Chapter 15

Eigenvalues and Eigenvectors

Suppose that A can be written $A = PDP^{-1}$, where $P = (\boldsymbol{c}_1 \dots \boldsymbol{c}_n)$ (so $\boldsymbol{c}_i$ is the ith column of P) and $D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$. The equation $A = PDP^{-1}$ can be rewritten as $AP = PD$. Since $PD = (\lambda_1 \boldsymbol{c}_1 \dots \lambda_n \boldsymbol{c}_n)$, we get

$$A(\boldsymbol{c}_1 \dots \boldsymbol{c}_n) = (\lambda_1 \boldsymbol{c}_1 \dots \lambda_n \boldsymbol{c}_n),$$

or

$$(A\boldsymbol{c}_1 \dots A\boldsymbol{c}_n) = (\lambda_1 \boldsymbol{c}_1 \dots \lambda_n \boldsymbol{c}_n).$$

This means that $A\boldsymbol{c}_1 = \lambda_1 \boldsymbol{c}_1$; i.e., multiply the column by A is the same as multiplying by a number. Similarly for the other columns.

Definition. Suppose A is a square matrix. The a number λ is called an ***eigenvector*** (or ***characteristic vector***) for A if

$$A\boldsymbol{x} = \lambda \boldsymbol{x}$$

for some $\boldsymbol{x} \neq 0$. Any such vector $\boldsymbol{x}$ is called and ***eigenvector*** (or ***characteristic vector***) for A corresponding to λ. □

Note that for any λ, the equation $A\boldsymbol{x} = \lambda \boldsymbol{x}$ has at least one solution; namely $\boldsymbol{x} = 0$. So for eigenvalues and eigenvectors, we are only interested in nonzero solutions.

When are there nontrivial solutions to $A\boldsymbol{x} = \lambda \boldsymbol{x}$? To help us find out, let's rewrite the equation as $0 = \lambda \boldsymbol{x} - A\boldsymbol{x}$, or $0 = (\lambda I - A)\boldsymbol{x}$. We already know that the homogeneous equation $0 = (\lambda I - A)\boldsymbol{x}$ has a nontrivial solution exactly when $\det(\lambda I - A) = 0$.

Definition. Let A be a square matrix. The ***characteristic polynomial*** of A is $p(\lambda) = \det(\lambda I - A)$. $\square$

We have shown the following.

Proposition. *The eigenvalues of a square matrix are exactly the roots of the characteristic polynomial.* $\square$

Example. ***Find the eigenvalues and corresponding eigenvectors for***

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}.$$

Since

$$\lambda I - A = \begin{pmatrix} \lambda - 1 & -1 \\ -4 & \lambda + 2 \end{pmatrix},$$

the characteristic polynomial is $p(\lambda) = \det(\lambda I - A) = (\lambda - 1)(\lambda + 2) - (-4)(-1) = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$. The eigenvalues are then $\lambda = -3$ and $\lambda = 2$.

To find the eigenvectors for $\lambda = -3$, we want to solve $(\lambda I - A)\boldsymbol{x} = 0$. Since $-3I - A = \left(\begin{smallmatrix} -4 & -1 \\ -4 & -1 \end{smallmatrix}\right)$, we put

$$\begin{pmatrix} -4 & -1 & 0 \\ -4 & -1 & 0 \end{pmatrix}$$

in echelon form, getting

$$\begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So we get the equation $x + \frac{1}{4}y = 0$. The variable y is free, and $x = -\frac{1}{4}y$. So the eigenvectors corresponding to $\lambda = -3$ are of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}y \\ y \end{pmatrix} = y\begin{pmatrix} -\frac{1}{4} \\ 1 \end{pmatrix}.$$

Similarly, the eigenvectors corresponding to $\lambda = 2$ are of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

☑

For any particular eigenvalue, the corresponding eigenvectors (with the zero vector tossed in) forms a subspace of the appropriate $\mathbb{R}^n$. This is clear if we recognize that the eigenvectors corresponding to the eigenvalue λ form the kernel of $\lambda I - A$.

