# 03 entropy, KL divergence,

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THE UNIVERSITY OF CHICAGO

# Office hours

- WT    Thursday 2-4 Crerar 346
- Amy 12:30-2:00 Tuesday  Maybe on Wednesday
- Qiming
- JungHo

# The norm definitions

$$L^1 - \text{norm} = |x|_1 = \sum_i |x_i|$$

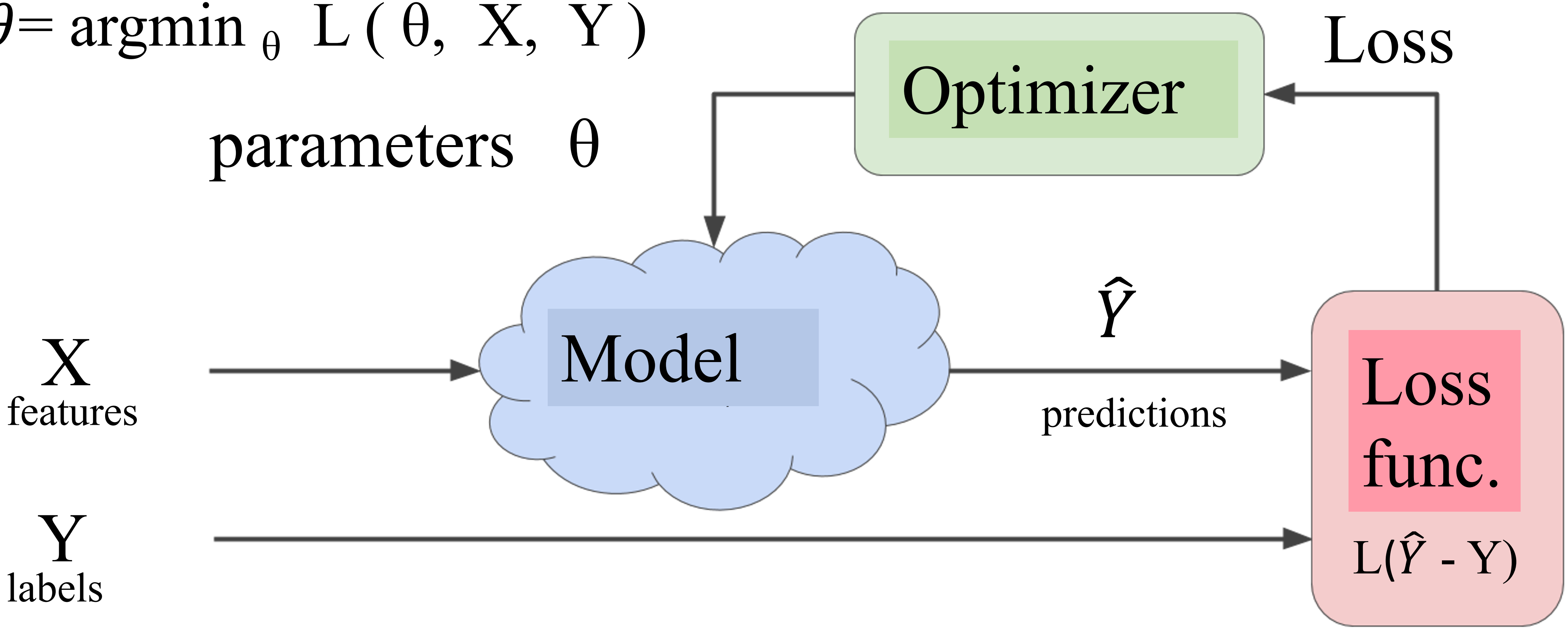
$$\ell^2 - \text{norm} = |x| = \sqrt{\sum_i |x_i|^2}$$

$$\ell^\infty - \text{norm} = |x|_\infty = \max_i |x_i|$$

"norms" always have same units as x!

