Section 06
More differentiation rules

If we want to take the derivative of the sum of functions, we simply take the derivative of each term separately. If, for example, $g(x) = x^2$ and $h(x) = x^3$, the the derivative of $f(x) = g(x) + h(x)$ is $f'(x) = g'(x) + h'(x) = 2x + 3x^2$. The derivative of a product, alas, isn't so simple. If $f(x) = g(x)h(x) = x^2x^3 = x^5$, then $f'(x)$ is *not* $g'(x)h'(x)$. We can compute $f'(x) = 5x^4$ and $g'(x)h'(x) = 2x3x^2 = 6x^3$. Not the same.

So what is the derivative of a product, $f(x) = g(x)h(x)$? Suppose we let $y = f(x)$, $u = g(x)$ and $v = h(x)$. If the value of x is changed by an amount Δx, to a new value of $x + \Delta x$, then we will get new values of y, u and v, namely $y + \Delta y$, $u + \Delta u$ and $v + \Delta v$. The amount that y changes will be

$$\begin{aligned}\Delta y &= (y + \Delta y) - y \\ &= (u + \Delta u)(v + \Delta v) - uv \\ &= uv + u\Delta v + v\Delta u + \Delta u\Delta v - uv \\ &= u\Delta v + v\Delta u + \Delta u\Delta v\end{aligned}$$

and so the rate of change of y with respect to x is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{u\Delta v + v\Delta u + \Delta u\Delta v}{\Delta x} \\ &= u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.\end{aligned}$$

For small values of Δx, we have

$$\begin{aligned}&\frac{\Delta v}{\Delta x} \text{ is approximately } \frac{dv}{dx} = h'(x) \\ &\frac{\Delta u}{\Delta x} \text{ is approximately } \frac{du}{dx} = g'(x) \\ &\Delta u \text{ is approximately } 0.\end{aligned}$$

So $\frac{\Delta y}{\Delta x}$ is approximately $u\frac{dv}{dx} + v\frac{du}{dx} + 0\frac{dv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, and so

$$f'(x) = g(x)h'(x) + h(x)g'(x).[1]$$

[1]We can look at this another way. Consider $f(x) = g(x)h(x)$, and we want $f'(x)$. Well,

Theorem (Product Rule)
Let $f(x) = g(x)h(x)$. Then $f'(x) = g'(x)h(x) + g(x)h'(x)$.

In words, the product rule states that to take the derivative of a product, you take the derivative of each factor, one at a time (and add the results).

Testing the product rule on the above example ($g(x) = x^2$, $h(x) = x^3$), we get $f'(x) = g'(x)h(x) + g(x)h'(x) = 2xx^3 + x^2 3x^2 = 2x^4 + 3x^4 = 5x^4$, as it should.

Example
Let $f(x) = x\sin(x)$. Then

$$\begin{aligned} f'(x) &= \left(\frac{d}{dx}x\right)\sin(x) + x\left(\frac{d}{dx}\sin(x)\right) \\ &= 1 \cdot \sin(x) + x\cos(x) \\ &= \sin(x) + x\cos(x) \end{aligned}$$

Example
Let $f(x) = x^2 e^x$. Then

$$\begin{aligned} f'(x) &= \left(\frac{d}{dx}x^2\right)e^x + x^2\left(\frac{d}{dx}e^x\right) \\ &= 2xe^x + x^2 e^x \end{aligned}$$

Example
Let $f(x) = e^x\sin(x) + 2x - 1$. Find the tangent line to the graph of f at $x = 0$.

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the derivative of 2 times $h(x)$ is $2h'(x)$, the derivative of 3 times $h(x)$ is $3h'(x)$, etc. So we might expect the derivative of $g(x)$ times $h(x)$ to be $g(x)h'(x)$. We'd be partially right, but this only takes into account that $h(x)$ is changing. To take into account that $g(x)$ is also changing, we add the counterpart to $g(x)h'(x)$, namely $g'(x)h(x)$. We get the product rule.

