Chapter 13

Shortest paths

Finding the shortest path between two nodes of a graph is an important problem that has many applications in practice. For example, a natural problem in a road network is to calculate the length of the shortest route between two cities, given the lengths of the roads.

In an unweighted graph, the length of a path equals the number of edges in the path and we can simply use breadth-first search to find the shortest path. However, in this chapter we concentrate on weighted graphs where more sophisticated algorithms are needed for finding shortest paths.

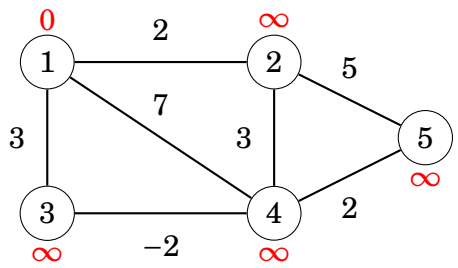
13.1 Bellman–Ford algorithm

The **Bellman–Ford algorithm**[1] finds the shortest paths from a starting node to all other nodes in the graph. The algorithm can process all kinds of graphs, provided that the graph does not contain a cycle with negative length. If the graph contains a negative cycle, the algorithm can detect this.

The algorithm keeps track of distances from the starting node to other nodes. Initially, the distance to the starting node is 0 and the distance to all other nodes in infinite. The algorithm reduces the distances by finding edges that shorten the paths until it is not possible to reduce any distance.

Example

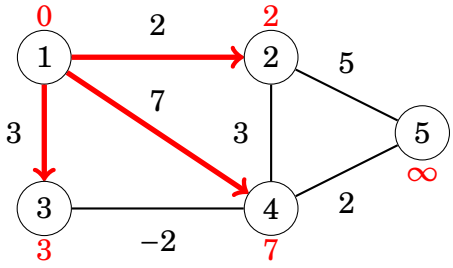
Let us consider how the Bellman–Ford algorithm works in the following graph:



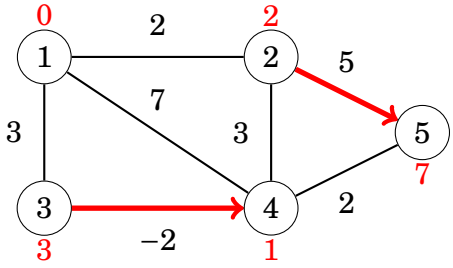
[1]The algorithm is named after R. E. Bellman and L. R. Ford who published it independently in 1958 and 1956, respectively [5, 21].

Each node in the graph is assigned a distance. Initially, the distance to the starting node is 0, and the distance to all other nodes is infinite.

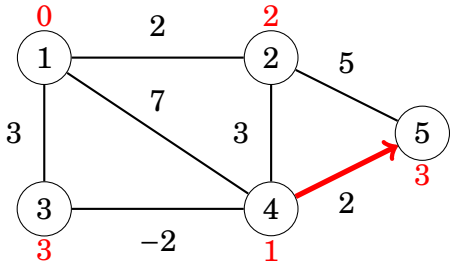
The algorithm searches for edges that reduce distances. First, all edges from node 1 reduce distances:



After this, edges $2 \to 5$ and $3 \to 4$ reduce distances:

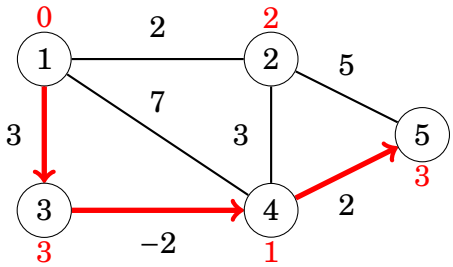


Finally, there is one more change:



After this, no edge can reduce any distance. This means that the distances are final and we have successfully calculated the shortest distance from the starting node to all other nodes.

For example, the shortest distance 3 from node 1 to node 5 corresponds to the following path:



Implementation

The following implementation of the Bellman–Ford algorithm finds the shortest distances from a node x to all other nodes in the graph. The code assumes that the graph is stored as adjacency lists in an array

```
vector<pair<int,int>> v[N];
```

as pairs of the form (x,w): there is an edge to node x with weight w.

The algorithm consists of $n-1$ rounds, and on each round the algorithm goes through all edges of the graph and tries to reduce the distances. The algorithm constructs an array `e` that will contain the distance from x to all nodes in the graph. The initial value 10^9 means infinity.

