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CONGESTION REDUX*

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Abstract. In this paper we analyze a class of second-order traffic models and show that these models support stable oscillatory traveling waves typical of the waves observed on a congested roadway. The basic model has trivial or constant solutions where cars are uniformly spaced and travel at a constant equilibrium velocity that is determined by the car spacing. The stable traveling waves arise because there is an interval of car spacing for which the constant solutions are unstable. These waves consist of a smooth part where both the velocity and spacing between successive cars are increasing functions of a Lagrange mass index. These smooth portions are separated by shock waves that travel at computable negative velocity.

Key words. conservation laws, traffic congestion, follow-the-leader

AMS subject classification. 35

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1. Introduction.

In the last several years a number of authors [1, 2, 3, 4, 5, 6, 7, 8, 9] have advanced "higher order" traffic models in an attempt to characterize strong permanent waves which appear in congested traffic. At the continuum level all of these authors have worked with models of the following form:

$$\frac{\partial s}{\partial t} - \frac{\partial u}{\partial m} = 0 \tag{1.1}$$

and

$$\epsilon \frac{\partial u}{\partial t} = \epsilon P'(s) \frac{\partial u}{\partial m} + V(s) - u. \tag{1.2}$$

Here $t \geq 0$ is time, m is a Lagrangian mass coordinate which gives the car index, and $\epsilon > 0$ has the interpretation of a relaxation time. The velocity of the mth car at time t is $u(m,t)$, and $s(m,t) \geq L > 0$ is a measure of the spacing between successive cars. Finally, the function $s \to V(s)$ has the interpretation of an "equilibrium" velocity, and the term $\epsilon P'(s)\frac{\partial u}{\partial m}$ appearing in (1.2) is typically referred to as the anticipatory acceleration. All authors assume that $P'(s) \geq 0$ on $s \geq L$. The parameter $L > 0$ has the interpretation of the length of a car on the roadway.

The trajectory of the mth car is given as the solution of

$$\frac{\partial x}{\partial t} = u \quad \text{and} \quad x(m,0) = x_0(m), \tag{1.3}$$

where $x_0(m)$ is the position of the mth car at $t = 0$. $s(m,t)$ is related to $x(m,t)$ by

$$s(m,t) = \frac{\partial x}{\partial m}(m,t) \tag{1.4}$$

and measures the spacing between successive cars.

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The hypothesis that $P'(s) \geq 0$ implies that the system (1.1) and (1.2) is hyperbolic with wave speeds $c = -P'(s) \leq 0$ and $c = 0$ and thus information propagates from right to left. This observation implies that when constructing finite difference schemes for (1.1) and (1.2) the appropriate spatial differences should be downwind, i.e., that

$$s(m,t) \doteq \frac{x(m+\Delta m, t) - x(m,t)}{\Delta m} \tag{1.5}$$

and

$$\frac{\partial u}{\partial m}(m,t) \doteq \frac{u(m+\Delta m, t) - u(m,t)}{\Delta m}. \tag{1.6}$$

If one chooses to discretize (1.1)–(1.4) spatially, keep time continuous, and, moreover, choose $\Delta m = 1$ (recalling that cars are really discrete entities), one is led to the classic follow-the-leader system

$$\frac{dx_m}{dt} = u_m \tag{1.7}$$

and

$$\epsilon \frac{du_m}{dt} = \epsilon P'(x_{m+1} - x_m)(u_{m+1} - u_m) + V(x_{m+1} - x_m) - u_m \tag{1.8}$$

studied by traffic engineers. On the other hand, if one lets

$$\rho(x,t) = \frac{1}{s(m,t)} \quad \text{and} \quad v(x,t) = u(m,t) \tag{1.9}$$

when

$$x = x(m,t), \tag{1.10}$$

one finds that as functions of x and t the functions ρ and v satisfy

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \tag{1.11}$$

and

$$\epsilon \left(\frac{\partial v}{\partial t} + (v + \rho \mathcal{R},_{\rho}(\rho)) \frac{\partial v}{\partial x} \right) = W(\rho) - v, \tag{1.12}$$

where

$$\mathcal{R}(\rho) \stackrel{def}{=} P(1/\rho) \quad \text{and} \quad W(\rho) \stackrel{def}{=} V(1/\rho). \tag{1.13}$$

