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# NONLINEAR EFFECTS IN THE DYNAMICS OF CAR FOLLOWING

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It is assumed that the velocity of a car at time $t$ is some (nonlinear) function of the spacial headway at time $t-\Delta$, so the equations of motion for a sequence of cars consists of a set of differential-difference equations. There is a special family of velocity-headway relations that agrees well with experimental data for steady flow, and that also gives differential equations which for $\Delta=0$ can be solved explicitly. Some exact solutions of these equations show that a small amplitude disturbance propagates through a series of cars in the manner described by linear theories except that the dependence of the wave velocity on the car velocity causes an accleration wave to spread as it propagates and a deceleration wave to form a stable shock. These conclusions are then shown to hold for quite general types of velocity-headway relations, and to yield a theory that in certain limiting cases gives all the results of the linear car-following theories and in other cases all the features of the nonlinear continuum theories, plus a detailed picture of the shock structure.

A NUMBER of papers[1-10] have now been published dealing with car following models in which the position $x_j(t)$ of a $j$th car at time $t$ is related to the position of the car ahead of it [the $(j-1)$th car] through a linear equation of the type

$$L_j\, x_j(t) = M_j\, x_{j-1}(t),$$

in which $L_j$ and $M_j$ represent suitably chosen linear differential-difference operators. Although these models have been very useful for describing the propagation and stability of small disturbances traveling through a line of cars, they must be interpreted only as linear approximations to more basic equations which are nonlinear. As has been pointed out by Gazis, Herman, and Potts,[11] a nonlinear model is necessary at least to explain the known fact that the steady-state relation between average velocity and average headway is a nonlinear one.

As yet very little has been done to explore the potential of nonlinear models. Some consequences of introducing into the car-following equations terms quadratic in the velocities have been investigated by Kometani and Sasaki,[12] and some effects of imposing finite bounds on velocities have been studied by Newell.[13,14] The continuum theories of Lighthill and Whitham[15] and Richards,[16] which should represent approximations for some kind of car-following theory, are also nonlinear, and because of this have led to the prediction of shocks. One of the early papers on traffic theory by Herrey and Herrey[17] also describes some effects that one would describe as being nonlinear.

Here we consider some further consequences of nonlinearities wherein we postulate that the velocity of the $j$th car at time $t$, $v_j(t)$, is some nonlinear function of the headway at time $t-\Delta$, i.e.,

$$v_j(t)=G_j[x_{j-1}(t-\Delta)-x_j(t-\Delta)] \qquad (1)$$

for some appropriate functions $G_j$. This will include as special cases most of the car-following models proposed so far, and will also yield in certain

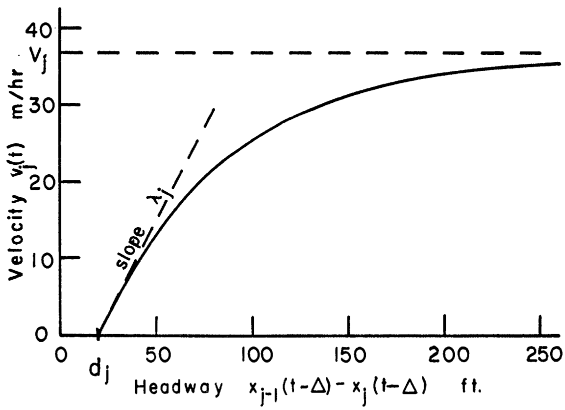


**Fig. 1.** The velocity-headway relation represented by equation (2).

limiting situations all the results of the continuum theories. We first treat a special functional form for $G_j$, which has the virtues of being physically reasonable and also of giving differential equations that can be solved explicitly at least for $\Delta=0$. We then show that the conclusions deduced from this special model are also valid qualitatively for a large class of functions $G_j$.

## A SPECIAL MODEL

Suppose that the function $G_j$ in (1) is chosen so that

$$v_j(t)=V_j-V_j\exp\{-\lambda_j V_j^{-1}[x_{j-1}(t-\Delta)-x_j(t-\Delta)-d_j]\} \qquad (2)$$

[provided that $v_j(t)\geqq 0$] with $V_j$, $\lambda_j$, and $d_j$ parameters associated with the $j$th car. This function is shown in Fig. 1 where one can see that $V_j$ represents the maximum velocity or free speed of the $j$th car, $d_j$ the minimum headway, and $\lambda_j$ the slope at $v_j(t)=0$. No motivation for this choice is proposed other than the claim that it has approximately the correct shape and is reasonably simple.

For a sequence of identical cars ($\lambda_j=\lambda$, $d_j=d$, and $V_j=V$ for all $j$) equation (2) yields a time-independent solution if the headway

$$x_{j-1}(t-\Delta)-x_j(t-\Delta)\equiv\rho^{-1}$$

is chosen independent of both $j$ and $t$ ($\rho$ represents the density of cars) and $v_j(t)=v(\rho)$ is evaluated from (2). Figure 2, broken line, shows this

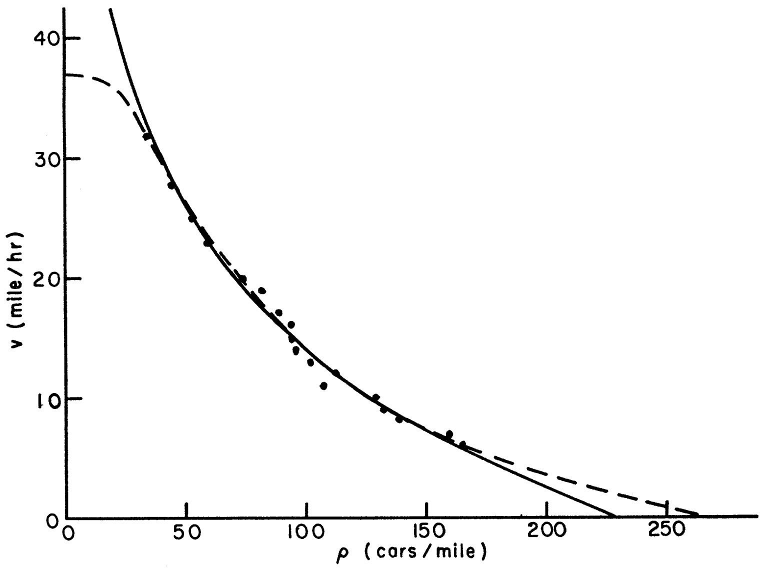


**Fig. 2.** The points represent the experimental points quoted by Greenberg and taken in the Lincoln Tunnel. The solid line is a fit to these data based upon Greenberg's theory and the broken line is obtained by fitting the present theory.

relation when $V=37$ miles/hr., $d=20$ ft., and $\lambda=0.79$ sec$^{-1}$ are chosen to fit the experimental data quoted by GREENBERG[18] for the average velocities and densities in the Lincoln Tunnel. Over the range of densities for which data are available, this curve is nearly identical with the theoretical curve, solid line, proposed by Greenberg, but differs from it outside this range. The present theory, however, has the advantage of giving finite velocities for $\rho\to 0$.

