601.433/633 Introduction to Algorithms **Fall 2024**
Homework #1 **Due:** September 3, 2024, 9am

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1 Asymptotic Notation (32 points)

For each of the following statements say if it true or false and prove your answer. The base of log is 2 unless otherwise specified, and ln is $\log_e$.

(a) $n \tan n = O(2^n)$

Rewrite the statement according to the definition:
$n \tan n \leq c \cdot 2^n$
$c \geq \frac{n \tan n}{2^n}$

According to the definition, there should exist a constant c and $n_0 > 0$ such that the above statement holds for all $n > n_0$. However, $\tan n$ goes to infinity when $n = (\pi/2) + \pi \cdot k$ for some integer k. Therefore, there doesn't exist a constant c because the statement cannot hold for all values of n ranging from n_0 to ∞, since $\tan n$ becomes infinity every $(\pi/2) + \pi \cdot k$ for some integer k.

$\therefore$ The statement is false. ■

(b) $e^n = \Theta(2^n)$

First, show that $e^n = O(2^n)$
Rewrite the statement according to the definition:
$e^n \leq c \cdot 2^n$
$c \geq \frac{e^n}{2^n}$
$c \geq (\frac{e}{2})^n$

Since $\frac{e}{2} > 1$, the expression $(\frac{e}{2})^n$ goes to infinity as n approaches infinity. According to the definition, there should exist a constant c when n ranges from n_0 to ∞. However, the above inequality cannot hold because c cannot be greater or equal to ∞.

$\therefore e^n \neq O(2^n)$, which means that at least one of the required conditions for $e^n = \Theta(2^n)$ doesn't hold. The statement is false. ■

(c) $n \cos n = O(n)$

Rewrite the statement according to the definition:
$n \cos n \leq c \cdot n$

$c \geq \frac{n \cos n}{n}$
$c \geq \cos n$

The range of $\cos n$ is $-1 \leq \cos n \leq 1$, for all n, therefore we can choose $c = 2$, $n_0 = 0$.

$\therefore$ The statement is true. ■

(d) $3^n = \Omega(3^{(n+2)})$

Rewrite the statement according to the definition:
$3^n \geq c \cdot 3^{(n+2)}$
$c \leq \frac{3^n}{3^{(n+2)}} = \frac{3^n}{3^n \cdot 3^2}$
$c \leq 1/9$

We can choose $c = 1/9$, $n_0 = 1$.

$\therefore$ The statement is true. ■

(e) $\log(n^{1/5}) = \Theta(\log(n^3))$

Show $\log(n^{1/5}) = O(\log(n^3))$
$\log(n^{1/5}) \leq c_1 \cdot \log(n^3)$
$c_1 \geq \frac{\log(n^{1/5})}{\log(n^3)}$
$c_1 \geq \frac{\frac{1}{5} \cdot \log(n)}{3 \cdot \log(n}$
$c_1 \geq 1/15$

We can choose $c_1 = 1/15$, $n_0 = 1$.

Show $\log(n^{1/5}) = \Omega(\log(n^3))$
$\log(n^{1/5}) \geq c_2 \cdot \log(n^3)$
Same logic as previous step, except inequality reversed.
$c_2 \leq 1/15$

We can choose $c_2 = 1/15$, $n_0 = 1$.

$\therefore$ Both $\log(n^{1/5}) = O(\log(n^3))$ and $\log(n^{1/5}) = \Omega(\log(n^3))$ holds. The statement is true. ■

(f) Let f, g be positive functions. Then $f(n) + g(n) = \Omega(\max(f(n), g(n)))$

Without loss of generality, let's assume $\max(f(n), g(n)) = f(n)$.

$f(n) + g(n) \geq c \cdot f(n)$
$c \leq \frac{f(n)+g(n)}{f(n)}$

$c \le \frac{f(n)}{f(n)} + \frac{g(n)}{f(n)}$
$c \le 1 + \frac{g(n)}{f(n)}$

By the assumption, $\frac{g(n)}{f(n)} \le 1$, therefore we can choose $c = 1$, $n = 1$.

Another intuitive way of solving this statement would be saying $f(n)+g(n) \ge f(n)$ (let $c = 1$) since f, g are positive functions, and adding two positive functions would be at least as big as only one of the functions by itself.

$\therefore$ The statement is true. ■

(g) Let f, g be positive functions, and let $g(n) = o(f(n))$. Then $f(n) + g(n) = \Theta(f(n))$

By definition, $g(n) = o(f(n))$ means that for all constant $c > 0$, there exists a constant $n_0 > 0$ such that $g(n) < c \cdot f(n)$ for all $n > n_0$.

Since $g(n) < c \cdot f(n)$, we can add $f(n)$ to both sides of the inequality:
$f(n) + g(n) < f(n) + c \cdot f(n)$
$f(n) + g(n) < (c + 1)f(n)$

To show that $f(n) + g(n) = \Theta(f(n))$, we must prove both O and Ω.

