Around the world traffic congestion causes a drain on the economy due to time and fuel wasted and collisions associated with traffic congestion. In 2011 congestion in America alone caused a loss of \$121 billion due to 5.5 billion hours of extra travel time and 2.9 billion gallons of wasted fuel. Although congestion levels in America are much higher now than several decades ago, they have dropped below the peak in 2005, but will increase when the economy improves[26]. As personal vehicle ownership increases globally, traffic congestion continues to be a persistent problem. It is no surprise then that research in the physics of traffic is developed with the ultimate goal of mitigating road traffic. Traffic control strategies such as ramp metering and variable speed limits are in place today, but a good model of traffic dynamics is needed for proper coordination of control strategies.

The topic of traffic models is vast, ranging from dense arterial networks to large scale freeways. In this article we focus on homogeneous road sections, in particular no ramps or intersections are present. The 1950's saw the development of the Lighthill-Whitham-Richards (LWR) model [16, 25], which became the seminal model for traffic flow, still studied today. This model is a conservation law for vehicles, based on fluid dynamics. Let ρ the lineic vehicular density (veh/m) and q the traffic flux (veh/s), dynamics follow the equation $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$. The simplicity of the model enabled the formulation of numerical discretization schemes such as Godunov ([12, 22]) or cell transmission models ([8, 9]). These are used in everyday's transportation research (e.g. Berkeley Advanced Traffic Simulation system).

Yet as a first-order model it has inherent shortcomings. Most of these have been pointed out in [7]. The LWR model fails to describe realistically what happens when a vehicle crosses a shock. Although the Rankine-Hugoniot condition guarantees macroscopic mass conservation, at the microscopic scale the speed of traveling particle would be discontinuous. It is also well known that this first order model fails to predict light traffic dynamics accurately. Considering all drivers are if contradictory with platoons dissolving because users' desired speeds vary from one person to another. Stop and go behavior otherwise named traffic oscillations cannot appear in the model with expanding amplitudes although this phenomenon is observed in practice. This phenomenon has attracted more an more attention in transportation research. Jamitons, traffic jam that appears without the presence of a bottleneck have been reproduced in small experiments ([28, 1]) and explained theoretically as the result of a particular configuration of the traffic system ([11]) as well as an outcome of fuzzy fundamental diagrams ([27]). Empirical studies have focused on detecting and quantifying this effect ([33]) on actual freeways in day-to-day traffic. Car-following behaviors ([19]) and lane-changing ([6, 2]) are often seen as the cause of oscillations. Several approaches have been developed so as to model these empirical facts.

This article will focus on macroscopic second order models as opposed to microscopic and mesoscopic framework. Payne and Whitham have developed in parallel an identical system of equations that aimed at a finer traffic modeling ([23, 29]). Their main aim was describing accurately what happens inside a traffic shock. To that end, they followed the same approach as in fluid dynamics and used a higher order model that would capture momentum related features. The Payne-Whitham model (PW) therefore consists of a mass conservation equation identical to that of LWR seconded by a momentum equation $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{p'(\rho)}{\rho}\frac{\partial \rho}{\partial x} = \frac{V(\rho)-v}{\tau} + \nu\frac{\partial^2 v}{\partial x^2}$ where the pressure function p is strictly increasing and the speed equilibrium V is decreasing. As Daganzo pointed later in his review of second order models ([7]), this approach was flawed both in the derivation of its equations and its predictions. The formulation of PW relies on the assumption that spacing and speed are slow varying which would yield negligible 2^{nd} and 3^{rd} derivatives for these quantities. This is contradictory with the observations of Newell ([21]) where his car following model predicts sharp and quick changes in these quantities. In terms of physics, modeling traffic by a gas-like model was highly unrealistic as fluids feature an anisotropic propagation of the information. Things are very different with traffic where cars will mostly react to downstream conditions. This is incompatible with the slope of one of the characteristic lines being greater than cars' speeds in the PW model. Also, at the upstream end of a jam with low density, traffic would flow in reverse if $\nu \neq 0$ which is completely unrealistic. Finally, gas particles are physically identical when drivers have different habits and desired speeds. The heterogeneous composition of traffic is not taken into account in the PW model, hence another flaw.

