Chapter 11

Rank

Recall, a system of m linear equations in n unknowns can be written

$$A\boldsymbol{x} = \boldsymbol{b},$$

where A is an $m \times n$ matrix, $\boldsymbol{x}$ is an $n \times 1$ column vector, and $\boldsymbol{b}$ is an $m \times 1$ column vector. Such a system may or may not have a solution, so we might ask when it does have a solution. Since $A\boldsymbol{x}$ is a linear combination of the columns of A, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution exactly when $\boldsymbol{b}$ is a linear combination of the columns of A.

Definition. Let A be an $m \times n$ matrix. The ***column space*** of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A. We similarly have the ***row space*** of A, which is the subspace of $\mathbb{R}^n$ spanned by the rows of A. □

Example. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 6 & 5 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 \\ 7 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 19 \end{pmatrix}$$

is in the column space of A, and

$$2 \cdot \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} + 3 \cdot \begin{pmatrix} 7 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 23 & 22 & 21 \end{pmatrix}$$

is in the row space of A. ☑

Since we're more familiar with working with rows than working with columns, we'll start there.

Proposition. *Elementary row operations do not change the row space of a matrix.* □

Example. The matrices

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{pmatrix}$$

have the same row space. ☑

Proposition. *If a matrix A is reduced to row-echelon form E, then E and A have the same row space.* □

Example. Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

Putting it in row-echelon form, we get

$$E = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So $\operatorname{span}\{(1,2,3,4),(5,6,7,8),(9,10,11,12)\} = \operatorname{span}\{(1,2,3,4),(0,1,2,3)\}$. ☑

It's not hard to show that the non-zero rows of a matrix in echelon form are linearly independent, so we get the following.

Corollary. *If a matrix A is reduced to row-echelon form E, then the non-zero rows of E are a basis for the row space of A.* □

Example. ***Find a basis for the row space of***

$$A = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{pmatrix}.$$

Since A can be row-reduced to the echelon matrix

$$\begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

a basis for the row space of A is $\{\begin{pmatrix} 1 & 3 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}\}$. ☑

Example. ***Find a basis for the span of*** **$\{(-1, 2, 5), (3, 0, 3), (5, 1, 8)\}$.**
We want a basis for the row space of

$$\begin{pmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{pmatrix}.$$

This matrix can be reduced to the echelon matrix

$$\begin{pmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

so the basis we want is $\{(1, -2, -5), (0, 1, 3)\}$. ☑

Example. ***Find a basis for the column space of***

$$A = \begin{pmatrix} 2 & 3 & 6 \\ 4 & 1 & 7 \\ 0 & 2 & 2 \\ 2 & 1 & 4 \end{pmatrix}.$$

Let's turn the columns into rows, and consider

$$A^T = \begin{pmatrix} 2 & 4 & 0 & 2 \\ 3 & 1 & 2 & 1 \\ 6 & 7 & 2 & 4 \end{pmatrix}.$$

This can be row reduced to

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -2/5 & 2/5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A basis for the row space of A^T is then $\{\begin{pmatrix} 1 & 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2/5 & 2/5 \end{pmatrix}\}$, and so a basis for the column space of A is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2/5 \\ 2/5 \end{pmatrix} \right\}$. ☑

The row space and column space of a matrix are not the same; in fact, they typically won't even be subspaces of the same space. We do, however, have the following.

Proposition. *For any matrix, the row space and the column space have the same dimension.* □

In fact, we can find both the row and column spaces at the same time.

Proposition. *For any matrix A, let E be an echelon form of A.*

1. *The non-zero rows of E form a basis for the row space of A.*
2. *The columns of A which correspond to the columns of E with leading* 1*s form a basis for the column space of A.*

Example. ***Find bases for the row and column spaces of***

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 3 \\ 4 & 5 & 2 \\ 2 & 3 & -4 \\ 5 & 6 & 5 \end{pmatrix}.$$

An echelon form for A is

$$E = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the column space of A is then given by the non-zero rows of E; namely, $\{(1 \quad 1 \quad 3), (0 \quad 1 \quad -10)\}$. Since the leading 1s of E are in the first and second columns of E, the first and second columns of A form a basis for the column space of A; the basis is $\left\{\begin{pmatrix} 1 \\ 4 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 3 \\ 6 \end{pmatrix}\right\}$. ☑

Definition. The ***rank*** of a matrix A, denoted $\operatorname{rank}(A)$, is the dimension of the column space of A. □

By the preceding proposition, the rank is also the dimension of the row space. Since a basis for the row space is the set of non-zero rows in the echelon form of the matrix, the dimension of the row space, and hence the rank of the matrix, is the number of non-zero rows in the echelon form of the matrix.

Example. ***Find rank(A), where***

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 4 \end{pmatrix}.$$

Since A can be row-reduced to the echelon matrix

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the rank of A is 2. ☑

Next, let's consider equations of the form

$$A\boldsymbol{x} = 0.$$

Recall, this type of system is called ***homogeneous***, and always has at least one solution, namely $\boldsymbol{x} = 0$.

