian elimination. For example, to avoid beginning by dividing the first row by 2, we could have started by switching the first two rows to get our leading 1 in the first row.

After Gaussian elimination, a matrix will look like

$$\begin{pmatrix} 1 & \cdots & & & & \\ 0 & 0 & 1 & \cdots & & \\ 0 & 0 & 0 & 1 & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

This is called row-echelon form.

Definition. A matrix is in ***row-echelon form*** if:

1. The first non-zero entry in each row is a 1. (This 1 is called a ***leading 1***.)
2. The leading 1 in any row is further to the right than the leading 1 of any higher row.
3. Any rows with all 0s are at the bottom. □

If a matrix A can be reduced to the row-echelon matrix E using elementary row operations, we write $A \rightsquigarrow E$. Note that the row-echelon form of a matrix is not unique: it is possible to have $A \rightsquigarrow E_1$ and $A \rightsquigarrow E_2$ with $E_1 \neq E_2$.

Once a matrix is in row-echelon form, the system can be solved with ***back-substitution***:

- Start with the last equation and find the value of the last variable x_n.
- Go up to the next equation and solve for x_{n-1}, using our value for x_n. Continue.
- If the column corresponding to a variable doesn't have a leading 1, the equations don't determine a value for that variable. That variable can take on any value, and is called a ***free variable***.

Example. To finish our example, our matrix after Gaussian elimination is

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

The corresponding equations are

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 + x_5 &= 3 \\ x_3 + 2x_4 + x_5 &= 1 \\ x_5 &= 2 \end{aligned}$$

From the last equation, we get $x_5 = 2$. The variable x_4 is a free variable. From the second equation from the bottom, $x_3 = 1 - 2x_4 - x_5 = 1 - 2x_4 - 2 = -1 - 2x_4$. The variable x_2 is a free variable. From the top equation, we get $x_1 = 3 - 2x_2 - x_3 - 3x_4 - x_5 = 3 - 2x_2 - (-1 - 2x_4) - 3x_4 - 2 = 2 - 2x_2 - x_4$. ☑

Remark. The back-substitution can be done as part of the elimination process, while in matrix form, by getting 0s above the leading 1s as well as below them. This is called ***Gauss-Jordan Elimination***.

Example. ***Perform back-substitution for the matrix in the previous example.***
The matrix in row-echelon form is:

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

There are no rows above the first leading 1, so we'll look at the leading 1 in the second row. We want a 0 above that, so we'll subtract row 2 from row 1:

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Next, we want zeros above the leading 1 in the third row. We only have to take care of the second row for this example, so we'll subtract row 3 from row 2:

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

This finishes the Gauss-Jordan elimination. This matrix corresponds to the equations:

$$\begin{aligned} x_1 + 2x_2 \quad + x_4 \quad &= 2 \\ x_3 + 2x_4 \quad &= -1 \\ x_5 &= 2 \end{aligned}$$

Without any more substitution, these give us

$$\begin{aligned} x_1 &= 2 - 2x_2 - x_4 \\ x_3 &= -1 - 2x_4 \\ x_5 &= 2 \end{aligned}$$

which are in terms of the free variables x_2 and x_4. ☑

Remark. So far, all of our calculations have been exact. When approximate calculations are used, errors can accumulate the computed answer can be far from the actual answer.

Example. ***Solve the system***

$$\begin{aligned} \mathbf{.4}\boldsymbol{x} &+ \mathbf{561.6}\boldsymbol{y} = \mathbf{562} \\ \mathbf{73.03}\boldsymbol{x} &- \mathbf{43.03}\boldsymbol{y} = \mathbf{30} \end{aligned}$$

where the result of each calculation is rounded to 4 significant digits.
We can tell by inspection that $x = 1$, $y = 1$ is a solution. (It is, in fact, the only solution.) However, let's solve this with Gaussian elimination. Our matrix is

$$\begin{pmatrix} .4000 & 561.6 & 562.0 \\ 73.03 & -43.03 & 30.00 \end{pmatrix}$$

Dividing row 1 by .4000, we get

$$\begin{pmatrix} 1.000 & 1404 & 1405 \\ 73.03 & -43.03 & 30.00 \end{pmatrix}$$

Subtracting 73.03 times row 1 from row 2, we get

$$\begin{pmatrix} 1.000 & 1404 & 1405 \\ 0.000 & -43.03 - 102500 & 30.00 - 102600 \end{pmatrix}$$

or

$$\begin{pmatrix} 1.000 & 1404 & 1405 \\ 0.000 & -102500 & -102600 \end{pmatrix}$$

Dividing row 2 by -102500, we get

$$\begin{pmatrix} 1.000 & 1404 & 1405 \\ 0.000 & 1.000 & 1.001 \end{pmatrix}$$

This corresponds to the equations

$$\begin{aligned} 1.000x \;+\; 1404y &= 1405 \\ 1.000y &= 1.001 \end{aligned}$$

So our computed solution would be

$$\begin{aligned} &y = 1.001 \\ &x = 1405 - 1404y = 1405 - 1404 \cdot 1.001 = 1405 - 1405 = 0 \end{aligned}$$

The computed y value is close to the y value of the true solution, but the computer x value is not close. ☑

One way to reduce this type of error in Gaussian elimination is with ***pivotal condensation***, where each leading 1 is obtained from the largest term in the corresponding column.

