ARZ model

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1 Non linearized models

1.1 Saint-Venant vs ARZ

Compare Saint-Venant and ARZ, different coupling behaviors.
Both models rely on the fundamental relation $q(x,t) = v(x,t)\rho(x,t)$.

(ρ, v)	Saint-Venant	ARZ
First order	$\frac{\partial a}{\partial t} + \frac{\partial av}{\partial x} = 0$	$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$
Second order	$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{g}{T}\frac{\partial a}{\partial x} = g\left(S_b - S_f(x,t)\right)$	$\frac{\partial v}{\partial t} + \left(v + \rho V^{'}(\rho)\right)\frac{\partial v}{\partial x} = \frac{V(\rho)-v}{\tau}$

1.2 Different expressions for the ARZ model

ARZ (ρ, v)	First order	$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$
	Second order	$\frac{\partial v}{\partial t} + \left(v + \rho V^{'}(\rho)\right)\frac{\partial v}{\partial x} = 0$
ARZ (ρ, q)	First order	$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$
	Second order	$\frac{\partial q}{\partial t} - \frac{q}{\rho}\left(\frac{q}{\rho} + \rho V^{'}(\rho)\right)\frac{\partial \rho}{\partial x} + \left(2\frac{q}{\rho} + \rho V^{'}(\rho)\right)\frac{\partial q}{\partial x} = \frac{\rho V(\rho)-q}{\tau}$
ARZ (v, q)	Second order	$\frac{\partial v}{\partial t} + \left(v + \frac{q}{v}V^{'}\left(\frac{q}{v}\right)\right)\frac{\partial v}{\partial x} = \frac{V\left(\frac{q}{v}\right)-v}{\tau}$
	Second order	$\frac{\partial q}{\partial t} + \frac{q}{v}\left(v + \frac{q}{v}V^{'}\left(\frac{q}{v}\right)\right)\frac{\partial v}{\partial x} + v\frac{\partial q}{\partial x} = \frac{\frac{q}{v}V\left(\frac{q}{v}\right)-q}{\tau}$

2 Structure of solutions and relaxation time

One considers the (ρ, v) linearized system for example: $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ -\frac{1}{\tau} & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$.

2.1 Solving for ξ_1 along its characteristic lines

Let $x \in \mathbb{R}$ and $F_x^1 : t \to \xi_1(t, x + \lambda_1 t)$.

$\frac{dF_x^1}{dt}(t) = -\frac{1}{\tau}\xi_1(t, x + t\lambda_1) = -\frac{1}{\tau}F_x^1(t)$

Therefore, $F_x^1(t) = K_x e^{-\frac{t}{\tau}}$ and

$$\xi_1(t, x + \lambda_1 t) = \xi_1(0, x)\, e^{-\frac{t}{\tau}}$$

2.2 Solving for ξ_2 along its characteristic lines

Let $x \in \mathbb{R}$ and $F_x^2 : t \to \xi_2(t, x + \lambda_2 t)$.

$\frac{dF_x^2}{dt}(t) = -\frac{1}{\tau}\xi_1(t, x + t\lambda_2)$

Therefore, $F_x^2(t) = -\frac{1}{\tau}\int_{u=0}^{t} \xi_1(u, x + u\lambda_2)\,du + F_x^2(0)$.

$\xi_1(t, x + \lambda_2 t) = \xi_1(t, x - (\lambda_1 - \lambda_2)t + \lambda_1 t) = \xi_1(0, x - (\lambda_1 - \lambda_2)t)\,e^{-\frac{t}{\tau}}$

$$\xi_2(t, x + \lambda_2 t) = -\frac{1}{\tau}\int\limits_{u=0}^{t} \xi_1(0, x - (\lambda_1 - \lambda_2)u)\,e^{-\frac{u}{\tau}}du + \xi_2(0, x)$$

The general expression of the solution is $\begin{pmatrix} \xi_1(t, x + \lambda_1 t) \\ \xi_2(t, x + \lambda_2 t) \end{pmatrix} = \begin{pmatrix} \xi_1(0, x)\,e^{-\frac{t}{\tau}} \\ -\frac{1}{\tau}\int_{u=0}^{t} \xi_1(0, x - (\lambda_1 - \lambda_2)u)\,e^{-\frac{u}{\tau}}du + \xi_2(0, x) \end{pmatrix}$.

3 Linearized velocity flow system

The linearized velocity flow system is

$$\begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}_t + \underbrace{\begin{pmatrix} v^* + \frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right) & 0 \\ \frac{q^*}{v^*}\left(v^* + \frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right)\right) & v^* \end{pmatrix}}_{A} \begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}_x = \underbrace{\begin{pmatrix} -\frac{(v^*)^2 + q^*V'\left(\frac{q^*}{v^*}\right)}{(v^*)^2\tau} & \frac{V'\left(\frac{q^*}{v^*}\right)}{v^*\tau} \\ -\frac{q^*\left((v^*)^2 + q^*V'\left(\frac{q^*}{v^*}\right)\right)}{(v^*)^3\tau} & \frac{q^*V'\left(\frac{q^*}{v^*}\right)}{(v^*)^2} \end{pmatrix}}_{B} \begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}$$

3.1 Preliminary simplifications

So calculations are easier, matrices A and B will be expressed with ρ^* and v^* exclusively. Indeed, one has $q^* = \rho^* v^*$.