Definition. Let λ be an eigenvalue of a square matrix A. Then the set $\{\boldsymbol{x} : A\boldsymbol{x} = \lambda \boldsymbol{x}\}$ is the ***eigenspace*** of A corresponding to λ. □

Example. ***Find the eigenvalues and corresponding eigenvectors for***

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Here, $\det(\lambda I - A) = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2$. The eigenvalues are then $\lambda = 1$ and $\lambda = 2$.

To find the eigenvectors for $\lambda = 1$, we want to solve $(\lambda I - A)\boldsymbol{x} = 0$. Since $I - A = \left(\begin{smallmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{smallmatrix}\right)$, we put

$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

in echelon form, getting

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This corresponds to the equations $x - z = 0$ and $y - z = 0$. So z is a free variable, and $y = z$, $x = z$. So the eigenvectors corresponding to $\lambda = 1$ are of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

In this case, the eigenspace corresponding to $\lambda = 1$ has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

To find the eigenvectors for $\lambda = 2$, we want to solve $(\lambda I - A)\boldsymbol{x} = 0$. Since $2I - A = \left(\begin{smallmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{smallmatrix}\right)$, we put

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

in echelon form, getting

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This corresponds to the equation $x - y + z = 0$. So y and z are free variables, and $x = y - z$. So the eigenvectors corresponding to $\lambda = 2$ are of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

In this case, the eigenspace corresponding to $\lambda = 2$ has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$. ☑

Remark. In the previous example, there was an eigenvalue which was a root of the characteristic polynomial of order one, and the corresponding eigenspace was one dimensional. There was also an eigenvalue which was a root of the characteristic polynomial of order two, and the corresponding eigenspace was two dimensional. If an eigenvalue is a root of the characteristic polynomial of order n, the corresponding eigenspace often n dimensional, but this is not a general rule. ⊠

Example. ***Find the eigenvalues and corresponding eigenvectors for***

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

Since the characteristic polynomial is $p(\lambda) = \det(\lambda I - A) = \lambda^2 - 4\lambda + 5$, which has no real roots, there are no (real) eigenvalues. While it may be useful to consider complex eigenvalues, we will not be doing that here. ☑

Definition. A square matrix A is ***diagonalizable*** if $A = PDP^{-1}$ for some diagonal matrix D. □

If A is diagonalizable, we saw earlier that the diagonal elements of D are eigenvalues of A, and the corresponding columns of P are eigenvectors. Since P is invertible, the columns of P form a basis for the appropriate $\mathbb{R}^n$. So if A is diagonalizable, then $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.

Conversely, suppose that A is an $n \times n$ matrix, and $\mathbb{R}^n$ has a basis consisting of eigenvectors of A. Let $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ be the eigenvectors, and $\lambda_1, \ldots, \lambda_n$ the corresponding eigenvalues. Then $A\boldsymbol{c}_i = \lambda_i \boldsymbol{c}_i$ for $i = 1, \ldots, n$, and so $A(\boldsymbol{c}_1 \ldots \boldsymbol{c}_n) = (\lambda_1 \boldsymbol{c}_1 \ldots \lambda_n \boldsymbol{c}_n)$. Letting $D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$ and $P = (\boldsymbol{c}_1 \ldots \boldsymbol{c}_n)$, we get $AP = PD$, and so $A = PDP^{-1}$.

Proposition. *An $n \times n$ matrix A is diagonalizable exactly when $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.* □

Given such a basis, the above discussion tells us how to find P and D such that $A = PDP^{-1}$.

Example. Let
$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$
Then, as we saw earlier, A has eigenvalues $\lambda = 1$ and $\lambda = 2$. The eigenspace corresponding to $\lambda = 1$ has basis
$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$
and the eigenspace corresponding to $\lambda = 2$ has basis
$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

So

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$ consisting of eigenvectors, with corresponding eigenvalues 1, 2 and 2. So if

$$P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

then $A = PDP^{-1}$. ☑