SAMPLE MIDTERM 1–STOCHASTIC PROCESSES

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- This document in PDF: https://github.com/bellecp/CC-BY-SA-teaching-material/blob/master/stochastic__processes/midterm-sample-exam-1.pdf
- Source code (markdown/latex): https://github.com/bellecp/CC-BY-SA-teaching-material/blob/master/stochastic__processes/midterm-sample-exam-1.md
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Length: 1h and 20 minutes.

Problem 1: Unicity of the optimal coupling. Consider two distributions on $\{1, 2, 3, 4\}$ given by

$$\mu = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \qquad \text{and} \qquad \nu = \left(\frac{1}{8}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}\right).$$

1. Provide an optimal coupling of (μ, ν) as a 4×4 matrix (it does not matter whether you provide the matrix or its transpose).
2. Provide another optimal coupling of (μ, ν), **different** than the one obtained in question 1. Highlight (with color, a circle or otherwise) the entries of the matrix that are different from your answer in question 1. Conclude that the optimal coupling is not always unique.
3. Provide, as a 4×4 matrix, the coupling of (μ, ν) for which the two coordinates are independent.

Problem 2: A metropolis chain. Consider a connected undirected graph $G = (V, E)$. We consider a base chain on $\mathcal{X} = V \times V$ that consists of two particles moving as a simple random walk on the graph, independently of each other. In other words

$$\Psi\Big((u_1, u_2), (v_1, v_2)\Big) = 1/\big(|N(u_1)|\ |N(u_2)|\big) \text{ if both } v_1 \in N(u_1), v_2 \in N(u_2),$$

and

$$\Psi\Big((u_1, u_2), (v_1, v_2)\Big) = 0 \text{ if } v_1 \notin N(u_1) \text{ or } v_2 \notin N(u_2),$$

for any four vertices u_1, u_2, v_1, v_2, where $N(v) \subset V$ is the set of neighbors of a vertex v and $|N(v)|$ the cardinality of $N(v)$. (A vertex v is not a neighbor of itself so that $v \notin N(v)$ always holds.) We are interested in constructing a Markov Chain on $\mathcal{X} = V \times V$ with stationary distribution

$$\forall (u_1, u_2) \in V \times V, \qquad \pi((u_1, u_2)) = \begin{cases} 0 & \text{if } u_1 = u_2, \\ 1/Z & \text{if } u_1 \neq u_2, \end{cases}$$

where $Z > 0$ is a normalizing constant not depending on u_1, u_2.

1. Is the base chain symmetric? (Prove or disprove).

2. If you apply the Metropolis scheme to construct a Chain with stationary distribution π from the base chain Ψ, what is the acceptance probability
$$a\Big((u_1,u_2),(v_1,v_2)\Big)$$
to accept a move from the base chain? Clearly write and simplify $a((u_1,u_2),(v_1,v_2))$ and verify that the detailed balance equations are satisfied.

Problem 3: Two irreducible and aperiodic chains that move independently will eventually meet. Consider a transition matrix P and the transition matrix Q over the set $\mathcal{X}\times\mathcal{X}$ defined by
$$Q\Big((x_1,x_2),(y_1,y_2)\Big) = P(x_1,y_1)P(x_2,y_2).$$
In words, in the chain Q, each coordinate is updated independently according to P. Denote by $(X_t,Y_t)_{t\geq 0}$ a Markov chain valued in $\mathcal{X}\times\mathcal{X}$ with transition matrix Q (X_t is the first coordinate, Y_t is the second coordinate).

We assume that P is aperiodic and irreducible, so that by a result of Chapter 1, there exists an integer $r>0$ and positive real $\epsilon>0$ such that
$$\forall t\geq r, \forall x,y\in\mathcal{X}, \quad P^t(x,y) = \mathbf{P}_x[X_t=y] \geq \epsilon > 0.$$

1. Show that Q is irreducible.
2. Show that Q is aperiodic. In the next question you may use without proof the following fact:
   Fact I: if $\bar{P}$ is an irreducible transition matrix of a Markov Chain $(M_t)_{t\geq 0}$ on a set $\bar{\mathcal{X}}$, then there exists an integer $\bar{r}>0$ and real number $\bar{\epsilon}\in(0,1)$ such that for any integer $k\geq 1$,
   $$\mathbf{P}_{\bar{x}}(\tau_{\bar{y}}^+ > k\bar{r}) \leq (1-\bar{\epsilon})^k \qquad \text{where} \qquad \tau_{\bar{y}}^+ = \min\{t\geq 1 : M_t = \bar{y}\},$$
   for any two states $\bar{x},\bar{y}\in\bar{\mathcal{X}}$.

3. Using the Markov Chain $(X_t,Y_t)_{t\geq 0}$ from the previous page, show that
   $$\forall k\geq 1, \quad \max_{x,y\in\mathcal{X}} \|P^{kr}(x,\cdot) - P^{kr}(y,\cdot)\|_{TV} \leq \alpha^k$$
   for some $\alpha\in(0,1)$ and some integer $r>0$.
4. (Extra credit question, if you have time). Prove **Fact I** above.