$A = \begin{pmatrix} v^* + \rho^*V'(\rho^*) & 0 \\ \rho^*(v^* + \rho^*V'(\rho^*)) & v^* \end{pmatrix}$

$B = \begin{pmatrix} -\frac{v^* + \rho^*V'(\rho^*)}{v^*\tau} & \frac{V'(\rho^*)}{v^*\tau} \\ -\frac{\rho^*\left(v^* + \rho^*V'(\rho^*)\right)}{v^*\tau} & \frac{\rho^*V'(\rho^*)}{v^*\tau} \end{pmatrix} = \frac{1}{v^*\tau}\begin{pmatrix} -\left(v^* + \rho^*V'(\rho^*)\right) & V'(\rho^*) \\ -\rho^*\left(v^* + \rho^*V'(\rho^*)\right) & \rho^*V'(\rho^*) \end{pmatrix}$

Let $\lambda_1 = v^*$ and $\lambda_2 = v^* + \rho^*V'(\rho^*) = v^* + \frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right)$. $V'(\rho^*) = {}^{\lambda_2 - \lambda_1}/_{\rho^*}$

The system simplifies as

$$\begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}_t + \begin{pmatrix} \lambda_2 & 0 \\ \rho^*\lambda_2 & \lambda_1 \end{pmatrix}\begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}_x = \frac{1}{v^*\tau}\begin{pmatrix} -\lambda_2 & \frac{\lambda_2 - \lambda_1}{\rho^*} \\ -\rho^*\lambda_2 & \lambda_2 - \lambda_1 \end{pmatrix}\begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}$$

$A = P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}P^{-1}$

With $P = \begin{pmatrix} 0 & \lambda_2 - \lambda_1 \\ 1 & \rho^*\lambda_2 \end{pmatrix}$.

As $P^{-1}BP = \frac{1}{v^*\tau}\begin{pmatrix} \frac{\rho^*\lambda_2}{\lambda_1 - \lambda_2} & 1 \\ -\frac{1}{\lambda_1 - \lambda_2} & 0 \end{pmatrix}\begin{pmatrix} -\lambda_2 & \frac{\lambda_2 - \lambda_1}{\rho^*} \\ -\rho^*\lambda_2 & \lambda_2 - \lambda_1 \end{pmatrix}\begin{pmatrix} 0 & \lambda_2 - \lambda_1 \\ 1 & \rho^*\lambda_2 \end{pmatrix} = \frac{1}{v^*\tau}\begin{pmatrix} -\lambda_1 & 0 \\ \frac{1}{\rho^*} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ \frac{1}{\tau q^*} & 0 \end{pmatrix}$

the system can be rewritten in the form

$$\begin{pmatrix} \widetilde{\xi_1} \\ \widetilde{\xi_2} \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \widetilde{\xi_1} \\ \widetilde{\xi_2} \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ \frac{1}{\tau q^*} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\xi_1} \\ \widetilde{\xi_2} \end{pmatrix}$$

with $\begin{pmatrix} \widetilde{\xi_1} \\ \widetilde{\xi_2} \end{pmatrix} = P^{-1} \begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} -\frac{\left(v^*+\rho^* V'(\rho^*)\right)}{V'(\rho^*)}\tilde{v} + \tilde{q} \\ \frac{1}{\rho^* V'(\rho^*)}\tilde{v} \end{pmatrix}$.

Therefore, with $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \widetilde{\xi_1} \\ -q^*\widetilde{\xi_2} \end{pmatrix} = \begin{pmatrix} -\frac{\left(v^*+\frac{q^*}{v^*} V'\left(\frac{q^*}{v^*}\right)\right)}{V'\left(\frac{q^*}{v^*}\right)}\tilde{v} + \tilde{q} \\ -\frac{v^*}{V'\left(\frac{q^*}{v^*}\right)}\tilde{v} \end{pmatrix}$.

The system becomes

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ -\frac{1}{\tau} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

We find the same relaxation time, as expected.

3.2 Preliminary study of the (v, q) system:

3.2.1 Diagonalized system:

One obtains $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ -\frac{1}{\tau} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$.

With $\begin{cases} \xi_1 = & -\frac{\left(v^*+\frac{q^*}{v^*} V'\left(\frac{q^*}{v^*}\right)\right)}{V'\left(\frac{q^*}{v^*}\right)}\tilde{v} + \tilde{q} \\ \xi_2 = & -\frac{v^*}{V'\left(\frac{q^*}{v^*}\right)}\tilde{v} \end{cases}$.

Once again the eigen values are $\begin{cases} \lambda_1 = & v^* \\ \lambda_2 = & v^* + \frac{q^*}{v^*} V'\left(\frac{q^*}{v^*}\right) \end{cases}$.

3.2.2 Froude number:

$F = \frac{q^* V'(\rho^*)}{(v^*)^2}$

$\lambda_2 > 0 \Leftrightarrow F < 1$

$\lambda_2 < 0 \Leftrightarrow F > 1$

3.2.3 Characteristics:

See figure

3.2.4 Laplace transform:

$\xi_t + A\xi_x = B\xi$ with $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Frequency domain:

Let $\mathcal{A}(s) = A^{-1}(B - sI) = \begin{pmatrix} -\frac{\frac{1}{\tau}+s}{\lambda_1} & 0 \\ -\frac{1}{\lambda_2 \tau} & -\frac{s}{\lambda_2} \end{pmatrix}$

$$\mathcal{B} = A^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix}$$

$$\frac{\partial \widehat{\xi}}{\partial x}(x,s) = \mathcal{A}(s)\,\widehat{\xi}(x,s) + \mathcal{B}\xi(x,t=0)$$

$$\mathcal{A}(s) = P(s)\begin{pmatrix} \nu_1(s) & 0 \\ 0 & \nu_2(s) \end{pmatrix} P^{-1}(s)$$

with

$$\nu_1(s) = -\frac{1+s\tau}{\lambda_1 \tau}$$

$$\nu_2(s) = -\frac{s}{\lambda_2}$$

and

$$P(s) = \begin{bmatrix} 0 & -\frac{s\lambda_1\tau - \lambda_2(1+s\tau)}{\lambda_1} \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\widehat{\xi}(x,s) = \Phi(x,s)\left(\widehat{\xi}(0,s) + \int_0^x \Phi^{-1}(v,s)\,\mathcal{B}\xi(v,0)\,dv\right)$$