Of course

$$\rho^2 \mathcal{R},_{\rho}(\rho) = -P'(s = 1/\rho) \leq 0 \quad \text{and} \quad \rho^2 W,_{\rho}(\rho) = -V'(s = 1/\rho) \leq 0. \tag{1.14}$$

References [2, 3, 4, 5] dealt primarily with the case where $P(\cdot)$ and $V(\cdot)$ were monotone increasing on $s \geq L$ and had the following additional properties:

$$V(L^+) = 0 \quad \text{and} \lim_{s \to \infty} V(s) = v_\infty \tag{1.15}$$

and

$$0 < V'(s) \leq P'(s),\ P''(s) < 0 \quad \text{and} \quad V''(s) < 0,\ L < s < \infty. \tag{1.16}$$

In that series of papers the authors established a variety of results about the system (1.1)–(1.4) and its discrete counterpart (1.7)–(1.8), notably that the constant solutions

$$s(m,t) \equiv s_0 > L \quad \text{and} \quad u(m,t) \equiv V(s_0) \tag{1.17}$$

were stable in the L_∞ norm.

Bando et al. [6] considered the discrete system (1.7)–(1.8) when $P'(\cdot) \equiv 0$ and found that the steady solutions (1.7)–(1.8) were linearly unstable if $0 < V'(s_0)$ was large enough and linearly stable otherwise. The continuous system (1.1)–(1.2) with $P'(\cdot) \equiv 0$ supports a stronger conclusion, namely, that all steady solutions are linearly unstable and this is a defect in that model. In that same paper, Bando and his coauthors also exhibited large amplitude oscillatory solutions to (1.7) and (1.8) in the case where

$$V(s) = \frac{v_\infty \left(\tanh\left(\frac{s-rL}{\delta}\right) + \tanh\left(\frac{(r-1)L}{\delta}\right)\right)}{\left(1 + \tanh\left(\frac{(r-1)L}{\delta}\right)\right)}, s \geq L \tag{1.18}$$

when

$$r > 1, \quad v_\infty > 0, \quad \text{and} \quad \delta > 0. \tag{1.19}$$

These solutions are reminiscent of the strong permanent waves seen in congested traffic. The authors' calculations gave no indication of the propagation speeds of these waves.

Finally, Greenberg, Klar, and Rascle [7] considered the system (1.1)–(1.2) when

$$P(s) = v_\infty \left(1 - L/s\right), \quad 0 < L \leq s \tag{1.20}$$

and

$$V(s) = \begin{cases} \mu v_\infty \left(1 - L/s\right), & L \leq s < s_* \\ v_\infty \left(1 - L/s\right), & s_* < s < \infty, \end{cases} \tag{1.21}$$

where $0 < \mu < 1$ and $v_\infty > 0$. The trivial equilibria for this model are

$$s = s_0 \quad \text{and} \quad u \equiv V(s_0) \quad \text{when} \quad s_0 \neq s_* \tag{1.22}$$

and

$$s = s_* \quad \text{and any} \quad u = u_* \text{ in the interval } \left(\mu v_\infty \left(1 - L/s_*\right),\ v_\infty \left(1 - L/s_*\right)\right). \tag{1.23}$$

The former equilibria are stable and the latter unstable. In [7] the authors established the existence of stable periodic traveling waves (the ring-road scenario) of large amplitude which propagate with speed $c = -P'(s_*)$. These waves were functions of $\xi = m - ct$ and were composed of a smooth increasing portion satisfying

$$s(-m_a^+) = s_a, \quad s(0) = s_*, \quad \text{and} \quad s(M_a^-) = S_a. \tag{1.24}$$

