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The Aw–Rascle and Zhang's model: Vacuum problems, existence and regularity of the solutions of the Riemann problem

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Abstract

In this paper we will discuss some aspects of the recent macroscopic models of the second-order proposed by [Aw, A., Rascle, M., 2000. Resurrection of second order models of traffic flow. SIAM Journal of Applied Mathematics 60 (3), 916–938] and [Zhang, H.M., 2002. A non-equilibrium traffic model devoid of gas-like behavior. Transportation Research Part B 36, 275–290]. These models were suggested after the publication of an article written by [Daganzo, C.F., 1995. Requiem for second-order fluid approximations of traffic flow. Transportation Research Part B 29, 277–286] showing that some classical second-order models can exhibit non-physical solutions. It is shown in this note that the ARZ (Aw–Rascle–Zhang) model respects the anisotropic character of traffic flow, that it yields physical solutions, and that vacuum problems can be solved satisfactorily, provided that the fundamental diagram (equilibrium speed–density relationship) is extended in a suitable fashion. It follows that the Riemann problem for the ARZ model with extended fundamental diagram always admits a solution, and that this solution depends continuously on the initial conditions.

Keywords: Analytical solutions; Riemann problem; Shock waves; Rarefaction waves; Contact discontinuities; Fundamental diagram; Vacuum problems

1. Introduction

During the last decade, important researches in traffic flow modeling have been devoted to higher order traffic models, aiming at the improvement of the description of non-equilibrium features of traffic flow. In the literature, it was Payne (1971), who proposed the first second-order traffic flow model. This model was criticized particularly in Daganzo (1995), proclaiming the "requiem" for second-order models on the grounds

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that they do not respect the anisotropic character of the traffic flow. In order to avoid such non-physical solutions, other models have been proposed: the model of Ross (1988), the model of Del Castillo et al. (1993) and the model of Zhang, 1998. These models (and others, not mentioned here) do not address the anisotropy problem satisfactorily. The main reason is that they exhibit characteristic speeds greater than the traffic flow speed, implying that a driver is liable to react to stimuli coming from the rear and move backwards.

Finally in Aw and Rascle (2000), the authors proclaimed the "resurrection" of second-order modeling and introduced a model devoid of gas-like behavior. Independently, and following a different rationale, Zhang (2002) introduced a similar model. This model will be called the ARZ model for short in the paper.

However, screening the theoretical development of both models, and in particular the resolution of the Riemann problem, it turns out that it is necessary to extend the fundamental diagram. Only by extending properly the fundamental diagram is it possible to solve the Riemann problem for all initial conditions and to avoid irregular behavior at low densities as analyzed in Aw and Rascle (2000). It is shown that the solutions of the ARZ model are endowed with desirable stability properties, in contrast to the earlier results.

2. The ARZ model: basic facts

Before analyzing the Riemann problem for the ARZ model, let us recall the mathematical development of this model. The basic equations of the model are the following:

1. Conservation equation:
$$\partial_t \rho + \partial_x(\rho v) = 0 \tag{1}$$
2. Momentum equations:
$$\text{Aw \& Rascle model}: \quad \partial_t v + (v - \rho P'(\rho))\partial_x v = \frac{A}{T}(V_e(\rho) - v) \tag{2}$$
$$\text{Zhang model}: \quad \partial_t v + (v + \rho V_e'(\rho))\partial_x v = 0 \tag{3}$$
where ρ and v denote the traffic density and speed respectively and ∂_t and ∂_x denote the partial derivative with respect to time and space respectively. $V_e(\rho)$ and $Q_e(\rho)$ denote the equilibrium speed– and flow–density relationships (fundamental diagram).

In Eq. (2), the quantity $P(\rho)$ represents the "traffic pressure", by analogy with gas dynamics. For the conservation part of the two models to coincide, it is necessary that the traffic pressure $P(\rho)$ be defined by

$$P(\rho) \stackrel{\text{def}}{=} V_{\max} - V_e(\rho)$$

since $P'(\rho) = -V_e'(\rho)$ and $P(\rho)_{\rho=0} = 0$.

Note that the Aw–Rascle model (2) includes on the right-hand-side a relaxation term which we will ignore for the Riemann-problem analysis.

Aw and Rascle suggest the following expression for the pressure:

$$P(\rho) \stackrel{\text{def}}{=} \rho^{\gamma}$$

with $\gamma > 0$ a parameter. This expression must be rescaled as $P(\rho) = V_{\max}(\rho/\rho_{\max})^{\gamma}$, yielding the following expression for the equilibrium speed–density relationship

$$V_e(\rho) = V_{\max}[1 - (\rho/\rho_{\max})^{\gamma}] \tag{4}$$

In order to study the elementary waves associated with the ARZ model and to discuss the analytical solutions, it is necessary to rewrite these models in conservative form. In the case of the Aw–Rascle model, the following variable y is introduced: $y = \rho(v + P(\rho))$ whereas in the case of Zhang's model the same variable is defined as: $y = \rho(v - V_e(\rho))$. We shall adopt Zhang's convention in this note. Physically, this variable represents the *relative flow*, i.e. the difference between the actual flow ($q = \rho v$) and the equilibrium flow $\rho V_e(\rho) \stackrel{\text{def}}{=} Q_e(\rho)$. Mathematical developments can be found in Zhang (2002) and Aw and Rascle (2000).

