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Formation of delta shocks and vacuum states in the vanishing pressure limit of Riemann solutions to the perturbed Aw–Rascle model ☆

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ABSTRACT

A traffic flow model describing the formation and dynamics of traffic jams was introduced by Berthelin et al., which consists of a constrained pressureless gas dynamics system and can be derived from the Aw–Rascle model under the constraint condition $\rho \leqslant \rho^*$ by letting the traffic pressure vanish. In this paper, we give up this constraint condition and consider the following form

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \big(\rho u + \varepsilon p(\rho)\big)_t + \big(\rho u^2 + \varepsilon u p(\rho)\big)_x = 0, \end{cases}$$

in which $p(\rho) = \rho^\gamma$ with $\gamma > 1$.
The formal limit of the above system is the pressureless gas dynamics system in which the density develops delta-measure concentration in the Riemann solution. However, the propagation speed and the strength of the delta shock wave in the limit situation are different from the classical results of the pressureless gas dynamics system with the same Riemann initial data.
In order to solve it, the perturbed Aw–Rascle model is proposed as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \left(\rho u + \dfrac{\varepsilon}{\gamma} p(\rho)\right)_t + \big(\rho u^2 + \varepsilon u p(\rho)\big)_x = 0, \end{cases}$$

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whose behavior is different from that of the Aw–Rascle model. It is proved that the limits of the Riemann solutions of the perturbed Aw–Rascle model are exactly those of the pressureless gas dynamics model.

1. Introduction

1.1. The Aw–Rascle model and its singular limit

The Aw–Rascle (AR) model in the conservative form [3] can be expressed as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \big(\rho\big(u + p(\rho)\big)\big)_t + \big(\rho u\big(u + p(\rho)\big)\big)_x = 0, \end{cases} \tag{1.1}$$

where ρ, u represent the density and the velocity, respectively; the velocity offset p takes the form $p(\rho) = \rho^\gamma$ with $\gamma > 0$. The AR model describes a single lane traffic model and therefore the velocity is assumed to be a bounded nonnegative function.

The AR model (1.1) is now widely used to study the formation and dynamics of traffic jams. It was proposed to remedy the deficiencies of second order models of car traffic pointed out by Daganzo [14] and has been independently derived by Zhang [49]. The derivation of the model from a microscopic Follow-the-Leader (FL) model through a scaling limit was given in [2]. The AR model resolves all the obvious inconsistencies and explains instabilities in car traffic flow, especially near the vacuum, i.e., for light traffic with few slow drivers. It is also the basis for the multi-lane traffic flow model [20,21], the model for a road network with unidirectional flow [19,24] and the hybrid traffic flow model [36].

Recently, the singular limit behavior has been investigated for the AR model (1.1) by changing p into εp and taking $p(\rho) = (\frac{1}{\rho} - \frac{1}{\rho^*})^{-\gamma}$ with the density constraint $\rho \leqslant \rho^*$, where the maximal density ρ^* corresponds to a bumper. In order to describe the formation and dynamics of traffic jams, a constrained pressureless gas dynamics (CPGD) model was proposed by Berthelin et al. [4] as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (u + \bar{p})_t + u(u + \bar{p})_x = 0, \end{cases} \tag{1.2}$$

with the condition

$$0 \leqslant \rho \leqslant \rho^*, \qquad \bar{p} \geqslant 0, \qquad \big(\rho^* - \rho\big)\bar{p} = 0. \tag{1.3}$$

In the above, ρ denotes the density of vehicles and it is assumed that the maximal density constraint ρ^* is independent of the velocity u. The quantity $\bar{p}$ can be regarded as the Lagrange multiplier of the constraint $\rho \leqslant \rho^*$ and is nonzero only when $\rho = \rho^*$, namely it arrives at the maximal density constraint which is in congested situations and cars are then forced to spread into clusters.

The CPGD model can be derived from the so-called rescaled modified Aw–Rascle (RMAR) model

$$\begin{cases} \rho^\varepsilon_t + \big(\rho^\varepsilon u^\varepsilon\big)_x = 0, \\ \big(u^\varepsilon + \varepsilon p\big(\rho^\varepsilon\big)\big)_t + u^\varepsilon\big(u^\varepsilon + \varepsilon p\big(\rho^\varepsilon\big)\big)_x = 0, \end{cases} \tag{1.4}$$

where $p(\rho^\varepsilon)$ takes the form

$$p\big(\rho^\varepsilon\big) = \left(\frac{1}{\rho^\varepsilon} - \frac{1}{\rho^*}\right)^{-\gamma}, \quad \rho^\varepsilon \leqslant \rho^*. \tag{1.5}$$

Furthermore, a more realistic model namely the second order model with constraint (SOMC) model was proposed in [5], where the dependence of the maximal density constraint on the velocity $\rho^* = \rho^*(u)$ was taken into account instead of the assumption that ρ^* was constant and independent of the velocity in [4]. It is worthy noticed that the SOMC model has the double-side behavior, namely it behaves like the Lighthill–Whitham first order model when the maximal density is attained, otherwise it behaves like the pressureless gas dynamics model in the free flow.

1.2. The perturbed Aw–Rascle model and main results of the paper

From the point of view of hyperbolic conservation laws, it is interesting to consider the limit behavior that $p(\rho)$ is not singular at $\rho = \rho^*$, i.e., we consider

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \big(\rho u + \varepsilon p(\rho)\big)_t + \big(\rho u^2 + \varepsilon u p(\rho)\big)_x = 0, \end{cases} \tag{1.6}$$

in which we replace $\rho p(\rho)$ with $p(\rho)$ in (1.1) for convenience and conciseness of computation in the sequel, namely $p(\rho)$ is denoted by $p(\rho) = \rho^\gamma$ with $\gamma > 1$ here instead of $\gamma > 0$ in (1.1). Now $p(\rho)$ in (1.6) can be regarded as the traffic pressure term and $\gamma > 1$ is analogous with the adiabatic gas constant in gas dynamics.

The formal limit of (1.6) would be the following so-called pressureless gas dynamics (PGD) model:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \big(\rho u^2\big)_x = 0, \end{cases} \tag{1.7}$$

which is also called the transport equations and can be obtained from Boltzmann equations [6], cold plasma [33] and the flux-splitting scheme of the full compressible Euler equations [1,32].

The PGD model (1.7) was also used to depict the process of the motion of free particles sticking under collision [7] and describe the formation of large scale in the universe [18,40]. From [4,5], we know that the clusters (or traffic jams) are formed in the Riemann solutions of the CPGD model (1.2) and the SOMC model, in which the clusters are defined as intervals where the density limit is reach. Compared with them, the cluster is compressed into a point and the density becomes a singular measure at this point in the Riemann solutions of the PGD model (1.7). Thus the delta shock wave can be regarded as the limit of the cluster as the interval distance tends to zero from the point of view of mathematics.

