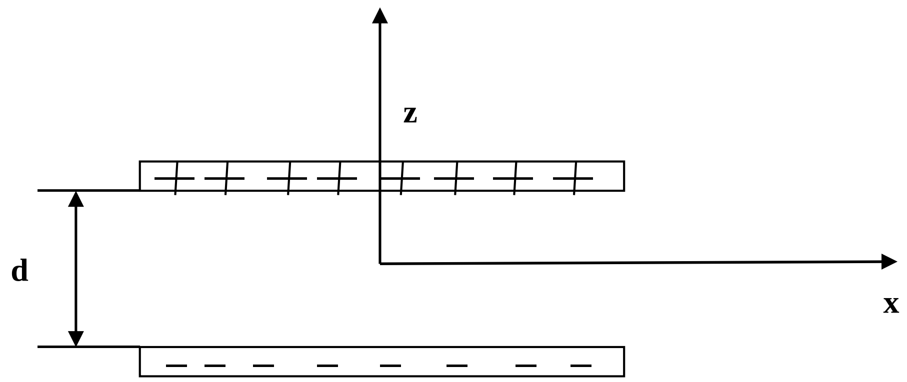
2^{nd} Recitation: Gauss' Law

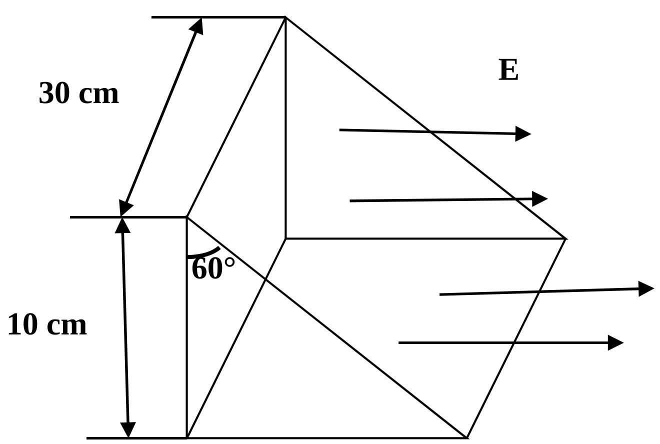
Exercise #1:
Find the electric field inside and outside a thin walled metal tube of radius **R = 3.0 cm** and carrying a positive charge per unit length λ of **2.0×10^{-8} C/m**

Exercise #2:
Two parallel infinite non-conducting plates lying in the **xy**-plane are separated by a distance **d**. Each plane is uniformly charged with equal but opposite surface charge densities as shown in the figure. Find the electric field everywhere in the space.



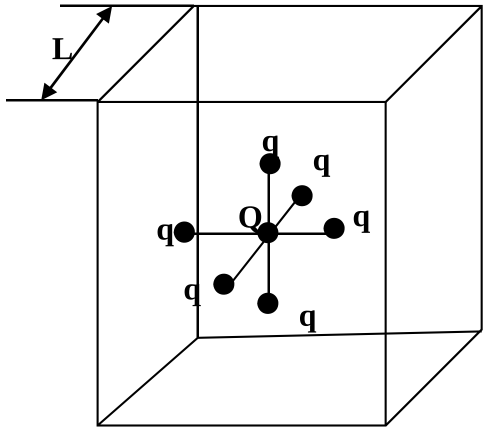
Exercice #3:
A sphere made with insulating of radius **R** has a charge density **ρ = a× r** where **a** is a constant. Let **r** be the distance from the center of the sphere. Find the electric field everywhere, both inside and outside the sphere.

Exercise #4:
Consider a close triangular box resting within a horizontal electric field of magnitude **E = 9.00×10^4 N/C**, as shown in the figure below.



Exercise #5:

A particle with charge **Q = 5.10μC** is located at the center of a cube of edge **L = 0.160 m**. In addition, six other identical charged particles having **q = -1.5μC** are positioned symmetrically around **Q** as shown in the figure below. Determine the electric flux through one face of the cube.



