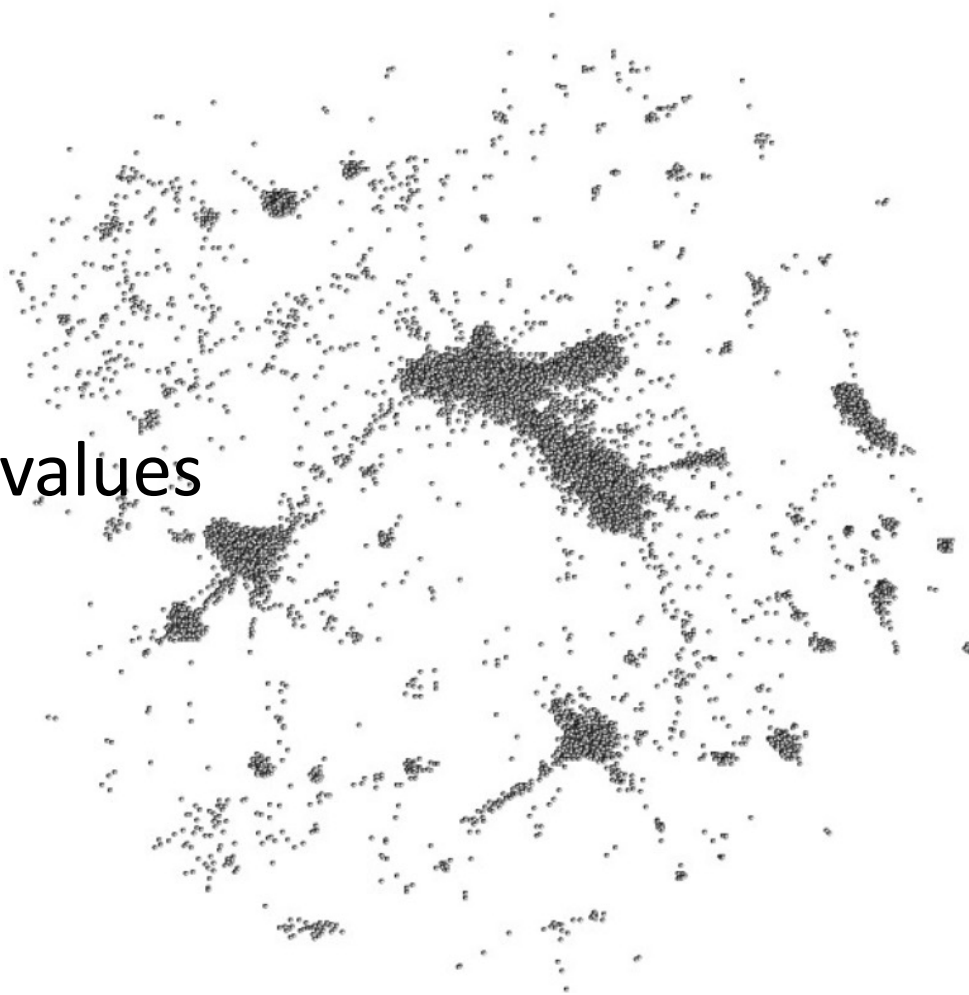


SVD, PCA, and dimension reduction

Roadmap

- Dimension reduction
- Absolute essentials of eigenvectors and eigenvalues
- SVD and PCA
- Examples



Consider genotype data

Individuals (few 1000s)

Genetic loci (few 100k)

	HGDP00448	HGDP00479	HGDP00985	HGDP00611	HGDP00623	HGDP00557	HGDP00569	HGDP00581
MitoA10045G	AA	AA	AA	AA	AA	AA	AA	AA
MitoA10551G	AA	AA	AA	AA	AA	AA	--	--
MitoA11252G	AA	AA	AA	AA	GG	AA	AA	AA
MitoA11468G	AA	AA	AA	AA	AA	AA	GG	GG
MitoA11813G	AA	AA	AA	AA	AA	AA	AA	AA
MitoA12309G	AA	AA	AA	AA	AA	AA	--	GG
MitoA13106G	GG	GG	GG	AA	AA	AA	AA	AA
MitoA13264G	AA	AA	AA	AA	AA	AA	AA	AA
MitoA13781G	AA	AA	AA	AA	AA	AA	AA	AA
rs10000543	CC	CC	TC	CC	CC	CC	TT	CC
rs10000918	GG	AG	GG	AG	AG	AG	AA	AG
rs10000929	AG	AA	AA	AA	AA	AG	AA	AA
rs10001378	TT	CC	TT	TC	TC	TT	TT	TC
rs10001548	TC	TT	TC	TT	TC	TC	TT	CC
rs10002472	GG	AG	GG	GG	AG	GG	GG	GG
rs10004399	AA	AA	AA	AG	AG	AA	AA	AG
rs1000459	TT	TC	TC	CC	CC	CC	https://hagsc.org/hgdp/files.html	
rs10005550	GG	GG	GG	GG	GG	GG	GG	GG

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MitoA10551G	AA	AA	AA	AA	AA	AA	--	--
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MitoA11468G	AA	AA	AA	AA	AA	AA	GG	GG
MitoA11813G	AA	AA	AA	AA	AA	AA	AA	AA
MitoA12309G	AA	AA	AA	AA	AA	AA	--	GG
MitoA13106G	GG	GG	GG	AA	AA	AA	AA	AA
MitoA13264G	AA	AA	AA	AA	AA	AA	AA	AA
MitoA13781G	AA	AA	AA					
rs10000543	CC	CC	TC					
rs10000918	GG	AG	GG					
rs10000929	AG	AA	AA					
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rs10002472	GG	AG	GG	GG	AG	GG	GG	GG
rs10004399	AA	AA	AA	AG	AG	AA	AA	AG
rs1000459	TT	TC	TC	CC	CC	CC	TC	CC
rs10005550	GG	GG	GG	GG	GG	GG	GG	GG

- Convert to numbers
- compare columns ?
- Would like to visualize in 2 dimesions:
 - 1000 x 2 ?

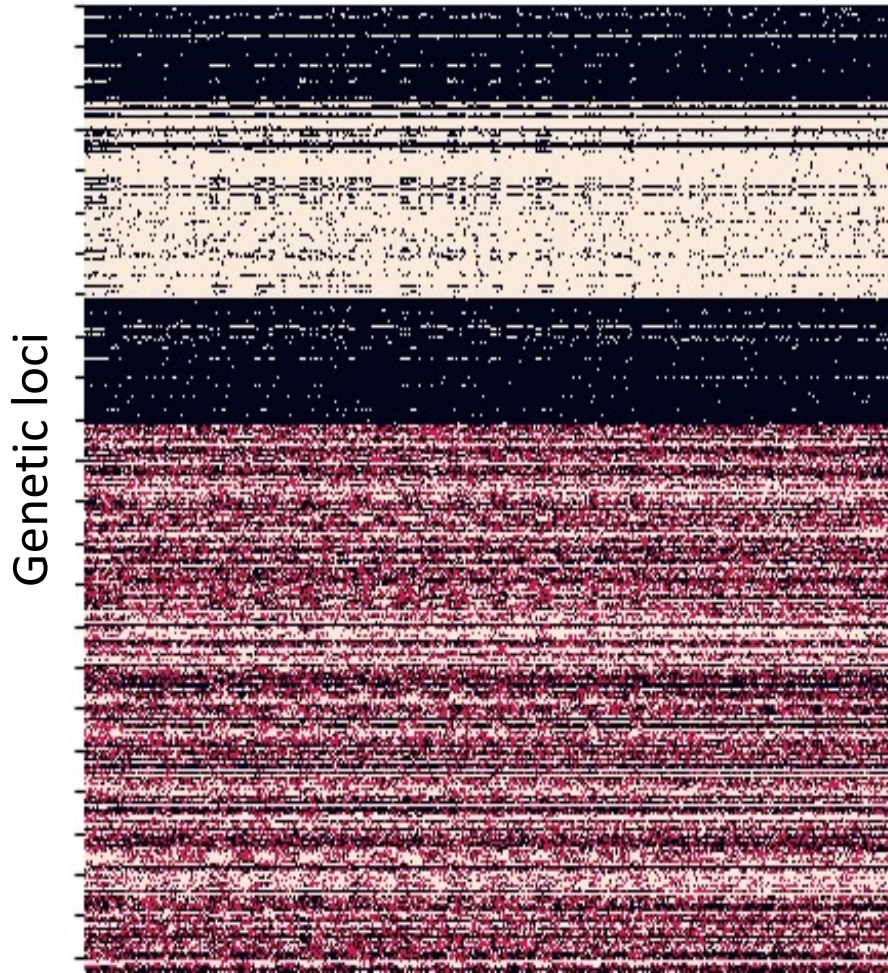
Suppose your data looks like genotype data

Individuals (few 1000s)

Genetic loci (few 100k)

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MitoA11813G	AA	AA	AA	AA	AA	AA	AA	AA
MitoA12309G	AA	AA	AA	AA	AA	AA	--	GG
MitoA13106G	GG	GG	GG	AA	AA	AA	AA	AA
MitoA13264G	AA	AA	AA	AA	AA	AA	AA	AA
MitoA13781G	AA	AA	AA	AA	AA	AA	AA	AA
rs10000543	CC	CC	TC	CC	CC	CC	TT	CC
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rs1000459	TT	TC	TC	CC	CC	CC	TC	CC
rs10005550	GG	GG	GG	GG	GG	GG	GG	GG

