ALM Project

Projects have to be chosen by replying to email to Ludovic.moreau@abeille-assurances.fr as soon as possible and no later than 3rd March 2023.

2 people max per project.

Projects have to be sent back to Ludovic.moreau@abeille-assurances.fr by 31st March 2023.

Projects have to be in doc or pdf format.

The results have to be documented via mathematical formula and reasoning, taking into account the assumptions to be used at each time step (this is part will be evaluated)

The quality of the redaction will be evaluated.

Project 1 – Simple Asset & Liability case

Part 1

Assume that the term structure of annually compounded interest rate $T \longmapsto Y(t_0, T)$ is given at initial time $t_0 = 0$:

Maturity (T)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Y(0,T)	-0.14%	-0.19%	-0.10%	0.03%	0.17%	0.30%	0.41%	0.50%	0.59%	0.66%	0.74%	0.80%	0.87%	0.93%	1.00%

Consider the following risk-free bonds (of annual frequency of payments):

- B^1: Nominal $N^1 = 100$, fixed coupon rate $c^1 = 4\%$, maturity $T^1 = 14$;
- B^2: Nominal $N^2 = 60$, fixed coupon rate $c^2 = 3\%$, maturity $T^2 = 10$;
- B^1: Nominal $N^3 = 80$, fixed coupon rate $c^3 = 2\%$, maturity $T^3 = 6$.

1. Compute the risk-free prices $P(0, T)$ of ZC bonds for $T \in \{1, \cdots, 20\}$.
2. For each of the previous bonds:
   a. Compute the risk-free prices B_0^1, B_0^2 and B_0^3.
   b. Compute the internal rates of return y^1, y^2 and y^3.
   c. Compute the durations d^1, d^2 and d^3.
3. Using the durations d^1, d^2 and d^3, approximate the prices of B^1, B^2 and B^3 in the case of a parallel shift of the interest rate curve of x, i.e. $Y'(t_0, T) = Y(t_0, T) + x$ for all T, for $x = 1\%, x = 2\%$ and $x = 4\%$. Compare it to the real prices of these bonds.
4. Recall the definition of a bond *issued-at-par*.
   a. Give the expression of the fixed coupon rate $c^{par,T}$ of a risk-free bond of annual frequency of payments, maturity T and nominal N issued-at-par.
   b. Compute $c^{par,T}$ for $T \in \{6; 10; 14\}$ and compare it to c^1, c^2 and c^3.
5. Assume now that the market prices of B^1, B^2 and B^3 are given by:

- $V_0^1 = 122.96;$
- $V_0^2 = 68.32;$
- $V_0^3 = 87.00.$

Compute the implied credit spreads s^1, s^2 and s^3 associated to these bonds.

6. Consider now that the market spreads are the following:

Rating	AAA	AA	A	BBB	BB	B	CCC
Average spread	0.16%	0.20%	0.27%	0.44%	0.89%	1.50%	2.55%

a. Give the expression of the fixed coupon rate $c^{par,T,r}$ of a risky bond of annual frequency of payments, maturity T, nominal N and fixed spread s^r associated to rating r issued-at-par.

b. Compute $c^{par,T,r}$ for $T \in \{6; 10; 14\}$ and $r \in \{AAA; AA; A; BBB; BB; B; CCC\}$.

Part 2

Assume that the term structure of annually compounded interest rate $T \longmapsto Y(t,T)$ evolves as time goes by as follows:

Maturity (T)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Y(0,T)	-0.14%	-0.19%	-0.10%	0.03%	0.17%	0.30%	0.41%	0.50%	0.59%	0.66%	0.74%	0.80%	0.87%	0.93%	1.00%
Y(1,T)	-0.04%	-0.04%	0.10%	0.28%	0.47%	0.65%	0.81%	0.95%	1.09%	1.21%	1.34%	1.45%	1.57%	1.68%	1.80%
Y(2,T)	0.06%	0.11%	0.30%	0.53%	0.77%	1.00%	1.21%	1.40%	1.59%	1.76%	1.94%	2.10%	2.27%	2.43%	2.60%
Y(3,T)	0.16%	0.26%	0.50%	0.78%	1.07%	1.35%	1.61%	1.85%	2.09%	2.31%	2.54%	2.75%	2.97%	3.18%	3.40%
Y(4,T)	0.26%	0.41%	0.70%	1.03%	1.37%	1.70%	2.01%	2.30%	2.59%	2.86%	3.14%	3.40%	3.67%	3.93%	4.20%
Y(5,T)	0.36%	0.56%	0.90%	1.28%	1.67%	2.05%	2.41%	2.75%	3.09%	3.41%	3.74%	4.05%	4.37%	4.68%	5.00%