Exercise #6:

A solid sphere of radius **40.0 cm** has a total positive charge of **28.0 μC** uniformly distributed through its volume. Calculate the magnitude of the electric field at the following distances: (a) **0 cm**; (b) **10.0 cm**; (c) **40.0 cm** and (d) **60.0 cm** from the center of the sphere

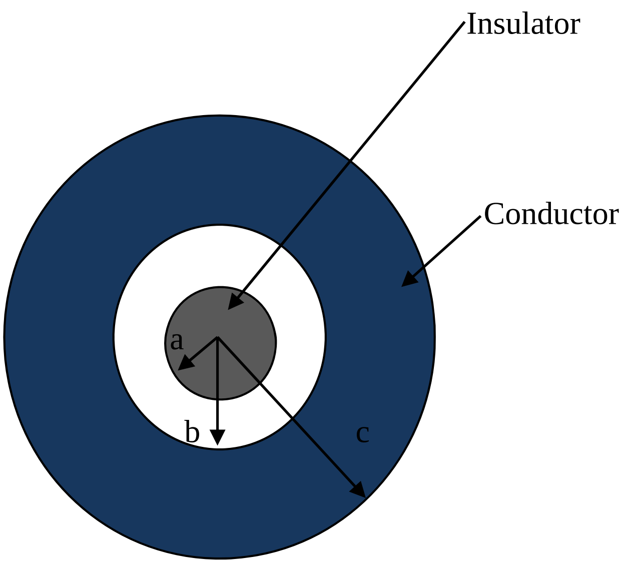
Exercise #7:

A solid insulating sphere of radius **a** has a uniform charge density of ρ and a total charge **Q**. Concentric with this sphere is an uncharged, conducting hollow sphere whose inner and outer radii are **b** and **c** as shown in the figure below.

1. Find the magnitude of the electric field in the following regions:

r < a; **a < r < b** ; **b < r < c** ; **r > c**

2. Determine the induced charge per unit area on the inner and outer surfaces of the hollow sphere.



Exercise 1;

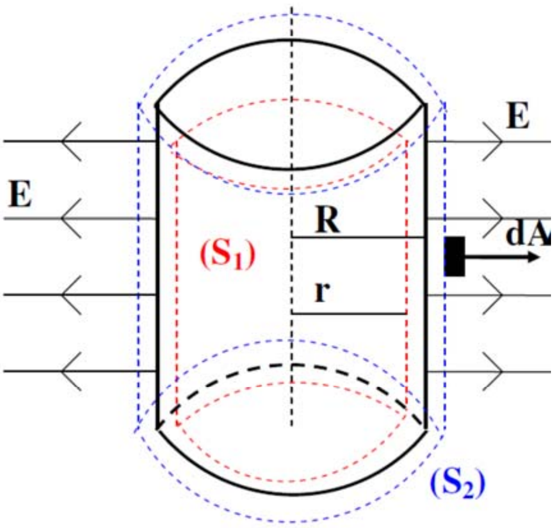
4. Gauss' law

Exercise 4.1: Find the electric field inside and outside a thin-walled metal tube of radius R = 3.0 cm and carrying a positive charge per unit length λ of 2.0 x 10^{-8} C/m.

Given: $\lambda = 2.0 * 10^{-8} \frac{C}{m}$

$R = 0.03\, m$

Find: $E(r)$ everywhere.



SOLUTION:

(a) For $r < R$, we consider an inner cylinder of radius r (a Gaussian surface S_1).

Because there is no change inside the cylinder:

$$\oint_{S_1} \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0} = 0 \Rightarrow E(r < R) = 0$$

(b) For $r \geq R$, we consider another cylindrical Gaussian surface S_2:

$$\oint_{S_2} \vec{E} d\vec{A} = \vec{E} \oint_{S_2} d\vec{A} = E \int dA = E(2\pi r h),$$ where h = tube's height.

↑ (because $\vec{E}$ is uniform) ↑ (because $\vec{E}$ is *perpendicular* on the lateral side of surface S_2 and is *parallel* with the two caps of S_2.)

