Chapter 3:
Linear independence / Basis / Dimension

2.4 Linear independence

The concept of linear independence is fundamental in linear algebra

Definition 23 *A set of vectors* $\{v_1, v_2, \cdots, v_k\}$ *of the vector space* $V(F)$ *is said to be linearly independent if and only if* $\sum_{i=1}^{k} \alpha_i v_i = \theta$ *implies* $\alpha_i = 0$ *for all* i *, otherwise the set is linearly dependent.*

This definition provides a test for linear independence: set a linear combination of these vectors to the zero vector and solve for α_i ; If all coefficients α_i are 0, the set is linearly independent, otherwise it is linearly dependent. Another way of looking at it is to rewrite $\sum_{i=1}^{k} \alpha_i v_i = \theta$ as $\theta = \sum_{i=1}^{k} \alpha_i v_i = \theta$ meaning writing the zero vector as a linear combination of the set $\{v_1, v_2, \cdots, v_k\}$.We can then reformulate the definition as follows: $\{v_1, v_2, \cdots, v_k\}$ is linearly independent if and only if the unique way of writing the zero vector θ as a linear combination of $\{v_1, v_2, \cdots, v_k\}$ is the trivial one $\theta = \sum_{i=1}^{k} 0 v_i$.

Example 24 *Consider the set* $\{e_1, e_2\}$ *of vectors of* $\mathbb{R}^2(\mathbb{R})$ *where* $e_1 = (1, 0)$ *and* $e_1 = (0, 1)$ *; Are they linearly independent? If we set* $\alpha_1 e_1 + \alpha_2 e_2 = \theta$, *we obtain* $\alpha_1(1, 0) + \alpha_2(0, 1) = (0, 0)$ *or* $(\alpha_1, \alpha_2) = (0, 0)$ *leading to the unique solution* $\alpha_1 = \alpha_2 = 0$ *and the set is linearly independent.*

Example 25 *Check the linear independence of* $\{v_1, v_2\}$ *where* $v_1 = (1, -2)$ *and* $v_2 = (-1, 2)$. *Setting* $\alpha_1 v_1 + \alpha_2 v_2 = \theta$ *leads to* $(\alpha_1 - \alpha_2, -2\alpha_1 + 2\alpha_2) = (0, 0)$, *hence* $\alpha_1 - \alpha_2 = 0$ *or* $\alpha_1 = \alpha_2$ *and the set is linearly dependent.*

Example 26 *Consider the space of real polynomials of degree* ≤ 2 *i.e.* $P_2 = \{p(x)/p(x) = a_0 + a_1 x + a_2 x^2\}$ *and the set* $\{1, x, x^2\}$. *This is a set of vectors of* P_2 *that can be tested for linear independence: setting* $\alpha_1.1 + \alpha_2.x + \alpha_3.x^2 = \theta$ *where* θ *is the zero vector of* P_2 *meaning the zero polynomial* $0.1 + 0.x + 0.x^2$ *leads to* $\alpha_1.1 + \alpha_2.x + \alpha_3.x^2 = 0.1 + 0.x + 0.x^2$ *or* $\alpha_1 = \alpha_2 = \alpha_3 = 0$ *and the set* $\{1, x, x^2\}$ *is linearly independent.*

Suppose the set $\{v_1, v_2, \cdots, v_k\}$ is linearly dependent , hence **at least** one scalar is nonzero in $\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_{i-1} v_{i-1} + \alpha_i v_i + \alpha_{i+1} v_{i+1} + \cdots \alpha_k v_k = \theta$. Suppose $\alpha_i \neq 0$ leading to $\alpha_i v_i =$ -$\alpha_1 v - \alpha_2 v_2 \cdots - \alpha_{i-1} v_{i-1} - \alpha_{i+1} v_{i+1} \cdots - \alpha_k v_k$ hence $v_i = -\frac{\alpha_1}{\alpha_i} v_1 - \frac{\alpha_2}{\alpha_i} v_2 \cdots - \frac{\alpha_1 - 1}{\alpha_i} v_{i-1} - \frac{\alpha_{i+1}}{\alpha_i} v_{i+1} \cdots - \frac{\alpha_k}{\alpha_i} v_k$ meaning that **at least** one vector of the set can expressed as a linear combination of the other vectors. This can be used to obtain another definition of linear dependence/independence : the set $\{v_1, v_2, \cdots, v_k\}$ is linearly dependent if and only if at least one vector of the set can be expressed as a linear combination of the other vectors and linearly independent otherwise (no vector of the set can be expressed as a linear combination of the other vectors).