# Fairy dust picture of optimization

$\hat{\theta} = \text{argmin}_{\theta}\ L(\theta, X, Y)$



# Inferring parameters

θ: value of model parameter

Y: training data

$$P(\theta \mid Y) = P(Y|\theta)\, P(\theta) \,/\, P(Y)$$

Distribution to help me find the parameter value

*posterior*

This is the likelihood

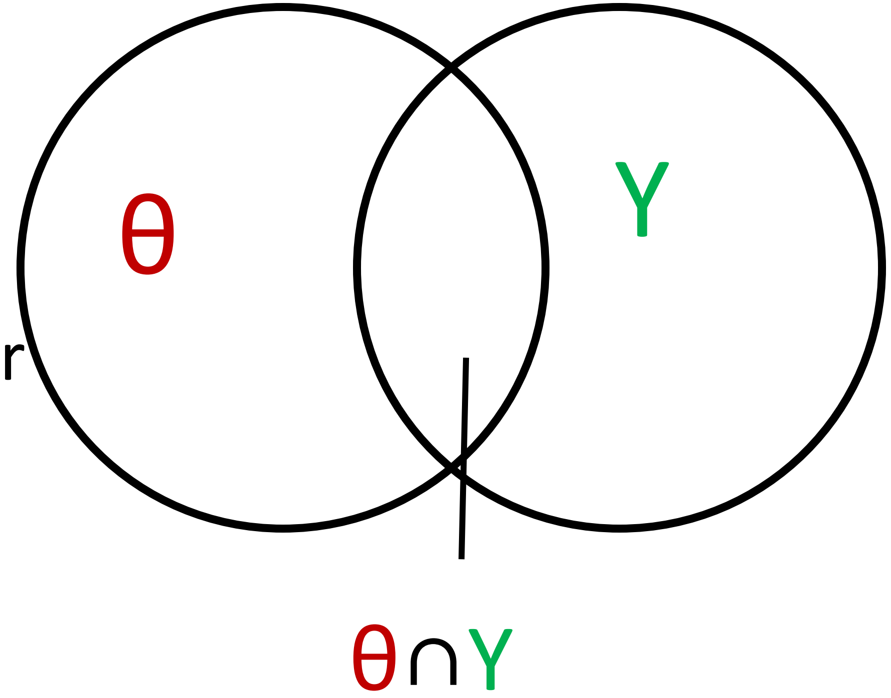
*likelihood*

Prior distribution for parameter

*prior*

Doesn't matter for us

*evidence*



# Inferring parameters

$\theta$: value of model parameter

$Y$: training data

$$\log P(\theta \mid Y) = \log P(Y \mid \theta) + \log P(\theta)$$

log parameter distribution

The log likelihood

Prior distribution on the parameter

*log posterior*

*log likelihood*

*log prior*

# Inferring parameters

θ: value of model parameter

Y: training data

$$\log P(\theta \mid Y) = \log P(Y \mid \theta) + \log P(\theta)$$

$$L(\theta \mid Y) = \sum \frac{(\hat{y}_i(\theta) - y_i)^2}{\sigma_i^2} + \sum \frac{(\theta_k)^2}{s_k^2}$$

*Additive normal errors on* ŷ

*Normal prior on each component of* θ

*log posterior*

*log likelihood*

*log prior*

# Dice problem

- Suppose I have two six-sided dice:

| | | | | | | |
|---|---|---|---|---|---|---|
| A: | 1 | 2 | 3 | 4 | 5 | 6 |
| B: | 1 | 2 | 3 | 4 | 5 | 1 |

$$P(O|A) = \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$$

$$P(O|B) = \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0$$

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Imagine the rolls of the dice as symptoms of a denial of service attack, symtoms of cancer, and the identity of the die (A or B) as the underlying state of nature to be estimated.

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The first time I see a six, the game is over.

