Section 2.5
Limits

In the previous section, we discussed the problems with finding the rate of change of a function at a point. Let's consider a concrete example. Let $f(x) = x^2$. The average rate of change of f from $x = 1$ to $x = 2$ will be $(f(2) - f(1))/(2 - 1) = 3$. More generally, the average rate of change of f from $x = 1$ to some unspecified value of x will be $(f(x) - f(1))/(x - 1) = (x^2 - 1)/(x - 1)$. If we are interested in this rate when $x = 1$, we are out of luck, because it is undefined. We may then wish to examine the rate for x close to 1 (but not, of course, equal to 1).

Let $R(x) = \frac{x^2-1}{x-1}$. While $R(x)$ is undefined for $x = 1$, we can still discuss the behavior of $R(x)$ for x near 1 (but not equal to 1). Note that

$$R(1.1) = 2.1$$
$$R(0.99) = 1.99$$
$$R(1.001) = 2.001$$

and so it appears that if x is close to 1 (but not equal to 1) then $R(x)$ will be close to 2. This is in fact the case, and gives us useful information about the behavior of $R(x)$ near $x = 1$.

Definition
Given a function R and a point a, we say that L is the ***limit*** of $R(x)$ as x approaches a if: whenever x is close to a, but not equal to a, $R(x)$ will be close to L. This is written

$$\lim_{x \to a} R(x) = L$$

When $\lim_{x \to a} R(x) = L$, we also say that $R(x)$ approaches L as x approaches a and write

$$R(x) \to L \text{ as } x \to a.$$

Example
Let $R(x) = \frac{x^2+x-6}{x-2}$. Then $R(2)$ is undefined. However, note that

$$R(2.1) = 5.1$$
$$R(1.99) = 4.99$$
$$R(2.001) = 5.001$$

and so it appears that if x is close to 2 (but not equal to 2) then $R(x)$ will be close to 5. This is in fact the case, and we have

$$\lim_{x \to 2} R(x) = 5.$$

This is an accurate, but not very precise, definition. For example "close" is not a precise term. We won't be worrying about being more precise. A less precise, but more dynamic, definition is that $\lim_{x\to a} f(x) = L$ means that as x gets closer and closer to a ($x \neq a$), $f(x)$ will get closer and closer to L.

In general, there's no reason why, for a given function R and a given number a, the limit $\lim_{x\to a} R(x)$ has to exist. It's perfectly possible that for x close to a, $R(x)$ doesn't stay close to any particular number. This is, in some sense, the normal situation. However, we will deal with nice functions and we won't worry about limits that don't exist.

Example
If x is close to 3, then $x + 2$ will be close to $3 + 2 = 5$. So $\lim_{x\to 3}(x + 2) = 5$.

The definition of $\lim_{x\to a} R(x)$ purposely ignores the value $R(a)$; $R(a)$ is irrelevant as far as the limit is concerned. This is because in important cases (such as when we define the instantaneous rate of change), the limit is used to describe the behavior of a function near a point where it is undefined. In the above example, though, the limit as x approaches 3 is equal to the expression $(x + 2)$ when x is replaced by 3. In many simple limits, that is how the limit ends up being computed — by evaluating an expression — but it's important to remember that the limit is not defined to be the value of the expression, it's only *sometimes* computed that way.

When a function f is nice enough that $\lim_{x\to a} f(x) = f(a)$, we say that f is ***continuous*** at a. Most (not all) functions that are given by simple formulas are continuous; in particular, polynomial functions are continuous everywhere and rational functions are continuous wherever they're defined.

Example
Since $f(x) = 2x^2 + 3x - 5$ is a polynomial,

$$\begin{aligned}\lim_{x\to 3} f(x) &= f(3) \\ &= 2 \cdot 3^2 + 3 \cdot 3 - 5 \\ &= 22.\end{aligned}$$

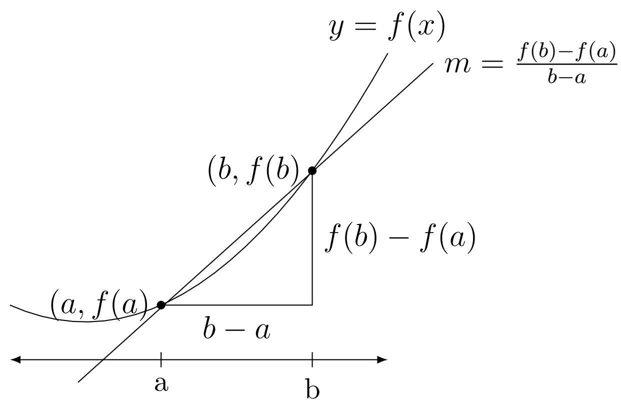

Figure 1: A continuous function on an interval and a discontinous function on an interval.

Example
Since $f(x) = (x^2 - 2x + 3)/(x - 3)$ is rational and $f(2)$ is defined,

$$\begin{aligned}\lim_{x\to 2} f(x) &= f(2)\\ &= \frac{2^2 - 2\cdot 2 + 3}{2-3}\\ &= \frac{3}{-1}\\ &= -3\end{aligned}$$

Since $f(x) = (x^2 - 2x + 3)/(x - 3)$ is not defined for $x = 3$, $\lim_{x\to 3} f(x)$ cannot be equal to $f(3)$. The limit $\lim_{x\to 3} f(x)$ may or may not exist, but all we can say right now is that it isn't the same as $f(3)$.

A function is said to be continuous on an interval if it is continuous at every point in the interval. Informally, a function is continuous on an interval if the graph of the function can be drawn on a sheet of paper without lifting the writing utensil off of the paper. (See figure **??**.)

Too often people get the idea that $\lim_{x\to a} R(x)$ is always $R(a)$ and if $R(a)$ is undefined then the limit doesn't exist. This of course is not the case; as mentioned previously, for some of the more interesting situations $\lim_{x\to a} R(x)$

has an important value while $R(a)$ is undefined. In this case, it just takes a little more work to find the limit (if it exists).

Example

Let $R(x) = \frac{x^2+x-2}{x-1}$. Then $R(1)$ is undefined, but we can still find $\lim_{x\to 1} R(x)$. We are interested in what $R(x)$ looks like if x is close to 1. Note that $R(x) = \frac{(x-1)(x+2)}{(x-1)} = x + 2$ everywhere *except* $x = 1$. For the limit as $x \to 1$, we don't care about $x = 1$, so

$$\lim_{x\to 1} R(x) = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1}(x + 2)$$

We have now turned the limit into something that we can evaluate. Since $\lim_{x\to 1}(x + 2) = 3$, we get

$$\lim_{x\to 1} R(x) = \lim_{x\to 1}(x + 2) = 3.$$

In the above example, it is important to note that $R(1)$ is undefined and that $R(x)$ is never equal to 3. The value of the *limit*, not the function, is equal to 3. It is also worth noting that the expressions $(x^2 + x - 2)/(x - 1)$ and $x + 2$ are not equivalent (one is defined for $x = 1$, one is not), but they have the same limit as x approaches 1.

Many results from Calculus become complicated when we are possibly dealing with limits that don't exist or functions which are not continuous. In order to focus on the ideas and avoid any extreme cases, we will only work with functions which are continuous and limits which exist.