Section 1
$\mathbb{R}^n$

If you've had Calculus, you may have come across vectors and $\mathbb{R}^n$, but even so a brief reminder may be helpful.

0.1 Definition of $\mathbb{R}^n$

Definition
An element of $\mathbb{R}^n$ is an ordered list of n real numbers.

We will write the ordered lists as columns. So, for example, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an element of $\mathbb{R}^2$, and $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ is an element of $\mathbb{R}^3$. If you learned about $\mathbb{R}^n$ in Calculus, the elements were probably given as rows rather than columns; an element of $\mathbb{R}^2$ would be written (x, y) rather than $\begin{pmatrix} x \\ y \end{pmatrix}$. While you should be familiar with both ways of representing elements of $\mathbb{R}^n$, when dealing with matrices it is usually more useful to think of elements of $\mathbb{R}n$ as columns.

An element of $\mathbb{R}^n$ is often represented by $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, although, to avoid subscripts, elements of $\mathbb{R}^2$ are usually represented as $\begin{pmatrix} x \\ y \end{pmatrix}$, and elements of $\mathbb{R}^3$ by $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The numbers x_1, x_2, $\ldots$, x_n (or x, y, z) are called the *coordinates*.

0.2 Points

The set $\mathbb{R}^1$ is simply the set of real numbers, which is viewed geometrically as a line. The line is then called the ***real line***. To make the identification between $\mathbb{R}^1$ and a line, we need a special point on the line, called the ***origin***, a unit length, and a positive direction. (Normally, the line is drawn horizontally

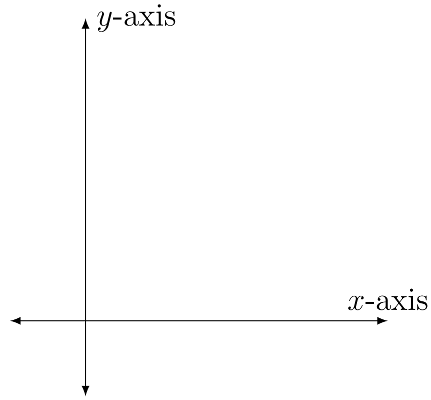

Figure 1: The x- and y-axes.

and the positive direction is to the right.) Any real number x is associated with the following point on the line (see figure

- If $x = 0$, then x is associated with the origin.
- If $x > 0$, then x is associated with the point which is x units from the origin in the positive direction.
- If $x < 0$, then x is associated with the point which is $|x|$ units from the origin in the negative direction (i.e., the opposite direction from the positive direction).

The set $\mathbb{R}^2$ can be viewed geometrically as a plane. This identification needs two copies of the real line which intersect at right angles at their origins. One copy of the real line is the x-`axis`, one is the y-`axis`, and they are oriented so that going about the origin from the positive x-axis to the positive y-axis is counterclockwise. (This is called the *positive orientation.*) The x-axis is usually drawn horizontally with the positive x direction to the right, meaning the y-axis is vertical and the positive y direction is up. (See figure **??**.)

An element $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ is associated with the point in the plane which is x units from the origin in the x direction and y units from the origin in the y direction. (See figure **??**.)

Continuing, the set $\mathbb{R}^3$ can be viewed geometrically as space. This identification needs three copies of the real line which intersect at right angles at

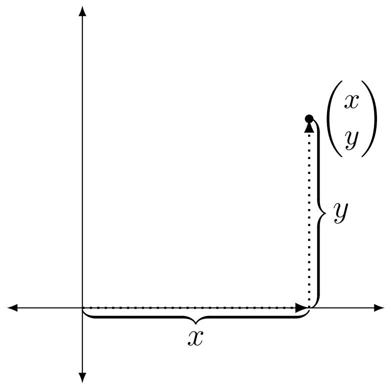

Figure 2: An element of $\mathbb{R}^2$ is associated with a point in the plane.

their origins. One copy of the real line is the x-`axis`, one is the y-`axis`, and one is the z-`axis`. They are oriented according to the `right-hand rule`, namely if you point the fingers on your right hand in the positive x direction so they bend in the positive y direction, your thumb will point in the positive z direction. (This is called the *positive orientation.*) The x-axis is usually indicated so that it is coming out the page, the y-axis is going to the right and the z axis is going up. (See figure **??**.)