Show $f(n) + g(n) = O(f(n))$
$f(n) + g(n) = (c + 1)f(n) \le c_1 \cdot f(n)$
Let $c_1 = c + 1$

Show $f(n) + g(n) = \Omega(f(n))$
$f(n) + g(n) = (c + 1)f(n) \ge c_2 \cdot f(n)$
Let $c_2 = c + 1$

Let $c = 1$, then $c_1 = 2$ and $c_2 = 2$, $n_0 = 1$. According to the definition, after algebraic manipulation, $f(n) + g(n) = \Theta(f(n))$ holds for all $n > n_0$, for a constant $n_0 > 0$.

$\therefore$ The statement is true. ■

(h) $2^{(5/2)\log n} = O(n^2)$

$2^{(5/2)\log n} \le c \cdot n^2$
$c \ge \frac{n^{(5/2)\log 2}}{n^2}$ (Inverse Property of Exponent rule)
$c \ge \frac{n^{(5/2)}}{n^2}$ (the base of log is 2, given in the problem statement)
$c \ge n^{(5/4)}$

Since $5/4 > 1$, the expression $n^{(5/4)}$ goes to infinity when n goes to infinity. According to the definition, there should exist a constant c when n ranges from n_0 to ∞. However, the above inequality cannot hold because c cannot be greater or equal to ∞.

$\therefore$ The statement is false. ■

2 Recurrences (32 pts)

Solve the following recurrences, giving your answer in Θ notation (so both an upper bound and a lower bound). For each of them you may assume $T(x) = 1$ for $x \leq 5$. Justify your answer (formal proof not necessary, but recommended).

(a) $T(n) = 5T(n-3)$

Unroll:
$5T(n-3)$
$5^2T(n-6)$
$5^3T(n-9)$
...
At level i (last level):
$5^iT(n-3i) = 5^i$ (since $T(n-3i) = 1$ for $(n-3i) \leq 5$)

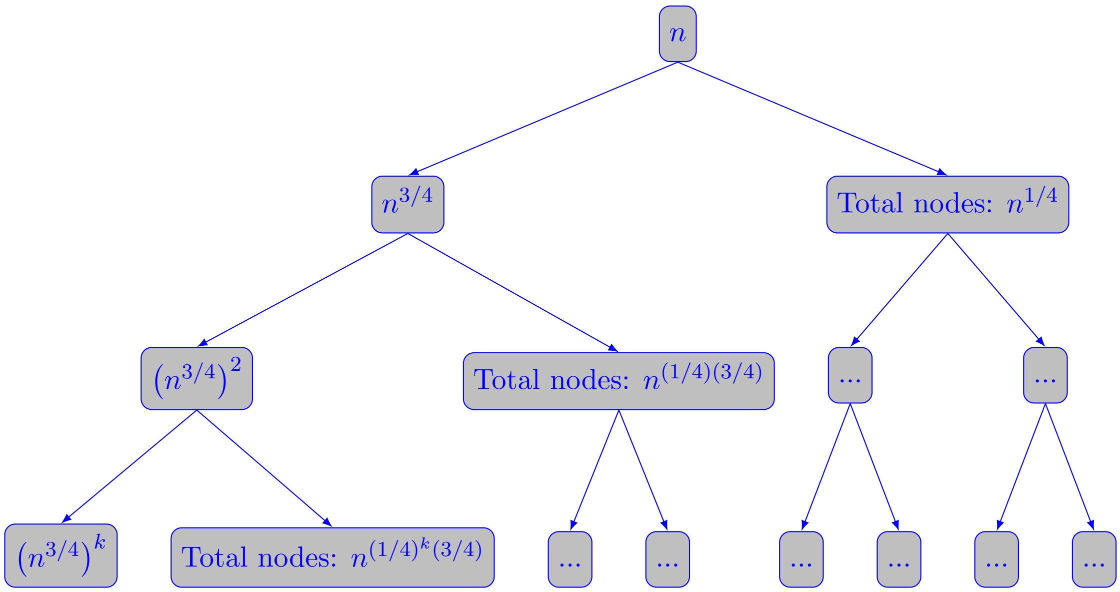
The height is $n/3$ because n decreases by 3 every time.

The last level dominates because 5^{height} has a higher degree than the rest of the terms.

$\therefore T(n) = 5T(n-3) = \theta(5^{n/3})$.

(b) $T(n) = n^{1/4}T(n^{3/4}) + n$

We can write the following recursion tree:



We must (1) find the amount of work per level, and (2) the number of levels there are.

(1) Prove by induction that each level has a total amount of work = n.

Base case: The first level has total work = n because of the + n in $n^{1/4}T(n^{3/4}) + n$

Let's assume an arbitrary level holds true. In this level, each box in the recursion tree is k. We assume that the total amount of this level is n. This means that there are n/k number of terms.