The numbers $s_a < s_* < S_a$ were not arbitrary. They satisfied

$$\frac{P(S_a) - P(s_a)}{S_a - s_a} = P'(s_*), \tag{1.25}$$

and the numbers m_a and M_a satisfied

$$k\,(m_a + M_a) = M \quad \text{and} \quad k \int_{m_a}^{M_a} s(\xi) d\xi = l. \tag{1.26}$$

Here M represents the number of cars on the ring-road, l is the length of the ring-road, and $k \geq 1$ is an integer which gives the number of increasing segments per period. These waves have jump discontinuities at the points $\{m_a \pm n\,(m_a + M_a)\}_{n=0}^{\infty}$ and (1.25) guarantees that the Rankine–Hugoniot conditions for (1.1)–(1.2) hold across the discontinuities. These waves also satisfy the Lax entropy condition across the shocks, namely, the condition that $S_a > s_a$.

Our goal in the remainder of this paper is to show that the results of [7] were no fluke; that is, they were not an artifact of the jump discontinuity in the equilibrium velocity function defined in (1.21) but rather were generic. In the remainder of this paper we shall limit ourselves to the analysis of (1.1)–(1.2) when $P(\cdot)$ and $V(\cdot)$ are both increasing on $[L, \infty)$ and satisfy the normalization conditions

$$P(L^+) = V(L^+) = 0 \quad \text{and} \quad \lim_{s \to \infty} V(s) = v_\infty > 0. \tag{1.27}$$

We shall assume that $V'(\cdot)$ has an isolated single maximum at $s_* > L$, that

$$V''(s) > 0, \quad L \leq s < s_* \quad \text{and } V''(s) < 0, \quad s_* < s < \infty, \tag{1.28}$$

that the difference $(P' - V')\,(\cdot)$ has two isolated zeros at points s_1 and s_2 satisfying $L < s_1 < s_* < s_2 < \infty$, and, finally, that $(P' - V')(\cdot) > 0$ on $(L, s_1) \cup (s_2, \infty)$.

In section 2 we shall give a simple argument showing that for s_0 in (s_1, s_2), the constant solution defined in (1.17) is unstable. We shall also show that if the initial data for s lies in this interval, then s approximately evolves via a convective backwards heat equation, thus confirming the instability of the constant solutions. This latter result will be established by using a Chapman–Enskog expansion of the solutions of (1.1) and (1.2). In section 3 we shall show how to construct the large amplitude periodic traveling wave solutions to (1.1)–(1.2) reminiscent of the waves seen in congested traffic. These solutions are similar in structure to those obtained in [7]. Section 4 will be devoted to numerical simulations. Here we shall limit ourselves to

$$P(s) = \lambda(1 - L/s), \quad L \leq s, \tag{1.29}$$

and $V(\cdot)$ given by (1.18). We shall demonstrate that for nonconstant initial data taking on values in the unstable interval (s_1, s_2), solutions converge to traveling waves. These simulations will be run on the follow-the-leader model (1.7)–(1.8). Comprehensive surveys on this vast subject may be found in Helbing [8] and Nagel, Wagner, and Woesler [9].

2. Linear stability of (1.17). We look for solutions of (1.1)–(1.2) of the form

$$s = s_0 + \delta_1 A \quad \text{and} \quad u = V(s_0) + \delta_1 W, \tag{2.1}$$

where $0 < \delta_1 \ll 1$. To leading order in δ_1 we find that A satisfies

$$\epsilon \left(\frac{\partial^2 A}{\partial t^2} - P'(s_0) \frac{\partial^2 A}{\partial t \partial m} \right) = V'(s_0) \frac{\partial A}{\partial m} - \frac{\partial A}{\partial t}. \tag{2.2}$$

If we look for solutions of (2.2) of the form

$$A = \exp(ikm + \lambda t), \tag{2.3}$$

we find that λ and k satisfy

$$\epsilon \lambda^2 + (1 - ik\epsilon P'(s_0))\, \lambda - ikV'(s_0) = 0. \tag{2.4}$$