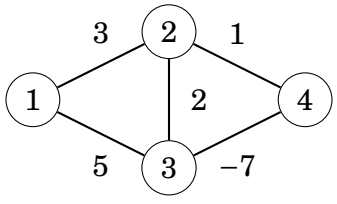
```
for (int i = 1; i <= n; i++) e[i] = 1e9;
e[x] = 0;
for (int i = 1; i <= n-1; i++) {
    for (int a = 1; a <= n; a++) {
        for (auto b : v[a]) {
            e[b.first] = min(e[b.first],e[a]+b.second);
        }
    }
}
```

The time complexity of the algorithm is $O(nm)$, because the algorithm consists of $n-1$ rounds and iterates through all m edges during a round. If there are no negative cycles in the graph, all distances are final after $n-1$ rounds, because each shortest path can contain at most $n-1$ edges.

In practice, the final distances can usually be found faster than in $n-1$ rounds. Thus, a possible way to make the algorithm more efficient is to stop the algorithm if no distance can be reduced during a round.

Negative cycle

The Bellman–Ford algorithm can be also used to check if the graph contains a cycle with negative length. For example, the graph



contains a negative cycle $2 \to 3 \to 4 \to 2$ with length -4.

If the graph contains a negative cycle, we can shorten a path that contains the cycle infinitely many times by repeating the cycle again and again. Thus, the concept of a shortest path is not meaningful in this situation.

A negative cycle can be detected using the Bellman–Ford algorithm by running the algorithm for n rounds. If the last round reduces any distance, the graph

contains a negative cycle. Note that this algorithm can be used to search for a negative cycle in the whole graph regardless of the starting node.

SPFA algorithm

The **SPFA algorithm** ("Shortest Path Faster Algorithm") [17] is a variant of the Bellman–Ford algorithm, that is often more efficient than the original algorithm. The SPFA algorithm does not go through all the edges on each round, but instead, it chooses the edges to be examined in a more intelligent way.

The algorithm maintains a queue of nodes that might be used for reducing the distances. First, the algorithm adds the starting node x to the queue. Then, the algorithm always processes the first node in the queue, and when an edge $a \to b$ reduces a distance, node b is added to the queue.

The following implementation uses a `queue` `q`. In addition, an array `z` indicates if a node is already in the queue, in which case the algorithm does not add the node to the queue again.

```
for (int i = 1; i <= n; i++) e[i] = 1e9;
e[x] = 0;
q.push(x);
while (!q.empty()) {
    int a = q.front(); q.pop();
    z[a] = 0;
    for (auto b : v[a]) {
        if (e[a]+b.second < e[b.first]) {
            e[b.first] = e[a]+b.second;
            if (!z[b]) {q.push(b); z[b] = 1;}
        }
    }
}
```

The efficiency of the SPFA algorithm depends on the structure of the graph: the algorithm is often efficient, but its worst case time complexity is still $O(nm)$ and it is possible to create inputs that make the algorithm as slow as the original Bellman–Ford algorithm.

13.2 Dijkstra's algorithm

Dijkstra's algorithm[2] finds the shortest paths from the starting node to all other nodes, like the Bellman–Ford algorithm. The benefit in Dijsktra's algorithm is that it is more efficient and can be used for processing large graphs. However, the algorithm requires that there are no negative weight edges in the graph.

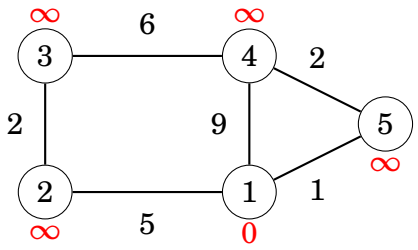
Like the Bellman–Ford algorithm, Dijkstra's algorithm maintains distances to the nodes and reduces them during the search. Dijkstra's algorithm is efficient,

[2] E. W. Dijkstra published the algorithm in 1959 [12]; however, his original paper does not mention how to implement the algorithm efficiently.

because it only processes each edge in the graph once, using the fact that there are no negative edges.

Example

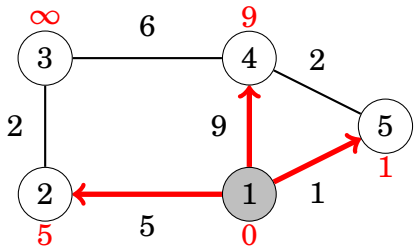
Let us consider how Dijkstra's algorithm works in the following graph when the starting node is node 1:



Like in the Bellman–Ford algorithm, initially the distance to the starting node is 0 and the distance to all other nodes is infinite.

