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TOWARDS THE MODELING OF VEHICULAR TRAFFIC AS A COMPLEX SYSTEM: A KINETIC THEORY APPROACH

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Kinetic theory methods are applied in this paper to model the dynamics of vehicular traffic. The basic idea is to consider each vehicular-driver system as a single part, or micro-system, of a large complex system, in order to capture the heterogeneous behavior of all the micro-systems that compose the overall system. The evolution of the system is ruled by nonlinearly additive interactions described by stochastic games. A qualitative analysis for the proposed model with discrete states is developed, showing well-posedness of the related Cauchy problem for the spatially homogeneous case and for the spatially nonhomogeneous case, the latter with periodic boundary conditions. Numerical simulations are also performed, with the aim to show how the model proposed is able to reproduce empirical data and to show emerging behavior as the formation of clusters.

Keywords: Traffic flow; empirical data; heterogeneity; nonlinearity; Cauchy problem.

AMS Subject Classification: 35L50, 35L65, 90B20

1. Introduction

This paper deals with a mathematical model for vehicular traffic flows based on the kinetic theory approach. From a mathematical point of view, traffic flow phenomena can be modeled at three different scales: microscopic, kinetic and macroscopic, as reported in the review papers.[8,10,16,24,31] As documented by the above literature, the microscopic description refers to vehicles individually identified and leads to systems

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of ordinary differential equations, while continuum mechanics assumptions lead to macroscopic models stated in terms of partial differential equations corresponding to fluid dynamic equations. The approach offered by the kinetic theory, developed after the pioneer contribution by Prigogine and Herman,[42] uses Boltzmann and/or Vlasov-type equations to model the complex system under consideration and needs a continuity assumption of the probability distribution defined over the variables of the phase space.

From the viewpoint of physics, the book by Kerner[36] reports a detailed interpretation of the physics of traffic, where various specific phenomena observed in traffic flow conditions are reported and critically analyzed, so that a valuable background to be used towards modeling is given. This book is based on a remarkable experience of the author documented in several papers, among others.[34,35,37] Additional bibliography can be obtained by the papers published in the special issue[5] and the bibliography cited therein.

This present paper focuses on the use of kinetic theory methods which are, in the authors' opinion, a useful approach that can be properly developed to capture the complexity of the physical reality of the dynamics of vehicular traffic. The basic idea is to consider each vehicular-driver system as a, so-called, active particle[9] of a large complex system, to model the heterogeneous behavior of the micro-systems that compose the overall system. The evolution of the system is ruled by interactions between the active particles described by stochastic games.

In details, Sec. 2 provides a critical analysis of the common features of vehicular traffic interpreted as complexity features. In addition, a brief outline on empirical data and their use for the validation of the models is given. Section 3 introduces the structures of the mathematical kinetic theory for active particles[9] that may act as paradigms for the derivation of specific models. More precisely, this section develops an analysis on models with discrete microscopic state, where the use of discrete states aims at capturing the granular behavior of vehicular traffic. Section 4 is devoted to some specific modeling strategies related to the granular structures introduced in the previous section. Sections 5 and 6 present a qualitative analysis, respectively for the spatially homogeneous case and for the spatially nonhomogeneous case, showing the well-posedness of the related Cauchy problem. Some numerical simulations are also given to capture emerging behaviors. Section 7 offers a critical analysis of the contents of this paper and looks ahead to research perspectives.

2. Traffic as a Complex System

This section presents a selection of the characteristics of vehicular traffic which should get the point to model such system as a complex system. Applied mathematicians involved in traffic modeling often refer to the criticisms, from the viewpoint of engineers, offered by the sharp paper of Daganzo,[23] who observes that the classical continuity assumption of mechanics cannot be applied to traffic flow.

Indeed, particle flows in fluid dynamics refer to thousands of particles, while only few vehicles are involved even in traffic jams. This remark can also be extended to the approach of the kinetic theory and specifically to the assumption of continuity of the distribution function. Furthermore, he observes that a vehicle is not a particle, but a system linking driver and mechanics, so that one has to take into account the reaction of the driver, who may be aggressive, timid, prompt etc. Finally, the aforesaid paper remarks that increasing the complexity of the model increases the number of parameters to be identified that may even be impossible due to the complexity of the setting measurements.

The basic idea is that micro-systems, vehicles-drivers on a road, constitute a large complex system of entities undergoing nonlinearly additive interactions that are specific of complexity. Mathematical models should attempt to capture the main aspects of such characteristics. Accordingly, let us select, among various ones and according to the authors' bias, some characteristics that will be considered as paradigms of the complexity.

2.1. *Heterogeneous distribution of the individual behaviors*

The individual behavior of the driver-vehicle micro-system is heterogeneously distributed among interacting entities. The shape of the distribution over the microscopic state has an influence over the strategy developed in the interactions among vehicles.

2.2. *Behaviors of complex systems*

Vehicles on roads should be regarded as complex living systems of entities, which interact in a nonlinear manner. Moreover, the output of interactions is related to specific strategies generated by the ability to communicate among them, and to organize the dynamics according to both their own strategy and interpretation of that of the others.[11] A further difficulty is generated by the fact that in several cases individual dynamics are not generally observable, while the emerging behaviors of the collective dynamics can be observed and geometrically interpreted.

2.3. *Granular dynamics*

The dynamics shows behavior of granular matter with aggregation and vacuum phenomena. Indeed, the continuity assumption of the distribution function in kinetic theory cannot be straightforwardly supposed.

2.4. *Influence of the environmental conditions*

The dynamics is remarkably affected by the quality of environment including quality of weather and road conditions. Therefore, the modeling approach needs to include this aspect into the mathematical equations by parameters to be tuned with respect to the variability of these conditions.

2.5. *Parameters*

The modeling approach needs suitable parameters to model some essential characteristics of the system under consideration. It is important that these parameters are related to specific different phenomena. Moreover, their identification should be technically pursued either by using existing experimental data, or by experiments to be properly designed.

Moreover, the modeling approach should refer to *empirical data* that can be, namely ought to be, used to validate mathematical models. On the other hand, the difficulty to obtain these data is due to the great variation of the environment and of the individual behaviors. This aspect reduces the amount of available data useful to validate theoretical models.

A deep analysis of empirical data is delivered in the book by Kerner,[36] which shows a variety of physical phenomena that characterize traffic flow. In particular, the author reports the *velocity diagram*, where the mean dimensionless velocity ξ is represented as a function of the dimensionless density ρ, showing that in steady flow conditions it is possible to identify two critical densities ρ_c and ρ_s characterizing three main different phases:

Phase I. $\rho \leq \rho_c$, where the mean velocity keeps its maximal value;

Phase II. $\rho_c < \rho \leq \rho_s$, where the mean velocity decays monotonically with ρ and shows a localized rapid decay for densities just above ρ_c;

Phase III. $\rho_s \leq \rho < 1$ that corresponds to the presence of stop and go phenomena.

Moreover, the velocity variance is small in Phase I, large in Phase II, while it is difficult to measure in Phase III. In particular, the existing models do not appear to have the ability of modeling the complex phenomena which characterize Phase III.[8]

Generally, empirical data are very sensitive to the quality of the road or of the surrounding environment, namely to the overall macro-system. Therefore, it is impossible to identify a unique deterministic representation for all roads. This characteristic has suggested to introduce, in vehicular traffic, a parameter suitable to describe the quality of the road.[18]

Several models use analytic approximations of empirical data and insert them artificially in the structure of the models instead of modeling a dynamics of interactions that generate a trend to a velocity diagram depending on some parameters. On the other hand, the modeling should invent a dynamics sufficiently close to physical reality such that the above empirical data are described by the model corresponding to steady flow conditions. Mathematical models should show the ability to depict emerging behaviors that are qualitatively observed although repeated experiments reproduce them, however with different quantitative results.

Bearing all the above in mind, we can state that the modeling approach should be consistent with the five requirements related to complexity features and with the objective of reproduce quantitatively the velocity diagram as an output at the macroscopic level of the microscopic interactions and with depicting, at least at a qualitative level, the emerging behaviors that are observed in several specific situations of real flow conditions.

3. Mathematical Representation and Structures

As already mentioned, this paper pursues the objectives that have been illustrated in the preceding sections, through the methods of the kinetic theory.

The first step of the modeling approach is the selection of the mathematical structure, among various conceivable ones, that can be considered the reference framework for the derivation of specific models. This section deals with such problem and looks for the structure that appears to be the most consistent with the complexity behaviors described in Sec. 2.

Several papers are offered by the existing literature on the kinetic modeling approach. Among others, we refer to Refs. 28, 33, 38–40, 42, 43 and to the review papers[8,31] for a rather complete bibliography. However, it has not yet been identified a unique structure to be used towards modeling, while a variety of models heuristically refer to very different structures, all derived from the classical mathematical kinetic theory.

The presentation is organized through two subsections: the first one deals with the representation of the system; the second one with the design of the framework suitable to model granular behaviors and nonlinearly additive interactions. The contents of this section take advantage of Ref. 2 where some preliminary ideas on the topics presented hereafter have been proposed.

3.1. *Kinetic theory representation*

Let us consider the representation by kinetic theory methods of a large system of interacting entities, namely vehicles with driver regarded as *micro-systems*, which interact in a suitable environment. Specifically we consider the one directional flow of vehicles along a road with length ℓ.

Since the *heterogeneous distribution of the individual behaviors* has to be taken into account, the microscopic state of the micro-system should include a suitable variable to depict this aspect. The mathematical kinetic theory for active particles[9] suggests to introduce a variable, called *activity*, to model the individual behaviors.

The following variables can be defined:

- $t \in \mathbb{R}^+$, which is the dimensionless time variable obtained referring the real time to a suitable critical time T_c, to be properly defined by a qualitative analysis of the differential models. Generally, it is convenient to identify the critical time T_c as the ratio between ℓ and the maximum admissible mean velocity V_M that is reached in free flow conditions.
- $x \in [0, 1]$, which is the dimensionless space variable obtained dividing the real space by the length ℓ of the road.

- $v \in [0,1]$, which is the dimensionless velocity variable related to the limit velocity V_ℓ reached by the fastest isolated vehicles.
- $u \in [0,1]$, which is the dimensionless activity variable that identifies the quality of the driver-vehicle micro-system. Specifically $u = 0$ corresponds to the worst quality, while $u = 1$ corresponds to the best quality.
- ρ_M, which is the maximum density of vehicles corresponding to bumper-to-bumper traffic jam.

The state of the whole system is defined by the statistical distribution of position, velocity and activity of the vehicles by means of the *generalized distribution function*

$$f = f(t,x,v,u) : \mathbb{R}^+ \times [0,1] \times [0,1] \times [0,1] \to \mathbb{R}^+, \tag{3.1}$$

where $f(t,x,v,u)$ is normalized with respect to ρ_M. The space, velocity, and activity variables define the microscopic state of each vehicle.

Classically, $f(t,x,v,u)dx\,dv\,du$ gives the number of vehicles which, at the time t, are in the elementary volume of the space of the microscopic states: $[x, x+dx] \times [v, v+dv] \times [u, u+du]$.

Macroscopic observable quantities can be obtained, under suitable integrability assumptions, by moments of the above distribution function. Due to normalization, all derived variables are in a dimensionless form. In particular, the *dimensionless local density*, referred to ρ_M, is given by

$$\rho[f](t,x) = \int_0^1 \int_0^1 f(t,x,v,u)dv\,du, \tag{3.2}$$

while the *total number of vehicles* at the time t is computed by integration over space.

In the same way, the local *dimensionless mean velocity* and the *flow* can be computed, respectively, as follows:

$$\xi[f](t,x) = \frac{1}{\rho[f](t,x)} \int_0^1 \int_0^1 vf(t,x,v,u)dv\,du \tag{3.3}$$

and

$$q[f](t,x) = \xi[f](t,x)\rho[f](t,x), \tag{3.4}$$

while the local *speed variance* is given by:

$$\sigma[f](t,x) = \frac{1}{\rho[f](t,x)} \int_0^1 \int_0^1 [v - \xi[f](t,x)]^2 f(t,x,v,u)dv\,du. \tag{3.5}$$

Moreover, the *speed pressure* is defined by the speed variance multiplied by the local density:

$$p[f](t,x) = \sigma[f](t,x)\rho[f](t,x). \tag{3.6}$$

Similarly, one can compute the local *mean value* and *variance* of the activity variable:

$$a[f](t,x) = \frac{1}{\rho[f](t,x)} \int_0^1 \int_0^1 u f(t,x,v,u) dv\, du \tag{3.7}$$

and

$$\mathrm{var}[f](a) = \frac{1}{\rho[f](t,x)} \int_0^1 \int_0^1 [u - a[f](t,x)]^2 f(t,x,v,u) dv\, du. \tag{3.8}$$

Remark 3.1. The above representations rely on the assumption that the number of interacting vehicles is large enough to justify the continuity assumption of the distribution function. Actually, this is not true in physical reality and an alternative approach needs to be developed as we shall see in the next sections. However, some preliminary reasonings, based on the above representation, are still useful towards a deeper analysis that will be developed in the sequel.