It is well known that the delta shock wave will appear in the Riemann solutions of the PGD model (1.7) when the Riemann initial data satisfy $u_+ < u_-$. For the case $u_+ < u_-$, we can see that the delta shock wave can also be obtained from the limit of Riemann solution of (1.6) as $\varepsilon \to 0$. However, the propagation speed and the strength of the delta shock wave in the limit situation of (1.6) are different from those of the PGD model (1.7) with the same Riemann initial data. Thus it is clear that the Riemann solutions of (1.6) do not converge to those of (1.7) as $\varepsilon \to 0$.

In order to solve this problem, we suppose that the pressure $p(\rho)$ converges to zero with different velocities in the AR model, i.e., we consider the perturbed Aw–Rascle (PAR) model as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \left(\rho u + \dfrac{\varepsilon}{\gamma} p(\rho)\right)_t + \big(\rho u^2 + \varepsilon u p(\rho)\big)_x = 0. \end{cases} \tag{1.8}$$

This perturbation does significantly alter the analytical properties of the AR model (1.1). Compared with (1.1) or (1.6), both the characteristic fields for (1.8) are genuinely nonlinear when $\rho, u > 0$ and ε sufficiently small. Thus the Riemann solution of (1.8) consists of shock wave or rarefaction wave for the second family instead of contact discontinuity.

In this paper, we are concerned with the phenomena of concentration and cavitation and the formation of δ-shocks and vacuum states in the Riemann solutions of the PAR model (1.8) as $\varepsilon \to 0$.

Thus we confine ourself into the Riemann problem only. The main purpose of this paper is to rigorously prove that the limits of Riemann solutions of the PAR model (1.8) are exactly the corresponding Riemann solutions of the PGD model (1.7) with the same Riemann data. It is shown that the existence of δ-shock waves is obtained as a limit of two shock waves and the existence of vacuum states is obtained as a limit of two rarefaction waves. These results show that the δ-shocks and the vacuum states result from the phenomena of concentration and cavitation as $\varepsilon \to 0$, respectively.

1.3. The delta shock wave

The PGD model (1.7) has been studied extensively since 1994. The existence of measure solutions of the Riemann problem was first proved by Bouchut [6] and the existence of the global weak solution was obtained by Brenier and Grenier [7] and E, Rykov and Sinai [18]. Sheng and Zhang [42] discovered that the δ-shocks and vacuum states do occur in the Riemann solutions to the PGD model (1.7) by the vanishing viscosity method. Huang and Wang [26] proved the uniqueness of the weak solution for the case when the initial data is a Radon measure.

Korchinski [28] introduced the concept of the Dirac function into the classical weak solution in his unpublished PhD thesis in 1977. In fact, the concept of the delta shock wave solution and the corresponding Rankine–Hugoniot condition were also presented by Zeldovich and Myshkis [48] in the case of the continuity equation in 1973. Tan, Zhang and Zheng [45] considered some one-dimensional reduced system and discovered that the form of Dirac delta functions supported on shocks was used as parts in their Riemann solutions for certain initial data. LeFloch et al. [15,22,29] applied the approach of nonconservative product to consider nonlinear hyperbolic systems in the nonconservative form. We can also refer to [27,31,39] for the related equations and results. Recently, the weak asymptotic method was widely used to study the δ-shock wave type solution by Danilov and Shelkovich et al. [16,17,37,41].

In [11], Chen and Liu considered the Euler equations of isentropic fluids with the perturbed pressure term as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \big(\rho u^2 + \varepsilon p(\rho)\big)_x = 0, \end{cases} \tag{1.9}$$

where $p(\rho) = \rho^\gamma/\gamma$ with $\gamma > 1$. They analyzed and identified the phenomena of concentration and cavitation and the formation of δ-shocks and vacuum states as $\varepsilon \to 0$, which checked the numerical observation for the two-dimensional situation by Chang, Chen and Yang [8,9]. They also pointed out that the occurrence of δ-shocks and vacuum states can be regarded as a result of resonance between two characteristic fields. In [12], they made a further step to generalize this result to the nonisentropic fluids.

For the related work, Li [30] considered the limit behavior of the isentropic Euler equations as the temperature T drops to zero for polytropic gas. Mitrovic and Nedeljkov [35] extended the results of [11] to the generalized pressureless gas dynamics model with the perturbed pressure term:

$$\begin{cases} \rho_t + \big(\rho g(u)\big)_x = 0, \\ (\rho u)_t + \big(\rho u g(u) + \varepsilon p(\rho)\big)_x = 0, \end{cases} \tag{1.10}$$

where $p(\rho) = \kappa\rho^\gamma$ for $1 < \gamma < 3$ and g is a non-decreasing function. They obtained the delta shock wave as a limit of two shock waves for (1.10). On the Riemann problem and the Cauchy problem for the generalized pressureless gas dynamics model, we can refer to [47] and [25] respectively.

From the point of view of hyperbolic conservation laws, the existence of δ-shock wave was proved by the vanishing viscosity method [13,42], where one added εu_{xx} or $\varepsilon t u_{xx}$ on the right of the PGD model (1.7) and then took a distributional limit. Now, we can obtain a strictly hyperbolic system by adding the perturbed pressure terms in (1.7) and then taking a distributional limit to prove it, which is easier to do.

1.4. Plan

The rest of the paper is organized as follows. In Section 2, we review the Riemann problem to (1.6) and the PGD model (1.7) respectively and then prove that the Riemann solutions of (1.6) do not converge to those of (1.7) as $\varepsilon \to 0$. In Section 3, we investigate the Riemann problem to the PAR model (1.8) in detail. In Section 4, we analyze the formation of vacuum states as the limit of two rarefaction waves in the Riemann solutions to (1.8) as the pressure vanishes. In Section 5, we analyze the formation of contact discontinuities as the limit of 1-shock wave and 2-rarefaction wave or 1-rarefaction wave and 2-shock wave in the Riemann solutions to (1.8) as the pressure vanishes. In Section 6, we analyze the formation of δ-shocks as the limit of two shock waves in the Riemann solutions to (1.8) as the pressure vanishes. Finally, the discussions are carried out in Section 7.

2. The limits of Riemann solutions for (1.6) as $\varepsilon \to 0$

In this section, we want to know whether the Riemann solutions of (1.7) are the limits of those of (1.6) as $\varepsilon \to 0$. At first we sketch the Riemann problem for (1.6), which is similar to the Riemann problem for the AR model (1.1) and the detailed study can be found in [3,44]. Then, we briefly review some results in the Riemann solutions to the PGD model (1.7). Finally, we consider the limits $\varepsilon \to 0$ of the Riemann solutions of (1.6) and compare them with the corresponding Riemann solutions of (1.7). For more details about the Riemann problem for hyperbolic conservation laws, see [10,38,43].