Example. The original matrix from the previous example was

$$\begin{pmatrix} .4000 & 561.6 & 562.0 \\ 73.03 & -43.03 & 30.00 \end{pmatrix}$$

We start by getting a leading 1 for the first row. This 1 will be in the first column, so instead of automatically dividing the first row by .4000, we look for the largest number in the first column. This is the 73.03 in the second row, so we switch the second row with the first to get

$$\begin{pmatrix} 73.03 & -43.03 & 30.00 \\ .4000 & 561.6 & 562.0 \end{pmatrix}$$

We now get the leading 1 for the first row by dividing by a larger, rather than smaller, number. Completing the problem as before, we get the solution $x = 1.001$, $y = 1.001$. This is a much better approximate solution than before. ☑

While this is an important practical concern, in the future we will pretend that all calculations are exact. ⊠

Let's do some more examples of Gaussian elimination. We will solve a system with exactly one solution, and one with no solutions.

Example. *Solve*

$$\begin{array}{rcrcrl} \boldsymbol{x} & + & \boldsymbol{y} & + & \mathbf{2}\boldsymbol{z} & = \mathbf{9} \\ \mathbf{3}\boldsymbol{x} & + & \boldsymbol{y} & + & \mathbf{2}\boldsymbol{z} & = \mathbf{11} \\ & & \mathbf{2}\boldsymbol{y} & + & \boldsymbol{z} & = \mathbf{7} \end{array}$$

The matrix for this system is

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 3 & 1 & 2 & 11 \\ 0 & 2 & 1 & 7 \end{pmatrix}$$

The Gaussian elimination will proceed as follows:
Subtract 3 times row 1 from row 2:

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & -2 & -4 & -16 \\ 0 & 2 & 1 & 7 \end{pmatrix}$$

Divide row 2 by -2:

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 1 & 7 \end{pmatrix}$$

Subtract 2 times row 2 from row 3:

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & -3 & -9 \end{pmatrix}$$

Finally, divide row 3 by -3:

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

This is now in row-echelon form, and corresponds to the equations

$$\begin{array}{rcrcrl} x & + & y & + & 2z & = 9 \\ & & y & + & 2z & = 8 \\ & & & & z & = 3 \end{array}$$

Back substitution gives us:

$$\begin{aligned}
&z = 3\\
&y = 8 - 2z = 8 - 2 \cdot 3 = 2\\
&x = 9 - y - 2z = 9 - 2 - 2 \cdot 3 = 1
\end{aligned}$$

So the solution is

$$x = 1, \quad , y = 2, \quad z = 3.$$

☑

Example. ***Solve***

$$\begin{array}{rcrcrl}
\mathbf{2x} & + & \mathbf{2y} & + & \mathbf{4z} & = \mathbf{12}\\
\mathbf{x} & + & \mathbf{y} & + & \mathbf{z} & = \mathbf{8}\\
\mathbf{3x} & + & \mathbf{3y} & + & \mathbf{2z} & = \mathbf{19}
\end{array}$$

Our matrix is

$$\begin{pmatrix} 2 & 2 & 4 & 12\\ 1 & 1 & 1 & 8\\ 3 & 3 & 2 & 19 \end{pmatrix}$$

Dividing row 1 by 2, we get

$$\begin{pmatrix} 1 & 1 & 2 & 6\\ 1 & 1 & 1 & 8\\ 3 & 3 & 2 & 19 \end{pmatrix}$$

Subtracting row 1 from row 2

$$\begin{pmatrix} 1 & 1 & 2 & 6\\ 0 & 0 & -1 & 2\\ 3 & 3 & 2 & 19 \end{pmatrix}$$

and then subtracting 3 times row 1 from row 3, we get

$$\begin{pmatrix} 1 & 1 & 2 & 6\\ 0 & 0 & -1 & 2\\ 0 & 0 & -4 & 1 \end{pmatrix}$$

Dividing row 2 by -1, we get

$$\begin{pmatrix} 1 & 1 & 2 & 6\\ 0 & 0 & 1 & -2\\ 0 & 0 & -4 & 1 \end{pmatrix}$$

Finally, we add 4 times row 2 to row 3

$$\begin{pmatrix} 1 & 1 & 2 & 6 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

While this isn't quite in row-echelon form (why not?), at this point we can see that the last row corresponds to the equation $0 = -7$, which is never true. So this system has no solutions. ☑