1. Compute the risk-free prices B_t^1, B_t^2 and B_t^3 as time goes by in this scenario.
2. Give the expression of the fixed coupon rate $c_t^{par,T,r}$ of a risky bond of annual frequency of payments, maturity T, nominal N and fixed spread s^r associated to rating r issued-at-par at time t (i.e. given interest rate curve $T \longmapsto Y(t,T)$.

a. Compute $c_t^{par,T,r}$ for $T \in \{6; 10; 14\}$, $r = BBB$ and $t \in \{1, \cdots, 5\}$.

b. Comment on the evolution of $c_t^{par,T,r}$ as time goes by.

Part 3

Consider now the case of an A/L manager which asset is composed of one unit of B^1, B^2 and B^3 and liabilities are given by the following cash-flows (to be paid to its clients):

Maturity (T)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
CF(T)	5.0	5.0	5.0	5.0	5.0	100.0	6.0	6.0	6.0	60.0	5.0	5.0	5.0	100.0	5.0

1. Compute the Liabilities values L_t and periodic liquidity gap LG_t for $t \in \{1, \cdots, 5\}$ and comment on the liquidity risk of the company.

Project 2 - The Life insurance case

We consider in this problem an insurer of initial portfolio as follows:

€m	Book value	Nominal	Coupon	Maturity	Spread
Cash	170		0,0%		
Bond 1	100	100	4,0%	1	0,17%
Bond 2	200	200	3,2%	2	0,23%
Bond 3	400	400	2,8%	4	0,26%
Bond 4	500	500	2,1%	5	0,24%
Bond 5	420	420	1,7%	7	0,40%
Bond 6	280	280	1,5%	8	0,50%
Bond 7	210	210	1,4%	10	0,70%
Bond 8	150	150	1,3%	12	0,80%
Bond 9	110	110	1,8%	14	1,10%
Bond 10	70	70	1,6%	15	1,00%

The term structure of interest rate is given at every time t by ($Y(t,T)$ denoting the interest rate of maturity T at time t):

Maturity (T)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Y(0,T)	-0,14%	-0,19%	-0,10%	0,03%	0,17%	0,30%	0,41%	0,50%	0,59%	0,66%	0,74%	0,80%	0,87%	0,93%	1,00%	1,06%	1,13%	1,19%	1,25%	1,31%
Y(1,T)	-0,04%	-0,04%	0,10%	0,28%	0,47%	0,61%	0,73%	0,83%	0,93%	1,01%	1,10%	1,17%	1,25%	1,32%	1,40%	1,47%	1,55%	1,62%	1,69%	1,76%
Y(2,T)	0,06%	0,11%	0,30%	0,53%	0,77%	0,92%	1,05%	1,16%	1,27%	1,36%	1,46%	1,54%	1,63%	1,71%	1,80%	1,88%	1,97%	2,05%	2,13%	2,21%
Y(3,T)	0,16%	0,26%	0,50%	0,78%	1,07%	1,23%	1,37%	1,49%	1,61%	1,71%	1,82%	1,91%	2,01%	2,10%	2,20%	2,29%	2,39%	2,48%	2,57%	2,66%
Y(4,T)	0,26%	0,41%	0,70%	1,03%	1,37%	1,54%	1,69%	1,82%	1,95%	2,06%	2,18%	2,28%	2,39%	2,49%	2,60%	2,70%	2,81%	2,91%	3,01%	3,11%
Y(5,T)	0,36%	0,56%	0,90%	1,28%	1,67%	1,85%	2,01%	2,15%	2,29%	2,41%	2,54%	2,65%	2,77%	2,88%	3,00%	3,11%	3,23%	3,34%	3,45%	3,56%
Y(6,T)	0,46%	0,71%	1,10%	1,53%	1,97%	2,16%	2,33%	2,48%	2,63%	2,76%	2,90%	3,02%	3,15%	3,27%	3,40%	3,52%	3,65%	3,77%	3,89%	4,01%
Y(7,T)	0,56%	0,86%	1,30%	1,78%	2,27%	2,47%	2,65%	2,81%	2,97%	3,11%	3,26%	3,39%	3,53%	3,66%	3,80%	3,93%	4,07%	4,20%	4,33%	4,46%
Y(8,T)	0,56%	0,86%	1,30%	1,78%	2,27%	2,47%	2,65%	2,81%	2,97%	3,11%	3,26%	3,39%	3,53%	3,66%	3,80%	3,93%	4,07%	4,20%	4,33%	4,46%
Y(9,T)	0,66%	1,01%	1,50%	2,03%	2,57%	2,78%	2,97%	3,14%	3,31%	3,46%	3,62%	3,76%	3,91%	4,05%	4,20%	4,34%	4,49%	4,63%	4,77%	4,91%
Y(10,T)	0,66%	1,01%	1,50%	2,03%	2,57%	2,78%	2,97%	3,14%	3,31%	3,46%	3,62%	3,76%	3,91%	4,05%	4,20%	4,34%	4,49%	4,63%	4,77%	4,91%