with

$$\Phi(x,s) = P(s)\begin{pmatrix} exp(\nu_1(s)\,x) & 0 \\ 0 & exp(\nu_2(s)\,x) \end{pmatrix} P^{-1}(s)$$

therefore

$$\Phi(x,s) = \begin{bmatrix} e^{-\frac{sx}{\lambda_1}} e^{-\frac{x}{\lambda_1\tau}} & 0 \\ \frac{\lambda_1\left(e^{-\frac{sx}{\lambda_1}} e^{-\frac{x}{\lambda_1\tau}} - e^{-\frac{sx}{\lambda_2}}\right)}{s\tau(\lambda_1-\lambda_2)-\lambda_2} & e^{-\frac{sx}{\lambda_2}} \end{bmatrix} = \begin{bmatrix} \phi_{11}(x,s) & \phi_{12}(x,s) \\ \phi_{21}(x,s) & \phi_{22}(x,s) \end{bmatrix}$$

3.3 Input-output and time domain response with $\lambda_2 > 0$:

3.3.1 Frequency domain:

with zero initial conditions the transition equation writes

$$\begin{pmatrix} \widehat{\xi_1}(x,s) \\ \widehat{\xi_2}(x,s) \end{pmatrix} = \Phi(x,s)\begin{pmatrix} \widehat{\xi_1}(0,s) \\ \widehat{\xi_2}(0,s) \end{pmatrix}$$

Let

$$Q = \begin{bmatrix} \frac{\rho^*\lambda_2}{\lambda_1-\lambda_2} & 1 \\ \frac{\rho^*\lambda_1}{\lambda_1-\lambda_2} & 0 \end{bmatrix}$$

One has

$$\begin{pmatrix} \widehat{\widehat{v}}(x,s) \\ \widehat{\widehat{q}}(x,s) \end{pmatrix} = \underbrace{Q^{-1}\Phi(x,s)\,Q}_{\Psi(x,s)}\begin{pmatrix} \widehat{\widehat{v}}(0,s) \\ \widehat{\widehat{q}}(0,s) \end{pmatrix}$$

with

- $\psi_{11}(x,s) = -\frac{e^{-\frac{(1+s\tau)x}{\lambda_1\tau}}\lambda_2 - e^{-\frac{sx}{\lambda_2}}s\tau(\lambda_1-\lambda_2)}{s\tau(\lambda_1-\lambda_2)-\lambda_2}$
- $\psi_{12}(x,s) = -\frac{(\lambda_1-\lambda_2)\left(e^{-\frac{(1+s\tau)x}{\lambda_1\tau}} - e^{-\frac{sx}{\lambda_2}}\right)}{\rho^*(s\tau(\lambda_1-\lambda_2)-\lambda_2)}$
- $\psi_{21}(x,s) = \frac{\rho^* s\tau\lambda_2\left(e^{-\frac{(1+s\tau)x}{\lambda_1\tau}} - e^{-\frac{sx}{\lambda_2}}\right)}{s\tau(\lambda_1-\lambda_2)-\lambda_2}$
- $\psi_{22}(x,s) = -\frac{e^{-\frac{sx}{\lambda_2}}\lambda_2 - e^{-\frac{(1+s\tau)x}{\lambda_1\tau}}s\tau(\lambda_1-\lambda_2)}{s\tau(\lambda_1-\lambda_2)-\lambda_2}$

Let $\alpha = -\frac{\lambda_2}{\tau(\lambda_1-\lambda_2)} = \frac{v^* + \frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right)}{\tau\frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right)}$

- $\psi_{11}(x,s) = \left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{sx}{\lambda_2}}\right)\frac{\alpha}{s+\alpha} + e^{-\frac{sx}{\lambda_2}}$
- $\psi_{12}(x,s) = \frac{(\lambda_1-\lambda_2)}{\lambda_2\rho^*}\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{sx}{\lambda_2}}\right)\frac{\alpha}{s+\alpha}$
- $\psi_{21}(x,s) = \frac{\rho^*\lambda_2}{(\lambda_1-\lambda_2)}s\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{sx}{\lambda_2}}\right)\frac{1}{s+\alpha}$
- $\psi_{22}(x,s) = -\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{sx}{\lambda_2}}\right)\frac{\alpha}{s+\alpha} + e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}}$

3.3.2 Constants

- Constants:
  - $\lambda_2 = v^* + \frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right) > 0$
  - $\lambda_1 = v^* > 0$
  - $\lambda_1 - \lambda_2 = -\frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right) > 0$
  - $\alpha = -\frac{\lambda_2}{\tau(\lambda_1-\lambda_2)} < 0$

3.3.3 Fundamental responses in diagonal form:

- Expression of Φ:
  - $\phi_{11}(x,s) = e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}}$
  - $\phi_{12}(x,s) = 0$
  - $\phi_{11}(x,s) = -\alpha\frac{\lambda_1}{\lambda_2}\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{sx}{\lambda_2}}\right)\frac{1}{s+\alpha}$
  - $\phi_{12}(x,s) = e^{-\frac{sx}{\lambda_2}}$
- $\begin{pmatrix} \widetilde{\xi}_1(0,t) \\ \widetilde{\xi}_2(0,t) \end{pmatrix} = \begin{pmatrix} H(t) \\ 0 \end{pmatrix}$:
  - $\widetilde{\xi}_1(0,t) = e^{-\frac{x}{\lambda_1\tau}}H\left(t - \frac{x}{\lambda_1}\right)$

- $\widetilde{\xi}_2(0,t) = -\frac{\lambda_1}{\lambda_2}\left(e^{-\frac{x}{\lambda_1\tau}}\left(1-e^{-\alpha\left(t-\frac{x}{\lambda_1}\right)}\right)H\left(t-\frac{x}{\lambda_1}\right)-\left(1-e^{-\alpha\left(t-\frac{x}{\lambda_2}\right)}\right)H\left(t-\frac{x}{\lambda_2}\right)\right)$