The tangent line will go through the point $(0, f(0)) = (0, -1)$ and have slope $f'(0)$. Since

$$\begin{aligned} f'(x) &= \left(\frac{d}{dx}e^x\right)\sin(x) + e^x\left(\frac{d}{dx}\sin(x)\right) + 2\cdot 1 - 0 \\ &= e^x\sin(x) + e^x\cos(x) + 2 \end{aligned}$$

the slope will be $m = f'(0) = 3$. The tangent line will then be

$$y - (-1) = 3(x - 0)$$

or

$$y = 3x - 1.$$

Corresponding to the product rule for multiplication is the quotient rule for division.

Theorem (Quotient Rule)
Let $f(x) = g(x)/h(x)$. Then

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}.$$

Note that the numerator of the quotient rule looks like the product rule (with a minus sign tossed in).[2] As we shall later see, that isn't a coincidence.

Example

[2] A mnemonic you might come across is the following: If you think of the quotient as *hi*/*ho* (*hi* for the top since it's higher than the bottom, I suppose, and *ho* for the bottom since, well, it rhymes with low, I guess) and use D to denote differentiation, then the product rule says that the derivative of *hi*/*ho* is (*ho*D*hi* - *hi*D*ho*)/ (*ho ho*). Whether or not this mnemonic is useful, it is at least fun to say.

Let $f(x) = \frac{x^2}{x^2+1}$. Then

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(x^2)(x^2+1) - x^2\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\ &= \frac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} \\ &= \frac{2x^3 + 2x - 2x^3}{(x^2+1)^2} \\ &= \frac{2x}{(x^2+1)^2}. \end{aligned}$$

Example
Find the derivative of $f(x) = \tan(x)$.

Since $\tan(x) = \sin(x)/\cos(x)$, we can use the quotient rule to find this derivative. We get

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(\sin(x))(\cos(x)) - \sin(x)\frac{d}{dx}(\cos(x))}{(\cos(x))^2} \\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} = \sec^2(x) \end{aligned}$$

Next, to get reasonably complicated functions, functions that might be useful for modeling real situations, we start with the basic functions (polynomials, trig functions, exponentials and logarithms). But it isn't enough to add, subtract, multiply and divide. We must also combine them with composition.

You should already be familiar with composition, but let's do a quick review. The ***composition*** of two functions g and h, denoted $g \circ h$, is given by $f(x) = g(h(x))$. It is useful to think of g and h as depending on different variables; g a function of u with values $g(u)$, h a function of x with values $h(x)$. The composition is then simply $g(u)$ where $u = h(x)$; i.e., the variable u in the formula for g is simply replaced by $h(x)$.

Example
Let $g(u) = \sin(u)$ and $h(x) = x^2 + 1$. The the composition $g \circ h$ is given by $g(h(x)) = \sin(h(x)) = \sin(x^2 + 1)$.

In the composition $g(h(x))$, the function g will be called the *outside* function and h will be called the *inside* function. Given a composition, we need to be able to recognize the outside and inside functions.

Example
Let $f(x) = \cos(\sqrt{x})$. In this composition, the outside function is $g(u) = \cos(u)$ and the inside function is $h(x) = \sqrt{x}$.

If we let $y = g(u)$, then y has a dual nature — we can consider y to be a function of u or, since $y = g(u) = g(h(x))$, we can consider y to be a function of x. In the above example, if $y = g(u)$, we can think of y as either representing $\sin(u)$ or $\sin(x^2 + 1)$.

Since this is a calculus class, we have to ask ourselves: if y is the composition $y = g(u) = g(h(x))$, what is the derivative of y as a function of x? Let's consider what happens to y if x changes by a value of Δx. Then $u = h(x)$ will also change, by an amount Δu, and $y = g(u) = g(h(x))$ will change by an amount Δy. We can now consider the rate of change of y with respect to u, which is $\frac{\Delta y}{\Delta u}$, as well as the rate of change of u with respect to x, which is $\frac{\Delta y}{\Delta x}$. Algebraic manipulation gives us a connection:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}.$$

This should make sense: if y changes 2 times as fast as u and u changes 3 times as fast as x, then y should change $2 \cdot 3 = 6$ times as fast as x. Relative rates of change are multiplied. The result for instantaneous rates of change is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

This rule behaves as if the expressions are fractions: cancelling the dus on the right side of the equation gives the left side.