3.2. *Mathematical frameworks for nonlinearly additive interactions*

As already mentioned, a variety of models belonging to different structures are known in the literature as reviewed in Ref. 8. Only recently, the need of using the kinetic theory for active particles has been postulated to model the heterogeneous behavior of the micro-systems that compose the overall system. This subsection focuses on the derivation of a specific structure in the case of nonlinearly additive interactions involving the micro-systems, generating the evolution of the generalized distribution function f.

In general, three types of particles are involved in the interactions:

Test particle which is representative of the whole system. The related distribution function is $f = f(t,x,v,u)$;

Field particles which interact with test and candidate particles. Their distribution function is $f^* = f(t,x^*,v^*,u^*)$;

Candidate particles which may acquire, in probability, the state of the test particle by interaction with the field particles. Their distribution function is $f_* = f(t,x_*,v_*,u_*)$.

Remark 3.2. All particles cannot be distinguished individually. Therefore, their state identifies them; more precisely, candidate particles are field particles whose state, after the identification, reaches that of the test particles.

Let us define, for each particle, an *interaction domain* $\Omega = \Omega(x)$ that is a subset of the vehicle visibility zone and can depend on the position x of the vehicle. We will consider *distributed interactions with stochastic games*: particles modify, in probability, their state according to a strategy based on the analysis of the position and state of all particles in Ω.

Let us just mention that different types of interactions have already been considered in the literature,[8] such as mean field binary interactions, where the resultant action on the test particle is the sum of all binary actions applied by the field particles belonging to Ω, and localized binary interactions, where the measure of Ω is negligible and just one free particle is in Ω.

The models with interactions ruled by stochastic games take into account the strategy based on an analysis that is weighted over the microscopic state of the other micro-systems present in the visibility zone. Such a structure can be written as follows:

$$\begin{aligned}
&\partial_t f(t,x,v,u) + v\partial_x f(t,x,v,u)\\
&\quad = J[f](t,x,v,u)\\
&\quad = \int_\Lambda \eta[\mathbf{K}[f](t,x^*),x]\mathcal{A}(v_* \to v; u_* \to u | v_*, v^*, u_*, u^*, \mathbf{K}[f](t,x^*))\\
&\qquad \cdot f(t,x,v_*,u_*)f(t,x^*,v^*,u^*)dv_*\, dx^*\, dv^*\, du_*\, du^*\\
&\qquad - f(t,x,v,u)\int_\Gamma \eta[\mathbf{K}[f](t,x^*),x] f(t,x^*,v^*,u^*)dx^*\, dv^*\, du^*,
\end{aligned} \tag{3.9}$$

where $\Lambda = \Gamma \times [0,1] \times [0,1]$, $\Gamma = \Omega \times [0,1] \times [0,1]$, while candidate particles modify their dynamics under two actions, namely interactions with the field particles and with the macroscopic action of the stream modeled by $\mathbf{K}$, with

$$\mathbf{K}[f](t,x^*) = \{\rho[f](t,x^*)\}, \quad x^* \in \Omega = [x, x+L], \quad L > 0.$$

The interactions of a candidate or a test micro-system in x (with velocities v_*, v, and activity u_*, u, respectively) with the field particles located in its interaction domain, i.e. $x^* \in \Omega$, with velocity v^* and activity u^*, are weighted by the term η, interpreted as an interaction rate. The candidate particle modifies its state according to $\mathcal{A}$, that denotes the probability density that the candidate particle with state (v_*, u_*) reaches the state (v,u) after the interaction with the field particles with state (v^*, u^*). On the other hand, the test particle loses its states v and u after interactions with field particles with velocity v^* and activity u^*.

Some recent papers[14,22,27] have proposed a modeling approach which takes into account the analysis by Daganzo in Ref. 23 and dealt with the lack of continuity of the distribution function with respect to the velocity variable. These models have been based on discretization methods of the velocity variable describing the evolution of groups of vehicles with velocity within suitable range.

In this section, following Ref. 30, we consider a modeling approach which takes into account not only the lack of continuity of the distribution function with respect to the velocity variable but also with respect to the activity variable.

In order to describe the method, at first we discretize the velocity variable and the activity variable by introducing the sets

$$I_v = \{0 = v_1, \ldots, v_i, \ldots, v_I = 1\}, \quad I_u = \{0 = u_1, \ldots, u_j, \ldots, u_J = 1\}$$

and define the distribution functions as a sum of Dirac distributions in the variable v and u, with coefficients depending on t and x

$$f(t,x,v,u) = \sum_{i=1}^{I}\sum_{j=1}^{J} f_{ij}(t,x)\delta(v-v_i)\delta(u-u_j), \tag{3.10}$$

where $f_{ij}(t,x) = f(t,x,v_i,u_j)$.

Thus, according to this mathematical representation, we define the local discrete density as follows:

$$\rho(t,x) = \sum_{i=1}^{I}\sum_{j=1}^{J} f_{ij}(t,x) \tag{3.11}$$

and introduce the discrete probability density

$$\mathcal{A}_{hk,pq}^{ij}(v_h \to v_i, u_k \to u_j | v_h, v_p, u_k, u_q, \rho(t,x^*)) \tag{3.12}$$

which denotes the probability density that the candidate particle (v_h, u_k) falls into the state (v_i, u_j) of the test particle after an interaction with a field particle (v_p, u_q), with the property that

$$\sum_{i=1}^{I}\sum_{j=1}^{J} \mathcal{A}_{hk,pq}^{ij} = 1, \quad \forall\, h,p = 1,\dots,I, \quad \forall\, k,q = 1,\dots,J.$$

In this way, the evolution equation (3.9) is reduced to the following system of $I \times J$ partial differential equations in the $I \times J$ unknowns f_{ij}

$$\begin{aligned}
&\partial_t f_{ij}(t,x) + v_i \partial_x f_{ij}(t,x) \\
&\quad = J_{ij}[f](t,x) = \sum_{h,p=1}^{I}\sum_{k,q=1}^{J} \int_x^{x+L} \eta[\rho(t,x^*),x] \\
&\qquad \cdot \mathcal{A}_{hk,pq}^{ij}(v_h \to v_i, u_k \to u_j | v_h, v_p, u_k, u_q, \rho(t,x^*)) f_{h,k}(t,x) f_{p,q}(t,x^*)\,dx^* \\
&\qquad - f_{ij}(t,x) \int_x^{x+L} \sum_{p=1}^{I}\sum_{q=1}^{J} \eta[\rho(t,x^*),x] f_{p,q}(t,x^*)\,dx^*.
\end{aligned} \tag{3.13}$$

Remark 3.3. A large variety of models include a term which describes a trend to an equilibrium. A simple approach is as follows:

$$\partial_t f(t,x,v,u) + v\partial_x f(t,x,v,u) = J[f](t,x,v,u) + c_r(\rho)(f_e - f), \tag{3.14}$$

where $f_e = f_e(v;\rho)$ denotes the equilibrium distribution function, that may be parametrized by the local density, and $c_r(\rho)(f_e - f)$ describes a trend to equilibrium analogous to the BGK model in kinetic theory, where the rate of convergence c_r depends on the local density. On the other hand, the dynamics of the interactions microscopic level should naturally generate, as direct predictions of the model, stationary solutions.

4. Modeling Strategy and Applications

The contents of the preceding sections naturally lead to the development of a modeling strategy which basically consists in implementing the mathematical structure (3.13) with the modeling of interactions at the microscopic scale. The aim pursued to achieve such objective consists in looking for a specific modeling of the interaction terms that characterize such structure, namely η and $\mathcal{A}_{hk,pq}^{ij}$, such that the five requirements presented in Sec. 2 are satisfied and a good agreement with empirical data, concerning both the fundamental diagram and the emerging behaviors in unsteady flow conditions, is provided.

As a first step, let us introduce a parameter that expresses the quality of the environment:

$$\alpha = \alpha(\rho_{\mathrm{c}}), \quad \rho_{\mathrm{c}} \in [0, \rho_{cM}], \tag{4.1}$$

where α is a positive monotone increasing function of ρ_c and ρ_{cM} is the maximal admissible value of ρ_c observed in the best quality road in optimal environmental conditions. Further details on the identification of $\alpha = \alpha(\rho_c)$ will be given in Sec. 5.

4.1. *Modeling of the encounter rate*

As already mentioned, the encounter rate $\eta[\rho(t, x^*), x]$ gives the number of interactions per unit time among the vehicles. Here we will consider the following expression

$$\eta[\rho(t, x^*), x] = \Psi[\rho(t, x^*)] w(x, x^*), \tag{4.2}$$

where $w(x, x^*)$ is a weight function and

$$\Psi[\rho(t, x^*)] = 1 + \frac{1}{\alpha}\rho^2 \tag{4.3}$$

with α defined in (4.1).

The function w models the weight of the interaction of the candidate particle with each field particle located in the visibility zone $\Omega(x) = [x, x+L]$, of the candidate particle. It is required that

$$w(x, x^*) \geq 0, \quad \int_x^{x+L} w(x, x^*) dx^* = 1, \quad \forall\, x \in [0, 1], \quad \forall\, x^* \in [x, x+L]. \tag{4.4}$$

Different choices of w can be considered. Among the others, we mention the piecewise constant function

$$w(x, x^*) = \begin{cases} \dfrac{1}{L}, & \text{for } x^* \in [x, x+L]; \\ 0, & \text{otherwise,} \end{cases} \tag{4.5}$$

which gives a standard average of the interactions in the visibility zone, and the following function

$$w(x, x^*) = \frac{e^{-\frac{(x-x^*)^2}{4L}}}{\sqrt{\pi L} \int_0^{\frac{\sqrt{L}}{2}} e^{-x^2} dx}. \tag{4.6}$$

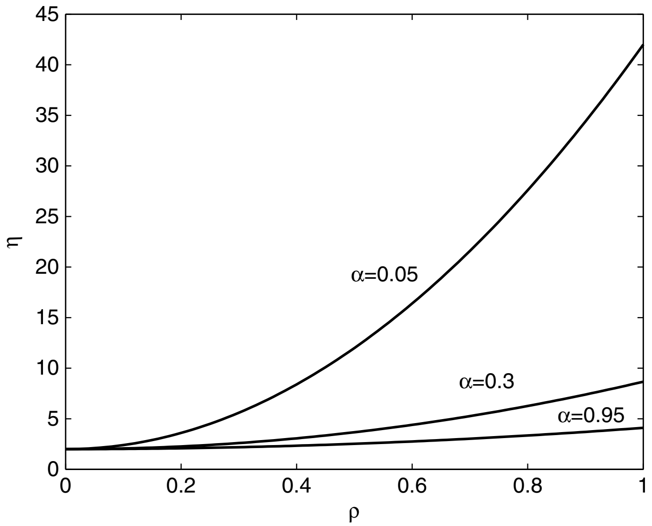

Fig. 1. Encounter rate vs. density.

We underline that, for both functions defined in (4.5) and (4.6), we have

$$\lim_{L\to 0} w(x,x^*) = \delta(x-x^*).$$

Focusing on (4.3), we note that Ψ is monotonically increasing with respect to ρ and depends on α. Accordingly, the local interaction becomes higher and higher as the density increases. Figure 1 reports, for different values of α, the behavior of η in the case when w is as in (4.5) with $L = 0.5$.