Because the charge enclosed by S_2 is $q = \lambda\, h$, then $\varepsilon_0 E(2\pi r h) = q$

Finally, we get $$E(r \geq R) = \frac{\lambda}{2\pi\varepsilon_0 r}$$

4.8.1 Two Parallel Infinite Non-Conducting Planes

Two parallel infinite non-conducting planes lying in the xy-plane are separated by a distance d. Each plane is uniformly charged with equal but opposite surface charge densities, as shown in Figure 4.8.1. Find the electric field everywhere in space.

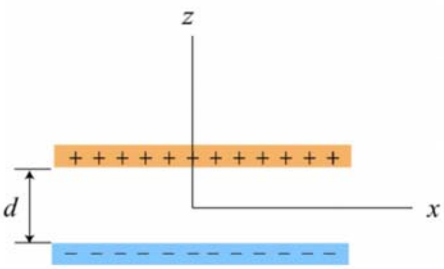


Figure 4.8.1 Positive and negative uniformly charged infinite planes

Solution:

The electric field due to the two planes can be found by applying the superposition principle to the result obtained in Example 4.2 for one plane. Since the planes carry equal but opposite surface charge densities, both fields have equal magnitude:

$$E_+ = E_- = \frac{\sigma}{2\varepsilon_0}$$

The field of the positive plane points away from the positive plane and the field of the negative plane points towards the negative plane (Figure 4.8.2)

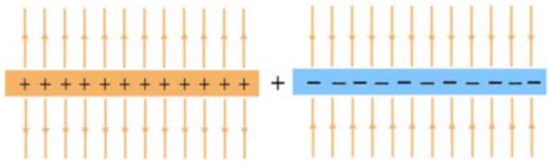


Figure 4.8.2 Electric field of positive and negative planes

Therefore, when we add these fields together, we see that the field outside the parallel planes is zero, and the field between the planes has twice the magnitude of the field of either plane.

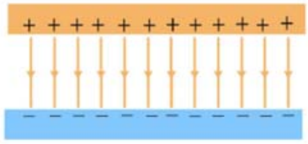


Figure 4.8.3 Electric field of two parallel planes

The electric field of the positive and the negative planes are given by

$$\vec{\mathbf{E}}_+ = \begin{cases} +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > d/2 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < d/2 \end{cases} , \qquad \vec{\mathbf{E}}_- = \begin{cases} -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > -d/2 \\ +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < -d/2 \end{cases}$$

Adding these two fields together then yields

$$\vec{\mathbf{E}} = \begin{cases} 0\,\hat{\mathbf{k}}, & z > d/2 \\ -\dfrac{\sigma}{\varepsilon_0}\hat{\mathbf{k}}, & d/2 > z > -d/2 \\ 0\,\hat{\mathbf{k}}, & z < -d/2 \end{cases} \tag{4.8.1}$$

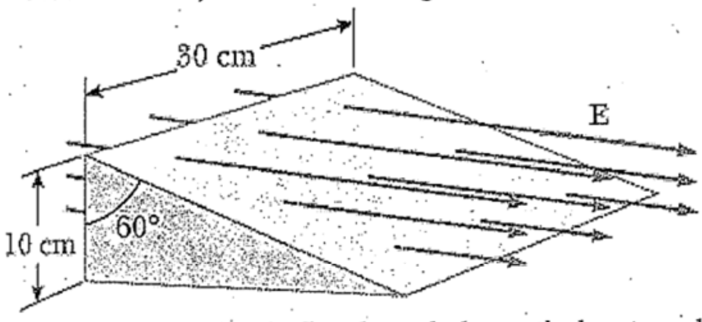
Note that the magnitude of the electric field between the plates is $E = \sigma/\varepsilon_0$, which is twice that of a single plate, and vanishes in the regions $z > d/2$ and $z < -d/2$.