Individuals

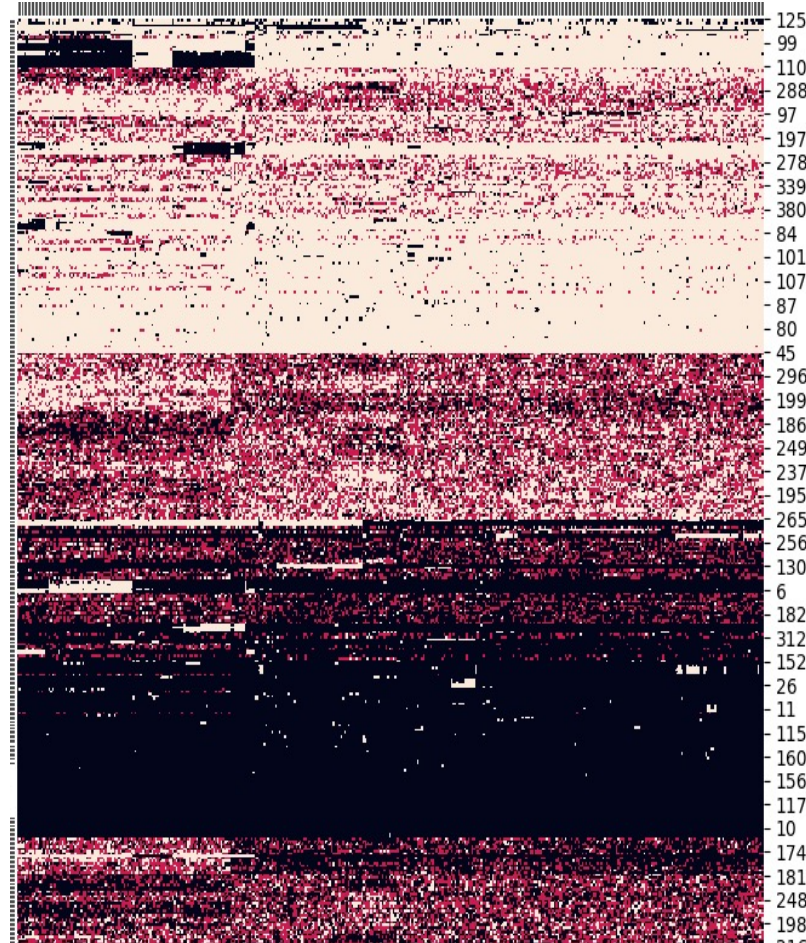


- Some rows are similar to each other, some columns are similar to each other
- The data are in some way “predictable”
- Can we build a low-complexity object that explains most of the data matrix.

Now that looks like structure

Individuals

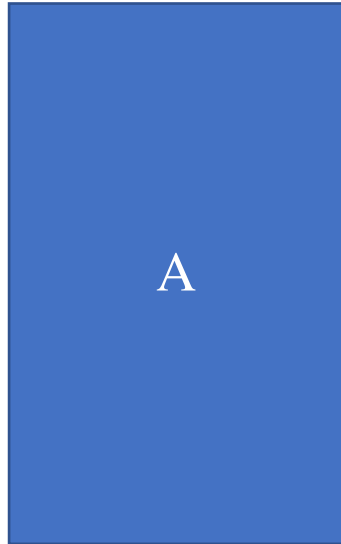
Genetic loci



- Reorder rows and columns to put similar rows next to each other?
- Libraries to make exactly this sort of eye-candy are mature. This one is `sanborn.clustermat`
- Organizing data like this is deeply satisfying—it makes us think we have discovered truth.
- What if I told you linear algebra could do this?

Why Reduce dimensions?

n columns
"samples"



m rows
"features"

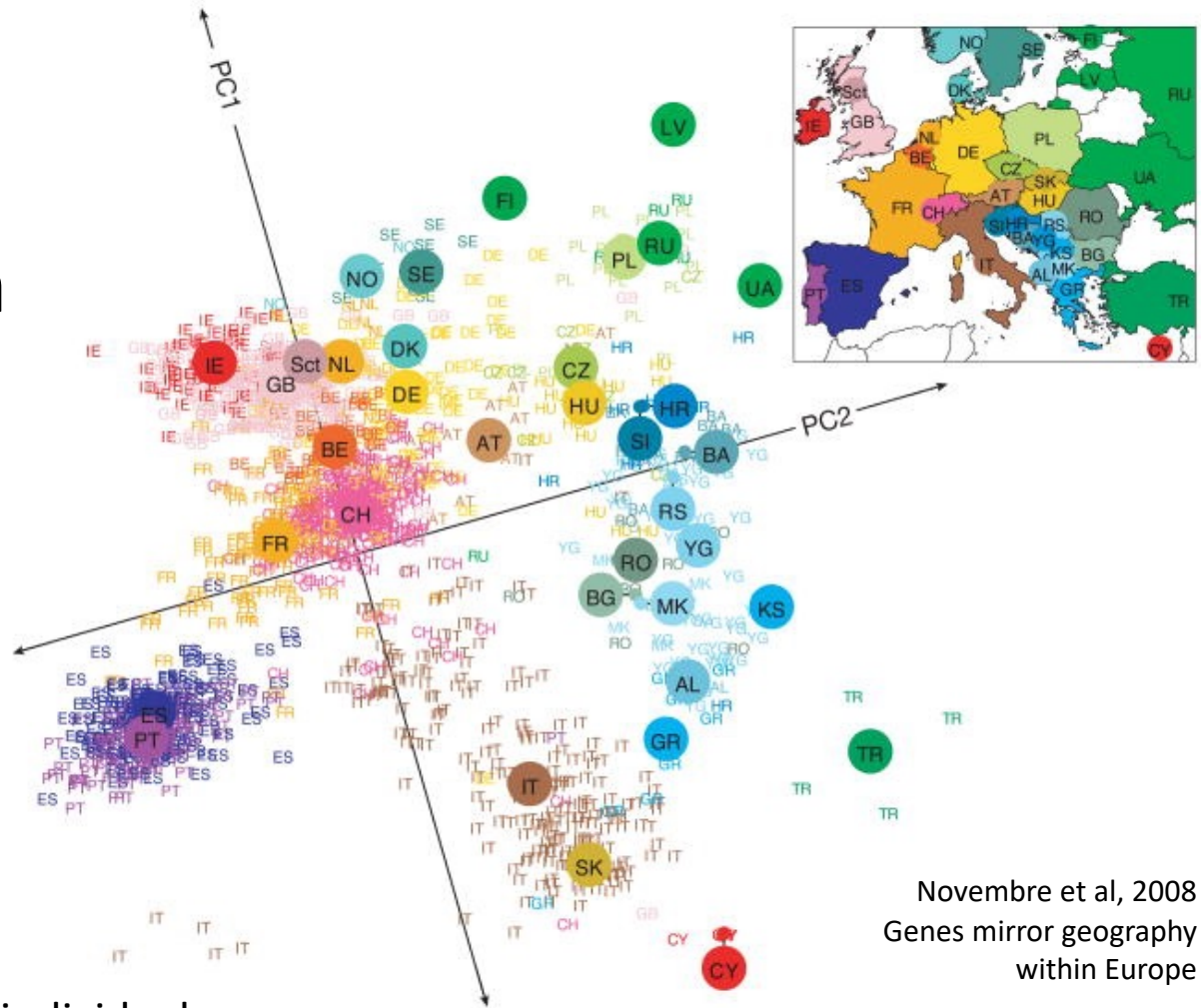
- Reducing the number of dimensions can make ML easier
- May reduce noise.
- Can exploit unlabeled data to make nicer inputs for ML
- We need dimension reduction and categorization to understand high-dimensional data
- Useful for making visualizations of high-dimensional data.

r
"reduced
features"

n samples



Example:
Genes mirror
geography within
Europe



Novembre et al, 2008
Genes mirror geography
within Europe

m x n = 200k genetic loci x 1400 individuals

10.1038/nature07331

How many dimensions again?

$n \times m$ can be large (compared to our computers and our displays):

Twitter (500M tweets / day, May 2020)

Youtube (2.5 billion views / day)

Natural Language processing (10^4 - 10^6 words)

Netflix prize was 500K users x 20K movies

Biochemical or genetic assays (10^4 - 10^7 features of interest)

Image processing (10^7 pixels easy)



Definition of eigenvectors and eigenvalues

There exists a set of vectors $\{e_i\}$
That has the following nice
property

$$\mathbf{A} \mathbf{e}_i = \lambda_i \mathbf{e}_i$$

Diagram illustrating the equation $\mathbf{A} \mathbf{e}_i = \lambda_i \mathbf{e}_i$ with annotations:

- square matrix** (points to \mathbf{A})
- $n \times n$** (points to \mathbf{A})
- i th eigenvalue** (points to λ_i)
- i th eigenvector** (points to \mathbf{e}_i)
- $n \times 1$** (points to \mathbf{e}_i)

Diagram illustrating the matrix equation $\mathbf{A} \mathbf{e} = \lambda_i \mathbf{e}$ with dimensions:

square matrix (points to \mathbf{A})

n (points to the height of \mathbf{A})

$n \times n$ (points to the width of \mathbf{A})

$n \times 1$ (points to the height of \mathbf{e})

$n \times 1$ (points to the height of \mathbf{e})

