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EE 178: Physics II

Electricity & Magnetism

Chapter 3: Gauss' Law

3.1 Introduction:

One way to solve complex problems in physics is the use of symmetry.

For certain charge distributions involving symmetry, the use of Gauss' Law, developed by German mathematician and physicist Carl Friedrich Gauss (1777-1855)

Instead of considering the fields $\overrightarrow{dE}$ of charge elements in a given charge distribution.

Gauss' law considers a hypothetical (imaginart) closed surface enclosing the charge distribution.

3.2 Flux:

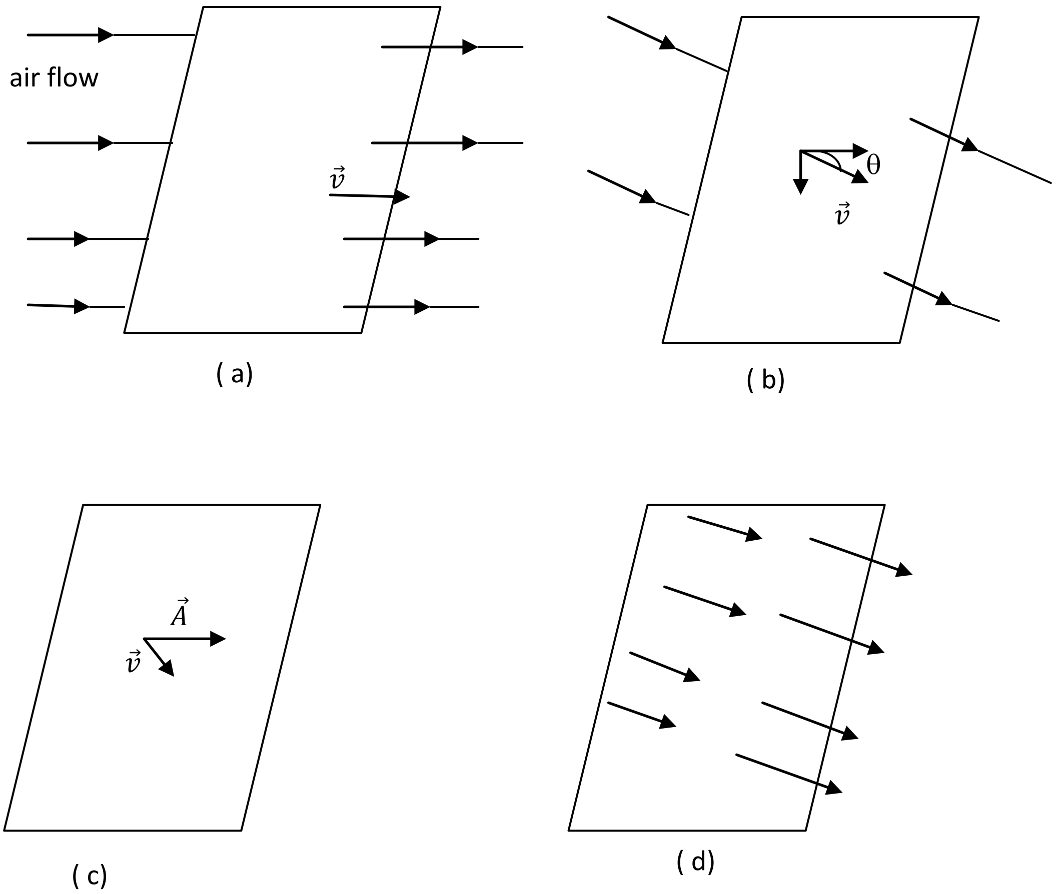
Suppose that, we want a wide airstream of uniform velocity $\vec{v}$ at a small square loop of area A. Let φ represent the volume flow rate (volume per unit time) at which air flows through the loop.

if $\vec{v} /\!\!/ \text{to the plane; so no air moves through the loop } \varphi = 0$

$$(\vec{v}, \vec{A}) = \theta \rightarrow \text{The rate } \varphi \text{ depends on the vertical component of } \vec{v}$$
$$: \ v\cos\theta. \qquad \varphi = (v\cos\theta)A$$

