Pergamon

Transpn Res.-B, Vol. 32, No. 7, pp. 485–498, 1998

Printed in Great Britain
0191-2615/98 $19.00 + 0.00

PII: S0191-2615(98)00014-9

A THEORY OF NONEQUILIBRIUM TRAFFIC FLOW

H. M. ZHANG*
The University of Iowa, Iowa City, IA 52242, U.S.A.

(Received 29 May 1997; in revised form 27 December 1997)

Abstract—This paper presents a new continuum traffic flow theory. The derivation of this new theory is based on both empirical evidence of traffic flow behavior and basic assumptions on drivers' reaction to stimuli. Central to the development of this theory is the existence of an equilibrium speed–concentration relationship and an introduction of a disturbance propagation speed. The new theory includes a well established continuum theory (the LWR theory) as a special case and removes certain deficiencies of this theory. Unlike existing higher order continuum models, the new theory does not exhibit the undesirable behavior of 'wrong-way travel' because in this theory traffic disturbances are always propagated against the traffic stream. © 1998 Elsevier Science Ltd. All rights reserved

Keywords: traffic flow, nonequilibrium traffic dynamics, continuum models

1. INTRODUCTION

Mathematical modeling of physical systems is essential to the development of an engineering field, and traffic engineering is no exception. A traffic system, comprised of drivers, vehicles, and roadways, exhibits extremely complex behavior derived from several sources—the heterogeneous nature of human behavior, highly nonlinear group dynamics, and large system dimensions, to name but a few. The mathematical modeling of traffic systems is, therefore, understandably diverse.

Two approaches, however, have dominated traffic flow modeling throughout the past few decades. One approach takes a microscopic view, studying individual movements of vehicles and the interactions between vehicle pairs (car-following). This approach considers driving behavior and develops vehicle pair dynamics that can be extended recursively to the modeling of traffic dynamics of vehicle streams. The best known example of this approach is the GM family of car-following models developed in the sixties (e.g. Gazis *et al.*, 1961). Despite its appeal from a behavioral perspective, this approach soon becomes mathematically intractable for modeling any traffic system of a realistic size.

The other Approach attempts to avoid the complications of modeling individual vehicle dynamics by studying the collective effects of vehicle interactions as measured by variables such as flow rate q, concentration ρ (also known as traffic density)—both terms will be used interchangeably in this paper—and travel speed υ, all of which are functions of space (x) and time (t). The first significant development in this direction is the continuum theory developed independently by Lighthill and Whitham (1955) and Richards (1956). This theory, referred to hereinafter as the LWR theory, relies on the existence of an *equilibrium* speed–concentration relationship $\upsilon = \upsilon_e(\rho)$, or equivalently a flow–concentration $q = \rho\upsilon \equiv q_e(\rho)$ or flow–speed relationship $q = Q_e(\upsilon)$ (because $q = \rho\upsilon$). With such a relationship, the LWR model becomes a first order, nonlinear partial differential equation (PDE) of hyperbolic type:

$$\frac{\partial\rho(x,t)}{\partial t} + q_e'(\rho)\frac{\partial\rho(x,t)}{\partial x} = 0 \tag{1}$$

where $q_e'(\rho) \equiv \frac{dq_e}{d\rho}$.

In essence, this model is a conservation equation with a special equation of state $\upsilon = \upsilon_e(\rho)$. It allows solutions in the weak sense (discontinuous $\rho(x,t)$ in x), and generates shocks where

*Fax: 001 319 335 6801; e-mail: mhzhang@blue.weeg.uiowa.edu

discontinuities occur. The weak solution satisfies the integral form of the conservation equation from which (1) is derived:

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho(x, t)\mathrm{d}x = q(x_1, t) - q(x_2, t) \tag{2}$$

The propagation of a shock depends on both flow and concentration in the neighboring (upstream l and downstream r) regions of a shock:

$$c_s = \frac{q_e(\rho_r) - q_e(\rho_l)}{\rho_r - \rho_l} \tag{3}$$

which is known as the *Rankine–Hugoniot jump condition* (LeVeque, 1992).

A shock arising from the LWR theory is preserved if $\rho_l < \rho_r$, which corresponds to congestion formation in traffic flow, because there is a unique weak solution for this case (Fig. 1). On the other hand, a shock arising from $\rho_l > \rho_r$ could be preserved (entropy violating weak solution) or dissolved (entropy satisfying solution), because there are multiple weak solutions. To pick from among weak solutions the physically meaningful one, a requirement called *entropy condition* is used (LeVeque, 1992):

$$\frac{q_e(\rho) - q_e(\rho_l)}{\rho - \rho_l} \geqslant c_s \geqslant \frac{q_e(\rho) - q_e(\rho_r)}{\rho - \rho_r} \tag{4}$$

The entropy-satisfying solution of the LWR model for $\rho_l > \rho_r$, is a fan of rarefaction waves emitting from the shock location x_0; these waves are bounded by the characteristic lines of $x(t) = x_0 + q_e'(\rho_l)t$ and $x(t) = x_0 + q_e'(\rho_r)t$ (Fig. 2).

With the assumption that there is a unique smooth relationship between speed and concentration, $\upsilon = \upsilon_e(\rho)$, one can obtain the acceleration of a traffic stream as prescribed by the LWR theory as follows:

$$\frac{\mathrm{d}\upsilon}{\mathrm{d}t} = \upsilon_e'(\rho)\left(\frac{\partial\rho}{\partial t} + \frac{\mathrm{d}x\partial\rho}{\mathrm{d}t\partial x}\right) \tag{5}$$

From the conservation of traffic flow, one gets:

$$\frac{\partial\rho(x, t)}{\partial t} = -q_e'(\rho)\frac{\partial\rho(x, t)}{\partial x} \tag{6}$$

by $q_e = \rho\upsilon_e(\rho)$, one has:

$$q_e' = \upsilon_e + \rho\upsilon_e'(\rho) \tag{7}$$

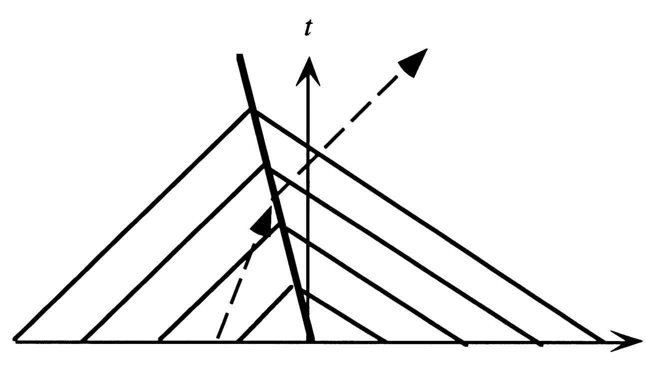

Fig. 1. Shock solution and vehicle trajectory.

