Chapter 3

Matrix Operations

Recall from calculus that if you have functions $f, g : \mathbb{R} \to \mathbb{R}$, then you can add them:

$$(f+g)(x) = f(x) + g(x)$$

and, if $c \in \mathbb{R}$, multiply a function by a number

$$(cf)(x) = cf(x).$$

If we have linear functions $L, T : \mathbb{R}^n \to \mathbb{R}^m$ and number $c \in \mathbb{R}$, we can add the functions

$$(L+T)(\boldsymbol{x}) = L(\boldsymbol{x}) + T(\boldsymbol{x})$$

and multiply them by numbers

$$(cL)(\boldsymbol{x}) = cL(\boldsymbol{x}),$$

and the results will still be linear functions. If $L(\boldsymbol{x}) = A\boldsymbol{x}$ and $T(\boldsymbol{x}) = B\boldsymbol{x}$ for matrices A and B, we would like to define matrix addition so that

$$(L+T)(\boldsymbol{x}) = (A+B)\boldsymbol{x},$$

and if $c \in \mathbb{R}$, we would like to define the product cA so that

$$(cL)(\boldsymbol{x}) = (cA)\boldsymbol{x}.$$

Definition. We can add two matrices A and B of the same size. This is done by adding the corresponding elements:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} = \tag{3.0}$$

$$\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix} \tag{3.0}$$

Example.

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 7 \end{pmatrix} + \begin{pmatrix} 4 & 3 & 5 \\ 2 & 1 & 9 \end{pmatrix} = \begin{pmatrix} 1+4 & 3+3 & 2+5 \\ 4+2 & 5+1 & 7+9 \end{pmatrix} = \begin{pmatrix} 5 & 6 & 7 \\ 6 & 6 & 16 \end{pmatrix}$$

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Example. The matrices

$$\begin{pmatrix} 2 & 1 & 2 \\ 4 & 2 & 1 \end{pmatrix} \tag{3.1}$$

and

$$\begin{pmatrix} 4 & 2 & 1 \\ 5 & 1 & 1 \\ 4 & 1 & 8 \end{pmatrix} \tag{3.2}$$

can't be added, since they are not the same size. ☑

Definition. We can multiply a number times a matrix by multiplying every element of the matrix by the number:

$$c \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{pmatrix}$$

□

Example.

$$4 \cdot \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 8 \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 & 4 \cdot 3 & 4 \cdot 2 \\ 4 \cdot 4 & 4 \cdot 6 & 4 \cdot 8 \end{pmatrix} = \begin{pmatrix} 4 & 12 & 8 \\ 16 & 24 & 32 \end{pmatrix}$$

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Matrix addition and scalar multiplication behave as you might expect; specifically, they have the following properties:

Proposition. *Let A, B and C be $m \times n$ matrices, with a and b numbers. Then*

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $a(bC) = (ab)C$
4. $(a + b)A = aA + bA$
5. $a(A + B) = aA + aB$ □

If we compose linear functions, the result will still be linear. If $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear and $T : \mathbb{R}^k \to \mathbb{R}^n$ is linear, then the composition $(L \circ T) : \mathbb{R}^k \to \mathbb{R}^m$ defined by

$$(L \circ T)(\boldsymbol{x}) = L(T(\boldsymbol{x}))$$

is also linear. If $L(\boldsymbol{x}) = A\boldsymbol{x}$ for an $m \times n$ matrix A and $T(\boldsymbol{x}) = B\boldsymbol{x}$ for an $n \times k$ matrix B, then we would like to define the matrix product AB so that $(L \circ T)(\boldsymbol{x}) = AB\boldsymbol{x}$. So we would want AB to be given by $((L \circ T)(\boldsymbol{e}_1) \ldots (L \circ T)(\boldsymbol{e}_k))$. Since $T(\boldsymbol{e}_i) = \boldsymbol{b}_i$, the ith column of B, we get $(L \circ T)(\boldsymbol{e}_i) = L(T(\boldsymbol{e}_i)) = L(\boldsymbol{b}_i) = A\boldsymbol{b}_i$. This motivates the following definition.