Before attempting to find time-dependent solutions of (2), we can eliminate the parameter $d_j$ from the equation through a substitution

$$y_j(t)=x_j(t)+\textstyle\sum_{k=1}^{k=j} d_k. \tag{3}$$

In effect we are cutting out of the highway an interval $d_j$ ahead of each $j$th car. Equation (2) then becomes

$$v_j(t)=dy_j(t)/dt=V_j-V_j\exp\{-\lambda_j V_j^{-1}[y_{j-1}(t-\Delta)-y_j(t-\Delta)]\}. \quad (4)$$

Although it is possible to construct the complete time-dependent solution of (4) for any $\Delta$, the exact solution is quite awkward to analyze. The solutions for $\Delta=0$, however, are relatively simple and since for $\Delta$ small enough to guarantee nonoscillatory transients,[3,5] $\Delta\lambda_j<1/e$, the solutions for $\Delta\neq 0$ are not expected to differ qualitatively from those for $\Delta=0$, we consider the case $\Delta=0$ first.

If, for $\Delta=0$, we define

$$z_j(t)=\exp[-\lambda_j V_j^{-1} y_j(t)] \quad (5)$$

so that

$$v_j(t)=dy_j(t)/dt=[-V_j/\lambda_j z_j(t)]\,dz_j(t)/dt, \quad (6)$$

substitution in (4) gives for $z_j(t)$ the linear differential equation

$$\lambda_j^{-1}\,dz_j(t)/dt+z_j(t)=[z_{j-1}(t)]^{\mu(j,j-1)} \quad (7)$$

with

$$\mu(j,j-1)=\lambda_j V_j^{-1}\lambda_{j-1}^{-1} V_{j-1}. \quad (8)$$

The formal solution of (7) is

$$z_j(t)=\lambda_j\exp(-\lambda_j t)\int_0^t dt'\,\exp(\lambda_j t')[z_{j-1}(t')]^{\mu(j,j-1)}+z_j(0)\exp(-\lambda_j t), \quad (9)$$

but this only gives $z_j(t)$ in terms of $z_{j-1}(t)$ and the initial conditions $z_j(0)$. To find $z_j(t)$ as a function of the motion of some lead car $z_0(t)$, one must iterate this formula.

The difficulty in iterating (9) stems from the fact that if $\mu(j,j-1)\neq 1$, equation (9) considered as a finite difference equation relative to $j$ is still nonlinear and even if only $\lambda_j$ varies with $j$, the difference equation is one with variable coefficients. The first complication can be removed if we choose $\lambda_j V_j^{-1}$ independent of $j$ and the second by choosing both $\lambda_j$ and $V_j$ independent of $j$.

If we permit $d_j$, $\lambda_j$, and $V_j$ all to vary independently with $j$, there appears to be only one class of solutions having a simple iteration, namely the trivial case in which all cars travel with the same constant velocity. This solution has the form

$$z_j(t)=B_j\exp(-\kappa_j t), \quad (10a)$$

with

$$\kappa_j=\kappa_0\lambda_j V_j^{-1}\lambda_0^{-1}V_0, \quad (10b)$$

and

$$B_j=\prod_{k=1}^{k=j}\lambda_k(\lambda_k-\kappa_k)^{-1}. \quad (10c)$$

Equation (10a) guarantees that the velocity $\kappa_j V_j/\lambda_j$ is independent of $t$, (10b) that it is independent of $j$, and (10c) that the headways are consistent with (2) for the given velocity.

## IDENTICAL CARS

IF WE postulate that $\lambda_j=\lambda$ and $V_j=V$ are the same for all cars (since $d_j$ is no longer in the equations we may still allow it to vary from car to car), then $\mu(j, j-1)=1$ and we may use $\lambda^{-1}$ as a unit of time through the substitution

$$\tau=\lambda t. \tag{11}$$

Equation (7) then becomes

$$dz_j(\tau)/d\tau+z_j(\tau)=z_{j-1}(\tau), \tag{12}$$

which is exactly the same equation as appears in the theory of REUSHEL [1] and PIPES [2] except that in these theories $z_j$ would be interpreted as the velocity.

Equation (12) can be solved in any number of ways. If, for example, we let

$$G(p,\ \tau)=\textstyle\sum_{j=1}^{j=\infty} z_j(\tau)\ p^j \tag{13}$$

be the generating function for the $z_j(\tau)$, then (12) gives for $G(p,\ \tau)$ the differential equation

$$dG(p,\tau)/d\tau+(1-p)\ G(p,\tau)=p\ z_0(\tau),$$

which has a solution

$$G(p,\tau)=\exp[-(1-p)\tau]\{G(p,0)+p\int_0^\tau d\tau'\exp[(1-p)\tau']\,z_0(\tau')\}.$$

The expansion of this in powers of $p$ gives $z_j(\tau)$,

$$z_j(\tau)=e^{-\tau}\sum_{n=0}^{j-1}\frac{\tau^n}{n!}z_{j-n}(0)+\int_0^\tau d\tau'\,\frac{(\tau')^{j-1}}{(j-1)!}z_0(\tau-\tau'), \tag{14}$$

in terms of the initial conditions $\{z_k(0)\}$ and the trajectory $z_0(\tau)$ of the lead car.

To illustrate the nature of this solution, we consider what happens if all cars travel with some constant velocity, which we designate by $(1-\alpha)V$, until time 0 and then the lead car suddenly changes velocity to a value $(1-\beta)V$. If we choose the origin of the spatial coordinates so that $y_0(0)=x_0(0)=0$, then according to (5) and (10) the conditions at time 0 are

$$z_j(0)=\alpha^{-j},$$

and the trajectory of the lead car is

$$z_0(\tau)=\exp[-(1-\beta)\ \tau]\quad \text{for } \tau>0.$$

Equation (14) then gives the solution

$$z_j(\tau)=\frac{e^{-(1-\alpha)\tau}\,\Gamma(j,\alpha\tau)}{\alpha^j\,\Gamma(j)}+\frac{e^{-(1-\beta)\tau}\,\gamma(j,\beta\tau)}{\beta^j\,\Gamma(j)}, \tag{15}$$

in which

$$\Gamma(a,b)=\int_b^\infty dt\, e^{-t}\, t^{a-1}, \qquad \gamma(a,b)=\int_0^b dt\, e^{-t}\, t^{a-1} \tag{16}$$

are the incomplete gamma functions and $\Gamma(a)=\Gamma(a,0)$ the complete gamma function.[19,20] The velocity of the $j$th car obtained from (6) is

$$v_j(\tau)=[1-\tfrac{1}{2}(\alpha+\beta)]\,V+\tfrac{1}{2}(\beta-\alpha)\,V\,(1-J)/(1+J), \tag{17}$$

with

$$J=\frac{(\tau\beta)^{-j}\,e^{\beta\tau}\,\gamma(j,\beta\tau)}{(\tau\alpha)^{-j}\,e^{\alpha\tau}\,\Gamma(j,\alpha\tau)}=\frac{(\tau\beta)^{-j}[e^{\beta\tau}-\sum_{k=0}^{j-1}\beta^k\,\tau^k/k!]}{(\tau\alpha)^{-j}\sum_{k=0}^{j-1}\alpha^k\,\tau^k/k!} \tag{18}$$