$$\varphi = v\,A\cos\theta = \vec{v}.\vec{A}$$

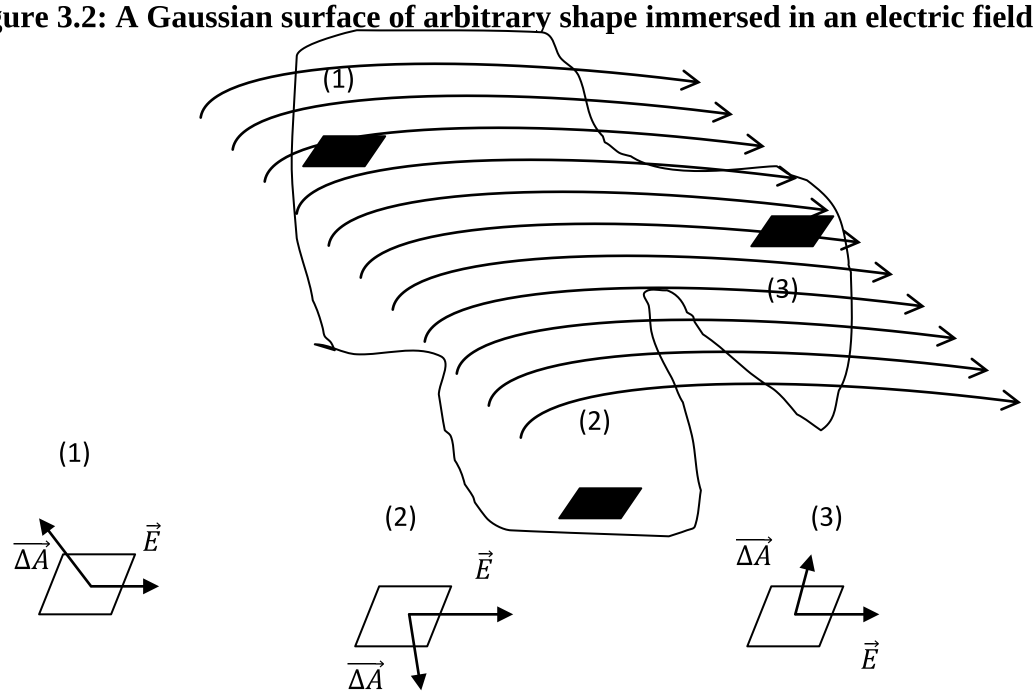
Figure 3.1: (a) A uniform airstream of velocity $\vec{v}$ is perpendicular to the plane of a square loop of area A; (b) The component of $\vec{v}$ perpendicular to the plane is $v\cos\theta$, where θ is the angle between $\vec{v}$ and a normal to the plane; (c) The area vector $\vec{A}$ is perpendicular to the plane of the loop and makes an angle θ with $\vec{v}$; (d) The velocity field intercepted by the area of the loop.



3.3. Flux of an electric field:

Let's take an asymmetric arbitrary Gaussian surface immersed in an electric field as shown in figure 3.2

Figure 3.2: A Gaussian surface of arbitrary shape immersed in an electric field



Because the squares have been taken to be arbitrarily small, the electric field $\vec{E}$ may be taken as constant over any given square.

The flux of the electric field for the Gaussian surface of the figure 3.2 is:

$$\varphi = \sum \vec{E}\,.\overrightarrow{\Delta A}$$

The exact defenition of the flux of the electric field through a closed surface is found by allowing the area of the squares shown in the figure 3.2 to become smaller and smaller, approaching a differential limit dA, so:

$$\varphi = \oint \vec{E}\,.\overrightarrow{dA}$$

(Electric flux through a Gaussian surface)

3.4- Gauss' Law:

Gauss' Law relates the net flux φ of an electric field through a closed surface (A Gaussian surface) to the net charge q_{enc} that is **enclosed by that surface**.

$$\varepsilon_0\,\varphi = q_{enc}$$

Replacing φ with the previous equation:

$$\varepsilon_0 \oint \vec{E}\,.\overrightarrow{dA} = q_{enc}$$

(Gauss' Law)

3.5- Gauss' Law and Coulomb Law:

The figure 3.3 represents a positive point charge, around which we have drawn a concentric spherical Gaussian surface of radius r. $\overrightarrow{dA}$ is perpendicular to that surface and directed outward from the interior. $\vec{E}$ is also perpendicular to that surface and directed outward from the interior. $\theta = \left(\vec{E},\overrightarrow{dA}\right) = 0$

Gauss' law:

$$\varepsilon_0 \oint \vec{E}\,.\overrightarrow{dA} = \varepsilon_0 \int E\,.dA = q_{enc}$$

