Chapter 3

Linear Mappings

In this chapter a class of relationships between vector spaces called **linear mappings** are examined.

3.1 General Mappings

A mapping between two sets A, B denoted $M : A \to B$ is a rule assigning to each element $a \in A$ an element $b \in B$ called the **image** of a under the mapping M written $b = M(a)$. The set A is the **domain** of M and the set B its **codomain**.

We define the **range** of M (or the image of A) as the set of images of all elements of A and is denoted as $\mathcal{R}(M) = \{b \in B/b = M(a)\}$. Clearly $\mathcal{R}(M)$ is a subset of A.

The rule of assigning a student number to every student is a simple example of a mapping. The real function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is another example of a mapping.

Mappings may enjoy some useful properties defined below:

Definition 35 *A mapping $M : A \to B$ is* ***one-to-one*** *or* ***injective*** *if and only if* ***distinct elements*** *have* ***distinct images****.*

This can be expressed as follows: M one-to-one $\Leftrightarrow$ For all $a \neq a'$ we have $M(a) \neq M(a') \Leftrightarrow$ For all $M(a) = M(a')$ we have $a = a'$.

A one-to-one mapping may also be termed as an invertible mapping: Every element of A has a **unique** image allowing to identifying it.

Example 36 *Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ where a, b are real constants. Let $x_1 \neq x_2$ and consider their images $f(x_1) = ax_1 + b$ and $f(x_2) = ax_2 + b$ leading to $f(x_1) - f(x_2) = a(x_1 - x_2)$. Clearly for $x_1 \neq x_2$ or $x_1 - x_2 \neq 0$, we have $f(x_1) \neq f(x_2)$ indicating that f is one-to-one provided $a \neq 0$.*

Example 37 *Consider now $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ which is not since $f(1) = f(-1)$.*

Definition 38 *A mapping $M : A \to B$ is **unto** or **surjective** if and only if every element of B is an image of an element of A.*

This can be expressed as: M unto $\Leftrightarrow$ For any $b \in B$, there exists $a \in A$ such that $b = M(a)$. It essentially states that **all** elements of B are **images** leading to $B = \mathcal{R}(M)$

Example 39 *Consider again $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$. For any $y \in \mathbb{R}$, we can find $x \in \mathbb{R}$ such that $y = ax + b$, hence $x = \frac{y-b}{a}$ for $a \neq 0$ indicating that the mapping is onto. On the other hand the mapping $f(x) = x^2$ is not since strictly negative real numbers cannot be images.*

Definition 40 *A mapping $M : A \to B$ is **bijective** if and only if it is both injective and surjective.*

For a bijective mapping every element of the domain has a unique image and every element of the codomain is the image of a unique element from the domain.

In linear algebra, we are interested in a special class of mappings: the so-called **linear mappings**.

3.2 Linear Mappings

Consider the two vector spaces $V_n(F)$ and $W_m(F)$ and a mapping $L : V_n(F) \to W_m(F)$

Definition 41 *The mapping $L : V_n(F) \to W_m(F)$ is said to be **linear** if and only if we have:*
i) For all $v_1, v_2 \in V_n(F)$, we have $L(v_1 + v_2) = L(v_1) + L(v_2)$
ii) For all $v \in V_n(F)$ and all $\alpha \in F$, we have $L(\alpha v) = \alpha L(v)$

The first property is called **additivity** while the second is called **homogeneity.** The additivity property requires that the image of a sum of vectors be the sum of their images. The homogeneity property requires that the image of a scaled vector be the its scaled image. These properties constitute what is called the **principal of superposition in engineering**.

Both properties can be combined into a single property: L linear if and only if $L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$ requiring that the image of a linear combination of images be the linear combination of their images. This can be written shortly as: L linear $\Leftrightarrow$ $\sum_{i=1}^{k} L(\alpha_i v_i) = \sum_{i=1}^{k} \alpha_i L(v_i)$.