Moreover, if we write $\lambda = \alpha + i\beta$ (with α and β real), we obtain

$$\epsilon(\alpha^2 - \beta^2) + \alpha + k\epsilon P'(s_0)\beta = 0 \tag{2.5}$$

and

$$2\epsilon\alpha\beta + \beta - k\epsilon P'(s_0)\alpha - kV'(s_0) = 0. \tag{2.6}$$

If we restrict our attention to the case where $0 < \epsilon \ll 1$, we find one root goes as

$$\lambda_1 = -\frac{1}{\epsilon} + ik(P' - V')(s_0) + 0(\epsilon) \tag{2.7}$$

and the other as

$$\lambda_2 = ikV'(s_0) - \epsilon k^2 V'(s_0)(P' - V')(s_0) + 0(\epsilon^2) \tag{2.8}$$

and it is the latter identity which allows us to conclude that the system is linearly stable when $(P' - V')(s_0) > 0$ and linearly unstable when $(P' - V')(s_0) < 0$.

A similar conclusion may be reached if we apply a Chapman–Enskog procedure to (1.1) and (1.2) when $0 < \epsilon \ll 1$. Specifically, we seek solutions to (1.1) and (1.2) where u is of the form

$$u = U^1 = u^0 + \epsilon u^1 \tag{2.9}$$

and u^0 and u^1 are independent of ϵ and functionals of s. Insertion of the ansatz (2.9) into (1.2) yields

$$\begin{aligned} &u^0 = V(s), u^1 = V'(s)\,(P'(s) - V'(s))\,\frac{\partial s}{\partial m} \quad \text{and} \\ &U^1 = V(s) + \epsilon V'(s)\,(P'(s) - V'(s))\,\frac{\partial s}{\partial m}. \end{aligned} \tag{2.10}$$

Then s is determined by solving

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial m} \left(V(s) + \epsilon V'(s)\,(P'(s) - V'(s))\,\frac{\partial s}{\partial m} \right). \tag{2.11}$$

This latter equation has a strong maximum principle so long as the initial data for s satisfies either

$$L \le s(m, 0) < s_1 \quad \text{for all } m \tag{2.12}$$

or

$$s_2 \le s(m, 0) < \infty \quad \text{for all } m \tag{2.13}$$

because in either of these cases the diffusion coefficient, $V'(s)(P'(s) - V'(s))$, is positive. On the other hand, when $s_1 < s < s_2$, the diffusion coefficient is negative and this yields explosive growth of the solution, confirming the instability of the constant solution (1.17) when $s_1 < s_0 < s_2$.

3. Large amplitude periodic traveling waves. In this section we seek solutions to (1.1) and (1.2) that are functions of

$$\xi = m + ct, \quad c > 0, \tag{3.1}$$

which are periodic in ξ with periodic M, the number of cars on the ring-road. The conversation structure of (1.1) implies that the $s(\cdot)$ component of the solution satisfies

$$\int_0^M s(\xi) d\xi = l, \tag{3.2}$$

where l is the length of the ring-road.

Insertion of the ansatz (3.1) into (1.1) implies that $u(\cdot)$ and $s(\cdot)$ satisfy

$$u(\xi) = u_{\#} + c(s(\xi) - s_{\#}), \tag{3.3}$$

and we insist that

$$u_{\#} = V(s_{\#}) \quad \text{and} \quad s(0) = s_{\#} \in (s_1, s_2). \tag{3.4}$$

The relations (3.3) and (3.4) further imply that

$$\epsilon c \left(c - P'(s)\right) \frac{ds}{d\xi} = \left(V(s) - V(s_{\#}) - c(s - s_{\#})\right). \tag{3.5}$$