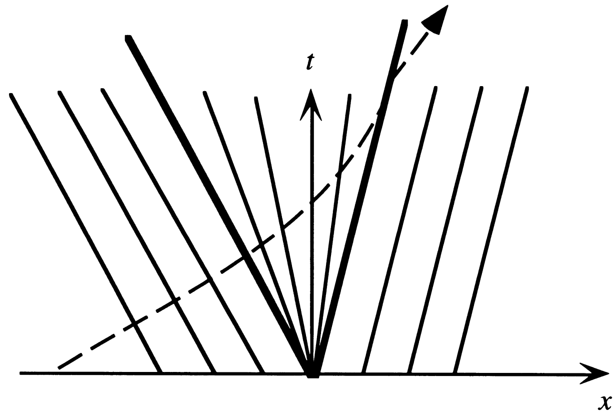

Fig. 2. Rarefaction wave fan solution and vehicle trajectory.

Substituting (6) into (5), and utilizing (7), one has:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\rho\left(v_e'\right)^2\frac{\partial\rho}{\partial x} \tag{8}$$

which is the traffic stream acceleration equation given by Pipes (1969).

Equation (8) reveals that the acceleration or deceleration of a vehicle stream is proportional to traffic concentration and, more importantly, to concentration gradient. When traveling, vehicles accelerate into a less denser stream and decelerate into a denser steam. The acceleration, however, is a delta function at location $\nabla^{-1}(\rho) \equiv -\frac{1}{\rho v_e' 2\frac{\partial\rho}{\partial x}} = 0$; i.e. it is infinite at this location (Fig. 3).

The LWR theory, as a first order approximate model of traffic flow dynamics, has a number of serious deficiencies: it predicts infinite deceleration when a vehicle crosses a shock, as revealed in eqn (8), and assumes that the equilibrium speed–concentration relationship also holds for nonequilibrium traffic. These deficiencies derive from the assumption that the equilibrium speed–concentration relationship is the sole manifestation of the collective effects among vehicle interactions. In reality, traffic flow is hardly in equilibrium, and its dynamics is a result of the retarded response of drivers to various *frontal* stimuli. In other words, drivers usually look ahead, but respond with a delay to changes of traffic conditions in front of them. It is the interplay between such *anticipation* and *inertia* that leads to the hysteresis phenomena observed in traffic flow (e.g. Treiterer and Myers, 1974).

Noticing the discrepancies between experimentally-obtained flow–concentration curves that contain hysteresis and the assumed smooth, single-valued flow–concentration curves, Lighthill and Whitham (1955), in their epochal paper on kinematic waves, suggested that higher-order terms be added to account for the effects of anticipation and inertia, which led to the following equation of motion:

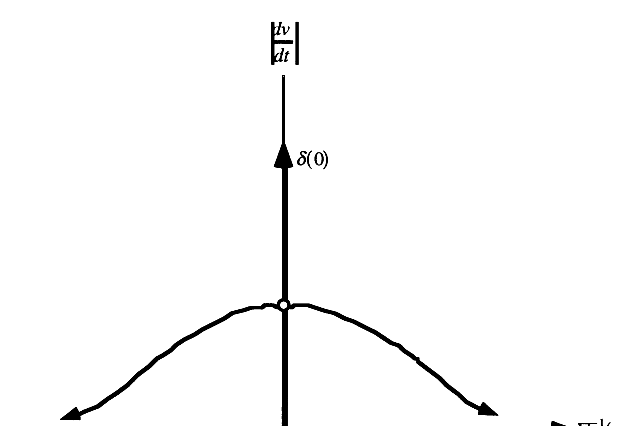

Fig. 3. Acceleration as a function of $\nabla^{-1}(\rho)$ in the LWR model.

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} + T\frac{\partial^2 q}{\partial t^2} - D\frac{\partial^2 q}{\partial x^2} = 0 \tag{9}$$

where T is a reaction time constant and D is the diffusion coefficient.

Because of a lack of experimental data to validate the model, this idea was not pursued further till 1971, when Payne (1971) derived an equation of motion from car-following theory:

$$\frac{\partial \upsilon}{\partial t} + \upsilon\frac{\partial \upsilon}{\partial x} = -\frac{\upsilon - \upsilon_e(\rho)}{T} - \frac{\mu}{T}\frac{1}{\rho}\frac{\partial \rho}{\partial x} \tag{10}$$

where $\mu = -\frac{\upsilon_e'(\rho)}{2}$.

There are also other higher-order continuum models. These models, unlike the one derived by Payne, are not generally 'behaviorally-based'. Examples of such models include the traffic model by Ross (1988):

$$\frac{\partial \upsilon}{\partial t} + \upsilon\frac{\partial \upsilon}{\partial x} = \frac{1}{T}(\upsilon_0 - \upsilon) \tag{11}$$

the higher-order continuum model proposed by Michaloupolos *et al.* (1993):

$$\frac{\mathrm{d}\upsilon}{\mathrm{d}t} = \frac{1}{T}\left(\upsilon_f - \upsilon\right) - G\frac{\partial \upsilon}{\partial t} - \nu\rho^\beta\frac{\partial \rho}{\partial x} \tag{12}$$

where $G = \mu\rho^\varepsilon g$, and μ, ν, ε and β are all constant parameters; and the following model by Kühne (1984, 1989):

$$\frac{\partial \upsilon}{\partial t} + \upsilon\frac{\partial \upsilon}{\partial x} = -\frac{\upsilon - \upsilon_e(\rho)}{T} - \frac{c_0^2}{\rho}\frac{\partial \rho}{\partial x} + \nu\frac{\partial^2 \upsilon}{\partial x^2} \tag{13}$$

where c_0 is the 'sound' speed, and ν is the 'viscosity' constant.