- $\begin{pmatrix}\widetilde{\xi}_1(0,t)\\ \widetilde{\xi}_2(0,t)\end{pmatrix} = \begin{pmatrix}0\\ H(t)\end{pmatrix}$:
  - $\widetilde{\xi}_1(0,t) = 0$
  - $\widetilde{\xi}_2(0,t) = H\left(t-\frac{x}{\lambda_2}\right)$
- $\begin{pmatrix}\widetilde{\xi}_1(0,t)\\ \widetilde{\xi}_2(0,t)\end{pmatrix} = \begin{pmatrix}sin(\omega t)\\ 0\end{pmatrix}$:
  - $\widetilde{\xi}_1(0,t) = e^{-\frac{x}{\lambda_1\tau}}sin\left(\omega\left(t-\frac{x}{\lambda_1}\right)\right)H\left(t-\frac{x}{\lambda_1}\right)$
  - $\widetilde{\xi}_2(0,t) = -\frac{\lambda_1\alpha}{\lambda_2}\left(e^{-\frac{x}{\lambda_1\tau}}\kappa^{sin}_{\alpha,\omega}\left(t-\frac{x}{\lambda_1}\right)-\kappa^{sin}_{\alpha,\omega}\left(t-\frac{x}{\lambda_2}\right)\right)$
- $\begin{pmatrix}\widetilde{\xi}_1(0,t)\\ \widetilde{\xi}_2(0,t)\end{pmatrix} = \begin{pmatrix}0\\ sin(\omega t)\end{pmatrix}$:
  - $\widetilde{\xi}_1(0,t) = 0$
  - $\widetilde{\xi}_2(0,t) = sin\left(\omega\left(t-\frac{x}{\lambda_2}\right)\right)H\left(t-\frac{x}{\lambda_2}\right)$
- $\begin{pmatrix}\widetilde{\xi}_1(0,t)\\ \widetilde{\xi}_2(0,t)\end{pmatrix} = \begin{pmatrix}cos(\omega t)\\ 0\end{pmatrix}$:
  - $\widetilde{\xi}_1(0,t) = e^{-\frac{x}{\lambda_1\tau}}cos\left(\omega\left(t-\frac{x}{\lambda_1}\right)\right)H\left(t-\frac{x}{\lambda_1}\right)$
  - $\widetilde{\xi}_2(0,t) = -\frac{\lambda_1\alpha}{\lambda_2}\left(e^{-\frac{x}{\lambda_1\tau}}\kappa^{cos}_{\alpha,\omega}\left(t-\frac{x}{\lambda_1}\right)-\kappa^{cos}_{\alpha,\omega}\left(t-\frac{x}{\lambda_2}\right)\right)$
- $\begin{pmatrix}\widetilde{\xi}_1(0,t)\\ \widetilde{\xi}_2(0,t)\end{pmatrix} = \begin{pmatrix}0\\ cos(\omega t)\end{pmatrix}$:
  - $\widetilde{\xi}_1(0,t) = 0$
  - $\widetilde{\xi}_2(0,t) = cos\left(\omega\left(t-\frac{x}{\lambda_2}\right)\right)H\left(t-\frac{x}{\lambda_2}\right)$

3.3.4 Fundamental responses to step and sinusoidal stimulations in (v,q) domain

- $\begin{pmatrix}\widetilde{v}(0,t)\\ \widetilde{q}(0,t)\end{pmatrix} = \begin{pmatrix}H(t)\\ 0\end{pmatrix}$:
  - $\widetilde{v}(x,t) = \left(1-e^{-\alpha\left(t-\frac{x}{\lambda_1}\right)}\right)e^{-\frac{x}{\lambda_1\tau}}H\left(t-\frac{x}{\lambda_1}\right)+e^{-\alpha\left(t-\frac{x}{\lambda_2}\right)}H\left(t-\frac{x}{\lambda_2}\right)$
  - $\widetilde{q}(x,t) = \frac{\rho\lambda_2}{(\lambda_1-\lambda_2)}\left(e^{-\alpha\left(t-\frac{x}{\lambda_1}\right)}e^{-\frac{x}{\lambda_1\tau}}H\left(t-\frac{x}{\lambda_1}\right)-e^{-\alpha\left(t-\frac{x}{\lambda_2}\right)}H\left(t-\frac{x}{\lambda_2}\right)\right)$
- $\begin{pmatrix}\widetilde{v}(0,t)\\ \widetilde{q}(0,t)\end{pmatrix} = \begin{pmatrix}0\\ H(t)\end{pmatrix}$:

- $\widetilde{v}(x,t)=\frac{(\lambda_1-\lambda_2)}{\lambda_2\rho}\left(\left(1-e^{-\alpha\left(t-\frac{x}{\lambda_1}\right)}\right)e^{-\frac{x}{\lambda_1\tau}}H\left(t-\frac{x}{\lambda_1}\right)-\left(1-e^{-\alpha\left(t-\frac{x}{\lambda_2}\right)}\right)H\left(t-\frac{x}{\lambda_2}\right)\right)$
- $\widetilde{q}(x,t)=\left(1-e^{-\alpha\left(t-\frac{x}{\lambda_2}\right)}\right)H\left(t-\frac{x}{\lambda_2}\right)+e^{-\frac{x}{\lambda_1\tau}}e^{-\alpha\left(t-\frac{x}{\lambda_1}\right)}H\left(t-\frac{x}{\lambda_1}\right)$