Remark 27 *Any set containing* θ *is dependent and any set consisting of a single vector is linearly independent if it is different from* θ

Writing v_i as a linear combination of the other vectors in $\{v_1, v_2, \cdots, v_k\}$ is equivalent to writing $v_i \in [v_1, v_2, \cdots, v_k]$ i.e. v_i is a vector of the subspace spanned by $\{v_1, v_2, \cdots, v_k\}$. It follows that linear independence of the set is equivalent to $v_i \notin [v_1, v_2, \cdots, v_k]$ for any $1 \le i \le k$.

Consider again the set $\{v_1, v_2, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_k\}$ and suppose that a subset , say, $\{v_i, v_{i+1}, \cdots, v_j\}$ is linearly dependent meaning that $\alpha_i v_i + \alpha_{i+1} v_{i+1} + \cdots \alpha_j v_j = \theta$ contains at least a nonzero scalar. Now we can write $0v_1 + 0v_2 + \cdots 0v_{i-1} + \alpha_i v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_j v_j + \cdots + 0v_k = \theta$ where at least one scalar is nonzero implying the dependence of the whole set. We conclude then that **if any subset of a given set is linearly dependent the whole set is linearly dependent.** In much the same manner we can show that **any subset of a linearly independent set is linearly independent.**

As an exercise (see recitations) show that $[v_1, v_2, \cdots, v_k, v_{k+1}]$ if and only if $v_{k+1} \in [v_1, v_2, \cdots, v_k]$. This result shows that the subspace spanned by a set vectors remains unchanged if we **add** or **remove** a **linearly dependent vector** or equivalently that the subspace spanned by a set of vectors **is affected only by the addition or removal of linearly independent vectors.** This outlines the importance of linearly independent vectors in linear algebra.

2.5 Basis and Dimension

For any vector space we can define a **special set** of vectors called a basis.

Definition 28 *Given the vector space $V(F)$, the set $B = \{v_1, v_2, \cdots, v_n\}$ of vectors of $V(F)$ is a basis for $V(F)$ if and only if*
i) $\{v_1, v_2, \cdots, v_n\}$ is linearly independent
ii) $V(F) = [v_1, v_2, \cdots, v_n] = [B]$

Clearly a subset of vectors of $V(F)$ qualifies to be a basis for $V(F)$ if and only if the set is linearly independent and spans the whole space $V(F)$. Suppose $\{v_1, v_2, \cdots, v_n\}$ is a basis. It follows, by definition, that $V(F) = [v_1, v_2, \cdots, v_n]$ hence **any** $v \in V(F)$ satisfies $v \in [v_1, v_2, \cdots, v_n]$ or $v = \sum_{i=1}^{n} \alpha_i v_i$. This tells us that any vector of $V(F)$ can be expressed as a linear combination of the vectors of the basis. Is this expression unique? Suppose we can also write $v = \sum_{i=1}^{n} \beta_i v_i$. and upon substraction we obtain $\theta = \sum_{i=1}^{n} (\alpha_i - \beta_i) v_i$. Since the vectors are linearly independent we conclude that $\alpha_i = \beta_i$ for all $1 \le i \le n$. We thus conclude that the expression of vectors in a given basis is **unique**. The expression $v = \sum_{i=1}^{n} \alpha_i v_i$. is conveniently written as $v_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ called the representation of v where the scalars called **components** of v are written in a column vector and the subscript B refers to the particular basis used. Since any set of linearly independent vectors spanning $V(F)$ qualifies to be a basis, we conclude that a basis is not unique. It follows that the representation of a vector is not unique: for every selection of a basis we obtain a (unique) representation with respect to this basis.