Payne's model, together with its computer implementations, aroused considerable interests in higher-order continuum traffic flow models. The applications of this model, however, reported mixed results (e.g. Hauer and Hurdle, 1979; Papageorgiou, 1983, Derzko *et al.*, 1983; Cremer and May, 1986; Papageorgiou *et al.*, 1990; Leo and Pretty, 1992). Some of the reported difficulties may have been caused by numerical methods used to convert a nonlinear system of hyperbolic PDEs to a set of FDEs (finite difference equations). It is believed, however, that most of these problems are attributable to a fundamental flaw in this model: it produces 'wrong way travel' (i.e. negative travel speed) under certain circumstances (Daganzo, 1995).

In this paper, we shall develop an enhanced continuum traffic flow theory that removes certain deficiencies of the LWR theory without introducing new flaws. Our presentation of this new theory is organized as follows: we first define the concept of nonequilibrium traffic flow, then develop the new continuum theory; next we examine the qualitative properties of this theory, and finally we compare it with some existing theories.

2. THEORETICAL DEVELOPMENT

To prepare for the development of our new nonequilibrium traffic flow theory, we shall first define what equilibrium means in traffic flow.

Definitions: equilibrium and nonequilibrium

Traffic is in *equilibrium* when the following temporal stationarity condition:

$$\frac{\mathrm{d}\upsilon}{\mathrm{d}t} = 0 \tag{14}$$

and spatial homogeneity condition:

$$\frac{\partial \rho}{\partial x} = 0 \tag{15}$$

hold for all x and t. Traffic is in *nonequilibrium* when one (or both) of the aforementioned condition is not satisfied.*

The development of the new nonequilibrium flow theory is based on drivers' car-following behavior, which is given by the following rule:

$$\text{delayed response} = \text{sensitivity} * \text{stimuli}$$

Rather than modeling interactions between vehicle pairs, we take a micro–macro approach; i.e. we assume that the dynamic properties of a vehicle n at a location x_n represents the *average* traffic condition at $[x_n - \Delta, x_n + \Delta]$, and is determined by the *average* traffic condition in the region $[x_n, x_n + 2\Delta]$, which we shall refer to hereafter as the reaction zone of a vehicle.

We start the development of the new continuum theory with a few basic assumptions:

1. There exists a unique smooth, equilibrium speed-concentration relationship $\upsilon_e(\rho)$ that satisfies $\upsilon_e'(\rho) \leqslant 0$, $(\rho\upsilon_e(\rho))'' < 0$, $\upsilon_e(0) = \upsilon_0$ and $\upsilon_e(\rho_j) = 0$, where υ_0 is the free flow speed and ρ_j is the traffic concentration when the road is packed with stopped vehicles.
2. Drivers react to traffic changes in their reaction zones with a time delay.
3. The speed of a vehicle at time $t + T$ is a function of the traffic concentration (which represents the *average stimuli* to the following vehicle by vehicles traveling in its reaction zone) at location $x + \Delta$ due to driver's anticipation of traffic conditions ahead.

These assumptions lead to the following micro–macro traffic model:

$$\frac{\mathrm{d}x_{n+1}(t+T)}{\mathrm{d}t} = \upsilon_e(\rho(x_{n+1} + \Delta, t)) \tag{16}$$

where T represents traffic relaxation time—the time needed for a following vehicle to assume a certain speed determined by leading vehicles.†

There are two types of coordinate systems for referencing the trajectory of a moving vehicle stream; one is the Euler coordinate system that has its origin located at a fixed point in space [(x, y) coordinates in Fig. 4], and the other is the Largrangian system that has its origin moving with the traffic stream itself [(x', y') coordinates in Fig. 4]. Depending on which coordinate system one uses to describe traffic flow, the same traffic stream dynamics could yield different system equations. We shall use the Euler coordinate system in our derivation of the new theory; some quantities, however, are given in the Largrangian coordinate system.

Using Taylor series expansion for eqn (16), and neglecting higher-order terms, we have:

$$\frac{\mathrm{d}x_{n+1}}{\mathrm{d}t} + T\frac{\mathrm{d}^2x_{n+1}}{\mathrm{d}t^2} = \upsilon_e\left(\rho(x_{n+1}, t) + \Delta\frac{\partial\rho(x_{n+1}, t)}{\partial x}\right)$$

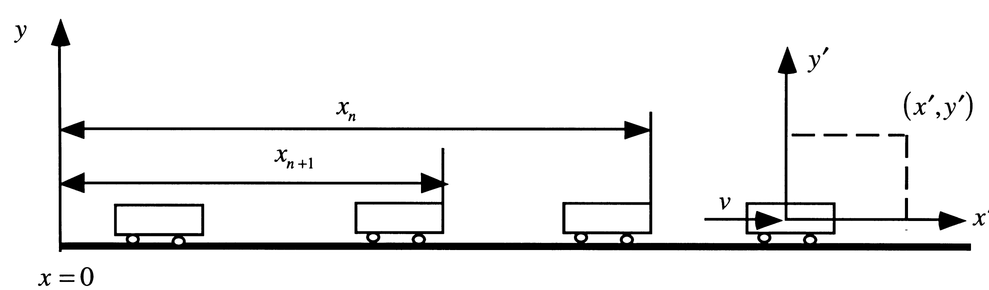

Fig. 4. Car following.

*It will become clear in later sections that our definition of equilibrium implies that $\upsilon = \text{constant}$, $\rho = \text{constant}$ and $\upsilon = \upsilon_e(\rho)$. An equivalent definition of traffic equilibrium is to require $\frac{\partial\upsilon}{\partial t} = 0$ and $\frac{\partial\rho}{\partial x} = 0$. It is noted that in the LWR theory, $\frac{\mathrm{d}\upsilon}{\mathrm{d}t} \neq 0$ leads to $\frac{\partial\rho}{\partial x} \neq 0$ and $\frac{\partial\rho}{\partial x} \neq 0$ implies that $\frac{d\upsilon}{dt} \neq 0$. We call traffic flow with such properties pseudo non-equilibrium flow and the LWR theory a pseudo non equilibrium theory.

†In current context this certain speed is determined by the function $\upsilon_e(\rho)$.

