Section 2
Instantaneous Rates of Change

Recall that if a ball's position at time t seconds is given by $y = f(t) = 16t^2$ feet, then:

- from $t = 3$ to $t = 4$ seconds, the average rate of change is $\frac{\Delta y}{\Delta t} = 112$ feet/sec.
- from $t = 3$ to $t = 3.1$ seconds, the average rate of change is $\frac{\Delta y}{\Delta t} = 97.6$ feet/sec.
- from $t = 3$ to $t = 3.01$ seconds, the average rate of change is $\frac{\Delta y}{\Delta t} = 96.16$ feet/sec.
- from $t = 3$ to $t = 3.001$ seconds, the average rate of change is $\frac{\Delta y}{\Delta t} = 96.016$ feet/sec.

What should the rate of change *at* $t = 3$ seconds be? Note that the average rates of change over a small interval *near* $t = 3$ seconds are close to 96 feet/sec. You might suggest that the average rates of change are also close to other numbers, such as 96.1 feet/sec, but over a small enough interval they will be closer to 96 feet/sec. The most useful way to *define* the rate of change *at* $t = 3$ seconds would be to have it be 96 feet/sec.

More generally, let $y = f(x)$ and suppose we want to know the rate of change of y at $x = a$. First, we'll start with an average rate of change from a to some nearby point $x \neq a$. We will have

$$\begin{aligned} \Delta x &= x - a \\ \Delta y &= f(x) - f(a) \end{aligned}$$

We get $x = a + \Delta x$, and we can view the nearby point as "a plus a little bit (Δx)". We then get $\Delta y = f(a + \Delta x) - f(a)$. The average rate of change will then be

$$\frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

Then we'll see how this behaves if the interval is small; i.e., if Δx is close to 0.

Definition
Let $y = f(x)$, and suppose that over a small interval, from $x = a$ to $x = a + \Delta x$, the average rates of change $\Delta y/\Delta x$ are close to some value $f'(a)$. We will define the ***instantaneous rate of change*** of $y = f(x)$ *at* $x = a$ to be this value $f'(a)$.

The instantaneous rate of change is usually called the ***derivative***.

This definition is not precise; for example, what is meant by "close" and what is meant by "small" aren't specified. The definition is usually made precise by introducing the idea of *limits*, but the definition given above will suffice for our purposes.

It is useful to think of the instantaneous rate of change as the rate of change over a very small interval. While the notation Δx is the change in x over an interval, the notation dx is (informally) the change in x over an indefinitely small interval. This is the motivation for the alternate notation

$$\frac{dy}{dx}(a) = f'(a)$$

for the derivative; we have

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

for small Δx.

Just as average rates of change tell us about relative changes, so do derivatives (instantaneous rates of change). So, for example, when $dy/dx = f'(a) = 2$, then $y = f(x)$ is changing about twice as fast as x (for x near a). The larger the derivative, the faster y is changing compared to x. If the derivative is negative, then y is changing in the opposite direction as x — the quantity y is getting smaller as x gets larger.

In general, there is no reason why the average rates of change $\frac{\Delta y}{\Delta x}$ have to be close to one value for small Δx, but we will only deal with nice functions; in this case, "nice" means that the derivatives will exist.

Example
Let $f(x) = x^2 + x$. Find $f'(2)$.

Let's begin by finding the average rate of change over a small interval of width Δx starting at $x = 2$; so the interval will go from $x = 2$ to $x = 2 + \Delta x$.

When $x = 2$, we have $y = f(2) = 2^2 + 2 = 6$, and when $x = 2 + \Delta x$,

$$\begin{aligned} y = f(2 + \Delta x) &= (2 + \Delta x)^2 + (2 + \Delta x) \\ &= 4 + 4\Delta x + (\Delta x)^2 + 2 + \Delta x \\ &= 6 + 5\Delta x + (\Delta x)^2 \end{aligned}$$

and so

$$\begin{aligned} \Delta y &= f(2 + \Delta x) - f(2) \\ &= (6 + 5\Delta x + (\Delta x)^2) - (6) \\ &= 5\Delta x + (\Delta x)^2. \end{aligned}$$

The average rate of change over this small interval will then be

$$\frac{\Delta y}{\Delta x} = \frac{5\Delta x + (\Delta x)^2}{\Delta x} = 5 + \Delta x.$$

For small Δx, namely for Δx close to 0, this will be close to $5 + 0 = 5$. So

$$f'(2) = 5.$$

Example
A ball is dropped so that in t seconds it will have fallen $f(t) = 16t^2$ feet. How fast is it falling in 3 seconds?