We seek a solution to (3.4) and (3.5) which is increasing on $-m_a < \xi < M_a$, where $-m_a < 0 < M_a$. For speeds $0 < c < V'(s_{\#})$, we see that the right-hand side of (3.5) satisfies

$$\text{sign } \left(V(s) - V(s_{\#}) - c(s - s_{\#})\right) = \text{sign } (s - s_{\#}) \tag{3.6}$$

for $|s - s_{\#}|$ small enough, and thus to obtain an increasing solution to (3.4) and (3.5) on some interval containing $\xi = 0$ in its interior we are compelled to choose

$$c = P'(s_{\#}). \tag{3.7}$$

This choice of c, together with the hypothesis that $P''(\cdot) < 0$, guarantees that

$$\text{sign } \left(P'(s_{\#}) - P'(s)\right) = \text{ sign } (s - s_{\#}), \tag{3.8}$$

and thus, with this choice of c, we are guaranteed a solution of (3.4) and (3.5) defined in some interval $-\tilde{m}_a < \xi < \tilde{M}_a$, where $-\tilde{m}_a < 0 < \tilde{M}_a$. Moreover, this solution satisfies

$$\frac{ds}{d\xi}(0) = \frac{-\left(V'(s_{\#}) - P'(s_{\#})\right)}{\epsilon P'(s_{\#}) P''(s_{\#})} > 0 \tag{3.9}$$

for $s_1 < s_{\#} < s_2$.

We shall now refine the observations of the preceding paragraphs. If

$$V(L) - V(s_2) - P'(s_2)(L - s_2) > 0, \tag{3.10}$$

we let $\bar{s}$ in (s_1, s_2) be the unique solution of

$$V(L) - V(\bar{s}) - P'(\bar{s})(L - \bar{s}) = 0, \tag{3.11}$$

whereas, if

$$V(L) - V(s_2) - P'(s_2)(L - s_2) \leq 0, \tag{3.12}$$

we let

$$\bar{s} = s_2. \tag{3.13}$$

In either case, for any $s_\#$ in $(s_1, \bar{s})$ we let $L < s_-(s_\#) < s_\# < s_+(s_\#)$ be the other two solutions of

$$V(s_\pm) - V(s_\#) - P'(s_\#)(s_\pm - s_\#) = 0. \tag{3.14}$$

We of course have

$$V(s) - V(s_\#) - P'(s_\#)(s - s_\#) < 0, \quad s_-(s_\#) < s < s_\#, \tag{3.15}$$

and

$$V(s) - V(s_\#) - P'(s_\#)(s - s_\#) > 0, \quad s_\# < s < s_+(s_\#). \tag{3.16}$$

For any s_a in $(s_-(s_\#), s_\#)$ we now let $S(s_a) > s_\#$ be the unique solution of

$$\frac{P(S(s_a)) - P(s_a)}{S(s_a) - s_a} = P'(s_\#) \tag{3.17}$$

and note that

$$\frac{dS(s_a)}{ds_a} = \frac{(P'(s_\#) - P'(s_a))}{(P'(s_\#) - P'(S(s_a)))} < 0. \tag{3.18}$$

We also let $\underline{s}(s_\#)$ be the smallest value of $s_a \geq s_-(s_\#)$ such that $S(s_a) \leq s_+(s_\#)$ and for any s_a in $(\underline{s}(s_\#), s_\#)$ we let

$$-m_a = \epsilon P'(s_\#) \int_{s_a}^{s_\#} \frac{(P'(r) - P'(s_\#))\, dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} < 0 \tag{3.19}$$

and

$$M_a = \epsilon P'(s_\#) \int_{s_\#}^{S(s_a)} \frac{(P'(s_\#) - P'(r))dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} > 0. \tag{3.20}$$