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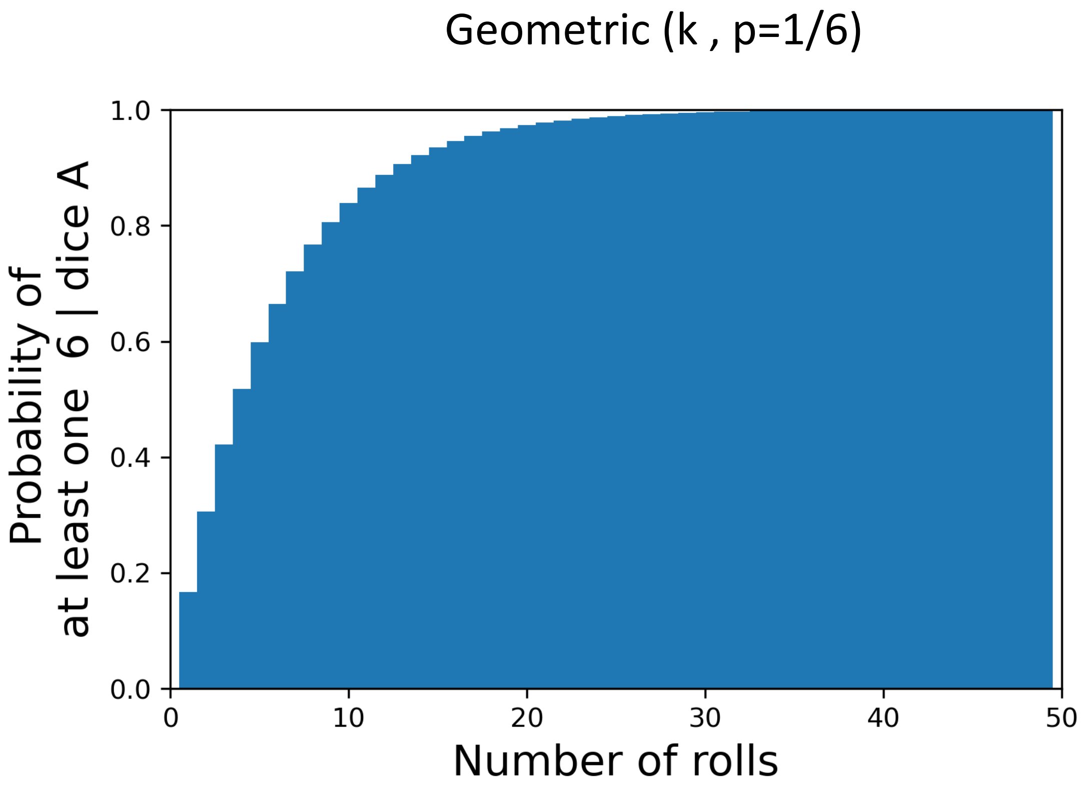

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P( A | roll ) = P(roll|A) P(A) / P(roll)

P( B | roll ) = P(roll|B) P(B) / P(roll)

Suggestion: Why don't I keep track of the ratio of the probability of A to the probability of B?

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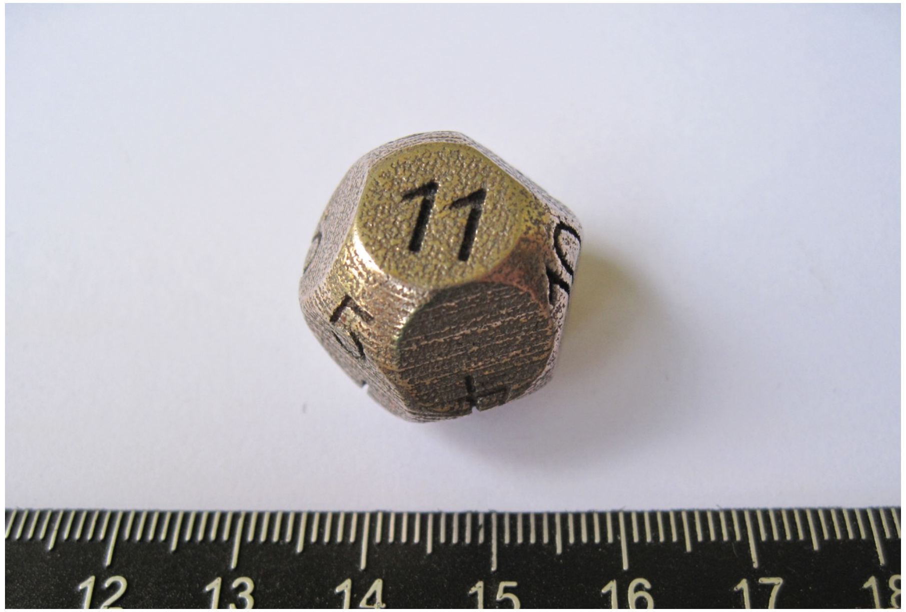
Suggestion: Why don't I keep track of the ratio of the probability of A to the probability of B?

Consequence: I'm going to need a lookup table for P(A|R ) / P(B|R) for all the relevant values of R.