2.1. The Riemann problem for (1.6)

The Riemann initial data are

$$(\rho, u)(x, 0) = (\rho_\pm, u_\pm), \quad \pm x > 0, \tag{2.1}$$

where $\rho_\pm > 0$, $u_\pm > 0$.

The eigenvalues and corresponding right eigenvectors of (1.6) are

$$\lambda_1 = u - \varepsilon(\gamma - 1)\rho^{\gamma-1}, \qquad \vec{r}_1 = \big(1, \varepsilon(1-\gamma)\rho^{\gamma-2}\big)^T, \tag{2.2}$$

$$\lambda_2 = u, \qquad \vec{r}_2 = (1, 0)^T. \tag{2.3}$$

Hence (1.6) is strictly hyperbolic when $\rho > 0$, and it is easy to see that λ_1 is genuinely nonlinear for $\rho > 0$ and λ_2 is always linearly degenerate. Therefore, the associated waves are rarefaction waves or shocks for the first family and contact discontinuities for the second family.

The Riemann invariants along the characteristic fields are

$$w = u + \varepsilon\rho^{\gamma-1}, \qquad z = u. \tag{2.4}$$

For a given left state (ρ_-, u_-), it is easy to check that the self-similar waves $(\rho, u)(\xi)(\xi = x/t)$ of the first family are the 1-rarefaction wave curves that can be connected on the right as follows:

$$R(\rho_-, u_-): \quad \begin{cases} \xi = \lambda_1 = u - \varepsilon(\gamma - 1)\rho^{\gamma-1}, \\ u - u_- = -\varepsilon\rho^{\gamma-1} + \varepsilon\rho_-^{\gamma-1}, \\ \rho < \rho_-, \quad u > u_-, \end{cases} \tag{2.5}$$

and the 1-shock wave curves that can be connected on the right are as follows:

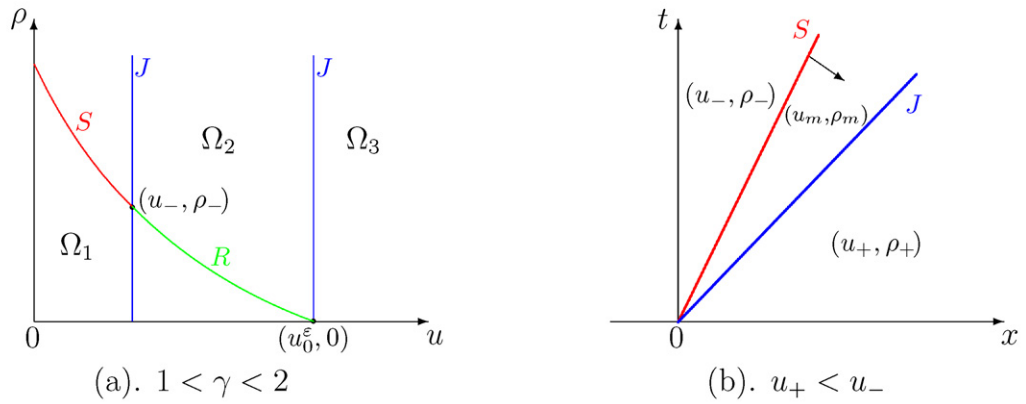

Fig. 1.

$$S(\rho_-,u_-):\quad \begin{cases} \sigma = u - \dfrac{\varepsilon\rho_-(\rho^{\gamma-1}-\rho_-^{\gamma-1})}{\rho-\rho_-}, \\ u-u_- = -\varepsilon\rho^{\gamma-1}+\varepsilon\rho_-^{\gamma-1}, \\ \rho>\rho_-, \quad u<u_-. \end{cases} \tag{2.6}$$

Here we notice that the shock wave curves coincide with the rarefaction wave curves in the phase plane, i.e., (1.6) still belongs to 'Temple class' [46]. Since λ_2 is linearly degenerate, the sets of states can be connected to a given left state (ρ_-,u_-) by a contact discontinuity on the right if and only if J: $\xi=u=u_-$.

Thus, we can summarize that the sets of states connected on the right consist of the 1-rarefaction wave curve $R(\rho_-,u_-)$, the 1-shock wave curve $S(\rho_-,u_-)$ and the 2-contact discontinuity $J(\rho_-,u_-)$ for a given left state (ρ_-,u_-). These curves divide the quarter phase plane $(\rho,u\geqslant 0)$ into three regions $\Omega_1=\{(\rho,u)\mid u<u_-\}$, $\Omega_2=\{(\rho,u)\mid u_-<u<u_0^\varepsilon\}$ and $\Omega_3=\{(\rho,u)\mid u>u_0^\varepsilon\}$, where $u_0^\varepsilon = u_-+\varepsilon\rho_-^{\gamma-1}$ (see Fig. 1(a)). According to the right state (ρ_+,u_+) in the different region, one can construct the unique global Riemann solution connecting two constant states $(\rho_\pm,u_\pm)$.

Remark 2.1. In Fig. 1(a), we draw the graph of Hugoniot locus for (1.6) by interchanging the ρ and u coordinates for convenience, also later figures for the graph of Hugoniot locus. Here we only depict the convex situation for the curve of the rarefaction and shock wave in $1<\gamma<2$ and the concave situation in $\gamma>2$ is similar.

2.2. The Riemann problem for the PGD model (1.7)

Now we consider the Riemann problem for the PGD model (1.7) with the Riemann initial data (2.1) and the detailed study can be found in [42,31]. We show some results briefly in the following.

The PGD model (1.7) has a double eigenvalue $\lambda=u$ and only one right eigenvector $\vec{r}=(1,0)^T$. Furthermore, we have $\nabla\lambda\cdot\vec{r}=0$, which means that λ is linearly degenerate. The Riemann problem (1.7) and (2.1) can be solved by contact discontinuities, vacuum or δ-shock wave connecting two constant states $(\rho_\pm,u_\pm)$.

By taking the self-similar transform $\xi=x/t$, the Riemann problem is reduced to the boundary value problem of the ordinary differential equations:

$$\begin{cases} -\xi\rho_\xi+(\rho u)_\xi=0, \\ -\xi(\rho u)_\xi+\big(\rho u^2\big)_\xi=0, \end{cases} \tag{2.7}$$

with $(\rho,u)(\pm\infty)=(\rho_\pm,u_\pm)$.

For the case $u_- < u_+$, there is no characteristic passing through the region $u_- < \xi < u_+$, therefore the vacuum should appear in the region. The solution can be expressed as

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_-), & -\infty < \xi \leqslant u_-, \\ (0, \xi), & u_- \leqslant \xi \leqslant u_+, \\ (\rho_+, u_+), & u_+ \leqslant \xi < +\infty. \end{cases} \tag{2.8}$$

For the case $u_- = u_+$, it is easy to see that the constant states $(\rho_\pm, u_\pm)$ can be connected by a contact discontinuity.

