SAMPLE MIDTERM 2, STOCHASTIC PROCESSES

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- This document in PDF: https://github.com/bellecp/CC-BY-SA-teaching-material/blob/master/stochastic__processes/midterm-sample-exam-2.pdf
- Source code (markdown/latex): https://github.com/bellecp/CC-BY-SA-teaching-material/blob/master/stochastic__processes/midterm-sample-exam-2.md
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Length: 1h and 20 minutes.

Contraction in expectation. Let $\mathcal{X}$ be the set of functions from $\{1, ..., n\}$ to $\{0, 1\}$, and for any two functions $x, y \in \mathcal{X}$, let $\rho(x, y)$ be the number of disagreements,

$$\rho(x, y) = |\{i = 1, ..., n : x(i) \neq y(i)\}| = \sum_{i=1}^{n} I_{x(i)\neq y(i)}.$$

Consider a transition matrix $P(\cdot, \cdot)$ on $\mathcal{X}$ and a random mapping representation, given by a deterministic function $f : \mathcal{X} \times [0, 1] \to \mathcal{X}$ and a random variable $Z \sim Unif[0, 1]$ with $\mathbf{P}(f(x, Z) = y) = P(x, y)$. We also consider an iid sequence $(Z_t)_{t\geq 0}$ of independent copies of Z and the Grand Coupling defined by $X_0^x = x$ and $X_t^x = f(X_{t-1}^x, Z_t)$ for all $x \in \mathcal{X}$ and all $t \geq 0$.

For the first four questions, we assume that there exists some $a > 0$ such that for all x, y with $\rho(x, y) = 1$ we have $\mathbf{E}[\rho(f(x, Z), f(y, Z)] \leq e^{-a}$.

1. Prove that ρ satisfies the triangle inequality $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for any states $x, y, z \in \mathcal{X}$.
2. Prove that $\mathbf{E}[\rho(f(x, Z), f(y, Z))] \leq \rho(x, y)e^{-a}$ for all $x, y \in \mathcal{X}$.
3. Prove that $\mathbf{E}[\rho(X_t^x, X_t^y)] \leq \rho(x, y)e^{-at}$ for any $t \geq 1$ and all $x, y \in \mathcal{X}$. Clearly explain in words every steps of your reasoning.
4. Provide an upper bound on the mixing time $t_{mix}(\epsilon)$ for any $\epsilon \in (0, 1)$.

Lower bound on the mixing time for a lazy random walk on regular graph. Let $d \geq 3$. Consider an undirected connected graph $G = (V, E)$ such that each vertex has **exactly** d neighbors. Let P be the **lazy** random walk on V (the chain stays on the current vertex with probability $1/2$, and jumps to a neighbor with probability $1/(2d)$).

1. Draw such a graph for $d = 3$ and $|V| = 8$ (these values of d and $|V|$ apply ONLY to this question).
2. Prove that the uniform distribution is stationary.
3. Why is the stationary distribution unique? Denote by π the stationary distribution.

4. Let $V_t^x = \{y \in V : P^t(x,y) > 0\}$ be the subset of V of states accessible from x after t steps. Show that $|V_t^x| \leq (d+1)^t$.
5. Prove that $\|P^t(x,\cdot) - \pi\|_{TV} \geq 1 - (d+1)^t/|V|$.
6. Prove a lower bound on $t_{mix}(1/4)$. Clearly explain your reasoning.