We want the instantaneous rate of change of f at $t = 3$ seconds; i.e., we want $f'(3)$.

$$\begin{aligned} f(3 + \Delta t) &= 16(3 + \Delta t)^2 \\ &= 16(9 + 6\Delta t + \Delta t^2) \\ &= 144 + 96\Delta t + 16\Delta t^2 \text{ feet} \\ f(3) &= 16 \cdot 3^2 = 16 \cdot 9 = 144 \text{ feet} \end{aligned}$$

so

$$\begin{aligned} \Delta y &= f(3 + \Delta t) - f(3) \\ &= (144 + 96\Delta t + 16\Delta t^2) - (144) \\ &= 96\Delta t + 16\Delta t^2 \text{ feet.} \end{aligned}$$

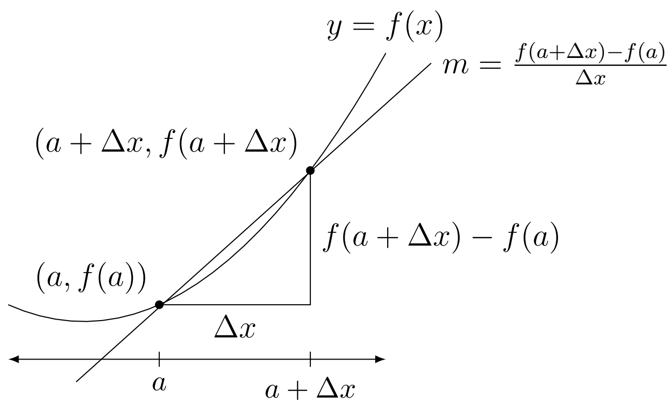

Figure 1: A secant line.
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Dividing by Δt, we have the average rates of change

$$\begin{aligned}\frac{\Delta y}{\Delta t} &= \frac{f(3 + \Delta t) - f(3)}{\Delta t} \\ &= \frac{96\Delta t + 16\Delta t^2}{\Delta t} \\ &= 96 + 16\Delta t \text{ feet/sec.}\end{aligned}$$

For small Δt, this will be close to $96 + 16 \cdot 0 = 96$ feet/sec, so this is our velocity:

$$f'(3) = 96 \text{ feet/sec.}$$

Recall that the average rate of change is the slope of a secant line. (See figure 1.) For small Δx, the slopes $\frac{\Delta y}{\Delta x}$ will be close to $f'(a)$, and the secant lines through $(a, f(a))$ will be close (in some unspecified sense) the line through $(a, f(a))$ with slope $m = f'(a)$. (See figure **??**.) This limiting line is called the ***tangent line***. Since we're not going to define what it means for the secant lines to be close to another line, we'll define the tangent line in a more concrete way.

Definition
The ***tangent line*** to the graph $y = f(x)$ at $x = a$ is the line through $(a, f(a))$ with slope $m = f'(a)$.

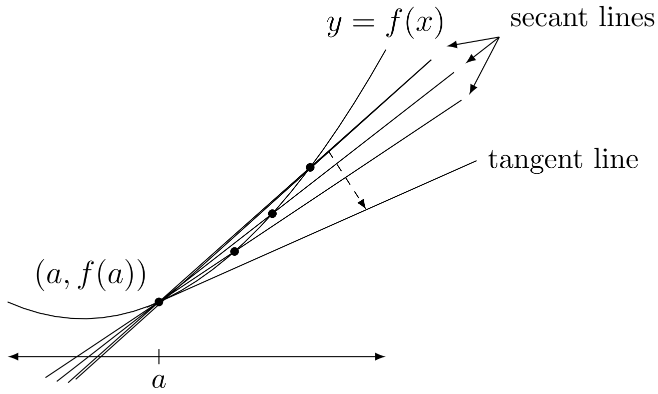

Figure 2: The tangent line.
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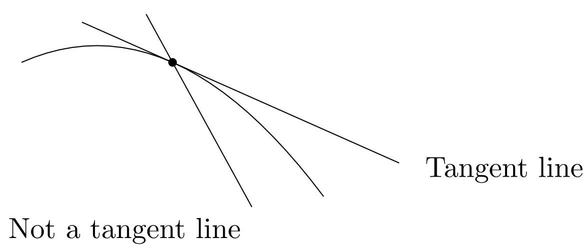

Figure 3: A tangent line and a non-tangent line.