4.10.3 Sphere with Non-Uniform Charge Distribution

A sphere made of insulating material of radius R has a charge density $\rho = ar$ where a is a constant. Let r be the distance from the center of the sphere.

(a) Find the electric field everywhere, both inside and outside the sphere.

Consider a closed triangular box resting within a horizontal electric field of magnitude $E = 9.00 \times 10^4$ N/C, as shown in the figure below.



(a) Calculate the electric flux through the vertical rectangular surface of the box.

kN·m²/C $\boxed{-2.7}$

(b) Calculate the electric flux through the slanted surface of the box.

kN·m²/C $\boxed{+2.7}$

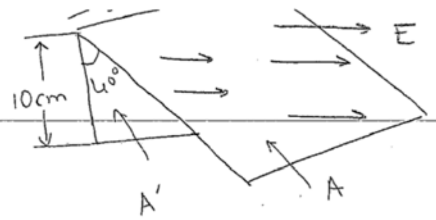
(c) Calculate the electric flux through the entire surface of the box.

0 kN·m²/C $\boxed{0}$

Given:

$$|E| = 9.00 \times 10^4 \frac{N}{C}$$

(a)



(length)(width)

A' = area vertical Rectangle = (10cm)(30cm)

$A' = 300\,cm^2 = 0.0300\,m^2$

By Def'n : Flux $\boxed{\Phi_E = |E|A'\cos\theta}$

$\Phi_E = \left|9\times10^4 \frac{N}{C}\right|(0.03m^2)\cos 180° = \boxed{-2.7\, \frac{kN\cdot m^2}{C}}$

where $\theta = 180°$ b/c the $\vec{E}$-field is perpendicular to the box at this location.

(b) Slant : $\boxed{\Phi_E = |E|A\cos\theta}$

here : $\theta = 60°$ and $A = (30.0cm)(W)$

$W = \frac{10cm}{\cos 60°}$

$\therefore A = (30.0cm)\frac{(10\,cm)}{\cos(60°)} = 600\,cm^2 = 0.06m^2$

$\Phi_E = \left(9\times10^4 \frac{N}{C}\right)(0.06m^2)\cos(60°) = \boxed{2.7\, \frac{kN\cdot m^2}{C}}$

c) Entire Box :

The bottom & the two triangular sides all lie parallel to $\vec{E}$, so $\Phi_E = 0$ for these. Thus,

$\boxed{\Phi_{E,total}} = -2.7\frac{kN\cdot m^2}{C} + 2.7\frac{kN\cdot m^2}{C} = \boxed{0}$

2. 12 points

The electric field everywhere on the surface of a thin, spherical shell of radius 0.790 m is measured to be 860 N/C and points radially toward the center of the sphere.

(a) What is the net charge within the sphere's surface?

nC $\boxed{-59.7\,nC}$

(b) What can you conclude about the nature and distribution of the charge inside the spherical shell?

- the positive charge has an asymmetric charge distribution
- the negative charge has an asymmetric charge distribution
- the negative charge has a spherically symmetric charge distribution
- the positive charge has a spherically symmetric charge distribution

*Given: $r = 0.790\,m$

$|E| = 860\,\frac{N}{C}$; "Points Radially toward Center of Sphere"

a) $E = \frac{k_e Q}{r^2}$; $Q = ?$ is what we are calculating.

$$860\,\frac{N}{C} = \frac{(8.99\times10^9)\,Q}{(0.79\,m)^2} \Rightarrow Q = 5.97\times10^{-8}\,C$$

But Q is negative since $\vec{E}$ points inward.

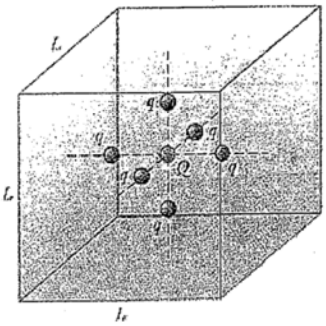
Therefore; $\boxed{Q = -59.7\,nC}$

b) The $\boxed{\text{negative}}$ charge has a $\boxed{\text{Spherically}}$ symmetric charge distribution.