If we consider very small changes in velocity by letting $\beta\to\alpha$ in (17) and keep only the first order terms in $(\beta-\alpha)$, the results should be equivalent to those given by a linear theory in which the velocity-headway relation of Fig. 1 is replaced by its tangent line at the velocity $(1-\alpha)V$. In this limit

$$J\to\gamma(j,\alpha\tau)/\Gamma(j,\alpha\tau),$$

$$v_j(\tau)\to(1-\alpha)\,V+(\beta-\alpha)\,V\,\gamma(j,\alpha\tau)/\Gamma(j), \tag{19}$$

$$dv_j/d\tau\to(\beta-\alpha)\,V\,\alpha\,e^{-\alpha\tau}\,(\alpha\tau)^{j-1}/(j-1)!. \tag{20}$$

With suitable change of notation, equations (19) and (20) are equivalent to those derived by Pipes for impulse acceleration or deceleration of the lead car provided that we take the slope of the linear velocity-headway relation in Pipes' theory to be the slope $\alpha\lambda$ of the curve in Fig. 1 at the velocity $(1-\alpha)V$.

The acceleration of the $j$th car, (20), has its maximum value when $\tau=(j-1)/\alpha$. Thus the peak acceleration propagates through the line of cars with a time lag of $(\alpha\lambda)^{-1}$ per car measured in real units of time. For large values of $j$ with $\tau-(j-1)/\alpha=O(j^{1/2})$,

$$\frac{dv_j}{d\tau}\sim\frac{(\beta-\alpha)\,V\alpha}{[2\pi(j-1)]^{1/2}}\exp\left[-\frac{(\alpha\tau-j+1)^2}{2(j-1)}\right], \tag{21}$$

which implies that the magnitude of the peak acceleration decreases as $(j-1)^{1/2}$ and the effective duration of the disturbance increases as $(j-1)^{1/2}$. These results are equivalent to those described by Herman et al. (reference 6, p. 97) if in their theory one sets $\Delta=0$ and again chooses the slope of the velocity-headway curve to be $\alpha\lambda$. The nonzero time lag also gives rise to a finite velocity of propagation for the acceleration wave in agreement with the continuum theories. As described in reference 14, the wave velocity is the intercept at the velocity axis in Fig. 1 of the tangent line to the velocity-headway curve at the velocity $(1-\alpha)V$. In contrast with the time lag $(\alpha\lambda)^{-1}$, the wave velocity depends also upon the parameters $d_j$ and it may be either positive or negative.

Although equations (19) to (21) obtain in the limit $\beta \to \alpha$ for any finite values of $\tau$ and $j$, they represent valid approximations only if $|\beta-\alpha|$ is small enough that

$$|\beta-\alpha|\ |\tau-j/\alpha| \ll 1, \tag{22}$$

$$(\beta-\alpha)^2 j \ll \alpha^2. \tag{23}$$

For $j \gg 1$, the interesting range of $\tau$ is that where $|\tau-j/\alpha|=0(j^{1/2})$ as in (21) and here, condition (22) is satisfied whenever (23) applies. Condition (23) implies, however, that for any given small value of $|\beta-\alpha|$, the above approximations are valid only for a finite range of $j$.

The restriction (23) arises from the fact that a small disturbance at a car velocity $(1-\alpha)V$ will reach the $j$th car at a time $\tau=(j-1)/\alpha$ whereas a disturbance at the car velocity $(1-\beta)V$ reaches the $j$th car at a time $\tau=(j-1)/\beta$. The sudden change in velocity of the lead car from $(1-\alpha)V$ to $(1-\beta)V$ can, therefore, be considered as a single small disturbance propagating with a time lag $(\alpha\lambda)^{-1}$ per car only so long as the duration of the disturbance $(j-1)^{1/2}/\alpha$, as represented in (21), is large compared with the time interval $(j-1)|\alpha^{-1}-\beta^{-1}|$ over which waves associated with velocities between $(1-\alpha)V$ and $(1-\beta)V$ will reach the $j$th car. The effects in question here, which arise from the dependence of the wave velocity upon the car velocity, are clearly the analogue of the nonlinear phenomena in the theories of Lighthill and Whitham and Richards that gave rise to shock formation and expanding waves.

## SHOCK WAVES

To INVESTIGATE the possible formation of shocks, we assume that $\beta>\alpha>0$ (deceleration of the lead car) and substitute in (18)

$$(\tau\beta)^{-j} e^{\beta\tau}/(\tau\alpha)^{-j} e^{\alpha\tau}=\exp[(\beta-\alpha)(\tau-j\delta)], \tag{24}$$

with

$$\delta=(\log \beta-\log \alpha)/(\beta-\alpha). \tag{25}$$

If we mark the two points on the velocity-headway curve associated with the velocities $(1-\alpha)V$ and $(1-\beta)V$ as shown in Fig. 3, then $\lambda^{-1}\delta$ is equal to the change in headway divided by the corresponding change in velocity, and thus is the recripocal slope of the straight line through these two points. The velocity at which this line intercepts the velocity axis is equivalent to the shock velocity in the continuum theory and the time $\lambda^{-1}\delta$ is, therefore, the time it takes a shock to travel from the $(j-1)$th to the $j$th car. In the limit $\beta \to \alpha$, the line of Fig. 3 becomes the tangent and $\lambda^{-1}\delta \to \lambda^{-1}\alpha^{-1}$, the time lag for propagation of a wave as described above.