The conservative form is expressed as

$$\partial_t U + \partial_x F(U) = 0 \quad \text{with } U = \begin{pmatrix} \rho \\ y \end{pmatrix}, \quad F(U) = \begin{pmatrix} y + \rho V_e(\rho) \\ \frac{y^2}{\rho} + y V_e(\rho) \end{pmatrix} \tag{5}$$

However, in the modeling approach of Zhang (2002) no indication is given about the conserved variables whereas Aw and Rascle emphasize the conservation of variables (ρ, y).

The system (5) is strictly hyperbolic (except at null density) because the calculation of the eigenvalues of the gradient of the flux matrix $F(U)$ yields the two following values:

$$\lambda_1(U) = \frac{y}{\rho} + Q_e'(\rho) = v + \rho V_e'(\rho) \leqslant v \quad \text{and} \quad \lambda_2(U) = \frac{y}{\rho} + V_e'(\rho) = v \tag{6}$$

The greatest eigenvalue is equal to the flow speed v. Consequently, the anisotropic character of the traffic is strictly preserved. The eigenvectors $r^1(U)$ and $r^2(U)$ associated to each eigenvalue are given as

$$r^1(U) = \begin{pmatrix} -\rho \\ -y \end{pmatrix}, \quad r^2(U) = \begin{pmatrix} \rho \\ y - \rho^2 V_e'(\rho) \end{pmatrix} \tag{7}$$

Before solving the Riemann Problem, it is necessary to compute first, the elementary waves associated to each eigenvalue. More details are given in Aw and Rascle (2000) and Zhang (1998, 2002).

The eigenvalue $\lambda_1(U)$ is strictly nonlinear: $\nabla\lambda_1(U) \cdot r^1(U) \neq 0$. The waves associated with this eigenvalue correspond to rarefaction or shock waves. On the contrary, the second eigenvalue $\lambda_2(U)$ is linearly degenerated: $\nabla\lambda_2(U) \cdot r^2(U) = 0$ and the corresponding waves are contact discontinuities. We can distinguish the following waves connecting two traffic states U_0 and U_1:

- 1-waves (shock or rarefaction) given by

  $$\frac{y_1}{\rho_1} - \frac{y_0}{\rho_0} = 0 \tag{8}$$

  which can be written in the coordinates (ρ, v) as

  $$v_1 - V_e(\rho_1) - v_0 + V_e(\rho_0) = 0 \tag{9}$$

  If U_1 and U_0 denote the downstream and upstream traffic states respectively, the 1-wave is a shock if $\rho_1 < \rho_0$ and a rarefaction wave if $\rho_1 > \rho_0$.
- 2-waves (contact singularity) characterized by

  $$\frac{y_1 + Q_e(\rho_1)}{\rho_1} - \frac{y_0 + Q_e(\rho_0)}{\rho_0} = 0 \tag{10}$$

  which can be written also in the coordinates (ρ, v) as

  $$v_1 - v_0 = 0 \tag{11}$$

In the figures depicting solutions of the ARZ model in the (ρ, v) plane, we will sometimes distinguish between 1-resp. 2-waves and 1-resp. 2-curves. The 1-resp. 2-curves prolong the 1-resp. 2-waves. By definition 1-resp. 2-curves are the sets of points in the (ρ, v) plane which can be connected to a given state say $U_c \stackrel{\text{def}}{=} (\rho_c, v_c)$ by a 1-resp. 2-wave. Following (9), the equation of a 1-curve is given by

$$v = V_e(\rho) + v_c - V_e(\rho_c)$$

and following (11) the equation of a 2-curve is given by

$$v = v_c$$

The data of the Riemann problem consists in the following initial condition (piecewise constant, two traffic states U_l and U_r, left and right)

$$\begin{cases} U(x,0) = U_l & \text{if } x < 0 \\ U(x,0) = U_r & \text{if } x > 0 \end{cases} \tag{12}$$

The resolution of the Riemann problem consists in finding self-similar solutions of (5) given the initial condition (11).

Self-similar solutions are function of the parameter $\xi = x/t$ and are constituted by 1- and 2-waves. In a transition from the left (U_l) to the right (U_r) states of traffic, ξ increases and ξ is equal to $\lambda[U(\xi)]$. Since $\lambda_1[U] < \lambda_2[U]$, for all U, 1-waves must precede 2-waves (Lax entropic condition) and the general solution of the Riemann problem includes:

- a 1-wave connecting the state U_l to an intermediate state U_0 to be defined,
- a 2-wave connecting the intermediate state U_0 to the state U_r.