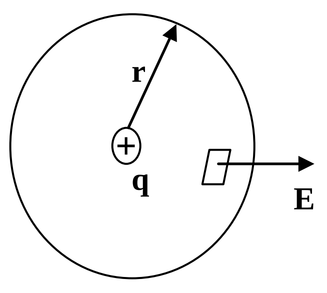
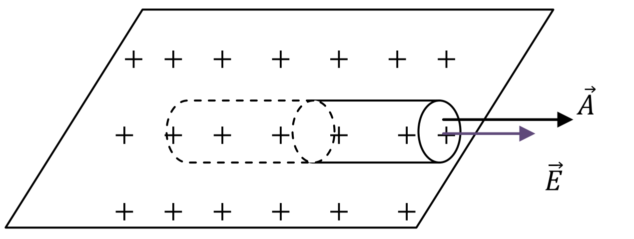


Figure 3.3: A spherical Gaussian surface centered on a point charge

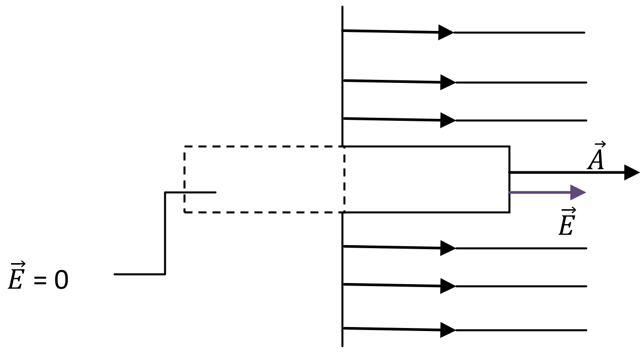
3.6- A charged isolated conductor:

Gauss' law permits to us to prove an important theorem about conductors:

If an excess charge is placed on an isolated conductor, that amount of charge will move, entirely to the surface of the conductor. None of the excess charge will be found within the body of the conductor



(a) There is flux only through the external end face



(b)

Figure 3.4: (a) Perspective view, (b) Side view of a tiny portion of a large, isolated conductor with excess positive charge on its surface.

Let us take a cylinder as shown in figure 3.4 as a Gaussian surface. *There is no flux through the internal cap because the electric field within the conductor is zero. There is no flux through the curved surface of the cylinder, because internally (in the conductor) there is no electric field and externally the electric field is parallel to the curved portion of the Gaussian surface.*

$$\varphi = E.A = \frac{q_{enc}}{\varepsilon_0} = \frac{\sigma\, A}{\varepsilon_0} \xrightarrow{yields} E = \frac{\sigma}{\varepsilon_0}$$

3.7 Applying Gauss' Law: Cylindrical symmetry:

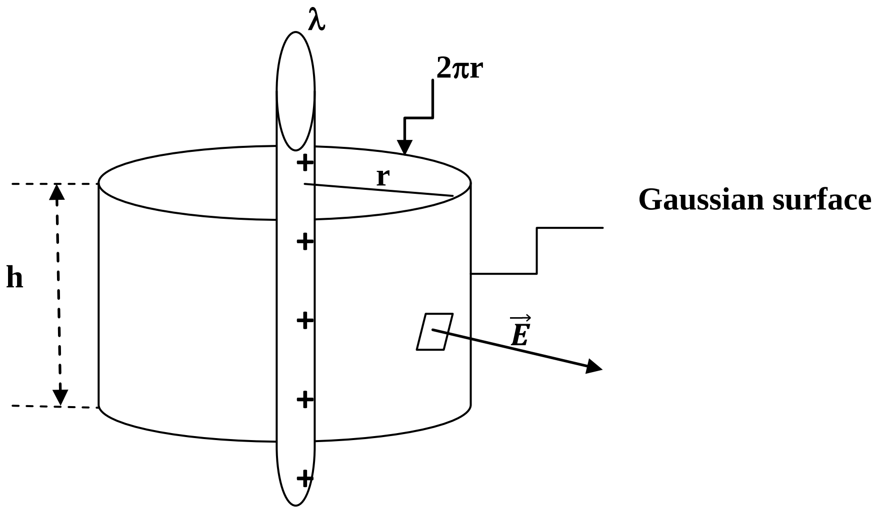


Figure 3.5: A Gaussian surface in the form of a closed cylinger surrounds a section of a very long, uniformly charged, cylindrical plastic rod.

As shown in the figure above, a section of an infinitely long cylindrical plastic rod with a uniform linear density λ. We want to find the electric field $\vec{E}$ at a distance r from the axis of the rod.

The chosen Gaussian surface should match the symmetry of the problem, which is cylindrical.

