# Chapter 2

# Vector Spaces and Subspaces

We already are familiar with the concept of **vectors** in geometry : entities with an origin, a direction and some length. These vectors are used in the plane or in the three dimensional space ( Euclidian space). In geometry we are also familiar with two basic operations on vectors: **vector addition** and **vector scaling** The Euclidian space happens to be only a special type of a vector space. In this course, the concept of vector and vector space is generalized.

## 2.1 Vector Spaces

To construct a general vector space we need a **non-empty** set, a **field,** two binary operations.( vector addition and vector scaling) and a number of conditions(axioms ) to be satisfied.

**Definition 11** *Given a non-empty set $V$, a field $F$ and two binary operations called* ***vector addition*** *and* ***scalar multiplication****, then $V$ is called a vector space over the field $F$ if and only if the following axioms are satisfied:*

*i) For all $v_1, v_2 \in V$, we have $v_1 + v_2 \in V$*

*ii) For all $v \in V$, we have $\alpha v \in V$ for any $\alpha \in F$*

*iii) For all $v_1, v_2, v_3 \in V$, we have $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$*

*iv) For all $v_1, v_2 \in V$, we have $v_1 + v_2 = v_2 + v_1$*

*v) There exists a vector $\theta \in V$, called the* ***zero vector****, satisfying $v+\theta = \theta+v$ for all $v \in V$*

*vi) For any $v \in V$, there exists $(-v) \in V$ such that $v + (-v) = \theta$*

*vii) For all $\alpha, \beta \in F$ and for all $u, v \in V$ we have $\alpha(u + v) = \alpha u + \alpha v$ and $(\alpha + \beta)v = \alpha v + \beta v$*

*viii) For all $\alpha, \beta \in F$ and for all $v \in V$, we have $\alpha(\beta v) = (\alpha\beta)v$*

*ix) For any $v \in V$, we have $1v = v$ where $1$ is the* ***unit element*** *of the field $F$*

The elements of $V$ are called **vectors** while those of $F$ are called **scalars**. It may be noticed that $V$ along with operation vector addition forms an Abelian group.

Vector addition and scalar multiplication are the **only operations** defined on vectors of a given vector space. We usually use the notation $V(F)$ to refer to the vector space $V$ over the field $F$.

The first axiom describes the property of **closedness under vector addition** while the second describes **closedness under scalar multiplication.**

**Example 12** *Consider the set of real numbers* $\mathbb{R}$ *and the arithmetic operations addition* $(+)$ *and multiplication* $(\times)$*; Is* $\mathbb{R}(\mathbb{R})$ *a vector space? Clearly* $\mathbb{R}$ *is a non-empty set since it is a field. It can be checked that all axioms are satisfied , thus* $\mathbb{R}(\mathbb{R})$ *is indeed a vector space where vectors are simply real numbers. What about* $\mathbb{R}(\mathbb{C})$*? We may notice right way that* $\mathbb{R}(\mathbb{C})$ *does not satisfy closedness under scalar multiplication: for* $v \in \mathbb{R}$ *and* $\alpha \in \mathbb{C}$*,* $\alpha v$ *is in general complex.*

**Example 13** *Consider the set* $S = \{f : \mathbb{R} \to \mathbb{R}/f$ *continuous in* $\begin{bmatrix} a & b \end{bmatrix}\}$*;Is* $S(\mathbb{R})$ *a vector space ? Notice here that we deal with real continuous functions and the operations of addition and scalar multiplication( multiplication by a real constant) are well defined. It can checked that all axioms are satisfied and .*$S(\mathbb{R})$ *is a vector space.*

**Example 14** *Consider now* $\mathbb{R}^2(\mathbb{R})$ *along with addition and scalar multiplication. It can checked that all axioms are satisfied making of* $\mathbb{R}^2(\mathbb{R})$*.a vector space. Here every vector consists of an ordered pair of real numbers corresponding to the coordinates of a given point in the plane. It follows that* $\mathbb{R}^2(\mathbb{R})$ *is simply the Euclidian plane.*

Given a vector space $V(F)$ and the two vectors $v_1, v_2$ we use vector addition to produce a new vector $v = v_1 + v_2$ called vector sum. We can extend this operation to more than two vectors as in $v = [\{(v_1 + v_2) + v_3\} + v_4] \cdots$

Combining vector addition and scalar multiplication we can obtain a new operation on vectors: the **linear combination.** Consider the vectors $v_1, v_2 \in V(F)$ and the scalars $\alpha_1, \alpha_2 \in F$. We can construct the vector $v = \alpha_1 v_1 + \alpha_2 v_2$ which is the linear combination of the $v_1, v_2$ vectors: $v$ is obtained by scaling $v_1$, scaling $v_2$ then summing the scaled vectors. This operation can be extended to more than two vectors such as in $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + v_n$ where $v$ is the vector resulting from the linear combination of the $n$ vectors $\{v_1, v_2, \cdots, v_n\}$. In this case $v$ is said to be **written** or expressed as a linear combination of the set of vectors $\{v_1, v_2, \cdots, v_n\}$. A short notation of the expression of $v$ is $v = \sum_{i=1}^{n} \alpha_i v_i$.

