1 ARZ model

Consider the full nonlinear ARZ model without relaxation [1, 2]:

$$\rho_t + (\rho v)_x = 0, \tag{1}$$
$$(v - V(\rho))_t + v(v - V(\rho))_x = 0. \tag{2}$$

2 Explicit numerical schemes

2.1 Godunov

The Godunov scheme requires a Riemann solver. To apply this scheme we first write the system in conservation form. The model is [3]:

$$U_t + [F(U)]_x = 0, \tag{3}$$

with conserved variables $U = \begin{pmatrix} \rho \\ y \end{pmatrix}$ and flux vector $F(U) = \begin{pmatrix} y + \rho V(\rho) \\ \dfrac{y^2}{\rho} + yV(\rho) \end{pmatrix}$.

The scheme is

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[F(u^*(U_j^n, U_{j+1}^n)) - F(u^*(U_{j-1}^n, U_j^n))\right], \tag{4}$$

where $u^*(U_j^n, U_{j+1}^n)$ is the solution to the Riemann problem with initial data U_j^n and U_{j+1}^n. The Riemann problem solution is discussed in [3].

2.2 Lax-Friedrichs

The Lax-Friedrichs scheme can be applied to equations in the form of (3) as well, but a Riemann solver is not needed. The partial derivatives are approximated using a forward difference in time and central difference in space, and U_j^n is replaced with its spatial average. The scheme is:

$$U_j^{n+1} = \frac{1}{2}[U_{j+1}^n + U_{j-1}^n] - \frac{\Delta t}{2\Delta x}[F(U_{j+1}^n) - F(U_{j-1}^n)]. \tag{5}$$

This method is first-order accurate and very dissipative [4]. The stencil for this method is shown in Figure 1.

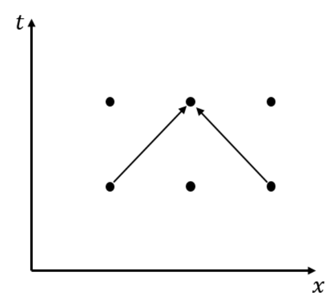

Figure 1: Stencil for Lax-Friedrichs method.

2.3 Lax-Wendroff

We look at a higher-order scheme with less numerical dissipation. For nonlinear conservation laws, the Lax-Wendroff scheme is:

$$\begin{aligned} U_j^{n+1} = U_j^n - \frac{\Delta t}{2\Delta x}[F(U_{j+1}^n) - F(U_{j-1}^n)] + \frac{(\Delta t)^2}{2(\Delta x)^2}[A_{j+1/2}(F(U_{j+1}^n) \\ - F(U_j^n)) - A_{j-1/2}(F(U_j^n) - F(U_{j-1}^n))], \end{aligned} \tag{6}$$

where $A_{j\pm 1/2}$ is the Jacobian matrix $F'(\cdot)$ evaluated at $\frac{1}{2}(U_j^n + U_{j\pm 1}^n)$. Evaluating the Jacobian matrix makes this method more expensive to use so Richtmyer proposed a two-step procedure to avoid using A. In the first step $u(x,t)$ is calculated at half time and space steps. In the second step these values are used to compute the solution at the next time step.
Richtmyer's two-step Lax-Wendroff method is:
First step:

$$U_{j+1/2}^{n+1/2} = \frac{1}{2}(U_{j+1}^n + U_j^n - \frac{\Delta t}{2\Delta x}[F(U_{j+1}^n - F(U_j^n)] \tag{7}$$

$$U_{j-1/2}^{n+1/2} = \frac{1}{2}(U_j^n + U_{j-1}^n - \frac{\Delta t}{2\Delta x}[F(U_j^n - F(U_{j-1}^n)] \tag{8}$$

Second step:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}[F(U_{j+1/2}^{n+1/2}) - F(U_{j-1/2}^{n+1/2})] \tag{9}$$

This method is second-order accurate and less dissipative than the Lax-Friedrichs method. However, as the Lax-Wendroff method does not preserve monotonicity, it produces oscillations near discontinuities [4]. The stencil for this method is shown in Figure 2.

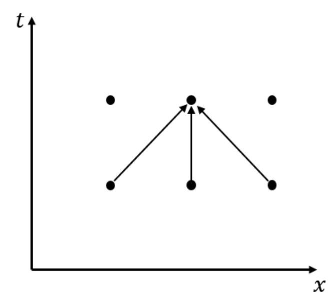

Figure 2: Stencil for Lax-Wendroff method.