An element $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$ is associated with the point in space which is x units from the origin in the x direction, y units from the origin in the y direction and z units from the origin in the z direction. (See figure **??**.)

This geometric association can be continued, but for $n > 3$, $\mathbb{R}^n$ becomes hard to visualize and even harder to draw. Because of this, when pictures are helpful, they will typically be in $\mathbb{R}^2$.

0.3 Vectors

The coordinates of an element of $\mathbb{R}^n$ can be thought of as directions to the point, starting at the origin. To get to the point in the plane given by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ from the origin, go 2 units in the x direction and 3 units in the y direction. These directions make sense even if we don't start at the origin; starting at the point given by $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$, after going 2 units in the x direction and 3 units in

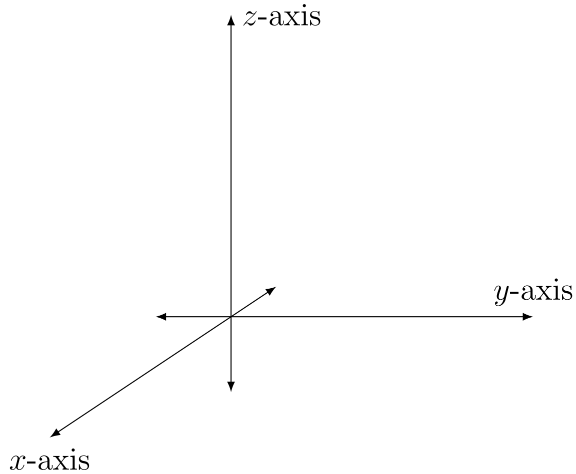

Figure 3: The x, y and z axes in space; the x axis is supposed to be coming out of the page.

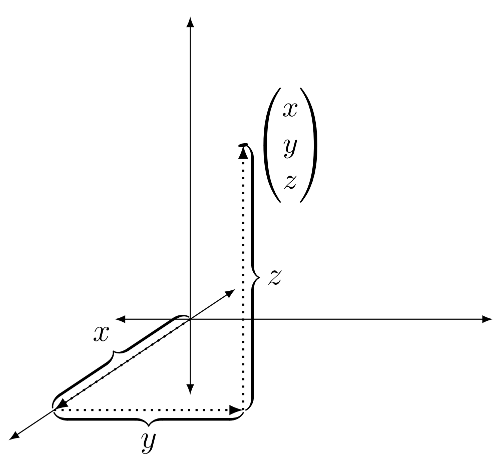

Figure 4: An element of $\mathbb{R}^3$ is associated with a point in space.

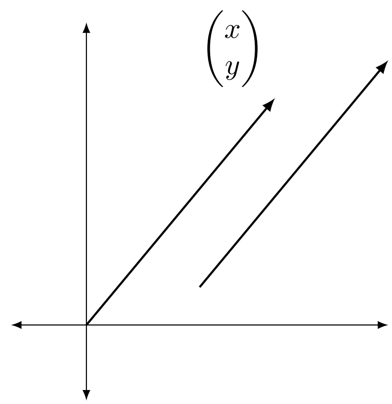

Figure 5: Both arrows represent the same vector in $\mathbb{R}^n$.

the y direction, we will end up at the point given by $\begin{pmatrix} 5+2 \\ 7+3 \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$.

Viewed as directions, an element of $\mathbb{R}^n$ is called a `vector`. Vectors are often represented by an arrow from the origin to the point. Any arrows with the same direction and length represent the same vector (see figure **??**), and so they don't need to begin at the origin. When written by hand, a vector is typically written with an arrow over it, as $\vec{x}$, but when typeset they are usually written in boldface, as $\mathbf{x}$.

The vector from $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ to $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ is given by $\mathbf{v} = \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \\ \vdots \\ y_n - x_n \end{pmatrix}$, abbreviated $\mathbf{v} = \mathbf{y} - \mathbf{x}$. For example, in $\mathbb{R}^2$, the vector from $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is $\mathbf{v} = \begin{pmatrix} 3-4 \\ 4-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (see figure **??**).