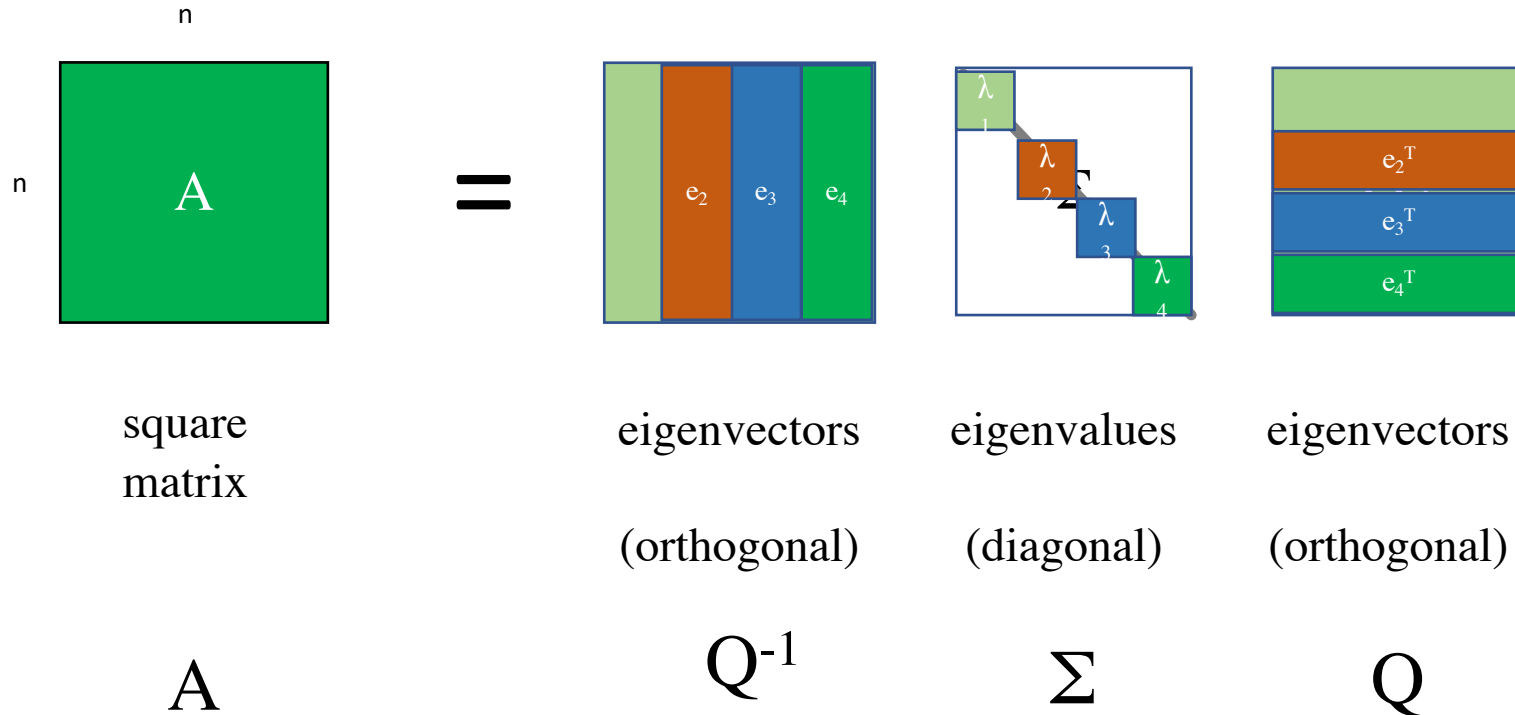
$n \times 1$ (points to the height of \mathbf{e})

Eigenvalue decomposition

These vectors can be constructed
orthogonal (zero inner products)

and normal ($x \cdot x = 1$)

$$\mathbf{A} = \text{Sum}_i \mathbf{e}_i^T \lambda_i \mathbf{e}_i$$



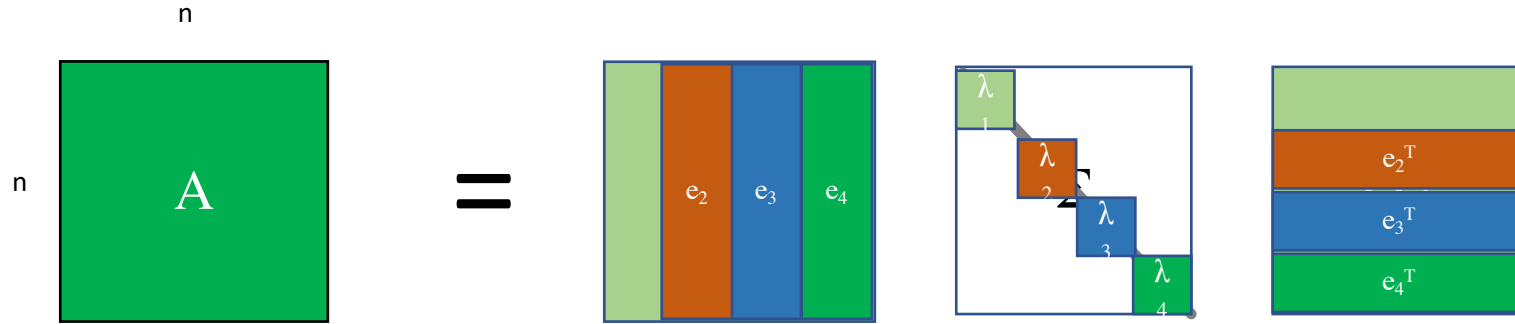
Eigenvalue decomposition

```
import numpy as np
from numpy import linalg as LA

# A library function will take a square matrix and give
# back a vector of eigenvalues and a matrix of eigenvectors.

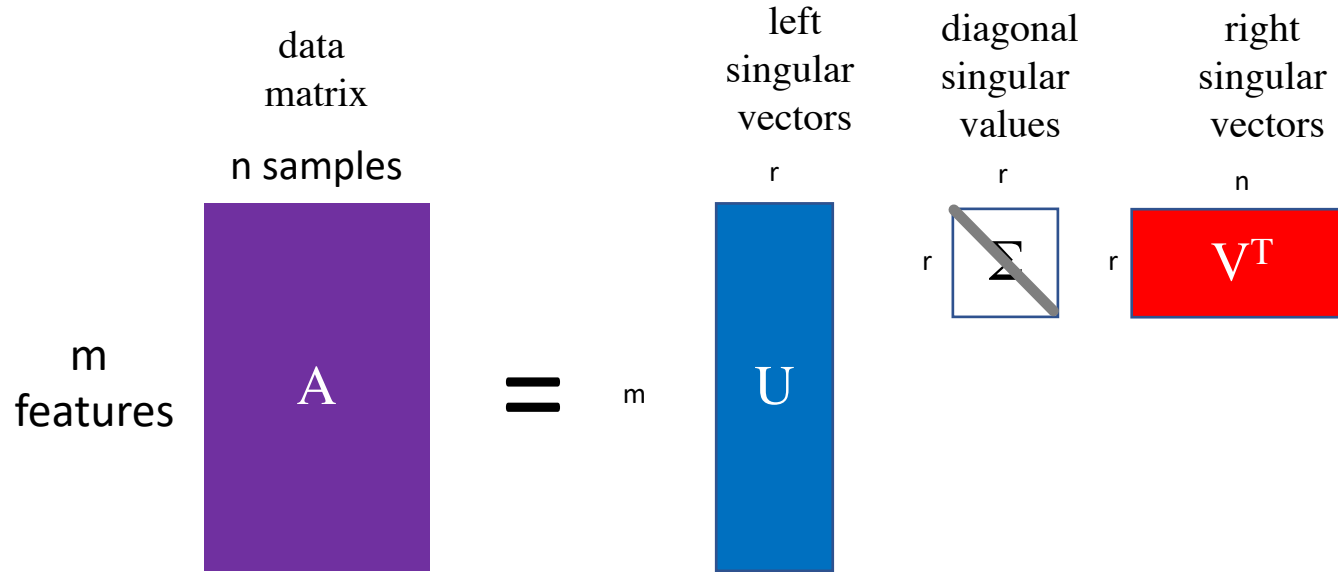
A=np.array( [ [16, 2, 3, 13 ],[5, 11, 10, 8], [9, 7, 6, 12],
[4, 14, 15, 1] ])
evalues, evectors = LA.eig(A)
print(evalues)
print(evectors)
B=np.dot(np.dot(evectors , evalues * np.eye(4)) ,
LA.inv(evectors))
```

Singular Value Decomposition (SVD)



- Eigenvalue decomposition is very nice, but only applies to square data matrices.
- Singular Value Decomposition (SVD) is a generalization for rectangular matrices

Singular Value Decomposition (SVD)



- Eigenvalue decomposition is very nice, but only applies to square data matrices.
- Singular Value Decomposition (SVD) is a generalization for rectangular matrices

Constructing SVD - rows

- Recipe for building SVD:

The diagram illustrates the construction of the matrix AA^T from matrix A and its transpose A^T . On the left, matrix A is represented by a purple rectangle with dimensions m (rows) by n (columns). To its right is matrix A^T , also a purple rectangle, with dimensions n (rows) by m (columns). An equals sign follows, leading to the resulting matrix AA^T , which is a light blue rectangle with dimensions m (rows) by m (columns).

$$\begin{matrix} n \\ \text{m} \end{matrix} \begin{matrix} A \end{matrix} \begin{matrix} n \\ m \end{matrix} \begin{matrix} A^T \end{matrix} = \begin{matrix} m \\ m \end{matrix} \begin{matrix} AA^T \end{matrix}$$

This is the matrix of inner products between rows;
summed over all the samples

Constructing SVD - rows

The diagram illustrates the SVD decomposition of the matrix AA^T . On the left is a light blue square representing AA^T , with a dimension m indicated above it and another m indicated to its left. This is followed by an equals sign. To the right of the equals sign are three matrices: a dark blue square labeled U , a white square with a diagonal line and a σ symbol in the top-left corner (representing the singular value matrix), and another dark blue square labeled U^T .