Let $x_{n+1} \to x$ and $\frac{dx_{n+1}}{dt} = \upsilon$, we get:

$$\upsilon(x, t) + T\frac{d\upsilon(x, t)}{dt} = \upsilon_e(\rho(x, t)) + \Delta\upsilon_e'(\rho(x, t))\frac{\partial\rho(x, t)}{\partial x} \tag{17}$$

which can be rearranged as:

$$\frac{d\upsilon(x, t)}{dt} = -\frac{1}{T}(\upsilon(x, t) - \upsilon_e(\rho(x, t))) + \frac{\Delta}{T}\upsilon_e'(\rho(x, t))\frac{\partial\rho(x, t)}{\partial x} \tag{18}$$

Both T and Δ measure the sensitivity of vehicle acceleration/deceleration to the change of traffic concentration *ahead of the vehicle*, with one capturing temporal and the other capturing spatial aspects of this response. Although in theory both Δ and T approach 0 as we take the limit, we argue that $\frac{\Delta}{T}$ is finite and define $\upsilon_c = -\frac{\Delta}{T}$. Clearly, υ_c unifies the spatial and temporal reactions of drivers to disturbances and is referred to hereafter as the disturbance propagation speed in a traffic stream. This speed equals $q'(\rho)$ for equilibrium flow when it is given in the Euler coordinate system, and equals $q'(\rho) - \upsilon = \rho\upsilon'(\rho)$ when it is given in the Lagrange coordinate system. Because the reaction zone $[x_n, x_n + 2\Delta]$ is moving with the traffic stream, υ_c should adopt the latter value $q'(\rho) - \upsilon$ for equilibrium flow. Furthermore, we argue that this disturbance speed for none-quilibrium flow should be in the same order of magnitude as its equilibrium counterpart, because numerable phase plots of speed against concentration (or occupancy) show that the deviations of (ρ, υ) from (ρ, υ_e) are moderate, suggesting that the dominant mode of nonequilibrium traffic flow dynamics is still governed by $\upsilon_e(\rho)$. Based on the above argument, we prescribe a functional form for υ_c as follows:

$$\upsilon_c = \alpha\rho\upsilon_e'(\rho) \tag{19}$$

where α is a positive parameter to be determined later. It will be seen that this new variable plays an essential role in the development of the new continuum theory.

With this new definition of a disturbance speed, we have:

$$\frac{d\upsilon(x, t)}{dt} = -\frac{1}{T}(\upsilon(x, t) - \upsilon_e(\rho(x, t))) - \alpha\rho(x, t)\big(\upsilon_e'(\rho(x, t))\big)^2\frac{\partial\rho(x, t)}{\partial x} \tag{20}$$

which is the equation of motion.

We know that:

$$\frac{d\upsilon(x, t)}{dt} = \frac{\partial\upsilon(x, t)}{\partial t} + \upsilon\frac{\partial\upsilon(x, t)}{\partial x} \tag{21}$$

Substitute eqn (21) into eqn (20), and we have:

$$\frac{\partial\upsilon(x, t)}{\partial t} + \upsilon\frac{\partial\upsilon(x, t)}{\partial x} = -\frac{1}{T}(\upsilon(x, t) - \upsilon_e(\rho(x, t))) - \alpha\rho(x, t)\big(\upsilon_e'(\rho(x, t))\big)^2\frac{\partial\rho(x, t)}{\partial x} \tag{22}$$

There are two parameters in the dynamic speed equation: relaxation time T and a positive α associated with the definition of disturbance propagation speed. Relaxation times can be measured from field experiments; the α parameter, however, cannot be directly measured. Fortunately, we can derive the proper value for α from the new theory itself, using its equivalence to the LWR theory for pseudo nonequilibrium traffic.

Assuming, as the LWR theory prescribes that drivers do not anticipate traffic conditions ahead, i.e. $\upsilon = \upsilon_e(\rho)$, we have:

$$\frac{\partial\upsilon}{\partial t} = \upsilon_e'(\rho)\frac{\partial\rho}{\partial t} \tag{23}$$

and

$$\frac{\partial\upsilon}{\partial x} = \upsilon_e'(\rho)\frac{\partial\rho}{\partial x} \tag{24}$$

and it follows that

$$\upsilon_e'(\rho)\frac{\partial\rho}{\partial t}+\upsilon\upsilon_e'(\rho)\frac{\partial\rho}{\partial x}+\alpha\rho\big(\upsilon_e'(\rho)\big)^2\frac{\partial\rho}{\partial x}=0 \tag{25}$$

If $\upsilon_e'(\rho)\neq 0$, we have

$$\frac{\partial\rho}{\partial t}+\big(\upsilon+\alpha\rho\upsilon_e'(\rho)\big)\frac{\partial\rho}{\partial x}=0 \tag{26}$$

Now if we take $\alpha=1$, and consider the equation of state in traffic flow, $q=\rho\upsilon$, we have

$$\upsilon+\rho\upsilon_e'(\rho)=\frac{\mathrm{d}(\rho\upsilon)}{\mathrm{d}\rho}=q'(\rho) \tag{27}$$

It follows that

$$\frac{\partial\rho}{\partial t}+q'(\rho)\frac{\partial\rho}{\partial x}=0 \tag{28}$$

This is the LWR model. By assuming $\upsilon=\upsilon_e(\rho)$, the new theory is reduced to the LWR theory. Clearly, the LWR model is a special case of the new model for traffic flow where drivers do not anticipate traffic conditions ahead. The LWR theory as a special case of the new theory is further verified by substituting $\alpha=1$ and $\upsilon=\upsilon_e(\rho)$ into eqn (20):

$$\frac{\mathrm{d}\upsilon}{\mathrm{d}t}=-\rho\big(\upsilon_e'(\rho)\big)^2\frac{\partial\rho}{\partial x}$$

This is the acceleration of the traffic stream for the LWR model that was derived in Section 1.

Now the disturbance speed that we defined earlier becomes

$$\upsilon_c=\rho\upsilon_e'(\rho)=q_e'(\rho)-\upsilon_e\leqslant 0$$

which is always non-positive, implying that a disturbance always travels against traffic stream. As a result, vehicles behind cannot influence vehicles ahead. The new theory has therefore removed a fundamental flaw associated with some existing higher-order continuum models.

In addition to the equation of motion and equation of state, the conservation of flow still holds:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho\upsilon)}{\partial x}=0 \tag{29}$$

With $\alpha=1$, the equation of motion now reads:

$$\frac{\partial\upsilon}{\partial t}+\upsilon\frac{\partial\upsilon}{\partial x}=\frac{1}{T}(\upsilon_e(\rho)-\nu)-\rho\big(\upsilon_e'(\rho)\big)^2\frac{\partial\rho}{\partial x} \tag{30}$$

Equations (29) and (30) form the governing equations of traffic flow, and constitute our new continuum theory for nonequilibrium traffic flow on highways.