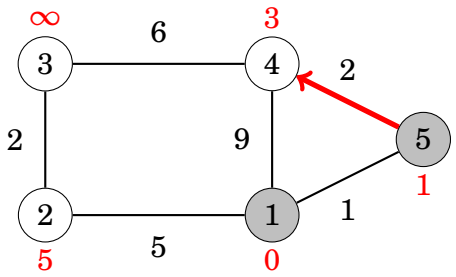
At each step, Dijkstra's algorithm selects a node that has not been processed yet and whose distance is as small as possible. The first such node is node 1 with distance 0.

When a node is selected, the algorithm goes through all edges that start at the node and reduces the distances using them:

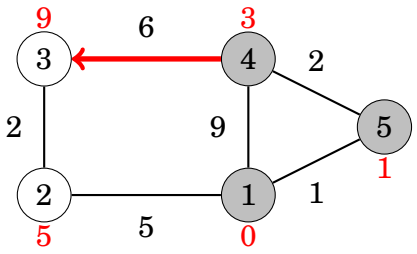


The edges from node 1 reduced distances to nodes 2, 4 and 5, whose distances are now 5, 9 and 1.

The next node to be processed is node 5 with distance 1:

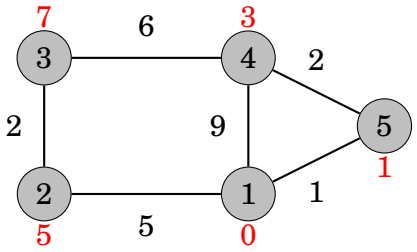


After this, the next node is node 4:



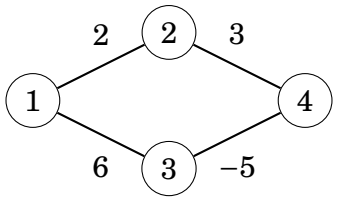
A remarkable property in Dijkstra's algorithm is that whenever a node is selected, its distance is final. For example, at this point of the algorithm, the distances 0, 1 and 3 are the final distances to nodes 1, 5 and 4.

After this, the algorithm processes the two remaining nodes, and the final distances are as follows:



Negative edges

The efficiency of Dijkstra's algorithm is based on the fact that the graph does not contain negative edges. If there is a negative edge, the algorithm may give incorrect results. As an example, consider the following graph:



The shortest path from node 1 to node 4 is $1 \to 3 \to 4$ and its length is 1. However, Dijkstra's algorithm finds the path $1 \to 2 \to 4$ by following the minimum weight edges. The algorithm does not take into account that on the other path, the weight -5 compensates the previous large weight 6.

Implementation

The following implementation of Dijkstra's algorithm calculates the minimum distances from a node x to all other nodes. The graph is stored in an array v as adjacency lists like in the Bellman–Ford algorithm.

An efficient implementation of Dijkstra's algorithm requires that it is possible to efficiently find the minimum distance node that has not been processed. An appropriate data structure for this is a priority queue that contains the nodes

ordered by their distances. Using a priority queue, the next node to be processed can be retrieved in logarithmic time.

In the following implementation, the priority queue contains pairs whose first element is the current distance to the node and second element is the identifier of the node.

```
priority_queue<pair<int,int>> q;
```

A small difficulty is that in Dijkstra's algorithm, we should find the node with the *minimum* distance, while the C++ priority queue finds the *maximum* element as default. An easy trick is to use *negative* distances, which allows us to directly use the C++ priority queue.

The code keeps track of processed nodes in an array z, and maintains the distances in an array e. Initially, the distance to the starting node is 0, and the distance to all other nodes is 10^9 (infinite).

```
for (int i = 1; i <= n; i++) e[i] = 1e9;
e[x] = 0;
q.push({0,x});
while (!q.empty()) {
    int a = q.top().second; q.pop();
    if (z[a]) continue;
    z[a] = 1;
    for (auto b : v[a]) {
        if (e[a]+b.second < e[b.first]) {
            e[b.first] = e[a]+b.second;
            q.push({-e[b.first],b.first});
        }
    }
}
```

The time complexity of the above implementation is $O(n + m \log m)$ because the algorithm goes through all nodes in the graph and adds for each edge at most one distance to the priority queue.

