Chapter 10

Bases

Definition. Let V be a subspace of $\mathbb{R}^n$. A ***basis*** for a vector space V is a set of linearly independent vectors which span V. $\square$

If $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is a basis for V, then any element $\boldsymbol{v}$ of V can be written as a linear combination of the elements of $\mathcal{B}$

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k,$$

since $\mathcal{B}$ spans V; and the coefficients $c_1, \ldots, c_n$ are unique, since $\mathcal{B}$ is linearly independent.

Proposition. *Let $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a basis for V. For any $\boldsymbol{v} \in V$, we can write*

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k$$

for unique values of $c_1, \ldots, c_k$. The numbers $c_1, \ldots, c_k$ are called the ***coordinates*** *of $\boldsymbol{v}$ with respect to the basis $\mathcal{B}$, and we write*

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}.$$

$\square$

Notice that above we implicity assumed that the basis was an ordered set.

Example. ***The set***

$$\boldsymbol{\mathcal{B}} = \left\{ \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$***. Let*** $v = \begin{pmatrix} \mathbf{3} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}$***, and let's find*** $[v]_{\boldsymbol{\mathcal{B}}}$***.***

We want to find a, b, and c such that

$$a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

The corresponding matrix

$$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{10.1}$$

has echelon form

$$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{10.2}$$

and so $a = 2$, $b = 1$, and $c = -1$. So

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

☑

Definition. The ***standard basis*** for $\mathbb{R}^n$ is $\mathcal{S} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$, where

$$\begin{aligned} \boldsymbol{e}_1 &= (1, 0, 0, \ldots, 0)^T \\ \boldsymbol{e}_2 &= (0, 1, 0, \ldots, 0)^T \\ \boldsymbol{e}_3 &= (0, 0, 1, \ldots, 0)^T \\ &\vdots \\ \boldsymbol{e}_n &= (0, 0, 0, \ldots, 1)^T \end{aligned}$$

□

Note that for any $\boldsymbol{v} \in \mathbb{R}^n$, $[\boldsymbol{v}]_{\mathcal{S}} = \boldsymbol{v}$.

Theorem. *For a given subspace of $\mathbb{R}^n$, any two bases have the same number of elements.* $\square$

Example. Since $\mathbb{R}^n$ has one basis with n elements, if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is any other basis, necessarily $m = n$. ☑

Definition. The ***dimension*** of a subspace V of $\mathbb{R}^n$, denoted $\dim(V)$, is the number of elements in any basis. $\square$

Example. Since $\mathcal{S} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a basis for $\mathbb{R}^3$, we have $\dim(\mathbb{R}^3) = 3$. ☑

If V is a subspace of $\mathbb{R}^n$, then a basis for a vector space V is a set of vectors which

- span V
- are linearly independent.

If $\dim(V) = n$ and there are exactly n vectors, then either one of the above conditions implies the other.

Proposition. *Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a set of vectors in V, where $n = \dim(V)$. If the vectors span V, then they are linearly independent, and so form a basis.* $\square$

Proposition. *Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a set of vectors in V, where $n = \dim(V)$. If the vectors are linearly independent, then they span V and so form a basis.* $\square$

Remark. You need to already know the dimension of the vector space to use on of the above propositions, but in many cases we already do. ⊠

Example. ***Is $\{v_1, v_2, v_3, v_4\}$ a basis for $\mathbb{R}^4$, where***

$$v_1 = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{1} \\ \mathbf{4} \end{pmatrix}$$

$$v_2 = \begin{pmatrix} \mathbf{2} \\ \mathbf{1} \\ \mathbf{4} \\ \mathbf{6} \end{pmatrix}$$

$$v_3 = \begin{pmatrix} \mathbf{3} \\ \mathbf{2} \\ \mathbf{1} \\ \mathbf{1} \end{pmatrix}$$

$$v_4 = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{1} \\ \mathbf{3} \end{pmatrix} \text{?}$$

Since we have 4 vectors in $\mathbb{R}^4$, which has dimension 4, it suffices to either show the vectors are independent or span $\mathbb{R}^4$. We can easily check to see if they are independent: the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 2 \\ 1 & 4 & 1 & 1 \\ 4 & 6 & 1 & 3 \end{pmatrix}$$

can be reduced to echelon form

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since there are no free variables, the vectors are independent, and so form a basis. ☑

Example. ***Is*** $\{v_1, v_2, v_3, v_4\}$ ***a basis for*** $\mathbb{R}^3$***, where***

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$
$$v_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$
$$v_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$
$$v_4 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} ?$$

Since there are 4 vectors in a space with dimension 3, the set is not a basis. ☑

If a subspace V has basis $\{v_1, \ldots, v_k\}$, then any element of V can be specified by choosing values of the coefficients $c_1, \ldots, c_k$ in the expression

$$c_1 v_1 + \cdots + c_k v_k.$$

The dimension, k in this case, can be informally thought of as the number of choices we need to make in order to specify an element of the space.

Example. The set

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + z = 0 \right\}$$

is a subspace of $\mathbb{R}^3$, any element is of the form $(xy - x)^T$. To specify an element of V, it suffices to choose a value for x and a value for y (note that there will then be no choice in the value for z). Since there are two choices, the dimension of V is 2.

A basis for V can be obtained by going through the choices one by one, letting each choice be 1 and the others 0. For V, if we let $x = 1$ and

$y = 0$, we get the vector $(1 \quad 0 \quad -1)^T$, and if we let $y = 1$ and $x = 0$, we get the vector $(0 \quad 1 \quad 0)^T$. A basis for V is then

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$