Definition. Let A be an $m \times n$ matrix, and B an $n \times k$ matrix. Then the product AB is the $m \times k$ matrix given by

$$AB = (A\boldsymbol{b}_1 \ldots A\boldsymbol{b}_k)\,.$$

□

Example. Let

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \\ 5 & 9 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}.$$

Then

$$A\begin{pmatrix}2\\3\end{pmatrix} = 2\begin{pmatrix}2\\4\\5\end{pmatrix} + 3\begin{pmatrix}3\\1\\9\end{pmatrix} = \begin{pmatrix}4\\8\\10\end{pmatrix} + \begin{pmatrix}9\\3\\27\end{pmatrix} = \begin{pmatrix}13\\11\\37\end{pmatrix}$$

and

$$A\begin{pmatrix}4\\2\end{pmatrix} = 4\begin{pmatrix}2\\4\\5\end{pmatrix} + 2\begin{pmatrix}3\\1\\9\end{pmatrix} = \begin{pmatrix}8\\16\\20\end{pmatrix} + \begin{pmatrix}6\\2\\18\end{pmatrix} = \begin{pmatrix}14\\18\\38\end{pmatrix},$$

so

$$AB = \begin{pmatrix}13 & 14\\11 & 18\\37 & 38\end{pmatrix}$$

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To multiply A and B, the only condition is that the number of columns of A equals the number of rows of B. We can also describe the product in another way.

Definition. Let A be an $m \times n$ matrix and B be an $n \times k$ matrix. Then their product AB is the $m \times k$ matrix whose element in the ith row and jth column is the ith row of A times the jth column of B. □

The following mnemonic is sometimes helpful:

$$\begin{array}{ccc} & & \begin{pmatrix} b_{11} & \cdots & \boldsymbol{b_{1j}} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & \boldsymbol{b_{nj}} & \cdots & b_{np} \end{pmatrix} = B \\ & & \downarrow \\ A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ \boldsymbol{a_{i1}} & \cdots & \boldsymbol{a_{in}} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} & & \left(\rightarrow \qquad ij\text{th element} \qquad \right) = AB \end{array}$$

Example. ***Find AB, where***

$$\boldsymbol{A = \begin{pmatrix}1 & 2 & 3\\4 & 2 & 1\end{pmatrix}, B = \begin{pmatrix}2 & 4 & 3\\1 & 2 & 2\\2 & 1 & 3\end{pmatrix}.}$$

Using the above mnemonic, we get

$$AB = \begin{array}{cc} & \begin{pmatrix} 2 & 4 & 3 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix} & \begin{pmatrix} 1\cdot 2+2\cdot 1+3\cdot 2 & 1\cdot 4+2\cdot 2+3\cdot 1 & 1\cdot 3+2\cdot 2+3\cdot 3 \\ 4\cdot 2+2\cdot 1+1\cdot 2 & 4\cdot 4+2\cdot 2+1\cdot 1 & 4\cdot 3+2\cdot 2+1\cdot 3 \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} 2+2+6 & 4+4+3 & 3+4+9 \\ 8+2+2 & 16+4+1 & 12+4+3 \end{pmatrix} = \begin{pmatrix} 10 & 11 & 16 \\ 12 & 21 & 19 \end{pmatrix}$$ ☑

Example.

$$\begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 3\cdot 4+2\cdot 3 & 3\cdot 1+2\cdot 7 \\ 4\cdot 4+1\cdot 3 & 4\cdot 1+1\cdot 7 \\ 5\cdot 4+6\cdot 3 & 5\cdot 1+6\cdot 7 \end{pmatrix} = \begin{pmatrix} 18 & 17 \\ 19 & 11 \\ 38 & 47 \end{pmatrix}$$ ☑

Matrix multiplication has many of the properties that you'd expect.

Proposition. *Let A, B, C and D be matrices, a a number. Then (whenever the relevant matrix products are defined):*

1. $A(B+C) = AB + AC$
2. $(B+C)D = BD + CD$
3. $a(AB) = (aA)B = A(aB)$
4. $A(BD) = (AB)D$ $\square$

However, it is *not* the case that $AB = BA$. Suppose, for example, that A is a 2×2 matrix and B is a 2×3 matrix. Then AB is a 2×3 matrix while BA isn't even defined. It is also possible that AB and BA both be defined but different sizes. Finally, even if AB and BA are both defined and the same size, they don't have to be equal.

Example. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$ but $BA = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$. ☑

Definition. If $AB = BA$, then A and B are said to ***commute***. $\square$

Matrix multiplication provides us with another way of using matrices to represent a linear system of equations. Consider the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = b_2 \\ \vdots & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = b_m \end{array}$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

be the matrix of coefficients,

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

be the column vector of unknowns, and

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

be the column vector of constants. Since

$$A\boldsymbol{x} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix},$$

the above system of equations is equivalent to the single matrix equation

$$A\boldsymbol{x} = \boldsymbol{b}.$$

Example. The system

$$\begin{array}{rcrcrl} 2x & + & y & + & 3y & = 4 \\ x & - & y & + & 2z & = 5 \end{array}$$

is equivalent to

$$A\boldsymbol{x} = \boldsymbol{b},$$

where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \end{pmatrix}$$

and

$$\boldsymbol{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

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