0.4 Vector operations

Two elements of $\mathbb{R}^n$ can be added by adding the corresponding elements.

Example

The sum of the two elements $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$ in $\mathbb{R}^3$ is $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} =$

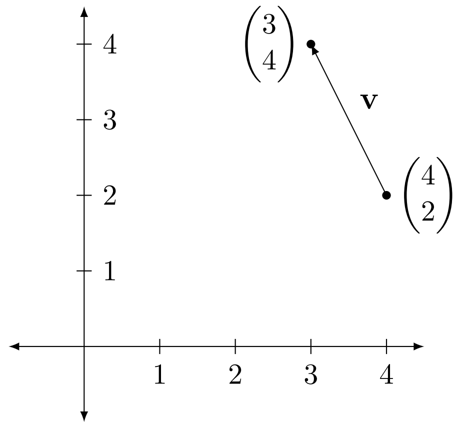

Figure 6: The vector between two points.

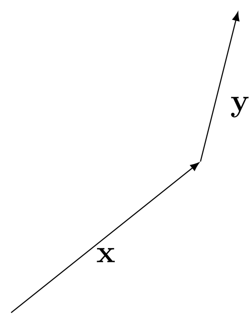

Figure 7: Placing vectors in $\mathbb{R}^n$ tip to tail.

$$\begin{pmatrix}2+3\\3+1\\5+7\end{pmatrix} = \begin{pmatrix}5\\4\\12\end{pmatrix}.$$

The sum $\mathbf{x}+\mathbf{y}$ of two vectors in $\mathbb{R}^n$ can be thought of as following the directions given by $\mathbf{x}$ and then following the directions given by $\mathbf{y}$. Geometrically, to add $\mathbf{x}+\mathbf{y}$, take the tail of an arrow representing $\mathbf{y}$ and place it on the tip of an arrow representing $\mathbf{x}$ (see figure **??**).

The arrow representing $\mathbf{x}+\mathbf{y}$ is then the arrow from the tail of the $\mathbf{x}$ arrow to the tip of the $\mathbf{y}$ arrow (see figure **??**).

Notice that $\mathbb{R}^n$ has the special element $\mathbf{0} = \begin{pmatrix}0\\\vdots\\0\end{pmatrix}$ which is the additive

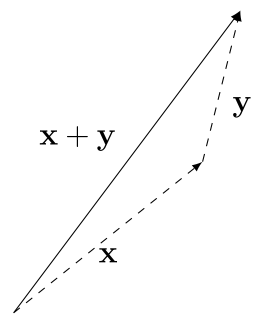

Figure 8: Adding two vectors in $\mathbb{R}^n$.

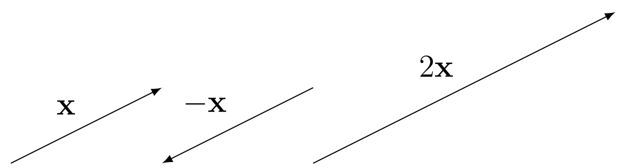

Figure 9: Multiplying a vector by a number.

identity: $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^n$.

Vectors can also be multiplied by numbers; this is done by multiplying each element of the vector by the number.

Example

The product of 5 and the element $\begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ of $\mathbb{R}^4$ is $\begin{pmatrix} 5 \cdot 2 \\ 5 \cdot 1 \\ 5 \cdot 2 \\ 5 \cdot 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 10 \\ 15 \end{pmatrix}$.

Multiplying a vector $\mathbf{x}$ by a number c can be thought of as following the directions given by $\mathbf{x}$, but going c times as far. Geometrically, multiplying $\mathbf{x}$ by c results in an arrow with the same direction as the arrow representing $\mathbf{x}$ (or reversing the direction, if c is negative), but changing the length by a factor of c (see figure **??**).

Since multiplying a vector by a number serves to scale the length of the vector, this type of multiplication is called ***scalar multiplication***, and numbers are called ***scalars***.

