SIAM J. APPL. MATH.
Vol. 60, No. 3, pp. 916–938

RESURRECTION OF "SECOND ORDER" MODELS OF TRAFFIC FLOW*

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Abstract. We introduce a new "second order" model of traffic flow. As noted in [C. Daganzo, *Requiem for second-order fluid with approximation to traffic flow,* Transportation Res. Part B, 29 (1995), pp. 277–286], the previous "second order" models, i.e., models with two equations (mass and "momentum"), lead to nonphysical effects, probably because they try to mimic the gas dynamics equations, with an unrealistic dependence on the acceleration with respect to the space derivative of the "pressure." We simply replace this space derivative with a *convective* derivative, and we show that this very simple repair completely resolves the inconsistencies of these models. Moreover, our model nicely predicts instabilities near the vacuum, i.e., for very light traffic.

Key words. traffic flow, conservation laws

AMS subject classifications. 35L65, 90B20

PII. S0036139997332099

1. Introduction.

The paper of Daganzo [7] is a neat study of the (severe) drawbacks of "second order"[1] models of car traffic, which essentially concludes by rejecting those models for (good) reasons, which are recalled below.

The goal of the present paper is to show that a very simple repair of these models immediately resolves all the obvious inconsistencies. Moreover, our model might explain instabilities in car traffic flow, especially near the vacuum, i.e., for very light traffic with few slow drivers.

The prototype of "first order models" (in other words, of scalar conservation laws) is the celebrated Lighthill–Whitham–Richards (LWR) model [19], [38], [32], [13]

$$\partial_t\rho + \partial_x(\rho\, V(\rho)) = 0, \tag{1.1}$$

where ρ is the density and $V(\rho)$, the corresponding preferred velocity, is a given nonincreasing function of ρ, nonnegative for ρ between 0 and some positive maximal density ρ_m (which corresponds to a total traffic jam). Clearly, such an equilibrium model is perfectly unable to describe flows such as the above mentioned light traffic with few slow drivers.

Let us now turn to "second order" models, whose prototype is the so-called Payne–Whitham (PW) model [28], [38]. Among other numerous references, see also [17], [1]. Those models are still macroscopic, but, in contrast with the LWR model, they consist of *two* equations and are usually based on the analogy to one-dimensional fluid flows. The first equation,

$$\partial_t\rho + \partial_x(\rho v) = 0, \tag{1.2}$$

*Received by the editors December 31, 1997; accepted for publication (in revised form) November 2, 1998; published electronically February 29, 2000. This research was supported by the EC TMR network HCL under grant ERB FMRX CT 96 0033.

http://www.siam.org/journals/siap/60-3/33209.html

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[1] Actually, from a mathematical point of view, it would be more accurate to say *"two-equation models"* rather than "second order models," since most of the examples considered in the literature involve only *first* order partial differential equations.

still expresses the conservation of mass, with ρ and v being, respectively, the density and the velocity, whereas the second equation mimics the momentum equation.

However, there are big differences between car traffic and fluid flows. In particular, there is no conservation of momentum in the former. Therefore, what would be the *pressure* term p in the momentum equation is in fact replaced by an *anticipation factor*, i.e., a term which is supposed to describe how an "average" driver would react to a variation in the concentration of cars with respect to *space*. We will come back to this essential point, but first let us briefly quote the main difference with fluid particles, as emphasized in [7]:

($\mathcal{P}$) "A fluid particle responds to stimuli from the front and from behind, but a car is an anisotropic particle that mostly responds to frontal stimuli."

The main critique in [7] is that second order models violate this principle. Precisely, we are going to show that the model introduced here satisfies this principle, but first we need additional mathematical comments.

As we will see below, the basic models are nonlinear hyperbolic systems of partial differential equations (PDEs). Practically, that means the following:

(i) At least in principle, the initial value problem is well-posed: given the initial velocity and concentration of cars at any point x, these two functions are defined for all future time $t \geq 0$ and any x, we hope in a unique way, and even—still in principle—depend on the initial data in a continuous way. In other words, a small perturbation of the initial data should produce only a small perturbation of the solution. However, as we will see below, this last property will not be satisfied near the vacuum because of lack of *strict* hyperbolicity.

(ii) The solution involves (simple) *waves*, i.e., solutions which essentially involve one eigenvalue $\lambda(U)$ of a suitable matrix $A(U)$ and the corresponding eigenvector $r(U)$, which depend on the unknown solution $U := (\rho, v)$.

(iii) Each eigenvalue $\lambda(U)$ physically represents a finite *propagation speed*. Moreover (see section 2), in practical situations, each eigenvalue $\lambda(U)$ is either "genuinely nonlinear" [18], in which case the associated waves are either compressive (shocks) or expansive (rarefaction waves), or $\lambda(U)$ is "linearly degenerate," which corresponds to contact discontinuities, e.g., to slip surfaces in compressible fluid flows.

Now let us come back to the above point ($\mathcal{P}$) stressed in [7].

Indeed, let us consider the PW type of models, defined by (1.2) (conservation of mass) and the acceleration equation, written here in nonconservative form:

$$\partial_t v + \ v \,\partial_x v + \ \rho^{-1} p'(\rho)\partial_x \rho = \ (\tau^{-1})\,(V(\rho) - v) + \ \nu \partial_x^2 v, \tag{1.3}$$

with a pressure law $p = p(\rho)$ inspired from gas dynamics, e.g., an isothermal law

$$p = p(\rho) = \rho. \tag{1.4}$$

In the above formulas, τ^{-1} and ν, etc. are nonnegative constants. Let us first consider the case where these two parameters vanish. Then the prototype of the initial value problem for this nonlinear hyperbolic problem is the celebrated Riemann problem, in which the initial datum for $U := (\rho,\ v)$ is

$$U(x, 0) = \begin{cases} U_- \text{ if } x < 0, \\ U_+ \text{ if } x > 0. \end{cases} \tag{1.5}$$

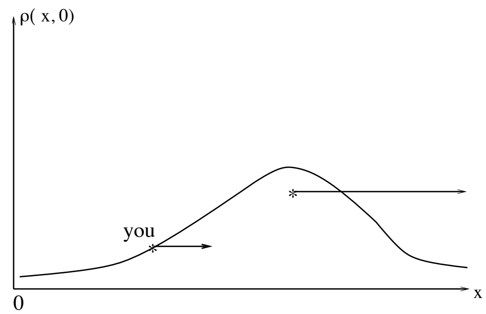

FIG. 1.1. *With such dense and fast traffic in front of you, would you brake, or accelerate?*

Indeed, in numerous cases, solving the above Riemann problem will produce intermediate states $U := (\rho,\ v)$ with *negative* speed! Moreover, with the same kind of initial data, adding a diffusion term in the acceleration equation would still make things worse (see again [7]), whereas adding a relaxation term would be harmless, but would not prevent the same kind of paradox, at least for a short time.

In addition, the characteristic speeds in (1.2), (1.3) are $v \pm \sqrt{p'(\rho)}$. Therefore some part of the information always travels *faster* than the velocity v of cars! The reason for this unacceptable drawback is clear: in (1.4) the *anticipation factor* involves the derivative of the pressure with respect to x. This is completely incorrect, as shown by the following example. Assume, for instance, that in front of a driver traveling with speed v the density is increasing with respect to x but decreasing with respect to $(x - v\ t)$. Then the PW type of models predicts that this driver would slow down, since the density ahead is increasing with respect to x! On the contrary, any reasonable driver would accelerate, since this denser traffic travels *faster* than him; see Figure 1.1 and the appendix.

The conclusion is clear: the correct dependence, such as the one introduced below, must involve the *convective derivative*

$$\partial_t + v\,\partial_x$$

of the "pressure," which we still take as an *increasing* function of the density:

$$p = p(\rho), \tag{1.6}$$

although a more complex "constitutive relation" could be envisioned. Precisely, the goal of this paper is to show that, with a convenient choice of function p in (1.6), this very simple (heuristic) fixing completely suppresses the above major inconsistency with $(\mathcal{P})$. Assuming no diffusion and—in this paper—no relaxation, we obtain the following system:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho v) = 0, \\ &\partial_t(v + p(\rho)) + v\,\partial_x(v + p(\rho)) = 0. \end{aligned} \tag{1.7}$$

Of course, we must prescribe initial data $U_0 := (\rho_0,\ v_0)$ whose components are two bounded nonnegative functions. In this paper we will restrict ourselves to the study of the Riemann problem, i.e., to the case where U_0 satisfies (1.5).