For every $j$, $\gamma(j,\beta\tau)$ in (18) is monotone increasing in $\tau$ from 0 at $\tau=0$ to $\Gamma(j)$ at $\tau \to \infty$ whereas $\Gamma(j,\alpha\tau)$ is monotone decreasing from $\Gamma(j)$ at $\tau=0$ to 0 at $\tau \to \infty$. For $\beta > \alpha$, (24) is monotone increasing and therefore $J$ is also monotone increasing from 0 at $\tau=0$ to $\infty$ at $\tau \to \infty$ and $v_j(\tau)$ is monotone decreasing from the initial velocity $(1-\alpha)\,V$ at $\tau=0$ to the final velocity $(1-\beta)\,V$ at $\tau \to \infty$. For $j \gg 1$, the interesting range of $\tau$, however, is that where $J$ is of order 1 and $v_j(\tau)$ has a value intermediate between its two limits. To locate this range we use the asymptotic formulas[19]

$$\gamma(j,\beta\tau) \sim \Gamma(j)\{\tfrac{1}{2} + \pi^{-1/2}\,\mathrm{Erf}[(\beta\tau - j + 1)/(2j-2)^{1/2}\} \quad \text{if } (\beta\tau - j + 1) = O[(j-1)^{1/2}] \tag{26a}$$

$$\gamma(j,\beta\tau) \sim \frac{e^{-\beta\tau}(\beta\tau)^j}{(j-1-\beta\tau)}\left\{1 - \frac{(j-1)}{(j-1-\beta\tau)^2} + \cdots\right\} \quad \text{if } (\beta\tau - j + 1) < O[(j-1)^{1/2}] \tag{26b}$$

$$\gamma(j,\beta\tau) \sim \Gamma(j) \quad \text{if } (\beta\tau - j + 1) > O[(j-1)^{1/2}]; \tag{26c}$$

$$\Gamma(j,\alpha\tau) \sim \Gamma(j)\{\tfrac{1}{2} - \pi^{-1/2}\,\mathrm{Erf}[(\alpha\tau - j + 1)/(2j-2)^{1/2}]\} \quad \text{if } (\alpha\tau - j + 1) = O[(j-1)^{1/2}] \tag{27a}$$

$$\Gamma(j,\alpha\tau) \sim \Gamma(j) \quad \text{if } (\alpha\tau - j + 1) < O[(j-1)^{1/2}] \tag{27b}$$

$$\Gamma(j,\alpha\tau) \sim \frac{e^{-\alpha\tau}(\alpha\tau)^j}{(\alpha\tau - j + 1)}\left\{1 - \frac{(j-1)}{(\alpha\tau - j + 1)^2} + \cdots\right\} \quad \text{if } (\alpha\tau - j + 1) > O[(j-1)^{1/2}]. \tag{27c}$$

If conditions (22) and (23) apply, the factor (24) is approximately one in the range of $\tau$ where (26a) and (27a) apply, and consequently the behavior of $J$ is determined almost entirely by the gamma functions as given in equations (19) to (21). When $(\beta-\alpha)j^{1/2}/\alpha$ is of order 1, however, the range of $\tau$ where (26a) applies overlaps that where (27a) applies and at the same time $(\beta-\alpha)(\tau - j\delta)$ is of order 1 in (24). Thus all three factors of $J$ contribute significantly to its variation.

For still larger $j$ with $j \gg \alpha^2/(\beta-\alpha)^2$, the ranges of $\tau$ for (26a) and (27a) no longer overlap. The range of $\tau$ that gives $J$ of order 1 is now that in which $(\beta-\alpha)(\tau - j\delta) = 0(1)$ and here (26c) and (27b) apply. The variation of $J$ is, therefore, determined almost entirely by the factor (24) and we obtain, for $j \to \infty$,

$$J \to \exp[(\beta-\alpha)(\tau - j\delta)],$$

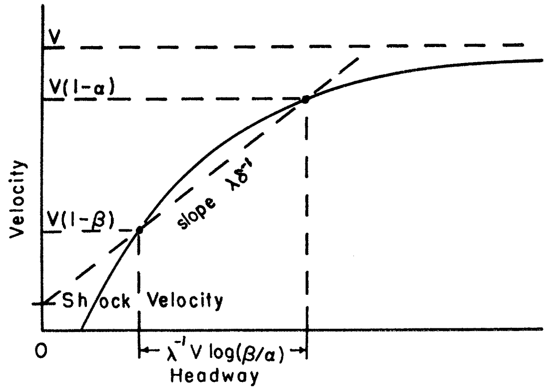


**Fig. 3.** The initial and final velocities of a car, $V(1-\alpha)$ and $V(1-\beta)$, determine two points on the velocity-headway curve (solid line). The straight line through these two points (broken line) defines $\delta$ in equation (25) and the intercept gives the shock velocity.

$$v_j(\tau)\rightarrow[1-\tfrac{1}{2}(\alpha+\beta)]\,V-\tfrac{1}{2}(\beta-\alpha)\,V\tanh[\tfrac{1}{2}(\beta-\alpha)(\tau-j\delta)], \quad (28)$$

$$dv_j(\tau)/d\tau\rightarrow-(\beta-\alpha)^2V/\{4\cosh^2[\tfrac{1}{2}(\beta-\alpha)(\tau-j\delta)]\}. \quad (29)$$

This is the shock wave including what in fluid dynamics would be called the shock profile. The feature which distinguishes it from other types of waves is the fact that the velocity is a function of $\tau-j\delta$ but is

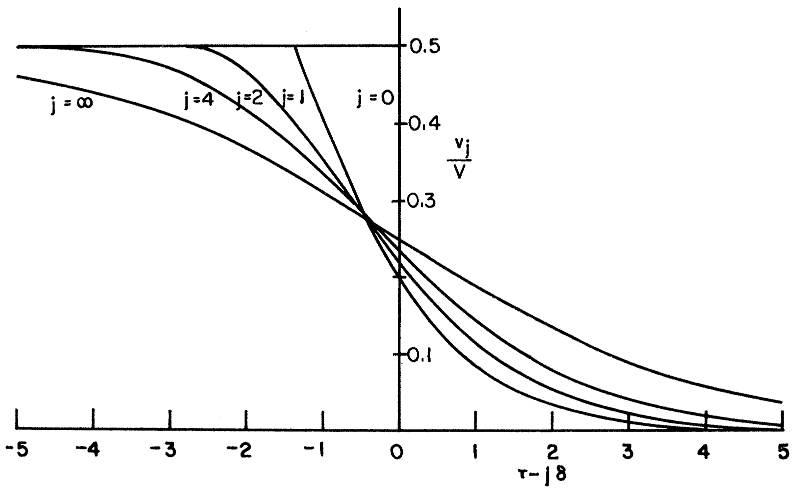


**Fig. 4.** A lead car ($j=0$) decelerates suddenly from the velocity $\frac{1}{2}V$ to 0. The velocities of succeeding cars are shown as a function of the displaced time coordinate $\tau-j\delta$. The curve for $j=\infty$ shows the shock profile.

otherwise independent of $j$. Each car undergoes exactly the same velocity change as its predecessor, except at a time $\delta$ later.

Whereas in the continuum theory, the shock is represented by a discontinuity in velocity, the present theory (because of its discrete representation of cars) gives the details of the motion through the shock. In particular we see that the maximum deceleration, measured in units of real time occurs when $\tau = j\delta$ and has a value

$$(dv/dt)_{\max} = -\tfrac{1}{4}\,(\beta-\alpha)^2\lambda V. \tag{30}$$

For the data used in Fig. 2, this gives a value $-11(\beta-\alpha)^2$ft/sec$^2$ or a maximum deceleration of 11 ft/sec$^2$ if the velocity changes from the free speed to zero.