- $\begin{pmatrix}\widetilde{v}(0,t)\\ \widetilde{q}(0,t)\end{pmatrix}=\begin{pmatrix}sin(\omega t)\\ 0\end{pmatrix}$:
  - $\widetilde{v}(x,t)=\alpha\left(e^{-\frac{x}{\lambda_1\tau}}\kappa^{sin}_{\alpha,w}\left(t-\frac{x}{\lambda_1}\right)-\kappa^{sin}_{\alpha,\omega}\left(t-\frac{x}{\lambda_2}\right)\right)+sin\left(\omega\left(t-\frac{x}{\lambda_2}\right)\right)$
  - $\widetilde{q}(x,t)=\frac{\rho^*\lambda_2}{(\lambda_1-\lambda_2)}\left(e^{-\frac{x}{\lambda_1\tau}}\iota^{sin}_{\alpha,\omega}\left(t-\frac{x}{\lambda_1}\right)-\iota^{sin}_{\alpha,\omega}\left(t-\frac{x}{\lambda_2}\right)\right)$
- $\begin{pmatrix}\widetilde{v}(0,t)\\ \widetilde{q}(0,t)\end{pmatrix}=\begin{pmatrix}0\\ sin(\omega t)\end{pmatrix}$:
  - $\widetilde{v}(x,t)=\frac{(\lambda_1-\lambda_2)}{\lambda_2\rho^*}\alpha\left(e^{-\frac{x}{\lambda_1\tau}}\kappa^{sin}_{\alpha,\omega}\left(t-\frac{x}{\lambda_1}\right)-\kappa^{sin}_{\alpha,w}\left(t-\frac{x}{\lambda_2}\right)\right)$
  - $\widetilde{q}(x,t)=-\alpha\left(e^{-\frac{x}{\lambda_1\tau}}\kappa^{sin}_{\alpha,\omega}\left(t-\frac{x}{\lambda_1}\right)-\kappa^{sin}_{\alpha,w}\left(t-\frac{x}{\lambda_2}\right)\right)+e^{-\frac{x}{\lambda_1\tau}}sin\left(\omega\left(t-\frac{x}{\lambda_1}\right)\right)$
- $\begin{pmatrix}\widetilde{v}(0,t)\\ \widetilde{q}(0,t)\end{pmatrix}=\begin{pmatrix}cos(\omega t)\\ 0\end{pmatrix}$:
  - $\widetilde{v}(x,t)=\alpha\left(e^{-\frac{x}{\lambda_1\tau}}\kappa^{cos}_{\alpha,w}\left(t-\frac{x}{\lambda_1}\right)-\kappa^{cos}_{\alpha,\omega}\left(t-\frac{x}{\lambda_2}\right)\right)+cos\left(\omega\left(t-\frac{x}{\lambda_2}\right)\right)$
  - $\widetilde{q}(x,t)=\frac{\rho^*\lambda_2}{(\lambda_1-\lambda_2)}\left(e^{-\frac{x}{\lambda_1\tau}}\iota^{cos}_{\alpha,\omega}\left(t-\frac{x}{\lambda_1}\right)-\iota^{cos}_{\alpha,\omega}\left(t-\frac{x}{\lambda_2}\right)\right)$
- $\begin{pmatrix}\widetilde{v}(0,t)\\ \widetilde{q}(0,t)\end{pmatrix}=\begin{pmatrix}0\\ cos(\omega t)\end{pmatrix}$:
  - $\widetilde{v}(x,t)=\frac{(\lambda_1-\lambda_2)}{\lambda_2\rho^*}\alpha\left(e^{-\frac{x}{\lambda_1\tau}}\kappa^{cos}_{\alpha,\omega}\left(t-\frac{x}{\lambda_1}\right)-\kappa^{cos}_{\alpha,w}\left(t-\frac{x}{\lambda_2}\right)\right)$
  - $\widetilde{q}(x,t)=-\alpha\left(e^{-\frac{x}{\lambda_1\tau}}\kappa^{cos}_{\alpha,\omega}\left(t-\frac{x}{\lambda_1}\right)-\kappa^{cos}_{\alpha,w}\left(t-\frac{x}{\lambda_2}\right)\right)+e^{-\frac{x}{\lambda_1\tau}}cos\left(\omega\left(t-\frac{x}{\lambda_1}\right)\right)$

3.4 Time domain response with $\lambda_2<0$:

$$\begin{pmatrix}\widehat{\xi_1}(x,s)\\ \widehat{\xi_2}(x,s)\end{pmatrix}=\Phi(x,s)\begin{pmatrix}\widehat{\xi_1}(0,s)\\ \widehat{\xi_2}(0,s)\end{pmatrix}$$

therefore

$$\begin{pmatrix}\widehat{\xi_1}(x,s)\\ \widehat{\xi_2}(x,s)\end{pmatrix}=\Phi(x,s)\begin{pmatrix}1 & 0\\ -\frac{\phi_{21}(L,s)}{\phi_{22}(L,s)} & \frac{1}{\phi_{22}(L,s)}\end{pmatrix}\begin{pmatrix}\widehat{\xi_1}(0,s)\\ \widehat{\xi_2}(L,s)\end{pmatrix}$$

let

$$\mathit{\Gamma}(x,s)=\begin{pmatrix}e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} & 0\\ \alpha\frac{\lambda_1}{\lambda_2}\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}}-e^{-\frac{L}{\lambda_1\tau}}e^{-\frac{s}{\lambda_2}\left(x-L\frac{\lambda_1-\lambda_2}{\lambda_1}\right)}\right)\frac{1}{s+\alpha} & e^{-\frac{s(x-L)}{\lambda_2}}\end{pmatrix}$$