4. –/1 points

A particle with charge $Q = 5.10\ \mu C$ is located at the center of a cube of edge $L = 0.160$ m. In addition, six other identical charged particles having $q = -1.5\ \mu C$ are positioned symmetrically around Q as shown in the figure below. Determine the electric flux through one face of the cube.

$\boxed{73.4}$ kN·m²/C



$Q = 5.1 \mu C$

$q_h = -1.5 \mu C\ (\times 6)$

$L = 0.160 m$

The total charge is $\boxed{Q - 6|q_h|}$

$\therefore Q_{tot} = Q - 6|q_h|$

The total outward flux from the cube can be found from $\boxed{\Phi_{cube} = \int \vec{E} \cdot d\vec{a}}$ $(K_e = 1/4\pi\epsilon_0)$

where $\vec{E}$ is found from charges positioned "symmetrically around" Q.

$\vec{E} = \frac{(K_e\ Q - 6|q_h|)}{R_2}$

$$\Phi_{cube} = \frac{K_e(Q - 6|q_h|)}{R^2}(4\pi R^2)$$

$$= \frac{Q - 6|q_h|}{4\pi\epsilon_0 R^2}(4\pi R^2)$$

Entire Flux $\boxed{\Phi_{cube} = \frac{Q - 6|q_h|}{\epsilon_0}}$

So the flux through one face is one-sixth through each face.

$$\boxed{\Phi_{one\text{-}face} = \frac{1}{6}\,\Phi_{cube}}$$

$$\boxed{\Phi_{one\text{-}face} = \frac{1}{6}\,\frac{(Q - 6|q_h|)}{\epsilon_0}}$$

Plugging in numbers & notice $|q_h| = 1.5\,\mu C$

$$\Phi_{one\text{-}face} = \frac{1}{6}\,\frac{(5.1 - 9.0)\times 10^{-6}\ C \cdot N\cdot m^2}{\left(8.854\times 10^{-12}\ \frac{C^2}{N\cdot m^2}\right)}$$

$$\Phi_{one\text{-}face} = -7.34\times 10^{4}\ \frac{N\cdot m^2}{C}$$

$$|\Phi| = 73.4\ kN\cdot m^2/C$$

5. --/4 points

A solid sphere of radius 40.0 cm has a total positive charge of 28.0 μC uniformly distributed throughout its volume. Calculate the magnitude of the electric field at the following distances.

(a) 0 cm from the center of the sphere $r = 0\text{cm}$

0 kN/C

(b) 10.0 cm from the center of the sphere $r = 10.0\text{cm}$

393 kN/C

(c) 40.0 cm from the center of the sphere $r = 40\text{ cm}$

1570 kN/C

(d) 60.0 cm from the center of the sphere $r = 60\text{cm}$

699 kN/C

$a = 40.0\text{ cm} = 40\times10^{-2}\text{m}$

$Q = 28\times10^{-6}\text{C}$

$k_e = 1/4\pi\epsilon_0$

Gauss's Law States:

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{q_{in}}{\epsilon_0}$$

Example $\boxed{24.3}$ works through parts (b) & (c) completely

a) Choose a spherical gaussian surface with $r = 0\text{cm}$. Calculate q_{in} by using $\boxed{q_{in} = \rho V'}$

$$q_{in} = \rho\left(\tfrac{4}{3}\pi r^3\right)$$

$$\oint E dA = E\oint dA = E(4\pi r^2) = \frac{q_{in}}{\epsilon_0}$$

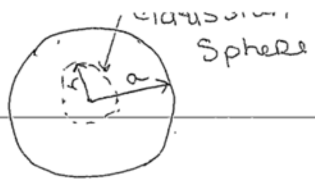
$$E = \frac{q_{in}}{4\pi r^2\epsilon_0} = \frac{\rho(4\pi r^3)}{4\pi r^2\epsilon(3)} = \frac{\rho r}{3\epsilon_0}$$

$$\rho = \frac{Q}{\frac{4}{3}\pi a^3} \quad \& \quad \epsilon_0 = \frac{1}{4\pi k_e} \quad \therefore \boxed{E = \frac{k_e Q}{a^3} r}$$

but $r = 0$

$$\boxed{E = 0}$$

(b)