4.2. *Modeling of the table of games*

In this subsection we propose a form for the so-called table of games $\mathcal{A}_{hk,pq}^{ij}$ which gives the probability that a vehicle with velocity v_h and state u_p reaches the velocity v_i and the state u_j, after an interaction with a vehicle traveling at velocity v_k and state u_q. To this end, following Ref. 9, we assume that it can be factorized as follows:

$$\begin{aligned}&\mathcal{A}_{hk,pq}^{ij}(v_h \to v_i, u_k \to u_j | v_h, v_p, u_k, u_q, \rho(t,x^*))\\ &\quad = \mathcal{B}_{hp}^{i}(v_h \to v_i | v_h, v_p, u_k, \rho(t,x^*))\mathcal{C}_{kq}^{j}(u_k \to u_j | u_k, u_q, \rho(t,x^*)),\end{aligned} \tag{4.7}$$

where $\mathcal{B}_{hp}^{i}$ models the probability density of a velocity transition from v_h to v_i, after an interaction with a field vehicle traveling at the speed v_p. $\mathcal{B}_{hp}^{i}$ satisfies the following requirement:

$$\mathcal{B}_{hp}^{i} \geq 0, \quad \sum_{i=1}^{I} \mathcal{B}_{hp}^{i} = 1, \quad \forall\, h,p = 1,\ldots,I \tag{4.8}$$

while $\mathcal{C}_{kq}^{j}$ express the probability of an activity transition from the state u_k to u_j, after an interaction with a field vehicle having state u_q and is such that

$$\mathcal{C}_{kq}^{j} \geq 0, \quad \sum_{j=1}^{J} \mathcal{C}_{kq}^{j} = 1, \quad \forall\, k, \;\; q = 1, \ldots, J. \tag{4.9}$$

Focusing on the term $\mathcal{B}_{hp}^{i}$, in the last years several authors[14,22,27] have proposed different ways to model it, based on an adaptive velocity grid,[22] on a fixed velocity grid[27] or based on both of them.[14] Moreover, in Ref. 22 the probabilities of transitions are supposed to be constant, while in Refs. 14 and 27 an explicit dependence on the traffic and road conditions via the macroscopic quantity ρ and the parameter α has been introduced. Furthermore, in Ref. 22 interactions among vehicles occur only if the velocities involved are sufficiently close; in Ref. 27 the interactions are not limited to vehicles with sufficiently close velocities but the transitions include velocity classes close to that involved; in Ref. 14 both interactions and transitions consider different velocity classes. We propose in this paper to use a fixed uniform grid defined as follows:

$$v_i = \frac{i-1}{I-1}, \quad i = 1, \ldots, I$$

and we give a model for $\mathcal{B}_{hp}^{i}$ taking into account that drivers adjust their velocities according to the road and traffic conditions, they can reach a new velocity not necessarily close to its speed class and interact nonlinearly with other drivers.

According to (4.7), the table of games $\mathcal{A}_{hk,pq}^{ij}$ introduces also an activity transition expressed by the term $\mathcal{C}_{kq}^{j}$. Recalling that the activity variable $u \in [0,1]$ identifies the quality of the driver-vehicle micro-system, where $u = 0$ corresponds to the worst quality and $u = 1$ corresponds to the best quality, we choose to model the term $\mathcal{C}_{kq}^{j}$ on the base of a uniform activity grid given by

$$u_j = 1 - \frac{j-1}{J}, \quad j = 1, \ldots, J.$$

In the following subsections, we give in details the model for $\mathcal{B}_{hp}^{i}$ and $\mathcal{C}_{kq}^{j}$.

4.2.1. *The term $\mathcal{B}_{hp}^{i}$*

Our model is based on the idea that the table of games $\mathcal{B}_{hp}^{i}$ should be defined according to the different phases of the densities.

- **Phase I:** $0 \leq \rho \leq \rho_{\mathrm{c}}$. In this case, the density, and consequently, the encounter rate, is very slow. Thus, the candidate vehicle has the tendency to keep the maximum velocity. Accordingly, we have:

$$\mathcal{B}_{hp}^{i} = \begin{cases} 1, & i = I; \\ 0, & \text{otherwise}. \end{cases} \tag{4.10}$$

- **Phase II:** $\rho_c < \rho \leq \rho_s$. Let us consider the candidate vehicle with velocity v_h and assume that it interacts with a field vehicle with velocity v_p. Thus, the cases $v_h > v_p$, $v_h = v_p$, $v_h < v_p$, are analyzed separately.

  — *Interaction with faster vehicles.* If $v_h < v_p$, the candidate vehicle maintains its current speed or accelerates. In the latter case, it can reach a new velocity $v_i \in \{v_{h+1}, \dots, v_p\}$ with a probability that depends not only on the road conditions and on the density, but also on the distance between the velocity classes involved and on its activity u_k. More precisely, the greater the distance between velocity classes the higher the acceleration required to reach that velocity and the lower the probability to actually reach it. Accordingly, the following table of games is proposed:

$$\mathcal{B}_{hp}^i = \begin{cases} 1 - \alpha u_k(\rho_s + \rho_{\mathrm{c}} - \rho), & i = h; \\ \alpha u_k \dfrac{1}{(i-h)} \dfrac{1}{\sum_{i=h+1}^{p} \frac{1}{i-h}} (\rho_s + \rho_{\mathrm{c}} - \rho), & i = h+1, \dots, p; \\ 0, & \text{otherwise}. \end{cases} \tag{4.11}$$

  Note that if $\rho \to \rho_{\mathrm{c}}$, which means that the density is very small, the probability to accelerate is greater than the probability to maintain its velocity. On the contrary if $\rho \to \rho_s$, the probability to reach a new velocity is very small.

  — *Interaction with slower vehicles.* If the candidate vehicle travels with a greater velocity, then we assume that it can maintain its velocity because it has enough free space to overtake, or it is forced to queue reaching a new velocity $v_i = \{v_p, \dots, v_{h-1}\}$. In both cases the probability depends on α and ρ and only in the last case it depends on the difference between the velocity classes involved, in the sense that the higher this difference the higher is the probability to force to queue. Consequently:

$$\mathcal{B}_{hp}^i = \begin{cases} \alpha u_k(\rho_s + \rho_{\mathrm{c}} - \rho), & i = h; \\ [1 - \alpha u_k(\rho_s + \rho_{\mathrm{c}} - \rho)]\,(h-i) \dfrac{1}{\sum_{i=p}^{h-1}(h-i)}, & i = p, \dots, h-1; \\ 0, & \text{otherwise}. \end{cases} \tag{4.12}$$

  Note that in the case when $\alpha \to 0$, the road conditions are such that it is not possible to overtake the field vehicle and thus the only possibility is to decelerate.

  — *Interaction with equally fast vehicles $h = p$.* If $h = p$, the candidate and the field vehicles travel to the same velocity. In this case, the result of the interaction is different, according to the density and the road conditions. More precisely, if the vehicles travel on a bad road (which means $\alpha \to 0$) the only possibilities for the candidate vehicle are to decelerate or to maintain its current velocity. Of course, the first possibility has more probability to happen if $\rho \to \rho_s$, while the second occurs with a high probability if $\rho \to \rho_{\mathrm{c}}$. On the contrary, in the case when $\alpha \to 1$, the candidate vehicle can accelerate or maintain its current velocity. More precisely, if $\rho \to \rho_c$ the probability to accelerate is greater than the probability to maintain its current speed, while if $\rho \to \rho_s$ the vehicle has a small probability to accelerate. Therefore, for $h = 2, \ldots, I-1$, the following table is given:

$$\mathcal{B}_{hp}^{i} = \begin{cases} (1-\alpha u_k)(h-i)\dfrac{1}{\sum_{i=1}^{h-1}(h-i)}(1-\rho_s-\rho_c+\rho), & i = 1, \ldots, h-1; \\ \alpha u_k + (1-2\alpha u_k)(\rho_s+\rho_c-\rho), & i = h; \\ \alpha u_k \dfrac{1}{(i-h)}\dfrac{1}{\sum_{i=h+1}^{I}\frac{1}{i-h}}(\rho_s+\rho_c-\rho), & i = h+1, \ldots, I. \end{cases} \tag{4.13}$$

When $h = 1$ or $h = I$ the candidate vehicle cannot respectively brake or accelerate. Thus, we merge the deceleration or the acceleration into the tendency to preserve the current velocity and the tables of games are as follows:

$$\mathcal{B}_{11}^{i} = \begin{cases} 1-\alpha u_k(\rho_s+\rho_c-\rho), & i = 1; \\ \alpha u_k(\rho_s+\rho_c-\rho), & i = 2; \\ 0, & \text{otherwise}. \end{cases} \tag{4.14}$$

$$\mathcal{B}_{II}^{i} = \begin{cases} \alpha u_k(1-\rho_s-\rho_c+\rho), & i = I-1; \\ 1-\alpha u_k(1-\rho_s-\rho_c+\rho), & i = I; \\ 0, & \text{otherwise}. \end{cases} \tag{4.15}$$

- **Phase III:** $\rho_s < \rho < 1$. In this case the density is very high and so regardless of road conditions, the candidate vehicle has a tendency to stop. Then we have:

$$\mathcal{B}_{hp}^{i} = \begin{cases} 1, & i = 1; \\ 0, & \text{otherwise}. \end{cases} \tag{4.16}$$

Note that all the terms $\mathcal{B}_{hp}^{i}$ defined in this subsection satisfy condition (4.8).

4.2.2. *The term* $\mathcal{C}_{kq}^{j}$

Concerning the modeling of the term $\mathcal{C}_{kq}^{j}$, it is assumed that when a candidate vehicle interacts with the field vehicle, the transition of the activity does not occur. Consequently, the table of games presents the following expression

$$\mathcal{C}_{kq}^{j} = \begin{cases} 1, & j = k; \\ 0, & \text{otherwise}. \end{cases} \tag{4.17}$$

Remark 4.1. As we shall see in the following sections, these tables of games ensure to get an upper bound for flux. We just mention here that in various biological frameworks a different modeling approach has been investigated, introducing in the model a nonlinear limited flux term.[1,4,19]

5. The Spatially Homogeneous Problem

This section deals with the theoretical and computational analysis of the spatially homogeneous problem associated to Eq. (3.13) in which the distribution function is independent of the variable x

$$f := f(t, v, u) = \sum_{i=1}^{I} \sum_{j=1}^{J} f_{ij}(t)\delta(v - v_i)\delta(u - u_j). \tag{5.1}$$

Consequently $\partial_x f_{ij}(t, x) = 0$ and the associated Cauchy problem can be written as

$$\begin{cases} \dfrac{df_{ij}}{dt} = \eta[\rho(t)] \left[\displaystyle\sum_{h,p=1}^{I} \sum_{k,q=1}^{J} \mathcal{A}_{hk,pq}^{ij} f_{h,k}(t) f_{p,q}(t) - f_{ij}(t) \sum_{p=1}^{I} \sum_{q=1}^{J} f_{p,q}(t) \right], \\ f_{ij}(0) = f_{ij}^0, \end{cases} \tag{5.2}$$

where

$$\mathcal{A}_{hk,pq}^{ij} := \mathcal{A}_{hk,pq}^{ij}(v_h \to v_i, u_k \to u_j | v_h, v_p, u_k, u_q, \rho(t)).$$

The initial value problem consists of $I \times J$ ordinary differential equations in the unknowns $f_{ij} : \mathbb{R}^+ \to \mathbb{R}^+$, supplemented by $I \times J$ initial conditions f_{ij}^0.

Let J be the operator given by

$$J[f] = J_{ij}[f] = \eta[\rho(t)] \left[\sum_{h,p=1}^{I} \sum_{k,q=1}^{J} \mathcal{A}_{hk,pq}^{ij}(\rho) f_{h,k}(t) f_{p,q}(t) - f_{ij}(t) \sum_{p=1}^{I} \sum_{q=1}^{J} f_{p,q}(t) \right]. \tag{5.3}$$

Considering that $\sum_{i=1}^{I} \sum_{j=1}^{J} J_{ij}[f] = 0$, one has $\partial_t \rho = 0$, by summing (5.2) over i, j. This gives $\rho(t) = \rho_0$ and the initial value problem (5.2) reduces to

$$\begin{cases} \dfrac{df_{ij}}{dt} = \eta[\rho_0] \left[\displaystyle\sum_{h,p=1}^{I} \sum_{k,q=1}^{J} \mathcal{A}_{hk,pq}^{ij} f_{h,k}(t) f_{p,q}(t) - f_{ij}(t)\rho_0 \right], \\ f_{ij}(0) = f_{ij}^0, \quad i = 1, \ldots, I, \quad j = 1, \ldots, J. \end{cases} \tag{5.4}$$

5.1. *Local existence*

The spatially homogeneous problem is a good benchmark to test the reliability of the theoretical predictions with respect to the available experimental data, since it provides some information on the trend of the system toward the equilibrium (the so-called fundamental diagram), that can be duly compared with the measurements performed under uniform flow conditions (see e.g. Kerner[36]).