For the case $u_- > u_+$, a solution containing a weighted δ-measure supported on a line will be constructed. Let $x = x(t)$ be a discontinuity curve, we consider a piecewise smooth solution of (1.7) in the form

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (w_1(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\rho_+, u_+), & x > x(t). \end{cases} \tag{2.9}$$

In order to define the measure solutions as above, like as in [11,12,42], the two-dimensional weighted δ-measure $p(s)\delta_S$ supported on a smooth curve $S = \{(x(s), t(s)):\ a < s < b\}$ should be introduced as follows:

$$\bigl\langle p(s)\delta_S, \psi\bigl(x(s), t(s)\bigr)\bigr\rangle = \int_a^b p(s)\psi\bigl(x(s), t(s)\bigr)\sqrt{x'(s)^2 + t'(s)^2}\, ds, \tag{2.10}$$

for any $\psi \in C_0^\infty(R \times R_+)$.

The measure solution satisfies the generalized Rankine–Hugoniot condition [31,47]

$$\begin{cases} \dfrac{dx}{dt} = \sigma, \\ \dfrac{d\beta(t)}{dt} = \sigma[\rho] - [\rho u], \\ \dfrac{d(\beta(t)u_\delta(t))}{dt} = \sigma[\rho u] - \bigl[\rho u^2\bigr], \end{cases} \tag{2.11}$$

where $\sigma = u_\delta(t)$ is the propagation speed of the δ-shock wave and $[\rho] = \rho(x(t)+0, t) - \rho(x(t)-0, t)$ denotes the jump of ρ across the discontinuity $x = x(t)$, etc.

Through solving (2.11), we obtain

$$\begin{cases} u_\delta(t) = \sigma = \dfrac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ x(t) = \sigma t, \\ \beta(t) = \sqrt{1+\sigma^2}\, w_1(t) = \sqrt{\rho_-\rho_+}(u_- - u_+)t. \end{cases} \tag{2.12}$$

The above constructed δ-measure solution (2.9) with (2.12) obeys

$$\langle \rho, \psi_t \rangle + \langle \rho u, \psi_x \rangle = 0, \qquad \langle \rho u, \psi_t \rangle + \bigl\langle \rho u^2, \psi_x \bigr\rangle = 0,$$

for any $\psi \in C_0^\infty(R \times R_+)$, in which

$$\langle \rho u^k, \psi \rangle = \int_0^\infty \int_{-\infty}^\infty \rho_0 u_0^k \psi \, dx\, dt + \langle w_k \delta_S, \psi \rangle,$$

for $k = 0, 1, 2$, where

$$\rho_0 u_0^k = \rho_- u_-^k + [\rho u^k] H(x - \sigma t),$$
$$w_k(t) = \frac{t}{\sqrt{1+\sigma^2}} \big(\sigma [\rho u^{k-1}] - [\rho u^k]\big).$$

The unique entropy solution (2.9) with (2.12) can be singled out from (2.11) which satisfies the δ-entropy condition: $u_+ < \sigma < u_-$.

Remark 2.2. In the definition of the delta shock wave solution, if (2.10) is replaced by

$$\big\langle p(s)\delta_S, \psi\big(x(s), t(s)\big)\big\rangle = \int_a^b p(s)\psi\big(x(s), t(s)\big)\, ds,$$

then we have $\beta(t) = w_1(t)$ and $w_k(t) = (\sigma[\rho u^{k-1}] - [\rho u^k])t$ for $k = 1, 2$.

2.3. The limits $\varepsilon \to 0$ of the Riemann solutions of (1.6)

Now we consider the limits of the Riemann solutions of (1.6) and then compare them with the corresponding Riemann solutions of the PGD model (1.7). Our discussion should be divided into the following three cases based on the ordering of u_- and u_+.

1. For $u_- < u_+$, if $\varepsilon > \frac{u_+ - u_-}{\rho_-^{\gamma-1}}$, then the Riemann solution of (1.6) consists of the rarefaction wave R and the contact discontinuity J: $u = u_+$ with the nonvacuum intermediate constant state $(\rho_*^\varepsilon, u_*^\varepsilon)$. The Riemann solution of (1.6) can be expressed as

$$\big(\rho^\varepsilon(\xi), u^\varepsilon(\xi)\big) = \begin{cases} (\rho_-, u_-), & -\infty < \xi < \lambda_1(\rho_-, u_-), \\ R, & \lambda_1(\rho_-, u_-) \leqslant \xi \leqslant \lambda_1(\rho_*^\varepsilon, u_*^\varepsilon), \\ (\rho_*^\varepsilon, u_*^\varepsilon), & \lambda_1(\rho_*^\varepsilon, u_*^\varepsilon) \leqslant \xi \leqslant u_+, \\ (\rho_+, u_+), & u_+ < \xi < +\infty, \end{cases} \tag{2.13}$$

where

$$\big(\rho_*^\varepsilon, u_*^\varepsilon\big) = \left(\sqrt[\gamma-1]{\rho_-^{\gamma-1} + \frac{u_- - u_+}{\varepsilon}}, u_+ \right), \tag{2.14}$$

and R consists of (ρ, u) which satisfies (2.5) under the condition $\lambda_1(\rho_-, u_-) \leqslant \xi \leqslant \lambda_1(\rho_*^\varepsilon, u_*^\varepsilon)$.

If ε is small enough to satisfy $0 < \varepsilon \leqslant \frac{u_+ - u_-}{\rho_-^{\gamma-1}}$, then a vacuum state appears in the Riemann solution of (1.6) as follows:

$$\big(\rho^\varepsilon(\xi), u^\varepsilon(\xi)\big) = \begin{cases} (\rho_-, u_-), & -\infty < \xi < \lambda_1(\rho_-, u_-), \\ R, & \lambda_1(\rho_-, u_-) \leqslant \xi \leqslant \lambda_1(0, u_0^\varepsilon), \\ (0, u_*^\varepsilon), & \lambda_1(0, u_0^\varepsilon) \leqslant \xi \leqslant u_+, \\ (\rho_+, u_+), & u_+ < \xi < +\infty, \end{cases} \tag{2.15}$$

where $u_0^\varepsilon = u_- + \varepsilon\rho_-^{\gamma-1}$ as before and $u_0^\varepsilon \leqslant u_*^\varepsilon \leqslant u_+$, here u_*^ε is fake velocity in order to make a distinction between two vacuum states which was proposed by Liu and Smoller [34] for compressible gas dynamics.

