Leontief Input-Output Systems

According to the Heckscher-Ohlin theory, a country will export the product that uses its abundant resources the most. There was no reason to doubt this seemingly common sensical theory, but Wassily Leontief used information about 50 types of US industries in 1947 to conclude that although the US was the most capital heavy country in the world, it nonetheless exported labor-intensive products and imported capital-intensive products. This counter-intuitive result is known as the ***Leontief Paradox***. In his investigations, Leontief developed techniques for investigating the inputs and outputs of industries in a system.

Let's look at a simple system, which we will refer to in examples throughout this section. Suppose a community has three industries: company G produces usable gas, company W produces clean water, and company E produces electricity. For the community to make money, the companies must sell their products to sources outside the community. However, each company in the community needs the products of the other companies in the communities, so they have to sell their products to each other before they can sell to outside sources.

More generally, suppose a community has n industries – call them I_1, $I_2, \ldots, I_n$ – producing distinct products. We can assume that the products are measured by a common unit, such as dollars. These companies will sell their products to each other as well to outside the community.

Consider industry I_i and the product that it makes. Some of it will be sold to the other companies — for each j, company I_j will use some of it, suppose for each unit of product that company I_j produces it needs c_{ij} units of the product I_i makes. Some of company I_i's product will be sold to outside sources — let d_i be the amount that is sold to outside sources. We then have

$$x_i = c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{in}x_n + d_i.$$

Example. Considering companies G (company 1), W (company 2) and E (company 3) again, suppose

- to produce \$1 of gas, it takes
  - \$.20 worth of gas
  - \$.10 worth of water
  - \$.15 worth of electricity.
- to produce \$1 of water, it takes
  - \$.40 worth of gas
  - \$.10 worth of water
  - \$.20 worth of electricity.
- to produce \$1 of electricity, it takes
  - \$.50 worth of gas
  - \$.10 worth of water
  - \$.10 worth of electricity.

We then have

$$\begin{aligned} x_1 &= .20x_1 + .40x_2 + .50x_3 + d_1 \\ x_2 &= .10x_1 + .10x_2 + .10x_3 + d_2 \\ x_3 &= .15x_1 + .20x_2 + .10x_3 + d_3 \end{aligned}$$

In general, we will have the system of equations

$$\begin{aligned} x_1 &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_1 \\ x_2 &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_2 \\ \vdots\ &\ \vdots \\ x_n &= c_{n1}x_1 + c_{n2}x_2 + \cdots + c_{nn}x_n + d_n \end{aligned}$$

This can be written in the form

$$\boldsymbol{x} = C\boldsymbol{x} + \boldsymbol{d},$$

where

- $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is the ***output vector***

- $C = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \dots & c_{nn} \end{pmatrix}$ is the ***input-output matrix***
- $\boldsymbol{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$ is the ***external demand vector***.

We can use the formulation

$$\boldsymbol{x} = C\boldsymbol{x} + \boldsymbol{d}$$

to answer various questions, such as

- if the outside demand is positive ($d_i > 0$ for all i), are all $x_i \geq 0$?
- can the community produce enough to fill internal needs as well as any outside demands?

Example. In our example, the input-output matrix is

$$\begin{pmatrix} .20 & .40 & .50 \\ .10 & .10 & .10 \\ .15 & .20 & .10 \end{pmatrix} \tag{7.1}$$

Note that the information we're given about each product determines a column of the matrix. ☑

As a straightforward example, suppose we know the external demand for the products a community makes. As long as we know the input-output matrix, we can find the amount each company needs to produce:

$$\begin{aligned} \boldsymbol{x} &= C\boldsymbol{x} + \boldsymbol{d} \\ \boldsymbol{x} - C\boldsymbol{x} &= \boldsymbol{d} \\ I\boldsymbol{x} - C\boldsymbol{x} &= \boldsymbol{d} \\ (I - C)\boldsymbol{x} &= \boldsymbol{d} \\ \boldsymbol{x} &= (I - C)^{-1}\boldsymbol{d} \end{aligned}$$

Example. Suppose, in our example, the outside demands are for \$100 of gas, \$200 of water, and \$150 of electricity, so

$$\boldsymbol{d} = \begin{pmatrix} 100 \\ 200 \\ 150 \end{pmatrix}.$$

Since the input-output matrix is

$$C = \begin{pmatrix} .20 & .40 & .50 \\ .10 & .10 & .10 \\ .15 & .20 & .10 \end{pmatrix},$$

we get

$$I - C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} .20 & .40 & .50 \\ .10 & .10 & .10 \\ .15 & .20 & .10 \end{pmatrix} = \begin{pmatrix} .80 & -.40 & -.50 \\ -.10 & .90 & -.10 \\ -.15 & -.20 & .90 \end{pmatrix},$$

so

$$(I - C)^{-1} = \begin{pmatrix} \frac{316}{205} & \frac{184}{205} & \frac{196}{205} \\ \frac{42}{205} & \frac{258}{205} & \frac{52}{205} \\ \frac{62}{205} & \frac{88}{205} & \frac{272}{205} \end{pmatrix}$$

and so

$$\boldsymbol{x} = \begin{pmatrix} \frac{316}{205} & \frac{184}{205} & \frac{196}{205} \\ \frac{42}{205} & \frac{258}{205} & \frac{52}{205} \\ \frac{62}{205} & \frac{88}{205} & \frac{272}{205} \end{pmatrix} \begin{pmatrix} 100 \\ 200 \\ 150 \end{pmatrix} = \begin{pmatrix} \frac{19560}{41} \\ \frac{12720}{41} \\ \frac{12920}{41} \end{pmatrix} \approx \begin{pmatrix} 477.073170732 \\ 310.243902439 \\ 315.12195122 \end{pmatrix}.$$

So the gas company needs to make about \$477.07 worth of gas, the water company needs to make about \$310.24 worth of water, and the electricity company needs to make about \$315.12 worth of electricity.