After this requiem, second order traffic models where not forgotten though. Zhang improved the mathematical structure of his first model ([31]). At first it was highly similar to that of PW in its momentum equation $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \rho\left(V_*^{'}(\rho)\right)^2 \frac{\partial \rho}{\partial x} = \frac{V(\rho)-v}{\tau}$ where $\rho V_*^{'}(\rho)$ represents the information propagation speed in traffic. This improvement was conducted independently of Aw and Rascle's work and eventually gave birth to an identical model (ARZ) ([3, 30]). The second order equation in that model, $\frac{\partial(v+p(\rho))}{\partial t} + v\frac{\partial(v+p(\rho))}{\partial x} = \frac{\rho(V(\rho)-v)}{\tau}$, does not present any wrong way travels. Indeed it corresponds to a setup of the PW model where $\nu = 0$. Also, it does not feature any gas-like behavior that contradicts the elementary physics of traffic. The key element for the model was formulating the evolution of the pseudo-pressure term $v + p(\rho)$ in a convective manner. This indeed changed the information propagation from a gas-like anisotropic behavior to a quantity that would be carried by the particles in the system at speed v. This model has since been thoroughly studied. [3] concluded that a relaxation term accounting for traffic equilibrium was needed so the speed of cars would be determined by the fundamental diagram and not the initial data. The effect of the relaxation has been studied in [24] where the author proved that, interestingly, the relaxed model converges towards LWR when $\tau \to 0$. It has been showed in [14] that the first ARZ model needed a fundamental diagram extended for negative and maximum speeds, zero and maximum densities so as to guarantee solutions to the Riemann and stability with low densities. In [20] discretization enabled extending the AR model to a network setup with junctions and traffic lights. The ARZ model now has a legacy that started with the extended ARZ model ([15]). An extension was presented in [4] as a pressure-less limit of the AR model where drivers will not significantly slow down if the congestion is not heavy. More recently a generalized model has been created ([10]) that corrects the fact that, in the ARZ model, several maximum densities co-exist which seems contradictory with the fact that this quantity is uniquely determined by the characteristics of the freeway. Finally, a Godunov descretization scheme has been exhibited for the ARZ model in [18]. With an extended fundamental diagram, this framework was able to solve the equations for any Riemann problem. This opened the way to a numerical comparison where ARZ fitted the data better than LWR.

It has been remarked in [10] that for congested regimes the LWR tends to offer a slightly better fit with respect to empirical measurements and is outperformed for low densities. Our aim here is using a model that is suitable for all regimes so as to establish control strategies for traffic. Therefore, the ARZ model seems like the best option as it offers reliable predictions in all regimes. Moreover, it manages to model traffic oscillations in dense traffic realistically ([13]). Laplace transform and spectral analysis are powerful tools for control problems that provide a simple yet holistic represent of a system. In that regard, it is important to have a model that accounts realistically for oscillatory phenomena. Models other than second order macroscopic could have been chosen. Behavioral models such as [21] and more recently [5] depict in a detailed fashion the effects of car-following and lane changing on freeway dynamics. Unfortunately their formulation makes them hard to use for control purposes. Recent data driven approaches ([32]) that have tried to model spectral features of traffic thanks to wavelet transform are also not suitable. They are part of an approach where other data mining strategies could be used and do not focus on the dynamics core to the system.

Our control focused analysis based on the ARZ equations is strongly inspired by the analysis of Saint-Venant equations in [17]. Let v the fluid velocity in a canal, q the flux, T the top width and y the water height. One has a mass conservation equation similar to LWR ($T\frac{\partial y}{\partial t} + \frac{\partial Q}{\partial x} = 0$) and a momentum equation structurally similar to ARZ ($\frac{\partial v}{\partial t} + v\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} = g\left(S_b - S_f(x,t)\right)$). $S_b - S_f(x,t)$ is a friction slope equilibrium equation analogous to $\frac{V(\rho)-v}{\tau}$. Linearizing this system about an equilibrium point enabled the design of efficient control strategies for canals leveraging the realism of Saint-Venant equations and the elegance of spectral analysis for linear systems. Approximations for the low frequency domain decompose the transfer matrix into a combination of delay and integration components. This leads to setting up efficient PI controllers while enabling a simpler theoretical analysis of the hydraulic system.

This article will follow the steps taken in ([17]) and adapt them to the analogous ARZ equations so as to achieve a two-fold objective. First we aim at developing strategies that enforce the readability and easiness of use of the ARZ model. Solutions to these non-linear equations are practically hard to derive in an efficient manner. Linearizing the system about an equilibrium is therefore a sound strategy to open the way to usual control schemes with multiple inputs and outputs. Although theoretically close, hydrodynamics and traffic belong to different contexts. It will therefore be necessary, in particular when considering the formulation of boundary conditions, to carefully guarantee the well-posedness of the problem. The other objective is assessing the fit quality of the model by practically comparing its output with actual data collected as part of the NGSIM project.