Let M_{IJ} be the set of the following matrices M_{IJ} endowed with the 1-norm:

$$\|f(t)\|_1 = \sum_{i=1}^{I}\sum_{j=1}^{J} |f_{ij}(t)|, \quad f = (f_{ij}) \in M_{IJ}. \tag{5.5}$$

Moreover, we introduce the linear space $X = C([0,T], M_{IJ})$ of the matrix-valued continuous functions $f = f(t) : [0,T] \to M_{IJ}$ for some $T > 0$, equipped with the infinity norm

$$\|f(t)\|_\infty = \sup_{t\in[0,T]} \|f(t)\|_1.$$

Note that $(X, \|\cdot\|_\infty)$ is a real Banach space. We will keep on denoting by ρ the sum of the components of any $f \in X$:

$$\rho(t) = \sum_{i=1}^{I}\sum_{j=1}^{J} f_{ij}(t).$$

Well-posedness of the spatially homogeneous problem means global in time existence and uniqueness of a solution $f = f(t)$ to the Cauchy problem (5.2). To show these results, it is usual to pass through two steps. The first one is to show the local existence and uniqueness in time of the solution f in X for a certain $T > 0$, while the second one is to extend the obtained result to a global solution defined for all $t > 0$. In order to prove this, we need the following assumptions.

Assumption H1. The transition probability $\mathcal{A}_{hk,pq}^{ij}[\rho]$ is such that

$$\sum_{i=1}^{I}\sum_{j=1}^{J} \mathcal{A}_{hk,pq}^{ij} = 1, \quad \forall\, h,p = 1,\dots,I, \quad \forall\, k,q = 1,\dots,J,$$

whenever $0 \le \rho \le 1$.

Assumption H2. There exists $C_\eta > 0$ such that $0 < \eta(\rho) \le C_\eta$, when $0 \le \rho \le 1$.

Under these assumptions, one obtains the following:

Theorem 5.1. *Let Assumptions* H1 *and* H2 *hold, and let in addition* $0 \le \rho_0 \le 1$. *Then there exists* $T > 0$ *such that problem* (5.2) *admits a unique non-negative local solution* $f \in X$ *satisfying the following estimate*

$$\|f(t)\|_1 = \rho_0, \quad \forall\, t \in [0,T]. \tag{5.6}$$

To prove this theorem, we need some preliminary estimates on the nonlinear operator J, defined by (5.3), which are given in the following lemma:

Lemma 5.1. *Let Assumptions* H1 *and* H2 *hold. Then one has the following estimates*: $\exists\, C_1 > 0$ *such that*

$$\|J(f,f)\|_1 \le C_1 \|f\|_1^2,$$

$$\|J(f,f) - J(g,g)\|_1 \le C_1(\|f\|_1 + \|g\|_1)\|f - g\|_1.$$

Let us define an operator ϕ by:

$$\phi(f)(t) = \int_0^t J(f,f)(s)ds.$$

Then the following lemma gives an estimate for ϕ:

Lemma 5.2. *Let Assumptions* H1 *and* H2 *hold. Then ϕ is a continuous map from X into X and $\exists\, C_1 > 0$ such that*:

$$\|\phi(f)\|_\infty \leq C_1 T \|f\|_\infty^2,$$
$$\|\phi(f) - \phi(g)\|_\infty \leq C_1 T(\|f\|_\infty + \|g\|_\infty)\|f-g\|_\infty.$$

Proof of Theorem 5.1. Equation (5.4) can be written in the form of the integral equation

$$f = N(f),$$

where $N(f)$ is given by

$$(N(f))_{ij} = f_{ij}^0 + \int_0^t \eta[\rho_0]\left[\sum_{h,p=1}^{I}\sum_{k,q=1}^{J} \mathcal{A}_{hk,pq}^{ij} f_{h,k}(s) f_{p,q}(s) - f_{ij}(s)\rho_0\right] ds.$$

The proof can be obtained by application of classical fixed point methods. Using Lemma 5.2 one has

$$\|N(f)\|_\infty \leq \|f_0\|_1 + C_1 T\|f\|_\infty^2, \tag{5.7}$$
$$\|N(f) - N(g)\|_\infty \leq C_1 T(\|f\|_\infty + \|g\|_\infty)\|f-g\|_\infty. \tag{5.8}$$

Let T such that

$$T < \frac{1}{4C_1\|f_0\|_1}$$

and a_0 the reel given by

$$a_0 = \frac{1 - \sqrt{1 - 4C_1 T\|f_0\|_1}}{2C_1 T\|f_0\|_1}.$$

Then from (5.7) and (5.8) one deduces easily that N is a contraction on a ball in X of radius a_0. Thus, there exists a unique local solution $f(t)$ of Eq. (5.2) on $[0,T]$. Moreover by summing over i,j and using Assumption H1, one gets (5.6). Positivity of solutions is now to be proved. Bearing this objective in mind, the solution f_{ij} satisfies the following identity:

$$(N(f))_{ij} = (f_{ij}) = \exp(-I^2J^2\eta[\rho_0]\rho_0 t) f_{ij}^0 + \int_0^t \exp(I^2J^2\eta[\rho_0]\rho_0(s-t))\eta[\rho_0]$$
$$\cdot\left[\sum_{h,p=1}^{I}\sum_{k,q=1}^{J} \mathcal{A}_{hk,pq}^{ij} f_{h,k}(s) f_{p,q}(s)\right] ds.$$

It is clear that N maps X_+ into itself if the initial datum (condition) is positive. Now the fixed point theorem in X_+ can be applied again using Lemma 5.2 and this completes the proof. □

5.2. *Global existence*

The existence of a local in time solution $f(t)$ to problem (5.2) and an *a priori* estimate (5.6) allow us to extend $f(t)$ on the whole positive real axis R_+:

Theorem 5.2. *Under the same hypotheses of Theorem* 5.1, *Problem* (5.2) *admits a unique non-negative global solution* $f \in C(R_+, M_{IJ})$ *satisfying estimate* (5.6).

Proof. We apply the same technique developed in the proof of Theorem 5.1 on the interval $(T, 2T]$, taking $f(T)$ as new initial condition. Since $f_{ij}(T) \geq 0$ for all $i = 1, \ldots, I$ and $j = 1, \ldots, J$, and $\sum_{h=1}^{I} \sum_{k=1}^{J} f_{hk} = \rho_0 \in [0, 1]$, we are in the same hypotheses of Theorem 5.1, hence we conclude the existence and uniqueness of a local in time continuous solution on $[T, 2T]$ satisfying

$$\|f(t)\|_1 = \rho_0, \quad \forall\, t \in [T, 2T]. \tag{5.9}$$

Iterating on all intervals of the form $[mT, (m+1)T]$, $m \in N$, we construct a global solution on R_+. This completes the proof. □

Remark 5.1. Let us point out that the model stated in Sec. 4 satisfies Assumptions H1 and H2.

5.3. *Simulations*

Some simulations are presented to show how our model is able to reproduce empirical data. In practice, we give the velocity diagram (Fig. 2) in which the mean velocity is represented as function of the macroscopic density ρ and the fundamental diagram (Fig. 3) which describes as the macroscopic flux varies with the density ρ. We have chosen to perform the simulations for different road conditions and this has meant simply to set different values of the parameter α appearing in our model. More precisely according to (4.1), we fix α as follows:

$$\alpha = \alpha_0 + \frac{\rho_{\mathrm{c}}}{\rho_{cM}}(1 - \alpha_0), \tag{5.10}$$

where α_0 is the minimal value of α identified by experiments. In our simulations we have fixed $\alpha_0 = 0.3$ and $\rho_{cM} = 0.1$. The problem under consideration has been numerically solved by means of the solver ode45 of Matlab by fixing six velocity classes $\{v_i\}_{i=1}^{6}$ and one activity class.

The consistency with experimented data is carefully respected. Indeed, one can observe that in free flow conditions, namely $\rho \in [0, \rho_{\mathrm{c}}]$, the mean velocity is constant and the growth of the flux is linear as experimentally described by Kerner.[36] Subsequently, in the phase transition between the regimes of free and congested

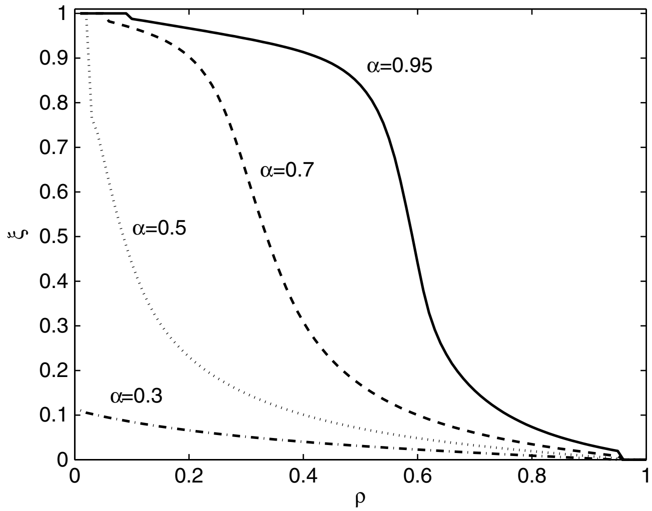

Fig. 2. Velocity diagram: mean velocity ξ vs. density ρ.

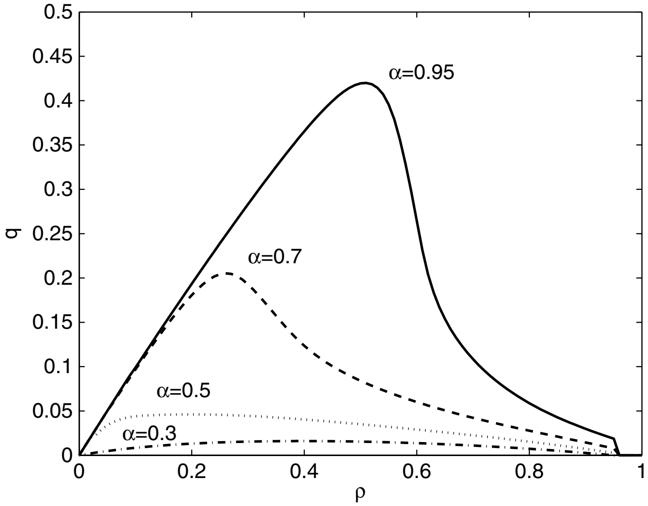

Fig. 3. Fundamental diagram: flux q vs. density ρ.

flow, the flux initially grows but subsequently decreases to zero and this confirms a critical change in the characteristics of traffic for high density.

It is worth putting in evidence the role of the parameter α, which models, within a minimal description, the overall quality of the environment. According to (5.10), the following specific features can be mentioned:

(i) If $\alpha = 1$ the critical density $\rho_c = \rho_{cM}$ and the mean velocity is constant and takes its maximal value.

(ii) Decreasing values of α imply a restriction of the interval $[0, \rho_c]$ where the mean velocity is still constant and takes its maximum value, but the decay to the congested flow in the second phase increases.
(iii) If $\alpha = \alpha_0$, the critical density is zero and the maximal mean velocity takes values much lower than the previous case.

Although these simulations appear worth to be further developed towards a deeper understanding, we feel that to state a good agreement with the empirical data has been achieved and this result has been obtained without artificial insertion of the velocity diagram into the model itself.

6. The Spatially Nonhomogeneous Problem

This section deals with the spatially nonhomogeneous problem with periodic conditions, consisting of an integro-differential system of $I \times J$ equations with linear hyperbolic advection part supplemented by a set of initial conditions $f_{ij}^0(x)$ and boundary conditions $\tilde{f}_{ij}^0(t)$

$$\begin{cases} \partial_t f_{ij}(t,x) + v_i \partial_x f_{ij}(t,x) = J_{ij}[f](t,x), \\ f_{ij}(0,x) = f_{ij}^0(x) \quad \forall\, x \in [0,1], \\ f_{ij}(t,0) = f_{ij}(t,1) = \tilde{f}_{ij}^0(t) \quad \forall\, t \in [0,T], \end{cases} \tag{6.1}$$

for $i = 1, \ldots, I, j = 1, \ldots, J$.

The proof of the existence theorems for this problem follows lines somehow analogous to those of Ref. 3, however some substantial technical differences are introduced to take into account the statement in bounded domains, while the problem treated in Ref. 3 was in an unbounded domain.