Since 1-waves can be either shock or rarefaction waves, there are two main types of solutions:

1. *Type 1*: 1-shock connecting U_l to the intermediate state U_0, followed by a 2-contact discontinuity wave connecting U_0 to U_r.
2. *Type 2*: 1-rarefaction wave connecting U_l to U_0, followed by a 2-contact discontinuity wave connecting U_0 to U_r.

The two types of solutions are depicted in the (x, t) plane by Fig. 1 and they are illustrated in the (ρ, v) plane by Fig. 2 (on the example of the fundamental diagram proposed by Aw–Rascle).

The intermediate state U_0 is computed using (11) and (9)

$$\begin{aligned} &v_0 = v_r \\ &v_0 + V_e(\rho_0) = v_l + V_e(\rho_l) \end{aligned}$$

from which we deduce

$$\begin{aligned} &v_0 = v_r \\ &\rho_0 = V_e^{-1}(v_r - v_l + V_e(\rho_l)) \end{aligned} \tag{13}$$

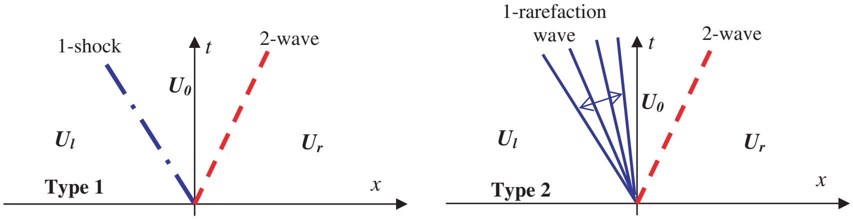

Fig. 1. Riemann problem solutions in the (x, t) plane.

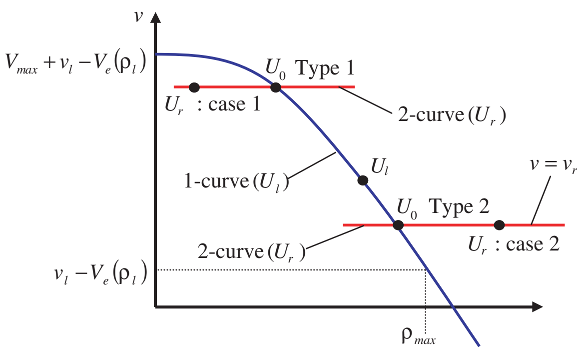

Fig. 2. Riemann problem solutions in the (ρ, v) plane.

This expression shows that the reciprocal function $V_e^{-1}(\cdot)$ plays a central role in the resolution of the Riemann problem.

3. Discussion of the existence of physical solutions of the ARZ model (5)

3.1. Existence of solutions to the Riemann problem

The existence of solutions to the Riemann problem is not always guaranteed. Inexistence of such solutions has two kinds of consequences:

- The model admits no solution for certain initial conditions which are nevertheless liable to represent real traffic situations.
- The model cannot be discretized using the Godunov scheme (the reader is referred to Kröner (1997) for background information on the application of this scheme to the discretization of systems of hyperbolic conservation laws).

Therefore it is important for the applicability of the ARZ model to specify it in such a way that the Riemann problem can always be solved.

Let us consider the fundamental diagram depicted in Fig. 3. This fundamental diagram is deduced from a piece-wise parabolic flow–density relationship

$$Q_e(\rho) = \begin{cases} \rho\left[V_{\max} - \frac{\rho}{\rho_{\text{crit}}}\left(V_{\max} - \frac{Q_{\max}}{\rho_{\text{crit}}}\right)\right] & \text{if } \rho \leqslant \rho_{\text{crit}} \\ Q_{\max} - W_{\max}(\rho - \rho_{\text{crit}}) - (W_{\max}(\rho_{\max} - \rho_{\text{crit}}) - Q_{\max})\left(\frac{\rho-\rho_{\text{crit}}}{\rho_{\max}-\rho_{\text{crit}}}\right)^2 & \text{if } \rho \geqslant \rho_{\text{crit}} \end{cases}$$

The basic parameters of this model are jam and critical densities $\rho_{\max}$, ρ_{crit}, free speed $V_{\max}$, capacity $Q_{\max}$ and congestion wave propagation speed $-W_{\max}$.

If $v_{\text{r}} < v_{\text{l}} - V_e(\rho_{\text{l}})$, the Riemann problem does not admit a solution, as can easily be checked by analyzing the problem in the (ρ, v) plane (see Fig. 4).

The traffic state U_{l} is characterized by a velocity v_{l} higher than the equilibrium speed $V_e(\rho_{\text{l}})$ when $v_{\text{r}} < v_{\text{l}} - V_e(\rho_{\text{l}})$. Such traffic conditions can occur in real traffic, therefore the Riemann problem must be solvable for such cases. Neither Aw and Rascle (2000) nor Zhang (2002) did consider this problem.