Same equation we calculated in part (a)

$$\boxed{E = \frac{k_e Q r}{a^3}} \quad \boxed{r < a}$$

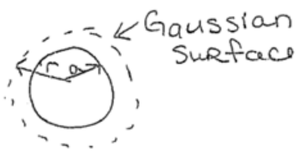
Plugging in values: $r = 10.0\,\text{cm}$

$$\boxed{E} = \frac{(8.99\times10^{9})(28\times10^{-6})(0.100)}{(0.40)^3} = \boxed{3.93\times10^{5}\,\frac{N}{C}}$$

c) $\boxed{r = a}$ $\quad E = \frac{k_e Q r}{a^3} = \boxed{\frac{k_e Q}{r^2}}$

$$E = \frac{(8.99\times10^{9})(28\times10^{-6})}{(0.40)^2} = \boxed{1.57\times10^{6}\,\frac{N}{C}}$$

d)



$$\oint \vec{E}\cdot d\vec{A} = \oint E\,dA = \frac{Q}{\epsilon_0}$$

$$E(4\pi r^2) = \frac{Q}{\epsilon_0}$$

$\boxed{r = 0.600\,m}$

$$\boxed{E = \frac{k_e Q}{r^2}} \quad r > a$$

$$E = \frac{(8.99\times10^{9})(28\times10^{-6})}{(0.60)^2}$$

$$\boxed{E = 6.99\times10^{5}\,\frac{N}{C}}$$

7. --/1 points

A particle with a charge of -60.0 nC is placed at the center of a nonconducting spherical shell of inner radius 20.0 cm and outer radius 24.0 cm. The spherical shell carries charge with a uniform density of -3.10 μC/m³. A proton moves in a circular orbit just outside the spherical shell. Calculate the speed of the proton.

6.99e5 m/s

$$\boxed{\begin{aligned} q_n &= -60.0\times10^{-9}\,C \\ R_{in} &= 0.20\,m \\ R_{out} &= 0.24\,m \\ \rho &= -3.10\times10^{-6}\,\frac{C}{m^3} \end{aligned}}$$

$V_{proton} = ?$

The volume of the spherical shell is:

$$V = \frac{4}{3}\pi\left[R_{out}^3 - R_{in}^3\right] = \frac{4}{3}\pi\left[(0.24m)^3 - (0.20m)^3\right]$$

$$V = 2.439\times10^{-2}\,m^3$$

Its charge is:

$$\rho V = \left(-3.10\times10^{-6}\,\frac{C}{m^3}\right)\left(2.439\times10^{-2}\,m^3\right)$$

$$Q = -7.56\times10^{-8}\,C$$

The net charge inside a sphere containing the proton's path as its equator is:

$$q_{net} = -60\times10^{-9}\,C - 7.56\times10^{-8}\,C = -1.36\times10^{-7}\,C$$

$$\left[\,Q_{net} = Q_{proton} + Q\,\right]$$

The electric field is Radially inward with magnitude :

$$|E| = \frac{k_e |q_n|}{r^2} = \frac{|q_n|}{\epsilon_0 4\pi R_{out}^2} = \frac{(8.99\times10^9 \, N\cdot m^2)|Q_{net}|}{C^2 (R_{out})^2}$$

$$|E| = 2.123\times10^4 \, N/C$$

For the proton :

$$\Sigma F = m\vec{a} = q_n \vec{E} \quad (q_n = e^-)$$

$$\boxed{eE = \frac{m_p v^2}{R_{out}}}$$ "Circular Path"

$$v = \left(\frac{eER_{out}}{m_p}\right)^{1/2} \quad ; \quad m_p \equiv \text{mass of proton}$$

$$= \left[\frac{1.60\times10^{-19} C (2.123\times10^4 N/C)(0.24 m)}{(1.67\times10^{-27} kg)}\right]^{1/2}$$

$$v = \left[4.88\times10^{11}\right]^{1/2} = \boxed{6.9857\times10^5 \, \frac{m}{s}}$$

10. --/5 points

A thin, square conducting plate 52.0 cm on a side lies in the *xy* plane. A total charge of $+4.00 \times 10^{-8}$ C is placed on the plate. Find the following. You may assume that the charge density is uniform.