3. EFFECTS OF HIGHER ORDER TERMS

In the derivation of the new theory, we neglected higher-order terms. There is concern, however, as to whether one can neglect higher-order terms when the concentration on a road is rapidly changing, as occurs near a shock path (Daganzo, 1995). We will show in this section that these higher-order terms can be neglected if the concentration function $\rho(x,t)$ is 'well-behaved' and temporal-spatial scales are properly treated.

Following the same procedures as in Section 2, we obtain in the following a 2nd order approximation for eqn (16):

$$\upsilon + T\frac{\mathrm{d}\upsilon}{\mathrm{d}t} = \upsilon_e(\rho) + \Delta\upsilon_e'(\rho)\frac{\partial\rho}{\partial x} + \frac{1}{2}\Delta^2\upsilon_e'2(\rho)\frac{\partial^2\rho}{\partial x^2} \tag{31}$$

which can be rearranged as:

$$\frac{\mathrm{d}\upsilon}{\mathrm{d}t} = \frac{1}{T}(\upsilon_e(\rho) - \upsilon) + \frac{\Delta}{T}\upsilon_e'(\rho)\frac{\partial\rho}{\partial x} + \frac{1}{2}T\left(\frac{\Delta}{T}\right)^2\upsilon_e'2(\rho)\frac{\partial^2\rho}{\partial x^2} \tag{32}$$

or

$$\frac{\mathrm{d}\upsilon}{\mathrm{d}t} = \frac{1}{T}(\upsilon_e(\rho) - \upsilon) + \upsilon_c\upsilon_e'\left(-\frac{\partial\rho}{\partial x} + \frac{1}{2}T\upsilon_c\upsilon_e'(\rho)\frac{\partial^2\rho}{\partial x^2}\right) \tag{33}$$

As $\rho(x, t)$ is a finite and suitably averaged* quantity that describes the physical distribution of vehicles on a road, it is reasonable to assume that $\rho(x, t)$ is a $C^n(n \geqslant 2)$ smooth function of bounded variation in x. Furthermore, if $\rho(x, t)$ is well-behaved, by which we mean

$$\frac{\frac{\partial^2\rho}{\partial x^2}}{\frac{\partial\rho}{\partial x}} < +\infty$$

then

$$\frac{\frac{1}{2}T\upsilon_c\upsilon_e'(\rho)\frac{\partial^2\rho}{\partial x^2}}{\frac{\partial\rho}{\partial x}} = O(T) + o\left(T^2\right)$$

which leads to

$$\frac{1}{2}T\upsilon_c\upsilon_e'(\rho)\frac{\partial^2\rho}{\partial x^2} = o\left(\frac{\partial\rho}{\partial x}\right)$$

for small T. The second order term can therefore be neglected as $T \to 0$. Following the same logic, one can show that the terms with orders higher than two are also negligible for well-behaved $\rho(x, t)$.

4. A QUALITATIVE EXAMINATION OF THE PROPERTIES OF THE NEW MODEL

The new model comprises two state equations: one is conservation of mass, and the other is the equation of motion, which is analogous to the conservation of momentum in fluid mechanics, but does not carry such a meaning in traffic flow. The conservation of mass is common to all continuum models; it is the equation of motion that distinguishes various continuum models. We shall examine some of the qualitative properties of the new theory for two typical cases: heavy and light traffic.

4.1. Heavy traffic

Considering $\rho \leqslant \rho_j$ and $|\upsilon_e'(\rho)| \leqslant c_m$ where c_m is a positive real number, and defining $C^2 = \rho_j c_m^2$, the behavior of (30) should be similar to the following model for heavy traffic flow, where ρ is large but $\frac{\partial\rho}{\partial x}$ is bounded:

$$\frac{\partial\upsilon}{\partial t} + \upsilon\frac{\partial\upsilon}{\partial x} = \frac{1}{T}(\upsilon_e(\rho) - \upsilon) - C^2\frac{\partial\rho}{\partial x} \tag{34}$$

*Due to the finiteness of vehicle length, $\rho(x, t)$ should be interpreted as the average intensity over a suitably chosen interval $[x - \Delta, x + \Delta]$.

It is noted that the first term on the right-hand side of (34) essentially constrains the system to behave as the LWR model, and the second term accounts for speed adjustment of a vehicle stream in anticipation of changing traffic conditions ahead.

If traffic becomes denser in front, the concentration gradient $\frac{\partial\rho}{\partial x}$ is positive, eqn (34) shows that vehicles tend to decelerate in anticipation of entering a denser stream; conversely, if traffic becomes lighter ahead, the concentration gradient $\frac{\partial\rho}{\partial x}$ is negative, and eqn (34) shows that vehicles tend to accelerate in anticipation of entering a lighter traffic stream. Where the change in concentration is drastic and $\frac{\partial\rho}{\partial x}$ therefore becomes large in magnitute, the change of speed will consequently be large, and the deviation of vehicle speed from equilibrium speed will be large, so is the speed correction (in the opposite direction as the anticipation correction) produced by the first term on the right-hand side of eqn (34). Conceptually the interaction between these two correction terms could lead to oscillatory behavior in the system and possibly explain the stop-and-go traffic phenomena that have eluded the grasp of transportation researchers for decades.

A remark needs to be made here: when the concentration gradient $\frac{\partial\rho}{\partial x}$ is not dominant, the effects of $\rho\left(\upsilon_e'(\rho)\right)^2$ should be treated as they are, which could capture the complexities of the traffic flow in transition regimes, where the effects of υ_c and $\upsilon_e'(\rho)$ are more prominent.

It should be noted that for the new model to have solutions in the strong sense (solutions that satisfy the PDEs literally), the concentration gradient has to be bounded; i.e. $|\frac{\partial\rho}{\partial x}| < \infty$. This is somewhat expected from its derivation—because of the anticipation and response delay introduced into the new theory, certain smoothness has been added to both the concentration and speed profiles. This is in fact more realistic than the discontinous solutions that the first order model produces, considering the finiteness of driver reaction times, vehicle speeds, and lengths. It is not fully clear whether the new model also allows weak (discontinuous) solutions. This issue will be explored in further analyses.