In the next sections, we will rewrite this system in different forms, both conservative and nonconservative, and we will study different examples of physically meaningful Riemann problems, as well as the initial value problem. As we will see, the above class of models satisfies (with a suitable choice of function p) the following principles that any reasonable model should satisfy.

Principles.

A. The system must be hyperbolic.

B. When solving the Riemann problem with *arbitrary* bounded nonnegative Riemann data (ρ, v) in a suitable region $\mathcal{R}$ of the plane, the density and the velocity must remain nonnegative and bounded from above.

C. In solving the same Riemann problem with arbitrary data $U_\pm := (\rho_\pm,\ v_\pm)$, all waves connecting any state $U := (\rho,\ v)$ to its left (i.e., behind it) must have a propagation speed (eigenvalue or shock speed) *at most* equal to the velocity v.

D. The solution to the Riemann problem must agree with the qualitative properties that each driver practically observes every day. In particular, braking produces shock waves, whose propagation speed can be either negative or nonnegative, whereas accelerating produces rarefaction waves which in any case satisfy Principle C.

E. Near the vacuum, the solution to the Riemann problem must be very sensitive to the data. In other words, there must be no continuous dependence with respect to the initial data at $\rho = 0$.

Clearly, any reasonable model should satisfy the above minimal requirements to avoid the previous inconsistencies. In particular, Principle C, perhaps less obvious at first glance, is nothing but the above point $(\mathcal{P})$: a car traveling at a velocity v receives no information from the rear.

Let us also point out the fact that the spirit of our model is also perfectly consistent with discrete car-following models; see, e.g., [10], [26], etc. Indeed, assuming a reaction of each driver to his distance to the previous car exactly means—at the macroscopic level—that the correct modeling involves the convective derivative of the density and not its derivative with respect to x.

Finally, before going into the details, let us remark that we have not mentioned the kinetic models; see, e.g., [29], [30], [15], [16], [39], [24], [25], and the references therein. It would certainly be interesting to see if there is some kinetic version of the model introduced here.

The outline of the paper is as follows. After this (long) introduction, we introduce the model and study the elementary waves in section 2. We then describe the solution to the Riemann problem in section 3, including the case of the vacuum. In section 4, we make remarks on the admissibility and on the stability of this solution, before our conclusion in section 5. The paper ends with a brief appendix in which we compare the solutions to a few typical Riemann problems for the new model introduced here with the one for the PW model. In each case, the PW model predicts unrealistic solution, with negative velocities, whereas the new model gives a correct answer.

2. The model.

2.1. The equations. As explained in the previous section, we are going to consider the purely hyperbolic case, in which no relaxation term is involved:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho v) = 0, \\ &\partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = 0. \end{aligned} \tag{2.1}$$

Here $\rho(x,t)$ and $v(x,t)$ are, respectively, the density and the velocity at point x and time t.

The function $p(.)$ is a smooth increasing function. For reasons which will appear below, the prototype of functions p that we will consider is

$$p(\rho) \;=\; \rho^{\gamma}, \qquad \gamma \;>\; 0. \tag{2.2}$$

In fact the only qualitatively important conditions are the behavior of this function p near the vacuum and the (strict) convexity of the function $\rho\, p(\rho)$. All our results will be valid, in particular under the following assumptions:

$$p(\rho) \;\sim\; \rho^{\gamma}, \quad \text{near} \quad \rho = \; 0, \quad \gamma \;>\; 0, \quad \text{and} \quad \forall\, \rho,\; \rho\, p''(\rho) + \; 2\, p'(\rho) > \; 0. \tag{2.3}$$

The first thing we have to check is Principle A. In other words, we must show that system (2.1) is hyperbolic. Indeed, multiplying the first equation of this system by $p'(\rho)$ and adding to the second one, we obtain

$$\begin{aligned} &\partial_t \rho + \; \partial_x(\rho v) = \; 0, \\ &\partial_t v + \; (v - \; \rho p'(\rho))\partial_x v = \; 0, \end{aligned} \tag{2.4}$$

which is clearly equivalent to (2.1) for smooth solutions. Setting $U := (\rho, v)$, we can rewrite it under the form

$$\partial_t U + \; A(U)\, \partial_x U = \; 0, \tag{2.5}$$

where

$$A(U) := \begin{pmatrix} v & \rho \\ 0 & v \;-\; \rho p'(\rho) \end{pmatrix}. \tag{2.6}$$

The eigenvalues of this matrix are

$$\lambda_1 \;=\; v \;-\; \rho\, p'(\rho) \leq \; \lambda_2 \;=\; v. \tag{2.7}$$

Therefore system (2.4) or, equivalently, system (2.1) is strictly hyperbolic, except for $\rho \,=\, 0$, where the two eigenvalues coalesce, and the matrix $A(U)$ is no longer diagonalizable.

We now turn to the other hyperbolic features of these equivalent systems. First, the right eigenvectors corresponding to λ_1 and λ_2 are, respectively, $r^1 = \binom{1}{-p'(\rho)}$ and $r^2 = \binom{1}{0}$.

For basic references on nonlinear hyperbolic systems of conservation laws we refer to [18], [35], [34]. Let ∇ denote the gradient with respect to U. We recall that an eigenvalue λ_k is genuinely nonlinear [18] if the function $\nabla \lambda_k(U) \cdot r^k(U)$ never vanishes, and it is linearly degenerate if, on the contrary, this function of U vanishes for all U. Moreover, the associated waves are contact discontinuities in the former case and are rarefactions or shocks (depending on the data) in the latter case.

An easy calculation shows that λ_2 is always linearly degenerate (LD), and λ_1 would be LD if and only if

$$p(\rho) \;=\; A - \;\frac{B}{\rho}, \tag{2.8}$$

where A and $B > 0$ are real constants.

In particular, under the assumption (2.2) or, more generally, (2.3) λ_1 is genuinely nonlinear (GNL). As a result, in this case the first eigenvalue is GNL and therefore will admit either shock waves or rarefactions. We recall that the propagation speed of a shock is *not* equal to the corresponding eigenvalue. In contrast, the second eigenvalue, *the faster one*, is always linearly degenerate and therefore corresponds to contact discontinuities, i.e., to waves whose propagation speed is always *equal* to the corresponding eigenvalue, exactly like a slip surface in continuum mechanics.

From (2.7) and the *linearly degenerate* nature of λ_2 we can already see another quite desirable feature of this model:

$$\lambda_1 \leq \lambda_2 = v. \tag{2.9}$$

In other words, all the waves propagate at a speed *at most equal to the velocity* v of the corresponding state; see Principle C in section 1.

Now, it is well known that nonlinear hyperbolic systems admit discontinuous solutions, which only makes sense if the system is written as a *conservation* law. In order to obtain such a conservative form, let us multiply the first equation of system (2.1) by $(v + p(\rho))$ and the second one by ρ, and then we add up these two equations to obtain

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho v) = 0, \\ &\partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0. \end{aligned} \tag{2.10}$$

Therefore the conservative variables for the model are ρ and

$$y := \rho\, v + \rho\, p(\rho). \tag{2.11}$$

Let us note that, in general, the choice of conserved quantities (mass, momentum, and energy) is far from being indifferent. In particular, for a scalar GNL conservation law such as the Burgers equation, discontinuous solutions cannot satisfy both this familiar equation

$$\partial_t u + \partial_x(u^2/2) = 0$$

and the conservation of energy

$$\partial_t(u^2/2) + \partial_x(u^3/3) = 0,$$

formally obtained by multiplying the Burgers equation by u. In other words, in general one must make a choice between the system of conservation laws and the additional conservation laws of the system. For instance, one can choose to conserve *exactly* mass and momentum, but *not* the energy, which would be conserved only for smooth solutions, and would *decrease* through a shock wave, according to the second principle of thermodynamics. Here our model is purely heuristic. There is no obvious physical interpretation of the "momentum" y. Conversely, (ρ, y) is the simplest and most natural pair of conservative variables. Naturally, we require that the admissible weak solutions to (2.10) satisfy the Lax entropy inequality [18]: for any convex entropy η associated with the flux q,

$$\partial_t \eta + \partial_x q \leq 0 \tag{2.12}$$

holds in the distribution sense so that the total energy

$$\int \eta(\rho,\ v)\, dx$$

decreases with respect to time.