Figure 4 shows the velocity of a sequence of cars when the lead car suddenly decelerates from $\tfrac{1}{2}V$ to 0. The velocity is plotted vs. $\tau - j\delta$ to show the convergence to the shock solution for $j \to \infty$.

## EXPANDING WAVES

FOR $\alpha > \beta$ (acceleration of the lead car) we still find that if (22) and (23) apply, the $\Gamma$-functions dominate the behavior of $J$ and we are led to equations (19) to (21). For $j \gg \alpha^2/(\beta-\alpha)^2$, however, in contrast with the case $\beta > \alpha$, the region of $\tau$ where $J$ is of order 1 is the region where (26b) and (27c) apply. We thus find that

$$J \to \frac{(\alpha\tau - j + 1)}{(j-1-\beta\tau)}\left\{1 - \frac{(j-1)}{(j-1-\beta\tau)^2} + \frac{(j-1)}{(j-1-\alpha\tau)^2} + \cdots\right\}, \tag{31}$$

$$v_j(\tau) \to V - \frac{V\,(j-1)}{\tau}\left\{1 - \frac{2\,(j-1)-(\beta+\alpha)\,\tau}{(j-1-\beta\tau)(\alpha\tau-j+1)} + \cdots\right\} \tag{32}$$

for $$j-1-\beta\tau > O(j^{1/2}) \quad \text{and} \quad \alpha\tau - j + 1 > O(j^{1/2}). \tag{33}$$

The last term of (32) is small if condition (33) is satisfied so to a first approximation

$$v_j(\tau) \sim \begin{cases} V(1-\alpha), & [\tau < (j-1)/\alpha] \\ V - V(j-1)/\tau, & [(j-1)/\alpha < \tau < (j-1)/\beta] \\ V(1-\beta). & [(j-1)/\beta < \tau] \end{cases} \tag{34}$$

This represents the analogue of the expanding wave in the continuum theory and is characterized by the fact that the dependence of velocity upon time is the same for all cars except for a rescaling of the units of time inversely with $j-1$.

Figure 5 shows the velocity for a sequence of cars when the lead car suddenly accelerates from $v=0$ to $v=\frac{1}{2}V$ at $\tau=0$. The velocity is plotted vs. $\tau/j$ to show the convergence of the velocity profile to the wave solution given in (34) for $j\to\infty$. The convergence is rather slow, particularly near the wave edges $\tau/j=1$ or 2, because the important parameter that determines the rate of convergence is $j^{1/2}$.

The above formulas [except the last part of (34)] apply even if the lead car acquires a velocity greater than the free speed of its successors ($\beta<0$). In this case for $j\geqq 1$, $v_j(\tau)\to V$ for $\tau\to\infty$ instead of $(1-\beta)V$.

To find the trajectories $x_j(\tau)$, we must evaluate $z_j(\tau)$ from (15) and use (3) and (5). If we assume that $d_j=d$, independent of $j$, then

$$x_j(\tau)=-V\lambda^{-1}\log z_j(\tau)-jd. \tag{35}$$

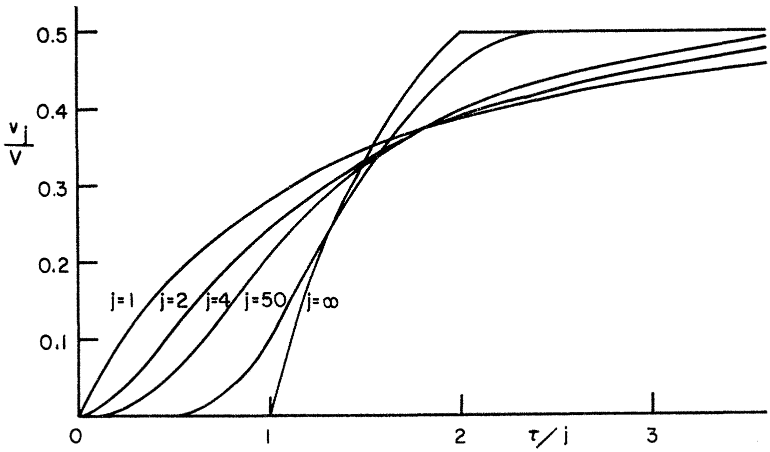


**Fig. 5.** A lead car ($j=0$) suddenly accelerates from velocity 0 to $\frac{1}{2}V$. The velocities of succeeding cars are shown as a function of a rescaled time coordinate $\tau/j$ to show convergence to the wave solution, $j=\infty$.

Figure 6 shows a few trajectories when a lead car instantaneously changes velocity from 0 to $V$ with $d\lambda/V=0.291$ chosen as in Fig. 2.

When a traffic light turns green, the lead car in the queue accelerates from velocity 0 to $V$. Although it does not reach the velocity $V$ instantaneously as car 0 of Fig. 6, it does perhaps accelerate in a manner comparable with that shown in Fig. 6 for car 3. If we disregard cars 0, 1, and 2, and think of car 3 as the lead car, we would still obtain the same trajectories for cars 4 and 5.

Regardless of which car we designate as the lead car, one can see from Fig. 6 that the time headways at or near $x=0$ are nearly independent of $j$ after the first two or three cars. This constant time headway is also the one that gives the maximum flow in the continuum theory.[15] In the artificial example in which car 0 is considered as the lead car, the time headways of the first few cars at $x=0$ are actually less than the asymptotic limit, which means that the 'flow' is temporarily higher than the maximum value given by the continuum theory. Here the rapid acceleration of the lead car tends to pull the succeeding cars past the point $x=0$ faster than one would expect. If, on the other hand, we think of the second or third

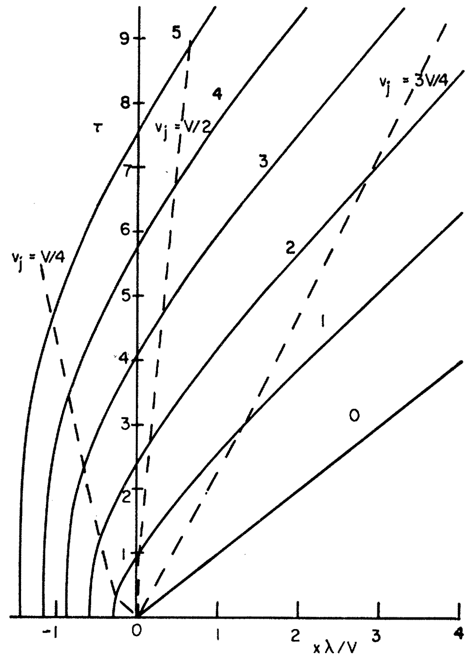


**Fig. 6.** A family of trajectories (solid lines), dimensionless time $\tau$ vs. dimensionless position $x\lambda V$, for a sequence of cars following a car 0 which suddenly accelerates from velocity 0 to $V$. The broken line joins points of these trajectories having the same constant velocity for velocities $\frac{1}{4}$ $V$, $\frac{1}{2}$ $V$, and $\frac{3}{4}$ $V$.

cars as being the lead car and measure time headways at the initial position of this car, we see that the time headways for the first few cars are longer than that given by the continuum theory. This latter type of behavior is in qualitative agreement with the observations of GREENSHIELDS, SCHAPIRO, AND ERICKSEN.[21]

The points in the $xt$-plane where each $j$th car acquires some given velocity $v$ asymptotically (for large $j$) lie on a straight line, the slope of which gives the wave velocity associated with the car velocity $v$. The points at which the various cars in Fig. 6 achieve the velocities $\frac{1}{4}$ $V$, $\frac{1}{2}$ $V$, and $\frac{3}{4}V$ have been joined by a broken line. These lines are not straight near the origin or, equivalently, one can say that the asymptotes do not emanate exactly from the origin. This fanning out of waves from the origin of a disturbance was displayed by HERREY and HERREY[17] and later in the continuum theories, except in these theories all waves were straight lines from the exact origin.