We note that one of the integrals (3.19) or (3.20) or both diverge as $s_a \to \underline{s}(s_\#)^+$. For any ξ in $(-m_a, M_a)$, the solution to (3.4) and (3.5) is given by the quadrature formula

$$\epsilon P'(s_\#) \int_{s_\#}^{s(\xi)} \frac{(P'(s_\#) - P'(r))\, dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} = \xi, \tag{3.21}$$

and the solution is extended to $(-\infty, \infty)$ by insisting that the periodicity condition

$$s\left(\xi \pm n(m_a + M_a)\right) = s(\xi), \quad n = 0, 1, \dots, \tag{3.22}$$

holds. As constructed, the solution has jump discontinuities as the points $M_a \pm n(m_a + M_a)$, $n = 0, 1, \dots$, and (3.17), (3.19), and (3.20) guarantee that the Rankine–Hugoniot condition for (1.1) and (1.2) holds across these discontinuities. The Lax entropy condition that $s^-(M_a \pm n(m_a+M_a)) > s^+(M_a \pm n(m_a+M_a))$ is also guaranteed since

$$s^-(M_a \pm n(m_a+M_a)) = S(s_a) > s_a = s^+(M_a \pm n(m_a+M_a)). \tag{3.23}$$

What remains to be shown is that for integers $k = 1, 2, \ldots$ we can choose s_a in $(\underline{s}(s_\#), s_\#)$ and $s_\#$ in $(s_1, \overline{s})$ so that

$$k(m_a+M_a) = M \tag{3.24}$$

and

$$\int_0^M s(\xi) d\xi = l. \tag{3.25}$$

The integer k represents the number of increasing segments per period.

We start by analyzing (3.24). Equations (3.19) and (3.20) imply that solving (3.24) is equivalent to solving

$$k\epsilon P'(s_\#) \int_{s_a}^{S(s_a)} \frac{(P'(s_\#) - P'(r))\, dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} \overset{def}{=} F(s_\#, s_a) = M. \tag{3.26}$$

We observe for any $s_\#$ in $(s_1, \overline{s})$ that

$$F(s_\#, s_\#) = 0, \tag{3.27}$$

$$\begin{aligned} \frac{\partial F}{\partial s_a}(s_\#, s_a) = k\epsilon P'(s_\#) \Bigg(& \frac{(P'(s_\#) - P'(S(s_a))S'(s_a))}{(V(S(s_a)) - V(s_\#) - P'(s_\#)\,(S(s_a) - s_\#))} \\ & + \frac{(P'(s_a) - P'(s_\#))}{(V(s_a) - V(s_\#) - P'(s_\#)(s_a - s_\#))} \Bigg) \end{aligned} \tag{3.28}$$

for any s_a in $(\underline{s}(s_\#), s_\#)$, and, finally, that

$$\lim_{s_a \to \underline{s}(s_\#)^+} F(s_\#, s_a) = +\infty. \tag{3.29}$$

Then (3.27)–(3.29) guarantee that for each $s_\#$ in $(s_1, \overline{s})$ there is a unique number $s_a(s_\#)$ in $(\underline{s}(s_\#), s_\#)$ satisfying (3.26). Thus, solving (3.24) and (3.25) is equivalent to finding an $s_\#$ in $(s_1, \overline{s})$ such that

$$k\epsilon P'(s_\#) \int_{s_a(s_\#)}^{S(s_a(s_\#))} \frac{(P'(s_\#) - P'(r))\, r dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} = l. \tag{3.30}$$

The last identity is a consequence of (3.25) and the fact that on $(-m_a, M_a)$

$$\frac{d\xi}{dr} = \frac{k\epsilon P'(s_\#)\,(P'(s_\#) - P'(r))}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))}. \tag{3.31}$$

If we exploit the fact that $s_a(s_\#)$ satisfies (3.26), we find that solving (3.30) is equivalent to solving

$$Ms_\# + k\epsilon P'(s_\#) \int_{s_a(s_\#)}^{S(s_a(s_\#))} \frac{(P'(s_\#) - P'(r))\,(r - s_\#) dr}{(V(r) - V(s_\#) - P'(s_\#)(r - s_\#))} = l. \tag{3.32}$$