- $\gamma_{11}(x,s)=e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}}$

- $\gamma_{12}(x,s) = 0$
- $\gamma_{21}(x,s) = \alpha\frac{\lambda_1}{\lambda_2}\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{L}{\lambda_1\tau}}e^{-\frac{s}{\lambda_2}\left(x-L\frac{\lambda_1-\lambda_2}{\lambda_1}\right)}\right)\frac{1}{s+\alpha}$
- $\gamma_{22}(x,s) = e^{-\frac{s(x-L)}{\lambda_2}}$

and as

$$\begin{pmatrix} \widehat{\xi}_1(0,s) \\ \widehat{\xi}_2(L,s) \end{pmatrix} = \begin{pmatrix} \frac{\rho\lambda_2}{\lambda_1-\lambda_2}\widehat{\widetilde{v}}(0,s) + \widehat{\widetilde{q}}(0,s) \\ \frac{\rho\lambda_1}{\lambda_1-\lambda_2}\widehat{\widetilde{v}}(L,s) \end{pmatrix}$$

one obtains

$$\begin{pmatrix} \widehat{\widetilde{v}}(x,s) \\ \widehat{\widetilde{q}}(x,s) \end{pmatrix} = Q^{-1}\Phi(x,s)\begin{pmatrix} 1 & 0 \\ -\frac{\phi_{21}(L,s)}{\phi_{22}(L,s)} & \frac{1}{\phi_{22}(L,s)} \end{pmatrix}\begin{pmatrix} \frac{\rho\lambda_2}{\lambda_1-\lambda_2} & 1 & 0 \\ 0 & 0 & \frac{\rho\lambda_1}{\lambda_1-\lambda_2} \end{pmatrix}\begin{pmatrix} \widehat{\widetilde{v}}(0,s) \\ \widehat{q}(0,s) \\ \widehat{\widetilde{v}}(L,s) \end{pmatrix}$$

which finally gives

$$\begin{pmatrix} \widehat{\widetilde{v}}(x,s) \\ \widehat{\widetilde{q}}(x,s) \end{pmatrix} = \Upsilon(x,s)\begin{pmatrix} \widehat{\widetilde{v}}(0,s) \\ \widehat{q}(0,s) \\ \widehat{\widetilde{v}}(L,s) \end{pmatrix}$$

where

- $\Upsilon_{11}(x,s) = -\frac{\lambda_2\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{L}{\lambda_1\tau}}e^{-\frac{s}{\lambda_2}\left(x-L\frac{\lambda_1-\lambda_2}{\lambda_1}\right)}\right)}{s\tau(\lambda_1-\lambda_2)-\lambda_2}$
- $\Upsilon_{12}(x,s) = -\frac{(\lambda_1-\lambda_2)}{\rho}\frac{\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{L}{\lambda_1\tau}}e^{-\frac{s}{\lambda_2}\left(x-L\frac{\lambda_1-\lambda_2}{\lambda_1}\right)}\right)}{s\tau(\lambda_1-\lambda_2)-\lambda_2}$
- $\Upsilon_{13}(x,s) = e^{-\frac{s(x-L)}{\lambda_2}}$
- $\Upsilon_{21}(x,s) = \frac{\rho\lambda_2}{(\lambda_1-\lambda_2)}\frac{\left(s\tau(\lambda_1-\lambda_2)e^{-\frac{-x}{\lambda_1\tau}}e^{-\frac{-sx}{\lambda_1}} - \lambda_2 e^{-\frac{L}{\lambda_1\tau}}e^{-\frac{s}{\lambda_2}\left(x-L\frac{\lambda_1-\lambda_2}{\lambda_1}\right)}\right)}{s\tau(\lambda_1-\lambda_2)-\lambda_2}$
- $\Upsilon_{22}(x,s) = \frac{s\tau(\lambda_1-\lambda_2)e^{-\frac{-x}{\lambda_1\tau}}e^{-\frac{-sx}{\lambda_1}} - \lambda_2 e^{-\frac{L}{\lambda_1\tau}}e^{-\frac{s}{\lambda_2}\left(x-L\frac{\lambda_1-\lambda_2}{\lambda_1}\right)}}{s\tau(\lambda_1-\lambda_2)-\lambda_2}$
- $\Upsilon_{23}(x,s) = -\frac{\rho\lambda_2}{(\lambda_1-\lambda_2)}e^{-\frac{s(x-L)}{\lambda_2}}$

A simplified version of these expressions is

- $\Upsilon_{11}(x,s) = \alpha\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{L}{\lambda_1\tau}}e^{-\frac{s}{\lambda_2}\left(x-L\frac{\lambda_1-\lambda_2}{\lambda_1}\right)}\right)\frac{1}{s+\alpha}$
- $\Upsilon_{12}(x,s) = -\frac{1}{\rho\tau}\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{L}{\lambda_1\tau}}e^{-\frac{s}{\lambda_2}\left(x-L\frac{\lambda_1-\lambda_2}{\lambda_1}\right)}\right)\frac{1}{s+\alpha}$
- $\Upsilon_{13}(x,s) = e^{-\frac{s(x-L)}{\lambda_2}}$
- $\Upsilon_{21}(x,s) = \frac{\rho\lambda_2}{\lambda_1-\lambda_2}\left(s\cdot e^{-\frac{-x}{\lambda_1\tau}}e^{-\frac{-sx}{\lambda_1}} + \alpha\cdot e^{-\frac{L}{\lambda_1\tau}}e^{-\frac{s}{\lambda_2}\left(x-L\frac{\lambda_1-\lambda_2}{\lambda_1}\right)}\right)\frac{1}{s+\alpha}$

- $\Upsilon_{22}(x,s) = \left(s \cdot e^{-\frac{-x}{\lambda_1 \tau}} e^{-\frac{-sx}{\lambda_1}} + \alpha \cdot e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2}\left(x - L\frac{\lambda_1 - \lambda_2}{\lambda_1}\right)} \right) \frac{1}{s+\alpha}$

- $\Upsilon_{23}(x,s) = \tau \rho \alpha \cdot e^{-\frac{s(x-L)}{\lambda_2}}$

3.4.1 Fundamental responses in diagonal form:

- $\begin{pmatrix} \widetilde{\xi}_1(0,t) \\ \widetilde{\xi}_2(0,t) \end{pmatrix} = \begin{pmatrix} H(t) \\ 0 \end{pmatrix}$:

  - $\widetilde{\xi}_1(0,t) = e^{-\frac{x}{\lambda_1 \tau}} H\left(t - \frac{x}{\lambda_1}\right)$
  - $\widetilde{\xi}_2(0,t) = \frac{\lambda_1}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} \left(1 - e^{-\alpha\left(t - \frac{x}{\lambda_1}\right)}\right) H\left(t - \frac{x}{\lambda_1}\right) - e^{-\frac{L}{\lambda_1 \tau}} \left(1 - e^{-\alpha\left(t - \frac{x - L\frac{\lambda_1 - \lambda_2}{\lambda_1}}{\lambda_2}\right)}\right) H\left(t - \frac{x - L\frac{\lambda_1 - \lambda_2}{\lambda_1}}{\lambda_2}\right) \right)$

- $\begin{pmatrix} \widetilde{\xi}_1(0,t) \\ \widetilde{\xi}_2(0,t) \end{pmatrix} = \begin{pmatrix} 0 \\ H(t) \end{pmatrix}$:

  - $\widetilde{\xi}_1(0,t) = 0$
  - $\widetilde{\xi}_2(0,t) = H\left(t - \frac{x-L}{\lambda_2}\right)$

- $\begin{pmatrix} \widetilde{\xi}_1(0,t) \\ \widetilde{\xi}_2(0,t) \end{pmatrix} = \begin{pmatrix} sin(\omega t) \\ 0 \end{pmatrix}$:

  - $\widetilde{\xi}_1(0,t) = e^{-\frac{x}{\lambda_1 \tau}} sin\left(\omega\left(t - \frac{x}{\lambda_1}\right)\right) H\left(t - \frac{x}{\lambda_1}\right)$
  - $\widetilde{\xi}_2(0,t) = \frac{\lambda_1 \alpha}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa^{sin}_{\alpha,\omega}\left(t - \frac{x}{\lambda_1}\right) - e^{-\frac{L}{\lambda_1 \tau}} \kappa^{sin}_{\alpha,\omega}\left(t - \frac{x - L\frac{\lambda_1 - \lambda_2}{\lambda_1}}{\lambda_2}\right) \right)$

- $\begin{pmatrix} \widetilde{\xi}_1(0,t) \\ \widetilde{\xi}_2(0,t) \end{pmatrix} = \begin{pmatrix} 0 \\ sin(\omega t) \end{pmatrix}$:

  - $\widetilde{\xi}_1(0,t) = 0$
  - $\widetilde{\xi}_2(0,t) = sin\left(\omega\left(t - \frac{x-L}{\lambda_2}\right)\right) H\left(t - \frac{x-L}{\lambda_2}\right)$

- $\begin{pmatrix} \widetilde{\xi}_1(0,t) \\ \widetilde{\xi}_2(0,t) \end{pmatrix} = \begin{pmatrix} cos(\omega t) \\ 0 \end{pmatrix}$:

  - $\widetilde{\xi}_1(0,t) = e^{-\frac{x}{\lambda_1 \tau}} cos\left(\omega\left(t - \frac{x}{\lambda_1}\right)\right) H\left(t - \frac{x}{\lambda_1}\right)$
  - $\widetilde{\xi}_2(0,t) = \frac{\lambda_1 \alpha}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa^{cos}_{\alpha,\omega}\left(t - \frac{x}{\lambda_1}\right) - e^{-\frac{L}{\lambda_1 \tau}} \kappa^{cos}_{\alpha,\omega}\left(t - \frac{x - L\frac{\lambda_1 - \lambda_2}{\lambda_1}}{\lambda_2}\right) \right)$

- $\begin{pmatrix} \widetilde{\xi}_1(0,t) \\ \widetilde{\xi}_2(0,t) \end{pmatrix} = \begin{pmatrix} 0 \\ cos(\omega t) \end{pmatrix}$:

  - $\widetilde{\xi}_1(0,t) = 0$
  - $\widetilde{\xi}_2(0,t) = cos\left(\omega\left(t - \frac{x-L}{\lambda_2}\right)\right) H\left(t - \frac{x-L}{\lambda_2}\right)$

4 Bode plots

4.1 Spatial transforms

Initial domain: (v, q).

Diagonalization basis: $\begin{cases} \xi_1 &= \frac{\rho^* \lambda_2}{\lambda_1 - \lambda_2} v + q \\ \xi_2 &= \frac{\rho^* \lambda_1}{\lambda_1 - \lambda_2} v \end{cases}$, $\begin{cases} v &= \frac{\lambda_1 - \lambda_2}{\rho^* \lambda_1} \xi_2 \\ q &= \lambda_1 \xi_1 - \lambda_2 \xi_2 \end{cases}$

4.2 Eigen values from fundamental diagram

$\lambda_1 = \frac{q(\rho*)}{\rho*}$, $\lambda_2 = q'(\rho^*)$

4.3 Values for parameters

$L = 100$ (meters)

4.3.1 Greenshields fondamental diagram

$q(\rho) = 4 \frac{q_m}{\rho_m^2} \rho (\rho_m - \rho)$

$q'(\rho) = 4 \frac{q_m}{\rho_m} \left(1 - 2 \frac{\rho}{\rho_m}\right)$

$\rho_m = 0.1$ veh/m

$q_m = 0.36$ veh/s

$\tau = 15$s

Annex A:

Useful frequency to time domain conversions

In the transfer functions that have been computed, several components are recurrent:

- $s \rightarrow \frac{1}{s+\alpha}$
- $s \rightarrow \frac{s}{s+\alpha}$
- $s \rightarrow e^{-\theta s}$

The last item is an usual θ delay.

