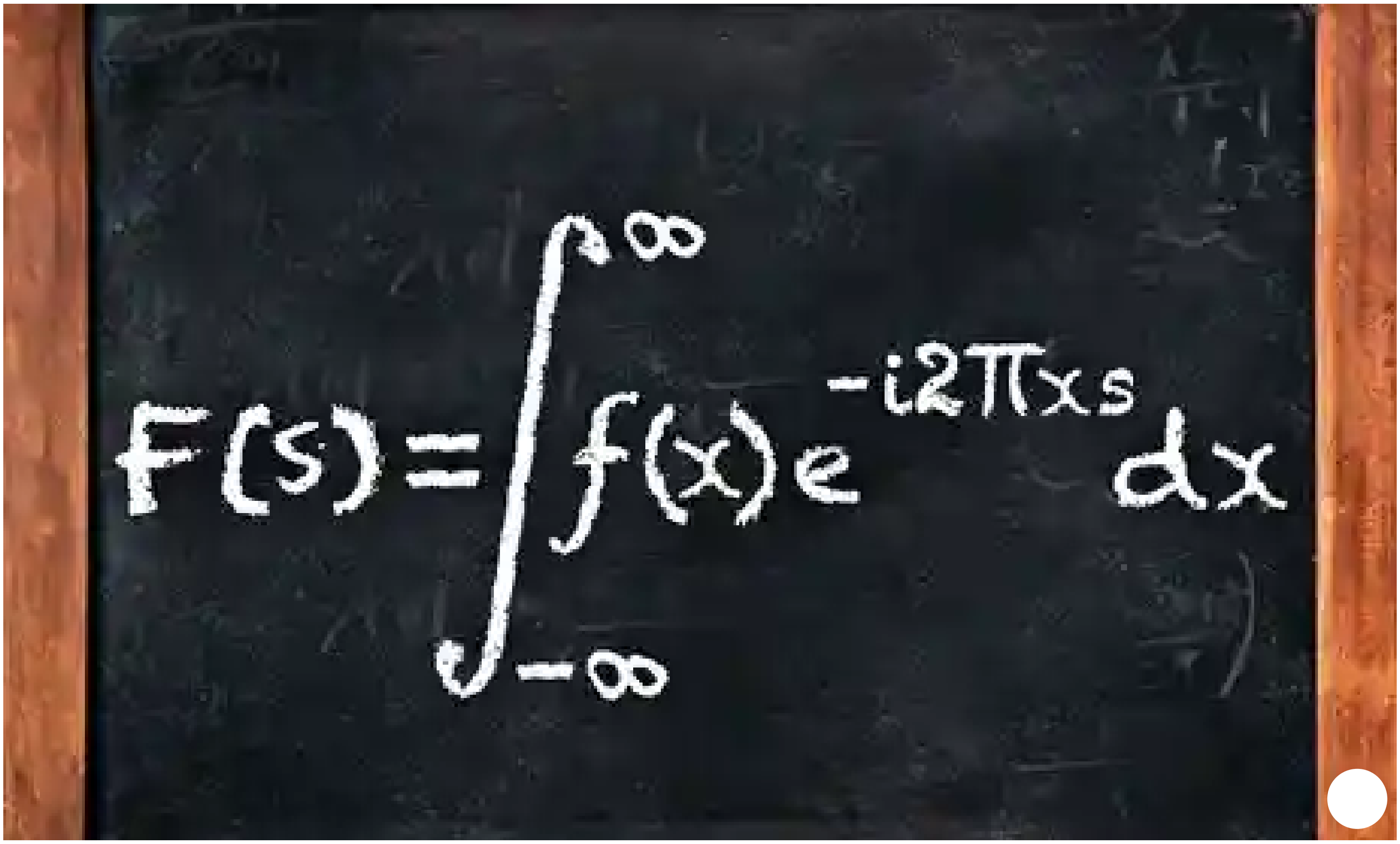

A short history of equations

The Fourier transform lets you have your cake and understand it

Teasing apart the ingredients of a jumble helps scientists to study complex things that change over time or space

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If there's a mathematical idea that applies itself to almost everything in everyday life but is almost unknown outside the scientific world, the Fourier transform has to be the most unsung contender. It pops up wherever scientists need to study complex things that fluctuate in the real world - sound, heat, light, stock prices - and has been used to separate the signal from the noise in data collected for astronomy, medicine, genetics and chemistry. It is also the main equation used in the compression of digital images and sound on the web.

The noted physicist Lord Kelvin wasn't exaggerating when he wrote, in 1867: "Fourier's theorem is not only one of the most beautiful results of modern analysis, but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics."

The equation was developed by the mathematician Baron Jean-Baptiste-Joseph Fourier and appeared in its final form in his 1822 book, *The Analytical Theory of Heat*. As the title suggests, Fourier was interested mainly in how heat flowed around materials but his mathematical tool turned out to be more fundamental than perhaps he realised.

In essence, it says that any complex wave-like signal you care to measure, that fluctuates over time or space, can be broken down into a sum of the familiar, regular, sine waves - the type that roll across the tops of oceans or vibrate along strings. Think of your complex signal as a cake. It contains flour, sugar, eggs and butter but you don't taste any of these things individually when you take a bite of the finished product. If the cake is the waveform, the recipe is the Fourier transform, a list of ingredients and how to combine them. If you want to adapt the final waveform (cake) somehow, it is much easier to do it by isolating the ingredients first.