Example 29 *Consider the set $\{e_1, e_2\}$ of vectors of $\mathbb{R}^2(\mathbb{R})$ where $e_1 = (1,0)$ and $e_1 = (0,1)$. This set was shown earlier to be linearly independent and to span $\mathbb{R}^2(\mathbb{R})$. It follows that it qualifies to be a basis for $\mathbb{R}^2(\mathbb{R})$; Representing the vector $(1,2)$ with respect to this basis amounts to expressing this vector as $(1,2) = \alpha_1(1,0) + \alpha_2(0,1) = (\alpha_1, \alpha_2)$ hence $\alpha_1 = 1$ and $\alpha_2 = 2$ and the column representation is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.*

If we select a new basis (check that it is!), say $\{v_1, v_2\}$ where $v_1 = (1,0)$ and $v_2 = (1,1)$ we have $(1,2) = \beta_1(1,0) + \beta_2(0,1)$ leading to $(1,2) = (\beta_1 + \beta_2, \beta_2)$ hence $\beta_1 + \beta_2 = 1$ and $\beta_2 = 2$ thus $\beta_1 = -1$ and $\beta_2 = 2$. The new representation of the vector is then $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Theorem 30 *Given a vector space $V(F)$, then any linearly independent subset of vectors of $V(F)$ can be **extended** to be a basis for $V(F)$.*

Proof. Let $S = S = \{v_1, v_2, \cdots, v_k\}$ be a set of linearly independent vectors of $V(F)$. If $[v_1, v_2, \cdots, v_k]$, then the current is already a basis. Otherwise, we pick a new vector v_{k+1} such that $v_{k+1} \notin [v_1, v_2, \cdots, v_k]$ ensuring that the new set $\{v_1, v_2, \cdots, v_k, v_{k+1}\}$ is linearly independent. Now if $[v_1, v_2, \cdots, v_k, v_{k+1}]$ $= V(F)$, we have a basis, otherwise we continue the process until we obtain a linearly independent spanning $V(F)$ and a basis is constructed. ■

The process described above assumes convergence in a **finite** number of steps leading to a basis with a finite number of vectors. If it does not converge, we are dealing with a basis consisting of an infinite number of vectors. Clearly the number of vectors in a basis indicates the **maximum** number of linearly independent vectors in the vector space.

Remark 31 *Since the number of vectors in the initial set is arbitrary, we can always start with the simplest linearly independent set consisting of a single nonzero vector.*

Definition 32 *The dimension of a vector space is given by the **number** of vectors in any basis of this space.*

The dimension of a vector space is then a positive integer $n \geq 0$ and all bases of a given vector space have the same number of vectors $n \geq 0$. If the dimension of $V(F)$ is n we write $\dim V(F) = n$ or $V_n(F)$ where the dimension appears as a subscript. If n is finite, $V_n(F)$ is said to be a **finite dimensional space** otherwise it is an **infinite dimensional space.**

Knowing the dimension of a vector space determines how many vectors will any basis contain. It follows that if $\dim V(F) = n$, all we need to construct a basis is to find n linearly independent vectors in $V(F)$.

Remark 33 *Given the representation of any vector of $V(F)$, we can deduce the dimension from the number of components of the representation as a column vector.*

A particular case is the zero subspace $[\theta]$. Since it consists of the zero vector θ only which is obviously linearly dependent, we conclude that $\dim[\theta] = 0$

The most commonly used vector spaces in Engineering and Science are $\mathbb{R}^n(\mathbb{R})$ and $\mathbb{C}^n(\mathbb{R})$ with dimension $n \geq 0$. A particularly simple and useful basis for $\mathbb{R}^n(\mathbb{R})$ is the so-called **standard basis** $\{e_1, e_2, \cdots, e_n\}$ where $e_1 = (1, 0, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0), \cdots, e_n = (0, 0, 0, \cdots, 1)$.

Example 34 *Consider the set $S = \{v_1, v_2, v_3\}$ in $\mathbb{R}^3(\mathbb{R})$ where $v_1 = (1, 0, 0), v_2 = (0, 1, 0)$ and $v_3 = (1, 1, 0\}$. What is a basis for $[S]$? Its dimension ? Clearly $[S] = [v_1, v_2, v_3]$ and as seen before $[S]$ remains unchanged if we remove linearly dependent vectors from the set S. It can easily be seen that $v_3 = v_1 + v_2$ indicating that v_3 is linearly dependent on $\{v_1, v_2\}$ so it can be removed from S without changing $[S]$.Hence we have $[S] = [v_1, v_2]$ where $\{v_1, v_2\}$ are linearly independent (standard vectors of $\mathbb{R}^3(\mathbb{R})$). It follows that $\{v_1, v_2\}$ are linearly independent and span $[S]$, thus they constitute a basis for $[S]$ by definition. Since the basis contains two vectors, we conclude that $\dim[S] = 2$.*