13.3 Floyd–Warshall algorithm

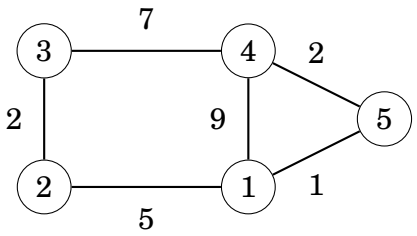
The **Floyd–Warshall algorithm**[3] is an alternative way to approach the problem of finding shortest paths. Unlike the other algorihms in this chapter, it finds all shortest paths between the nodes in a single run.

The algorithm maintains a two-dimensional array that contains distances between the nodes. First, the distances are calculated only using direct edges between the nodes. After this the algorithm reduces the distances by using intermediate nodes in the paths.

[3]The algorithm is named after R. W. Floyd and S. Warshall who published it independently in 1962 [20, 60].

Example

Let us consider how the Floyd–Warshall algorithm works in the following graph:



Initially, the distance from each node to itself is 0, and the distance between nodes a and b is x if there is an edge between nodes a and b with weight x. All other distances are infinite.

In this graph, the initial array is as follows:

	1	2	3	4	5
1	0	5	∞	9	1
2	5	0	2	∞	∞
3	∞	2	0	7	∞
4	9	∞	7	0	2
5	1	∞	∞	2	0

The algorithm consists of consecutive rounds. On each round, the algorithm selects a new node that can act as an intermediate node in paths from now on, and the algorithm reduces the distances in the array using this node.

On the first round, node 1 is the new intermediate node. There is a new path between nodes 2 and 4 with length 14, because node 1 connects them. There is also a new path between nodes 2 and 5 with length 6.

	1	2	3	4	5
1	0	5	∞	9	1
2	5	0	2	**14**	**6**
3	∞	2	0	7	∞
4	9	**14**	7	0	2
5	1	**6**	∞	2	0

On the second round, node 2 is the new intermediate node. This creates new paths between nodes 1 and 3 and between nodes 3 and 5:

	1	2	3	4	5
1	0	5	**7**	9	1
2	5	0	2	14	6
3	**7**	2	0	7	**8**
4	9	14	7	0	2
5	1	6	**8**	2	0

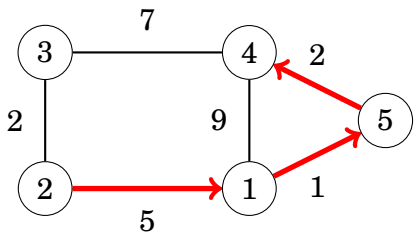
On the third round, node 3 is the new intermediate round. There is a new path between nodes 2 and 4:

	1	2	3	4	5
1	0	5	7	9	1
2	5	0	2	**9**	6
3	7	2	0	7	8
4	9	**9**	7	0	2
5	1	6	8	2	0

The algorithm continues like this, until all nodes have been appointed intermediate nodes. After the algorithm has finished, the array contains the minimum distances between any two nodes:

	1	2	3	4	5
1	0	5	7	3	1
2	5	0	2	9	6
3	7	2	0	7	8
4	3	9	7	0	2
5	1	6	8	2	0

For example, the array tells us that the shortest distance between nodes 2 and 4 is 8. This corresponds to the following path:



Implementation

The advantage of the Floyd–Warshall algorithm that it is easy to implement. The following code constructs a distance matrix `d` where `d[`a`][`b`]` is the shortest distance between nodes a and b. First, the algorithm initializes `d` using the adjacency matrix `v` of the graph (10^9 means infinity):

```
for (int i = 1; i <= n; i++) {
    for (int j = 1; j <= n; j++) {
        if (i == j) d[i][j] = 0;
        else if (v[i][j]) d[i][j] = v[i][j];
        else d[i][j] = 1e9;
    }
}
```

After this, the shortest distances can be found as follows:

```
for (int k = 1; k <= n; k++) {
    for (int i = 1; i <= n; i++) {
        for (int j = 1; j <= n; j++) {
            d[i][j] = min(d[i][j], d[i][k]+d[k][j]);
        }
    }
}
```

The time complexity of the algorithm is $O(n^3)$, because it contains three nested loops that go through the nodes in the graph.

Since the implementation of the Floyd–Warshall algorithm is simple, the algorithm can be a good choice even if it is only needed to find a single shortest path in the graph. However, the algorithm can only be used when the graph is so small that a cubic time complexity is fast enough.