Chapter 5

Elementary Matrices

The *identity* matrix will play an important role in our analyses; starting now.

Definition. The matrix

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called the $n \times n$ ***identity matrix***. When the size is implicit, the identity matrix is also simply written I. □

The *identity* in identity matrix stands for multiplicative identity. If I is an identity matrix, then whenever IA is defined, $IA = A$, and whenever AI is defined, $AI = A$.

Example.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

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To keep track of the steps in Gaussian elimination, we can use elementary matrices.

Definition. An ***elementary matrix*** is the result of apply an elementary row operation to an identity matrix. □

Example. Starting with

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

add three times row 1 to row 3. The result,

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix},$$

is an elementary matrix. ☑

Let E be an elementary matrix. If EA is defined for some matrix A, then EA is equal to the matrix A after the corresponding elementary row operation has been performed on A.

Example. Let

$$A = \begin{pmatrix} 1 & 2 \\ 5 & 7 \\ 9 & 10 \end{pmatrix}.$$

The 3×3 matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

corresponds to taking the identity matrix and adding three times row 1 to row 3. So EA is the result of taking A and adding three times row 1 to row 3:

$$EA = \begin{pmatrix} 1 & 2 \\ 5 & 7 \\ 9 + 3 \cdot 1 & 10 + 3 \cdot 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 5 & 7 \\ 12 & 16 \end{pmatrix}.$$

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So performing an elementary row operation is equivalent to multiplying (on the left) by an elementary matrix. Performing a sequence of elementary row operations is the equivalent to multiplying by a sequence of elementary matrices:

$$E_n E_{n-1} \dots E_1 A = EA,$$

or by a single matrix E, where

$$E = E_n E_{n-1} \dots E_1.$$

Note that

$$E = EI = E_n E_{n-1} \dots E_1 EI$$

is merely the result of applying the elementary row operations to the identity matrix. We can the find E by performing the same row operations to I that we do to A. This is done by augmenting A with the identity matrix (writing I to the right of A)

$$(A|I)$$

and performing the row operations on the augmented matrix. We will use the following proposition often.

Proposition. *Let a matrix A be augmented with the identity matrix:*

$$(A|I)\,.$$

If elementary row operations are performed on this augmented matrix, resulting in

$$(B|E)\,,$$

then

$$B = EA. \qquad \square$$

Example. ***Let***

$$A = \begin{pmatrix} \mathbf{0} & \mathbf{2} & \mathbf{2} & \mathbf{7} \\ \mathbf{4} & \mathbf{9} & \mathbf{5} & \mathbf{11} \\ \mathbf{3} & \mathbf{6} & \mathbf{3} & \mathbf{6} \end{pmatrix}.$$

Find a matrix E such that EA is in row echelon form.
We'll begin by augmenting A with the identity:

$$\left(\begin{array}{cccc|ccc} 0 & 2 & 2 & 7 & 1 & 0 & 0 \\ 4 & 9 & 5 & 11 & 0 & 1 & 0 \\ 3 & 6 & 3 & 6 & 0 & 0 & 1 \end{array}\right)$$

We'll switch rows 1 and 3:

$$\left(\begin{array}{cccc|ccc} 3 & 6 & 3 & 6 & 0 & 0 & 1 \\ 4 & 9 & 5 & 11 & 0 & 1 & 0 \\ 0 & 2 & 2 & 7 & 1 & 0 & 0 \end{array}\right)$$

and divide row 1 by 3:

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 2 & 0 & 0 & 1/3 \\ 4 & 9 & 5 & 11 & 0 & 1 & 0 \\ 0 & 2 & 2 & 7 & 1 & 0 & 0 \end{array}\right).$$

Next, we can subtract 4 times row 1 from row 2:

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 2 & 0 & 0 & 1/3 \\ 0 & 1 & 1 & 3 & 0 & 1 & -4/3 \\ 0 & 2 & 2 & 7 & 1 & 0 & 0 \end{array}\right).$$

Finally, we'll subtract 2 times row 2 from row 3:

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 2 & 0 & 0 & 1/3 \\ 0 & 1 & 1 & 3 & 0 & 1 & -4/3 \\ 0 & 0 & 0 & 1 & 1 & -2 & 8/3 \end{array}\right).$$

The matrix is now in row echelon form; the beginning part is a row echelon form of A:

$$EA = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where E is the second part of the augmented matrix:

$$E = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 1 & -4/3 \\ 1 & -2 & 8/3 \end{pmatrix}.$$

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