## 2.2 Subspaces

Given a vector space $V(F)$, we can construct from it other vector spaces if some conditions are satisfied.

**Definition 15** *A non-empty subset of $S$ of $V$ is a subspace of $V(F)$ over the field $F$ , denoted $S(F)$, if and only if it satisfies the axioms of a vector space under the operations inherited from $V(F)$.*

Clearly $S(F)$ is a vector space on its own under the operations vector addition and scalar multiplication defined for $V(F)$. It follows that $V(F)$ is always a subspace of itself. It may be noticed that, since all elements of $S$ are vectors of $V(F)$, some of the axioms are automatically satisfied. It follows that not all axioms are to be checked. For example if $v_1, v_2, v_3 \in S$, then $v_1, v_2, v_3 \in V$ and by the associativity axiom we have $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ meaning that associativity is also satisfied for $S(F)$. It can be shown ( Do it as an exercise ) that we can reduce the number of axioms to be satisfied to only two axioms, namely **closedness under vector addition** and **closedness under scalar multiplication**. In this case, we say that $S(F)$ is **closed** under vector addition and scalar multiplication. This leads to the following result:

**Theorem 16** *A non-empty subset $S$ of the vector space $V(F)$ is a subspace denoted $S(F)$ if and only if*
*i) For all $v_1, v_2 \in S(F)$, we have $v_1 + v_2 \in S(F)$*
*ii) For all $v \in S(F)$ and all $\alpha \in F$, we have $\alpha v \in S(F)$*

**Proof.** : Do it as an exercise ■

This result tells us how to prove that $S(F)$ is a subspace of $V(F)$: Show that the non-empty set $S$ satisfies $S \subset V$ and that $S(F)$ is closed under vector addition and scalar multiplication. It must be noted that a subspace must always contain the zero vector $\theta$: clearly if $S(F)$ is closed under scalar multiplication then $v \in S(F)$ implies that $\alpha v \in S(F)$ for any $\alpha \in F$, in particular for $\alpha = 0$, hence $\alpha v = 0v = \theta \in S(F)$.

The two conditions on closedness under vector addition and scalar multiplication can be combined into a single one: **closedness under linear combinations**.

This can be summarized as : $S(F)$ subspace of $V(F) \Leftrightarrow S \subset V$ and for all $v_1, v_2 \in S(F)$ and all $\alpha_1, \alpha_2 \in F$, we have $\alpha_1 v_1 + \alpha_2 v_2 \in S(F)$

**Example 17** *Consider $S = \{(x, y, z)/x + y + z = 0\}$. Clearly $S$ is a non-empty subset of $\mathbb{R}^3(\mathbb{R})$, it follows that we need only check for closedness under linear combinations. Let $v_1, v_2 \in S$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, then $v_1 = (x_1, y_1, z_1)$ where $x_1 + y_1 + z_1 = 0$ and $v_2 = (x_2, y_2, z_2)$ where $x_2 + y_2 + z_2 = 0$. Now $\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1(x_1, y_1, z_1) + \alpha_2(x_2, y_2, z_2) = (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2, \alpha_1 z_1 + \alpha_2 z_2)$ where the sum of the components is $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_1 z_1 + \alpha_2 z_2 = \alpha_1(x_1 + y_1 + z_1) + \alpha_2(x_2 + y_2 + z_2) = \alpha_1.0 + \alpha_2 0 = 0$. It follows that the linear combination $\alpha_1 v_1 + \alpha_2 v_2$ satisfies the characteristic property of $S$, thus $\alpha_1 v_1 + \alpha_2 v_2 \in S(\mathbb{R})$ and $S(\mathbb{R})$ is a subspace of $\mathbb{R}^3(\mathbb{R})$.*

**Example 18** *Let $T = \{(x, y, z)/x + y + z = 1\}$. Clearly $T \subset \mathbb{R}^3(\mathbb{R})$ but it can be noticed right away that $T(\mathbb{R})$ is not closed under vector addition, hence $T(\mathbb{R})$ is not a subspace of $\mathbb{R}^3(\mathbb{R})$.*

