CPSC 542G Assignment 2

Due Thursday, March 29, 2018

1. Interpolate the function

$$f(x) = cos(30\pi x), \quad -1 \le x \le 1,$$

using a polynomial of degree n at Chebyshev points, given by

$$x_i = \cos\left(\frac{2i+1}{2(n+1)}\pi\right), \quad i = 0, \dots, n.$$

Do this for n from 10 to 170 in increments of 10. Calculate the maximum interpolation error on the uniform evaluation mesh `x = -1:.001:1` and plot the error vs. polynomial degree using `semilogy`. Observe the error decline profile, including spectral accuracy, and tell us about it.

2. Consider using C^2 cubic splines to interpolate the function

$$f(x) = e^{3x}\sin(200x^2)/(1+20x^2), \quad 0 \le x \le 1,$$

featured in Figure 10.8 (p. 318) of the text, as well as Slide 41 of Chapter 10.

Write a short MATLAB script using `spline`, interpolating this function at equidistant points $x_i = i/n,\ i = 0, 1, \dots, n$. Repeat this for $n = 2^j,\ j = 4, 5, \dots, 14$. For each such calculation record the maximum error at the points `x = 0:.001:1`. Plot these errors against n using `loglog`.

Make observations regarding both computational efficiency (in terms of ball-park flop count) and accuracy in comparison to Figure 10.8.

3. (a) Carry out the integration described in Example 17.8 of the ODE notes (and slides 7 and 25) using the implicit trapezoidal method. Determine if the solution spirals in, spirals out, or forms an approximate circle as desired.

   [To simplify notation, let us define $x(t) = y_1(t)$ and $y(t) = y_2(t)$, so the ODE system is written as $x' = -y,\ y' = x$.]

   (b) Prove the observed results for the trapezoidal method as well as those for forward Euler and backward Euler observed in Example 17.8.

   [Hint: Check what happens to $x^2 + y^2$ during *one* time step.]

4. Consider the following ODE system

$$\begin{aligned} x' &= px - qy, \\ y' &= qx + py, \end{aligned} \tag{1}$$

defined for $0 \le t \le T$ and subject to initial conditions $x(0) = y(0) = 1/\sqrt{2}$. For each t, let $r(t) = (x(t)^2 + y(t)^2)^{1/2}$. The functions $p = p(r)$ and $q = q(r)$ appearing in Equation (1) are known to satisfy $p(1) = 0$ and $q(1) = 1$. Thus, if $(x(t), y(t))$ is the position of a particle on the plane in Cartesian coordinates, then $r(t)$ is the Euclidean distance to the origin. The nonlinear functions p and q depend on the particle's position only through this distance.

   (a) Show that, writing the ODE in polar coordinates $x = r\cos(\phi)$, $y = r\sin(\phi)$, $r \neq 0$, we obtain the ODE system

$$r' = pr, \quad \phi' = q.$$

   (b) Conclude from Part (a) that $r(t) = 1$ and $\phi'(t) = 1$ for all t. Thus, this system describes motion on the unit circle with constant angular velocity.

   (c) Solve the initial value problem for the nonlinear ODE (1) numerically for $T = 10$ using the explicit trapezoidal method (RK2) and the explicit classical Runge-Kutta method RK4 with a constant step size h. Employ the functions

$$p = r^{-2}(1 - r^2)^{\mu}, \quad q = 1 + (1 - r^2)^{\beta},$$

   for β, μ positive integers. Set $\mu = 3$, $\beta = 2$.

   Run your program for $h = 2^{-\ell}$, $\ell = 2, 3, \ldots, 6$ for a total of 10 runs. Record for each h the maximum value of $|r_i - 1|$, where $r_i = (x_i^2 + y_i^2)^{1/2}$, and the maximum value of $|\frac{\phi_i - \phi_{i-1}}{h} - 1|$, where $\phi_i = \arctan(y_i/x_i)$. For the latter, exclude those values where $|\frac{\phi_i - \phi_{i-1}}{h} - 1|$ exceeds 1, because such a large error indicates that (x_i, y_i) and (x_{i-1}, y_{i-1}) are not in the same quadrant and as such are not part of the investigation.

   Compute and record also the observed rates of convergence, as in Slide 33 (and the corresponding table in the text). Make as many (reasonable) observations as you can.

   (d) Now, repeat (c) for the larger integration interval $T = 10,000$. Make another table for this as in Part (c), observe changes/differences and report.

5. The TR-BDF2 is a one-step method for the ODE $y' = f(t, y)$ consisting of applying first the trapezoidal scheme over half a step $h/2$ to approximate the midpoint value, and then the BDF2 scheme (with $h/2$) over one step:

$$y_{i+1/2} = y_i + \frac{h}{4}(f(y_i) + f(y_{i+1/2})), \tag{2a}$$

$$y_{i+1} = \frac{1}{3}[4y_{i+1/2} - y_i + hf(y_{i+1})]. \tag{2b}$$

One advantage is that only two systems of the original size need be solved per time step. Another is that if we are using BDF2 and suddenly need to halve the step size, then here is a way for obtaining the missing "initial" value.

(a) Write the method (2) as a Runge-Kutta method in standard tableau form (i.e., find A and $\mathbf{b}$). This is an instance of a *diagonally implicit Runge-Kutta* (DIRK) method: please explain this name.

(b) Show that both the order and the stage order equal 2.

(c) Show that the stability function satisfies $R(-\infty) = 0$: this method is L-stable and has stiff decay.

(d) Can you construct an example where this method would fail where the BDF2 method would not?