The tangent line will then have equation

$$y - f(a) = f'(a)(x - a),$$

or, in slope-intercept form,

$$y = f(a) + f'(a)(x - a).$$

The importance of the tangent line is that it is the line through the point $(a, f(a))$ on the graph $y = f(x)$ which best approximates the graph near $x = a$. (See figure 3.) Typically a graph looks nothing like a line. If you zoom in on a graph near a point, however, the graph begins to flatten. The more you zoom in, the flatter the graph becomes. The graph begins to look like the tangent line. (See figure 4.)

Example
If $f(x) = x^2 + x$, then we have already computed $f'(2) = 5$. The tangent line

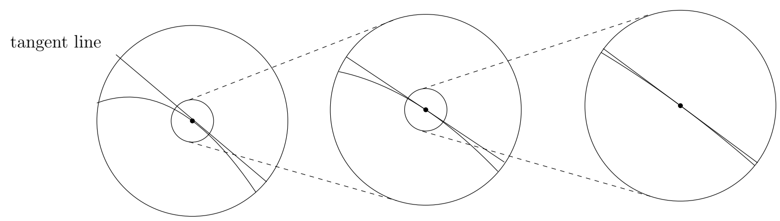

Figure 4: Zooming in on the tangent line.

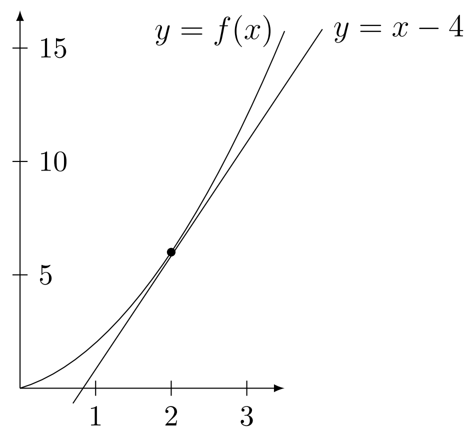

Figure 5: A graph and its tangent line.

to the graph of f at $x = 2$ is then the line through the point $(2, f(2)) = (2, 6)$ with slope $m = f'(2) = 5$. This line is

$$y - 6 = 5(x - 2)$$

or, in slope-intercept form,

$$y = 5x - 4.$$

(See figure 5.)

Example
Find the tangent line to the graph of $f(x) = 2x^2 + 1$ at $x = 1$.

This will be the line through $(1, f(1)) = (1, 3)$ with slope $m = f'(1)$. To find $f'(1)$, we first find

$$\frac{\Delta y}{/\Delta} = \frac{f(1 + \Delta x) - f(1)}{\Delta x}$$

Since

$$\begin{aligned}
f(1) &= 2 \cdot 1^2 + 1 \\
&= 2 \cdot 1 + 1 = 3 \\
f(1 + \Delta x) &= 2(1 + \Delta x)^2 + 1 \\
&= 2(1 + 2\Delta x + \Delta x^2) + 1 \\
&= 2 + 4\Delta x + 2\Delta x^2 + 1 \\
&= 3 + 4\Delta x + 2\Delta x^2
\end{aligned}$$

and so

$$\begin{aligned}
\Delta y &= f(1 + \Delta x) - f(1) \\
&= (3 + 4\Delta x + \Delta x^2) - (3) \\
&= 4\Delta x + \Delta x^2 \\
\frac{f(1 + \Delta x) - f(1)}{\Delta x} &= \frac{4\Delta x + \Delta x^2}{\Delta x} \\
&= 4 + \Delta x.
\end{aligned}$$

For small Δx, this will be close to $4 + 0 = 4$, and so So the slope will be

$$m = f'(1) = 4.$$

The line is then

$$y - 3 = 4(x - 1)$$

or, in slope-intercept form,

$$y = 4x - 1.$$