In particular (see, e.g., [31]), for any convex function $F(.)$ of the Riemann invariant w, $\eta \equiv \rho F(w)$ is an entropy convex with respect to ρ, y. Therefore we might expect that

$$\partial_t(\rho F(w)) + \partial_x(\rho v F(w)) \leq 0 \tag{2.13}$$

in the distribution sense.

However (see below), (2.10) is a system with *coinciding* rarefactions and shock curves. Therefore, using the Rankine–Hugoniot relations (2.25), (2.13) turns out to be an equality for any function F. As a result, even if we replaced $y \equiv \rho w$ with $\widetilde{y} \equiv \rho F(w)$, we would still recover (2.10).

2.2. The elementary waves. In order to solve the Riemann problem, we need to compute the Riemann invariants in the sense of Lax (in short, the RI–Lax) associated with each eigenvalue $\lambda_k,\ k = 1,\ 2$. We will say (see, e.g., [18]) that a scalar function w (resp., z) of $U := (\rho, v)$ is a 1-RI–Lax (resp., a 2–RI Lax) if $\nabla w \cdot r^1 \equiv 0$ (resp., $\nabla z \cdot r^2 \equiv 0$).

Here, we easily obtain

$$w(U) = v + p(\rho), \qquad z(U) = v. \tag{2.14}$$

We have already seen that under assumption (2.3) the first eigenvalue λ_1 is GNL, whereas the second eigenvalue $\lambda_2 = v$ is always LD. Therefore, depending on the data, the waves of the first family will be either rarefaction waves or shocks, while the waves of the second family will always be contact discontinuities. We first describe the former.

The 1-rarefaction waves. We again refer to [18] for more details. In the $(x,\ t)$ plane, an arbitrary given state U_- on the left can be connected to any other state U_0 on the right by a 1-rarefaction wave if and only if

$$w(U_0) = w(U_-) \quad \text{and} \quad \lambda_1(U_0) > \lambda_1(U_-). \tag{2.15}$$

We will draw curves such as $w(U) = w(U_-)$ either in the $U := (\rho,\ v)$ or in the $M := (\rho,\ m) := (\rho,\ \rho\, v)$ plane, or even in the $Y := (\rho,\ y)$ plane.

We note that the correspondence between U and M or Y is no longer one to one at the vacuum $\rho = 0$, where v is physically not defined. However, as in gas dynamics it is sometimes more convenient to treat this case in the $U = (\rho,\ v)$ plane.

In this plane, these curves are depicted in Figure 2.1 with a pressure law such as (2.2). Naturally, the concavity of these curves would be the opposite if $\gamma > 1$.

In particular, we see that the only points U which can be connected to a given U_- by a 1-rarefaction wave satisfy

$$w(U) = w(U_-) \quad \text{and} \quad 0 \leq v \leq w(U_-) = v_- + p(\rho_-).$$

If this condition on the velocities is not satisfied, we will have to introduce the vacuum as an intermediate state; see further.

In the M plane, the corresponding curves provide a nice geometrical interpretation of the eigenvalues λ_k; see Figure 2.2. Indeed, $\lambda_2(U)$ is the velocity v and therefore is the slope of the secant OM, where O is the origin and $M = (\rho,\ \rho\, v)$. Conversely, one

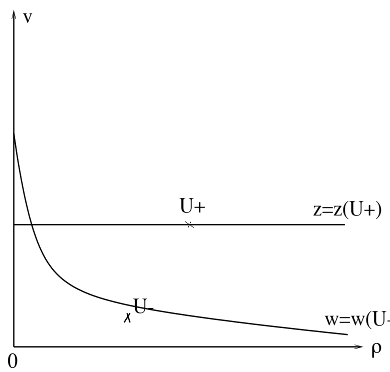

FIG. 2.1. *Riemann invariants in the* $(\rho,\ v)$ *plane for* $\gamma\ >\ 1$.

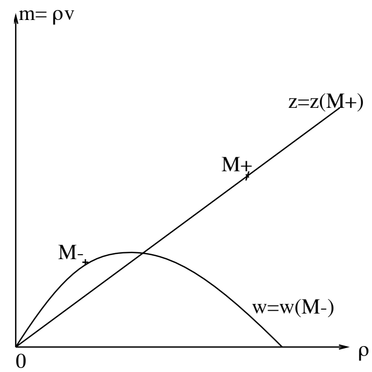

FIG. 2.2. *Riemann invariants in the* $(\rho,\ \rho v)$ *plane.*

can easily check that in the same plane $\lambda_1(U)$ is the slope of the tangent to the curve $w(U)=\ w(U_-)$, or equivalently to the curve

$$m=\ \rho\, v=\ -\rho\, p(\rho)+\ \rho\, w(U_-), \tag{2.16}$$

passing through M. Indeed, differentiating the momentum q with respect to ρ in (2.16) and using (2.14), we get

$$\frac{dm}{d\rho}=\ -\rho\, p'(\rho)-\ p(\rho)+\ w(U_-)=\ v-\ \rho\, p'(\rho)=\ \lambda_1(U). \tag{2.17}$$

Now, these curves $w(U)=\ w(U_-)$ are strictly concave in the $M=\ (\rho,\ \rho\, v)$ plane, in view of (2.3). Therefore the derivative $\frac{dm}{d\rho}=\ \lambda_1(U)$ is strictly decreasing with respect to ρ. Consequently, the admissible part of such a curve given by (2.15) corresponds to

$$w(U_0)=\ w(U_-)\quad\text{and}\quad \rho_0<\ \rho_-. \tag{2.18}$$

Finally, we can also study these curves $w(U)=\ w(U_-)$ in the plane of conservative variables $Y=\ (\rho,\ y)$, where they are even simpler (see Figure 2.3), but where there is no geometric interpretation of the propagation speeds. Indeed, in view of (2.11), (2.14), and (2.15),

$$y=\ \rho\, v+\ \rho\, p(\rho))=\ \rho\, w(U_-)\quad\text{and}\quad \rho_0<\ \rho_-. \tag{2.19}$$

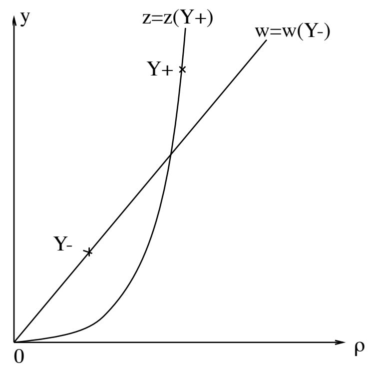

FIG. 2.3. *Riemann invariants in the* $(\rho,\ y)$ *plane.*

Therefore the rarefaction curves corresponding to (2.15) in the $(\rho,\ y)$ plane are segments emanating from the origin.

We now turn to waves of the second family.

The 2-contact discontinuities. These are very simple in the U plane as well as in the $M = (\rho,\ m)$ plane. Indeed, a state U on the left in the (x, t) plane can be connected to a given state U_+ on the right if and only if

$$v = v_-,$$

i.e., if the corresponding point M belongs to the line given by

$$m = \rho\, v = z(U_+)\, \rho \tag{2.20}$$

and no admissibility condition is required, since the corresponding eigenvalue $\lambda_2 = v$ is linearly degenerate. We recall that these waves, the *faster* ones, travel at the speed v, i.e., exactly at the same speed as the corresponding cars.

We also remark that the curves corresponding to each family are straight lines (passing through the origin) in *one* of the planes $(\rho,\ m)$ or $(\rho,\ y)$, but not in the other one. Indeed, if we now study the same 2-waves in the $(\rho,\ y)$ plane, the corresponding curves are no longer straight lines, but curves, given by

$$y = \rho\, v + \rho\, p(\rho) = \rho\, v_+ + \rho\, p(\rho), \tag{2.21}$$

which are strictly convex, due to (2.3).

Classically, these contact discontinuities satisfy both the relation $z = z(U_+)$ and the Rankine–Hugoniot relations, since the solution is discontinuous through such a wave. We now turn to the case of shock waves.

The shock waves. In this case, we must definitely consider the system in conservative form; in other words system (2.10). Any discontinuity traveling with speed s satisfies the Rankine–Hugoniot relations, namely,

$$[F(Y)] = s\, [Y], \tag{2.22}$$

where $[f] := f_r - f_l$ denotes the jump of any quantity f through the discontinuity, and where we have set

$$Y := (\rho,\ y) \quad \text{and} \quad F(Y) := (\rho\, v, y\, v). \tag{2.23}$$

Therefore, as in gas dynamics,

$$[\rho\, v] = s\,[\rho] \quad \text{and} \quad [v(\rho\,(v + p(\rho)))] = s[\rho\,(v + p(\rho))]. \tag{2.24}$$

Setting

$$V := v - s$$

we obtain

$$\begin{aligned} \rho_r\, V_r &= \rho_l\, V_l := Q, \\ Q[v + p(\rho)] &= 0, \end{aligned} \tag{2.25}$$

where Y_l and Y_r play the same role as U_- and U_+ above.