If the encounter rate $\eta[\rho]$ satisfies the decomposition (4.3), the initial nonhomogeneous problem (3.13) can be written as:

$$\partial_t f_{ij}(t,x) + v_i \partial_x f_{ij}(t,x) = \Gamma_{ij}[\mathbf{f},\mathbf{f}](t,x) - f_{ij}(t,x)\Lambda[\mathbf{f}](t,x), \tag{6.2}$$

supplemented by the set of initial and boundary conditions already stated in (6.1), where the gain and loss operators are respectively given by

$$\begin{aligned} \Gamma_{ij}[\mathbf{f},\mathbf{f}](t,x) = \int_x^{x+L} \sum_{h,p=1}^{I} \sum_{k,q=1}^{J} \Psi[\rho(t,x^*)] w(x,x^*) \mathcal{A}_{hk,pq}^{ij}(\rho(t,x^*)) \\ \cdot f_{h,k}(t,x) f_{p,q}(t,x^*) dx^* \end{aligned} \tag{6.3}$$

and

$$\Lambda[\mathbf{f}](t,x) = \int_x^{x+L} \sum_{p=1}^{I} \sum_{q=1}^{J} \Psi[\rho(t,x^*)] w(x,x^*) f_{p,q}(t,x^*) dx^*. \tag{6.4}$$

To state precisely our results, let us define the functional space

$$L^1_{M_{IJ}} = \left\{ f = (f_{pq}) \in M_{IJ} : \|f(t)\|_1 = \sum_{p=1}^{I} \sum_{q=1}^{J} \int_0^1 |f_{p,q}(t,x)| dx < \infty \right\},$$

and let us introduce, for $T > 0$, the Banach space

$$X_T = X[0,T] = C([0,T], L^1_{M_{IJ}})$$

of the matrix-valued functions

$$f = f(t,x) : [0,T] \times R \to M_{IJ},$$

such that for all fixed $t \in [0,T]$ the function $x \to f(t,x)$ belongs to $L^1_{M_{IJ}}$, where the norm in X_T is given by

$$\|f\|_{X_T} = \sup_{t \in [0,T]} \|f(t)\|_1.$$

The qualitative analysis developed hereafter takes advantage by the assumptions listed below:

Assumption H3. Both the encounter rate $\Psi[\rho]$ and the transition probability $\mathcal{A}^{ij}_{hk,pq}(\rho)$ are Lipschitz continuous functions of the macroscopic density ρ, i.e. there exist constants $L_\eta, L_\mathcal{A}$, such that

$$\begin{aligned} |\Psi[\rho_1] - \Psi[\rho_2]| &\leq L_\Psi |\rho_1 - \rho_2|, \\ |\mathcal{A}^{ij}_{hk,pq}[\rho_1] - \mathcal{A}^{ij}_{hk,pq}[\rho_2]| &\leq L_\mathcal{A} |\rho_1 - \rho_2|, \end{aligned} \tag{6.5}$$

whenever $0 \leq \rho_1 \leq \rho_c$, $0 \leq \rho_2 \leq \rho_c$, and for all $i, h, p = 1, \ldots, I$ and $j, k, q = 1, \ldots, J$.

Assumption H4.

(1) There exists a constant c_w such that:

$$0 \leq w(z) \leq c_w, \quad z \in [0, L]; \tag{6.6}$$

(2) There exists $C_\Psi > 0$ such that

$$0 < \Psi(\rho) \leq C_\Psi, \tag{6.7}$$

whenever $0 \leq \rho \leq \rho_c$.

Remark 6.1. Let us point out that the model stated in Sec. 4 satisfies Assumptions H3 and H4, which are consistent with the physical reality.

The following results can be proved under the above assumptions:

Theorem 6.1. (Local existence) *Let $f^0_{ij} \in L^\infty \cap L^1_{M_{IJ}}$, $f^0_{ij} \geq 0$, $i = 1, \ldots, I$, $j = 1, \ldots, J$. Then there exist f^0_0, T, ρ_c and a_0, such that, if $\|f^0\|_1 \leq f^0_0$, there* *exists a unique non-negative solution to the corresponding initial value problem for* (6.2) *satisfying*: $f \in X_T$,

$$\sup_{t\in[0,T]} \sum_{p=1}^{I}\sum_{q=1}^{J}\int_0^1 |f_{p,q}(t,x)|dx \leq a_0\|f^0\|_1, \quad \forall\, t \in [0,T]$$

and $\rho(t,x) \leq \rho_c$.

Theorem 6.2. (Global existence) *Let assumptions of Theorem* 6.1 *be satisfied. Then there exist* f_1^0, ρ_c *and* $a_r(r = 1,\ldots,n-1)$, *such that if* $\|f^0\|_1 \leq f_1^0$, *there exists a unique non-negative solution to the initial value problem for* (6.2) *satisfying for any* $r \leq n-1$

$$f(t) \in X[0,nT], \tag{6.8}$$

$$\sup_{t\in[0,T]} \|f(t+(r-1)T)\|_1 \leq a_{r-1}\|f^0\|_1, \quad \forall\, t \in [0,T], \tag{6.9}$$

$$\rho(t+(r-1)T,x) \leq \rho_c, \quad \forall\, t \in [0,T], \quad \forall\, x \in [0,1]. \tag{6.10}$$

Moreover if

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\|f_{ij}^0\|_\infty \leq 1, \tag{6.11}$$

one has

$$\rho(t+(r-1)T,x) \leq 1, \quad t \in [0,T], \quad x \in [0,1]. \tag{6.12}$$

6.1. *Proof of the local existence*

This subsection deals with the proof of Theorem 6.1 concerning the local existence to the initial value problem for (6.2). Pursuing this aim, let us introduce the function

$$\psi_{ij}(t,x) = \exp(\lambda t) f_{ij}(t,x), \quad \text{for } \lambda > 0. \tag{6.13}$$

Obviously, the problem can be written in equivalent form in terms of the functions $\psi(t,x) = \psi_{ij}(t,x)$, where $\psi_{ij}(i = 1,\ldots,I, j = 1,\ldots,J)$ are solutions of the following initial value problem:

$$\begin{cases} \dfrac{\partial\widehat{\psi_{ij}}}{\partial t} = \lambda\widehat{\psi_{ij}} + \exp(-\lambda t)(\widehat{\Gamma_{ij}}[\psi,\psi](t,x) - \widehat{\psi_{ij}}(t,x)\widehat{\Lambda}[\psi](t,x)), \\ \psi_{ij}(t=0,x) = f_{ij}^0(x), \quad \psi_{ij}(t,0) = \psi_{ij}(t,1), \end{cases} \tag{6.14}$$

where, for a given matrix $(f_{ij}(x,t))$, we denote:

$$(\widehat{f_{ij}}(t,x)) = (f_{ij}(t,x+v_i t)), \quad i = 1,\ldots,I, \quad j = 1,\ldots,J,$$

$$\begin{aligned} \widehat{\Gamma_{ij}[\mathbf{f},\mathbf{f}]}(t,x) = \int_x^{x+L} \sum_{h,p=1}^{I}\sum_{k,q=1}^{J} &\Psi[\rho(t,x^*+v_i t)]w(x+v_i t, x^*+v_i t) \\ &\cdot \mathcal{A}_{hk,pq}^{ij}(v_h \to v_i, u_k \to u_j | v_h, v_p, u_k, u_q, \rho(t,x^*+v_i t)) \\ &\cdot \widehat{f_{h,k}}(t,x+(v_i-v_h)t)\widehat{f_{p,q}}(t,x^*+(v_i-v_p)t)dx^* \end{aligned} \tag{6.15}$$

and

$$\widehat{\Lambda[\mathbf{f}]}(t,x) = \int_x^{x+L} \sum_{p=1}^{I}\sum_{q=1}^{J} \Psi[\rho(t,x^*)] w(x+v_i t, x^* + v_i t) \cdot \widehat{f_{p,q}}(t, x^*(v_i - v_p)t) dx^*. \tag{6.16}$$

Integrating (6.14) over time t, with $t \leq T$, yields the mild formulation of the spatially nonhomogeneous problem that we will refer to in the sequel:

$$\widehat{\psi_{ij}}(t,x) = f_{ij}^0(x) + \int_0^t (\Gamma_{ij}\widehat{[\psi,\psi]}(s,x)\exp(-\lambda s) + \widehat{\psi_{ij}}(s,x)\{\lambda - \widehat{\Lambda[\psi]}(s,x)\exp(-\lambda s)\})ds. \tag{6.17}$$

Moreover, let us consider the operator N acting on X_T whose components are

$$(\widehat{N(\psi)})_{ij}(t,x) = f_{ij}^0(x) + \int_0^t (\Gamma_{ij}\widehat{[\psi,\psi]}(s,x)\exp(-\lambda s) + \widehat{\psi_{ij}}(s,x)\{\lambda - \widehat{\Lambda[\psi]}(s,x)\exp(-\lambda s)\})ds. \tag{6.18}$$

The contraction mapping principle will be applied to (6.18) to prove Theorem 6.1. However, some preliminary results, presented by the following lemmas, are necessary.

Lemma 6.1. *Let $f_{ij}, g_{ij} \in L_1$, such that $f_{ij}, g_{ij} \geq 0$ and*

$$\sum_{i=1}^{I}\sum_{j=1}^{J} \widehat{f_{ij}}(t, x - v_i s) \leq \rho_{\mathrm{c}} \exp(\lambda t), \quad \sum_{i=1}^{I}\sum_{j=1}^{J} \widehat{g_{ij}}(t, x - v_i s) \leq \rho_{\mathrm{c}} \exp(\lambda t),$$

for $x \in [0,1], s, t \geq 0$. Then

$$\|\widehat{\Gamma}(f,f) - \widehat{\Gamma}(g,g)\|_1 \leq (c_w c_\Psi + IJc_w\rho_{\mathrm{c}}(c_\Psi L_{\mathcal{A}} + L_\Psi)) \cdot (\|f\|_1 + \|g\|_1)\|f-g\|_1, \tag{6.19}$$

$$\|\widehat{f}\,\widehat{\Lambda}(f) - \widehat{g}\widehat{\Lambda}(g)\|_1 \leq (c_w c_\Psi + L_\Psi \rho_{\mathrm{c}} c_w)(\|f\|_1 + \|g\|_1)\|f-g\|_1. \tag{6.20}$$

Proof. First write $\widehat{\Gamma_{ij}}(f,f)(t,x) - \widehat{\Gamma_{ij}}(g,g)(t,x)$ as follows:

$$\begin{aligned}
&\widehat{\Gamma_{ij}}(f,f)(t,x) - \widehat{\Gamma_{ij}}(g,g)(t,x) \\
&\quad= \sum_{h,p=1}^{I}\sum_{k,q=1}^{J} \int_x^{x+L} \Psi[\rho(t,x^*+v_i t)] \mathcal{A}_{hk,pq}^{ij}[(\rho_f, x^* + v_i t);\alpha] w(x+v_i t, x^*+v_i t) \\
&\qquad \cdot [\widehat{f_{h,k}}(t, x+(v_i - v_h)t)\widehat{f_{p,q}}(t, x^* + (v_i - v_p)t) \\
&\qquad - \widehat{g_{h,k}}(t, x+(v_i - v_h)t)\widehat{g_{p,q}}(t, x^* + (v_i - v_p)t)]dx^*
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{h,p=1}^{I}\sum_{k,q=1}^{J}\int_{x}^{x+L}\widehat{g_{h,k}}(t,x+(v_i-v_h)t),\widehat{g_{p,q}}(t,x^*+(v_i-v_p)t)\\
&\cdot[\Psi[\rho_f(t,x^*+v_it)]\mathcal{A}_{hk,pq}^{ij}[\rho_f(t,x^*+v_it)]\\
&-\Psi[\rho_g(t,x^*+v_it)]\mathcal{A}_{hk,pq}^{ij}[\rho_g(t,x^*+v_it)]]dx^*\\
=\;& A_{ij}+B_{ij},
\end{aligned}
$$

Using Assumptions H1, H3 and H4 and summing over i, j, yields

$$
\begin{aligned}
\sum_{i=1}^{I}\sum_{j=1}^{J}\|A_{ij}\|_1 \le\;& c_\Psi c_w\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{h,p=1}^{I}\sum_{k,q=1}^{J}\int_{R\times R}\mathcal{A}_{hk,pq}^{ij}|\widehat{f_{h,k}}(t,x+(v_i-v_h)t)|\\
&\cdot|\widehat{f_{p,q}}(t,x^*+(v_i-v_p)t)-\widehat{g_{p,q}}(t,x^*+(v_i-v_p)t)|dx^*dx\\
&+c_\Psi c_w\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{h,p=1}^{I}\sum_{k,q=1}^{J}\int_{R\times R}\mathcal{A}_{hk,pq}^{ij}|\widehat{g_{p,q}}(t,x^*+(v_i-v_p)t)|\\
&\cdot|\widehat{f_{h,k}}(t,x+(v_i-v_h)t)-\widehat{g_{h,k}}(t,x+(v_i-v_h)t)|dx^*dx\\
\le\;& c_\Psi c_w\|f-g\|_1(\|f\|_1+\|g\|_1).
\end{aligned}\tag{6.21}
$$

On the other hand, considering that

$$
|\rho_f-\rho_g|\le\exp(-\lambda t)\sum_{r=1}^{I}\sum_{\nu=1}^{J}|\widehat{f_{r\nu}}(t,x^*+v_it)-\widehat{g_{r\nu}}(t,x^*+v_it)|,
$$

$$
\sum_{p=1}^{I}\sum_{q=1}^{J}|\widehat{g_{pq}}(t,x^*+(v_i-v_p)t)|\le\rho_{\mathrm{c}}\exp(\lambda t),
$$

and using Assumptions H3–H4, noting that $\mathcal{A}_{hk,pq}^{ij}\le 1$, and summing over i, j, yields:

$$
\sum_{i=1}^{I}\sum_{j=1}^{J}\|B_{ij}\|_1\le IJc_w\rho_{\mathrm{c}}(L_\Psi+c_\Psi L_A)\|g\|_1\|f-g\|_1,\tag{6.22}
$$

which gives, by using (6.21), estimate (6.19).