Since 2πr is cylinder's circumference and h is its length, the area A of the cylindrical surface is 2π r h. The flux of $\vec{E}$ through this cylindrical surface is:

$$\varphi = E\ A\cos\theta = \vec{E}.\vec{A} = E\ (2\pi rh)\cos 0 = E.(2\pi rh)$$

$$\varphi = 0\ through\ the\ end\ caps\ because\ \vec{E}\ is\ radial\ and\ vertical\ to\ \vec{A}$$

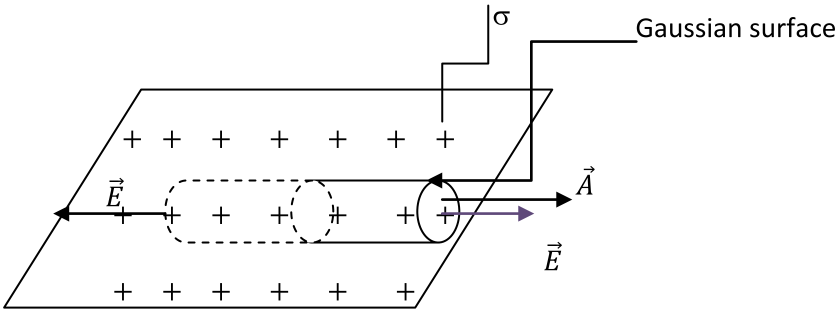
The charge enclosed by the surface is λh

$$\varphi = E.(2\pi rh) = \frac{q_{enc}}{\varepsilon_0} = \frac{\lambda\, h}{\varepsilon_0}$$

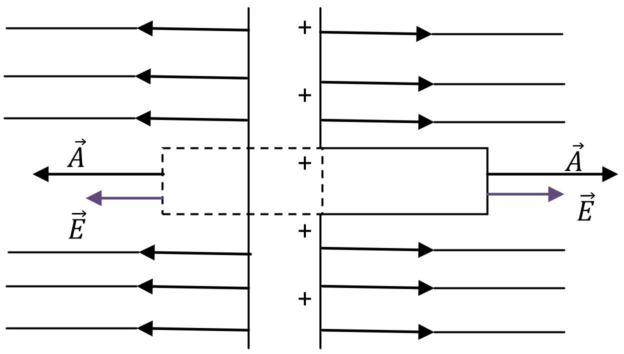
$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

(Line of charge)

3.8 Applying Gauss' Law: Planar symmetry non-conducting sheet:



(a) There is flux only through the two end faces



(b)

Figure 3.6: (a) Perspective view and (b) side view of a portion of very large, thin plastic sheet, uniformly charged on one side to surface charge density σ. A closed cylindrical Gaussian surface passes through the sheet and perpendicular to it.

Since the charge is positive n the non-conducting sheet, $\vec{E}$ is directed away from the sheet.

$\vec{E}$ is parallel to $\vec{A}$ in the curved surface, no flux

$$\varphi = \oint \vec{E}\,.\,\overrightarrow{dA} = \frac{q_{enc}}{\varepsilon_0} = E.A + E.A = \sigma.A$$

σ.A is the charge enclosed by the Gaussian surface

$$E = \frac{\sigma}{2\varepsilon_0}$$

(Non-conducting sheet)

Two conducting plates:

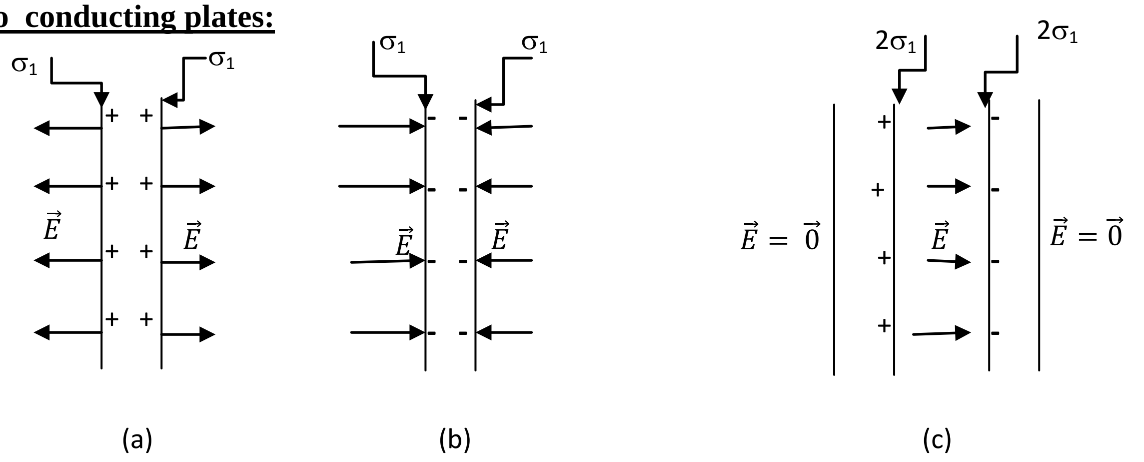


Figure 3.7: (a) A thin, very large conducting plate with excess positive charge, (b) an identical plate with excess negative charge, (c) The two plates arranged so they are parallel and close

Since the plates are conductors, when we bring them into this arrangement, the excess charge on one plate attracts the excess charge on the other, and all the excess charge moves onto tne inner faces of the plates.