3.2. Non-physical solutions to the Riemann problem

Let us consider now the same initial conditions for the Riemann problem as above but with the Aw–Rascle fundamental diagram (4).

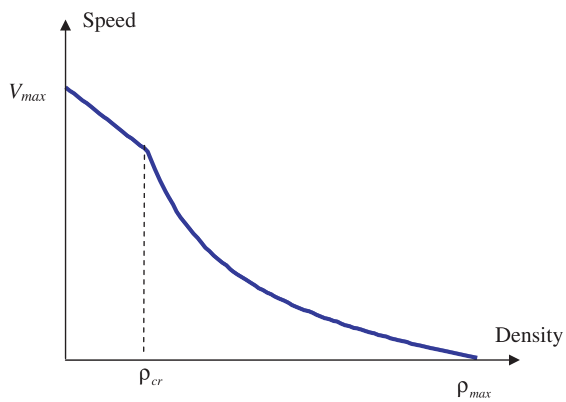

Fig. 3. Speed–density relationship (FD).

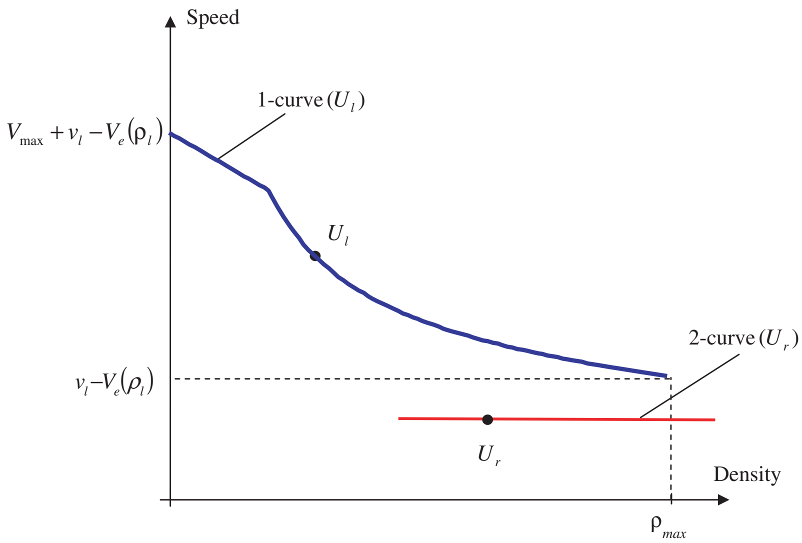

Fig. 4. Case of no solution to the Riemann problem.

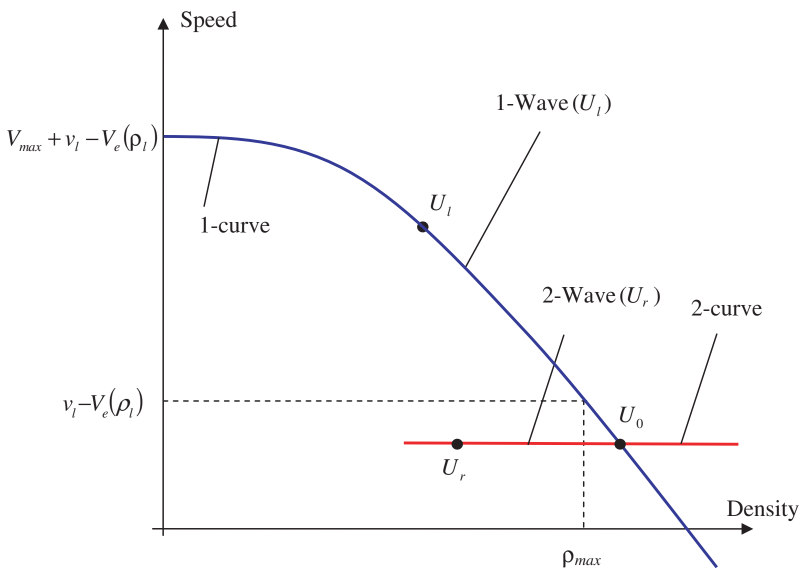

Fig. 5. Non-physical intermediate state (Aw–Rascle fundamental diagram).

Since the fundamental diagram is defined for $\rho > \rho_{\max}$ (see Fig. 2), the Riemann problem admits a solution, but the density ρ_0 of the intermediate state U_0 is greater than the maximum density $\rho_{\max}$ (see Fig. 5). This solution is not physical.

4. How to guarantee physical solutions to the Riemann problem for the ARZ model

4.1. Proper extension of the fundamental diagram

In order to find a physical solution of the Riemann problem for all possible initial traffic conditions, the fundamental diagram must be extended as depicted in Fig. 6.