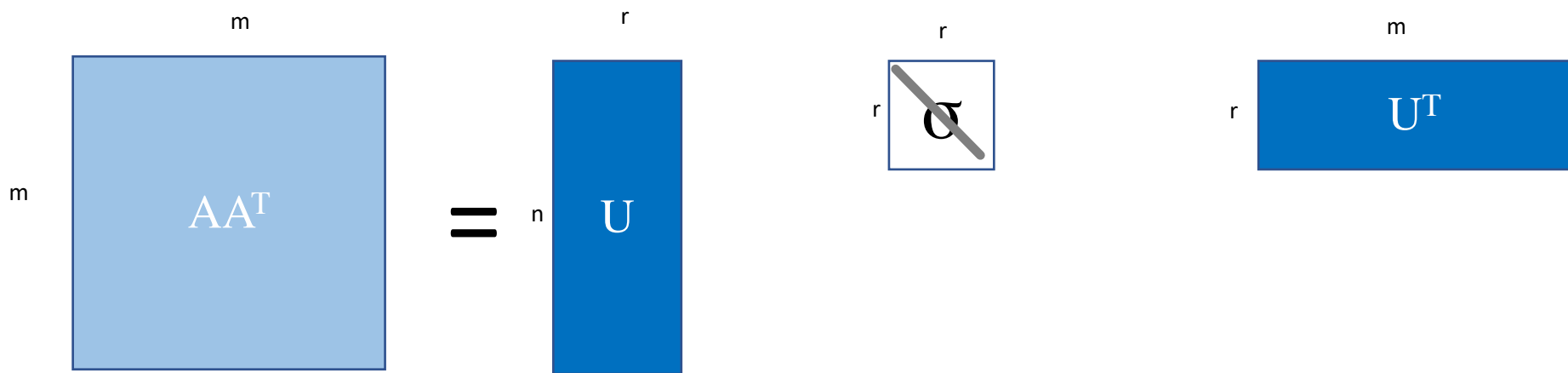
- Now take the eigenvalue decomposition of AA^T

Constructing SVD - rows

Diagram illustrating the SVD decomposition of the matrix AA^T . The matrix AA^T is shown as a light blue square with dimensions m by m . It is equal to the product of three matrices: U (a blue rectangle of size m by n), a diagonal matrix of singular values σ (a square with a diagonal line and σ), and U^T (a blue rectangle of size n by m). The zero blocks are shown as white rectangles.

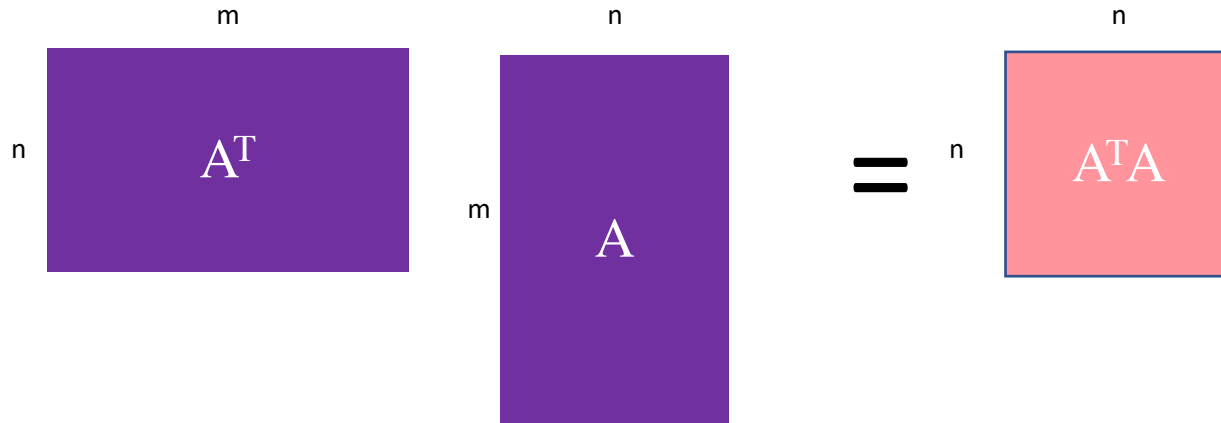
- Now take the eigenvalue decomposition of AA^T
- I have a bunch of zero eigenvalues, so I have a bunch of eigenvectors that can't contribute to the sum

Constructing SVD - rows



- Now take the eigenvalue decomposition of AA^T
- U ($n \times r$), σ (r), U^T ($r \times n$)

Constructing SVD - columns



The diagram illustrates the construction of the matrix $A^T A$ from two input matrices A^T and A . Matrix A^T is represented by a purple rectangle with dimensions n (height) and m (width). Matrix A is represented by a purple rectangle with dimensions m (height) and n (width). An equals sign indicates the multiplication of these two matrices. The result is a pink square matrix labeled $A^T A$ with dimensions n (height) and n (width).

- This matrix has dimensions $n_{\text{samples}} \times n_{\text{samples}}$.

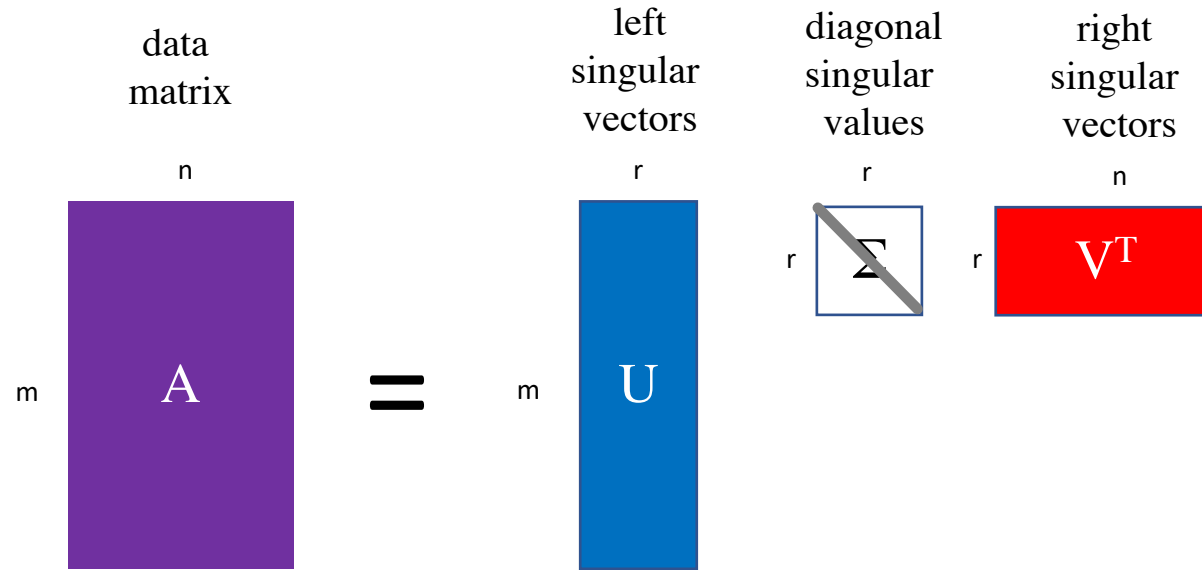
Constructing SVD - columns

Eigenvalue decomposition again

The diagram illustrates the eigenvalue decomposition of the matrix $A^T A$. On the left, a pink square represents the matrix $A^T A$, with its dimensions labeled as n by n . This is followed by an equals sign. To the right of the equals sign, three matrices are shown: a red vertical rectangle labeled V with dimensions n by r ; a small blue square labeled Σ with dimensions r by r , containing a diagonal line and the Greek letter σ ; and a red horizontal rectangle labeled V^T with dimensions r by n .

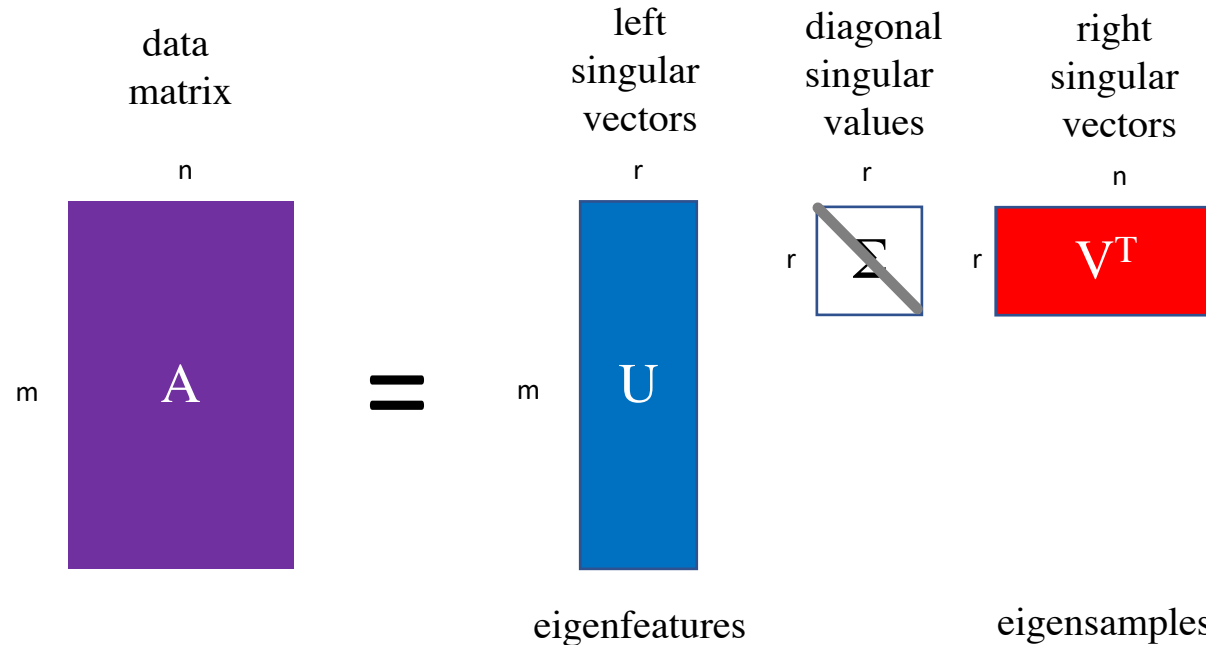
- Because of the relationship between $A^T A$ and $A A^T$ these two matrices have the same eigenvalues.
- Some of the eigenvalues have to be zero in the larger matrix