**Example 19** *Consider again the vector space $S(\mathbb{R})$ where $S = \{f : \mathbb{R} \to \mathbb{R}/f$ continuous in $\begin{bmatrix} a & b \end{bmatrix}\}$ and the set $S = \{f : \mathbb{R} \to \mathbb{R}/f \quad$ continuous $\}$; Let $f \in T$ it follows that $f$ is continuous in $\mathbb{R}$, hence continuous in $\begin{bmatrix} a & b \end{bmatrix} \subset \mathbb{R}$ implying that $f \in S$ or $T \subset S$. Furthermore if $f_1, f_2 \in T$, then $f_1, f_2$ continuous in $\mathbb{R}$ and $\alpha_1 f_1 + \alpha_2 f_2$ is also continuous in $\mathbb{R}$. Hence $T(\mathbb{R})$ is closed under linear combination constituting then a subspace of $S(\mathbb{R})$*

Let $S(F), T(F)$ be subspaces of the vector space $V(F)$ and consider the set defined by $S(F) + T(F) = \{v \in V(F)/v = s + t \quad \text{where } s \in S(F), t \in T(F)\}$. Is $S(F) + T(F)$ another subspace of $V(F)$? To answer the question, we must test if $S + T$ is a subset of $V$ and if is closed under linear combination. Let $u \in S+T$, hence $u = s+t$ where $s \in S$ and $t \in T$. Now $s \in S$ , $t \in T \Rightarrow s, t \in V$ since $S, T$ are subspaces of $V$. It follows that $s + t \in V$ since $V(F)$ is a vector space and $S+T \subset V$. Let us test now for closedness under linear combination: consider $v_1, v_2 \in S(F) + T(F)$ and $\alpha_1, \alpha_2 \in F$ and the linear combination $\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1(s_1 + t_1) + \alpha_2(s_2 + t_2)$ which can be rewritten as $\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_1 t_1 + \alpha_2 t_2$ which is a sum of $\alpha_1 s_1 + \alpha_2 s_2 \in S$ and $\alpha_1 t_1 + \alpha_2 t_2 \in T$ hence $\alpha_1 v_1 + \alpha_2 v_2 \in S(F) + T(F)$ concluding that $S(F) + T(F)$ is a subspace of $V(F)$.It follows that **sum of two subspaces is also a subspace.**

Consider now the intersection $S(F) \cap T(F)$. It is clear that $S(F) \cap T(F)$ is a subset of $V(F)$. Is it closed under linear combinations? Let $v_1, v_2 \in S(F) \cap T(F)$ and $\alpha_1, \alpha_2 \in F$, hence $v_1, v_2 \in S(F)$ and $v_1, v_2 \in T(F)$. From $v_1, v_2 \in S(F)$, we conclude that $\alpha_1 v_1 + \alpha_2 v_2 \in S(F)$ since $S(F)$ is a subspace by definition. In the same manner $v_1, v_2 \in T(F)$ implies $\alpha_1 v_1 + \alpha_2 v_2 \in T(F)$ and we conclude that $\alpha_1 v_1 + \alpha_2 v_2 \in S(F) \cap T(F)$ making of $S(F) \cap T(F)$ a subspace of $V(F)$. Hence the conclusion: **the intersection of two subspaces is also a subspace.**

**Definition 20** *The subspaces $S(F), T(F)$ are said to constitute the* ***direct sum*** *of the vector space $V(F)$, denoted $V(F) = S(F) \oplus T(F)$, if and only if*
*i)* $V(F) = S(F) + T(F)$
*ii)* $S(F) \cap T(F) = [\theta]$

The notation $[\theta]$ refers to the **zero subspace**: the subset of $V(F)$ constituted of the single vector $\theta$ and which is obviously closed under linear combination