Addition and scalar multiplication of vectors satisfy the following properties.

Proposition Properties of addition and scalar multiplication

Commutivity of addition. *For all vectors* $\mathbf{x}$ *and* $\mathbf{y}$*, we have* $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$.

Associativity of addition. *For all vectors* $\mathbf{x}$*,* $\mathbf{y}$ *and* $\mathbf{z}$*, we have* $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

Additive identity. *For the vector* $\mathbf{0}$*, we have* $\mathbf{x} + \mathbf{0} = \mathbf{x}$ *for all vectors* $\mathbf{x}$.

Additive inverses. *For any vector* $\mathbf{x}$*, there is a vector* $-\mathbf{x}$ *such that* $\mathbf{x} + -\mathbf{x} = \mathbf{0}$.

Associativity of multiplication. *For any vector* $\mathbf{x}$ *and scalars* a *and* b*, we have* $a(b\mathbf{x}) = (ab)\mathbf{x}$.

Multiplicative identity. *For any vector* $\mathbf{x}$*, we have* $1\mathbf{x} = \mathbf{x}$.

Distributivity I. *For any scalar* a *and vectors* $\mathbf{x}$ *and* $\mathbf{y}$*, we have* $a(\mathbf{x}+\mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

Distributivity II. *For any scalars* a *and* b *and vector* $\mathbf{x}$*,we have* $(a+b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

Addition and scalar multiplication are the two ***vector space operations***. Although we will (mostly) restrict our attention to $\mathbb{R}^n$, any set in which there is defined addition and scalar multiplication is called a ***vector space***.

Definition
A ***vector space*** is a set V on which there are two operations:

1. Addition: if $\mathbf{x}, \mathbf{y} \in V$, then $\mathbf{x} + \mathbf{y} \in V$.
2. Scalar multiplication: if $\mathbf{x} \in V$ and $c \in \mathbb{R}$, then $c\mathbf{x} \in V$.

These operations must satisfy the properties of addition and scalar multiplication listed above.

An example of a vector space (besides $\mathbb{R}^n$) is the set of all polynomials. Any two polynomials can be added to get another polynomial, and a polynomial can be multiplied by a number to get another polynomial. For any

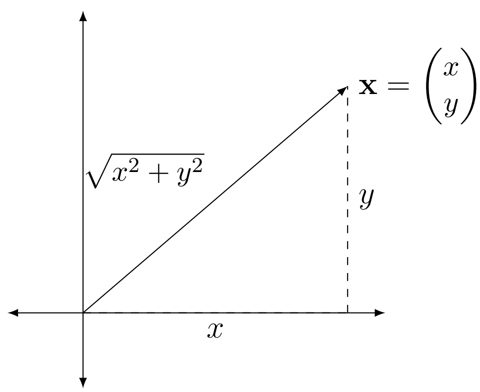

Figure 10: The length of $\mathbf{x}$.

fixed positive integer n, the set $\mathcal{P}_n$ of polynomials of degree n or less — the set of polynomials of the form $p(x) = a_n x^n + \cdots + a_0$ — is also a vector space. What's more, the set $\mathcal{P}_n$ can be identified with $\mathbb{R}^{n+1}$; the element $\begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_0 \end{pmatrix}$ of $\mathbb{R}^{n+1}$ can be considered an alternate way of writing the polynomial $p(x) = a_n x^n + \cdots + a_0$. Under this association, addition and scalar multiplication behave properly. For example, adding two polynomials in $\mathcal{P}_2$, $(a_1x^2+b_1x+c_1)+(a_2x^2+b_2x+c_2) = (a_1+a_2)x^2+(b_1+b_2)x+(c_1+c_2)$, corresponds to addition in $\mathbb{R}^3$, $(a_1 \quad b_1 \quad c_1)^T + (a_2 \quad b_2 \quad c_2)^T = (a_1+a_2 \quad b_1+b_2 \quad c_1+c_2)^T$. Similarly, scalar multiplication in $\mathcal{P}_2$ corresponds to scalar multiplication in $\mathbb{R}^3$.