Therefore there are two cases: (i) $Q = 0$. In other words,

$$\rho_r\,(v_r - s) = \rho_l\,(v_l - s) = 0, \tag{2.26}$$

and then

$$v_r - s = v_l - s = 0,$$

except if one of the two densities ρ_l, ρ_r is zero (vacuum). Even in that case, the velocity is not defined in the M or in the Y plane, but is perfectly defined in the U plane. We can say that

$$v_r = v_l,$$

so that this case corresponds to a contact discontinuity of the second family. We have already treated this case.

(ii) $Q \neq 0$. This case corresponds to the case of a shock of the first family. Therefore we assume that

$$U_l = U_-, \qquad U_r = U,$$

and we see that

$$v + p(\rho) = v_- + p(\rho_-).$$

In other words, compared to (2.14), shock curves of the slower family *coincide* with the corresponding rarefaction curves; see, e.g., [37], [23], [8], [9]. This is due to the special form of system (2.10) that we rewrite under the form

$$\partial_t Y + \partial_x (v\, Y) = 0, \tag{2.27}$$

where we can invert relation (2.11) to express v by

$$v = y/\rho - p(\rho),$$

and v and y are considered as independent variables. In other words, the flux $F(Y)$ is *proportional* to Y. We will come back to this point later.

Finally, 1-shock waves of speed s which connect U_- to U are called admissible in the sense of Lax if they satisfy

$$\begin{aligned} s &< \lambda_1(U_-), \\ \lambda_1(U_+) < s &< \lambda_2(U_+). \end{aligned} \tag{2.28}$$

Consequently, a state U can be connected to U_- by a Lax shock of the first family if and only if the corresponding point in the $M = (\rho,\ m)$ plane belongs to the following piece of curve:

$$\rho\,(v + p(\rho)) = \rho\,(v_- + p(\rho_-)) = \rho\, w(U_-)\,, \quad \rho > \rho_-. \tag{2.29}$$

Of course, the shock curves also coincide with rarefaction curves in the plane U or in the Y plane, where they are given by

$$y = \rho\, w(U_-), \qquad \rho > \rho_-. \tag{2.30}$$

Therefore, in this plane, like the rarefaction waves, shock curves are pieces of straight lines passing through the origin; see Figure 2.3. Again, this situation is typical of systems with coinciding rarefactions and shock waves; see, for instance, the above mentioned references.

We also note that in the $M = (\rho,\ \rho\, v)$ plane the first relation (2.24) has the same nice graphical interpretation as for the case of rarefaction waves: in that plane the speed s of a 1-shock or a 2-contact discontinuity connecting M_l to M_r is now the slope of the secant $M_l M_r$. In view of the concavity of the curves $w = w(U_l)$, the first eigenvalue $\lambda_1(U)$ is decreasing with respect to ρ and bigger than the slope of any corresponding secant. Therefore, in most cases (see Figure 2.2) it is very easy to check geometrically the Lax inequalities, such as (2.24), between eigenvalues and shock speeds. Now we are able to consider the Riemann problem.

3. The Riemann problem. Description. This problem is of course the prototype of many practical situations in traffic flows. In order to handle all cases, we consider the conservative variables

$$Y = (\rho, y) = (\rho,\ \rho(v + p(\rho))) = (\rho,\ \rho\, w),$$

where the system rewrites

$$\begin{aligned} \partial_t \rho + \partial_x(\rho v) &= 0,\\ \partial_t y + \partial_x(y\, v) &= 0. \end{aligned} \tag{3.1}$$

However, most of the discussion will take place in the plane M or in the U plane. Indeed, it is sometimes more convenient—and equivalent—to express the results either in the $U = (\rho,\ v)$ or in the $M = (\rho,\ m)$ variables:

$$U = (\rho,\ v)(x,0) = U_\pm \quad \text{for} \quad \pm\ x > 0. \tag{3.2}$$

Here U_- and U_+ are arbitrarily given vectors in $\mathbb{R}^2$ satisfying the constraints

$$\rho \geq 0,\ v \geq 0\ w = v + p(\rho) \leq v_m, \tag{3.3}$$

where w is the 1-RI in the sense of Lax as defined in section 2, and $v_m := w_m$ is a given positive maximal speed.

Let us call $\mathcal{R}$ the set of vectors U—or of corresponding vectors M or Y—which satisfy (3.3). In the M or in the Y plane, this set is convex. If $\gamma > 1$ this set is also convex in the U plane.

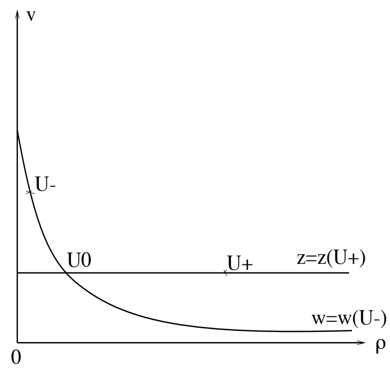

FIG. 3.1. *Riemann problem. Case* 1.

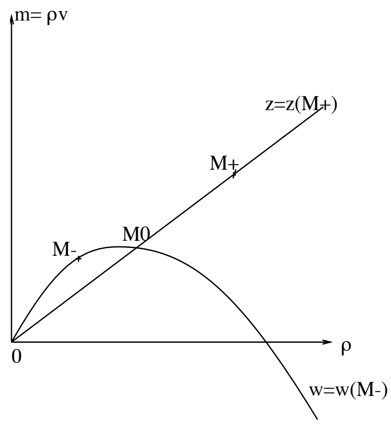

FIG. 3.2. *Riemann problem. Case* 1.

Such vectors will be called *"physical"* and we will also have to check that, starting with *arbitrary* initial data in $\mathcal{R}$, the solution remains in $\mathcal{R}$. In other words, we will have to show that the set $\mathcal{R}$ is an *invariant region* for the Riemann problem.

The goal is to first connect the left state U_- to a middle state U_0 by a wave of the first family and then to connect this intermediate state U_0 to the right state U_+ by a contact discontinuity of the second family. Of course, it would be equivalent to go the other way around, i.e., to start from U_+ and to arrive at U_-. Either way can be more convenient, in particular if one of the Riemann data is at the vacuum; see further. In any case, the 1-wave is a rarefaction wave if $\rho_0 < \rho_-$ or is a Lax shock wave in the opposite case, except if the vacuum state is involved, since at the vacuum the two eigenvalues coalesce so that some of the inequalities in (2.28) become equalities; see below. We start with the general cases.

Case 1. $\rho_- > 0,\ \rho_+ > 0,\ 0 \leq v_+ \leq v_-$.

Clearly in this case, the solution is unique. Indeed, in the U plane let U_0 be the unique intersection of the (strictly decreasing) curve $w(U) = w(U_-)$ with the straight line $v = v_+$; see Figure 3.1.

The solution clearly consists of a 1-shock connecting U_- to U_0, followed by a 2-contact discontinuity connecting U_0 to U_+. We can easily check (see Figure 3.2) in the M plane that any other solution would involve at some point a 1-wave which

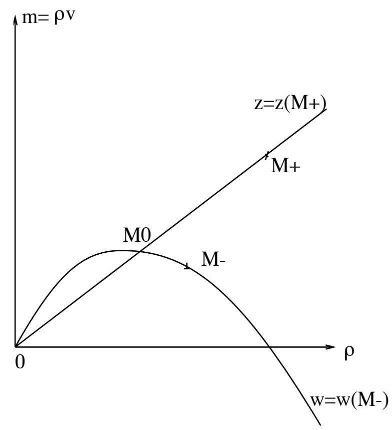

FIG. 3.3. *Riemann problem. Case* 2.

would be faster than the subsequent 2-wave, and this is of course inconsistent.

Case 2. $\rho_- > 0$, $\rho_+ > 0$, $v_- \leq v_+ \leq v_- + p(\rho_-)$.