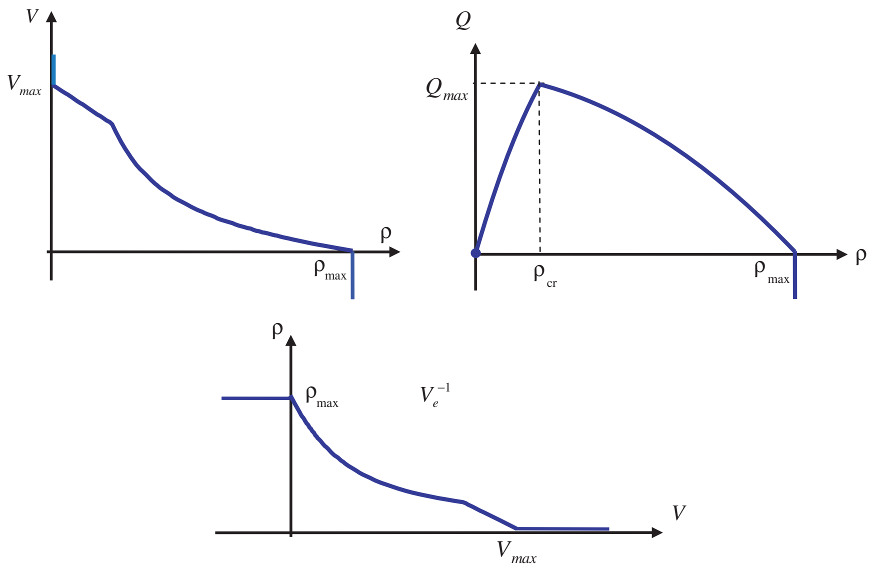

Fig. 6. Extension of the fundamental diagram.

More specifically, it is the inverse equilibrium speed–density relationship $V_e^{-1}(\cdot)$ that must be extended according to

$$\begin{cases} V_e^{-1}(v) = 0 & \text{if } v \geqslant V_{\max} \\ V_e^{-1}(v) = \rho_{\max} & \text{if } v \leqslant 0 \end{cases} \tag{14}$$

Any equilibrium speed–density relationship should be extended in such a way.

For Q_e the extension is given by

$$\begin{cases} Q_e(\rho) = 0 & \text{if } \rho = 0 \\ Q_e(\rho) = (-\infty, 0] & \text{if } \rho = \rho_{\max} \end{cases} \tag{15}$$

This extension resolves any difficulties inherent to the Riemann Problem at high densities as shown in Fig. 7, for the same initial data as led to no solution in Fig. 4.

To check this assertion let us for example assume the initial data satisfies to

$$v_\mathrm{r} < v_\mathrm{l} - V_e(\rho_\mathrm{l})$$

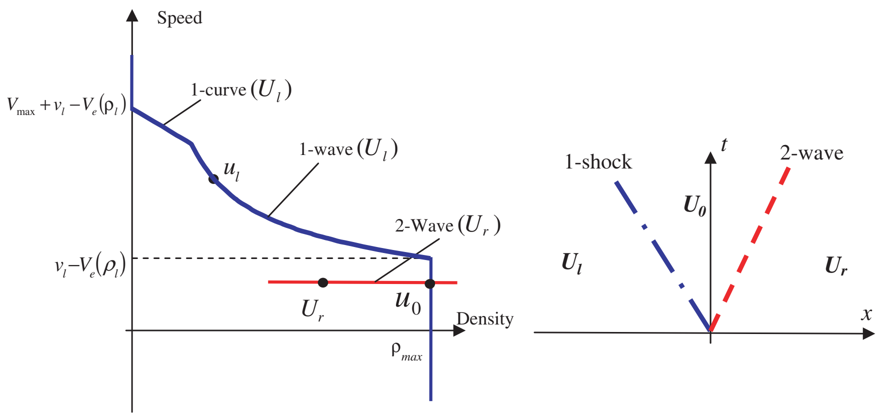

Fig. 7. Riemann problem solution for the extended fundamental diagram.

as in Section 2. From (13) and (14) we deduce

$$\begin{cases} \rho_0 = V_e^{-1}(v_\mathrm{r} - v_\mathrm{l} + V_e(\rho_\mathrm{l})) = \rho_{\max} \\ v_0 = v_\mathrm{r} \end{cases} \tag{16}$$

and (U_0) is uniquely determined.

Remarks

1. It has sometimes been suggested in the mathematical literature that the method of *domain invariance* might be suitable to deal with the non-physical solution problem. Basically, the idea is to restrict initial conditions (and boundary conditions) in such a way that solutions stay well-behaved at all times. This idea is not appropriate in the present case, because if the model is fed with real data, there is no reason to assume that the measurement data will comply with the restrictions imposed by domain invariance. The model must be able to deal with any kind of input.
2. The solution is physical in the sense that the density of the solution always stays below jam density, whatever the initial condition, and the speed of the solution is positive if the speed of the initial condition is positive. The 1-shock propagates at speed

$$v = \frac{q_\mathrm{l} - \rho_{\max} v_\mathrm{r}}{\rho_\mathrm{l} - \rho_{\max}}$$

In the next section we shall show that when the density is vanishingly small the extended fundamental diagram also yields consistent and physical solutions.