Remark 42 *A linear mapping where the domain and codomain are identical is called a **linear transformation** as in $T : V_n(F) \to V_n(F)$*

Example 43 *Consider the mapping* $L : \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R})$ *defined by* $L(x, y) = (0, y)$*. To check linearity, consider any* $v_1, v_2 \in \mathbb{R}^2(\mathbb{R})$ *and any* $\alpha_1, \alpha_2 \in \mathbb{R}$*, then*

$L(\alpha_1 v_1 + \alpha_2 v_2) = L[\alpha_1(x_1, y_1) + \alpha_2(x_2, y_2)] = L(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2] = (0, \alpha_1 y_1 + \alpha_2 y_2) = \alpha_1(0, y_1) + \alpha_2(0, y_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2)$ establishing that L is linear.

This mapping represents an orthogonal projection onto the $y-$axis of the $y - x$ plane.

Example 44 *Consider the vector space* $V(\mathbb{R}) = \{V(\mathbb{R}) : \mathbb{R} \to \mathbb{R}/\ f$ *differentiable* $\}$ *and the mapping* $D : V(\mathbb{R}) \to V(\mathbb{R})$ *defined by* $D[f] = \frac{df(x)}{dx}$*. Let* $f_{\ 1},\ f_{\ 2} \in V(\mathbb{R})$ *and* $\alpha_1, \alpha_2 \in \mathbb{R}$ *and consider* $D[\alpha_1 f_1 + \alpha_2 f_2] = \frac{d}{dx}[\alpha_1 f_1(x) + \alpha_2 f_2(x)]$*. It follows that* $\frac{d}{dx}[\alpha_1 f_1(x) + \alpha_2 f_2(x)] = \alpha_1 \frac{df_1(x)}{dx} + \alpha_1 \frac{df_1(x)}{dx} = \alpha_1 D[f_1] + \alpha_2 D[f_2]$*,hence* $D[\alpha_1 f_1 + \alpha_2 f_2] = \alpha_1 D[f_1] + \alpha_2 D[f_2]$ *and the mapping is linear.*

Example 45 *Consider the mapping* $L : \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R})$ *defined by* $L(x, y) = (0, xy)$*. Clearly* $L(\alpha v) = L[\alpha(x, y)] = L(\alpha x, \alpha y) = (0, \alpha^2 xy) \neq \alpha L(v) = \alpha L(x, y) = \alpha(0, xy) = (0, \alpha xy)$ *and the mapping is not linear.*

A linear mapping is also called a **space homomorphism** and if bijective also, it is called a **space isomorphism**.

Remark 46 *From the homogeneity property* $L(\alpha v) = \alpha L(v)$ *for any* $\alpha \in F$ *and any* $v \in V_n(F)$*,then for the particular case* $\alpha = 0$ *we have* $L(0v) = 0L(v)$ *or* $L(\theta) = \theta$ *meaning that a linear mapping always maps the zero vector of* $V_n(F)$ *into the zero vector of* $W_m(F)$*.*

Consider the linear mapping $L : V_n(F) \to W_m(F)$ and suppose we select a basis $\{v_1, v_2, \cdots, v_n\}$ for $V_n(F)$. If v is any vector of $V_n(F)$, we can express it uniquely as $v = \sum_{i=1}^n \alpha_i v_i$ and its image as $L(v) = \sum_{i=1}^n L(\alpha_i v_i)$ and by linearity $L(v) = \sum_{i=1}^n \alpha_i L(v_i)$ meaning that the image of v is expressed uniquely as a linear combination of the images of the vectors of the selected basis. It follows that the effect of the mapping L is described uniquely by the effect of the mapping on the vectors of any selected basis.

3.3 Fundamental Subspaces

Consider the linear mapping $L : V_n(F) \to W_m(F)$ and suppose that S is a subspace of $V_n(F)$. Is the image of S i.e. $L(S)$ also a subspace? Clearly $L(S)$ is a nonzero subset of $W_m(F)$, thus we need only check $L(S)$ for closedness of under linear combinations. Let $w_1, w_2 \in L(S)$, hence there exits $v_1, v_2 \in V_n(F)$ such that $w_1 = L(v_1)$ and $w_2 = L(v_2)$. Now $\alpha_1 w_1 + \alpha_2 w_2 = \alpha_1 L(v_1) + \alpha_2 L(v_2)$ and by linearity $\alpha_1 w_1 + \alpha_2 w_2 = L(\alpha_1 v_1 + \alpha_2 v_2)$ implying that $\alpha_1 w_1 + \alpha_2 w_2$ is an image or $\alpha_1 w_1 + \alpha_2 w_2 \in L(S)$.Thus, $L(s)$ is closed under linear combination therefore a subspace. We can thus conclude that under a linear mapping every subspace of the domain is mapped into a subspace of the codomain or **the image of a subspace is also a subspace**.