0.5 Length and direction of vectors

Vectors in $\mathbb{R}^n$ can be used to indicate distance and direction. The distance indicated by a vector is called its *length*. Using the Pythagorean Theorem, we can show that the distance from the tail to the tip of a vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ is $\sqrt{x^2+y^2}$. (See figure **??**.)

Definition

The ***length*** of a vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

in $\mathbb{R}^n$ is

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example

Let $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$. Then $\mathbf{x} = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$.

Related to the length of a vector is the *dot product*. (Since the result of the dot product is a scalar, this is also called the *scalar product*.)

Definition

The ***dot product*** (or ***scalar product***) of two vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

is

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Example

Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^n.$$

Then

$$\mathbf{x} \cdot \mathbf{y} = 1 \cdot 3 + 2 \cdot 2 + 1 \cdot 1 = 8.$$

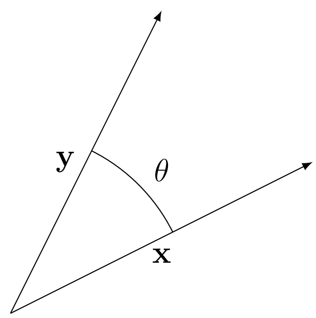

Figure 11: The angle between two vectors.

Note that for any $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$.

The dot product has a geometric interpretation. If $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $\mathbb{R}^n$, let θ be the angle between them. (See figure **??**.)

Then

$$\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta).$$

Note that when the angle is $\theta = 90°$, then $\cos(\theta) = 0$ and so $\mathbf{x}\cdot\mathbf{y} = 0$. This is a useful characterization of what it means for two vectors to be perpendicular.

Remark

Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are ***orthogonal*** (or ***perpendicular***) if $\mathbf{x}\cdot\mathbf{y} = 0$

Example

The vectors $\mathbf{x} = \begin{pmatrix}1\\2\end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix}-2\\1\end{pmatrix}$ in $\mathbb{R}^2$ are orthogonal, since $\mathbf{x}\cdot\mathbf{y} = 1\cdot(-2) + 2\cdot 1 = 0$. (See figure **??**.)

The dot product satisfies the following properties.

Proposition Properties of the dot product

Commutivity. *For all vectors* $\mathbf{x}$ *and* $\mathbf{y}$*, we have* $\mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$.

Distributivity. *For any scalar* a *and vectors* $\mathbf{x}$ *and* $\mathbf{y}$*, we have* $a(\mathbf{x}\cdot\mathbf{y}) = (a\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot(a\mathbf{y})$.

Distributivity II. *For any vector* $\mathbf{x}$*, we have* $\mathbf{x}\cdot\mathbf{x} \geq 0$*, and* $\mathbf{x}\cdot\mathbf{x} = 0$ *exactly when* $\mathbf{x} = \mathbf{0}$.

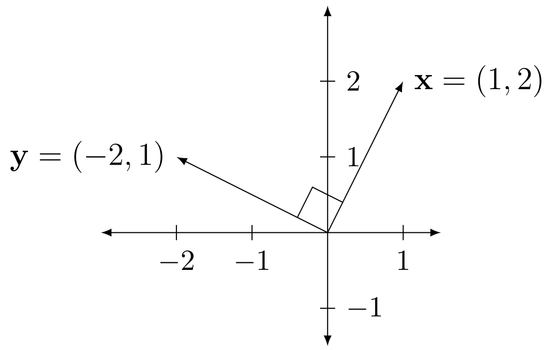

Figure 12: Orthogonal vectors.

Any vector space with an operation which satisfies these properties is called an ***inner product space***, since the dot product is sometimes called the inner product. (We are just using real numbers; if we were using complex numbers the definition of the inner product would be slightly different, but we won't worry about that.)

0.6 Subspaces of $\mathbb{R}^n$

Even when we restrict our attention to $\mathbb{R}^n$, we will come across more vector spaces than $\mathbb{R}^n$ itself. A subset of $\mathbb{R}^n$ will have a special status if addition and scalar multiplication will preserve it. Specifically:

Definition
A non-empty subset S of a vector space is a ***subspace*** if:

1. For any $\mathbf{x}, \mathbf{y} \in S$, their sum $\mathbf{x}+\mathbf{y}$ is also in S. (We say that S is ***closed under addition***.)