From (2.5), it is easy to see that

$$\lim_{\varepsilon\to 0} \lambda_1(\rho_-, u_-) = \lim_{\varepsilon\to 0} \lambda_1\big(0, u_0^\varepsilon\big) = \lim_{\varepsilon\to 0} u_0^\varepsilon = u_-, \tag{2.16}$$

which means that the rarefaction wave R will degenerate to be the contact discontinuity J: $u = u_-$ as $\varepsilon \to 0$. Meanwhile the vacuum state will fill up the region between the two contact discontinuities, which is exactly identical with the corresponding Riemann solutions of the PGD model (1.7).

2. If $u_- = u_+$, then $(\rho_\pm, u_\pm)$ can be directly connected by a contact discontinuity J: $u = u_- = u_+$ for arbitrary $\varepsilon > 0$. As $\varepsilon \to 0$, the limit of the Riemann solution of (1.6) is still the same contact discontinuity J and is also the corresponding Riemann solutions of (1.7).

3. If $u_- > u_+$, then the Riemann solution of (1.6) consists of the shock wave S and the contact discontinuity J with the nonvacuum intermediate constant state $(\rho_*^\varepsilon, u_*^\varepsilon)$ as follows:

$$\big(\rho^\varepsilon(\xi), u^\varepsilon(\xi)\big) = \begin{cases} (\rho_-, u_-), & -\infty < \xi < \sigma^\varepsilon, \\ (\rho_*^\varepsilon, u_*^\varepsilon), & \sigma^\varepsilon < \xi < \tau^\varepsilon, \\ (\rho_+, u_+), & \tau^\varepsilon < \xi < +\infty. \end{cases} \tag{2.17}$$

Here σ^ε and τ^ε are the speeds of the shock wave S and the contact discontinuity J respectively, which can be calculated by

$$\sigma^\varepsilon = u_*^\varepsilon - \frac{\varepsilon\rho_-((\rho_*^\varepsilon)^{\gamma-1} - \rho_-^{\gamma-1})}{\rho_*^\varepsilon - \rho_-}, \qquad \tau^\varepsilon = \tau = u_+, \tag{2.18}$$

and the intermediate state $(\rho_*^\varepsilon, u_*^\varepsilon)$ has the same expression with (2.14).

From (2.14), it is easy to see that $\lim_{\varepsilon\to 0} \rho_*^\varepsilon = +\infty$ for $u_- > u_+$. Then, it follows that

$$\lim_{\varepsilon\to 0} \sigma^\varepsilon = u_+ - \lim_{\varepsilon\to 0} \frac{\rho_-(u_- - u_+)}{\rho_*^\varepsilon - \rho_-} = u_+, \tag{2.19}$$

which means that S and J coincide to form a new type of nonlinear hyperbolic wave (see Fig. 1(b)), which is called as the delta shock wave in [45]. Compared with the corresponding Riemann solutions of (1.7), it is clear to see that the propagation speed of the delta shock wave here is $\tau = u_+$ which is different from that of (1.7).

Now, we make a further step to consider the strength of the delta shock wave for the limit situation of (1.6). It follows from (2.18) and (2.14) that

$$\lim_{\varepsilon\to 0} \big(\tau - \sigma^\varepsilon\big)\rho_*^\varepsilon = \lim_{\varepsilon\to 0} \frac{\rho_-\rho_*^\varepsilon(u_- - u_+)}{\rho_*^\varepsilon - \rho_-} = \rho_-(u_- - u_+). \tag{2.20}$$

Furthermore, we have

$$\bar{\beta}(t) = \lim_{\varepsilon\to 0} \int_{\sigma^\varepsilon t}^{\tau t} \rho_*^\varepsilon \, dx = \rho_-(u_- - u_+)t. \tag{2.21}$$

Based on the definition of (2.10), the strength of the delta shock wave for the limit situation of (1.6) can be obtained by

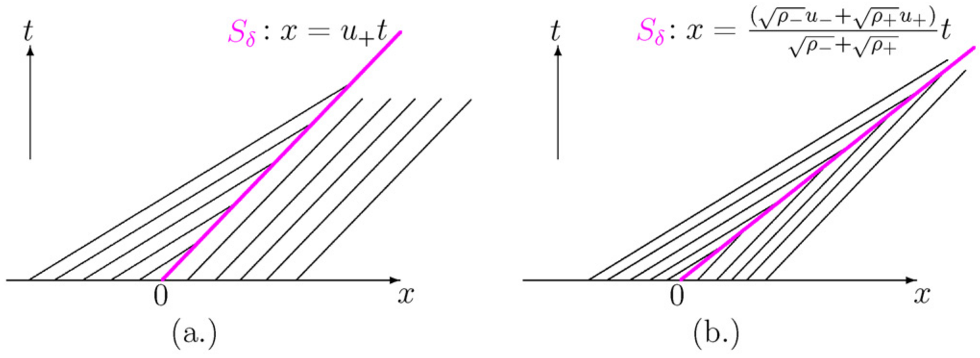

Fig. 2.

$$\overline{w}_1(t) = \frac{\rho_-(u_- - u_+)t}{\sqrt{1+u_+^2}}. \tag{2.22}$$

It is interesting to see that the strengths are also different for the limit situation of (1.6) and the PGD model (1.7), this maybe due to the different propagation speeds of the delta shock waves for them. For the limit situation of (1.6), the characteristics on the left side of the delta shock wave will come into the delta shock wave line $x = \tau t$ while the characteristics on the right side of it will parallel to it. For the PGD model (1.7), the characteristics will come into the delta shock wave curve $x = \sigma t$ from both sides. It is worthy noticed that the Riemann solution for the limit situation of (1.6) is not the entropy solution of the PGD model (1.7) since it does not satisfy the δ-entropy condition: $u_+ < \sigma < u_-$ defined before.

Thus, we can see that the Riemann solutions of (1.6) do not converge to those of (1.7) as $\varepsilon \to 0$. See Fig. 2 for details, Fig. 2(a) is the limit of the Riemann solution of (1.6) as $\varepsilon \to 0$ and Fig. 2(b) is the Riemann solution of the PGD model (1.7) when $u_+ < u_-$.

Remark 2.3. If we replace the classical δ-entropy condition: $u_+ < \sigma < u_-$ by the special δ-entropy condition: $u_+ < \tau = u_-$, namely we require that only the characteristic lines on the left side of the delta shock wave run into the delta shock wave line $x = \tau t$ instead of the characteristic lines on both sides, then the limit of the Riemann solution of (1.6) and (2.1) is

$$\begin{cases} u_\delta(t) = u_-, \\ x(t) = \tau t = u_+ t, \\ \overline{\beta}(t) = \sqrt{1+\tau^2}\,\overline{w}_1(t) = \rho_-(u_- - u_+)t, \end{cases}$$

which satisfies the generalized Rankine–Hugoniot condition (2.11) and is also the entropy solution of the PGD model (1.7) under the special δ-entropy condition: $u_+ < \tau = u_-$.