By the same arguments, from Assumptions H1 and H3–H4, one deduces (6.20) and this completes the proof of Lemma 6.1. □

Lemma 6.2. *Let $T>0$ and $\psi^1=(\psi_{ij}^1),\psi^2=(\psi_{ij}^2)\in X_T$, such that*

$$
\sum_{i=1}^{I}\sum_{j=1}^{J}\widehat{\psi_{ij}^1}(t,x-v_is)\le\rho_{\mathrm{c}}\exp(\lambda t),\quad \sum_{i=1}^{I}\sum_{j=1}^{J}\widehat{\psi_{ij}^2}(t,x-v_is)\le\rho_{\mathrm{c}}\exp(\lambda t),
$$

for $x \in [0,1], s, t \in [0,T]$. *Then we have*:

(1) $N(\psi^1) \in X_T$ *and* $\exists\, C_1 > 0$ *such that*

$$\|N(\psi^1)\|_{X_T} \le \|f^0\|_1 + \frac{C_1}{\lambda}\|\psi^1\|^2_{X_T} + \lambda T\|\psi^1\|_{X_T}. \tag{6.23}$$

(2) *There exists* $C_2 > 0$ *such that*

$$\|N(\psi^1) - N(\psi^2)\|_{X_T} \le \left(\frac{C_2}{\lambda}(\|\psi^1\|_{X_T} + \|\psi^2\|_{X_T}) + \lambda T\right)\|\psi^1 - \psi^2\|_{X_T}, \quad \forall\, t \in [0,T]. \tag{6.24}$$

(3) *Moreover if* $\psi^1_{ij} \ge 0$ *then* $\exists\, \lambda_0$ *such that if* $\lambda \ge \lambda_0$ *and if* $f^0_{ij} \ge 0$ *one has* $(\widehat{N(\psi)})_{ij} \ge 0$.

(4) *Let* $f^0_{ij} \in L^\infty$. *Then for* $\rho_{\rm c}$ *large enough* $(\rho_{\rm c} \ge \rho_1)$, *there exists* T *such that*

$$\sum_{i=1}^{I}\sum_{j=1}^{J}(\widehat{N(\psi)})_{ij}(t, x - v_i s) \le \rho_{\rm c}\exp(\lambda t), \quad \forall\, t \in [0,T], \quad \forall\, x \in [0,1]. \tag{6.25}$$

Proof. Taking into account Assumption H4 on Ψ and summing over i,j yields:

$$\|\Gamma[\widehat{\psi^1, \psi^1}]\|_1 \le c_\Psi c_w \|\psi^1\|^2_1. \tag{6.26}$$

Moreover, the same estimate holds for the loss operator $\widehat{\Lambda}$:

$$\|\widehat{\Lambda[\psi^1]}\widehat{\psi^1}\|_1 \le c_\Psi c_w \|\psi^1\|^2_1. \tag{6.27}$$

Therefore using (6.18), (6.26), (6.27), and by noting $C_1 = 2c_\Psi c_w$ one gets (6.23).

From (6.18) and Lemma 6.1, by the same arguments, one gets easily (6.24) with $C_2 = c_w c_\Psi + IJc_w\rho_{\rm c}(c_\Psi L_{\mathcal{A}} + L_\Psi)$.

Since $\widehat{\Gamma_{ij}}[\psi^1, \psi^1](t,x) \ge 0$ if $\psi \ge 0$, then the non-negativity of $(\widehat{A(\psi^1)})_{ij}$ depends on the possibility to find $\lambda > 0$ such that

$$\lambda - \widehat{\Lambda[\psi^1]}(s,x)\exp(-\lambda s) \ge 0. \tag{6.28}$$

Noting that from (4.4) and (6.16) we obtain

$$\begin{aligned}
\widehat{\Lambda[\psi^1]}(s,x) &= \int_x^{x+L}\sum_{p=1}^{I}\sum_{q=1}^{J}\Psi[\rho(s,x^*)]w(x+v_i s, x^* + v_i s) \\
&\quad \cdot \widehat{\psi^1_{p,q}}(s, x^* + (v_i - v_p)s)dx^* \\
&\le \exp(\lambda s)\rho_{\rm c} c_\Psi \int_{(x+v_i s)}^{(x+v_i s)+L} w(x + v_i s, x^*)dx^* \\
&\le \exp(\lambda s)\rho_{\rm c} c_\Psi.
\end{aligned}$$

The non-negativity of $(\widehat{\Lambda(\psi)})_i$ is then achieved by choosing $\lambda \geq \lambda_0 = \rho_{\rm c} c_\Psi$. To deal with (4) first let us see that

$$\begin{aligned}\widehat{\Gamma}(\psi^1, \psi^1)(t, x - v_i s) = {} & \sum_{h,p=1}^{I} \sum_{k,q=1}^{J} \int_{x-v_i s}^{x-v_i s+L} \Psi[\rho(t, x^* + v_i s)] \mathcal{A}_{hk,pq}^{ij} \\ & \cdot \widehat{\psi_{h,k}^1}(t, x + v_i(t-s) - v_h t) \\ & \cdot \widehat{\psi_{p,q}^1}(t, x^* + (v_i - v_p)t) w(x + v_i(t-s), x^* + v_i t) dx^* \\ \leq {} & c_\Psi \rho_{\rm c}^2 \exp(2\lambda t) \int_{x+v_i(t-s)}^{x+v_i(t-s)+L} w(x + v_i(t-s), x^*) dx^*, \\ \leq {} & c_\Psi \rho_{\rm c}^2 \exp(2\lambda t).\end{aligned}$$

Then one has the following estimate:

$$\begin{aligned}& \sum_{i=1}^{I} \sum_{j=1}^{J} (\widehat{N(\psi)})_{ij}(t, x - v_i s) \\ & \quad \leq \sum_{i=1}^{I} \sum_{j=1}^{J} \|f_{ij}^0\|_\infty + \frac{IJ c_\Psi \rho_{\rm c}^2}{2\lambda}(\exp(2\lambda t) - 1) + \rho_{\rm c}(\exp(\lambda t) - 1),\end{aligned}$$

which gives (6.25) if

$$\rho_{\rm c} > \sum_{i=1}^{I} \sum_{j=1}^{J} \|f_{ij}^0\|_\infty = \rho_1 \tag{6.29}$$

and

$$t \leq T = \frac{1}{2\lambda} \ln\left(1 + \frac{2\lambda}{IJ c_\Psi \rho_{\rm c}^2}\left(\rho_{\rm c} - \sum_{i=1}^{I} \sum_{j=1}^{J} \|f_{ij}^0\|_\infty\right)\right). \tag{6.30}$$

□

Lemma 6.3. *Let T be given by* (6.30) *with $\lambda = \rho_{\rm c} c_\Psi$. Then there exist f_0^0 and ρ_2 such that if $\|f^0\|_1 \leq f_0^0, \rho_{\rm c} \geq \rho_2$ one has*

$$(\lambda T - 1)^2 \geq \frac{4C}{\lambda} \|f^0\|_1, \tag{6.31}$$

where $C = \mathrm{Max}(C_1, C_2)$ is given by Lemma 6.2.

Proof. Using the inequality $\ln(1 + x) \leq x, x \geq 0$, we get

$$\lambda T \leq \frac{\lambda}{IJ c_\eta \rho_{\rm c}^2}\left(\rho_{\rm c} - \sum_{i=1}^{I} \sum_{j=1}^{J} \|f_{ij}^0\|_\infty\right) \leq \frac{1}{IJ}$$

which gives

$$(\lambda T - 1)^2 \geq \left(\frac{IJ-1}{IJ}\right)^2.$$

Then, if f^0 and $\rho_{\rm c}$ are such that

$$\|f^0\|_1 \leq \frac{(IJ-1)^2}{(IJ)^2} \frac{c_\Psi}{8IJc_w(c_\eta L_{\mathcal{A}} + L_\Psi)} = f_0^0 \tag{6.32}$$

and

$$\rho_{\rm c} \geq \frac{8c_w\|f^0\|_1(IJ)^2}{(IJ-1)^2} = \rho_2, \tag{6.33}$$

(6.31) is obtained. □

Proof of Theorem 6.1. Let now $\rho_{\rm c}$, and f_0 such that (6.29), (6.32), (6.33) hold, and let T be given by (6.30). Consider now the following subset in X_T defined as follows:

$$\begin{aligned} B_T = \Bigg\{ &\psi = (\psi_{ij}) \in X_T : \psi_{ij} \geq 0, \|\psi(t)\|_1 \leq a_0\|f^0\|_1, \\ &\sum_{i=1}^{I}\sum_{j=1}^{J} \widehat{\psi_{ij}}(t, x - v_i t) \leq \rho_{\rm c}\exp(\lambda t), \\ &\psi_{ij}(t,0) = \psi_{ij}(t,1),\ t \in [0,T],\ x \in [0,1] \Bigg\}. \end{aligned}$$

Let $\psi^1, \psi^2 \in B_T$, and let $\rho_{\rm c} \geq \rho_0 = \mathrm{Max}(\rho_1, \rho_2)$ then from Lemmas 6.1 and 6.2, there exists T such that $(\widehat{N(\psi^1)})_{ij} \geq 0$ and satisfying (6.25) for $\lambda = \rho_{\rm c} c_\Psi$, $t \in [0,T]$. Moreover from Lemma 6.2 one has:

$$\|N(\psi_1)\|_{X_T} \leq \|f^0\|_1 + \frac{1}{\lambda}Ca_0^2\|f^0\|_1^2 + \lambda a_0 T\|f^0\|_1, \tag{6.34}$$

$$\|N(\psi^1) - N(\psi^2)\|_{X_T} \leq \left(\frac{2}{\lambda}Ca_0\|f^0\|_1 + \lambda T\right)\|\psi^1 - \psi^2\|_{X_T}, \tag{6.35}$$

where C is the constant defined in Lemma 6.3. Let

$$\Delta_0 = (\lambda T - 1)^2 - \frac{4C}{\lambda}\|f^0\|_1. \tag{6.36}$$

Then by the previous lemma one has $\Delta \geq 0$, for $\|f^0\|_1 \leq f_0^0$. Now let a_0 be the positive quantity given by:

$$a_0 = \lambda\frac{(1-\lambda T) - \sqrt{\Delta_0}}{2C\|f^0\|_1}, \tag{6.37}$$

which is a solution of

$$\|f^0\|_1 + \frac{1}{\lambda}Ca_0^2\|f^0\|_1^2 + \lambda a_0 T\|f^0\|_1 = a_0\|f^0\|_1,$$

and using also (6.34), we obtain that $N(\psi^1) \in B_T$. Moreover,

$$\left(\frac{2}{\lambda}Ca_0\|f^0\|_1 + \lambda T\right) = 1 - \sqrt{\Delta_0} < 1,$$

which shows, from (6.35), that the mapping N is a contraction in a ball B_T. Using the fixed point theorem, the proof of Theorem 6.1 is complete, which concerns the local existence. □

6.2. *Proof of existence for large times*

The aim of this subsection consists in proving Theorem 6.2 concerning the global existence of solutions to the initial value problem for (6.2). It will be proved that the solution can be extended in each interval $[0, nT]$, for $n \in N$. As in the preceding section some preliminary results are needed:

Lemma 6.4. *Let f^0 satisfies the conditions of Theorem 6.1. Consequently, there exist a_1, f_1^0 and ρ_0^1 such that if $\|f^0\|_1 \le f_1^0$, $\rho_c \ge \rho_0^1$, the solution to the initial value problem for (6.2) can be extended in the interval $[T, 2T]$ and satisfies the estimate:*

$$\|\psi(t+T)\|_1 \le a_1\|f^0\|_1, \quad t \in [0, T]. \tag{6.38}$$

Proof. Let

$$\Delta_1 = (\lambda T - 1)^2 - \frac{4}{\lambda}Ca_0\|f^0\|_1. \tag{6.39}$$

Noting that from (6.37) one has

$$a_0 \le \frac{\rho_c c_\Psi}{2C\|f^0\|_1} \le \frac{c_\Psi}{2c_w L_\Psi}.$$

Then, by the previous arguments, one gets

$$\Delta_1 \ge \left(\frac{IJ-1}{IJ}\right)^2 - 2C\frac{\|f^0\|_1}{\rho_c c_w L_\Psi}.$$

If we choose f^0 and ρ_c such that

$$\rho_c \ge \rho_0^1 = \mathrm{Max}\left(\rho_0, \frac{4(IJ)^2}{(IJ-1)^2}\frac{c_\Psi}{L_\Psi}\|f^0\|_1\right),$$

$$\|f^0\|_1 \le f_1^0 = \mathrm{Min}\left(f_0^0, \frac{(IJ-1)^2 c_\Psi}{4(IJ)^3 c_w(c_\Psi L_{\mathcal{A}} + L_\Psi)}\right),$$

one gets $\Delta_1 \ge 0$.