4.2. Light traffic

Now we analyze a simple case using our new traffic flow theory—traffic dynamics for non-crowded roads. For very light traffic, we have $\upsilon_e'(\rho) \approx 0$; therefore $\upsilon_e(\rho) \approx \upsilon_0$, and the new model is reduced to:

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\upsilon)}{\partial x} = 0 \tag{35}$$

$$\frac{\partial\upsilon}{\partial t} + \upsilon\frac{\partial\upsilon}{\partial x} = \frac{1}{T}(\upsilon_0 - \upsilon) \tag{36}$$

This model has been proposed by Ross (1988) as a new higher-order continuum model for all traffic conditions, and was critized by Newell (1989) for lacking a mechanism to model vehicle deceleration.

It is clear that in this light traffic model the speed equation is decoupled from the conservation equation and can therefore be solved independently.

Equation (36) is equivalent to

$$\frac{\mathrm{d}(\upsilon_0 - \upsilon)}{\upsilon_0 - \upsilon} = -\frac{\mathrm{d}t}{T} \tag{37}$$

which yields

$$\upsilon(x, t) = \upsilon_0 - (\upsilon_0 - g_0(x))e^{\frac{-t}{T}} \tag{38}$$

where $g_0(x)$ is the initial speed distribution of traffic on a road.

With eqn (38) the concentration $\rho(x, t)$ is the solution of the following PDE:

$$\frac{\partial\rho}{\partial t} + \upsilon\frac{\partial\rho}{\partial x} = -\frac{\partial\upsilon_0}{\partial x} + \mathrm{e}^{\frac{-t}{T}}\frac{\partial(\upsilon_0 - g_0(x))}{\partial x} \tag{39}$$

which is different from the solution obtained by assuming $\upsilon = \upsilon_e(\rho)$.

Examining eqn (38), it can be seen that when traffic is light ($\upsilon_e' \approx 0$) and no vehicle interaction occurs ($\upsilon_c = 0$), traffic eventually settles at the equilibrium speed υ_0 (which is also the free flow speed) regardless of the initial speed distribution. This accords with experimental observations and provides further evidence of the versatility of the new theory.

It is also clear that eqns (35) and (36) allow only acceleration movement in a traffic stream. It is therefore inapproriate to use this model to describe heavy traffic flow where interactions among vehicles axe prominent. A slight variation of the light traffic model, which is given below, might be a resonable approximation to the new model for heavy traffic conditions. This, however, needs futher analysis and is left for future work.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \upsilon)}{\partial x} = 0 \tag{40}$$

$$\frac{\partial \upsilon}{\partial t} + \upsilon \frac{\partial \upsilon}{\partial x} = \frac{1}{T}(\upsilon_e - \upsilon) \tag{41}$$

5. A COMPARISON WITH PAYNE'S MODEL

This paper is not complete without comparing the new theory with some existing higher-order continuum theories, and checking whether the new theory removes the flaws of existing theories. In the sections that follow, we shall compare our theory with Payne's theory, one of the most widely studied higher-order continuum models, and show that the new theory does not exhibit the 'wrong-way travel' behavior.

Before we proceed with our comparision, we simplify (30) for heavy traffic conditions, in which $\upsilon_e'(\rho) \approx -c$. (30) then becomes

$$\frac{\partial \upsilon}{\partial t} + \upsilon \frac{\partial \upsilon}{\partial x} = \frac{1}{T}(\upsilon_e(\rho) - \upsilon) - c^2 \rho \frac{\partial \rho}{\partial x} \tag{42}$$

Notice that the simplified equation of motion greatly resembles Payne's model. The major difference between these two models lies in the anticipation term—the anticipation in (42) is proportional to density but (10) is proportional to the inverse of density. This minor difference, however, has dramatic consequences for the behavior of these two models, as we shall next demonstrate.

Suppose that at a time instant t, the distribution of $\rho(x, t)$ is as shown in Fig. 5. We can always construct a *smooth* concentration distribution (at least initially) such that at one point the concentration ρ is a small number:

$$\rho = \varepsilon = o(1) > 0$$

and the concentration gradient at that point takes the value:

$$\frac{\partial \rho}{\partial x} = \frac{1}{\varepsilon}$$

If we substitute these values into both Payne's model and our model, we have respectively:

$$\frac{\mathrm{d}\upsilon}{\mathrm{d}t} = -\frac{\upsilon - \upsilon_e(\rho)}{T} - \frac{c_0^2}{\varepsilon^2} \text{ (Payne's model)} \tag{43}$$

$$\frac{\mathrm{d}\upsilon}{\mathrm{d}t} = -\frac{\upsilon - \upsilon_e(\rho)}{T} - c^2 \text{ (the new model)} \tag{44}$$

It is clear that the second term in the RHS of reduced Payne's model approaches $-\infty$ when $\varepsilon \to 0$, while that of our model is finite. To keep the acceleration/deceleration finite in Payne's model, one needs an infinite negative speed to balance the infinite concentration gradient ('wrong-way travel'), which is one of the root causes of many reported numerical problems with this model.

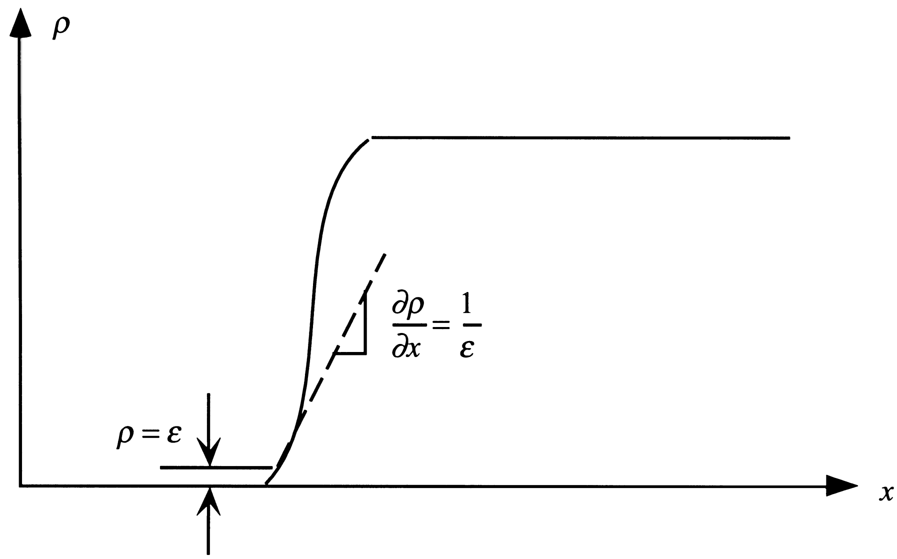

Fig. 5. An example concentration distribution.