## GENERALIZATIONS

ALTHOUGH we have described here only the consequences of a specific type of velocity-headway relation, that given by (2) with identical cars and $\Delta=0$, most of the qualitative conclusions that we have deduced for this particular model apply also to a general class of relations of the type described by (1).

If $G_j$ in (1) has a continuous first derivative, then for small variations in its argument it can always be approximated by a linear function and the solution of (1) with any linear approximation for $G_j$ will give the results described in references 3 to 7. In such a procedure, however, one is approximating the velocity or the time derivative of $x_j(t)$, and when one integrates equation (1) to obtain $x_j(t)$ one can guarantee the validity of the approximation to $x_j(t)$ only for a finite length of time or finitely many iterations on $j$, because the time integral of a small error over an arbitrarily long period of time or the iteration of a small error after many iterations may not be small. Thus we found in the above sections that the linear theory broke down when (23) failed. This type of approximation can, however, be used to describe rapidly varying fluctuations.

Another type of approximation will apply if we postulate that $v_j(t)$ is slowly varying with respect to both $j$ and $t$ but not necessarily confined to a small range of values. We first write (1) in the form

$$x_j(t)=x_{j-1}(t)-H_j[v_j(t+\Delta)], \tag{36}$$

in which the function $H_j$ is the inverse of $G_j$, $G_j^{-1}$. If we differentiate this equation with respect to time we obtain

$$v_j(t)=v_{j-1}(t)-H_j'[v_j(t+\Delta)]\,v_j'(t+\Delta), \tag{37}$$

with the prime denoting differentiation with respect to the argument of the function. Slowly varying conditions shall now be interpreted to mean that the second term of (37) is small, so that to a first approximation $v_j(t)\sim v_{j-1}(t)$. To this approximation (36) then gives $x_j(t)$ explicitly in terms of the trajectory of the $(j-1)$th car. A scheme similar to this was used previously in reference 14 to obtain piecewise linear approximations to the trajectories of a sequence of cars.

To obtain better approximations, one can make successive substitutions in which any expression involving the trajectory of the $j$th car, $v_j(t), v_j'(t)$, etc. in the second term of (37) is replaced by the appropriate expression derived from the left side of this equation. Thus in the next approximation we would write

$$\begin{aligned} v_j(t) = v_{j-1}(t) - H_j'\{v_{j-1}(t+\Delta) - H_j'[v_j(t+2\Delta)]\, v_j'(t+2\Delta)\} \\ \times\{v_{j-1}'(t+\Delta) - H_j''[v_j(t+2\Delta)][v_j'(t+2\Delta)]^2 \\ - H_j'[v_j(t+2\Delta)]\, v_j''(t+2\Delta)\}. \end{aligned} \tag{38}$$

We do not expect that an infinite repetition of this procedure will be convergent, but only that the error after each approximation is successively of lower order in the limit as the time scale for variation in $v_j(t)$ becomes infinite (asymptotic approximation).

If we now neglect $\Delta$ and the difference $v_j(t) - v_{j-1}(t)$ in the second term of (37) or (38) we obtain

$$v_j(t) \sim v_{j-1}(t) - H_j'[v_{j-1}(t)]\, v_{j-1}'(t) \sim v_{j-1}\{t - H_j'[v_{j-1}'(t)]\}. \tag{39}$$

This approximation already gives results that are equivalent to those described by the continuum theory, since (39) states that the $j$th car acquires the same velocity as its predecessor, but at a time $H_j'[v_{j-1}'(t)]$ later. If we substitute (39) into (36) we obtain a corresponding approximation for the spacing between the cars, from which one can compute the effective velocity with which the car velocity of the $(j-1)$th car is propagated to the $j$th car, i.e., the wave velocity. Since (39) describes the same type of solution as obtained from the continuum theory, one can derive the partial differential equations of the continuum theory from this solution. One could also obtain these differential equations by replacing finite differences $v_{j-1}(t) - v_j(t)$ in (37) by partial derivatives with respect to a continuous variable $j$, and then making suitable transformations of the variables to those used in the continuum theory.

If $H_j[v_j]$ changes with $v_j$ so also does the wave velocity, and (39) will lead to shocks or expanding waves. As a shock starts to develop, however, the assumption that the velocities are slowly varying breaks down and (39) is no longer a valid approximation. It is desirable, therefore, to carry the approximation scheme at least one step further. If we expand (38) in powers of $H_j$ and $\Delta$ keeping terms up to second order and make some rearrangements we obtain

$$\begin{aligned} v_j(t) \sim v_{j-1}\{t - H_j'[v_{j-1}(t+\Delta - 2H_j'\{v_{j-1}(t)\})]\} \\ + \tfrac{1}{2}\, H_j'[v_{j-1}(t)]\{H_j'[v_{j-1}(t)] - 2\Delta\}\, v_{j-1}''(t). \end{aligned} \tag{40}$$

The first term of (40) is similar to the right side of (39) in that the velocity $v_{j-1}$ is evaluated at a time $H_j'$ earlier, except that now $H_j'$ must be evaluated from the velocity $v_{j-1}$ that obtains at a displaced time

$$t+\Delta-2H_j'[v_{j-1}(t)]$$

rather than at the time $t$ as in (39). If only this first term of (40) were important we would, however, still expect results of essentially the same kind as given in the continuum theory, particularly one would be forced to admit discontinuities (shocks) to salvage the paradox of intersecting waves.

It is the second term of (40) that gives the shock structure and plays a role analogous to the viscosity in fluid dynamics. To illustrate the effects of this term let us first take a linear function for $G_j$ in (1) so that

$$H_j'(v_j)=\lambda^{-1} \qquad (41)$$

independent of both $v_j$ and $j$. Equation (40) then becomes

$$v_j(t)-v_{j-1}(t-\lambda^{-1})\sim(2\lambda^2)^{-1}\,(1-2\,\Delta\lambda)\,v''_{j-1}(t), \qquad (42)$$

which is essentially a finite difference approximation to the diffusion equation.