Now we must prove that it holds for the level right after that arbitrary level. Each box in this level is $k^{3/4}$. Each box in the level above has $k^{1/4}$ children. Thus, the amount of total work that this level has is (number of terms) $\cdot$ (work per box) $= (k^{1/4} \cdot n/k) \cdot k^{3/4} = n$.

By induction, we have proven that the total work per level is n.

(2) We can find the number of total levels. We know that the base case is reached when $x \leq 5$, and that each box is $n^{(\frac{3}{4})^k}$, thus:

$$n^{(\frac{3}{4})^k} \leq 5$$
$$\log_5 n^{(\frac{3}{4})^k} \leq \log_5 5$$
$$(\tfrac{3}{4})^k \cdot \log_5 n \leq 1$$
$$\log_5 n \leq (\tfrac{4}{3})^k$$
$$\log_{\frac{4}{3}} \log_5 n \leq \log_{\frac{4}{3}} (\tfrac{4}{3})^k$$
$$k \cdot \log_{\frac{4}{3}} \tfrac{4}{3} \geq \log_{\frac{4}{3}} \log_5 n$$
$$k \geq \log_{\frac{4}{3}} \log_5 n$$

$\therefore T(n) = n^{1/4}T(n^{3/4}) + n =$ (work per level) $\cdot$ (number of levels) $= \theta(n \log_{\frac{4}{3}} \log_5 n)$

(c) $T(n) = 6T(n/4) + n$

Level i: $6^{i-1}(\frac{n}{4^{i-1}})^{i-1} = n(\frac{6}{4})^{i-1}$
Height of tree: $\log_4 n + 1$

Last level dominates:
$= \theta(n \cdot (\frac{6}{4})^{\log_4 n+1-1})$
$= \theta(n \cdot \frac{6^{\log_4 n}}{4^{\log_4 n}})$
$= \theta(n \cdot 6^{\log_4 n} \cdot \frac{1}{n})$
$= \theta(6^{\log_4 n})$

$\therefore T(n) = 6T(n/4) + n = \theta(6^{\log_4 n}) = \theta(n^{\log_4 6})$.

(d) $T(n) = T(n-2) + 10$

Unroll:
$T(n-2) + 10$
$T(n-4) + 20$
$T(n-6) + 30$
...
At level i (last level):
$T(n-2i) + 10i = 10i$ (since $T(n-2i) = 1$ for $(n-2i) \leq 5$)

Each level does an equal amount, which is 10. The height is $n/2$ because n decreases by 2 every time.

$\therefore T(n) = T(n-2) + 10 = \theta(n/2 \cdot 10) = \theta(5n) = \theta(n)$.

3 Basic Proofs (36 pts)

(a) There are currently 157 students registered for the class (92 undergrad, 65 grad). Prove that there are at least 14 students who have birthdays in the same month.

We can use a similar logic to the Pigeonhole Principle. The theorem states that if we put n items in m containers, and if $n > m$, then there is at least one container that has more than one element in it.

In this problem, the 12 months represent 12 containers. We can equally distribute 13 students into each container. $13 \cdot 12 = 156$ students. However, there are 157 students. The 157th student must still fit into one of the containers. Each container already has 13 students, so after putting the last student in, one of the containers must contain 14 students. Therefore, there are at least 14 students who have birthdays in the same month.

(b) Prove **by induction** that $\sum_{i=1}^{n}(2i-1) = n^2$ for all positive integers n.

Base case: n = 1

$$\sum_{i=1}^{1}(2i-1) = 1^2$$
$$2 - 1 = 1$$

Let's assume that the statement $\sum_{i=1}^{n}(2i-1) = n^2$ holds true for some value k.

$$\sum_{i=1}^{k}(2i-1) = k^2$$

Now we prove that it holds for n = k + 1.

$$\sum_{i=1}^{k+1}(2i-1)$$
$$= \sum_{i=1}^{k}(2i-1) + [2(k+1)-1]$$
$$= k^2 + 2(k+1) - 1$$
$$= k^2 + 2k + 1$$
$$= (k+1)^2$$

$\therefore$ Since the statement holds for the base case $n = 1$, and true for $k \implies$ true for $k+1$, by induction, $\sum_{i=1}^{n}(2i-1) = n^2$ is true for all positive integers n. ■

(c) I have a bucket with 32 balls, 20 of which are white and 12 of which are black. If I draw 9 balls at random from the bucket (all at one time), what is the probability that exactly three of them are white?

Since we are drawing the 9 balls all at one time, order doesn't matter. The number of ways we can choose 3 out of the 20 white balls is $\binom{20}{3}$, and so, 6 of the 9 balls we drew must be black, which can be picked in $\binom{12}{6}$ ways. The total number of ways to draw 9 balls from 32 is $\binom{32}{9}$.

$\therefore$ The probability that exactly 3 of them are white is $\frac{\binom{20}{3}\cdot\binom{12}{6}}{\binom{32}{9}}$.