Chapter 9

Span and Independence

Suppose $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$. If we perform the vector space operations on them, we can multiply them by numbers, to get the vectors $c_1 v_1, \ldots, c_k v_k$, and add them, to get $c_1 v_1 + \cdots + c_k v_k$. The result is called a ***linear combination*** of the original vectors.

Definition. Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. A ***linear combination*** of these vectors is a vector of the form

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

for some numbers $c_1, c_2, \ldots, c_k$. $\square$

Example. In $\mathbb{R}^3$, the vector $\begin{pmatrix} 10 \\ 8 \\ 14 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, since

$$\begin{pmatrix} 10 \\ 8 \\ 14 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 4 \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

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Notice that the linear combination

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

can be rewritten

$$\begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = A\boldsymbol{c},$$

where A is the matrix whose columns are the vectors $\boldsymbol{v}_1,\dots,\boldsymbol{v}_k$ and $\boldsymbol{c}$ is the vector consisting of the numbers $c_1, \dots, c_k$. So a linear combination of the columns of a matrix is merely the product of the matrix with a vector.

Definition. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$ be vectors in $\mathbb{R}^n$. The ***span*** of these vectors, denoted $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$, is the set of all linear combinations of them. □

Example. The vector $\begin{pmatrix} 5 \\ 8 \\ 5 \end{pmatrix}$ is in $\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$, since

$$\begin{pmatrix} 5 \\ 8 \\ 5 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

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It should be clear that the span of a set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, from which we get the following.

Proposition. *The span of a set of vectors in $\mathbb{R}^n$ is the smallest subspace of $\mathbb{R}^n$ containing the vectors.* □

Example. ***Let*** $\boldsymbol{v}_1 = \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{1} \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} \mathbf{5} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}$, ***and*** $\boldsymbol{w} = \begin{pmatrix} \mathbf{7} \\ \mathbf{7} \\ \mathbf{7} \end{pmatrix}$. ***Is*** $\boldsymbol{w} \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$***?***

To determine if $\boldsymbol{w}$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we can use the characterization of linear combinations given above and see if there's a solution to $A\boldsymbol{c} = \boldsymbol{w}$, where

$$A = \begin{pmatrix} 1 & 5 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solving $A\boldsymbol{c} = \boldsymbol{w}$, we get the augmented matrix

$$\begin{pmatrix} 1 & 5 & 7 \\ 3 & 1 & 7 \\ 1 & 0 & 7 \end{pmatrix} \tag{9.1}$$

When put in row-echelon form, we get

$$\begin{pmatrix} 1 & 5 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{9.2}$$

Since the last row corresponds to the equation $0 = 1$, we see that there is no solution, so $\boldsymbol{w}$ is not in $\text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2)$. ☑

In the previous example, we merely put the matrix with columns $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{w}$ in echelon form. Let's do another example.

Example. ***Let*** $\boldsymbol{v_1} = \begin{pmatrix} \mathbf{2} \\ \mathbf{1} \\ \mathbf{3} \end{pmatrix}$***,*** $\boldsymbol{v_2} = \begin{pmatrix} \mathbf{4} \\ \mathbf{2} \\ \mathbf{1} \end{pmatrix}$***, and*** $\boldsymbol{w} = \begin{pmatrix} \mathbf{2} \\ \mathbf{1} \\ \mathbf{8} \end{pmatrix}$***. Is*** $\boldsymbol{w} \in$ ***span***$\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$***?***
Starting with the matrix

$$\begin{pmatrix} 2 & 4 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 8 \end{pmatrix} \tag{9.3}$$

we put it into echelon form to get

$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.4}$$

Since no row of this echelon matrix corresponds to an inconsistent equation, the corresponding equations have a solution, and so $\boldsymbol{w}$ is in the span of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. ☑

When discussing linear combinations of vectors, we may want to know if there are "too many" vectors; i.e., if some of the vectors are redundant. For example, since $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, it follows that any linear combination of $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$,

$\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$,

$$a \cdot \begin{pmatrix}1\\2\\0\end{pmatrix} + b \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + c \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} \tag{9.4}$$

$$= a\left(1 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + 2 \cdot \begin{pmatrix}0\\1\\0\end{pmatrix}\right) + b \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + c \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} \tag{9.4}$$

$$= (a+b) \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + (2a+c) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} \tag{9.4}$$

is also a combination of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ alone. With respect to spans, the set $\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix})\right\}$ contains an unnecessary vector. In this situation, when a set of vectors contains one vector which is a linear combination of the others, we say the vectors are linearly dependent. We have the following more useful characterization.

Definition. A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is ***linearly independent*** if the only solution of

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \mathbf{0}$$

is

$$c_1 = c_2 = \cdots = c_n = 0.$$

If the vectors are not linearly independent, they are ***linearly dependent***.□

We will equivalently say "the vectors are linearly independent" and "the set of vectors is linearly independent".