We solve the initial value problem for (6.2) in $[T, 2T]$ with initial condition given by $\psi(T, x)$. Then, for any $t \in [0, T]$, one has:

$$\widehat{\psi_{ij}}(t+T, x) = \widehat{\psi_{ij}}(T, x) + \int_0^t (\widehat{\Gamma_{ij}[\psi,\psi]}(s+T, x)\exp(-\lambda(s+T)) \\ + \widehat{\psi_{ij}}(s+T, x)\{\lambda - \widehat{\Lambda[\psi]}(s+T, x)\exp(-\lambda(s+T))\})ds.$$

Consider now the ball

$$B_T^1 = \Bigg\{ g(t) = \psi(t+T) \in X_T : \psi \geq 0, \|\psi(t+T)\|_1 \leq a_1 \|f^0\|_1,$$
$$\sum_{i=1}^{I} \sum_{j=1}^{J} \widehat{\psi_{ij}}(t+T, x - v_i(t+T)) \leq \rho_c \exp(\lambda t),\ t \in [0,T], x \in [0,1] \Bigg\}.$$

Using the same technique as in the proof of Theorem 6.1 yields:

$$\|A(\psi^1)(t+T)\|_{X_T} \leq a_0 \|f^0\|_1 + \frac{Ca_1^2}{\lambda} \|f^0\|_1^2 + \lambda a_1 T \|f^0\|_1,$$
$$\|A(\psi^1)(t+T) - A(\psi^2)(t+T)\|_{X_T} \leq \left(\frac{2Ca_1}{\lambda} \|f^0\|_1 + \lambda T \right) \cdot \|\psi^1(t+T) - \psi^2(t+T)\|_{X_T}$$

when $\psi^1, \psi^2 \in B_T^1$, and by the same arguments as in the proof of Theorem 6.1, by choosing a_1 given by

$$a_1 = \lambda \frac{(1-\lambda T) - \sqrt{\Delta_1}}{2C\|f^0\|_1} \geq 0,$$

the fixed point theorem gives the existence of a solution in $[T, 2T]$. This solution is continuous in $[T, 2T]$ and in particular it satisfies (6.38) and (6.12). This completes the proof of Lemma 6.4.

The iteration process is applied to prove global existence in $[0, \infty[$. Suppose that the solution exists and is continuous $[0, (n-1)T]$ satisfying:

$$\|\psi(t + (r-1)T)\|_1 \leq a_{r-1} \|f^0\|_1, \quad r = 1, \ldots, n-1, \quad t \in [0,T]$$

and

$$\|\psi(rT)\|_1 \leq a_{r-1} \|f^0\|_1, \quad r = 1, \ldots, n-1,$$

where the reals a_0 and Δ_0 are given respectively by (6.37) and (6.36), and a_r, Δ_r are given by

$$a_r = \lambda \frac{(1-\lambda T) - \sqrt{\Delta_r}}{2C\|\mathbf{f}^0\|_1}, \quad r = 1, \ldots, n-1, \tag{6.40}$$

where

$$\Delta_r = (\lambda T - 1)^2 - 4Ca_{r-1} \frac{\|f^0\|_1}{\lambda}, \quad r = 1, \ldots, n-1. \tag{6.41}$$

It can be proved that we can extend the solution in $[(n-1)T, nT]$ satisfying, for any $t \in [0,T]$,

$$\|\psi(t + (n-1)T)\|_1 \leq a_{n-1} \|f^0\|_1, \tag{6.42}$$
$$\|\psi(nT)\|_1 \leq a_{n-1} \|f^0\|_1. \tag{6.43}$$

Let $\psi = (\psi_{ij})$ be the solution of the problem

$$\begin{aligned}\widehat{\psi_{ij}}(t+(n-1)T,x) &= \widehat{\psi_{ij}}((n-1)T,x)\\ &\quad + \int_{(n-1)T}^{t+(n-1)T} (\widehat{\Gamma_i[\psi,\psi]}(s,x)\exp(-\lambda s)\\ &\quad + \widehat{\psi_{ij}}(s,x)\{\lambda - \widehat{\Lambda[\psi]}(s,x)\exp(-\lambda s)\})ds. \qquad (6.44)\end{aligned}$$

By the same arguments as in the proof of Theorem 6.1, one easily has

$$\begin{aligned}&\|A(\psi^1)(t+(n-1)T)\|_{X_T}\\ &\quad \le a_{n-2}\|f^0\|_1 + \frac{Ca_{n-1}^2}{\lambda}\|f^0\|_1^2 + \lambda a_{n-1}T\|f^0\|_1, \qquad (6.45)\\ &\|A(\psi^1)(t+(n-1)T) - A(\psi^2)(t+(n-1)T)\|_{X_T}\\ &\quad \le \left(\frac{2Ca_{n-1}}{\lambda}\|f^0\|_1 + \lambda T\right)\|\psi^1(t+(n-1)T) - \psi^2(t+(n-1)T)\|_{X_T}. \qquad (6.46)\end{aligned}$$

The proof is complete if we choose a_{n-1} such that

$$a_{n-1} = \lambda\frac{(1-\lambda T) - \sqrt{\Delta_{n-1}}}{2C\|f^0\|_1}, \qquad (6.47)$$

where

$$\Delta_{n-1} = (\lambda T - 1)^2 - \frac{4Ca_{n-2}}{\lambda}\|f^0\|_1, \qquad (6.48)$$

which give the solution in $[(n-1)T, nT]$, satisfying (6.42) and (6.43). Moreover, it is easy to see that if f^0 satisfies (6.11), we get (6.12) and this completes the proof. □

6.3. *Simulations*

Let us consider simulations of the flow dynamics in the following special case: we assume that we have two clusters of vehicles traveling with different speeds on a closed road (for instance a ring) and having different density. More precisely, we assume that the first cluster travels with a velocity v_I, that is the maximum admissible velocity, and has an initial density

$$\rho(0,x) = 100\sin^2(10\pi(x-0.2)(x-0.3)), \quad x \in [0.2, 0.3],$$

while the other one travels with a velocity v_{I-1} and has a density

$$\rho(0,x) = 50\sin^2(10\pi(x-0.5)(x-0.6)), \quad x \in [0,5,0.6].$$

In order to solve the above problem, we have applied a high resolution method based on a slope-limiter technique, in which the first-order upwind flux is combined with a correction term given by the Superbee limiter. The simulations were performed, for different values of α, by using six velocity classes and one activity class,

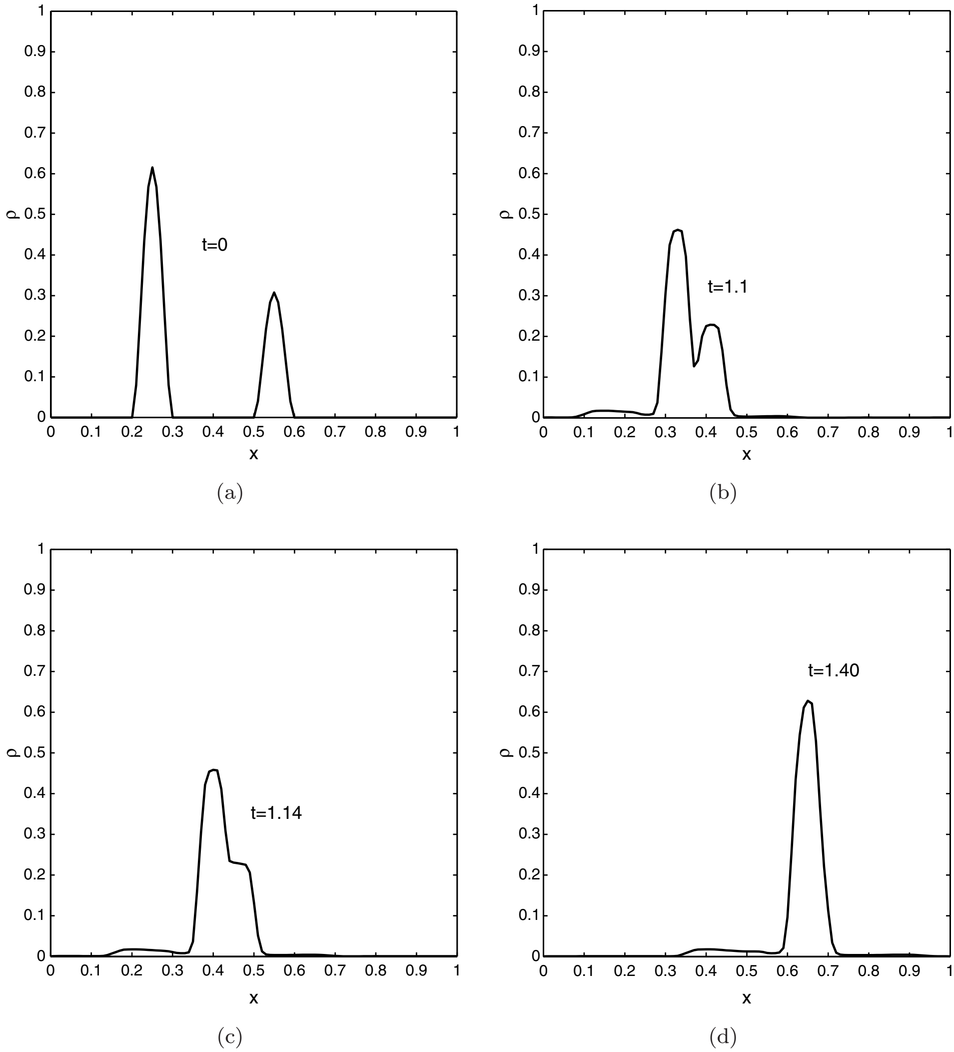

Fig. 4. Evolution of two clusters of vehicles in the case of bad road condition: a merging case.

by fixing an interaction length $L = 0.05$ and by choosing as weight function those defined in (4.5).

The first simulation in Fig. 4 refers to $\alpha = 0.3$, namely to a road with low quality features: fast vehicles after having reached the slow ones, as indicated in Fig. 4(b), have a mixing period shown in Fig. 4(c) and subsequently cluster as shown in Fig. 4(d), while a small group of vehicles is left behind.