If $y = g(h(x)) = g(u)$ where $u = h(x)$, then $dy/du = g'(u) = g'(h(x))$ and $du/dx = h'(x)$. This gives us the standard form of the differentiation rule, called the ***chain rule***.

Example

Let $F(x) = \sin(2x+1)$. We can write $y = F(x)$ as $y = F(x) = \sin(2x+1) = \sin(u)$ where $u = 2x+1$. So $dy/du = \cos(u)$ and $du/dx = 2$, so

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \cos(u) \cdot 2 \\ &= 2\cos(2x+1)\end{aligned}$$

Theorem (Chain Rule)

Let $f(x) = g(h(x))$. Then $f'(x) = g'(h(x)) \cdot h'(x)$.

Note that both g and g' are evaluated at $h(x)$, both h and h' are evaluated at x. Let's do an example carefully.

Example

Let $f(x) = \cos(x^3+1)$. Then the outside function is $g(u) = \cos(u)$ and the inside function is $h(x) = x^3+1$. Then $g'(u) = -\sin(u)$, so $g'(h(x)) = -\sin(x^3+1)$, and $h'(x) = 3x^2$. So

$$f'(x) = g'(h(x))h'(x) = -\sin(x^3+1)(3x^2).$$

Warning:

A common mistake when applying the chain rule is to leave off the factor of $h'(x)$. In the above example, this would be letting $f'(x) = -\sin(x^3+1)$. (WRONG!) Another common mistake is to evaluate g' at $h'(x)$. In the above example, this would be $f'(x) = -\sin(3x^2)$ (WRONG!).

After a little practice, using the chain rule should become fairly natural and it isn't necessary to right out all the steps.

Example

Let $f(x) = \sin(2x^2)$. Then $f'(x) = \cos(2x^2) \cdot \frac{d}{dx}(2x^2) = \cos(2x^2) \cdot (4x)$.

Example
Let $f(x) = e^{\sin(x)}$. Then $f'(x) = e^{\sin(x)} \cos(x)$.

One important special case of the chain rule is when the outside function is a power function, $g(u) = u^n$. This special case is called the ***power rule***.

Theorem (Power rule)
Let $f(x) = (h(x))^n$. Then

$$f'(x) = n(h(x))^{n-1} h'(x).$$

Example
Let $f(x) = \sqrt{\sin(x)} = (\sin(x))^{1/2}$. Then

$$f'(x) = \frac{1}{2}(\sin(x))^{-1/2} \cos(x).$$

The chain rule can naturally be used with other rules.

Example
Let $f(x) = x \sin(x^2)$. Then

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x) \sin(x^2) + x \frac{d}{dx}(\sin(x^2)) \\ &= 1 \cdot \sin(x^2) + x \cdot \cos(x^2) \cdot 2x \\ &= \sin(x^2) + 2x^2 \cos(x^2). \end{aligned}$$

The quotient rule is a combination of the product rule and power rule. If $f(x) = g(x)/h(x) = g(x)(h(x))^{-1}$, then

$$\begin{aligned} f'(x) &= \frac{d}{dx}(g(x))(h(x))^{-1} + g(x) \frac{d}{dx} h(x)^{-1} \\ &= g'(x)(h(x))^{-1} + g(x)(-1)(h(x))^{-2} h'(x). \end{aligned}$$

Putting this back in terms of fractions, we get

$$
\begin{aligned}
f'(x) &= \frac{g'(x)}{h(x)} - \frac{g(x)h'(x)}{h(x)^2} \\
&= \frac{g'(x)h(x)}{h(x)^2} - \frac{g(x)h'(x)}{h(x)^2} \\
&= \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}.
\end{aligned}
$$

Some important special cases are when the inside function is a constant multiple of x.

Proposition
Let k be a constant.

- *Let $f(x) = \sin(kx)$. Then $f'(x) = k\cos(kx)$.*
- *Let $f(x) = \cos(kx)$. Then $f'(x) = -k\sin(kx)$.*
- *Let $f(x) = e^{kx}$. Then $f'(x) = ke^{kx}$.*

Example
Let $f(x) = \sin(4x)$. Then $f'(x) = 4\cos(4x)$.