Singular Value Decomposition (SVD)



- I have two sets of basis vectors and one set of r singular values.
- Two powers of $A \rightarrow$ elements of Σ are square roots of eigenvalues

Singular Value Decomposition (SVD)



- I have two sets of basis vectors and one set of r singular values.
- Two powers of $A \rightarrow$ elements of Σ are square roots of eigenvalues

SVD

```
import numpy as np
from numpy import linalg as LA
```

```
#
https://numpy.org/doc/stable/reference/generated/numpy.linalg.svd.html
```

```
U, SIGMA, VT = LA.svd(A)
```

We could calculate SVD
ourselves in two lines –
but we shouldn't

Approximate, truncated SVD

```
import numpy as np
from numpy import linalg as LA
from sklearn.utils.extmath import
randomized_svd
```

```
# https://scikit-learn.org/stable/modules/generate\_d/sklearn.utils.extmath.randomized\_svd.html
```

```
U, SIGMA, VT = randomized_svd(A,
n_components=50)
```

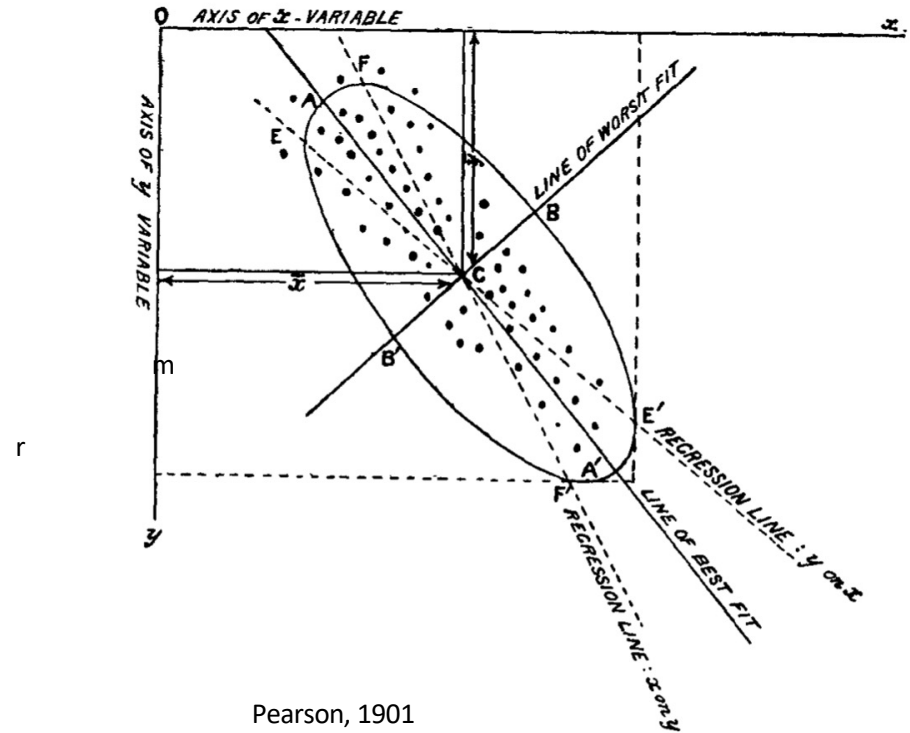
The library functions use convergence or optimization to find approximate matrix factorizations.

Does not actually build or store $A^T A$

Approximate and truncated decompositions are cheaper to compute.

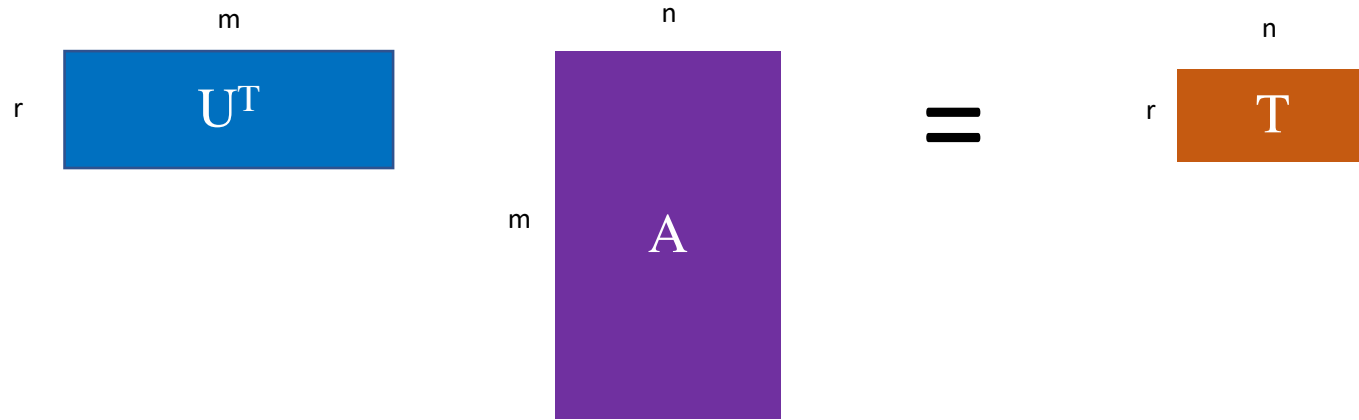
Principal Component analysis

- SVD is a just linear algebra + geometry interpretation of the columnwise & rowwise inner products.
- PCA is the doctrine of using vectors derived from SVD to interpret data.
- The data points are rotated onto the singular vectors, renamed PCA coordinates
- $AV = T$ (new coordinates)
- Can make pretty pictures



Pearson, 1901
doi: 10.1080/14786440109462720

Principal Component analysis



The diagram illustrates the Principal Component Analysis (PCA) equation. It consists of three matrices and an equals sign. The first matrix is a blue rectangle labeled U^T with dimensions r (height) and m (width). The second matrix is a purple rectangle labeled A with dimensions m (height) and n (width). The third matrix is an orange rectangle labeled T with dimensions r (height) and n (width). The equation is represented as $U^T A = T$.

This is the dimension reduction we might have asked for

Table 1
Principal Components Analysis of BIS-11 Items (Oblique Rotation)

BIS-11 items	First-order factors					
	1	2	3	4	5	6
11. I "squirm" at plays or lectures.	.84	.17	-.08	-.03	.03	.02
32. I am restless at the theater or lectures.	.84	.19	-.12	-.06	-.00	-.03
5. I don't "pay attention."	.57	.04	.16	-.02	.27	.02
17. I act "on impulse."	.15	.74	.08	-.02	-.20	.06
20. I act on the spur of the moment.	.12	.72	.19	-.10	-.19	.01
23. I buy things on impulse.	-.08	.59	-.04	.28	.10	.11
12. I am a careful thinker. ^a	.17	-.13	.64	.17	-.18	.05
1. I plan tasks carefully. ^a	-.05	.16	.64	-.04	.11	-.10
8. I am self-controlled. ^a	.10	.00	.63	-.24	.08	-.17

Patton et al. 1995 Factor structure of the Barratt Impulsiveness Scale
doi: 10.1002/1097-4679(199511)51:6<768::AID-JCLP2270510607>3.0.CO;2-1