2.4 ENO and WENO

The ENO (essentially non-oscillatory) schemes were developed by Harten, Engquist, Osher, and Chakravarthy to solve the problem of finding higher-order schemes that do not produce oscillations near discontinuities [4]. The idea is to use a high-degree polynomial to interpolate the solution U, then compute $[F(U)]_x$. The stencil is chosen depending on the upwind direction. Points added for higher-order polynomials are chosen so that the interpolant has the least oscillation.

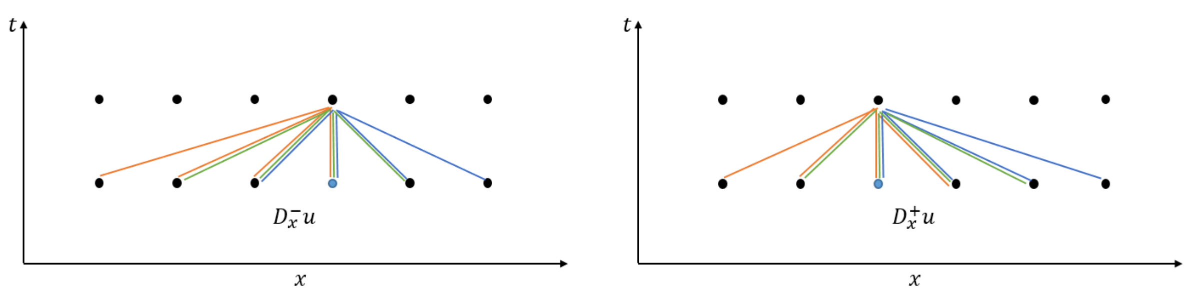

Figure 3: Possible stencils for ENO represented by different colors. $D_x^- u$ is used if information propagates from left to right and $D_x^+ u$ is used if information propagates from right to left.

The WENO (weighted ENO) scheme, introduced by Liu, Osher, and Chan [5], uses a convex combination approach rather than picking the smoothest stencil in order to achieve the ENO property. Instead of using stencils as in ENO, WENO uses a weighted combination of higher-order reconstructions to approximate U. The weights depend on a smoothness indicator which estimates the smoothness of the solution.
Both ENO and WENO schemes discretize in space. Essentially these schemes reconstruct U, then compute $[F(U)]_x$. TVD Runge-Kutta schemes can be used to solve in time for the solution.

3 Implementation of explicit schemes

A comparison of the scheme with actual data is shown below.

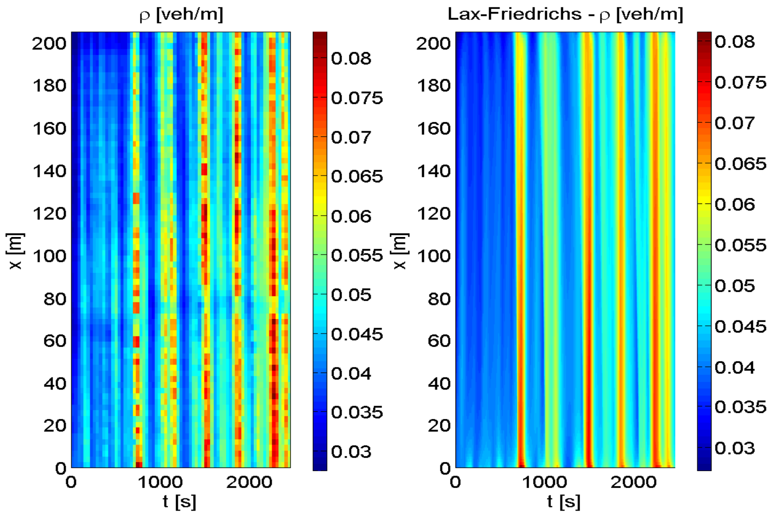

Figure 4: Lax-Friedrichs scheme.

