Matrix Algebra
Inverses
Homework 6

1. Let A and B be square invertible matrices of the same size. Show that $(AB)^{-1} = B^{-1}A^{-1}$. (Recall, $C = (AB)^{-1}$ is the matrix which satisfies $C(AB) = (AB)C = I$. You need to show that $C = B^{-1}A^{-1}$ satisfies this.) Provide an example showing that even when A and B are both invertible, $(AB)^{-1}$ is not necessarily $A^{-1}B^{-1}$.
2. Find the inverses (if they exist) of the following matrices.

a)

$$\begin{pmatrix} -3 & 1 \\ -2 & 1 \end{pmatrix}$$

Answer: The inverse is

$$\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}$$

b)

$$\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$$

Answer: The inverse is

$$\begin{pmatrix} \frac{2}{7} & \frac{1}{14} \\ -\frac{1}{7} & \frac{3}{14} \end{pmatrix}$$

c)

$$\begin{pmatrix} 5 & -2 \\ 2 & -1 \end{pmatrix}$$

Answer: The inverse is

$$\begin{pmatrix} 1 & -2 \\ 2 & -5 \end{pmatrix}$$

d)

$$\begin{pmatrix} 1 & 3 & -2 \\ -2 & 4 & 1 \\ 5 & 1 & -3 \end{pmatrix}$$

Answer: The inverse is

$$\begin{pmatrix} -\frac{13}{28} & \frac{1}{4} & \frac{11}{28} \\ -\frac{1}{28} & \frac{1}{4} & \frac{3}{28} \\ -\frac{11}{14} & \frac{1}{2} & \frac{5}{14} \end{pmatrix}$$

e)

$$\begin{pmatrix} 3 & -2 & 4 \\ 5 & 3 & 3 \\ 2 & 5 & -2 \end{pmatrix}$$

Answer: The inverse is

$$\begin{pmatrix} \frac{21}{19} & -\frac{16}{19} & \frac{18}{19} \\ -\frac{16}{19} & \frac{14}{19} & -\frac{11}{19} \\ -1 & 1 & -1 \end{pmatrix}$$

f)

$$\begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 2 & -3 & 1 & 3 \\ -1 & 3 & -2 & 0 \end{pmatrix}$$

Answer: The inverse is

$$\begin{pmatrix} \frac{7}{2} & \frac{3}{2} & -\frac{2}{3} & \frac{7}{6} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 0 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{2}{3} & -\frac{1}{6} \end{pmatrix}$$

3. For each of the following systems of equations, write it in the form $A\boldsymbol{x} = \boldsymbol{b}$. Then find A^{-1} and use it to solve the system.

a)

$$\begin{aligned} x + 3y &= 3 \\ 3x - y &= 2 \end{aligned}$$

Answer: This is $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{pmatrix}$$

The solution of the system is $x = 9/10$, $y = 7/10$.

b)

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + 3y + 2z &= 2 \\ 4x - 2y + 3z &= 1 \end{aligned}$$

Answer: This is $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ 4 & -2 & 3 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{13}{33} & -\frac{4}{33} & \frac{7}{33} \\ \frac{2}{33} & \frac{7}{33} & -\frac{4}{33} \\ -\frac{16}{33} & \frac{10}{33} & -\frac{1}{33} \end{pmatrix}$$

The solution of the system is $x = 25/33$, $y = 14/33$, $z = -13/33$.