Example. The vectors $\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ are linearly dependent, since

$$1 \cdot \begin{pmatrix}1\\2\\0\end{pmatrix} - 1 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} - 2 \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

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Example. The vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ are linearly independent, since if

$$c_1\begin{pmatrix}1\\0\\0\end{pmatrix}+c_2\begin{pmatrix}0\\1\\0\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

then

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

and so

$$c_1=c_2=c_3=0.$$

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Recall that the equation

$$c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\cdots+c_n\boldsymbol{v}_n=\mathbf{0}$$

can be written

$$A\boldsymbol{c}=\mathbf{0},$$

where $A=(\boldsymbol{v}_1 \quad \ldots \quad \boldsymbol{v}_n)$ and $\boldsymbol{c}=(c_1 \quad \ldots \quad c_n)$ Since this is a homogeneous system, there is always one solution, namely $\boldsymbol{c}=\mathbf{0}$. So the possibilities are that this is the only solution, in which case the vectors are independent, or there is more than one solution, in which case the vectors are dependent. If we solve the system of equations corresponding to our vector equation, there will be more than one solution exactly when there are free variables. So the test for linear independence becomes a check for free variables: if there are free variables, the vectors are dependent, otherwise they are independent.

Example. ***Are the vectors*** $\begin{pmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{1}\end{pmatrix}$, $\begin{pmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{pmatrix}$ ***and*** $\begin{pmatrix}\mathbf{4}\\\mathbf{5}\\\mathbf{6}\end{pmatrix}$ ***linearly independent?***

We are interested in the equation

$$c_1\begin{pmatrix}1\\2\\1\end{pmatrix}+c_2\begin{pmatrix}1\\2\\3\end{pmatrix}+c_3\begin{pmatrix}4\\5\\6\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

which can be rewritten

$$\begin{pmatrix}1&1&4\\2&2&5\\1&3&6\end{pmatrix}\boldsymbol{x}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

The augmented matrix

$$\begin{pmatrix} 1 & 1 & 4 & 0 \\ 2 & 2 & 5 & 0 \\ 1 & 3 & 6 & 0 \end{pmatrix} \tag{9.5}$$

has echelon form

$$\begin{pmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{9.6}$$

There are no free variables, since the column corresponding to each variable has a leading one, so there is only the solution $c_1 = c_2 = c_3 = 0$. The vectors are linearly independent. ☑

Notice that the last column is all 0s, which gives us no new information and so can be omitted.

Example. ***Are the vectors*** $\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$, $\begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{pmatrix}$ ***and*** $\begin{pmatrix} \mathbf{5} \\ \mathbf{8} \\ \mathbf{11} \\ \mathbf{14} \end{pmatrix}$ ***linearly independent?***

We start with the matrix

$$\begin{pmatrix} 1 & 1 & 5 \\ 1 & 2 & 8 \\ 1 & 3 & 11 \\ 1 & 4 & 14 \end{pmatrix} \tag{9.7}$$

After Gaussian elimination, we get

$$\begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.8}$$

Since the third column has no leading 1, there is a free variable, and so the vectors are linearly dependent. ☑

Notice that if we have m vectors in $\mathbb{R}^n$ with $m > n$, then we will be working with a matrix which has more columns than rows. Since there can be at most one leading one per row, there must be some columns without leading ones; in other words, there will be free variables. The vectors must be dependent.

Proposition. *If $m > n$, then any m vectors in $\mathbb{R}^n$ will be linearly dependent.* $\square$

Example. The vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\begin{pmatrix}2\\4\\5\end{pmatrix}$ and $\begin{pmatrix}3\\8\\9\end{pmatrix}$ are linearly dependent, since they are 4 vectors in $\mathbb{R}^3$ and $4 > 3$. ☑

Next, suppose we have exactly n vectors, $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$, in $\mathbb{R}^n$. Then they are independent exactly when

$$A\boldsymbol{c} = \mathbf{0}$$

has a unique solution $\boldsymbol{c}$, where

$$A = \begin{pmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_n\end{pmatrix}.$$

We know that this has a unique solution exactly when the determinant of the matrix is non-zero.

Proposition. *The n vectors $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$, in $\mathbb{R}^n$ are linearly independent exactly when* $\det\begin{pmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_n\end{pmatrix} \neq 0$. $\square$

Example. The vectors $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\9\end{pmatrix}$ are linearly independent, since

$$\det\begin{pmatrix}1 & 3\\2 & 9\end{pmatrix} = 9 - 6 \neq 0.$$

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Since finding the determinant of a matrix involves Gaussian elimination as well as keeping track of the steps, it is typically easier to check for linear independence by performing Gaussian elimination and checking for free variables.