To get some idea about the range of the function defined by the left-hand side of (3.32) we note that

$$\operatorname{sign}\left(\frac{(P'(s_\#)-P'(r))\,(r-s_\#)}{(V(r)-V(s_\#))-P'(s_\#)(r-s_\#)}\right)=\operatorname{sign}\,(r-s_\#) \tag{3.33}$$

and that for r close to $s_\#$

$$\frac{(P'(s_\#)-P'(r))\,(r-s_\#)}{(V(r)-V(s_\#)-P'(s_\#)(r-s_\#))}\sim\frac{-P''(s_\#)(r-s_\#)}{(V'-P')\,(s_\#)}. \tag{3.34}$$

So long as $s_1 < s_\# < \overline{s}$ we have $s_-(s_\#) < s_a(s_\#)$ and $S(s_a(s_\#)) < s_+(s_\#)$ and the integrand $\frac{\left(P'(s_\#)-P'(r)\right)(r-s_\#)}{(V(r)-V(s_\#)-P'(s_\#)(r-s_\#))}$ is nonsingular. In this case the function defined by the left-hand side of (3.32) is approximately given by

$$Ms_\# - \frac{k\epsilon P'(s_\#)P''(s_\#)(S(s_a(s_\#))-s_a(s_\#))(S(s_a(s_\#))+s_a(s_\#)-2s_\#)}{2(V'-P')(s_\#)}. \tag{3.35}$$

This last identity is instructive, especially in the situation where $P''(\cdot)$ is approximately constant. In that case $S(s_a(s_\#))+s_a(s_\#)-2s_\#$ is approximately zero and thus the function defined by (3.35) approximately reduces to $Ms_\#$. Equation (3.30) then approximately becomes

$$Ms_\# = l. \tag{3.36}$$

This sort of analysis on the function defined by the left-hand side of (3.30) is all we could manage with the degree of generality allowed on the functions $P(\cdot)$ and $V(\cdot)$. Though not particularly sharp it gives a fair indication of when (3.24) and (3.25) are solvable.

4. Simulations. All computations in this section were run with the follow-the-leader model (1.7) and (1.8) when

$$P(s)=\lambda\left(1-\frac{L}{s}\right),\quad L\le s, \tag{4.1}$$

and

$$V(s)=v_\infty\frac{\left(\tanh\left(\frac{s-rL}{\delta}\right)+\tanh\left(\frac{(r-1)L}{\delta}\right)\right)}{\left(1+\tanh\left(\frac{(r-1)L}{\delta}\right)\right)}. \tag{4.2}$$

The specific parameters used were

$$L = 15 \text{ feet}, \tag{4.3}$$

$$\lambda = 150 \text{ feet/sec } = 102.2727\ldots \text{ mph}, \tag{4.4}$$

$$v_\infty = 100 \text{ feet/sec } = 68.1818\ldots \text{ mph}, \tag{4.5}$$

$$\delta = 15 \text{ feet}, \tag{4.6}$$

and

$$r = 3. \tag{4.7}$$

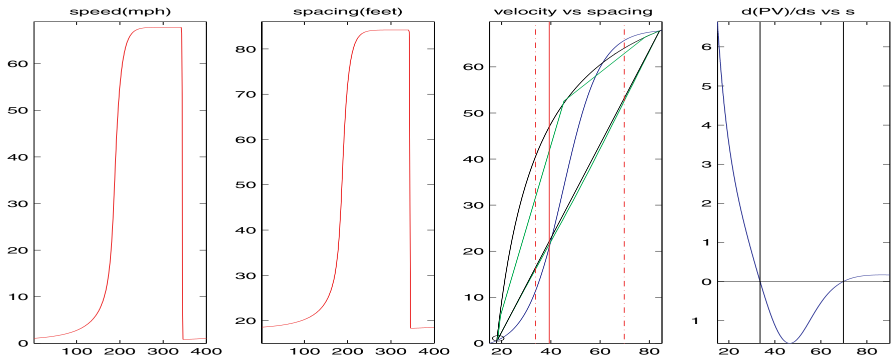

Fig. 1.

