Matrix Algebra
Orthonormal Bases
Homework 13

1. Which of the following sets are orthogonal? Which are orthonormal?

   a) $\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\-1\\0\end{pmatrix}\right\}$

   b) $\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\-1\\2\end{pmatrix}\right\}$

   c) $\left\{\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix},\begin{pmatrix}-1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{pmatrix}\right\}$

   d) $\left\{\begin{pmatrix}1/2\\-1/2\\1/2\\1/2\end{pmatrix},\begin{pmatrix}1/(2\sqrt{3})\\-1/(2\sqrt{3})\\-3/(2\sqrt{3})\\1/(2\sqrt{3})\end{pmatrix},\begin{pmatrix}-1/\sqrt{3}\\0\\0\\1/\sqrt{3}\end{pmatrix}\right\}$

   e) $\left\{\begin{pmatrix}12\\5\\9\\-4\end{pmatrix},\begin{pmatrix}1\\1\\-1\\2\end{pmatrix},\begin{pmatrix}10\\-13\\-7\\-2\end{pmatrix}\right\}$

   Answer: (b), (c), (d) and (e) are orthogonal; (c) is also orthonormal.

2. For each of the following, $\mathcal{B}$ is an orthogonal set. Find the coefficients $c_1, c_2, \ldots$, in the expansions $\boldsymbol{x} = c_1\boldsymbol{w}_1 + c_2\boldsymbol{w}_2 + \ldots$

   a) $\mathcal{B} = \left\{\begin{pmatrix}2\\-6\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix}\right\}$, $\boldsymbol{x} = \begin{pmatrix}1\\2\end{pmatrix}$

   Answer: $\boldsymbol{x} = \frac{-1}{4}\cdot\begin{pmatrix}2\\-6\end{pmatrix} + \frac{1}{2}\begin{pmatrix}3\\1\end{pmatrix}$, so $c_1 = \frac{-1}{4}$ and $c_2 = \frac{1}{2}$

   b) $\mathcal{B} = \left\{\begin{pmatrix}2\\2\\1\end{pmatrix},\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\-4\end{pmatrix}\right\}$, $\boldsymbol{x} = \begin{pmatrix}1\\2\\3\end{pmatrix}$

Answer: $\boldsymbol{x} = 1 \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$, so $c_1 = 1, c_2 = \frac{1}{2}$ and $c_3 = -\frac{1}{2}$.

c) $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix}$

Answer: $\boldsymbol{x} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, so $c_1 = \frac{1}{2}, c_2 = 0, c_3 = -\frac{1}{2}$ and $c_4 = 2$.

3. For each $\mathcal{B}$ in the previous problem, find an orthonormal basis.

Answer:

(b) $\mathcal{B}' = \left\{ \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{18} \\ 1/\sqrt{18} \\ -4/\sqrt{18} \end{pmatrix} \right\}$

(c) $\mathcal{B}' = \left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right\}$