If we let
$$t^*=t-j\lambda^{-1} \qquad (43)$$

then for each $j$, $t^*$ measures the time coordinate for the $j$th car displaced by an amount $\lambda^{-1}$ relative to that of the $(j-1)$th car. Thus a wave traveling with a time lag $\lambda^{-1}$ per car reaches each car at the same value of $t^*$. If we also define

$$v^*(j,t^*)\equiv v^*(j,t-j\lambda^{-1})\equiv v_j(t), \qquad (44)$$

then
$$v_j(t)-v_{j-1}(t-\lambda^{-1})=v^*(j,t^*)-v^*(j-1,t^*),$$

and the left side of (42) becomes a finite difference of $v^*$ with respect to $j$ for fixed $t^*$. By considering $j$ as a continuous variable and replacing finite differences by derivatives, we obtain from (42)

$$\partial v^*(j,t^*)/\partial j\sim[(1-2\,\Delta\lambda)/2\lambda^2]\,\partial^2 v^*(j,t^*)/\partial t^{*2}. \qquad (45)$$

This is the classic diffusion equation, except that in the usual interpretation of the diffusion equation the coordinate $j$ plays the role of time and $t^*$ the role of position. The function $v_j(t)$ can also be interpreted as a solution of a diffusion equation for a moving fluid or for a stationary fluid as viewed from a moving coordinate system.

The 'diffusion constant' in (45), $(2\lambda^2)^{-1}\,(1-2\,\Delta\lambda)$, is positive only if $2\,\Delta\lambda<1$. This is the condition for stability derived by Chandler, Herman, and Montroll[3] and by Kometani and Sasaki[5] for a discrete linear theory to guarantee that fluctuations are not amplified as they propagate. It is also a condition for stability of (45) in an equivalent sense.

The solution of (45) for arbitrary motion of the lead car $v^*(0,t^*)=v_0(t)$ and for $2\,\Delta\lambda<1$ can be constructed from the Green's function. Thus

$$v^*(j,t^*)\sim\int_{-\infty}^{+\infty} dt_0^*\, G(t^*,t_0^*)\, v^*(0,t_0^*), \tag{46}$$

with
$$G(t^*,t_0^*)=\frac{\lambda}{[2\pi j\,(1-2\,\Delta\lambda)]^{1/2}}\exp\left[-\frac{\lambda^2(t^*-t_0^*)^2}{2j\,(1-2\,\Delta\lambda)}\right]. \tag{47}$$

This Green's function (47) agrees exactly with the asymptotic formula (large $j$) derived by a quite different method in reference 3, p. 177 and reference 6, p. 97 for the response to a sudden change in behavior of the lead car. It also describes the same type of behavior as (21) above.

If instead of (41) we now assume that $H_j'(v_j)$ is monotonically increasing with $v_j$, then we must consider the competition between the nonlinear effects of the first term of (40) and the diffusive type influence of the second term. Although the complete solution of (40) is awkward to construct, one can easily see qualitatively what is implied.

If some $(j-1)$th car accelerates from one velocity to a higher velocity during some given time interval, the change in velocity of the $j$th car will be spread over a longer time interval first because of the variation of $H'$, which gives a shorter time lag for the propagation of the low velocity than for the higher velocity, and second because of the diffusive effects of the second term. For rapid accelerations the latter effect dominates but for slow accelerations the former dominates, and in all cases the accelerations will become slow for sufficiently large $j$ and the nonlinear effects will dominate.

If, however, the $(j-1)$th car decelerates, the nonlinear effects of the first term of (40) tend to compress the time interval of deceleration, whereas the second term tends to expand it. If the deceleration is rapid the latter effect dominates, but it if is slow the former dominates. In either case the tendency is to approach an equilibrium situation in which these two competing effects balance. It is this solution that one would call a shock.

## GENERAL SHOCK EQUATIONS

SUPPOSE we have a sequence of identical cars, $G_j=G$ for all $j$, with $G$ concave downwards and $\Delta$ not too large. If the cars are initially traveling in equilibrium at a constant velocity $U_1$ and the lead car decides to decelerate in any manner so as to eventually acquire another constant velocity $U_2$, then the above analysis suggests that the deceleration of the

$j$th car in the limit $j \to \infty$ will be a shock from the velocity $U_1$ to $U_2$, the properties of which depend upon $U_1$ and $U_2$ but are otherwise independent of the trajectory of the lead car.

The shock is characterized by the condition

$$x_{j-1}(t) = x_j(t+T) + C, \tag{48}$$

with $T$ and $C$ constants, i.e., the $j$th car has exactly the same trajectory as the $(j-1)$th car except for a displacement in time by an amount $T$ and in space by an amount $C$. Substitution of (48) into (1) gives the single differential-difference equation

$$v_j(t) = G[x_j(t+T-\Delta) - x_j(t-\Delta) + C] \tag{49}$$

for the shock trajectory of an arbitrary $j$th car. The two constants $T$ and $C$ are determined by the conditions that when $t \to -\infty, v_j(t) \to U_1$ and so $x_j(t+T-\Delta) - x_j(t-\Delta) \to TU_1$. Similarly for $t \to +\infty, v_j(t) \to U_2$ and $x_j(t+T-\Delta) - x_j(t-\Delta) \to TU_2$. These conditions with (49) imply

$$U_1 = G[U_1 T + C], \qquad U_2 = G[U_2 T + C]. \tag{50}$$

The solutions of (50) for $T$ and $C$ can be found graphically from the velocity-headway curve. If, as in Fig. 3, one marks the two points on the velocity-headway curve corresponding to velocities $U_1$ and $U_2$ and draws a straight line through these points, this line will, according to (50), have slope $T^{-1}$ and headway intercept $C$.

For 'weak' shocks we can derive an approximate solution of (49) by expanding (49) in powers of $\Delta$ and $T$, thus

$$\begin{aligned} H[v_j(t)] &= G^{-1}[v_j(t)] = x_j(t+T-\Delta) - x_j(t-\Delta) + C \\ &\sim C + T\, v_j(t) + \frac{T(T-2\Delta)}{2} \frac{dv_j(t)}{dt} \\ &\qquad + \frac{T\,(T^2 - 3\Delta T + 3\Delta^2)}{3!} \frac{d^2 v_j(t)}{dt^2} + \cdots. \end{aligned} \tag{51}$$

The lowest approximation obtains from neglecting the second derivative term and solving

$$-\tfrac{1}{2}\, T\, (T-2\Delta)\, dv_j(t)/dt \sim C + Tv_j(t) - H[v_j(t)]. \tag{52}$$

By virtue of (50), the right side of (52) represents the difference in headway between the velocity-headway curve and a straight line passing through the two points of this curve at the velocities $U_1$ and $U_2$ (in Fig. 3, the horizontal distance from the curve to the straight line). It vanishes at $v_j(t) = U_1$ or $U_2$ and is positive for any intermediate velocities. Equation

(52), therefore, has a solution for $U_2 < U_1$ provided $T > 2\Delta$, a condition that is essentially equivalent to our previous stability condition.