3. The Riemann problem for the PAR model (1.8)

In this section, we solve the Riemann problem for the PAR model (1.8) with the Riemann initial data (2.1) in detail and examine the dependence of the Riemann solutions on the parameter $\varepsilon > 0$. The eigenvalues of the PAR model (1.8) are

$$\lambda_1 = u - \sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1}u}, \qquad \lambda_2 = u + \sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1}u}, \tag{3.1}$$

so it is strictly hyperbolic in the quarter phase plane $(\rho, u > 0)$.

The corresponding right eigenvectors are

$$\vec{r}_1 = \big(\rho, -\sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1}u}\big)^T, \qquad \vec{r}_2 = \big(\rho, \sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1}u}\big)^T. \tag{3.2}$$

Since $\nabla\lambda_i \cdot \vec{r}_i \neq 0$ $(i = 1, 2)$ for $\rho \neq 0$ and $(\gamma+1)\sqrt{u} \pm \sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1}} \neq 0$, both the characteristic fields are genuinely nonlinear for $\rho, u > 0$ and ε sufficiently small.

The Riemann invariants along the characteristic fields are

$$w = \sqrt{u} + \sqrt{\frac{\varepsilon}{\gamma-1}\rho^{\gamma-1}}, \qquad z = \sqrt{u} - \sqrt{\frac{\varepsilon}{\gamma-1}\rho^{\gamma-1}}. \tag{3.3}$$

For (1.8) and (2.1) are invariant under uniform stretching of coordinates: $(x,t) \to (\alpha x, \alpha t)$ where the constant $\alpha > 0$, we seek the self-similar solution

$$(\rho, u)(x,t) = \big(\rho(\xi), u(\xi)\big), \quad \xi = x/t. \tag{3.4}$$

Then the Riemann problem (1.8) with (2.1) is reduced to the boundary value problem of the ordinary differential equations:

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi\left(\rho u + \dfrac{\varepsilon}{\gamma}\rho^\gamma\right)_\xi + \big(\rho u^2 + \varepsilon\rho^\gamma u\big)_\xi = 0, \end{cases} \tag{3.5}$$

with $(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm)$.

For smooth solutions, setting $U = (\rho, u)^T$ we can rewrite (3.5) under the form

$$A(U)U_\xi = 0, \tag{3.6}$$

where

$$A(\rho, u) = \begin{pmatrix} u - \xi & \rho \\ -\xi u - \varepsilon\xi\rho^{\gamma-1} + u^2 + \varepsilon\gamma\rho^{\gamma-1}u & -\xi\rho + 2\rho u + \varepsilon\rho^\gamma \end{pmatrix}. \tag{3.7}$$

Besides the constant state solutions, it provides a rarefaction wave which is a continuous solution of (3.6) in the form $(\rho, u)(\xi)$. Then, for a given left state (ρ_-, u_-), the rarefaction wave curves in the phase plane, which are the sets of states that can be connected on the right by 1-rarefaction wave or 2-rarefaction wave, are as follows:

1-rarefaction wave $\boldsymbol{R_1(\rho_-, u_-)}$:

$$\begin{gathered} \xi = \lambda_1 = u - \sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1}u}, \\ \sqrt{u} - \sqrt{u_-} = -\sqrt{\frac{\varepsilon}{\gamma-1}\rho^{\gamma-1}} + \sqrt{\frac{\varepsilon}{\gamma-1}\rho_-^{\gamma-1}}; \end{gathered} \tag{3.8}$$

2-rarefaction wave $\boldsymbol{R_2(\rho_-, u_-)}$:

$$\begin{gathered} \xi = \lambda_2 = u + \sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1}u}, \\ \sqrt{u} - \sqrt{u_-} = \sqrt{\frac{\varepsilon}{\gamma-1}\rho^{\gamma-1}} - \sqrt{\frac{\varepsilon}{\gamma-1}\rho_-^{\gamma-1}}. \end{gathered} \tag{3.9}$$

For 1-rarefaction wave, through differentiating u with respect to ρ in the second equation in (3.8), we get

$$u_\rho = -\sqrt{\varepsilon(\gamma-1)\rho^{\gamma-3}u} < 0, \tag{3.10}$$

$$u_{\rho\rho} = \frac{1}{2}\sqrt{\varepsilon(\gamma-1)}\left(\sqrt{\varepsilon(\gamma-1)}\rho^{\gamma-3} - (\gamma-3)\sqrt{\rho^{\gamma-5}u}\right). \tag{3.11}$$

Thus, it is easy to get $u_{\rho\rho} > 0$ for $1 < \gamma < 3$ and ε sufficiently small, i.e., the 1-rarefaction wave curve is convex for $1 < \gamma < 3$ and ε sufficiently small in the quarter phase plane $(\rho, u > 0)$.

Through differentiating ξ with respect to ρ and u in the first equation in (3.8), it yields

$$\left(2 - \sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1}u^{-1}}\right)u_\xi - (\gamma-1)\sqrt{\varepsilon(\gamma-1)\rho^{\gamma-3}u}\,\rho_\xi = 2.$$

Combining (3.10) and $u_\rho = u_\xi/\rho_\xi$, we have

$$\left(\gamma+1 - \sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1}u^{-1}}\right)u_\xi = 2. \tag{3.12}$$

Thus, we have $u_\xi > 0$ for ε sufficiently small, i.e., the set (ρ, u) which can be joined to (ρ_-, u_-) by the 1-rarefaction wave is made up of the half-branch of $R_1(\rho_-, u_-)$ with $u \geqslant u_-$ for ε sufficiently small. It is obvious to see that $R_1(\rho_-, u_-)$ intersects with the u-axis and the intersection can be immediately derived from (3.8).

With the similar computation to the 2-rarefaction wave curve, it comes to $u_\rho > 0$, $u_{\rho\rho} < 0$ and $u_\xi > 0$ for ε sufficiently small, respectively. Thus, we can draw the conclusion that it is concave for $1 < \gamma < 3$ in the quarter phase plane $(\rho, u > 0)$ and the set (ρ, u) which can be joined to (ρ_-, u_-) by the 2-rarefaction wave is made up of the half-branch of $R_2(\rho_-, u_-)$ with $u \geqslant u_-$ for ε sufficiently small.

For a bounded discontinuity at $x = x(t)$, the Rankine–Hugoniot condition holds:

$$\begin{cases} -\sigma[\rho] + [\rho u] = 0, \\ -\sigma\left[\rho u + \dfrac{\varepsilon}{\gamma}\rho^\gamma\right] + \left[\rho u^2 + \varepsilon\rho^\gamma u\right] = 0, \end{cases} \tag{3.13}$$

where $\sigma = \frac{dx}{dt}$ and $[u] = u_r - u_l$ with $u_l = u(x(t) - 0, t)$ and $u_r = u(x(t) + 0, t)$, etc.