Again, in this case, the solution is clearly unique; see Figure 3.3. For instance, in the M plane let M_0 be the unique intersection point of the strictly concave curve $w(U) = w(U_-)$ with the straight line $v = v_+$. Again, it would be equivalent to consider in the U plane the unique intersection U_0 of the (strictly decreasing) curve $w(U) = w(U_-)$ with the horizontal line $v = v_+$. Then the solution clearly consists of a 1-rarefaction wave connecting U_- to U_0, followed by a 2-contact discontinuity connecting U_0 to U_+.

Let us now move to cases where the solution to the Riemann problem involves the vacuum $\rho = 0$. As we said, at this point the system is *not* strictly hyperbolic, and the velocity is neither physically defined nor mathematically defined in the plane M or Y.

Mathematically, as in [22] for compressible gas dynamics, we can also study the problem in the U plane, i.e., make a distinction between two vacuum states with different (fake) velocities. This is of course nonphysical, but a priori makes sense if we are interested in solving the Riemann problem in a continuous way with respect to the initial data. However, as we will see below, the solution does *not* depend continuously on the data when the density ρ vanishes; see further.

Case 3. $\rho_- > 0$, $\rho_+ > 0$, $v_- + p(\rho_-) < v_+$.

This case is similar to Case 2. The solution is also unique, but now the intermediate state U_0 is the vacuum. This is obvious in the M plane (see Figure 3.4) or in the Y plane.

In the U plane, let U_1 be the unique intersection of the (strictly decreasing) curve $w(U) = w(U_-)$ with the axis $\rho = 0$ and U_2 be the point $\rho = 0, v = v_+$.

We see that the solution consists of a 1-shock connecting U_- to U_0, followed by a fake *vacuum wave* which connects the two vacuum states U_1 and U_2, followed by a 2-contact discontinuity connecting U_2 to U_+.

For any v, $\rho \equiv 0$ is a solution. Therefore the above (illusory) vacuum wave can be viewed either as a rarefaction wave or as a discontinuous solution—shock or contact discontinuity—which satisfies the Rankine–Hugoniot relations and satisfies—for a suitable range of arbitrary values of s—the only admissibility condition that makes

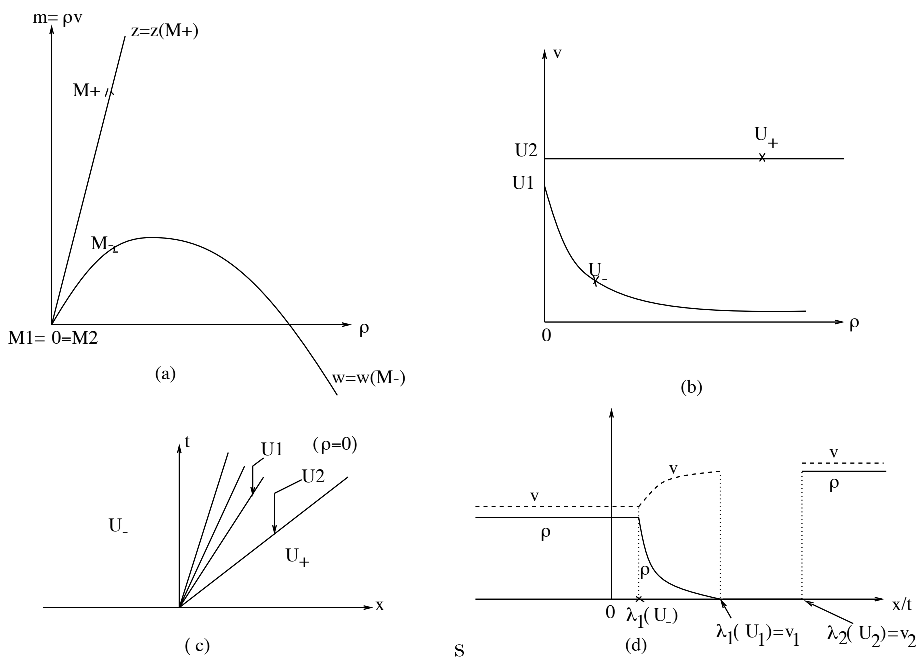

FIG. 3.4. *Riemann problem. Case* 3*: in* (b)*,* $0 < \gamma < 1$*. We show in* (c) *the corresponding picture in the* $(x,\ t)$ *plane and in* (d) *the qualitative aspect of the solution. The velocity is undefined for* $\rho = 0$*.*

sense here, namely,

$$\lambda_1(U_1) = v_1 - \rho_1\, p'(\rho_1) = v_1 < s < \lambda_2(U_2) = v_2. \tag{3.4}$$

We now turn to the two cases where one of the Riemann data is the vacuum $\rho = 0$. As we said, it is more delicate to define the solution properly at this point. Therefore, following classical ideas, we connect the other Riemann datum—for which the density is positive—to the vacuum, either on its left or on its right.

Case 4. $\rho_- > 0,\ \rho_+ = 0$.

With the above rule, we connect M_- on the left to the origin $M_+ = 0$, on the right. In fact, this case is similar to Case 3 (see Figure 3.5), but now there is no need to add a contact discontinuity, since the state M_+, in other words the origin, is on the rarefaction curve issued from M_-. Therefore there is only the above 1-rarefaction wave.

Case 5. $\rho_- = 0,\ \rho_+ > 0$.

Now we connect the Riemann datum M_+ on the right to the vacuum state on the left by a 2-contact discontinuity. This case is similar to Case 1 (see Figure 3.6), but now the wave of the first family has disappeared.

We now briefly discuss mathematically the admissibility of the waves constructed here and their stability with respect to perturbations of the Riemann data.

4. Admissibility. Stability near the vacuum. Let us start with a few remarks on the admissibility of the above mentioned waves. The classical question is, of course, whether nonlinear hyperbolic systems of conservation laws admit too many

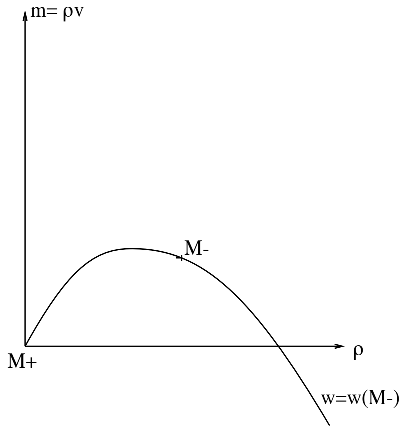

FIG. 3.5. *Riemann problem. Case* 4: $\rho_+ = 0$.

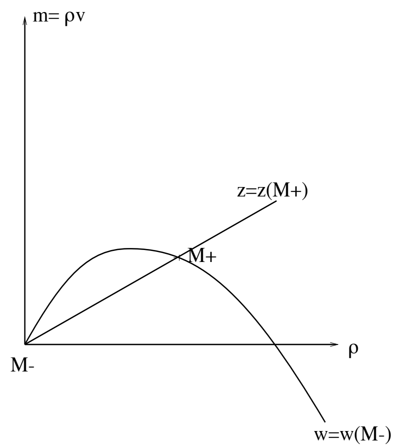

FIG. 3.6. *Riemann problem. Case* 5: $\rho_- = 0$.

weak (discontinuous) solutions; see, e.g., [18], [27]. Therefore one must add *admissibility conditions* to reject these "nonphysical" solutions.

We do not intend to give a complete discussion here. We just want to make a few remarks which show that the solution constructed here is the most reasonable.

First of all, there is no big problem as to the solutions corresponding to Cases 1, 2, and 3, which do not involve the vacuum (except in Case 3, but only as an intermediate state between M_- and M_+). In particular, the "vacuum wave" introduced in Case 3 just mimics the same (classical) in gas dynamics; see, e.g., [22]. Conversely, the shock wave involved in Case 1 is a Lax shock of the first family, i.e., satisfies

$$\begin{aligned} s &< \lambda_1(U_-)\ , \\ \lambda_1(U_+) < s &< \lambda_2(U_+)\ , \end{aligned} \tag{4.1}$$

where U_- and U_+ are the values of U on both sides of the shock. Therefore [18], it satisfies the Lax entropy condition, already mentioned in (2.13).