Every linear mapping defines two special subspaces called **fundamental subspaces**: one in the domain and one in the codomain.

Definition 47 *The range space of the linear mapping* $L : V_n(F) \to W_m(F)$ *is defined as* $\mathcal{R}(L) = \{w \in W_m(F)/w = L(v) \text{ where } v \in V_n(F)\}$*; The dimension of* $\mathcal{R}(L)$ *is called the rank of* L *and denoted* $rank(L)$ *or* $r(L)$*.*

The range space $\mathcal{R}(L)$ is the image of $V_n(F)$ and since the image of subspace is also a subspace, then $\mathcal{R}(L)$ is a subspace of $W_m(F)$.

Example 48 *Consider the linear mapping* $D : P_3 \to P_3$ *, where* $P_3 = \{p(x)/p(x) = a_0+a_1x+a_2x^2+a_3x^3\}$ *is the space of real polynomials of degree* ≤ 3 *and defined by* $D[p(x)] = \frac{dp(x)}{dx}$*. The range space is* $\mathcal{R}(L) = \{p'(x)/p'x) = \frac{dp(x)}{dx}\} = \{p'x) = a_1 + 2a_2x + 3a_3x^2\}$*,hence the space of all real polynomials of degree* ≤ 2*. Clearly a basis for this subspace is* $\{1, x, x^2\}$ *and* $r(D) = 3$*.*

The rank of a mapping is important concept in linear algebra and we are often required to compute it. Since this rank is the dimension of $\mathcal{R}(L)$, it is determined by the number of vectors in any basis of $\mathcal{R}(L)$. In the following we attempt to find bases for it.

Theorem 49 *Consider the linear mapping* $L : V_n(F) \to W_m(F)$ *and suppose* $\{v_1, v_2, \cdots, v_n\}$ *is a basis for* $V_n(F)$*,then* $\mathcal{R}(L)$ *is spanned by the set* $\{L(v_1), L(v_2), \cdots, L(v_n)\}$ *or equivalently* $\mathcal{R}(L)$ $=[L(v_1), L(v_2), \cdots, L(v_n)]$*.*

Proof. We must show that $\mathcal{R}(L) \subset [L(v_1), L(v_2), \cdots, L(v_n)]$.and $[L(v_1), L(v_2), \cdots, L(v_n)].\subset \mathcal{R}(L)$.

i) Showing $\mathcal{R}(L) \subset [L(v_1), L(v_2), \cdots, L(v_n)]$: Let $w \in \mathcal{R}(L)$, hence there exits $v \in V_n(F)$ such that $w = L(v)$. Expressing v in the basis $B = \{v_1, v_2, \cdots, v_n\}$ gives uniquely $w = L(\sum_{i=1}^n \alpha_i v_i) = \sum_{i=1}^n \alpha_i L(v_i)$, hence $w \in [L(v_1), L(v_2), \cdots, L(v_n)]$ implying $\mathcal{R}(L) \subset [L(v_1), L(v_2), \cdots, L(v_n)]$

ii) Showing $[L(v_1), L(v_2), \cdots, L(v_n)].\subset \mathcal{R}(L)$.: Let $w \in [L(v_1), L(v_2), \cdots, L(v_n)]$., it follows that $w = \sum_{i=1}^n \alpha_i L(v_i)$ and using linearity $w = L(\sum_{i=1}^n \alpha_i v_i)$ implying that w is the image of the vector $\sum_{i=1}^n \alpha_i v_i$ of $V_n(F)$, hence a vector of $\mathcal{R}(L)$.leading to $[L(v_1), L(v_2), \cdots, L(v_n)].\subset \mathcal{R}(L)$. ■

This result states that $\mathcal{R}(L)$ can be generated from the images of the vectors of any basis for the domain $V_n(F)$.

