Section 4
Matrix Operations

Like vectors, matrices can be added and multiplied by scalars.

- We can add two matrices of the same size. This is done by adding the corresponding elements:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 7 \end{pmatrix} + \begin{pmatrix} 4 & 3 & 5 \\ 2 & 1 & 9 \end{pmatrix} = \begin{pmatrix} 1+4 & 3+3 & 2+5 \\ 4+2 & 5+1 & 7+9 \end{pmatrix} = \begin{pmatrix} 5 & 6 & 7 \\ 6 & 6 & 16 \end{pmatrix}$$

Example
The matrices

$$\begin{pmatrix} 2 & 1 & 2 \\ 4 & 2 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 4 & 2 & 1 \\ 5 & 1 & 1 \\ 4 & 1 & 8 \end{pmatrix}$$

can't be added, since they are not the same size.

- We can multiply a number (scalar) times a matrix by multiplying every element of the matrix by the number:

$$c \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{pmatrix}$$

  Example

$$4 \cdot \begin{pmatrix} 1 & 3 & 2 \\ 4 & 6 & 8 \end{pmatrix} = \begin{pmatrix} 4\cdot 1 & 4\cdot 3 & 4\cdot 2 \\ 4\cdot 4 & 4\cdot 6 & 4\cdot 8 \end{pmatrix} = \begin{pmatrix} 4 & 12 & 8 \\ 16 & 24 & 32 \end{pmatrix}$$

Matrix addition and scalar multiplication behave as you might expect; namely, they obey the properties of addition and scalar multiplication mentioned in the first section.

In some cases, we can multiply matrices by other matrices (rather than numbers). Let's look at a simple case.

Definition
Let A be a $1 \times n$ matrix (a row vector) and B be an $n \times 1$ matrix (a column vector);

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Then their product is defined

$$\begin{aligned} A \cdot B &= \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{aligned}$$

Note that while A is a row vector and B is a column vector, their product is essentially the dot product.

Example

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 38$$

Example

The product

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

is not defined, since the width of the first vector doesn't match the height of the second.

Now we can define a more general matrix product.

Definition

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then their product AB is the $m \times p$ matrix whose element in the ith row and jth column is the ith row of A times the jth column of B.

The following mnemonic is sometimes helpful:

$$\begin{array}{cc} & \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = B \\ A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} & \begin{pmatrix} & & \downarrow & & \\ & & & & \\ \rightarrow & & ij\text{th element} & & \\ & & & & \\ & & & & \end{pmatrix} = AB \end{array}$$

Note that in order to be able to multiply A times B, each row of A must have the same number of elements as each column of B; in other words, the number of columns of A must equal the number of rows of B.

Example
Find AB, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix}.$$

Using the above mnemonic, we get

$$AB = \begin{array}{cc} & \begin{pmatrix} 2 & 4 & 3 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix} & \begin{pmatrix} 1\cdot 2 + 2\cdot 1 + 3\cdot 2 & 1\cdot 4 + 2\cdot 2 + 3\cdot 1 & 1\cdot 3 + 2\cdot 2 + 3\cdot 3 \\ 4\cdot 2 + 2\cdot 1 + 1\cdot 2 & 4\cdot 4 + 2\cdot 2 + 1\cdot 1 & 4\cdot 3 + 2\cdot 2 + 1\cdot 3 \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} 2+2+6 & 4+4+3 & 3+4+9 \\ 8+2+2 & 16+4+1 & 12+4+3 \end{pmatrix} = \begin{pmatrix} 10 & 11 & 16 \\ 12 & 21 & 19 \end{pmatrix}$$

Example

$$\begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 3\cdot 4 + 2\cdot 3 & 3\cdot 1 + 2\cdot 7 \\ 4\cdot 4 + 1\cdot 3 & 4\cdot 1 + 1\cdot 7 \\ 5\cdot 4 + 6\cdot 3 & 5\cdot 1 + 6\cdot 7 \end{pmatrix} = \begin{pmatrix} 18 & 17 \\ 19 & 11 \\ 38 & 47 \end{pmatrix}$$

With matrix multiplication defined, the importance of the identity matrix should be clear. If I is an identity matrix, then $IA = A$ whenever IA is defined, and $BI = B$ whenever BI is defined. The identity matrix is the multiplicative identity.

Matrix multiplication has many of the properties that you'd expect.

Proposition
Let A, B, C and D be matrices, a a number. Then (whenever the relevant matrix products are defined):

1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$

3. $a(AB) = (aA)B = A(aB)$

4. $A(BD) = (AB)D$

However, it is *not* the case that $AB = BA$. Suppose, for example, that A is a 2×2 matrix and B is a 2×3 matrix. Then AB is a 2×3 matrix while BA isn't even defined. It is also possible that AB and BA both be defined but different sizes. Finally, even if AB and BA are both defined and the same size, they don't have to be equal.

Example

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$ but $BA = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$.

Definition

If $AB = BA$, then A and B are said to ***commute***.

Matrix multiplication provides us with another way of using matrices to represent a linear system of equations. Consider the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

be the matrix of coefficients,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

be the column vector of unknowns, and

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

be the column vector of constants. Since

$$A\mathbf{x} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix},$$

the above system of equations is equivalent to the single matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

Example
The system

$$\begin{array}{ccccccc} 2x & + & y & + & 3y & = & 4 \\ x & - & y & + & 2z & = & 5 \end{array}$$

is equivalent to

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Xcas

Adding and multiplying matrices in Xcas is done simply by using "+" and "*".

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A := [[1,2,3],[4,5,6]]; B := [[3,2,1],[4,7,8]]
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$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 4 & 7 & 8 \end{pmatrix}$$

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2*A
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$$\begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

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A+B
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$$\begin{pmatrix} 4 & 4 & 4 \\ 8 & 12 & 14 \end{pmatrix}$$

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C := [[2,3,1],[3,2,2],[4,3,1]]
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$$\begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 2 \\ 4 & 3 & 1 \end{pmatrix}$$

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A*C
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$$\begin{pmatrix} 20 & 16 & 8 \\ 47 & 40 & 20 \end{pmatrix}$$

You will get an appropriate error if you try to operate on mis-matched matrices.

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A*B
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$$\mathrm{Error : Invaliddimension}$$