# Dice problem




- Suppose I have two 11-sided dice, one of which is unfair, but they have the same faces:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A: | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | (fair 11-sided die) |
| C: | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | ( sum of 2 six-sided dice) |

P(O|A) = $\frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}, \frac{1}{11}$

P(O|C) = $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$

# So let's calculate the odds ratio A:B

It is convenient to look at the ratio of probabilities (odds) favoring A over B:

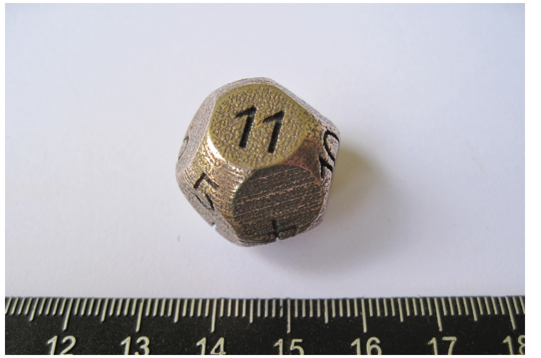
$$\frac{P(A|O)}{P(B|O)} = \frac{P(O|A)}{P(O|B)} \quad \frac{P(A)}{P(B)}$$

and its logarithm, log-odds A over B:

$$\log\left(\frac{P(A|O)}{P(B|O)}\right) = \log(P(O|A) - \log(P(O|B)) + (\log P(A) - \log P(B))$$

Why? Because products become sums, and I can just add a term to my log odds each time the die is thrown.

# How long will it take?

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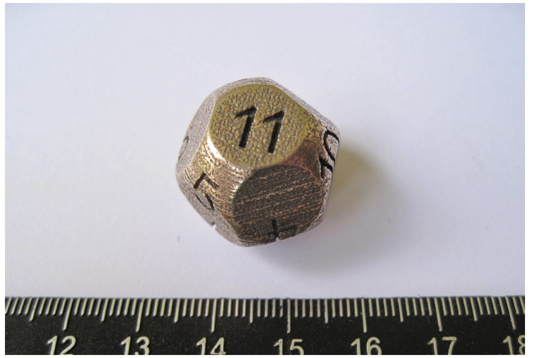
$$\log_2\left(\frac{P(A|O)}{P(C|O)}\right) = 1.71, 0.71, 0.13, -0.29, -0.61, -0.97, -0.61, -0.29, 0.13, 0.71, 1.77$$

Mean log-odds | A = 0.22 bits / roll

Mean log-odds | B = -0.19 bits / roll

$\log_2 99 = 6.63$ bits (threshold 99:1)

# How long will it take?



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$\log_2 99$ = 6.63 bits (threshold 99:1)

6.63 bits / 0.22 bits / roll = 30.2 rolls |A

6.63 bits / 0.19 bits / roll = 35.8 rolls |B

# Wait, what assumptions did I make here?

- 0.4047 bits per roll is the difference between the expected value of $\log_2\left(\frac{P(A\,|O)}{P(C|\,O)}\right)$ in the universe where A is true its value when C is true.

Why can I trust this ? Central limit theorem.

Sums of 30 independent likelihood terms? Well-behaved mean.

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- CLT requires independence of random variables.

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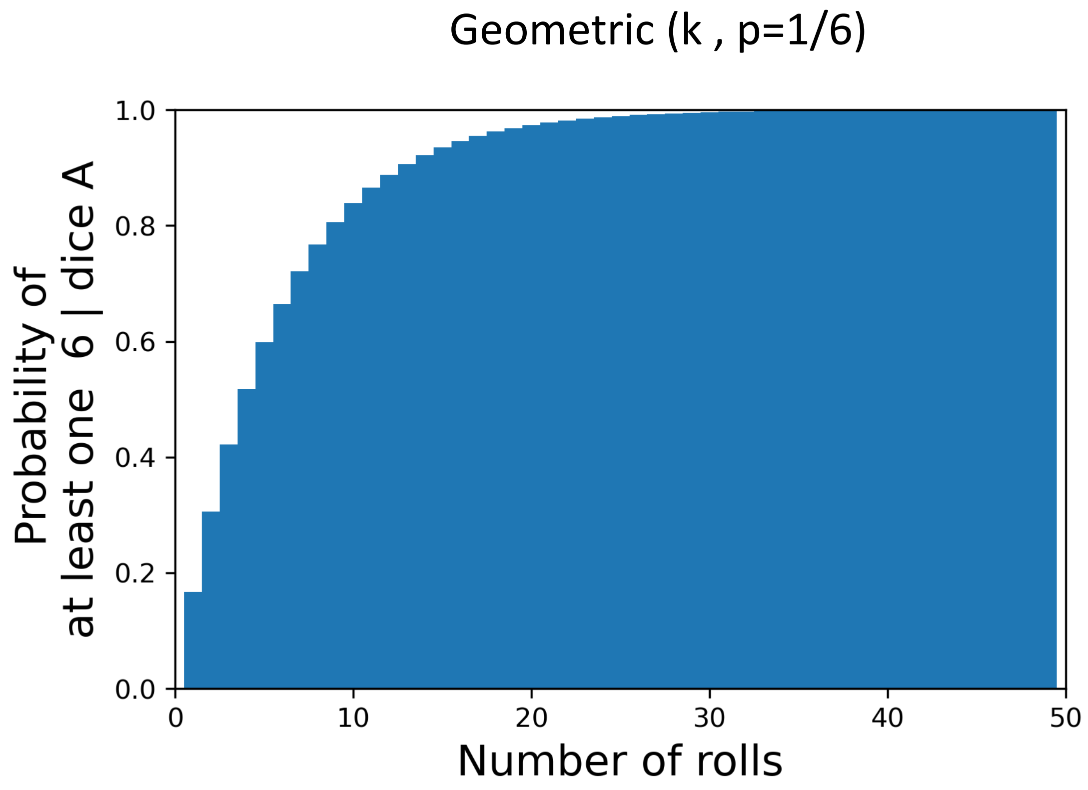