We have seen previously that a span remains unchanged if we remove linearly dependent vectors from the spanning set. Clearly if we remove linearly dependent vectors from $\{L(v_1), L(v_2), \cdots, L(v_n)\}$., we will not change $[L(v_1), L(v_2), \cdots, L(v_n)]$. It follows that the **remaining vectors** in $\{L(v_1), L(v_2), \cdots, L(v_n)\}$ are linearly independent and span $\mathcal{R}(L)$ constituting then a basis for $\mathcal{R}(L)$ by definition.

This suggests an approach to construct a basis for $\mathcal{R}(L)$: Select any basis $\{v_1, v_2, \cdots, v_n\}$ for $V_n(F)$, obtain their images $\{L(v_1), L(v_2), \cdots, L(v_n)$, remove from this set the linearly dependent vectors , then the remaining vectors constitute a basis. The number of vectors in this basis is by definition the dimension of $\mathcal{R}(L)$, hence the rank of the linear mapping L.

Definition 50 *The null space or Kernel of the mapping $L : V_n(F) \to W_m(F)$, denoted $\mathcal{N}(L)$, is defined as $\mathcal{N}(L) = \{v \in V_n(F)/L(v) = \theta\}$; The dimension of $\mathcal{N}(L)$ is called the **nullity** of L and denoted $n(L)$.*

$\mathcal{N}(L)$ consists then of the set of all vectors of $V_n(F)$ mapping into the zero vector of $W_m(F)$. This subset of $V_n(F)$ can easily be shown to be closed under linear combinations making of it a subspace of $V_n(F)$.

Example 51 *Consider again the linear mapping $D : P_3 \to P_3$ defined by $D[p(x)] = \frac{dp(x)}{dx}$ where $P_3 = \{p(x)/p(x) = a_0 + a_1x + a_2x^2 + a_3x^3\}$. It follows that $\mathcal{N}(D) = \{p(x)/D[p(x)] = \theta]$ where θ is the zero polynomial in this case. $D[p(x)] = \frac{dp(x)}{dx} = \theta$ leads to $\frac{d}{dx}(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_2^2 = 0.1 + 0.x + 0.x^2 + 0.x^3$ hence $a_1 = a_2 = a_3 = 0$ and $\mathcal{N}(D) = \{p(x) = a_0\}$ or the set of all zero degree polynomials.The nullity $n(D)$ is 1 since a basis for $\mathcal{N}(D)$ is $\{1\}$.*

From the last two examples we notice that $\dim[\ n(D)] + \dim[\ \mathcal{R}(D)] = \dim[\ P_3]$. Can this be generalized?

Theorem 52 *Given the linear mapping $L : V_n(F) \to W_m(F)$, the rank and nullity of L are related by $n(L) + r(L) = n$ i.e. the dimension of the domain.*

Proof. Suppose $n(L) = k$ and let $\{v_1, v_2, \cdots, v_k\}$ be a basis for $\mathcal{N}(L)$. Now these vectors constitute a linearly independent subset of vectors of $V_n(F)$ and since any linearly independent subset can be extended to be a basis (shown earlier), we can extend $\{v_1, v_2, \cdots, v_k\}$ to be a basis for $V_n(F)$ as $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ where $(n-k)$ linearly independent vectors were added. Consider now the set of images of the added vectors $\{L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)\}$ which can be shown to be a basis for $\mathcal{R}(L)$. To be so, the set must be linearly independent and must span $\mathcal{R}(L)$.

i) $\{L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)\}$ linearly independent: let $\sum_{i=k+1}^{n} \alpha_i L(v_i) = \theta$ and show $\alpha_i = \theta$ for $k+1 \leq i \leq n$. By linearity we can write $L(\sum_{i=k+1}^{n} \alpha_i v_i) = \theta$ meaning that $(\sum_{i=k+1}^{n} \alpha_i v_i)$ is a vector of $\mathcal{N}(L)$. Thus it can expressed uniquely using the basis $\{v_1, v_2, \cdots, v_k\}$ as $\sum_{i=k+1}^{n} \alpha_i v_i = \sum_{i=1}^{k} \alpha_i v_i$ or $\sum_{i=1}^{k} \alpha_i v_i - \sum_{i=k+1}^{n} \alpha_i v_i = \theta$ obtaining a linear combination of $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ set to zero. Since these vectors are linearly independent by construction, we conclude that $\alpha_i = 0$ for all $1 \leq i \leq n$, thus $\alpha_i = 0$ for $k+1 \leq i \leq n$ and the set $\{L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)\}$ is linearly independent.