**Example 21** *Consider the subspaces $S(\mathbb{R}) = \{(x,, y)/y = 0\}$ and $T((\mathbb{R}) = \{(x,, y)/x = 0\}$ of the vector space $\mathbb{R}^2(\mathbb{R})$; It can be shown that $\mathbb{R}^2(\mathbb{R}) = S(\mathbb{R}) \oplus T((\mathbb{R})$. We must show first that $\mathbb{R}^2(\mathbb{R}) = S(\mathbb{R}) + T((\mathbb{R})$ or that $S(\mathbb{R}) + T((\mathbb{R}) \subset \mathbb{R}^2(\mathbb{R})$ and $\mathbb{R}^2(\mathbb{R}) \subset S(\mathbb{R}) + T((\mathbb{R})$. We need only show $\mathbb{R}^2(\mathbb{R}) \subset S(\mathbb{R}) + T((\mathbb{R})$ since the first part is obvious. Let $v \in \mathbb{R}^2(\mathbb{R})$, it follows that $v = (x, y)$ which can be rewritten as $v = (x, 0) + (0, y)$ where $(x, 0) \in S(\mathbb{R})$ and $(0, y) \in T((\mathbb{R})$, hence $\mathbb{R}^2(\mathbb{R}) \subset S(\mathbb{R}) + T((\mathbb{R})$ and consequently $\mathbb{R}^2(\mathbb{R}) = S(\mathbb{R}) + T((\mathbb{R})$. Now consider $S(\mathbb{R}) \cap T((\mathbb{R})$ and let $v \in S(\mathbb{R}) \cap T((\mathbb{R})$, it follows that $v \in S(\mathbb{R})$ or $v = (x, 0)$* ***and*** *$v \in T(\mathbb{R})$ or $v = (0, y)$ leading to $v = (0, 0) = \theta$. Hence $\mathbb{R}^2(\mathbb{R}) = S(\mathbb{R}) \oplus T((\mathbb{R})$.*

## 2.3 Spanning Subsets/Spanned Subspaces

We have seen previously that we can generate new subspaces from existing ones using sums and intersections. In this section we will see another way of generating subspaces but from subsets of vectors of a given vector space.

Consider the vector space $V(F)$, a subset $S = \{v_1, v_2, \cdots, v_k\}$ of vectors of $V(F)$ and define the set $[S]$ as $[S] = \{v / \sum_{i=1}^{k} \alpha_i v_i$ where $\alpha_i \in F\}$. Every vector of $[S]$ is then some linear combination of the vectors of $S$ and $[S]$ is the set of **all possible linear combinations** of the vectors of $S$. Is $[S]$ a subspace of $V(F)$ ? Clearly $[S] \subset V(F)$, hence we need only check if is closed under linear combination. Let $v, w \in [S]$ , it follows that $v = \sum_{i=1}^{k} \alpha_i v_i$ and $w = \sum_{i=1}^{k} \beta_i v_i$ . Now consider the linear combination $av + bw = a \sum_{i=1}^{k} \alpha_i v_i + b \sum_{i=1}^{k} \beta_i v_i$ $= \sum_{i=1}^{k} a\alpha_i v_i + b\beta_i v_i = \sum_{i=1}^{k} (a\alpha_i + b\beta_i) v_i$ where $(a\alpha_i + b\beta_i) \in F$ showing clearly that $av + bw$ is a linear combination of the vectors of $S$ and therefore $[S]$ is closed under linear combinations , thus a subspace of $V(F)$.

This tells us essentially that any subset of vectors of a given vector space can be used to generate a subspace by taking all the possible linear combinations of these vectors. The subset $S$ is called a **spanning subset** and the generated subspace $[S]$ is called a **spanned subspace**. Clearly a subspace $[S]$ can always be generated from any subset $S$ regardless of the number of vectors in $S$ ( Excluding for obvious reasons the empty subset $\varnothing$ ).

Let us consider first the simplest possible non-empty subset of $V(F)$ i.e. $S = \{\theta\}$ consisting of the single ( zero) vector. Clearly $[S] = [\theta]$ will be the simplest generated ( spanned) subspace from the zero vector. This subspace is called the **zero subspace** and denoted as $[\theta]$ .

**Example 22** *Consider the vector space* $\mathbb{R}^2(\mathbb{R})$ *and the vectors* $e_1 = (1, 0)$ *and* $e_2 = (0, 1)$

*i) The subspace spanned by* $e_1$ *is* $[\ e_1] = \{\alpha e_1\} = \{(\alpha, 0)\}$ *for all possible* $\alpha \in \mathbb{R} ::$ *it is the* $y = 0$ *axis in the* $y$ *vs* $x$ *plane*

*ii) The subspace spanned by* $e_2$ *is* $[\ e_2] = \{\beta e_2\} = \{(\alpha, 0)\}$ *for all possible* $\beta \in \mathbb{R} ::$ *it is the* $x = 0$ *axis in the* $y$ *vs* $x$ *plane*

*iii) The subspace spanned by* $\{e_1, e_2\}$ *is* $[e_1, e_2\ ] = \{\alpha e_1 + \beta e_2\} = \{(\alpha, \beta)\}$ *for all possible* $\alpha, \beta \in \mathbb{R} ::$ *it is the whole* $y$ *vs* $x$ *plane or* $\mathbb{R}^2(\mathbb{R})$ *itself.*

## 2.4 Linear independence

The concept of linear independence is fundamental in linear algebra