Integration of (52) gives the relation

$$\frac{2t}{T\,(T-2\Delta)} \sim \int^{v_j(t)} \frac{dv}{C+Tv-H(v)}, \tag{53}$$

the inverse of which determines $v_j(t)$ uniquely except for an arbitrary translation of the time coordinate. Since exact solutions of (2) have already been found for $\Delta=0$, the dependence of the approximate solution (53) upon $\Delta$ is of particular interest, and we see from (53) that to this approximation $v_j(t)$ is independent of $\Delta$ except for a rescaling of the time coordinate by a factor $1-2\Delta/T$. Thus accelerations within the shock are uniformly higher by a factor $(1-2\Delta/T)^{-1}$ for $\Delta \neq 0$ than for $\Delta=0$.

If $U_1-U_2$ is sufficiently small, then for $U_2 \leqq v \leqq U_1$ one can approximate $H(v)$ by a quadratic function so that $C+Tv-H(v)$ vanishes at $v=U_1$ and $U_2$, i.e.,

$$C+Tv-H(v) \sim \tfrac{1}{2}\,D\,(v-U_1)(U_2-v), \tag{54}$$

with $D$ equal to the second derivative of $H(v)$ evaluated at some point between $U_1$ and $U_2$. The right side of (53) can then be integrated to give

$$v_j(t) = U_1+U_2+(U_1-U_2)\tanh\gamma t, \tag{55}$$

with

$$\gamma = D\,(U_2-U_1)\,T^{-1}\,(T-2\Delta)^{-1}. \tag{56}$$

This is essentially the same as (28) except for a change in the time origin, some trivial differences in notation, plus the fact that the time constant $\gamma$ in (55) and (56), now depends upon $\Delta$.

The validity of (51) to (55) requires that the second and higher derivative terms in (51) be small compared with the first derivative term. For $\Delta < \tfrac{1}{2}T$, this will be true provided that

$$T^2\gamma\,(T-2\Delta)^{-1} \ll 1. \tag{57}$$

Since $\gamma^{-1}$ is a measure of the time required for a car to travel through the shock and $T$ measures the time for a wave to travel from one car to the next, condition (57) means that the shock must encompass many cars simultaneously. This will be true if $\gamma$ is sufficiently small, or equivalently if either $D$ (thus the curvature of the velocity-headway curve) or $U_2-U_1$ (the amplitude of the shock) are small.

Chandler, Herman, and Montroll have pointed out an interesting mechanism by which a driver can cause an accident that he never sees. If $T < 2\Delta$, the motion of a sequence of cars is unstable in the sense that a small fluctuation of the lead car is amplified as it propagates, so that it may cause an accident 10 or 20 cars back. Here we are confronted with another type of amplification. If the lead car decelerates in any manner whatsoever from a velocity $U_1$ to $U_2$, the resulting disturbance will eventually propagate as a shock. If the lead car decelerates very gradually, then the decelerations of subsequent cars must increase until they reach the values associated with the shock trajectory for the velocities $U_1$ and $U_2$. This phenomenon has nothing to do with instability and takes place even if $\Delta=0$. As $\Delta$ increases, however, so also does the deceleration in the shock; and for weak shocks, at least, the estimates of the shock deceleration given in (30) are too low by a factor of approximately

$$(1-2\Delta/T)^{-1}.$$

Whereas the decelerations estimated from (30) for $\Delta=0$ were well within the limits of braking power of most cars, experiments indicate that the factor $(1-2\Delta/T)^{-1}$ may be quite large, and we are confronted with the possibility that the present theory will produce impossibly high decelerations.

If we can assume the decelerations in the shock are always positive even for $\Delta\neq 0$, then we can find upper bounds on the decelerations from (49). If we differentiate (49) we obtain

$$dv_j(t)/dt = G'[x_j(t+T-\Delta)-x_j(t-\Delta)+C][v_j(t+T-\Delta)-v_j(t-\Delta)],$$

but $$v_j(t+T-\Delta)\geqq U_2; \qquad v_j(t-\Delta)\leqq U_1$$

and $$G'[x_j(t+T-\Delta)-x_j(t-\Delta)+C]\leqq G'[U_2T+C];$$

therefore $$|dv_j/dt|\leqq G'[U_2T+C]\,[U_1-U_2]. \tag{58}$$

For $U_2=0$ and the velocity-headway relation of Fig. 2, this gives an upper bound of 43 $U_1/V$ ft/sec$^2$ which, however, exceeds the braking power of cars by about a factor of 2 if $U_1$ is equal to the free speed $V$.

To obtain more accurate estimates of the decelerations in strong shocks, it will probably be necessary to solve equation (49) numerically.

## CONCLUSIONS

THE MAIN purpose of this paper is to show that with a nonlinear car-following model of the type given by (1), it is possible to incorporate into a single theory all the results previously derived for linear car-following models and the nonlinear phenomena previously obtained from continuum theories. Thus the model includes practically everything that has been contained in any older models for dense traffic flow and, in addition, allows one to investigate such things as the development of shocks, shock profiles, the range of validity of previous theories (under the assumption that the present theory is valid), and the spreading of an acceleration wave. Although we have not tried to display here the consequences of permitting different cars to have different modes of behavior [$G_j$ in (1) dependent upon $j$] such things can also be investigated with this theory. This would be of particular interest in studying the motion of a pack of cars temporarily stopped by a traffic light, for in addition to the effects described above for identical cars, one also will find some additional spreading of the pack caused by differences in the free speed of different cars.

The fact that the present model includes almost everything included in any previous models of traffic flow with no passing does not mean, however, that the model is correct. Indeed, the theory has a number of serious deficiencies. It contains the implication that drivers follow one another under nonstationary conditions by adhering to the same velocity-headway relation they would choose under steady conditions. Although experimental evidence lends some support to this conjecture, the data are not extensive enough to give more than some hints as to the manner in which the theory fails. It has been found, for example, that if a lead driver travels at a constant velocity, then performs some maneuver, and returns to his original speed, a follower will not always reacquire the same headway that it had initially. Thus the relation between velocity and headway is not unique. Some data taken in the Holland Tunnel[22] also describes some odd effects that have no obvious explanation within the framework of the present theory.

In addition to these things that the theory fails to describe, there are also predictions of the theory that have not been observed and may not exist. Although experimentalists speak frequently of shocks being propagated, no one has yet established the existence of shocks as something different from a wave. Also the present theory predicts that if one creates two disturbances one of which is an acceleration and the other a deceleration, then the spreading of the wave generated by the acceleration will always cause the wave to overtake the shock generated by the deceleration, and the two disturbances will eventually annihilate each other. The experimental truth of this has not been established, and there are indications that it may not be true.

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