ii) $\{L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)\}$ spans $\mathcal{R}(L)$: we must show $[L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)] = \mathcal{R}(L)$ equivalently $[L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)] \subset \mathcal{R}(L)$ and $\mathcal{R}(L) \subset [L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)]$. Now $[L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)] \subset \mathcal{R}(L)$ is obvious since every vector of $[L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)]$ is a linear combination of images hence a vector of $\mathcal{R}(L)$. Let $w \in \mathcal{R}(L)$, it follows that there exists $v \in V_n(F)$ such that $w = L(v)$; Expressing v using the basis $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ leads to

$v = \sum_{i=1}^{n} \alpha_i L(v_i)$, hence $w = L(\sum_{i=1}^{n} \alpha_i v_i)$ and by linearity $w = \sum_{i=1}^{n} \alpha_i L(v_i)$ which can be split into $w = \sum_{i=1}^{k} \alpha_i L(v_i) + \sum_{i=k+1}^{n} \alpha_i L(v_i)$. Clearly $\sum_{i=1}^{k} \alpha_i L(v_i) = \theta$ since v_i for $1 \leq i \leq k$ are vectors of $\mathcal{N}(L)$ whose images are zero. It follows that $w = \sum_{i=k+1}^{n} \alpha_i L(v_i)$ i.e. w is a linear combination of the set $\{L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)\}$,thus $w \in [L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)]$ and $\mathcal{R}(L) \subset [L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)]$. Finally we conclude that $\{L(v_{k+1}), L(v_{k+2}), \cdots, L(v_n)\}$ is indeed a basis for $\mathcal{R}(L)$ establishing that $\dim \mathcal{N}(L) + \dim \mathcal{R}(L) = \dim V_n(F) = n$ ■

This result is useful in the sense that the nullity can determined from the rank ana vice-versa.

In this section, we have seen that every linear mapping L defines two fundamental subspaces: $N(L)$ in the domain and $\mathcal{R}(L)$ in the codomain whose dimensions are related. Furthermore $\mathcal{R}(L)$ is spanned by the images of the vectors of any basis for the domain of L.

Example 53 *For the linear mapping $D : P_3 \to P_3$ defined by $D[p(x)] = \frac{dp(x)}{dx}$ seen earlier, we found $n(D) = 1$ and $r(D) = 3$ leading to $n(D) + r(D) = 4$ which is the dimension of P_3.*

3.4 Nonsingular Linear Mappings

Among linear mappings, there exists a special class of mappings called **nonsingular** or **invertible** mappings having the important characteristic that a vector can be identified from its image. In fact, these linear mappings form a subclass of general one-to-one mappings and are considered in this section. Before defining a singular mapping, we need to define a special mapping called the **identity mapping.** Given the vector space $V_n(F)$, we define the identity mapping I as the mapping $I : V_n(F) \to V_n(F)$ such that $I(v) = v$ for any $v \in V_n(F)$.

Definition 54 *A linear mapping $L : V_n(F) \to W_m(F)$ is said to be nonsingular or invertible if and only if there exists a mapping $L^{-1} : \mathcal{R}(L) \to V_n(F)$ such that $L^{-1} \circ L = I$ where I is the identity mapping.*

Clearly if w is the image of v under the mapping L i.e. $w = L(v)$ and L is nonsingular we have $(L^{-1} \circ L)(v) = L^{-1}[L(v)] = L^{-1}(w) = I(v) = v$ allowing to determine v from its image w. It can easily be shown (do it as an exercise) that L^{-1} is also linear.

This definition tells us what is a nonsingular mapping but it is not very helpful in testing whether or not a given linear mapping is nonsingular. The following result offers several easier ways for this testing.