2. For any $\mathbf{x} \in S$, $c \in \mathbb{R}$, the scalar product $c\mathbf{x}$ is also in S. (We say that S is ***closed under scalar multiplication***.)

Every $\mathbb{R}^n$ has two somewhat trivial subspaces. The entire space $\mathbb{R}^n$ is a subspace of itself, and the one element subset $\{\mathbf{0}\}$ is also a subspace of $\mathbb{R}^n$. Every other subspace is between these two: If S is a subspace of $\mathbb{R}^n$, then by definition S must be contained in $\mathbb{R}^n$, but also the element $\mathbf{0}$ must be an element of S.

Example
The set of vectors in $\mathbb{R}^3$ whose second component is 0 (i.e., elements of the form $\begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$) is a subspace of $\mathbb{R}^3$. This is straightforward to check:

1. We can add two elements of this type

$$\begin{pmatrix} a_1 \\ 0 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ 0 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ 0 \\ b_1 + b_2 \end{pmatrix}$$

   to get another element of this type. So this set is closed under addition.

2. We can multiply an element of this type by a number

$$c \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} ca \\ 0 \\ cb \end{pmatrix}$$

   to get another element of this type. So this set is closed under scalar multiplication.

So this is indeed a subspace.

Example
The set of vectors in $\mathbb{R}^4$ whose first element is 1 (i.e., elements of the form $\begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}$) is not a subspace of $\mathbb{R}^4$, since the zero element $\begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is not in this set.

Example
Let $\mathcal{S}$ be the set of vectors in $\mathbb{R}^3$ in which the first *or* second component is 0 (i.e., the set of vectors of the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ where either $a = 0$ or $b = 0$).

This set $\mathcal{S}$ is not a subspace of $\mathbb{R}^3$. Note that $\mathcal{S}$ contains the zero vector, but that is not enough to ensure the set is a subspace. In this case, we can add two elements of this set and get an element not in this set:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

so the set is not closed under addition, and so is not a subspace of $\mathbb{R}^3$.

We will later discuss a convenient way of characterizing subspaces of $\mathbb{R}^n$. If we want to visualize the subspaces of $\mathbb{R}^n$, let's start with $\mathbb{R}^2$.

1. The trivial subspace $\{\mathbf{0}\}$ is itself a subspace.
2. If a subspace contains $\mathbf{x} \neq 0$, then it must contain all multiples of $\mathbf{x}$, $\{c\mathbf{x} : c \in \mathbb{R}\}$. The set $\mathbf{x}$, $\{c\mathbf{x} : c \in \mathbb{R}\}$ is itself a subspace of $\mathbb{R}^2$, corresponding to a line through the origin.
3. If a subspace contains $\mathbf{x} \neq 0$ and $\mathbf{y}$ which is not a multiple of $\mathbf{x}$, then it will have to be all of $\mathbb{R}^2$.

So, geometrically, the subspaces of $\mathbb{R}^2$ are:

1. The trivial subspace $\{\mathbf{0}\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$
2. Lines through the origin
3. $\mathbb{R}^2$ itself.

Similarly, the subspaces of $\mathbb{R}^3$ are:

1. The trivial subspace $\{\mathbf{0}\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$
2. Lines through the origin.
3. Planes through the origin.
4. $\mathbb{R}^3$ itself.

Xcas

`Xcas/giac` (`https://www-fourier.ujf-grenoble.fr/~parisse/giac.html`) is a computer algebra system freely available and easily installable on most computer operating systems and smartphones. (Windows, Mac OSX, Linux, Android, iOS, for example.) What's more, it can be run inside a javascript enabled browser (it works well with Firefox).

Computers and smartphones are not usually allowed on tests, but `Xcas` is also available on some calculators. It is the built-in computer algebra system for the HP Prime, and can be installed on some other calculators, such as the TI Nspire calculator and the lower end and less expensive Casio fx-CG50 (see `https://www-fourier.ujf-grenoble.fr/~parisse/casio/khicasioen.html`), which is roughly equivalent to the TI-84+.