As is well known, in general (weak) Lax shocks admit *viscous profiles*. In other words, here we consider the system (2.10) in the plane of conservative variables $Y =$

$(\rho,\ y)$ and perturb it by (artificial) viscosity. Then, for all positive ε there exists a traveling wave solution

$$(\rho,\ y) := (\rho,\ y)((x - st)/\varepsilon) := (\rho,\ y)(\xi)$$

to the parabolic perturbed system:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho v) = \varepsilon\, \partial_x^2 \rho, \\ &\partial_t y + \partial_x(y\, v) = \varepsilon\, \partial_x^2 y, \end{aligned} \tag{4.2}$$

such that

$$(\rho,\ y)(x,t) \to (\rho_\pm,\ y_\pm) \quad \text{when} \quad x \to \pm\infty. \tag{4.3}$$

Of course, such profiles converge to the corresponding shock wave when ε goes to 0. In general, for a Lax shock the existence of viscous profiles follows from techniques of dynamical systems; see, e.g., [7], [12]. Moreover, such profiles are stable to small perturbations; see, e.g., [21], [36].

Here the existence of such profiles can be checked directly. In the Y variables they are radially symmetric:

$$(\rho,\ y)(\xi) = \rho(\xi)\ (\rho_r,\ y_r).$$

Moreover, they exist even in the case where $Y_- = 0$, although this discontinuity—described in Case 5 of section 3 as a contact discontinuity of the *second* family—is no longer a Lax shock of the first family. Therefore, mathematically, we can consider this 2-contact discontinuity as a 1-shock, which admits a viscous profile. We also recall that its speed is the slope of the chord $M_-M_+ = OM_+$ in the plane M; see Figure 3.6. Of course, physically, we strongly prefer to view this wave as a contact discontinuity, since there is no car behind the drivers, and therefore no compression coming from the rear.

We now turn to the question of stability with respect to perturbations of the Riemann data. Clearly, the waves described in the previous section depend continuously on the data, except possibly near the vacuum.

Let us fix the right state M_+ in Case 5 of section 3, and let us slightly perturb the left state, so that $M_- = (\rho_-,\ \rho_-\ v_-),\ \rho_- << 1$. Then we are back to one of the three cases, Case 1, 2, or 3 of section 3, depending on the velocity v_-:

(i) If $v_- + p(\rho_-) < v_+$, then the solution consists of a small rarefaction wave connecting M_- to the origin, followed by a vacuum wave and then by a contact discontinuity (Case 3).

(ii) Similarly, if $v_- < v_+ < v_- + p(\rho_-)$, then the solution consists again of a small rarefaction, followed by a contact discontinuity (Case 2). Therefore, in these two cases the solution still depends continuously on the data.

Now the case $v_+ < v_-$ (Case 1) can be split into the following two subcases:

(iii) If $v_+ < v_- < v_- + p(\rho_-) < v_+ + p(\rho_+)$, then the solution consists of a shock wave connecting M_- to M_0, with $w(M_-) = w(M_0) < w(M_+)$, followed by a contact discontinuity connecting M_0 to M_+. In this case, the original contact discontinuity $0M_+$ has been replaced by a shock wave M_-M_0, with $M_- \neq O$ and a speed $s \neq v_+$, followed by the contact M_0M_+; see Figure 4.1.

There is still a continuous dependence of the solution with respect to the data.

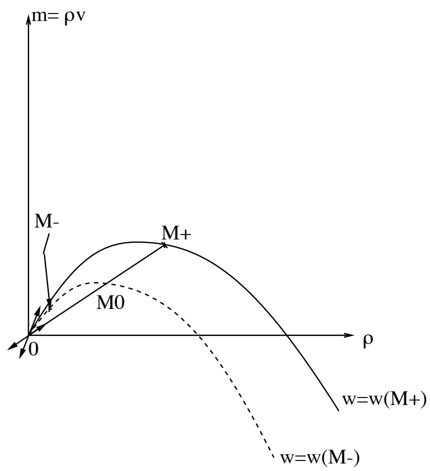

FIG. 4.1. *Perturbation* (iii) *of* M_- *in Case* 5.

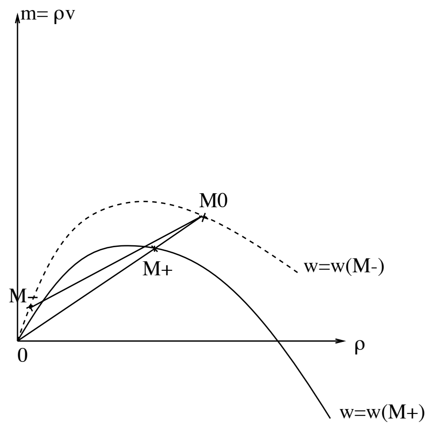

FIG. 4.2. *Perturbation* (iv) *of* M_- *in Case* 5.

(iv) In contrast, if now the velocity v_- is so big that $v_+ < v_- < v_+ + p(\rho_+) < v_- + p(\rho_-)$, then the solution has the same structure as in the previous case, except that now $w(M_-) = w(M_0) > w(M_+)$. Therefore the shock M_-M_0 now has a much larger amplitude than the original contact discontinuity and is almost immediately followed by the contact M_0M_+; see Figure 4.2. Therefore, in this case a big oscillation has appeared, and the solution has dramatically changed, under a small perturbation of the Riemann data.

Let us now come back to Case 4 of section 3, and let us slightly perturb the right state. Now $M_+ = (\rho_+, \rho_+ v_+)$, $\rho_+ << 1$, and we keep the other state M_- fixed. Then the perturbed solution strongly depends on the velocity v_+:

(v) If $v_+ > v_- + p(\rho_-)$, then the perturbed solution consists of the same unperturbed rarefaction wave (Case 3), which still connects M_- to the vacuum, followed by a fake vacuum wave and then by a small contact discontinuity. In this case, the solution depends continuously on the data.

(vi) However, if $v_- + p(\rho_-) \geq v_+ \geq v_-$, then the new solution consists

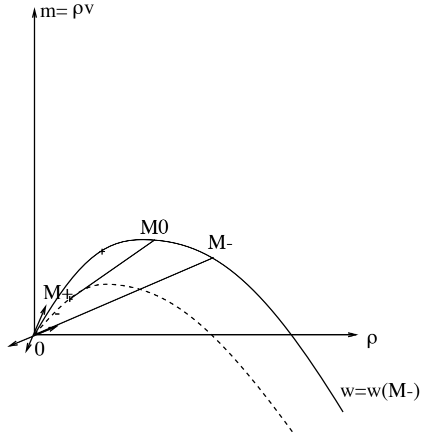

FIG. 4.3. *Perturbation* (vi) *of* M_+ *in Case* 5.

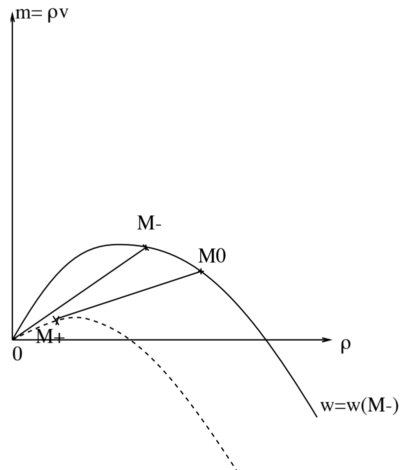

FIG. 4.4. *Perturbation* (vii) *of* M_+ *in Case* 5.

(Case 2) of a rarefaction wave connecting M_- to an intermediate state M_0, followed by a contact discontinuity. As we see in Figure 4.3 these two waves can be large. In particular, if $v_+ = v_-$, then the unperturbed rarefaction wave completely disappears and is entirely replaced by a contact discontinuity directly connecting M_- to M_+. This is therefore our second example of discontinuous dependence with respect to the data.

(vii) Finally, if $v_+ < v_-$, then the perturbed solution is still more dramatically different from the original one; see Figure 4.4. Indeed, as in Case 1 of section 3, the new solution consists of a (possibly large) shock wave, followed by a large contact discontinuity.

The three examples (v), (vi), and (vii) are certainly spectacular examples of discontinuous dependence with respect to the Riemann data and therefore of possible instabilities in traffic flow. In particular the latter is frequently observed by any driver in everyday life, each time there is ahead very light traffic of very slow drivers.

We have also noted that similar systems present a lack of continuous dependence (in L^1) of the solution with respect to the initial data [23], [8], [9]. However, the system considered here seems slightly less pathological. In particular, there is no overcompressive shock here.

Finally, we conclude this section by observing that the convex and bounded region $\mathcal{R}$ defined in the M plane by (3.3) is clearly *invariant* for the Riemann problem. In other words, if we start with arbitrary initial data in $\mathcal{R}$, and if we construct an approximate solution by using the Riemann problem and taking averages, e.g., by the Godunov scheme, then this approximate solution will remain in $\mathcal{R}$ for all time, which implies, in particular,

$$0 \leq \ v \ \leq v_m; \qquad 0 \leq \ \rho \, p^{-1}(v_m). \tag{4.4}$$

We also note that the image of this region $\mathcal{R}$ in the Y plane is also convex and bounded. Therefore [4] the same region would also be invariant for the artificial viscosity perturbation (4.2).