Theorem 55 *Given the linear mapping $L : V_n(F) \to W_m(F)$, then the following statements are equivalent:*
i) L nonsingular
ii) L^{-1} exists

iii) $\mathcal{N}(L) = [\theta]$ *or* $n(L) = 0$
iv) $r(L) = n$
v) L *preserves linear independence*

Proof. Since all statements are equivalent, we must prove that $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv) \Rightarrow v) \Rightarrow i)$

$i) \Rightarrow ii)$: Suppose L nonsingular , then L^{-1} exists by definition.

$ii) \Rightarrow iii)$: Suppose L^{-1} exists and let $v \in \mathcal{N}(L)$ meaning $L(v) = \theta$, then $L^{-1}[L(v)] = L^{-1}(\theta) = \theta$ or $(L^{-1} \circ L)(v) = \theta$ leading to $I(v) = v = \theta$. It follows that any vector of $\mathcal{N}(L)$ is the zero vector θ, hence $\mathcal{N}(L) = [\theta]$ and $\dim \mathcal{N}(L) = \dim[\theta]$ or $n(L) = 0$

$iii) \Rightarrow iv)$: Suppose $n(L) = 0$, then by $n(L) + r(L) = n$ we conclude that $r(L) = 0$

$iv) \Rightarrow v)$: Let $\{v_1, v_2, \cdots, v_k\}$ be a set of linearly independent vectors and consider the set $\{L(v_1), L(v_2), \cdots, L(v_k)\}$. Setting $\sum_{i=1}^k \alpha_i L(v_i) = \theta$ and using linearity we have $L(\sum_{i=1}^k \alpha_i v_i) = \theta$ meaning that $(\sum_{i=1}^k \alpha_i v_i) \in \mathcal{N}(L)$ and since $\mathcal{N}(L) = [\theta]$, we conclude that $\sum_{i=1}^k \alpha_i v_i = \theta$. Now by assumption $\{v_1, v_2, \cdots, v_k\}$ are linearly dependent implying that $\alpha_i = 0$ for $1 \leq i \leq k$. It follows that $\{L(v_1), L(v_2), \cdots, L(v_k)\}$ are also linearly independent showing that L preserves a nonsingular mapping preserves linear independence.

$v) \Rightarrow i)$: Consider $u, v \in V_n(F)$ such that $u \neq v$ and let $\{v_1, v_2, \cdots, v_n\}$ be a basis for $V_n(F)$. We can write uniquely $u = \sum_{i=1}^n \alpha_i v_i$ and $v = \sum_{i=1}^n \beta_i v_i$. Now $L(u) = L(\sum_{i=1}^n \alpha_i v_i)$ and $L(v) = L(\sum_{i=1}^n \beta_i v_i)$ leading to $L(u) - L(v) = L(\sum_{i=1}^n (\alpha_i - \beta_i) v_i$ or $L(u) - L(v) = \sum_{i=1}^n (\alpha_i - \beta_i) L(v_i)$. Now suppose $L(u) = L(v)$, then $L(u) - L(v) = \sum_{i=1}^n (\alpha_i - \beta_i) L(v_i) = \theta$. Now since $u \neq v$, at least one $(\alpha_i - \beta_i)$ is nonzero meaning that the the set $\{L(v_1), L(v_2), \cdots, L(v_k)\}$ is linearly dependent contradicting the hypothesis of linear independence preservation. Hence the supposition $L(u) = L(v)$ is necessarily wrong meaning that $u \neq v$ implies $L(u) \neq L(v)$ and the mapping L is one-to-one or nonsingular. ■

Any of the above statements can be used to test the nonsingularity of a given linear mapping. Perhaps the simplest is to set $L(v) = \theta$ and show that the **unique solution** is $v = \theta$ since $\mathcal{N}(L) = [\theta]$ for nonsingular mappings.

If $\{v_1, v_2, \cdots, v_n\}$ is a basis for $V_n(F)$, then for a nonsingular mapping $\{L(v_1), L(v_2), \cdots, L(v_n)\}$ is a basis for $\mathcal{R}(L)$: this is so because L preserves linear independence and $r(L) = n$.