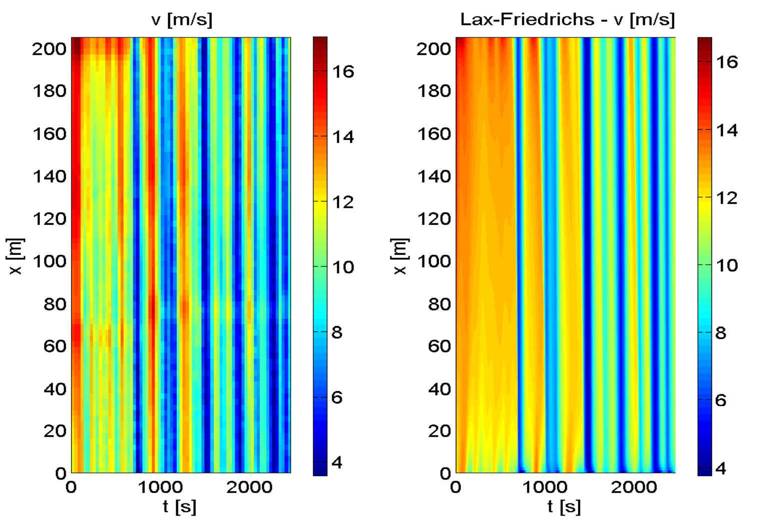

Figure 5: Lax-Friedrichs scheme.

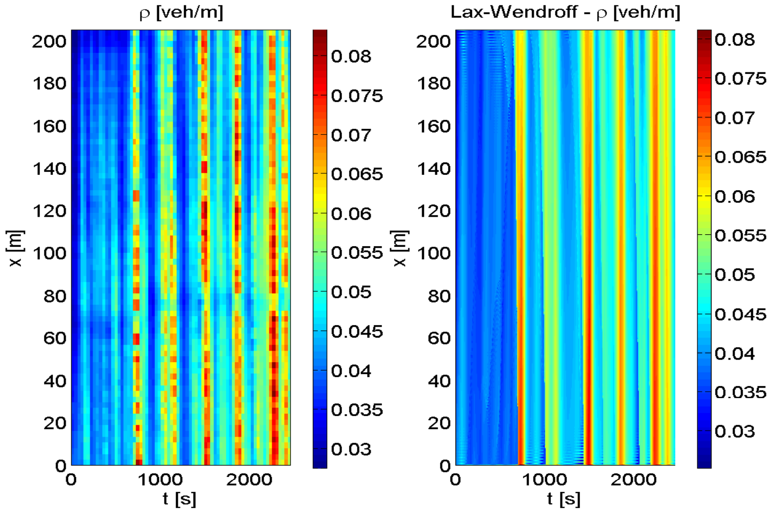

Figure 6: Lax-Wendroff scheme.

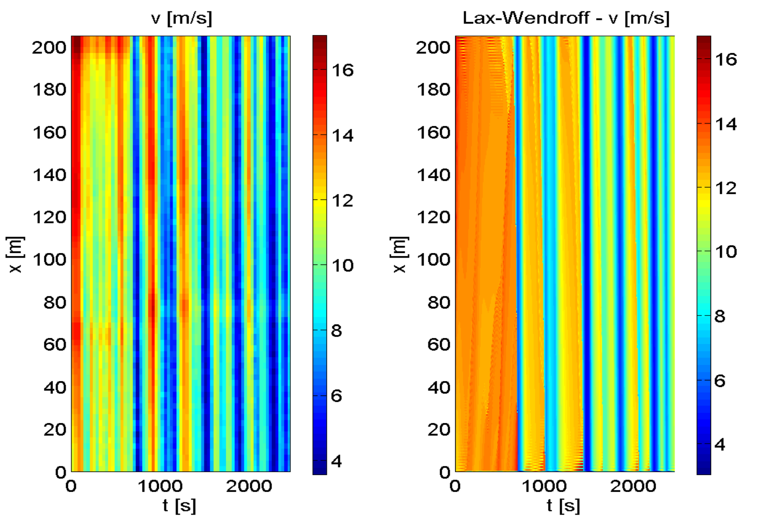

Figure 7: Lax-Wendroff scheme.

4 Implicit schemes

We apply the following implicit schemes on the linearized ARZ equation.

4.1 Crank-Nicolson

For a PDE system of the form $U_t + AU_x = 0$, the Crank-Nicolson scheme is

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = -\frac{A}{2}(D_x^0 u_j^n + D_x^0 U_j^{n+1}). \tag{10}$$

Letting $R = A\frac{\Delta t}{\Delta x}$, we can rearrange to write

$$\frac{1}{4}RU_{j+1}^{n+1} + U_j^{n+1} - \frac{1}{4}RU_{j-1}^{n+1} = -\frac{1}{4}RU_{j+1}^n + U_j^n + \frac{1}{4}RU_{j-1}^n. \tag{11}$$