In addition, since this system is a system with *coinciding* rarefaction and shock waves, [37], [14], it also has—at least away from the origin—very special properties of stability in the space BV of functions with bounded variation so that the Glimm scheme [11] converges and provides the global existence of an entropy weak solution. We do not pursue this direction here.

5. Conclusion.

Let us rewrite system (1.7):

$$\begin{aligned} &\partial_t \rho + \ \partial_x(\rho v) = \ 0, \\ &\partial_t(v + \ p(\rho)) + \ v \, \partial_x(v + \ p(\rho)) = \ 0. \end{aligned} \tag{5.1}$$

Then we can summarize our results as follows.

THEOREM 5.1. *We consider the above system* (5.1) *or its formally equivalent forms* (2.4) *or* (2.10) *in the M or Y plane. Then*

(i) *any of these systems is strictly hyperbolic, except at the origin (Principle* A*);*

(ii) *for any function p which satisfies* (2.3), *and for any Riemann data $M_\pm = (\rho_-, \ \rho_- \ v_-)$ in the region*

$$\mathcal{R} := \ (\rho, \ v) \mid 0 \leq \ v \leq \ v_m - \ p(\rho), \qquad \rho \geq 0, \ \ 0 \leq \ v \leq \ v_m,$$

there exists a unique solution to the Riemann problem associated with (2.10) *and the data $M_\pm$. This solution satisfies all the principles stated in section* 1:

(iii) *The solution remains for all times with values in the invariant region $\mathcal{R}$. In particular, the velocity and the density remain nonnegative and bounded from above (Principle* B*).*

(iv) *The propagation speed of any wave involving a state U is at most equal to its velocity v: no information travels faster than the velocity of cars (Principle* C*).*

(v) *The qualitative properties of the solution are as expected: braking corresponds to a shock, accelerating to a rarefaction etc. (Principle* D*).*

(vi) *When one of the Riemann data is near the vacuum, the solution presents the instabilities described in section* 4 *(Principle* E*).*

In conclusion, the heuristic model introduced here respects *all* the natural requirements stated in the introduction. *All* the inconsistencies of "second order" models have disappeared. Moreover, the model nicely predicts instabilities near the vacuum, i.e., for very light traffic.

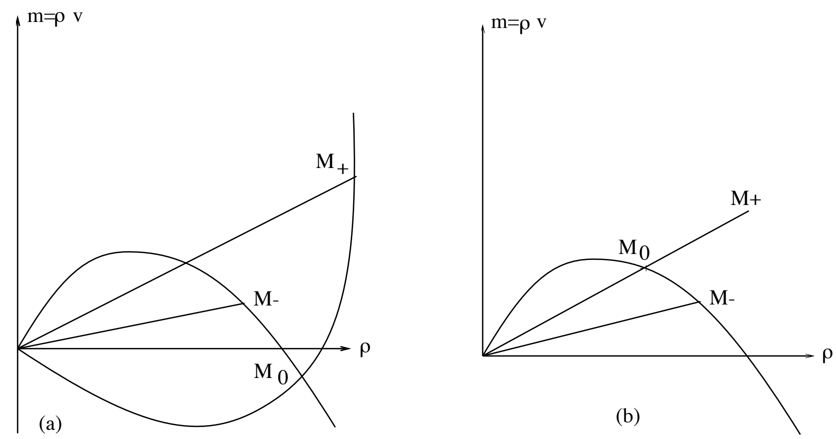

FIG. 5.1. *Examples of Riemann problems for the PW model* (a)*, and for our model* (b).

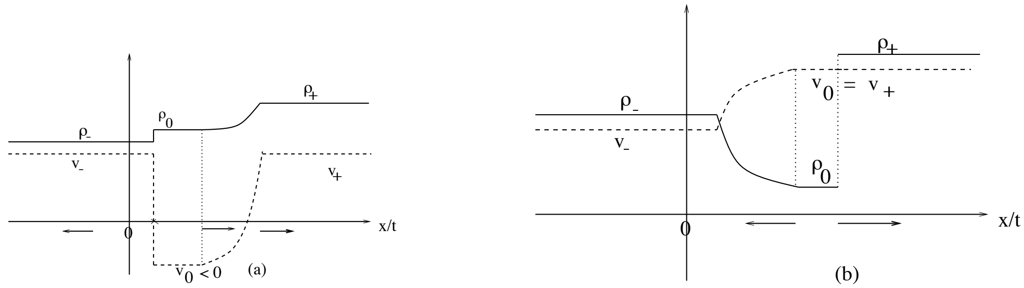

FIG. 5.2. *Negative velocities appear with the PW model* (a)*, not with our model* (b).

Moreover, we have already noted that its *only* new ingredient, namely, the anticipation factor viewed as a response to the *convective* derivative of the density, is a nice continuous version of discrete car-following models.

Finally, as far as we can see, there is one small drawback to this model, which appears in Case 4 of section 3, i.e., when there is a rarefaction wave connecting a state M_- to the vacuum: $M_+ = 0$. In this case, the maximal velocity v reached by the cars is the slope of the tangent to the curve $w(M) = w(M_-)$ in the M plane, i.e.,

$$v_{max} = v_- + p(\rho_-).$$

In other words, this maximal speed reached by the cars on an empty road *depends* on the initial data M_-! This is clearly wrong. Therefore a more complete model should include a relaxation term of the form

$$(\tau)^{-1} \, (V(\rho) - v)$$

in the momentum equation, where τ is a relaxation time and $V(\rho)$ a suitable, preferred (nonincreasing) velocity corresponding to each density ρ. This is easy to do without losing the above nice properties of the model introduced here. Of course another natural question is whether to let τ go to 0 to see if we then recover the LWR model (1.1) as the zero-relaxation limit; see, e.g., [33], [3], [2], [5]. We will come back to these questions in a future paper.

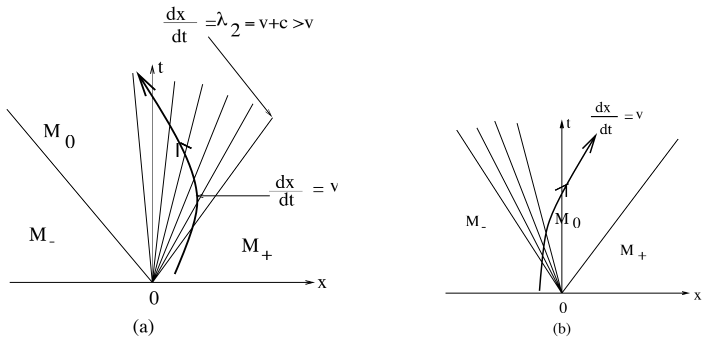

FIG. 5.3. *The trajectories* $\frac{dx}{dt} = v$ *cross the* 2 (*resp.,* 1) *rarefaction wave from the right to the left with PW model* (a) *and from the left to the right with our model* (b).

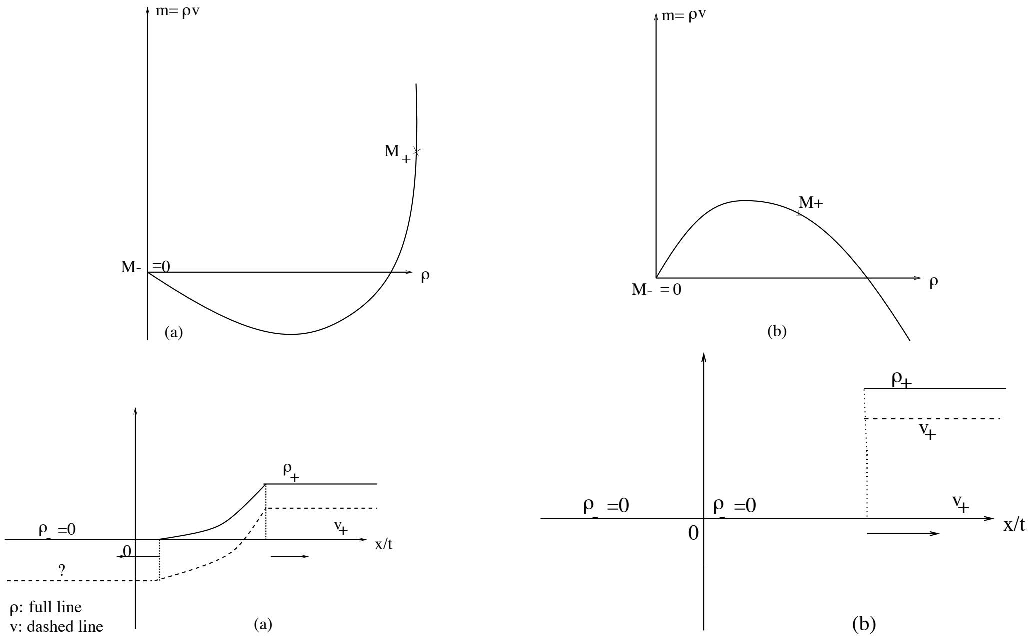

FIG. 5.4. *if* $M_- = M_+ = 0$ *(vacuum,* v_- *is undefined), then negative velocities appear with the PW model* (a)*, whereas our model predicts the correct solution* (b).