(a) the charge density on the plate

7.4e-8 C/m^2

(b) the electric field just above the plate

magnitude 8.36e3 N/C

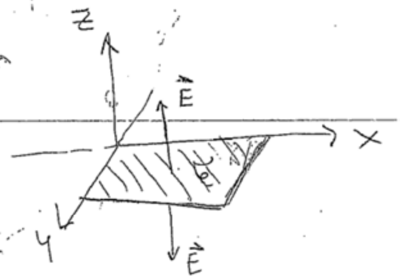
direction ---Select--- upward

(c) the electric field just below the plate

magnitude N/C

direction ---Select---

$$\boxed{\begin{aligned} x &= 52.0 \times 10^{-2}\,m \\ q &= 4.0 \times 10^{-8}\,C \end{aligned}}$$



a) $\sigma = ? \Rightarrow \frac{\text{charge}}{\text{area}}$

The charge density on each of the surfaces (upper & lower) of the plate is:

$$\sigma = \frac{1}{2}\left(\frac{q}{A}\right) = \frac{1}{2}\,\frac{(4.0 \times 10^{-8}\,C)}{(0.52\,m)^2} = \boxed{7.40 \times 10^{-8}\,\frac{C}{m^2}}$$

b) [Refer to Example 24.5]

$$\Phi = EA = \frac{q_{in}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{E_{above} = \frac{\sigma}{\epsilon_0}\,\hat{z}} = \frac{(7.4 \times 10^{-8}\,C/m^2)}{(8.85 \times 10^{-12}\,C^2/N \cdot m^2)}\,\hat{z}$$

$$E_{above} = \boxed{8.36 \times 10^{3}\,\frac{N}{C}} \text{ upward}$$

(c) same Electric field as in (b) $\Rightarrow$ i.e. magnitude but pointing in the $(-\hat{z})$ direction.

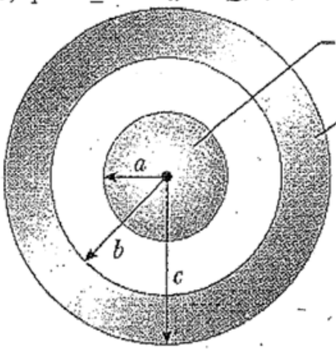
Therefore;

$$|E_{below}| = |E_{above}|$$

$$= 8.36 \times 10^3 \ \frac{N}{C}$$

Direction is "downward"

√ A solid, insulating sphere of radius a has a uniform charge density of ρ and a total charge of Q. Concentric with this sphere is an uncharged, conducting hollow sphere whose inner and outer radii are b and c, as shown in the figure below. (Use rho for ρ, epsilon_0 for ε_0, and Q, r, a, b and c, as needed.)



*Note: Must use capital "Q" in answers

(a) Find the magnitude of the electric field in the following regions.

$r < a$

(rho r)/(3 epsilon_0)

$a < r < b$

Q/(4pi r^2 epsilon_0)

$b < r < c$

0

$r > c$

Q/(4 pi r^2 epsilon_0)

(b) Determine the induced charge per unit area on the inner and outer surfaces of the hollow sphere.

inner surface

-Q/(4 pi b^2)

outer surface

Q/(4 pi c^2)

$$\boxed{\oint \vec{E} \cdot d\vec{a} = E(4\pi r^2) = \frac{q_{in}}{\epsilon_0}}$$

Gauss's Law

2) For $\boxed{r<a}$ $q_{in} = \rho\left(\frac{4}{3}\pi r^3\right)$

so $\boxed{E = \frac{\rho r}{3\epsilon_0}}$

$\boxed{a<r<b}$ and $\boxed{c<r}$; $q_{in} = Q$

So $$\boxed{E = \frac{Q}{4\pi r^2 \epsilon_0}}$$

For $\boxed{b \le r \le c}$; $E = 0$; since $\boxed{E = 0}$ inside a conductor

(b) $\sigma = \frac{\text{charge}}{\text{area}} = ?$

Let q_1 = induced charge on the inner surface of the hollow sphere.