Eliminating σ from (3.13), we obtain

$$[\rho]\left[\rho u^2 + \varepsilon\rho^\gamma u\right] - [\rho u]\left[\rho u + \frac{\varepsilon}{\gamma}\rho^\gamma\right] = 0. \tag{3.14}$$

Simplifying (3.14) yields

$$I_1 + \frac{\varepsilon}{\gamma}I_2 = 0, \tag{3.15}$$

where

$$I_1 = -\rho_r\rho_l(u_r - u_l)^2,$$
$$I_2 = \rho_r\rho_l\left((\gamma-1)\left(\frac{1}{\rho_l} - \frac{1}{\rho_r}\right)\left(\rho_r^\gamma u_r - \rho_l^\gamma u_l\right) + (u_l - u_r)\left(\rho_r^{\gamma-1} - \rho_l^{\gamma-1}\right)\right).$$

Thus, we have

$$(u_r-u_l)^2-\frac{\varepsilon}{\gamma}\left((\gamma-1)\left(\frac{1}{\rho_l}-\frac{1}{\rho_r}\right)\left(\rho_r^{\gamma}u_r-\rho_l^{\gamma}u_l\right)+(u_l-u_r)\left(\rho_r^{\gamma-1}-\rho_l^{\gamma-1}\right)\right)=0,$$

which means that

$$(\gamma-1)\left(\frac{1}{\rho_l}-\frac{1}{\rho_r}\right)\left(\rho_r^{\gamma}u_r-\rho_l^{\gamma}u_l\right)+(u_l-u_r)\left(\rho_r^{\gamma-1}-\rho_l^{\gamma-1}\right)>0$$

should be true if $u_r\neq u_l$. Then, we have

$$u_r-u_l=\pm\sqrt{\frac{\varepsilon}{\gamma}\left((\gamma-1)\left(\frac{1}{\rho_l}-\frac{1}{\rho_r}\right)\left(\rho_r^{\gamma}u_r-\rho_l^{\gamma}u_l\right)+(u_l-u_r)\left(\rho_r^{\gamma-1}-\rho_l^{\gamma-1}\right)\right)}. \tag{3.16}$$

Using the Lax entropy inequalities, the 1-shock wave satisfies

$$\sigma<\lambda_1(U_l),\qquad \lambda_1(U_r)<\sigma<\lambda_2(U_r), \tag{3.17}$$

while the 2-shock wave satisfies

$$\lambda_1(U_l)<\sigma<\lambda_2(U_l),\qquad \lambda_2(U_r)<\sigma. \tag{3.18}$$

From the first equation in (3.13), we have

$$\sigma=\frac{\rho_r u_r-\rho_l u_l}{\rho_r-\rho_l}=u_r+\frac{\rho_l(u_r-u_l)}{\rho_r-\rho_l}=u_l+\frac{\rho_r(u_r-u_l)}{\rho_r-\rho_l}. \tag{3.19}$$

Based on (3.1), (3.17) and (3.19), we can obtain that the 1-shock wave should satisfy the following entropy condition

$$-\sqrt{\varepsilon(\gamma-1)\rho_r^{\gamma+1}u_r}<\frac{\rho_r\rho_l(u_r-u_l)}{\rho_r-\rho_l}<-\sqrt{\varepsilon(\gamma-1)\rho_l^{\gamma+1}u_l}, \tag{3.20}$$

which implies that $\rho_r^{\gamma+1}u_r>\rho_l^{\gamma+1}u_l$ and $\frac{u_r-u_l}{\rho_r-\rho_l}<0$, thus we can obtain $u_r<u_l$ and $\rho_r>\rho_l$.

Similarly, taking into account (3.1), (3.18) and (3.19), the 2-shock wave should obey the entropy condition as

$$\sqrt{\varepsilon(\gamma-1)\rho_r^{\gamma+1}u_r}<\frac{\rho_r\rho_l(u_r-u_l)}{\rho_r-\rho_l}<\sqrt{\varepsilon(\gamma-1)\rho_l^{\gamma+1}u_l}, \tag{3.21}$$

which means that $\rho_r^{\gamma+1}u_r<\rho_l^{\gamma+1}u_l$ and $\frac{u_r-u_l}{\rho_r-\rho_l}>0$, thus it is easy to get $u_r<u_l$ and $\rho_r<\rho_l$.

Through the above analysis, for a given left state $(\rho_l,u_l)=(\rho_-,u_-)$, the sets of states which can be connected to (ρ_-,u_-) by a 1-shock wave on the right are as follows:

1-shock wave $S_1(\rho_-,u_-)$:

$$u-u_-=-\sqrt{\frac{\varepsilon}{\gamma}\left((\gamma-1)\left(\frac{1}{\rho_-}-\frac{1}{\rho}\right)\left(\rho^{\gamma}u-\rho_-^{\gamma}u_-\right)+(u_--u)\left(\rho^{\gamma-1}-\rho_-^{\gamma-1}\right)\right)},\quad \rho>\rho_-. \tag{3.22}$$

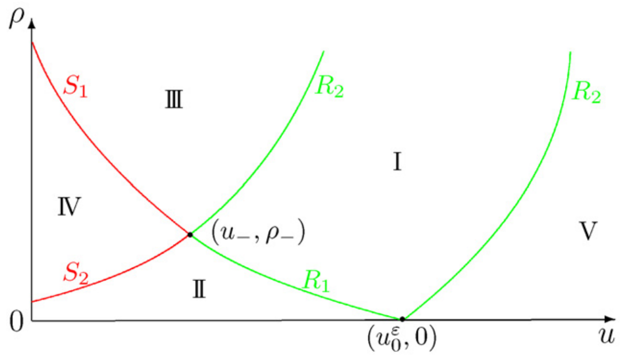

Fig. 3.

The curve S_2 consisting of all those states which can be connected to a given left state (ρ_-, u_-) by a 2-shock wave on the right is as follows:

2-shock wave $S_2(\rho_-, u_-)$:

$$u - u_- = -\sqrt{\frac{\varepsilon}{\gamma}\left((\gamma-1)\left(\frac{1}{\rho_-} - \frac{1}{\rho}\right)\left(\rho^\gamma u - \rho_-^\gamma u_-\right) + (u_- - u)\left(\rho^{\gamma-1} - \rho_-^{\gamma-1}\right)\right)}, \quad \rho < \rho_-. \tag{3.23}$$

For the 1-shock wave, through differentiating u with respect to ρ in (3.22), we get

$$\left(2(u-u_-) - \frac{\varepsilon}{\gamma}(\gamma-1)\left(\frac{1}{\rho_-} - \frac{1}{\rho}\right)\rho^\gamma + \frac{\varepsilon}{\gamma}\left(\rho^{\gamma-1} - \rho_-^{\gamma-1}\right)\right)u_\rho$$
$$= \frac{\varepsilon(\gamma-1)}{\gamma\rho^2\rho_-}\left(\left(\rho^\gamma - \rho_-^\gamma\right)\rho_- u_- + \gamma(\rho - \rho_-)\rho^\gamma u\right).$$

The above calculation shows that $u_\rho < 0$ for the 1-shock wave for ε sufficiently small, and that the 1-shock wave curve is starlike with respect to (ρ_-, u_-) in the region $\rho > \rho_-$. Similarly, we can get $u_\rho > 0$ for the 2-shock wave for ε sufficiently small, and that the 2-shock wave curve is starlike with respect to (ρ_-, u_-) in the region $\rho < \rho_-$. From (3.13) and (3.23), it is not difficult to check that $S_2(\rho_-, u_-)$ has the u-axis as its asymptote when $\rho \to 0$.

