Cardinality of the constrained observation set

Let d denote the dimension of the square grid-word, let $n_c := d^2$ denote the number of cells and let n_h denote the number of hiding spots. Let $\mathcal{C} := \{0, 1, 2\}$ denote the set of possible cell colors, where 0 encodes a black cell color (cell has not been investigated by drilling), 1 encodes a gray cell color (cell has been investigated by drilling and identified as not a hiding spot) and 2 encodes a blue cell color (cell has been investigated by drilling and identified as a hiding spot). As a single cell can only take on a single color at any trial during the game, the unconstrained observation set is given by

$$\mathcal{O} := \mathcal{C}^{n_c} \tag{1}$$

with cardinality

$$|\mathcal{O}| := |\mathcal{C}|^{n_c}. \tag{2}$$

Let $o \in \mathcal{O}$ denote an observation and let n_g^o and n_b^o denote the number of 1s (gray cells) and 2s (blue cells) of this observation. Then by the rules of the game, the constrained observation set is defined as

$$O := \{o \in \mathcal{O} | 0 \leq n_b^o \leq n_h^o, 0 \leq n_g^o \leq n_c - n_h\} \subset \mathcal{O}. \tag{3}$$

In words, the elements of the constrained observation set are those elements of the unconstrained observation set for which the number of blue cells is maximally the number of hiding spots (there cannot be more hiding spots identified than there are actual hiding spots) and for which the number of gray cells is maximally the number of non-hiding spot cells of the grid world (all cells may have been investigated by drilling, if so, the hiding spots are colored blue and all remaining cells are colored gray).

Let

$$\mathcal{O}_b := \{o \in \mathcal{O} | n_b^o > n_h\} \text{ and } \mathcal{O}_g := \{o \in \mathcal{O} | n_g^o > n_c - n_h\} \tag{4}$$

denote the observation sets that violate the constraints on the maximal number of blue cells and the maximal number of gray cells, respectively. Then, the constrained observation set can be written as

$$O = \mathcal{O} \setminus \left(\mathcal{O}_b \cup \mathcal{O}_g\right) \tag{5}$$

and by the inclusion-exclusion principle, the cardinality of the constrained observation set is given by

$$|O| = |\mathcal{O}| - |\mathcal{O}_b| - |\mathcal{O}_g| + |\mathcal{O}_b \cap \mathcal{O}_g| \tag{6}$$

We are thus required to analyze the cardinalities of $\mathcal{O}_b$, $\mathcal{O}_g$ and $\mathcal{O}_b \cap \mathcal{O}_g$.

Cardinality of $\mathcal{O}_b$

$\mathcal{O}_b$ is the set of all observations in the unconstrained observation set, which contain more blue cell entries than there are hiding spots. These are all those observations in $o \in \mathcal{O}$ with $k_b = n_h + 1, n_h + 2, ..., n_c - 1, n_c$ blue entries among the n_c available cells. For each of those number of blue cells k_b, there exist the possibilities of $k_g = 0, 1, ..., n_c - k_b - 1, n_c - k_b$ gray cell entries among the $n_c - k_b$ non-allocated cells. We thus have

$$|O_b| = \sum_{k_b=n_h+1}^{n_c} \binom{n_c}{k_b} \left(\sum_{k_g=0}^{n_c-k_b} \binom{n_c - k_b}{k_g} \right) \tag{7}$$

Cardinality of $\mathcal{O}_g$

$\mathcal{O}_g$ is the set of all observations in the unconstrained observation set, which contain more gray cells than there are non-hiding spot grid world cells. These are all those observations in $o \in \mathcal{O}$ with $k_g = n_c - n_h + 1, n_c - n_h + 2, ...n_c - 1, n_c$ gray entries among the n_c available cells. For each of those number of gray cells k_g, there exist the possibilities of $k_b = 0, 1, ..., n_c - k_g - 1, n_c - k_g$ blue cell entries among the $n_c - k_g$ available cells. We thus have

$$|O_g| = \sum_{k_g=n_c-n_h+1}^{n_c} \binom{n_c}{k_g} \left(\sum_{k_b=0}^{n_c-k_g} \binom{n_c - k_g}{k_b} \right) \tag{8}$$

Cardinality of $\mathcal{O}_g \cap \mathcal{O}_b$

$\mathcal{O}_g \cap \mathcal{O}_b$ is the set of all observations in the unconstrained observation set, which contain more blue entries than there are hiding spots and more gray entries than there a non-hiding spot grid cells. Such observations thus fulfil

$$n_b^o > n_h \wedge n_g^o > n_c - n_h, \tag{9}$$

and, by means of addition of the inequalities

$$n_b^o + n_g^o > n_h + n_c - n_h \Leftrightarrow n_b^o + n_g^o > n_c \tag{10}$$

and are thus not admissible. The set $\mathcal{O}_g \cap \mathcal{O}_b$ is thus the empty set with cardinality

$$|\mathcal{O}_g \cap \mathcal{O}_b| = |\emptyset| = 0. \tag{11}$$