# Entropy

$$H = -\sum_{i} p_i \, log(p_i)$$

Entropy is a property of a probability distribution (or of an element within a probability distribution, like an outcome) that describes how concentrated the distribution is.

# Entropy

$$H = -\sum_i p_i \, log(p_i)$$

It has been constructed to have some nice properties (entropy of independent events is additive) and is, up to a multiplicative constant (which is the same as choosing the base of logarithm) the unique measure for which products are sums.

# Entropy – discrete formulation

$$H = -\sum_i p_i \, log(p_i)$$

Expected value of the –log probability

# Entropy – continuous formulation

$$H = -\int p(\boldsymbol{x}) log\ p(\boldsymbol{x}) d\boldsymbol{x}$$

Expected value of the –log probability

# Entropy – parts of the definition

- Expected value
- log – entropies of independent events additive
- negative – probabilities are all < 1; logs of all these probabilities are < 0; convention makes low-probabilities high entropy and high probabilities low-entropy.

| letter | | freq p | -log(p) |
|---|---|---|---|
| 1 | a | 0.0575 | 4.13 |
| 2 | b | 0.0128 | 6.28 |
| 3 | c | 0.0263 | 5.25 |
| 4 | j | 0.0006 | 10.70 |
| 5 | d | 0.0285 | 5.13 |

# Properties of entropy

- If all of the probability is in one point, $H = 0$
- Bernoulli distribution: entropy is maximum at $p=0.5$
- Uniform categorical distribution $\{1/n, 1/n \dots 1/n\}$ has entropy $\log n$

- Watch out for reporting entropy; logarithms have "units" that communicate the base of the log. log10 "ban" log 2 "bits" natural log "nats" Antilog entropy (sometimes called "perplexity") doesn't have that problem, since it's a unitless number.

Tool to characterize and compare distributions; tool for use in loss function.

# Kullback-Leibler divergence

also called relative entropy

"Expectation of the logarithmic difference between the two probability distributions:"

$$D_{KL}(P \| Q) = \sum P(x) \log\left(\frac{P(x)}{Q(x)}\right)$$

But we just saw this in the nonstandard dice example:

This is the expected odds favoring the P distribution over the Q distribution assuming the P distribution is correct.

# Properties of the Kullback-Liebler divergence

- Why is it called a divergence? Because it's not symmetrical, and the word "distance" is reserved for metrics that are.
- $D_{KL}(P \| Q) = \sum P(x) \log(\frac{P(x)}{Q(x)})$
- $D_{KL}(Q \| P) = \sum Q(x) \log(\frac{Q(x)}{P(x)})$
- That logarithm term is the same, but the point of measuring the divergence between two distributions is that P and Q are different.
- If P and Q are the same, divergence is zero.
- Otherwise, $D_{KL}$ is always positive

# So about that K-L divergence asymmetry…

- But we saw an asymmetry with the dice:
  - There was different qualitative behavior between the two true states of the dice:
  - The die with only 5 different sides causes odds to accrue slowly
  - The six-sided die will stumble into infinite odds in favor after an average of 6 rolls.
- Doesn't handle 0 probability elegantly; the infinite negative logarithm makes the expected value infinite, which is inconvenient.
- There is a symmetrized version that satisfies the triangle inequality, goes by the name Jensen-Shannon divergence if you need it. It imagines an equal mixture model of P and Q.