5. Stability of the solution for small densities

5.1. Smooth extension of the fundamental diagram

Aw and Rascle (2000) (Section 4 of their paper) examined what they called the problem of "stability at the origin", i.e. stability of the solution for vanishingly small densities. They discussed extensively the lack of stability of the solutions. Such problems are sometimes called *vacuum problems*. With the extended fundamental diagram there is no instability of solutions. As an example we shall treat first the case of the Riemann problem where

$$v_\mathrm{r} > V_{\max} + v_\mathrm{l} - V_e(\rho_\mathrm{l})$$

By construction the state U_0, has null density and velocity v_r; we also define the state $\widetilde{U}_0$, with null density and velocity $V_{\max} + v_\mathrm{l} - V_e(\rho_\mathrm{l})$ (see Fig. 8).

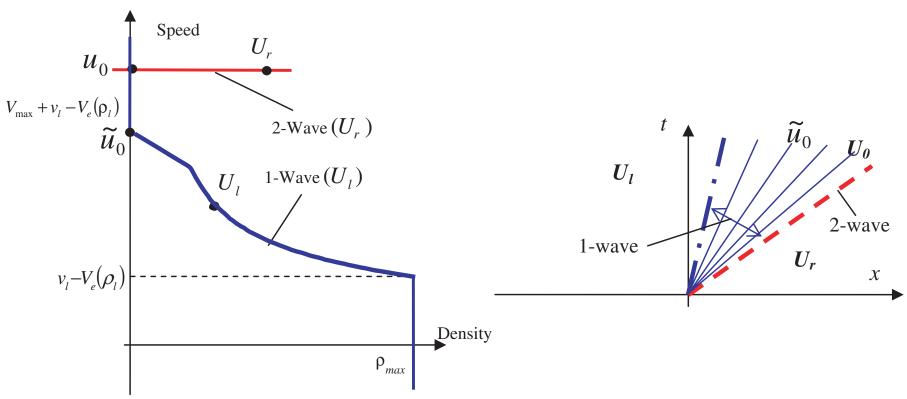

Fig. 8. Riemann problem solution: special case of a 1-wave empty of traffic.

The solution is described by a 1-wave connecting state U_l to state U_0, and a 2-wave connecting U_0 to U_r. The velocity of a characteristic is $\lambda_1(U) = v + \rho V_e'(\rho)$ (by Eq. (6)). This value is well-defined only up to the state $\widetilde{U}_0$, but not on the segment $\widetilde{U}_0 - U_0$ along which $\rho = 0$ but $V_e'(\rho) = -\infty$. We *define* the characteristic speed $\lambda_1(U) = v + \rho V_e'(\rho)$ for states $U = (\rho, v)$ such that $\rho = 0$ and $v \geqslant V_{\max} + v_l - V_e(\rho_l)$ by

$$\lambda_1(U) \stackrel{\text{def}}{=} v \quad \text{if } \rho = 0 \text{ and } v \geqslant V_{\max} + v_l - V_e(\rho_l) \tag{17}$$

All states on segment $\widetilde{U}_0 - U_0$ have a null density and a velocity ranging between $V_{\max} + v_l - V_e(\rho)$ and v_r: there are no vehicles between the characteristics $\widetilde{U}_0$ and U_0 (the vehicles U_r are too fast). All states on segment $\widetilde{U}_0 - U_0$ satisfy (17). This convention can be justified by considering a smooth approximation $\widehat{V}_e(\rho)$ to the extended fundamental diagram, such that

$$\begin{aligned}
&\widehat{V}_e(\rho) > 0, \quad \widehat{V}_e'(\rho) < 0 \quad \forall \rho > 0 \\
&\lim_{\rho \to 0} \widehat{V}_e(\rho) = +\infty \\
&\lim_{\rho \to 0} \rho \widehat{V}_e(\rho) = 0 \quad \text{i.e. } \widehat{Q}_e(\rho)_{|\rho=0} = 0 \\
&\lim_{\rho \to 0} \rho \widehat{V}_e'(\rho) = 0 \quad \text{i.e. (17)}
\end{aligned} \tag{18}$$

as illustrated by Fig. 9.