So, the new surface charge density σ on each inner face is twice σ_1.

$$E.A = \frac{2\,\sigma_1}{\varepsilon_0}\,A = \frac{\sigma}{\varepsilon_0}\,A$$

$$E = \frac{\sigma}{\varepsilon_0} = \frac{2\,\sigma_1}{\varepsilon_0}$$

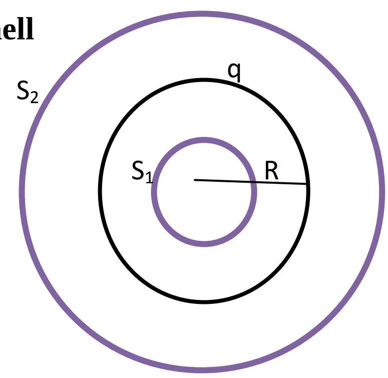
From the positively charged plate to the negatively charged plate

E = 0 to the left and right of the plates

3.9: Applying Gauss' Law: Spherical symmetry

(conducting sphere)

Figure 3.8: A thin, uniformly charged, spherical shell with total charge q, in cross section. Two Gaussian surfaces S_1 and S_2 encloses the shell.



We apply Gauss' Law:

1- To surface S_2, so $r \geq R$

$$r \geq R: \quad E.A = \frac{q}{A}$$

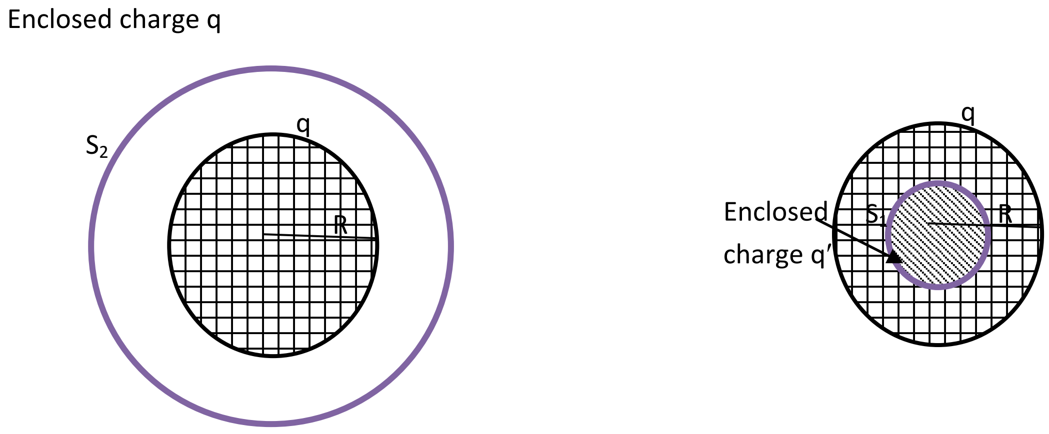
$$E.4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad (spherical\ shell, field\ at\ r \geq R)$$

2- To surface S_1, r<R: no charge inside

$$E.4\pi r^2 = 0 \rightarrow \quad E = 0\ (spherical\ shell, field\ at\ r < R)$$

(Non-conducting sphere)`



(a) (b)

Figure 3.9: Non-conducting spherical charge with volume distribution. A concentric Gaussian surface with r>R is shown in (a). A similar Gaussian surface with r<R is shown in (b)

For r<R: $\varphi = E.A = E.4\pi r^2 = \frac{q_{enc}}{\varepsilon_0} = \frac{\rho\frac{4\pi}{3}r^3}{\varepsilon_0}$

$$E = \frac{\rho r}{3\varepsilon_0}$$

For r>R: $\varphi = E.A = E.4\pi r^2 = \frac{q_{enc}}{\varepsilon_0} = \frac{q}{\varepsilon_0} = \frac{\rho\frac{4\pi}{3}R^3}{\varepsilon_0}$

$$E = \rho\frac{R^3}{3\varepsilon_0 r^2}$$