Fourier's insight was to isolate the regularity contained in what looks, in the real world, a tangled mess of complexity. The term F(s) is called the transform and gives you the properties of the specific sine-wave component of your signal at the frequency "s". Specifically, it gives the amplitude (the maximum height of the wave above and below zero) and phase (how much of a wave's cycle has been completed relative to a fixed point).

On the right of the equation, f(x) represents how your signal fluctuates over space (x is the co-ordinate of location). This is multiplied by e (a widely used constant in maths, equal to 2.718) raised to the power of x multiplied by s, i (the square root of -1) and pi. This gives the properties of the sine wave at frequency s but, since we want the sum of all the frequencies, the expression on the right is added up across all space for steadily increasing values of s (the adding up is represented by the integration sign, by the elongated "S" on the left and the "dx" on the right).

To get a good visual of the Fourier transform, look at a graphical equaliser on an old hi-fi system. The bouncing lines give you the strengths of individual frequencies in the music at any point in time, as a spectrum. That spectrum is the real-time (albeit crude) Fourier transform of the sound coming out of the speakers.

Indeed, your ear performs Fourier transforms all the time (not mathematically, but you know what I mean). A sound wave is, physically, a collection of pressure waves moving through the air, but it isn't the vibrations they create in your eardrum that you "hear".

Instead the components of the inner ear perform a sort of Fourier transform to untangle the composite pressure wave, teasing out each specific frequency and their associated amplitudes in order to covert them into a spectrum of electrical signals which is sent to the brain. Your brain interprets this changing spectrum as sound.

The Fourier transform represents the same information as the original signal, but in a way that makes it simpler for engineers and physicists to work with.

Say you want to compress music or speech files. While vinyl records store analogue sound in surface pits and grooves that mimic the waveform, an MP3 lists frequencies and amplitudes - the Fourier transform - present in every moment in the track. This makes it easy to filter and compress the file - you can delete the information about frequencies outside the range of human hearing. This method is also useful if you want to clean up a recording - if there is a background hum, the Fourier transform information allows you to isolate and delete the main frequencies of that hum and preserve the rest.

The transform can also be applied to clean up pictures which are, in essence, just two-dimensional signals where colour and brightness fluctuate over space. A photograph taken in dim light, for example, might have lots of noise in the form of random spots of light - take the Fourier transform of the image and you can filter them out. This is especially useful in cleaning up astronomical images from space probes, sharpening the focus onto an area of interest.

Some of the most important uses of the Fourier transform have been made in understanding the precise arrangement of atoms in molecules. The science of crystallography works by shining X-rays at a crystal of the substance you want to examine (salt, say, or DNA or a protein). The X-rays come in from one side, are scattered by the molecules within and produce a pattern of spots on the other side. The intensity and position of each spot gives the amplitude and frequency information for the Fourier transform of the crystal's molecular structure. In essence, the scattering of X-rays by the crystal give the Fourier transform of the molecule within. Working the transform backwards, scientists can use the scattering pattern to infer the original 3D molecular structure. This method was used to work out the structure of DNA in the 1950s and has now advanced to giving us insights into the structures of complex proteins and even viruses.

Fourier transforms are also used to process information in medical imaging, including MRI and CAT scans, and to stop buildings falling down in earthquakes. When the ground shakes, a Fourier transform can show you which frequencies of vibration impart the most energy to their surroundings. A building will also have its own natural modes of vibration (a complex version of how a string vibrates at preferred frequencies) and, if the building's natural frequency matches that of an earthquake, it will shake harder and is more likely to fall down. Architects and building engineers can work out their proposed design's preferred vibrational frequency using computer models and, using information from the Fourier transform of local earthquakes, tune the building's preferred frequency so that it has the best possible chance of remaining standing, should disaster strike.

For more on the wonders of the Fourier transform check out Professor Stephen Curry's lecture at the Royal Institution: http://bit.ly/ri-fed

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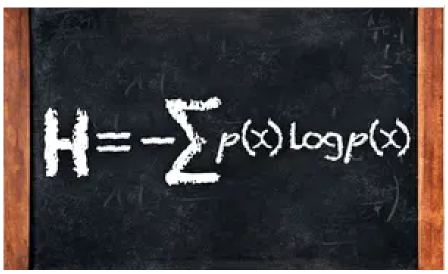

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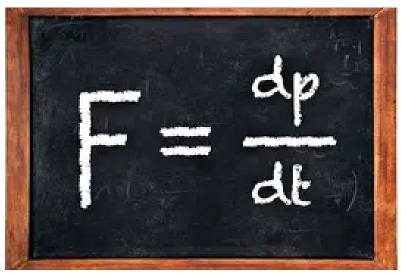

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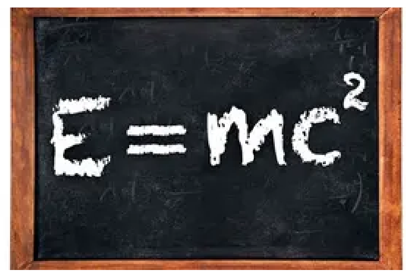

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