Why, we may ask, does the LWR theory not exhibit 'wrong-way' travel as Payne's model does with an infinite concentration gradient? The answer to this question lies in the treatment of scale: Payne's model uses relaxation in a finite time T to restore traffic flow to its equilibrium states (which requires a large negative speed), while the LWR theory forces the traffic flow to change its speed to the equilibrium speed in *zero* time (which requires an infinite deceleration). Although infinite deceleration is as unrealistic as negative speed, it is not as severe a problem to traffic flow modeling as negative speed. The LWR theory can therefore still provide qualitatively correct predictions under extreme conditions, while Payne's theory cannot.

6. THE ANISOTROPIC PROPERTY OF TRAFFIC FLOW

This section discusses the anisotropic property of traffic flow, where an example constructed by Daganzo (1995) will be used to illustrate the essential ideas. This example is given by the following initial conditions:

$$\upsilon = 0, \rho = \rho_j H(x); \forall x \leqslant A, t = 0 (A > 0) \tag{45}$$

$$\upsilon = 0; x = A, t > 0 \tag{46}$$

where $H(x)$ is a Heaviside unit step function* (Fig. 6).

This initial condition depicts a traffic scenario where there is initially stopped traffic (say in front of a signal) and no vehicles join the queue afterwards (say due to an eternal blocking upstream). The correct equilibrium solution to this problem is therefore the initial condition itself.

Daganzo (1995) showed that any higher-order continuum model derived from the following form (with zero relaxation time)

$$\upsilon(x, t) = \upsilon_e(\rho) - \frac{\mu}{\rho}\frac{\partial\rho}{\partial x} \tag{47}$$

moves vehicles back from the stopped queue at large times t, as shown in Fig. 6.

This is not difficult to see because $\upsilon(x, t) \geqslant 0$ requires

$$\frac{\partial\rho}{\partial x} \leqslant \frac{\rho\upsilon_e(\rho)}{\mu} = \frac{q}{\mu}$$

*One might argue that Heaviside-type density distributions do not exist in reality, because p is a spatial average of concentration, and vehicles have finite length. This mathematical abstraction is justified, however, when one is interested in the general properties of shock propagation, not the detailed structure of the shock itself, and the length of the region where concentration changes drastically is much shorter than that of the road.

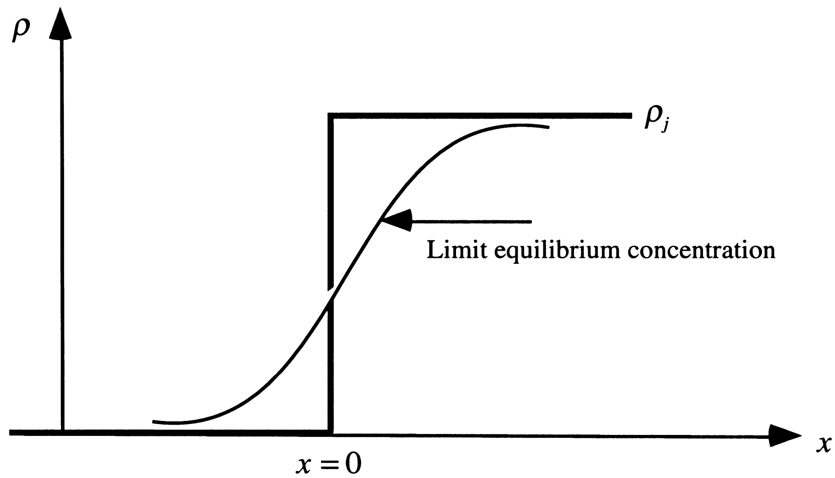

Fig. 6. Incorrect equilibrium solution of the stopping jam problem.

a finite concentration gradient. Any concentration gradient that is greater than $\frac{q}{\mu}$ leads to negative travel speed (wrong-way travel) for increasing concentration. There are two ways to avoid the wrong-way travel problem: (1) inserting a discontinuity at an appropriate location, as the LWR theory does, and (2) allowing the relaxation time T or diffusion constant μ approaches zero as the concentration gradient approaches infinity.

Note that if one allows for $\mu \to 0$ rather than uses $\mu = -\frac{v_e'}{2T}$, then one would have discontinuous equilibrium solution for Model (47). Next we use a limit argument to demonstrate how one can construct a smooth non-equilibrium concentration profile that approaches the discontinuous equilibrium profile when relaxation time goes to zero, without moving vehicles back.

Consistent with our assumption (i.e. zero relaxation time), we have $T \to 0$. Furthermore we argue that the anticipation effect is dominant in traffic flow when the concentration gradient is large, then (30) can be approximated by:

$$v = v_e - T\rho\left(v_e'\right)^2 \rho_x \tag{48}$$

which can be rewritten as

$$\frac{\partial \rho}{\partial x} = \frac{v_e - v}{T\rho\left(v_e'\right)^2} \tag{49}$$

Rather than starting from the initial condition shown in Fig. 6, we imagine that there is an infinitesimally small amount of flow joining the stopped queue such that there is a smooth transition concentration curve in the region $(-\varepsilon, 0)$ (Fig. 7). We argue that v_e is slightly greater than v because of a braking wave, and let $\varepsilon = v_e - v$ be the small speed difference in the transition region $(-\varepsilon, 0)$. Because the product of $T \to 0$ and $\rho \to 0$ is of higher order than $\varepsilon \to 0$, we conclude $\frac{\partial \rho}{\partial x} \to +\infty$ as $\varepsilon \to 0$ (or equivalently $x_- \to 0$). On the other hand, $v_e = v$ in the regions $(-\infty, -\varepsilon) \bigcup (0, \infty)$, leading to $\frac{\partial \rho}{\partial x} = 0$. Combining these two arguments, we conclude that the limit solution of the new theory in its approximate form approaches the initial condition as $T, \varepsilon \to 0$ and $t \to \infty$.