These notes will discuss how to use `Xcas` to do the matrix operations that we discuss in class. To find more about the capabilities of `Xcas`, see the manual (`https://www-fourier.ujf-grenoble.fr/~parisse/giac.html#doc`).

`XCas` can function as a simple calculator. If you start the program, you will be met with a prompt

```
>>
```

You can evaluate a mathematical expression by typing it in and hitting enter.

```
2+4
```

$$6$$

You can enter more than one expression on the same line by separating any expressions with semicolons; the results will be separated by commas.

```
2+4; 5*3
```

$$6, 15$$

Giac has all of the usual operators (`+`, `-`, `*`, `/`,`^`, etc.), functions (`sin`, `cos`, `tan`, `exp`, `ln`, etc.) and constants (`e`, `pi`, etc.).

```
2^10
```

$$1024$$

```
sin(pi/4)
```

$$\frac{\sqrt{2}}{2}$$

`Xcas` will give you exact results when possible. If the input is approximate (as indicated by a number with a decimal point), you will get a decimal approximation.

```
sin(pi/4)
```

$$\frac{\sqrt{2}}{2}$$

```
sin(pi/4.0)
```

$$0.707106781187$$

You can also ensure a decimal approximation by using the `evalf` function.

```
evalf(sin(pi/4))
```

$$0.707106781187$$

The `evalf` function takes a second argument indicating the desired precision. For example, you can get a decimal approximation for π:

```
evalf(pi)
```

$$3.14159265359$$

or you can get a decimal approximation with 50 digits to π:

```
evalf(pi,50)
```

$$3.1415926535897932384626433832795028841971693993751$$

(The default number of decimals for `evalf` is configurable; see the manual for more details.)

You can assign values to variable names with the `:=` operator.

```
a := 5
```

$$5$$

```
a^2
```

$$25$$

You can create a list of objects (such as numbers, variables or expressions) by putting them between square brackets, separated by commas.

```
[1,2.3,x,x+y]
```

$$[1, 2.3, x, x + y]$$

The number of elements in a list is given by the size command.

```
size([1,2.3,x,x+y])
```

$$4$$

If you have a list, you can extract an element from the list by following the list with the index within square brackets. Indexing begins at 0, so to get the first element of the list, you can enter it followed with `[0]`.

```
a := [5,10,15,20]
```

$$[5, 10, 15, 20]$$

```
a[0]
```

$$5$$

```
a[2]
```

$$15$$

Indexing typically begins with 0 in computer programs, but normally in mathematics it begins with 1. You can use indices beginning with 1 in `Xcas` by enclosing the index in double square brackets, meaning that you can get the first element of a list by following it with `[[1]]`.

```
a := [5,10,15,20]
```

$$[5, 10, 15, 20]$$

```
a[[1]]
```

$$5$$

```
a[[3]]
```

$$15$$

A row vector can be represented by a list of numbers. For reasons that will come up later, to represent a column vector, you use a list where each number is in square brackets.

```
[1,2,3]
```

$$[1, 2, 3]$$

```
[[1],[2],[3]]
```

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

To find the length of a vector, you can use the `norm` command. (The `length` command will give you the number of elements in the list. The length of a vector is also called the ***norm***, so the `norm` command returns the length/norm.)

```
A := [1,2,3]
```

$$[1, 2, 3]$$

```
norm(A)
```

$$\sqrt{14}$$

```
B := [[1],[2],[3]]
```

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

```
norm(B)
```

$$3.74165738677$$

The `dot` command will find the dot product of two vectors.

```
dot([1,2,2],[3,2,1])
```

$$9$$

```
dot([[2],[3],[2]],[[3],[1],[1]])
```

$$11$$

To find the dot product of two row vectors, you can also simply multiply them.

```
[1,2,2]*[3,2,1]
```

$$9$$

Vectors can be added and multiplied by scalars using the ordinary addition and multiplication operators.

```
[2,3,4] + [5,3,2]
```

$$[7, 6, 6]$$

```
3*[[2],[4],[6]]
```

$$\begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix}$$