Section 8
Graphs

Functions are often used to represent things: $p(t)$ might represent the position of an object at time t, for example, or $P(x)$ might represent the profit a company makes if it charges $\$x$ dollars per unit.

Depending on the interpretation of the function, the following information might be useful:

- Where is the function increasing?
  (This might tell you when an object is moving away from you, or when increasing the price of a product would increase the profits.)

- Where is the function decreasing?
  (This might tell you when an object is moving toward you, or when increasing the price of a product would decrease the profits.)

This information can be read easily, if imprecisely, from the graph of a function.

Recall, a function f is ***increasing*** over an interval if the values $f(x)$ are increasing as the variable x increases. Graphically, this means that the graph $y = f(x)$ goes up as it goes toward the right. (See figure 1.) Also, a

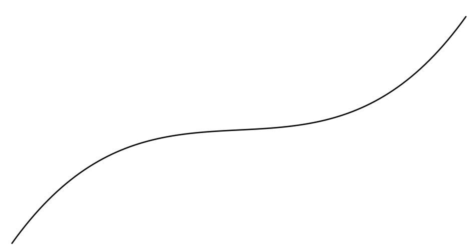
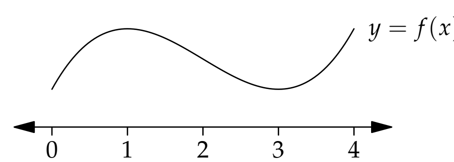


Figure 1: The graph of an increasing function.

function f is ***decreasing*** over an interval if the values $f(x)$ are decreasing as the variable x increases. Graphically, this means that the graph $y = f(x)$ goes down as it goes toward the right. (See figure 2.)

Example

Let f be the function with the following graph.



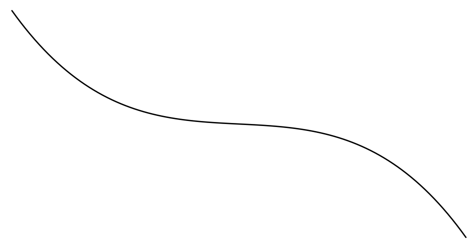


Figure 2: The graph of a decreasing function.

Then f is increasing for x from 0 to 1, f is decreasing for x from 1 to 3, and f is increasing again for x from 3 to 4.

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While we can get a pretty good idea of the behavior of a function by looking at its graph, we would like to get the information precisely. This is where the Calculus comes in. Recall that near $x = a$, the graph of f is very close to the tangent line, which has slope $f'(a)$. Since a line with positive slope is increasing, if $f'(a) > 0$ then near $x = a$ the graph is very close to an increasing line and so the graph itself must be increasing. (See figure 3.) Similarly, if $f'(a) < 0$, then near $x = a$, the graph is close to a decreasing

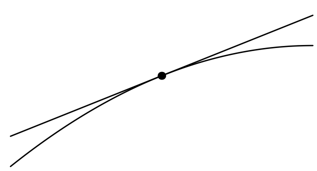

Figure 3: A graph with an increasing tangent line.

line and must be decreasing itself. (See figure 4.) These statements can be

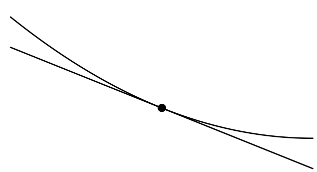

Figure 4: A graph with a decreasing tangent line.

shown to follow from the definition of the derivative, but we won't worry about that. We have the following.

Proposition

If $f'(a) > 0$, then f is increasing near $x = a$.
If $f'(a) < 0$, then f is decreasing near $x = a$.

$\square$

Example
Let $f(x) = x^3 - 3x^2 + 1$. Then $f'(x) = 3x^2 - 6x$. Since $f'(1) = 3 \cdot 1^2 - 6 \cdot 1 = 3 - 6 = -3 < 0$, we know that f is decreasing near 1. Since $f'(3) = 3 \cdot 3^2 - 6 \cdot 3 = 27 - 18 = 9 > 0$, we know that f is increasing near 3. (See figure 5.)

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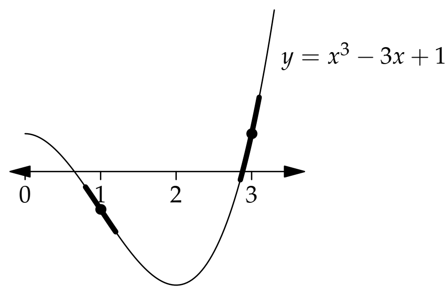


Figure 5: Decreasing and increasing parts of a graph.

The second derivative of a function also gives us information about the graph. Suppose that $f''(x)$ is positive on an interval. Since $f''(x)$ is the derivative of $f'(x)$, this just means that the derivative of $f'(x)$ is positive on the interval, which means that $f'(x)$ is increasing. But what does it mean for $f'(x)$ to be increasing? This has nothing to do with whether or not $f(x)$ is increasing. Since the values of $f'(x)$ give the slope of the graph at different points, it means that the slope of the graph is increasing. (See figure 6.) In

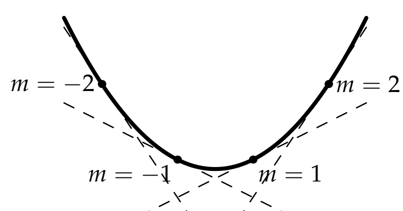
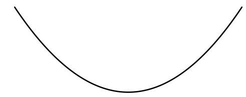
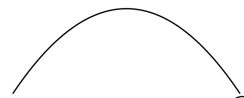


Figure 6: Increasing Slopes.

this case, the graph is said to be ***concave up***. We have something similar for the case when $f''(x) < 0$.

Definition
The graph of a function f is ***concave up*** on an interval if $f''(x) > 0$ on the interval. In this case, the graph will look like 

The graph is ***concave down*** on an interval if $f''(x) < 0$ on the interval. In this case, the graph will look like 

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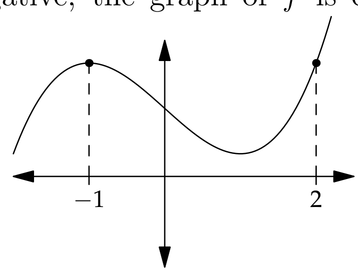
Remark
If $f''(a) > 0$, then the graph of f will be concave up near $x = a$. Similarly if $f''(a) < 0$.

$\triangle$

Example
Let $f(x) = x^3 - 3x + 2$. Then $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. Since $f'(2) = 12$ is positive, the graph of f is concave up near $x = 2$. Since $f'(-1) = -6$ is negative, the graph of f is concave down near $x = -1$.



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