The spread are assumed constant over time and given by:

Rating	**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
Average spread	0.16%	0.20%	0.27%	0.44%	0.89%	1.50%	2.55%

We shall assume in all this problem that the book values remain constant over time.

We shall assume that the market value of the cash is always equal to its book value and that the cash carries no interest.

We assume that the Solvency Capital Requirement (SCR) for the interest rate risk consist in the computation of the market value of the portfolio in the worst of the cases between an interest rate up and down scenarios, where these scenario are given by:

$$Y^{up}(t_0,T) = Y(t_0,T) \times (1+\delta^{\text{up}}) \quad \text{and} \quad Y^{up}(t_0,T) = Y(t_0,T) \times \left(1+\delta^{\text{down}}\right)$$

with

maturity \ choc	shock up	shock down
1	70%	-75%
2	70%	-65%
3	68%	-63%
4	65%	-61%
5	63%	-59%
6	60%	-57%
7	58%	-55%
8	55%	-53%
9	53%	-51%
10	50%	-49%
11	48%	-47%
12	46%	-45%
13	43%	-43%
14	41%	-41%
15	38%	-39%
16	36%	-37%
17	33%	-35%
18	31%	-33%
19	28%	-31%
20	26%	-29%

We assume now that the liability of this insurer is given by an initial Mathematical Reserve (book value) $MR_{t_0} = 2088$ and that this mathematical reserve is given at any time by:

$$MR_{t+1} = MR_t \times (1 - \mathrm{sr}) \times (1 + \mathrm{ps})$$

where sr and ps denote respectively the surrender rate (including mortality and lapse) and profit sharing of the portfolio. The outflows of the insurer are given at any time $t > 0$ by

$$CF_t^L = \mathrm{sr} \times MR_t.$$

<u>Part 1</u>

1. Compute the total market values of the asset side of this balance sheet at time $t_0 = 0$.
2. Compute the duration and the internal rate of return of this portfolio (asset side).
3. Compute for each time t the coupon rate of a bond of *time-to-maturity* 7 years and rating A such that this bond is issued at par at time t.
4. Assuming that every coupon or nominal payments are invested in cash as time goes by, what is at every time interest rate of the portfolio (excluding nominal payments and compared to the total book value of the portfolio for every time $t > 0$.
5. Assuming that coupon payments are invested in cash and that the nominal payments are invested in a bond of rating A, *time-to-maturity* 7 years issued at par (of book value equal to its market value at origination), what is at every time the interest rate of the portfolio.
6. Compute the initial value of the portfolio in the interest rate up and down scenario described above and deduce the SCR of the (asset) portfolio.

<u>Part 2</u>

We consider in this part that the surrender rate is fixed and constant at any time, $\mathrm{sr} = 7\%$ and the profit sharing credited by the insurer on the mathematical reserves is equal to 85% of the interest rate of the portfolio as obtained by a re-investment of the coupon in cash and the nominal payments in a 7 years bond of rating A issue at par (cf. Part 1.3).

We assume that the cash outflows are paid with the cash of the insurer.

1. Compute at each time $t > 0$ the mathematical reserve MR_t and the cash outflows of the insurer.
2. Assuming (for simplicity) that the last mathematical reserve MR_{10} is paid at maturity 15 with no intermediary payments, compute the market value of the liability.
3. Following the same reasoning of the previous question, compute the SCR of the company on its own funds (asset – liability) using appropriate market assumptions.

<u>Part 3</u>

We will consider now that the liability of the insurer are subject to dynamic lapse and offers options and guarantees.

1. We consider that the surrender rate of the insurer at every time t is subject to an additional dynamic lapse rate given by

$$\mathrm{sr_t} = 7\% + 4\% \times 1_{\{ps_t < Y(t,10)\}}.$$

   Compute at each time $t > 0$ the mathematical reserve MR_t and the cash outflows of the insurer.