Appendix: Riemann problem: Comparison with the PW model. We present here a few examples of Riemann problems, with same data $M_\pm := (\rho_\pm,\ \rho_\pm v_\pm)$.

In the first case (Figures 5.1 and 5.2) $v_- < v_+$. Nevertheless, drivers in the rear brake(!), and even reach negative velocities with the PW model, (a), whereas they accelerate until they reach the velocity v_+ with the model introduced here (b). The trajectories $\frac{dx}{dt} = v$ corresponding to each case are depicted in Figure 5.3. The situation is still clearer in Figure 5.4, where $M_- = O$. Again with the PW model (a), cars start going backward(!), whereas our model (b) predicts the correct behavior, i.e., the obvious traveling wave solution.

Acknowledgments. We thank C. Buisson and O. Lesort for several stimulating discussions, and the references they suggested, among which [6] has been one of the

motivations of this work.

REFERENCES

[1] A. Brenneis, P. Konhauser, and B.S. Kerner, *Comparison of one and two-equation models for traffic flow calculations*, in Proc. 26th AIAA Fluid Dynamics Conference, June 19–22, San Diego, CA, 1995.

[2] G.Q. Chen, C.D. Levermore, and T.P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math., 47 (1994), pp. 787–830.

[3] G.Q. Chen and T.P. Liu, *Zero relaxation and dissipation limits for hyperbolic conservation laws*, Comm. Pure Appl. Math., 46 (1993), pp. 755–781.

[4] K.N. Chueh, C.C. Conley, and J. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana Univ. Math. J., 26 (1977), pp. 372–411.

[5] J.F. Collet and M. Rascle, *Convergence of the relaxation approximation to a scalar nonlinear hyperbolic equation arising in chromatography*, Z. Angew. Math. Phys., 47 (1996), pp. 399–409.

[6] C. Daganzo, *Requiem for second-order fluid approximation to traffic flow*, Transportation Res., Part B, 29 (1995), pp. 277–286.

[7] L.R. Foy, *Steady state solutions of hyperbolic systems of conservation laws with viscosity terms*, Comm. Pure Appl. Math., 17 (1964), pp. 177–188.

[8] H. Freistuhler, *Dynamical stability and vanishing viscosity: A case study of a non-strictly hyperbolic system*, Comm. Pure Appl. Math., 45 (1992), pp. 167–188.

[9] H. Freistuhler, *Contributions to the mathematical theory of magnetohydrodynamic shock waves,* in Nonlinear Evolutionary Partial Differential Equations in AMS/IP Stud. Adv. Math. 3, AMS, Providence, RI, 1997, pp. 175–187.

[10] H.-T. Fritsche, *A model for traffic simulation*, in Traffic Engineering & Control, 35 (1994).

[11] J. Glimm, *Solutions in the for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math., 18 (1965), pp. 697–715.

[12] J. Goodman, *Nonlinear asymptotic stability of viscous shocks and phase transitions*, Arch. Rational Mech. Anal., 83 (1983), pp. 333–361.

[13] D. Helbing, *Improved fluid dynamic model for vehicular traffic*, Phys. Rev. E, 51 (1995), pp. 31–64.

[14] B.L. Keyfitz and H.C. Kranzer, *A System of non-strictly hyperbolic conservation laws arising in elasticity theory*, Arch. Rational Mech. Anal., 72 (1980), pp. 219–241.

[15] A. Klar, R.D. Kuhne, and R. Wegener, *Mathematical models for vehicular traffic*, Surveys Math. Indust., 6 (1996), pp. 215–239.

[16] A. Klar and R. Wegener, *Enskog-like kinetic models for vehicular traffic*, J. Statist. Phys., 87 (1997), pp. 91–114.

[17] R.D. Kuhne and R. Beckschulte, *Non-linearity stochastics in unstable traffic flow*, in Transportation and Traffic Theory, C.F. Daganzo, ed., Proc. 12th Int. Symp. Trans. and Traffic Theory, Berkeley, CA, Elsevier, New York, 1993, pp. 367–386.

[18] P.D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, CBMF-NSF Regional Conf. Series in Appl. Math., SIAM, Philadelphia, PA, 1973.

[19] M.J. Lighthill and J.B. Whitham, *On kinematic waves.* I: *Flow movement in long rivers.* II: *A Theory of traffic flow on long crowded roads*, Proc. Royal Soc. London Ser. A, 229 (1955), pp. 281–345.

[20] T.-P. Liu, *The entropy condition and the admissibility of shocks*, J. Math. Anal. Appl., 53 (1976), pp. 78–88.

[21] T.-P. Liu, *Nonlinear stability of shock waves for viscous conservation laws*, Mem. Amer. Math. Soc., 56 (1985), no. 328.

[22] T.-P. Liu and J.A. Smoller, *On the vacuum state for the isentropic gas dynamics equations*, Adv. Appl. Math., 1 (1980), pp. 345–359.

[23] T.-P. Liu and C.-H. Wang, *On a nonstrictly hyperbolic system of conservation law*, J. Differential Equations, 57 (1983), pp. 1–14.

[24] P. Nelson, *A kinetic model of vehicular traffic and its associated bimodal equilibrium solutions*, Transp. Theory Stat. Phys., 24 (1995), pp. 383–409.

[25] P. Nelson and D.D. Bui, *A novel traffic stream model deriving from a bimodal kinetic equilibrium*, submitted.

[26] G.F. Newell, *Nonlinear effects in the dynamics of car following*, Oper. Res., 9 (1961), pp. 209–229.

[27] O. Oleinik, *On the uniqueness of the generalized solution of the Cauchy problem for non-linear*

system of equations occurring in mechanics, Uspekhi. Math. Nauk (N.S.), 12 (1957), pp. 169–176.

[28] H.J. Payne, *Models of Freeway Traffic and Control*, Math. Models Publ. Sys., Simulation Council Proc. 28, Vol. 1, 1971, pp. 51–61.

[29] I. Prigogine, *A Boltzmann-like approach to the statical theory of traffic flow*, in Theory of Traffic Flow, General Motors Research Laboratory, Warren, MI, 1959–1961, pp. 158–164.

[30] I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic*, Elsevier, New York, 1971.

[31] M. Rascle, *On the static and dynamic study of oscillations for some hyperbolic systems of conservation laws*, Ann. Inst. H. Poincare, 8 (1991), pp. 333–350.

[32] P.I. Richards, *Shock waves on the highway*, Oper. Res., 4 (1956), pp. 42–51.

[33] S. Schochet, *The instant-response limit in Whitham's nonlinear traffic-flow model: Uniform well-posedness and global existence*, Asymptotic Anal., 1 (1988), pp. 263–282.

[34] D. Serre, *Systemes de lois de conservation* I *et* II, Arts et Sciences, Diderot, 1996.

[35] J.A. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, Berlin, 1983.

[36] A. Szepessy and X. Zhouping, *Nonlinear stability of viscous shock waves*, Arch. Rational Mech. Anal., 122 (1993), pp. 53–103.

[37] B. Temple, *Systems of conservation laws with coinciding shock and rarefaction curves*, Contemp. Math., 17 (1983), pp. 143–151.

[38] G.B. Whitham, *Linear and Nonlinear Waves*, Pure and Applied Math., Wiley-Interscience, New York, 1974.

[39] R. Wegener and A. Klar, *A kinetic model for vehicular traffic derived from a stochastic microscopic model*, Transp. Theory Stat. Phys., to appear.