The first main contribution of the article is therefore deriving the linearized equations of the ARZ model in a format that highlights the main properties the model should feature. An equivalent of the Froude number is created for traffic that separates free-flowing and congested regimes. The linear system will be diagonalized which provides a very easy to use set of equations for the Riemann invariants. Deriving time domain responses after formulating spectral transfer matrices will prove that the linearized system is unstable in the free-flowing regime and accounts for non-linear wave propagation giving rise to jamitons. An important contribution here is that these waves occur for an entire set of values of velocity, density and flux and lead the linearized system away from its equilibrium point in free-flowing regime. The spectral form we obtain is a formulation of a second order model consistent with the fact that information does not propagate faster than cars travel that is ready to use for control. In order to guarantee that linearization does not destroy the realistic properties of ARZ a numerical experiment is conducted. NGSIM data has been used previously so as to assess the quality of second order models' predictions ([18, 10]). However these studies focus on averaged errors and only display predictions at a couple of points on the freeway. Here we show an entire map of the states and conduct the model assessment in a holistic manner. This yields a complete analysis of the strengths and weaknesses of the model that would be used for control. An estimation procedure is also conducted on the NGSIM data that, unlike [10], does not rely on any assumption about the typical vehicle length or the safety distance factor. A new way of validating the estimation of macroscopic is presented that validates the whole data aggregation procedure. It is also described how in a linear system, Fast Fourier transform allows fast and simple numerical resolution for most boundary conditions. With this new prediction technique, no discretization scheme is needed and no grid size condition needs to be fulfilled. This procedure will prove that the linearized model successfully accounts for traffic oscillations and will also provide simple and consistent methods to estimate the parameters of the model.

The article is organized in two sections. The first one derives the linearized ARZ equation and their spectral domain form. It shows that this procedure can be conducted for any of the (v, q), (ρ, q), (ρ, v) couple of variables. Velocity and flux being the easiest values that can be observed and controlled in traffic, we will mainly focus on that representation and its diagonalized form. Conditions distinguishing regimes will be highlighted and the traffic Froude number exhibited. We will then derive distributed transfer matrices and analyze their properties thanks to Bode plots. The numerical analysis will be conducted in the second section. After presenting the data, estimation procedures for (v, q, ρ) and the parameters of the model, this study will confront empirical estimates with the numerical values predicted by the linearized model. We will explain how decomposing boundary input signals into fundamental elements thanks to the Fast Fourier Transform turns the spectral domain diagonalized form into a prediction tool.

References

[1]

[2] Soyoung Ahn and Cassidy Michael J. Freeway traffic oscillations and vehicle lane-change maneuvers. *Transportation and traffic theory 2007 : papers selected for presentation at ISTTT17, a peer reviewed series since 1959.*, pages 691–710, 2007.

[3] A. Aw and M. Rascle. Resurrection of second order models of traffic flow. *SIAM Journal of Applied Mathematics*, 60(3):916–938, 2000.

[4] Florent Berthelin, Pierre Degond, Marcello Delitala, and Michel Rascle. A model for the formation and evolution of traffic jams. *Archive for Rational Mechanics and Analysis*, 187(2):185–220, 2008.

[5] Danjue Chen, Jorge Laval, Zuduo Zheng, and Soyoung Ahn. A behavioral car-following model that captures traffic oscillations. *Transportation Research Part B: Methodological*, 46(6):744 – 761, 2012.

[6] B. Coifman, S. Krishnamurthy, and X. Wang. Lane-change maneuvers consuming freeway capacity. In SergeP. Hoogendoorn, Stefan Luding, PietH.L. Bovy, Michael Schreckenberg, and DietrichE. Wolf, editors, *Traffic and Granular Flow 03*, pages 3–14. Springer Berlin Heidelberg, 2005.

[7] C. Daganzo. Requiem for second-order fluid approximations of traffic flow. *Transportation Res. Part B*, 29(4):277–286, 1995.

[8] Carlos F Daganzo. The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory. *Transportation Research Part B: Methodological*, 28(4):269–287, 1994.

[9] Carlos F Daganzo. The cell transmission model, part ii: network traffic. *Transportation Research Part B: Methodological*, 29(2):79–93, 1995.

[10] S. Fan, M. Herty, and B. Seibold. Comparative model accuracy of a data-fitted generalized aw-rascle-zhang model. *Networks and Heterogeneous Media*, 9(2):239–268, 2014.