2. In the context of the previous question, what is the option at the hand of the insurer on the credited rate ps to reduce its exposure to the dynamic lapse?
3. In the context of Question 1, we now consider that 20% of the engagement of the insurer has an additional guarantee of 3% of the mathematical reserve at any time, i.e. the mathematical reserve are credited at any time of a credit rate:

$$\mathrm{cr}_t = \min(pf_t; 3\% \times MR_{t-1}).$$

   Compute at each time $t > 0$ the mathematical reserve MR_t and the cash outflows of the insurer.

Project 3 – Portfolio simulation and hedging

Consider the following market conditions (index t denotes the time of valuation whereas τ denotes the time-to-maturity, so that $Y(t,\tau)$ denotes the actuarially compounded interest rate of time-to-maturity τ at time t).

time t	0	1	2	3	4	5
Y(t,1)	-0,14%	-0,24%	-0,08%	0,09%	0,25%	0,39%
Y(t,2)	-0,19%	0,08%	0,25%	0,41%	0,55%	0,66%
Y(t,3)	-0,10%	0,42%	0,58%	0,71%	0,80%	0,88%
Y(t,4)	0,03%	0,73%	0,85%	0,93%	0,99%	1,06%
Y(t,5)	0,17%	0,95%	1,02%	1,06%	1,14%	1,18%
Y(t,6)	0,30%	1,30%	1,36%	1,38%	1,41%	1,43%
Y(t,7)	0,41%	1,60%	1,64%	1,67%	1,71%	1,74%
Y(t,8)	0,50%	1,91%	1,92%	1,97%	2,01%	2,06%
Y(t,9)	0,59%	2,21%	2,20%	2,26%	2,31%	2,37%
Y(t,10)	0,66%	2,51%	2,48%	2,55%	2,61%	2,69%
Spread of A bond	0,27%	0,25%	0,27%	0,27%	0,25%	0,26%
Spread of BBB bond	0,44%	0,42%	0,44%	0,44%	0,42%	0,43%
Spread of BB bond	0,89%	0,87%	0,89%	0,89%	0,87%	0,88%
Corporate stock market	1	1,01	1,00	0,99	1,01	1,00

Consider an insurer initial portfolio (at time $t = 0$):

€m	Book value	Fair Value	Nominal	Coupon	Maturity	Spread	Rating
Cash	79,1	79,1		0,00%			
Stock (100% correlated to Stock Market index)	158,2	180,8					
Bond 1	100	TBC	100	4,0%	1	0,17%	A
Bond 2	200	TBC	200	3,2%	2	0,23%	A
Bond 3	400	TBC	400	2,8%	4	0,26%	A
Bond 4	500	TBC	500	2,1%	5	0,24%	A
Bond 5	420	TBC	420	1,7%	7	0,40%	BBB
Bond 6	280	TBC	280	1,5%	8	0,50%	BBB
Bond 7	210	TBC	210	1,4%	10	0,70%	BB
Bond 8	150	TBC	150	1,3%	12	0,80%	BB

We shall assume in all this problem that the book values remains constant over time.

We shall assume that the market value of the cash is always equal to its book value and that the cash carries no interest.

1. **Valuation of the In-Force** (no New Business)
    a. Compute the market value, duration and internal rate of return of each of the bonds in portfolio.
    b. Assuming that the corporate stock market evolves in the previous conditions and carries a dividend of 2% of the market value at any time, project the cash statement of the company for each time $t \in \{1; 2; 3; 4; 5\}$.
    c. Assuming that the multiplicative variations of the spreads of the bonds are 100% correlated to the multiplicative variations of the associated market spread, compute the market values of the portfolio at any time $t \in \{1; 2; 3; 4; 5\}$.
    d. Compute the duration of the portfolio.

2. **With New Business** - Assume now that the Strategic Asset Allocation (SAA) of the insurer is given at any time by the initial portfolio mix, SAA target for asset class i is given by $x^i := \frac{VM_0^i}{\sum_i VM_0^i}$.
    a. Compute x^i for $i \in \{Cash, Stock, A\ rated\ bond, BBB\ rated\ bonds, BB\ rated\ bonds\}$ and compare with the evolution of the asset mix computed in Question 1.
    b. In the market conditions of the central scenario, compute the coupon rate c_t^j for any time $t \in \{1; 2; 3; 4; 5\}$ and rating $j \in \{A, BBB, BB\}$ (c_t^j denoting the coupon of a 10 years maturity bond of rating j issued at par at time t).
    c. Describes the amount to be sold or purchased for all asset class at any time to respect the insurer SAA of Question 2.a (purchases of bond are made at any time on 10-year maturity bonds issued at par) and compute the correspond realized gains and losses at any time (difference between market and book values).
3. **Under Market Stress** - Redo Questions 2.b and 2.c under the following market stressed conditions:

time t	**0**	**1**	**2**	**3**	**4**	**5**
Y(t,1)	-0,14%	-0,24%	1,92%	2,09%	2,25%	2,39%
Y(t,2)	-0,19%	0,08%	2,25%	2,41%	2,55%	2,66%
Y(t,3)	-0,10%	0,42%	2,58%	2,71%	2,80%	2,88%
Y(t,4)	0,03%	0,73%	2,85%	2,93%	2,99%	3,06%
Y(t,5)	0,17%	0,95%	3,02%	3,06%	3,14%	3,18%
Y(t,6)	0,30%	1,30%	3,36%	3,38%	3,41%	3,43%
Y(t,7)	0,41%	1,60%	3,64%	3,67%	3,71%	3,74%
Y(t,8)	0,50%	1,91%	3,92%	3,97%	4,01%	4,06%
Y(t,9)	0,59%	2,21%	4,20%	4,26%	4,31%	4,37%
Y(t,10)	0,66%	2,51%	4,48%	4,55%	4,61%	4,69%
Spread of A bond	0,27%	0,25%	0,57%	0,47%	0,35%	0,26%
Spread of BBB bond	0,44%	0,42%	0,94%	0,74%	0,52%	0,43%
Spread of BB bond	0,89%	0,87%	1,79%	1,49%	1,17%	0,88%
Corporate stock market	1	0,95	0,88	0,95	1,03	1,06

4. **Hedging position** – Consider now the case of an insurer having designed an equity hedge portfolio based on the stressed scenario of Question 3.
    a. Compute the Black-Scholes price at initial time $t = 0$ of a 2 years European put option, which underlying is the corporate stock market index and money ness is 88% ($r = -0.19\%$ and $\sigma = 12\%$).
    b. Compute the Black-Scholes price at time $t = 1$ and $t = 2$ of this put option in the market conditions of Question 3 ($r = -0.24\%$ at time $t = 1$). What is the impact at time $t \in \{1,2\}$ on the insurer portfolio if the insurer purchased a notional 158.2 of this put option at initial time?

Project 4 – Interest Rate internal model and interest rate hedging

Perform a Principal Component Analysis (PCA) on 30 years of monthly interest rate term structure additive changes, using up to 3 eigen vectors. Comment on the result.

Propose a simulation scheme (based on temporal series) allowing a distribution of Interest rate term structure in 1 year, using monthly simulations

Consider now the case where initial interest rate term structure, spread, asset portfolio and liability cash flows are the following:

Maturity	t_0 IR
0	0.2800000%
1	-0.1361838%
2	-0.1924160%
3	-0.1034311%
4	0.0313587%
5	0.1694032%
6	0.2952019%
7	0.4053898%
8	0.5014824%
9	0.5864976%
10	0.6634972%
11	0.7350479%
12	0.8030989%
13	0.8690314%
14	0.9337643%
15	0.9978650%

Rating	AAA	AA	A	BBB	BB	B	CCC
Average spread	0.16%	0.20%	0.27%	0.44%	0.89%	1.50%	2.55%

€m	Book value	Nominal	Coupon	Maturity	Spread
Cash	170		0,0%		
Bond 1	100	100	4,0%	1	0,17%
Bond 2	200	200	3,2%	2	0,23%
Bond 3	400	400	2,8%	4	0,26%
Bond 4	500	500	2,1%	5	0,24%
Bond 5	420	420	1,7%	7	0,40%
Bond 6	280	280	1,5%	8	0,50%
Bond 7	210	210	1,4%	10	0,70%
Bond 8	150	150	1,3%	12	0,80%
Bond 9	110	110	1,8%	14	1,10%
Bond 10	70	70	1,6%	15	1,00%

Time	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Liability cash flows	122.00	146.40	146.40	170.80	195.20	219.60	244.00	268.40	244.00	219.60	170.80	146.40	73.20	48.80	24.40

Compute the initial market values of asset and liability and assess the shareholders equity value.

Using the Interest rate term structure distribution and considering the spread are fixed, compute the distribution of the shareholder equity value in 1 year. What is the loss of probability 1/200.

Assess the exposure to interest rate and propose an interest rate hedging strategy allowing the hedge of this exposure. Assess the distribution of shareholder equity value in case such a strategy is chosen.