For such a smooth extension $\widehat{V}_e(\rho)$, (17) is satisfied for vanishingly small densities, and the system (5) is strictly hyperbolic, which avoids all difficulties in the resolution of the model. Let us note that, as a consequence of (18), the flow–density relationship will hardly be affected by the approximation. An example of such an approximation is

$$\begin{aligned}
&\widehat{V}_e(\rho) = V_e(\rho) \quad \text{if } \rho \geqslant \varepsilon \\
&\widehat{V}_e(\rho) = V_e(\rho) + \left(a \log\left(\frac{\varepsilon}{\rho}\right)\right)^{1/2} \quad \text{if } \rho \leqslant \varepsilon
\end{aligned} \tag{19}$$

with a and ε two small parameters. $\widehat{V}_e$ which is defined by (19), satisfies (18), and $\widehat{V}_e^{-1}$ is arbitrarily close to V_e^{-1} (it suffices to choose the parameters a and ε small enough).

By geometric construction and by Eqs. (9), (11) and (13) using the closeness of $\widehat{V}_e^{-1}$ and V_e^{-1}, it follows that the solutions to the Riemann problem are obviously stable with respect to initial conditions. They are also stable with respect to small perturbations of the fundamental diagram that satisfy (18). Further the solutions of the Riemann Problem exhibit positive densities unconditionally and positive speeds given that the speeds of the initial conditions are positive. The speed of the solution of the Riemann Problem lies in the interval $[\mathrm{Min}(v_l, v_r), \mathrm{Max}(v_l, v_r)]$. The Riemann Problem solutions are always "physical".

In order to illustrate this stability, let us consider now an example, the perturbation (iv) of case 5 in Aw and Rascle (2000) (Fig. 12 of that paper).

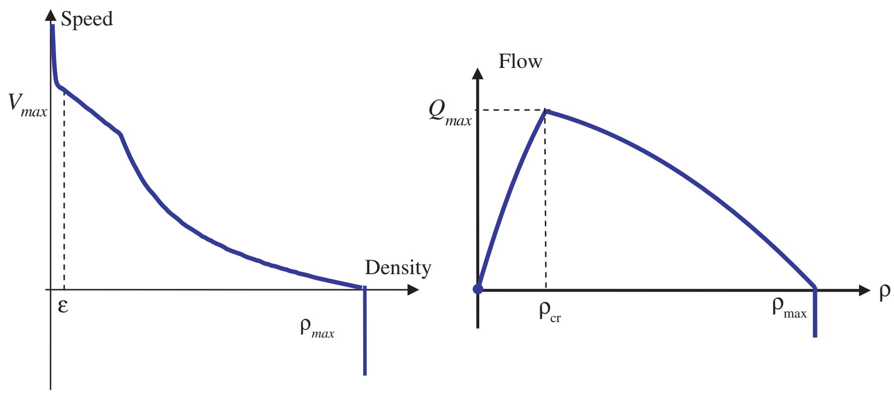

Fig. 9. Smooth extension of the fundamental diagram.

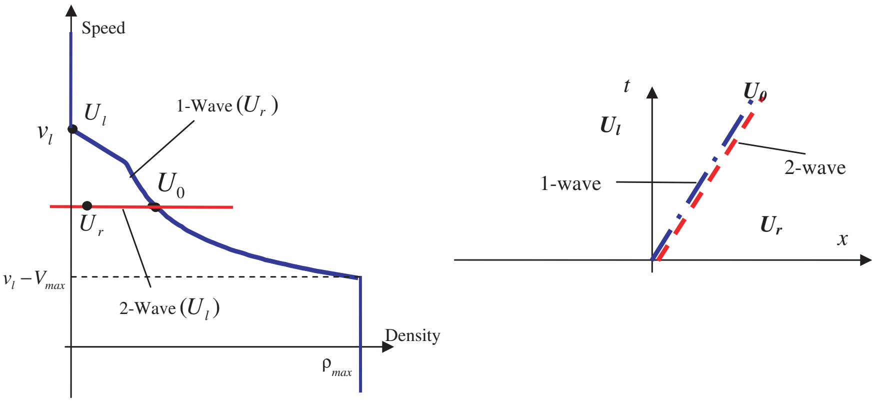

Fig. 10. Riemann problem, with no traffic on the left side.

5.2. Study of an example of perturbation

Let us first consider a Riemann problem with no traffic on the left: $\rho_l = 0$. The initial data on the right-hand side satisfies to

$$v_r < v_l$$

The solution of the Riemann problem is: a 1-shock $U_l \to U_0$ and a 2-wave $U_0 \to U_r$ (see Fig. 10). The velocity of the 2-wave is v_r, and the velocity of the 1-shock is (Rankine–Hugoniot formula)

$$v = \frac{q_0 - q_l}{\rho_0 - \rho_l} = \frac{q_0}{\rho_0} = v_0 = v_r$$

In the above formula, q, ρ, v denote the flow, density, speed of the various traffic states. The indices refer to the traffic states. Thus the 1-shock and the 2-wave propagate at the same speed. This is the expected result: since there are no vehicles on the left, the vehicles on the right move on at speed v_r. The reader can easily check that other values of v_l yield a qualitatively identical solution.