[11] M. R. Flynn, A. R. Kasimov, J. c. Nave, R. R. Rosales, and B. Seibold. Self-sustained nonlinear waves in traffic flow. *Physical Review E*, page 056113, 2009.

[12] S. K. Godunov. A difference scheme for numerical solution of discontinuous solution of hydrodynamic equations. *Math. Sbornik*, 47:271–306, 1969.

[13] J. M. Greenberg. Congestion redux. *SIAM J. Appl. Math*, 64:1175–1185.

[14] Jean-Patrick Lebacque, Salim Mammar, and Habib Haj-Salem. The aw-rascle and zhang's model: Vacuum problems, existence and regularity of the solutions of the riemann problem. *Transportation Research Part B: Methodological*, 41(7):710 – 721, 2007.

[15] Jean-Patrick Lebacque, Salim Mammar, and Habib Haj Salem. Generic second order traffic flow modelling. In *Transportation and Traffic Theory 2007. Papers Selected for Presentation at ISTTT17*, 2007.

[16] M. J. Lighthill and J. B Whitham. On kinematic waves. II: A theory of traffic flow on long crowded roads. *Proc. Royal. Soc.*, pages 317–345, 1955.

[17] Xavier Litrico and Vincent Fromion. *Modeling and control of hydrosystems*. Springer, 2009.

[18] Salim Mammar, Jean-Patrick Lebacque, and Habib Haj Salem. Riemann problem resolution and godunov scheme for the aw-rascle-zhang model. *Transportation Science*, 43(4):531–545, 2009.

[19] Michael Mauch and Michael J. Cassidy. Freeway traffic oscillations: Observations and predictions. In *In Proc. 15th Int. Symp. on Transportation and Traffic Theory*, pages 653–674. Elsevier, 2002.

[20] S. Moutari and M. Rascle. A hybrid lagrangian model based on the aw-rascle traffic flow model. *SIAM Journal on Applied Mathematics*, 68(2):413–436, 2007.

[21] G. F. Newell. Nonlinear effects in the dynamics of car following. *Operations Research*, 9(2):pp. 209–229, 1961.

[22] S. Osher. Riemann solvers, the entropy condition, and difference. *SIAM Journal on Numerical Analysis*, 21(2):217–235, 1984.

[23] H. J. Payne. *Models of Freeway Traffic and Control.* Simulation Councils, Incorporated, 1971.

[24] M. Rascle. An improved macroscopic model of traffic flow: derivation and links with the lighthill-whitham model. *Mathematical and computer modelling*, 35:581–590, 2002.

[25] Paul I. Richards. Shock waves on the highway. *Operations Research*, 4(1):pp. 42–51, 1956.

[26] D. Schrank, B. Eisele, , and T. Lomax. Urban mobility report. Technical report, Texas A and M Transportation Institute, 2012.

[27] B. Seibold, M. R. Flynn, A. R. Kasimov, and R. Ruben Rosales. Constructing set-valued fundamental diagrams from jamiton solutions in second order traffic models. *ArXiv e-prints*, April 2012.

[28] Yuki Sugiyama, Minoru Fukui, Macoto Kikuchi, Katsuya Hasebe, Akihiro Nakayama, Katsuhiro Nishinari, Shin ichi Tadaki, and Satoshi Yukawa. Traffic jams without bottlenecks, experimental evidence for the physical mechanism of the formation of a jam. *New J. Phys.*, 2008.

[29] G. B. Whitham. *Linear and Nonlinear Waves.* A Wiley-Interscience publication. Wiley, 1974.

[30] H. M. Zhang. A non-equilibrium traffic model devoid of gas-like behavior. *Transportation Res. Part B*, 36:275–290, 2002.

[31] H.M. Zhang. A theory of nonequilibrium traffic flow. *Transportation Research Part B: Methodological*, 32(7):485 – 498, 1998.

[32] Zuduo Zheng, Soyoung Ahn, Danjue Chen, and Jorge Laval. Freeway traffic oscillations: Microscopic analysis of formations and propagations using wavelet transform. *Procedia - Social and Behavioral Sciences*, 17(0):702 – 716, 2011. Papers selected for the 19th International Symposium on Transportation and Traffic Theory.

[33] Benjamin A Zielke, Robert L Bertini, and Martin Treiber. Empirical measurement of freeway oscillation characteristics: an international comparison. *Transportation Research Record: Journal of the Transportation Research Board*, 2088(1):57–67, 2008.