Definition. Let A be an $m \times n$ matrix. The ***nullspace*** of A, denoted $\mathcal{N}(A)$, is the set of all $\boldsymbol{x} \in \mathbb{R}^n$ with $A\boldsymbol{x} = 0$. $\square$

Example. ***Let***

$$\begin{pmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{pmatrix}.$$

Find the nullspace of A.
The augmented matrix corresponding to $A\boldsymbol{x} = 0$ is

$$\begin{pmatrix} 1 & 2 & -2 & 1 & 0 \\ 3 & 6 & -5 & 4 & 0 \\ 1 & 2 & 0 & 3 & 0 \end{pmatrix}.$$

Reduced to row-echelon form, this becomes

$$\begin{pmatrix} 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which corresponds to the equations

$$\begin{array}{ccccccccl} x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & = 0 \\ & & & & x_3 & + & x_4 & = 0. \end{array}$$

Solving this with back-substitution, we get

$$\begin{aligned} &x_4 \ \text{ is a free variable} \\ &x_3 = -x_4 \\ &x_2 \ \text{ is a free variable} \\ &x_1 = -2x_2 - 3x_4. \end{aligned}$$

So the solutions are vectors of the form

$$\begin{aligned}
\boldsymbol{x} &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\
&= \begin{pmatrix} -2x_2 - 3x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} \\
&= x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix},
\end{aligned}$$

so the solution set is

$$\operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Notice that augmenting the matrix with the column of 0s was unnecessary; the column will remain all 0s under the elementary row operations. Also, notice that when writing the solution set as a span, we got one vector for each free variable. These vectors will always be independent, and so will form a basis for the nullspace of A.

Definition. The ***nullity*** of a matrix A, denoted $\mathcal{N}(A)$, is the dimension of the nullspace of A. □

So in the previous example, the nullity of A was 2.

Example. ***Find the rank and nullity of***

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & 12 \end{pmatrix}.$$

The row-echelon form of this matrix is

$$\begin{pmatrix} \mathbf{1} & 0 & -2 & 0 & 1 \\ 0 & \mathbf{1} & 3 & 0 & -4 \\ 0 & 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since there are 3 non-zero rows, the rank is 3. Now, each non-zero row has a leading 1, each in a different column. The columns without these 1s correspond to free variables. The number of free variables, which is the nullity, is the number of columns minus the number of these leading 1s. So the nullity is 2. ☑

Proposition. *Let A be an $m \times n$ matrix. Then*

$$rank(A) + \mathcal{N}(A) = n.$$

The nullspace of an $m \times n$ matrix A, the solution set to the homogeneous system

$$A\boldsymbol{x} = 0,$$

is a subspace of $\mathbb{R}^n$. If $\boldsymbol{b} \neq 0$, the solution set of

$$A\boldsymbol{x} = \boldsymbol{b}$$

is *not* a subspace of $\mathbb{R}^n$. (Why not?)

Now, given a system

$$A\boldsymbol{x} = \boldsymbol{b},$$

let $\boldsymbol{x}_p$ be a specific (particular) solution, so

$$A\boldsymbol{x}_p = \boldsymbol{b}.$$

If $\boldsymbol{x}$ is any other solution, then

$$A\boldsymbol{x} = \boldsymbol{b} = A\boldsymbol{x}_p,$$

so

$$\begin{aligned} A\boldsymbol{x} - A\boldsymbol{x}_p &= 0 \\ A(\boldsymbol{x} - \boldsymbol{x}_p) &= 0. \end{aligned}$$

This means exactly that $\boldsymbol{x} - \boldsymbol{x}_p \in \mathcal{N}(A)$.

Proposition. *Let $\boldsymbol{x}_p$ be a particular solution of*

$$A\boldsymbol{x} = \boldsymbol{b}.$$

Then $\boldsymbol{x}$ is a solution of the system exactly when

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_h$$

for some $\boldsymbol{x}_h \in \mathcal{N}(A)$. $\square$

Example. One solution of

$$\begin{array}{rrrrl} x & +2y & -2z & +w & = 6 \\ 3x & +6y & -5z & +4w & = 19 \\ x & +2y & & +3w & = 8 \end{array}$$

is

$$\boldsymbol{x}_p = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Since the nullspace of

$$A = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

has basis $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$, the elements of the nullspace are the vectors of the form $\boldsymbol{x}_h = a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix}$. The solutions of the original system are then the vectors of the form

$$\boldsymbol{x}_p + \boldsymbol{x}_h = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

Recall that a system of linear equations

$$A\boldsymbol{x} = \boldsymbol{b}$$

has either zero, one, or infinitely many solutions. We can now see how they can happen.

- If $\boldsymbol{b}$ is not in the column space of A, then there are no solutions.
- If $\boldsymbol{b}$ is in the column space of A, then there are solutions, and they will be of the form $\boldsymbol{x}_p + \boldsymbol{x}_h$. The set of possible $\boldsymbol{x}_h$ is the nullspace of A. If the nullspace is trivial, equal to $\{0\}$, then the only possible value of $\boldsymbol{x}_h$ is 0, and the only solution is $\boldsymbol{x}_p$. If the nullspace of A is not trivial, it will be infinite, and there will be infinitely many values of the form $\boldsymbol{x}_p + \boldsymbol{x}_h$.