Let us now consider the following *perturbed Riemann problem*. The left-hand side initial data is now a traffic state U_l' with the same speed as U_l, v_l, but with a density ρ_l' such that $\rho_l' > 0$ but ρ_l' small. The right-hand side data U_r is unchanged.

The solution of the Riemann problem is (see Fig. 11): a 1-shock $U_l' \to U_0'$ and a 2-wave $U_0' \to U_r$. The velocity of the 2-wave is v_r, and the velocity of the 1-shock is

$$v' = \frac{q_0' - q_l'}{\rho_0' - \rho_l'} = v_r - \rho_l' \frac{v_r - v_l}{\rho_0' - \rho_l'}$$

q, ρ, v denote the flow, density, speed of the various traffic states, and the indices refer to the traffic states. Following (13), the density ρ_0' and speed v_0' of the state U_0' is given by

$$v_0' = v_r$$
$$\rho_0' = V_e^{-1}(v_r - v_l + V_e(\rho_l'))$$

Thus $\rho_0' \approx \rho_0$ since the density ρ_l' is very small, and the speed of the 1-shock can be approximated by (first-order approximation in ρ_l')

$$v' \approx v_r - \rho_l' \frac{v_r - v_l}{\rho_0}$$

The speed of the 1-shock is slightly less than v_r. The solution of the perturbed problem is nearly identical to the solution of the original problem, which shows the stability of the solution with respect to perturbations of the initial data in this particular case (see Figs. 11 and 12).

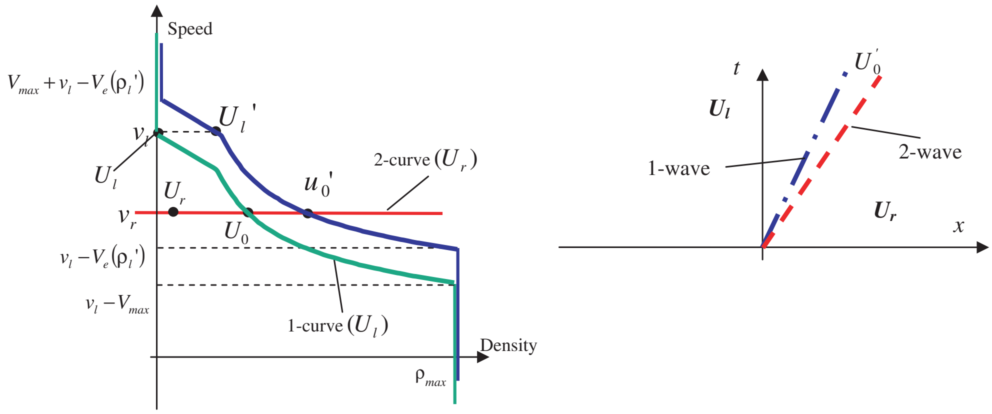

Fig. 11. Solution of the perturbed problem.

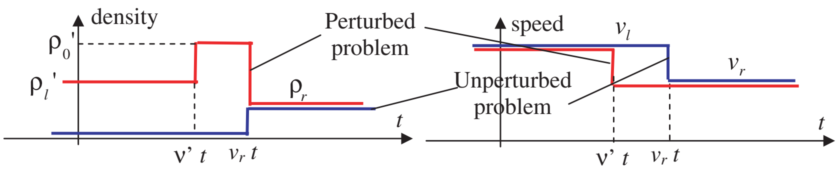

Fig. 12. Density, speed for the solutions of the perturbed and unperturbed problem.

The traffic state U_0' occupies at time t a stretch of length

$$t(v'-v_r)\approx t\rho_1'\frac{v_r-v_l}{\rho_0}$$

which is proportional to ρ_1' and becomes vanishingly small as $\rho_1'\to 0$. Further, the density ρ_0' of state U_0' cannot exceed ρ_{max}, a consequence of the extension of V_e and formula (13).

The speed associated to the state U_0' is equal to v_r, also a consequence of (13). It follows that the solution of the perturbed Riemann problem has the solution of the unperturbed Riemann problem as a limit as $\rho_1'\to 0$, in the L^1_{loc} (integral) sense.

6. Conclusion

The proposed extension of the equilibrium speed- and flow–density relationships V_e and Q_e by (14) or (19), and (15), and the induced extension of V_e^{-1}, enable us to obtain solutions to the Riemann problem for all possible initial data. These solutions are physical in the sense that they satisfy $0\leqslant\rho\leqslant\rho_{max}$ and the speed is positive (provided the initial conditions satisfy this condition). The solutions depend continuously on the initial data.

Ongoing research based on these ideas includes the general solution of the Riemann problem, leading to the development of the Godunov scheme for the ARZ model, the solution of the inhomogeneous Riemann problem and the definition of proper boundary conditions for the ARZ model. Preliminary publications on this work are under way (Mammar et al., 2005; Lebacque et al., 2005).

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