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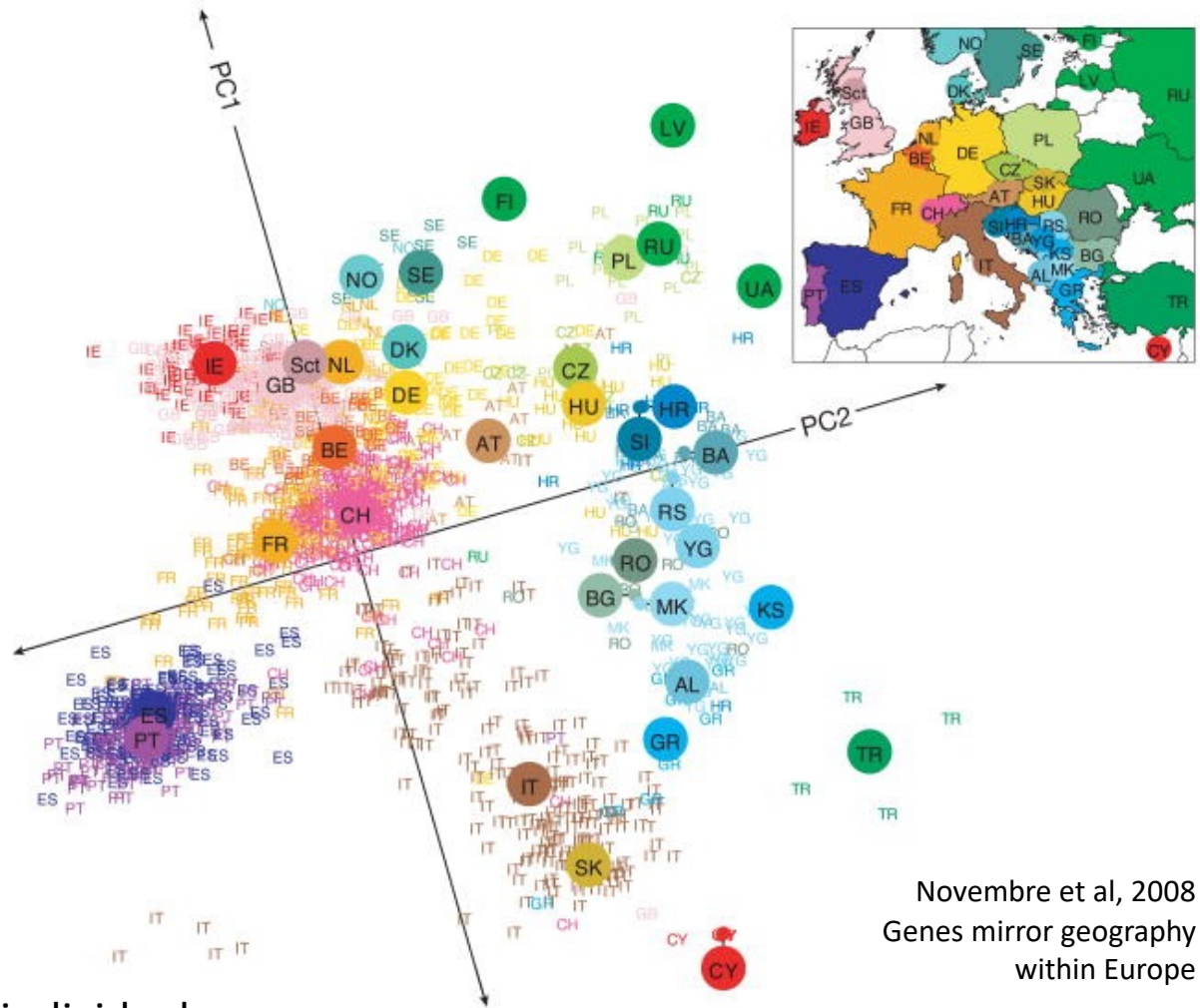
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Data correlations here used to inform reduction of dimension from 35-question survey to 6 scores.

Genes mirror geography within Europe



Novembre et al, 2008
Genes mirror geography
within Europe

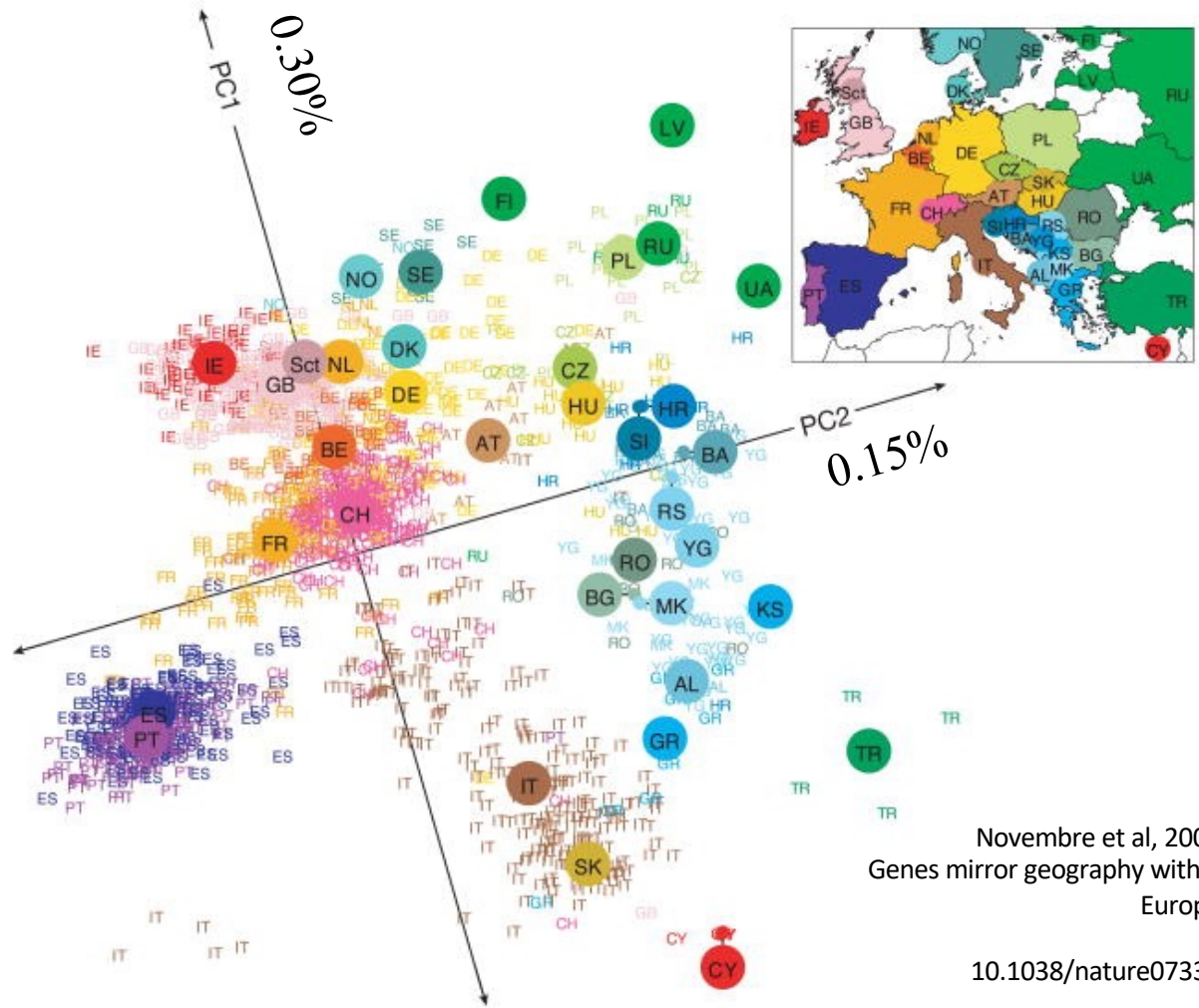
m x n = 200k genetic loci x 1400 individuals

10.1038/nature07331

Genes mirror geography within Europe

Principal components and eigenvalues come back from largest to smallest magnitude.

It is conventional to report principal components along with the fraction of variance explained by each



Novembre et al, 2008
Genes mirror geography within
Europe

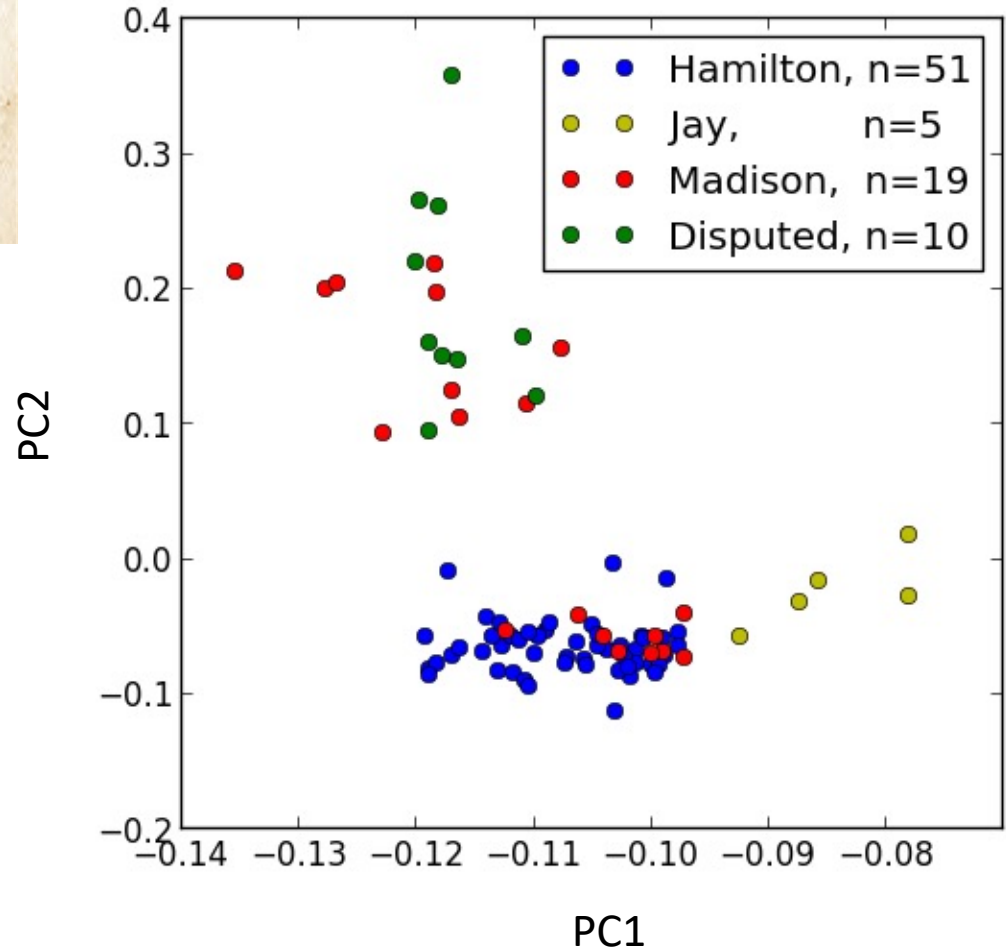
10.1038/nature07331

THE FEDERALIST:

A COLLECTION OF

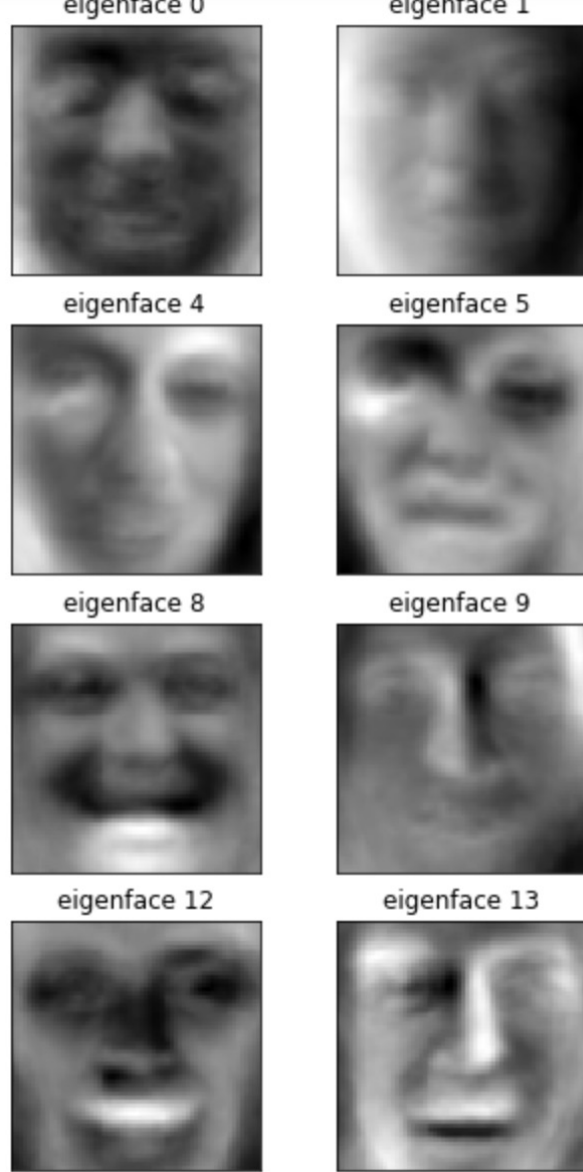
ESSAYS,

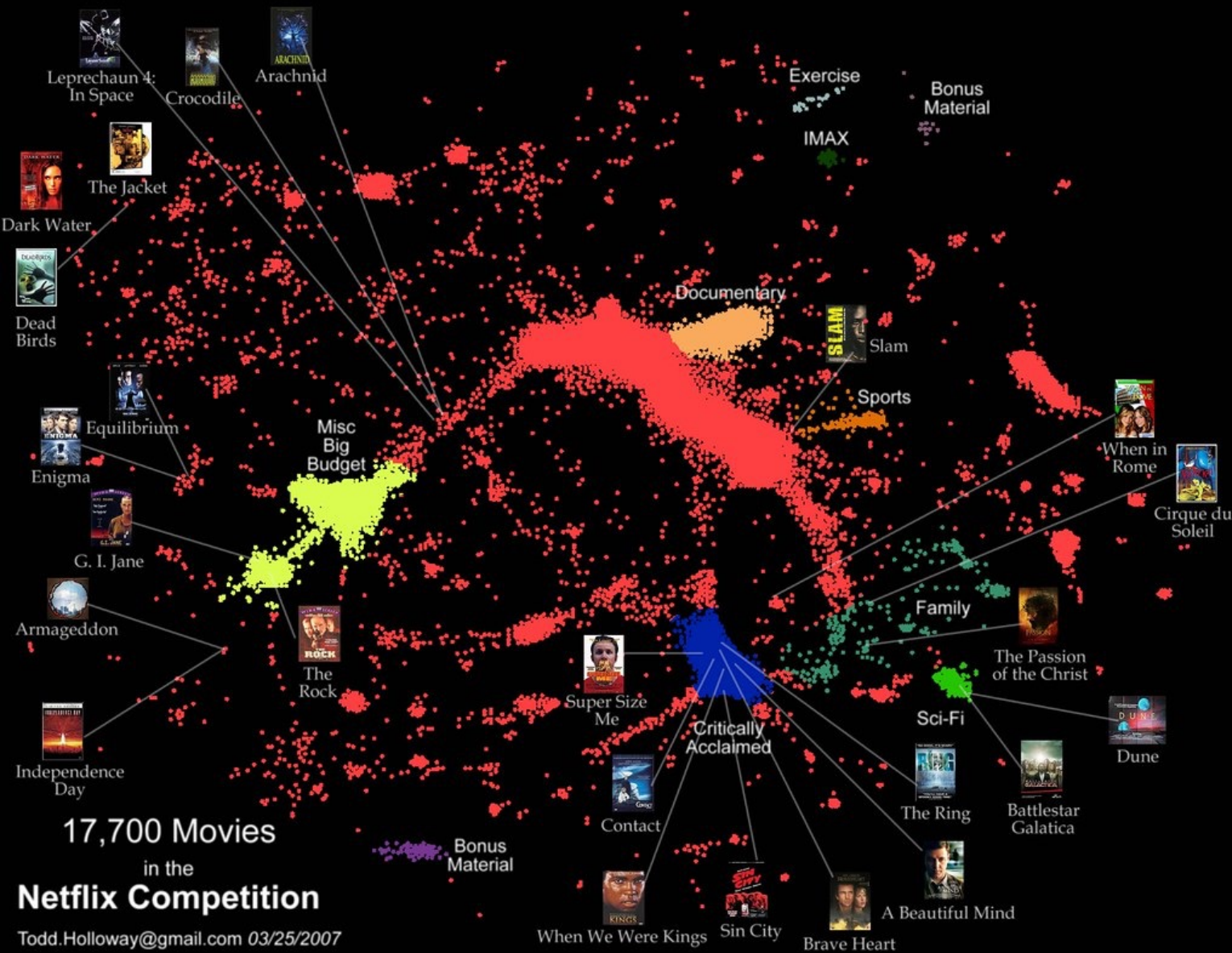
- Sometimes $\sum (x - \bar{x})(y - \bar{y})$ is not an appropriate distance score for your data.
- You can run PCA on an $n \times n$ distance matrix evaluated using a domain-specific columnwise comparison.
- Resulting low-dimensional coordinates visualize vocabulary/style relationships



Caveats

- SVD/PCA don't know anything but the cloud of points and its correlations.
- Each new sample builds a different set of basis vectors
- Hard to interpret: positive and negative values in the eigenvectors
- Does not handle non-numerical values—have to change the objective function for that (Netflix)





Linear algebra objects

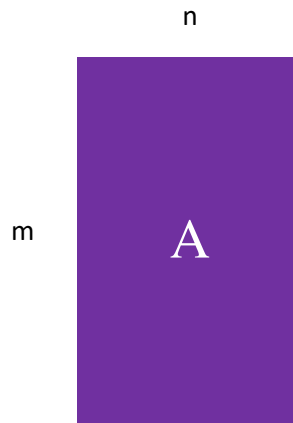
λ

number
“scalar”