On the other hand, the simulation of Fig. 5 refers to a high quality road-environment, $\alpha = 1$: when fast vehicles reach the slow ones as shown in Fig. 5(b), a mixing period follows as indicated in Fig. 5(c). Finally, fast vehicles leave behind the slow ones (Fig. 5(d)). This figure also shows that a few first vehicles, that were

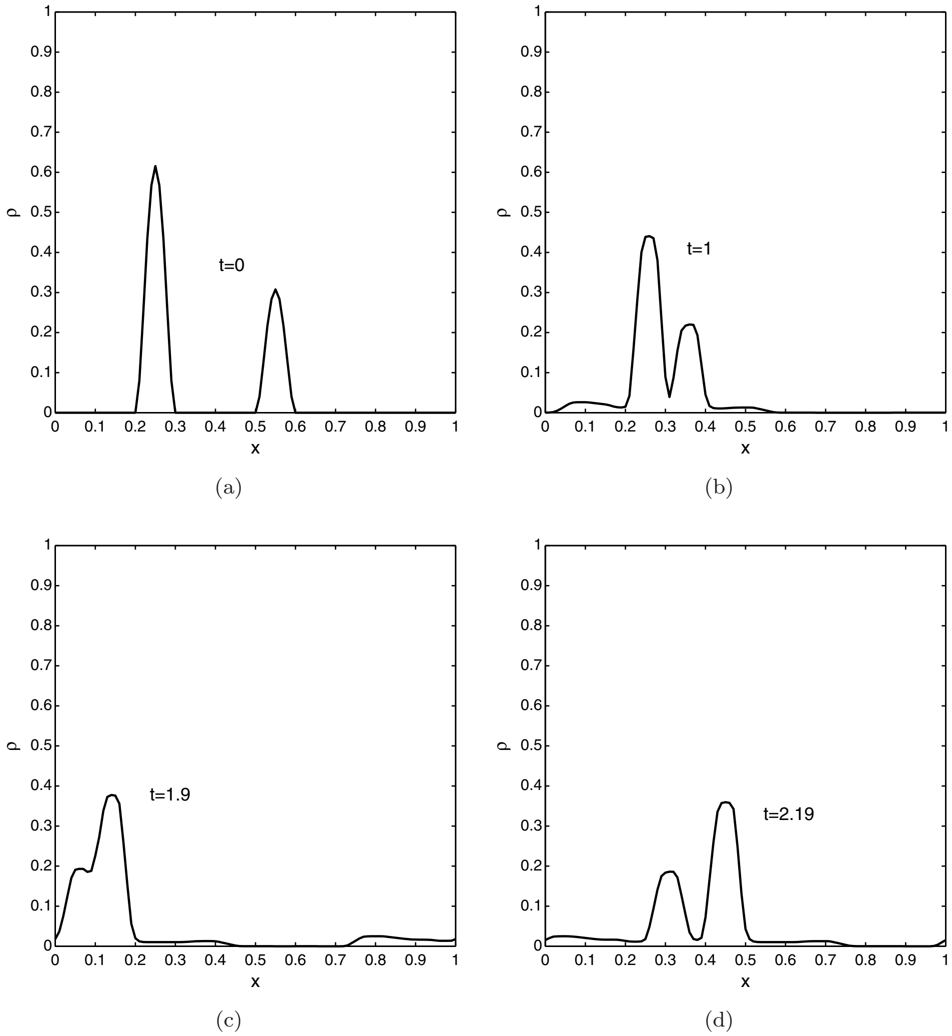

Fig. 5. Evolution of two clusters of vehicles in the case of optimal road condition.

fast to pass, precede the fast cluster, while a few slow ones are left behind. This happens because the first slow vehicles to be passed continue with their velocity, while the clustering gives a small advantage to those slow vehicles mixed with the fast ones.

What we have shown is just one of the several emerging behaviors that can be depicted by the model proposed in this paper.

7. Critical Analysis and Perspectives

The contribution of this paper has been focused on the modeling vehicular traffic looking at the collection of driver-vehicle subsystems as a living, hence complex, system. Therefore, after having identified the main complexity features of the class of systems under consideration, we have selected in Sec. 3 the mathematical structure considered as the reference framework for the derivation of the specific models, as described in Sec. 4. Starting from the additional objective of modeling granular flow, the approach has been focused on models with discrete states. Comparison with empirical data has given encouraging results that support further research activity along the path opened by this paper.

Looking ahead, we mention that a detailed analysis of interactions at the microscopic level can contribute to relatively simpler models derived at the microscopic scale. For instance to the closure of mass conservation equations[6] of second-order models.[12,13,26] Particularly important is the modeling of interactions with breaking ability also in the presence of density gradients.[17,25] In this case the driver adapts the velocity of the vehicle to the perceived density rather than to the real one, where the perceived velocity is higher/lower than the real one if the density gradient is positive/negative.

Finally, it is worth stating that some of the results proposed in this paper can be properly developed to model crowd dynamics. The present state-of-the-art mainly consists in modeling at the microscopic scale.[7,21,41,44] However, recent studies[2] have introduced some models by the kinetic theory approach, which needs to take carefully into account the complexity features in a line analogous to that of this paper, that in crowds are even more important than in vehicular dynamics. In fact, the heterogeneity of the micro-system individual-body can be very important as documented in the report,[20] while the strategy developed by pedestrians can be modified by external events such as transition to panic conditions.[32]

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References

1. F. Andreu, J. Calvo, J. M. Mazónand and J. Soler, On a nonlinear flux-limited equation arising in the transport of morphogens, to appear in *J. Differential Equations*
2. N. Bellomo and A. Bellouquid, On the modeling of crowd dynamics: Looking at the beautiful shapes of swarms, *Netw. Heterog. Media* **6** (2011) 383–399.
3. N. Bellomo and A. Bellouquid, Global solution to the Cauchy problem for discrete velocity models of vehicular traffic, to appear on *J. Differential Equations*.
4. N. Bellomo, A. Bellouquid, J. Nieto and J. Soler, Multiscale biological tissue models and flux-limited chemotaxis for multicellular growing systems, *Math. Models Methods Appl. Sci.* **20** (2010) 1179–1207.
5. N. Bellomo and F. Brezzi, Traffic, crowds and swarms, *Math. Models Methods Appl. Sci.* **18**(Suppl.) (2008) 1145–1148.
6. N. Bellomo and V. Coscia, First order models and closure of the mass conservation equation in the mathematical theory of vehicular traffic flow, *C. R. Mecanique* **333** (2005) 843–851.

7. N. Bellomo and C. Dogbé, On the modeling crowd dynamics from scaling to hyperbolic macroscopic models, *Math. Models Methods Appl. Sci.* **18**(Suppl.) (2008) 1317–1345.
8. N. Bellomo and C. Dogbé, On the modeling of traffic and crowds — A survey of models, speculations, and perspectives, *SIAM Rev.* **53** (2011) 409–463.
9. N. Bellomo, C. Bianca and M. Delitala, Complexity analysis and mathematical tools towards the modeling of living systems, *Phys. Life Rev.* **6** (2009) 144–175.
10. N. Bellomo, A. Marasco and A. Romano, From the modeling of driver's behavior to hydrodynamic models and problems of traffic flow, *Nonlinear Anal. Real World Appl.* **3** (2002) 339–363.
11. N. Bellomo, H. Berestycki, F. Brezzi and J. P. Nadal, Mathematics and complexity in human and life sciences, *Math. Models Methods Appl. Sci.* **20** (2010) 1391–1395.
12. F. Berthelin, P. Degond, M. Delitala and M. Rascle, A model for the formation and evolution of traffic jams, *Arch. Rational Mech. Anal.* **187** (2008) 185–220.
13. F. Berthelin, P. Degond, V. Le Blanc, S. Moutari, M. Rascle and J. Royer, A traffic-flow model with constraints for the modeling of traffic jams, *Math. Models Methods Appl. Sci.* **18** (2008) 1269–1298.
14. C. Bianca and V. Coscia, On the coupling of steady and adaptive velocity grids in vehicular traffic modeling, *Appl. Math. Lett.* **24** (2011) 149–155.
15. I. Bonzani and L. M. Gramani Cumin, Modeling and simulations of multilane traffic flow by kinetic theory methods, *Comput. Math. Appl.* **56** (2008) 2418–2428.
16. I. Bonzani and L. M. Gramani Cumin, Critical analysis and perspectives on the hydrodynamic approach for the mathematical theory of vehicular traffic, *Math. Comput. Model.* **50** (2009) 526–541.
17. I. Bonzani and L. Mussone, From experiments to hydrodynamic traffic flow models. I. Modeling and parameter identification, *Math. Comput. Model.* **37** (2003) 1435–1442.
18. I. Bonzani and L. Mussone, From the discrete kinetic theory of vehicular traffic flow to computing the velocity distribution at equilibrium, *Math. Comput. Model.* **49** (2009) 610–616.
19. Y. Brenier, Extended Monge–Kantorovich theory, in *Optimal Transportation and Applications* (Martina Franca, 2001), Lecture Notes in Math., Vol. 1813 (Springer, 2003), pp. 91–121.
20. S. Buchmuller and U. Weidman, Parameters of pedestrians, pedestrian traffic and walking facilities, *ETH Rep.* **132** (2006).
21. V. Coscia and C. Canavesio, First order macroscopic modeling of human crowds, *Math. Models Methods Appl. Sci.* **18**(Suppl.) (2008) 1217–1247.
22. V. Coscia, M. Delitala and P. Frasca, On the mathematical theory of vehicular traffic flow. II. Discrete velocity kinetic models, *Internat. J. Non-Linear Mech.* **42** (2007) 411–421.
23. C. F. Daganzo, Requiem for second order fluid approximations of traffic flow, *Transp. Res. B* **29B** (1995) 277–286.
24. S. Darbha, K. R. Rajagopal and V. Tyagi, A review of mathematical models for the flow of traffic and some recent results, *Nonlinear Anal. TMA* **69** (2008) 950–970.
25. E. De Angelis, Nonlinear hydrodynamic models of traffic flow modeling and mathematical problems, *Math. Comput. Model.* **29** (1999) 83–95.
26. P. Degond and M. Delitala, Modeling and simulation of vehicular traffic jam formation, *Kinetic Relat. Models* **1** (2008) 279–293.
27. M. Delitala and A. Tosin, Mathematical modeling of vehicular traffic: A discrete kinetic theory approach, *Math. Models Methods Appl. Sci.* **17** (2007) 901–932.

28. J. Dolbeault and R. Illner, Entropy methods for kinetic models of traffic flow, *Commun. Math. Sci.* **1** (2003) 409–421.
29. E. Godlewski and P. A. Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Applied Mathematical Sciences, Vol. 118 (Springer-Verlag, 1996).
30. L. M. Gramani Cumin, On the modeling of granular traffic flow by the kinetic theory for active particles trend to equilibrium and macroscopic behavior, *Int. J. Non-Linear Mech.* **44** (2009) 263–268.
31. D. Helbing, Traffic and related self-driven many-particle systems, *Rev. Mod. Phys.* **73** (2001) 1067–1141.
32. D. Helbing, I. J. Farkás, P. Molnár and T. Vicsek, Simulation of pedestrian crowds in normal and evacuation situations, in *Pedestrian and Evacuation Dynamics*, eds. M. Schreckenberg and S. D. Sharma (Springer, 2002), pp. 21–58.
33. M.-O. Hongler and R. Filliger, Mesoscopic derivation of a fundamental diagram of one-lane traffic, *Phys. Lett. A* **301** (2002) 408–412.
34. B. S. Kerner, Experimental features of self-organization in traffic flow, *Phys. Rev. Lett.* **81** (1998) 3797–3800.
35. B. S. Kerner, Synchronized flow as a new traffic phase and related problems for traffic flow modeling, *Math. Comput. Model.* **35** (2002) 481–508.
36. B. S. Kerner, *The Physics of Traffic, Empirical Freeway Pattern Features, Engineering Applications and Theory* (Springer, 2004).
37. B. S. Kerner and H. Rehborn, Experimental properties of complexity in traffic flow, *Phys. Rev. E* **53** (1996) R4275–R4278.
38. A. Klar and R. Wegener, Enskog-like kinetic models for vehicular traffic, *J. Statist. Phys.* **87** (1997) 91–114.
39. A. Klar and R. Wegener, Kinetic derivation of macroscopic anticipation models for vehicular traffic, *SIAM J. Appl. Math.* **60** (2000) 1749–1766.
40. A. Klar and R. Wegener, Kinetic traffic flow models, in *Modeling in Applied Sciences: A Kinetic Theory Approach*, eds. N. Bellomo and M. Pulvirenti (Birkhäuser, 2000).
41. B. Piccoli and A. Tosin, Time-evolving measures and macroscopic modeling of pedestrian flow, *Arch. Rational Mech. Anal.* **199** (2011) 707–738.
42. I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic* (Elsevier, 1971).
43. F. Siebel and W. Mauser, On the fundamental diagram of traffic flow, *SIAM J. Appl. Math.* **66** (2006) 1150–1162.
44. F. Venuti, L. Bruno and N. Bellomo, Crowd dynamics on a moving platform: Mathematical modeling and application to lively footbridges, *Math. Comput. Model.* **45** (2007) 252–269.