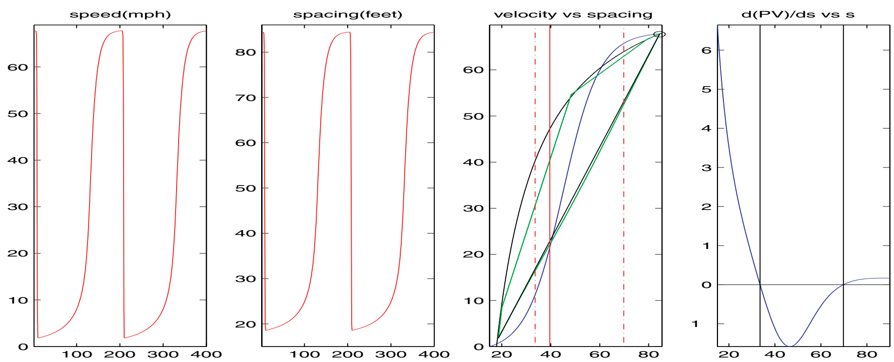

Fig. 2.

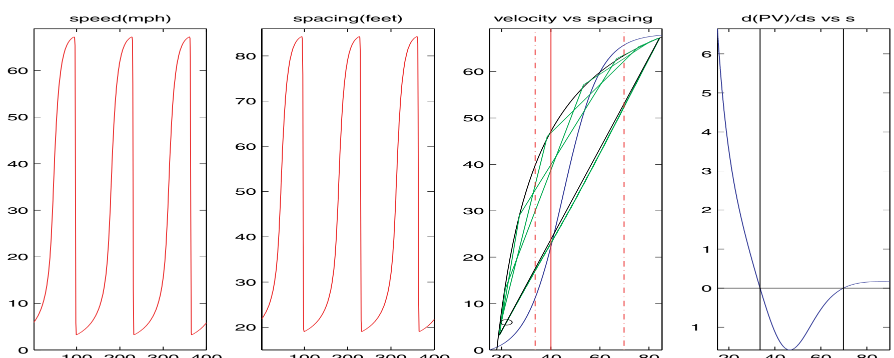

Fig. 3.

For initial data, we choose three sets of data

$$x_m^{(k)}(0) = 45m + 30 \sum_{j=0}^{m-1} \sin\left(\frac{kj\pi}{200}\right) \tag{4.8}$$

and

$$u_m^{(k)}(0) = 35 \text{ feet/sec} \tag{4.9}$$

for $m = 0, \pm 1, \pm 2, \dots$ and $k = 1, 2$, and 3. The observation that

$$x_{m+400}^{(k)}(0) = x_m^{(k)}(0) + 18000 \tag{4.10}$$

implies that we may interpret the data as initial data for a ring-road with 400 cars which is of length 18000 feet.

For our choice of parameter values the unstable region for $(P' - V')(\cdot)$ is the interval $33.59625\ldots < s < 69.8215$ and our data has initial car spacings

$$s_m^{(k)}(0) = x_{m+1}^{(k)}(0) - x_m^{(k)}(0) \tag{4.11}$$

which lie in that interval. A graph of $s \to (P' - V')(s)$ is shown in the fourth panel of Figures 1–3. Simulations were run with relaxation times

$$\epsilon = 1,\ 5,\ \text{and } 10. \tag{4.12}$$

We show the spatially periodic solutions at time $t = 1$ hour when $\epsilon = 10$ seconds. Figures 1, 2, and 3 correspond to the initial data indexed by $k = 1, 2$, and 3, respectively. The solution indexed by each particular k has k discontinuities per period after one hour. Run over a longer period, they all revert to a solution with one discontinuity per period.

The first two frames in each figure are self-explanatory. In the third frame of each figure we plot the curve $m \to (s_m = x_{m+1} - x_m, u_m)$. This curve is shown in green. The blue curve is the equilibrium curve $s \to (s, V(s))$ and the black concave curve is a suitably normalized image of $P(\cdot)$. The circle -o- is the image of (s_1, u_1).

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