It is important to note that the treatment of scale is critical to the solution of traffic problems with discontinuities. One important fact is that the propagation speed of a disturbance in a traffic stream is always finite. One therefore cannot let the spatial scale of response approach zero and still keep relaxation time finite.* Such a practice will lead to strange behavior such as negative flow at a backward moving shock. This also has significant implications in the numerical solutions of higher-order continuum models, which will be investigated in future research.

*It should be noted that as a general rule drivers reaction time (a part of the relaxation time) is not infinitesmially small, one therefore would expect that observed shock profiles are smooth and have a finite width. Such a smooth shock profile can be approximated by a discontinuous one when one is not interested in the detailed shock structure but the general propagation characteristics of the shock. To make the first approximation self-consistent, a corresponding approximation that the relaxation time be zero is also required.

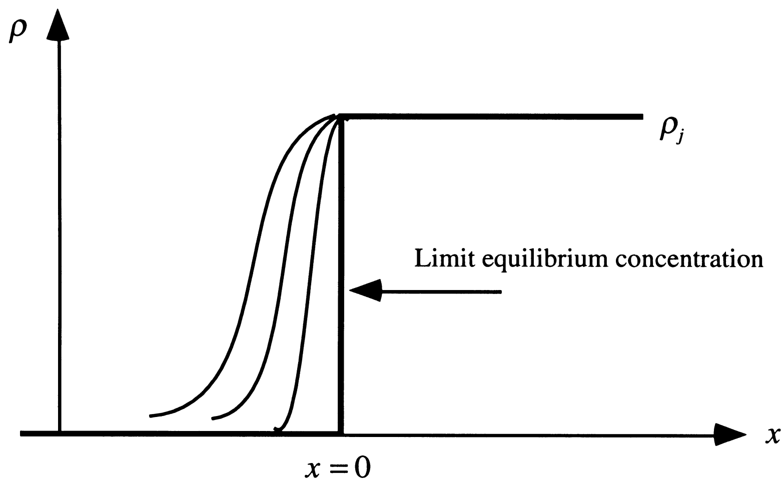

Fig. 7. Obtaining the discontinuous equilibrium solution as a limit of smooth solutions.

7. CONCLUDING REMARKS

The continuum theory of Lighthill and Whitham, and Richards, although it works rather well under many circumstances, has two serious deficiencies: it lacks a mechanism for traffic to accelerate or decelerate at a finite speed when the concentration gradient is large, and its assumption that the equilibrium speed–concentration relationship holds for nonequilibrium speed and concentration observations. The 'improved' models that attempt to remove these deficiencies, however, introduce a new flaw—wrong-way travel. We have in this paper presented an enhanced continuum theory. The derivation of this new theory is based on the assumptions that drivers adjust their vehicle speeds according to traffic conditions ahead of them, that the disturbance propagation speed (relative to moving vehicles) in the nonequilibrium theory is proportional to that of the LWR theory, and that an unique speed–concentration relationship exists for equilibrium traffic flow. We have shown that the new theory includes the LWR theory as a special case and removes certain deficiencies of that theory without introducing the undesirable property of wrong-way travel.

REFERENCES

Cremer, M. and May, A. D. (1986) An extended traffic flow model for inner urban freeways. *Preprints 5th IFAC/IFIP/IFORS International Conference on Control in Transportation Systems*, pp. 383–388. Vienna, Austria.

Daganzo, C. (1995) Requiem for second-order approximations of traffic flow. *Transportation Research-B* **29**(4), 277–286.

Derzko, N. A., Ugge, A. J. and Case, E. R. (1983) Evaluation of dynamic freeway flow model by using field data. *Transportation Research Record* **905**, 52–60.

Gazis, D. C, Herman, R. and Rothery, R. W. (1961) Nonlinear follow-the-leader models of traffic flow. *Operations Research* **9**(4), 545–567.

Hauer, E. and Hurdle, V. F. (1979) Discussions in Payne, H. J., FREFLO: a macroscopic simulation model of freeway traffic. *Transportation Research Record* **722**, 75–76.

Kühne, R. D. (1984) Macroscopic freeway model for dense traffic—stop-start waves and incident detection. *Ninth International Symposium on Transportation and Traffic Theory*, pp. 20–42.

Kühne, R. D. (1989) Freeway control and incident detection using a stochastic continuum theory of traffic flow. Proceedings 1st International Conference on Applied Advanced Technology in Transportation Engineering, San Diego, CA, pp. 287–292.

Leo, C. J. and Pretty, R. L. (1992) Numerical simulations of macroscopic continuum traffic models. *Transportation Research-B* **26**(3), 207–220.

LeVeque, R. J. (1992) Numerical methods for conservation laws, 2nd edn. Lectures in Mathematics ETH Zurich, Birkhauser Verlag, Basel, Switzerland.

Lighthill, M. J. and Whitham, G. B. (1955) On kinematic waves: II. a theory of traffic flow on long crowded roads. *Proceedings Royal Society, London, Series A* **229**(1178)), 317–345.

Michalopoulos, R, Yi, P. and Lyrintzis, A. S. (1993) Continuum modeling of traffic dynammics for congested freeways. *Transportation Research-B* **27**(4), 315–332.

Newell, G. F. (1989) Comments on traffic dynamics. *Transportation Research-B* **23**, 386–389.

Papageorgiou, M. (1983) Applications of automatic control concepts to traffic flow modeling and control. In *Lecture Notes in Control and Information Sciences*, eds A. V. Balakrishnan and M. Thoma. Spring-Verlag, Berlin.

Papageorgiou, M., Blosseville, J. and Hadj-Salem, H. (1990) Modeling and real time control of traffic flow on the southern part of Boulevard Peripherique in Paris: Part I: modeling. *Transportation Research-A* **24**(5), 345–359.

Payne, H. J. (1971) Models of freeway traffic and control. *Simulation Councils Proc. Series: Mathematical Models of Public Systems*, Vol 1, No. 1, ed. George A. Bekey.

Pipes, L. A. (1969) Vehicle acceleration in the hydrodynamic theory of traffic flow. *Transportation Research* **3**, 229–234.

Richards, P. I. (1956) Shock waves on the highway. *Operations Research* **4**, 4251.

Ross, P. (1988) Traffic dynamics. *Transportation Research-B* **22**(6), 421–435.

Treiterer, J. and Myers, J. A. (1974) The hysteresis phenomena in traffic flow. In *Proceedings of the Sixth Symposium on Transportation and Traffic Theory*, ed. D. J. Buckley, pp. 13–38.