The inputs we will use are:

- $s \rightarrow \frac{1}{s}$ (step function)
- $s \rightarrow \frac{\omega}{s^2+\omega^2}$ ($sin(\omega t) H(t)$ in the time domain)
- $s \rightarrow \frac{s}{s^2+\omega^2}$ ($cos(wt) H(t)$ in the time domain)

Therefore, we need to compute the inverse transforms of the following functionals in the frequency domain: $s \rightarrow \frac{1}{s(s+\alpha)}$, $s \rightarrow \frac{1}{s+\alpha}$, $s \rightarrow \frac{1}{s+\alpha} \frac{\omega}{s^2+\omega^2}$, $s \rightarrow \frac{s}{s+\alpha} \frac{\omega}{s^2+\omega^2}$, $s \rightarrow \frac{1}{s+\alpha} \frac{s}{s^2+\omega^2}$ and $s \rightarrow \frac{s}{s+\alpha} \frac{s}{s^2+\omega^2}$.

Classically one has the inverse transforms for the following funtionals:

- $s \to \frac{1}{s(s+\alpha)} \overset{\text{time domain}}{\Longrightarrow} t \to \frac{1}{\alpha}\left(1-e^{-\alpha t}\right) H(t)$
- $s \to \frac{1}{s+\alpha} \overset{\text{time domain}}{\Longrightarrow} t \to e^{-\alpha t} H(t)$

For the other functionals, one can compute the convolution products (denoted $\star$) in the time domain.

First sinusoidal:

$s \to \frac{1}{s+\alpha}\frac{\omega}{s^2+\omega^2} \overset{\text{time domain}}{\Longrightarrow} t \to \left(e^{-\alpha \cdot} H(\cdot) \star sin(\omega \cdot) H(\cdot)\right)(t)$

One has $\left(e^{-\alpha \cdot} H(\cdot) \star sin(\omega \cdot) H(\cdot)\right)(t) = \int_{u=-\infty}^{+\infty} e^{-\alpha u} H(u)\, sin(w(t-u)) H(t-u)\, du = \int_{u=0}^{t} e^{-\alpha u} sin(\omega(t-u))\, du$

and $\int_{u=0}^{t} e^{-\alpha u} sin(\omega(t-u))\, du = \frac{\omega e^{-\alpha t} - \omega cos(\omega t) + \alpha sin(\omega t)}{\alpha^2+\omega^2}$

Therefore

$$s \to \frac{1}{s+\alpha}\frac{\omega}{s^2+\omega^2} \overset{\text{time domain}}{\Longrightarrow} t \to \frac{\omega e^{-\alpha t} - \omega cos(\omega t) + \alpha sin(\omega t)}{\alpha^2+\omega^2} = \kappa^{sin}_{\alpha,\omega}(t)$$

Second sinusoidal:

$s \to \frac{s}{s+\alpha}\frac{\omega}{s^2+\omega^2} \overset{\text{time domain}}{\Longrightarrow} t \to \frac{d}{dt}\left(\frac{\omega e^{-\alpha t} - \omega cos(\omega t) + \alpha sin(\omega t)}{\alpha^2+\omega^2}\right)(t) + \frac{\omega e^{-\alpha t} - \omega cos(\omega t) + \alpha sin(\omega t)}{\alpha^2+\omega^2}\Big|_{t=0^-}$

As, $\frac{\omega e^{-\alpha t} - \omega cos(\omega t) + \alpha sin(\omega t)}{\alpha^2+\omega^2}\Big|_{t=0^-} = 0$, we obtain

$$s \to \frac{s}{s+\alpha}\frac{\omega}{s^2+\omega^2} \overset{\text{time domain}}{\Longrightarrow} t \to \frac{-\alpha\omega e^{-\alpha t} + \omega^2 sin(\omega t) + \alpha\omega cos(\omega t)}{\alpha^2+\omega^2} = \iota^{sin}_{\alpha,w}(t)$$

First cosinusoidal:

$s \to \frac{1}{s+\alpha}\frac{s}{s^2+\omega^2} \overset{\text{time domain}}{\Longrightarrow} t \to \left(e^{-\alpha \cdot} H(\cdot) \star cos(\omega \cdot) H(\cdot)\right)(t)$

Therefore

$$s \to \frac{1}{s+\alpha}\frac{s}{s^2+\omega^2} \overset{\text{time domain}}{\Longrightarrow} t \to -\frac{\alpha e^{-\alpha t} - \alpha cos(\omega t) - \omega sin(\omega t)}{\alpha^2+\omega^2} = \kappa^{cos}_{\alpha,\omega}(t)$$

Second cosinusoidal:

$s \to \frac{s}{s+\alpha}\frac{s}{s^2+\omega^2} \overset{\text{time domain}}{\Longrightarrow} t \to \frac{d}{dt}\left(-\frac{\alpha e^{-\alpha t} - \alpha cos(\omega t) - \omega sin(\omega t)}{\alpha^2+\omega^2}\right)(t) - \frac{\alpha e^{-\alpha t} - \alpha cos(\omega t) - \omega sin(\omega t)}{\alpha^2+\omega^2}\Big|_{t=0^-}$

As, $\frac{\alpha e^{-\alpha t} - \alpha cos(\omega t) - \omega sin(\omega t)}{\alpha^2+\omega^2}\Big|_{t=0^-} = 0$, we obtain

$$s \to \frac{s}{s+\alpha}\frac{s}{s^2+\omega^2} \overset{\text{time domain}}{\Longrightarrow} t \to -\frac{-\alpha^2 e^{-\alpha t} + \alpha\omega sin(\omega t) - \omega^2 cos(\omega t)}{\alpha^2+\omega^2} = \iota^{cos}_{\alpha,\omega}(t)$$

5 Practical values for parameters

5.1 Relaxation time

Values of $\tau \in [10, 30]\, s$

5.2 Typical length

Values of $L \in [100, 1000]\, m$

5.3 Triangular diagram

$\rho_{max} =$