Since $E = 0$ inside the conductor, the total charge enclosed by a spherical surface of Radius $b \le r \le c$ must be zero.

Therefore, $q_1 + Q = 0$

$$\boxed{\sigma_1} = \frac{q_1}{4\pi b^2} = \boxed{\frac{-Q}{4\pi b^2}}$$

Let q_2 = induced charge on the outside surface of the hollow sphere.

Since the hollow sphere is uncharged, we require

$$q_1 + q_2 = 0$$

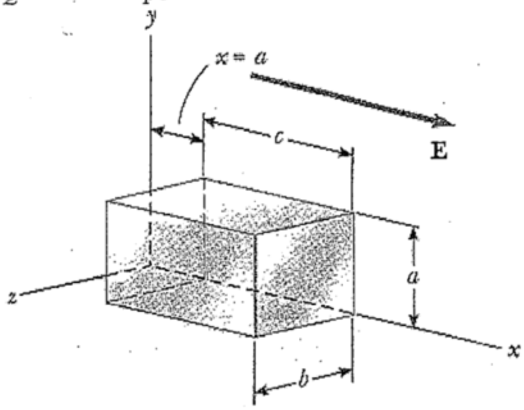
$$\boxed{\sigma_2} = \frac{q_1}{4\pi c^2} = \boxed{\frac{Q}{4\pi c^2}}$$

12. **--/2 points**

A closed surface with dimensions $a = b = 0.400$ m and $c = 0.660$ m is located as shown in the figure below. The left edge of closed surface is located at position $x = a$. The electric field throughout the region is nonuniform and given by $\vec{E} = (5.0 + 2.0x^2)\,\hat{\imath}$ N/C, where x is in meters. Calculate the net electric flux leaving the closed surface. What net charge is enclosed by the surface?

$\varphi_E =$ 0.308 $N \cdot m^2/C$

$Q =$ 2.73 pC



The electric field throughout the region is directed along x; therefore, $\vec{E}$ will be perpendicular to "dA" over the four faces of the surface. These surfaces are perpendicular to the yz-plane. Also E will be parallel to "dA" over the two faces which are parallel to the yz-plane.

Given:

$a = b = 0.40m$

$c = 0.660m$

$\vec{E} = (5.0 + 2.0x^2)\hat{\imath}\ \frac{N}{C}$

a) $\Phi_E = -\left(E_x\Big|_{x=a}\right)A + \left(E_x\Big|_{x=a+c}\right)A$

$$= -\left(5.0 + 2a^2\right)(ab) + \left(5.0 + 2(a+c)^2\right)ab$$

$$= -2a^3b + 2ab(a^2 + 2ac + c^2)$$

$$= -2a^3b + 2a^3b + 4a^2bc + 2abc^2$$

$$\boxed{\Phi_E = 2abc(2a + c)}$$

substituting the given values for a, b, & c.

$$\Phi_E = 2(0.4)(0.4)(0.66)\left[2(0.4) + 0.66\right]$$

$$\Phi_E = (2.112\times10^{-1})[1.46] = \boxed{0.308\,\frac{N\cdot m^2}{C}}$$

$$Q = \epsilon_0\Phi_E = \left(8.85\times10^{-12}\frac{C^2}{N\cdot m^2}\right)\left(0.308\,\frac{N\cdot m^2}{C}\right)$$

$$= 2.73\times10^{-12}\,C$$

$$\boxed{Q = 2.73\ pC}$$