We can also write

$$
\underbrace{\begin{bmatrix} 1 & \frac{1}{4}R & 0\cdots \\ -\frac{1}{4}R & \ddots & \ddots \\ 0 & \ddots & \\ \vdots & & \end{bmatrix}}_{A_1} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -\frac{1}{4}R & 0\cdots \\ \frac{1}{4}R & \ddots & \ddots \\ 0 & \ddots & \\ \vdots & & \end{bmatrix}}_{A_2} \begin{bmatrix} U_1^{n} \\ U_2^{n} \\ U_3^{n} \\ \vdots \end{bmatrix}. \tag{12}
$$

Then solving the scheme essentially involves inverting a matrix:

$$
\mathbf{U}^{n+1} = A_1^{-1} A_2 \mathbf{U}^n. \tag{13}
$$

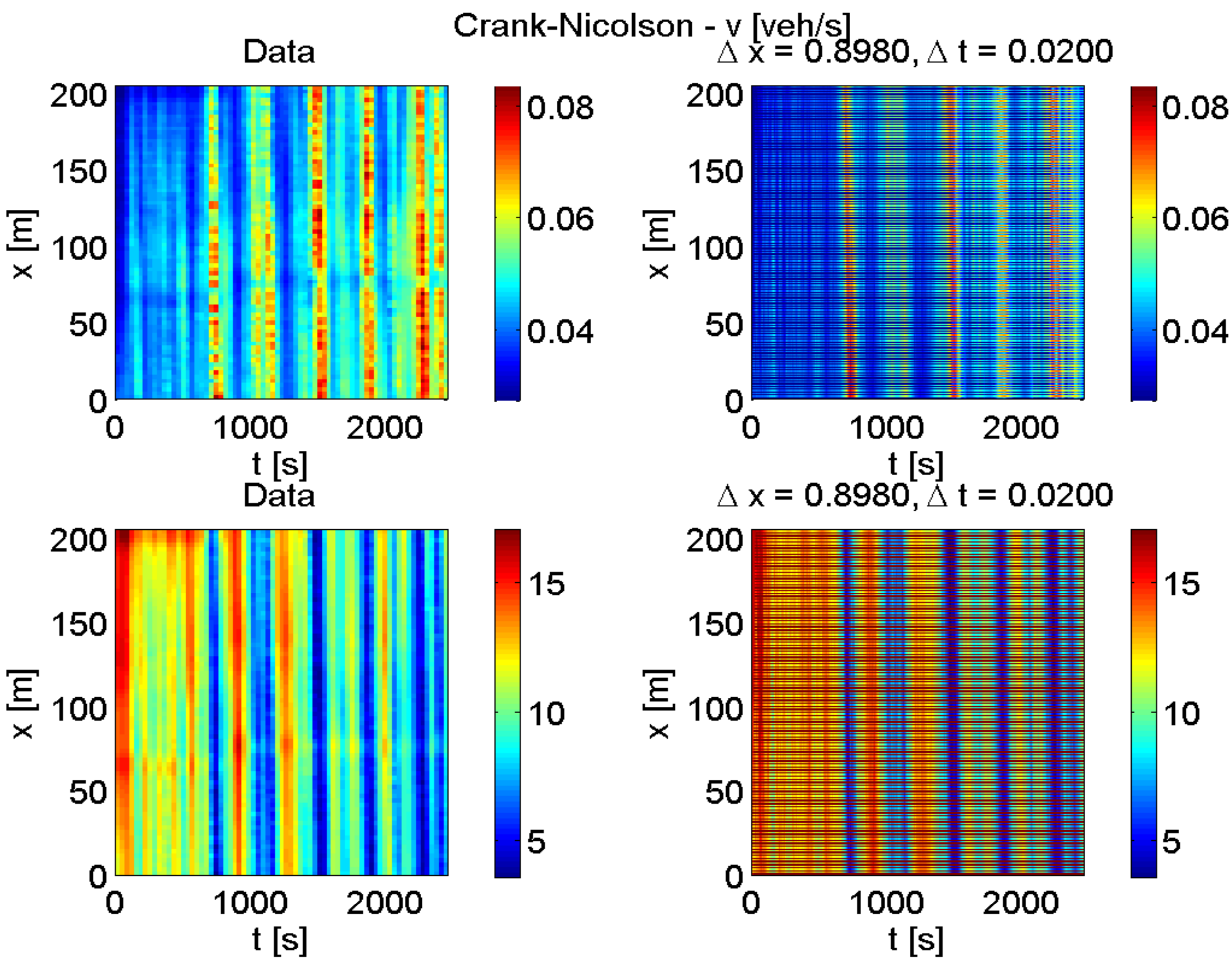

Figure 8: Although the Crank-Nicolson scheme is stable, it produces spurious oscillations.

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