$m \times 1$

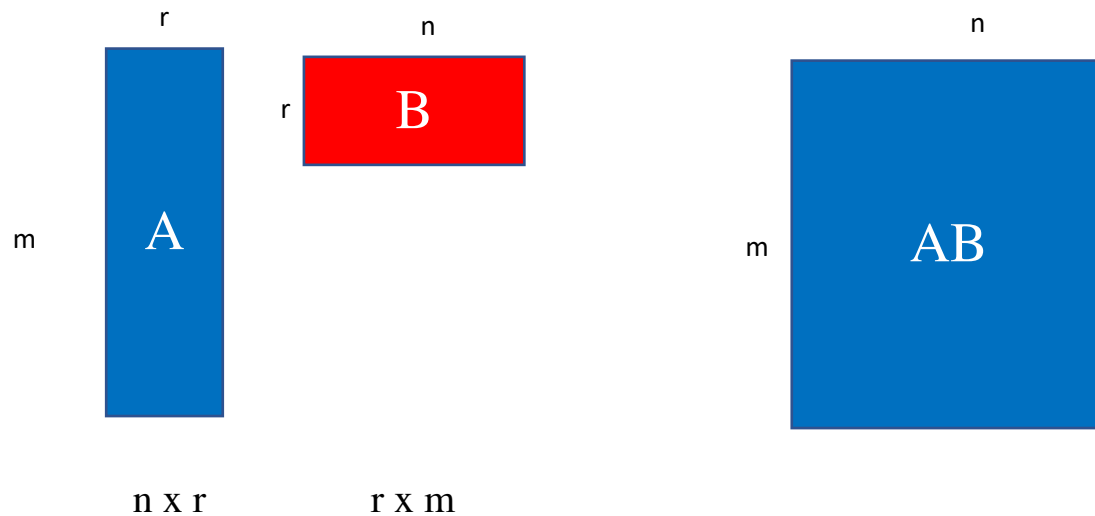
vector



$m \times n$

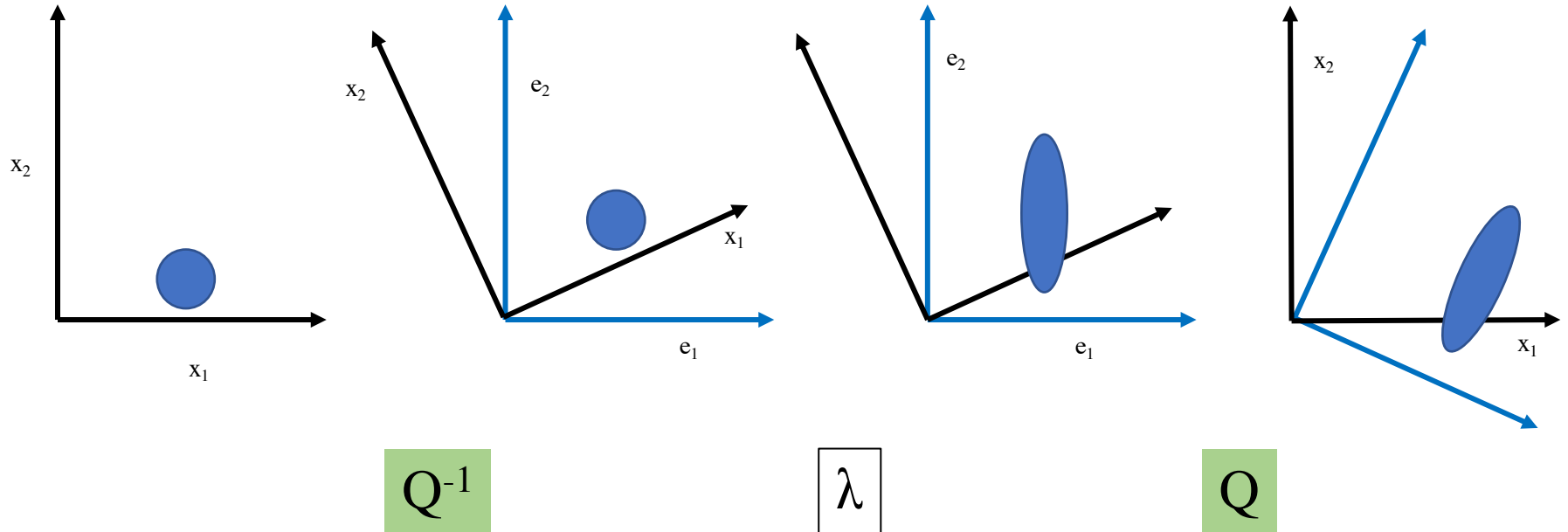
2d matrix

Matrix multiplication



2d matrix

Transformation of x_1, x_2 using A



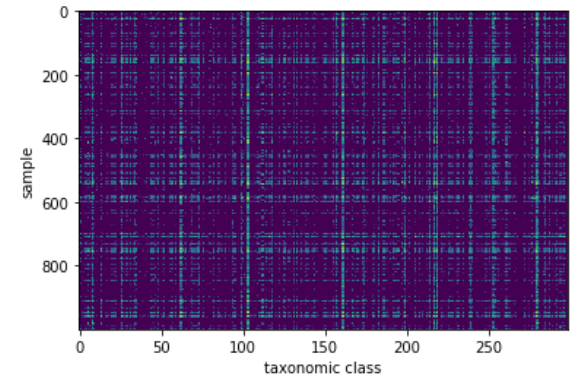
Dimension reduction

For a general matrix representing n observations of m traits:

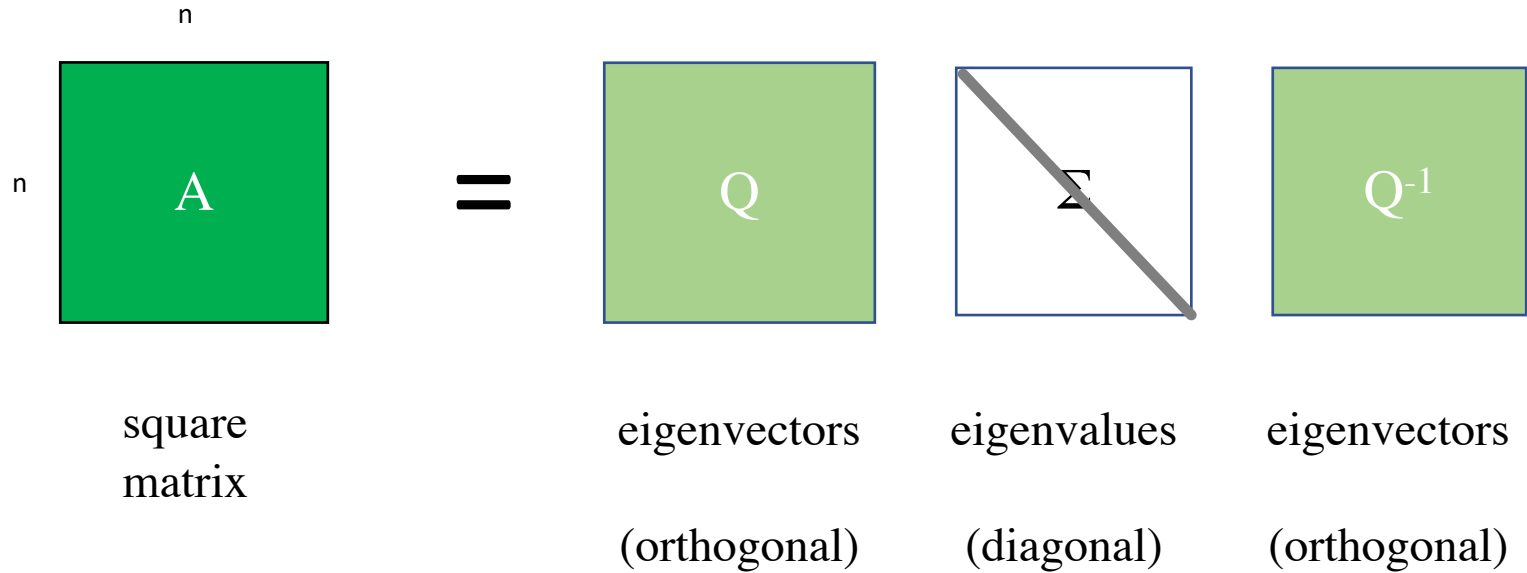
$$A (m \times n)$$

It is unlikely that all of the elements in this matrix are independent; there is likely some “structure”

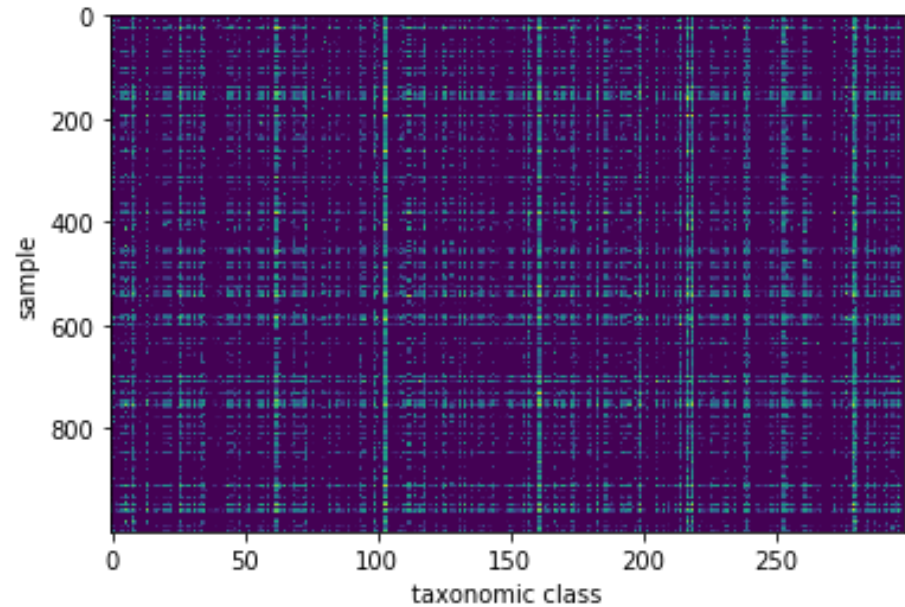
If I could replace A with an approximation of A that has fewer than $n \times m$ numbers, A will be cheaper to store, faster to copy, can run on computers with less memory.



Eigenvalue decomposition



$A (n \times m)$
 (rows x columns)
 (n “samples” by m “features”)



	Hydrozoa	Spingobacteriia	Marattiopsida	Pedinophyceae	Chrysiogenetes (class)	Fusobacteria (class)	Amphibia
(Index,)							
mgm4703051.3.mg	2	354	0	0	0	0	0
mgm4755207.3.mg	3430	0	25	14	33111	101765	5094
mgm4473196.3.mg	150	0	0	10	1265	487614	11598
mgm4679127.3.mg	0	245	0	0	0	0	0
mgm4529798.3.mg	2	4588	0	0	25	0	56

Now this looks like “structure”

Organizing data like this is deeply satisfying—it makes us think we have discovered truth.

Mature libraries to make exactly this sort of eye-candy graphs. This one is `sanborn.clustermat`

What if I told you linear algebra could do this?