Through the above analysis, it can be concluded that the set of states connected on the right consists of the 1-rarefaction wave curve $R_1(\rho_-, u_-)$, the 2-rarefaction wave curve $R_2(\rho_-, u_-)$, the 1-shock wave curve $S_1(\rho_-, u_-)$ and the 2-shock wave curve $S_2(\rho_-, u_-)$ for the given left state (ρ_-, u_-). These curves divide the quarter phase plane into five parts I, II, III, IV and V. For fixed (ρ_-, u_-), the regions are determined by the location of the above curves, namely determined by the formulae (3.8), (3.9), (3.22) and (3.23).

According to the right state (ρ_+, u_+) in the different part, one can construct the unique global Riemann solution connecting two constant states (ρ_-, u_-) and (ρ_+, u_+). Now, we put all of these curves together in the quarter phase plane $(\rho, u > 0)$ to obtain a picture as in Fig. 3. It follows from (3.8) that $u_0^\varepsilon = (\sqrt{u_-} + \sqrt{\frac{\varepsilon}{\gamma-1}\rho_-^{\gamma-1}})^2$ in Fig. 3.

From Fig. 3, we can see the structure of the Riemann solution which contains a 1-rarefaction wave, 2-rarefaction wave and a nonvacuum intermediate constant state when $(\rho_+, u_+) \in \mathrm{I}$; which contains a 1-rarefaction wave, 2-shock wave and a nonvacuum intermediate constant state when $(\rho_+, u_+) \in \mathrm{II}$; which contains a 1-shock wave, 2-rarefaction wave curve and a nonvacuum intermediate constant

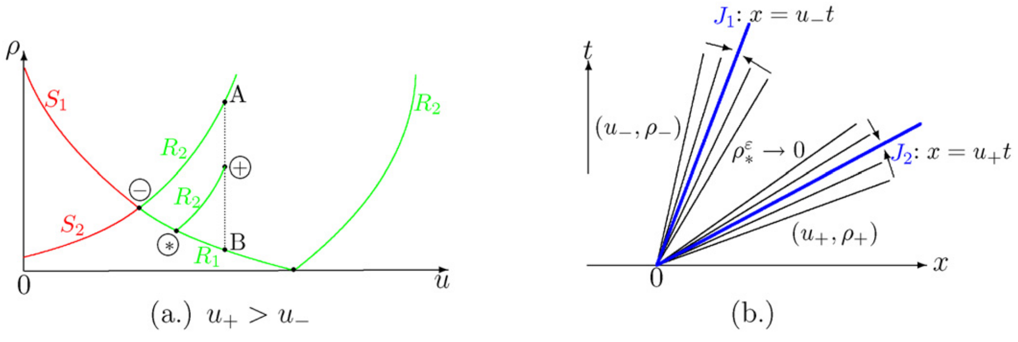

Fig. 4.

state when $(\rho_+, u_+) \in \mathrm{III}$; which contains a 1-shock wave, 2-shock wave and a nonvacuum intermediate constant state when $(\rho_+, u_+) \in \mathrm{IV}$; which contains a 1-rarefaction wave, 2-rarefaction wave and a vacuum intermediate state when $(\rho_+, u_+) \in \mathrm{V}$.

4. The vanishing pressure limit of (1.8) for $u_- < u_+$

In this section, we study the formation of vacuum state in the Riemann problem (1.8) and (2.1) when $u_- < u_+$ as $\varepsilon \to 0$.

Lemma 4.1. *If $u_- < u_+$, then there exists an $\varepsilon_0 > 0$ such that $(\rho_+, u_+) \in \mathrm{I}(\rho_-, u_-) \cup \mathrm{V}(\rho_-, u_-)$ for $0 < \varepsilon < \varepsilon_0$.*

Proof. If ε is small enough to satisfy $\rho_B < \rho_+ < \rho_A$ (see Fig. 4(a)), then $(\rho_+, u_+) \in \mathrm{I}(\rho_-, u_-) \cup \mathrm{V}(\rho_-, u_-)$. Here ρ_A and ρ_B can be obtained from (3.9) and (3.8) respectively as

$$\sqrt{u_+} - \sqrt{u_-} = \sqrt{\frac{\varepsilon}{\gamma-1}\rho_A^{\gamma-1}} - \sqrt{\frac{\varepsilon}{\gamma-1}\rho_-^{\gamma-1}}, \tag{4.1}$$

$$\sqrt{u_+} - \sqrt{u_-} = -\sqrt{\frac{\varepsilon}{\gamma-1}\rho_B^{\gamma-1}} + \sqrt{\frac{\varepsilon}{\gamma-1}\rho_-^{\gamma-1}}. \tag{4.2}$$

The conclusion is obviously true for $\rho_- = \rho_+$. When $\rho_- \neq \rho_+$, if the following inequality holds

$$\sqrt{\frac{\varepsilon}{\gamma-1}\rho_-^{\gamma-1}} - (\sqrt{u_+} - \sqrt{u_-}) < \sqrt{\frac{\varepsilon}{\gamma-1}\rho_+^{\gamma-1}} < \sqrt{\frac{\varepsilon}{\gamma-1}\rho_-^{\gamma-1}} + (\sqrt{u_+} - \sqrt{u_-}), \tag{4.3}$$

then we have $(\rho_+, u_+) \in \mathrm{I}(\rho_-, u_-) \cup \mathrm{V}(\rho_-, u_-)$.

Thus, the conclusion can be drawn by taking

$$\varepsilon_0 = (\gamma-1)\left(\frac{\sqrt{u_+} - \sqrt{u_-}}{\sqrt{\rho_+^{\gamma-1}} - \sqrt{\rho_-^{\gamma-1}}}\right)^2. \qquad \square \tag{4.4}$$

Lemma 4.1 shows that 1-rarefaction wave curve $R_1(\rho_-, u_-)$ and 2-rarefaction wave curve $R_2(\rho_-, u_-)$ become steeper when ε tends to zero. It can be concluded that the Riemann problem (1.8) with the Riemann initial data (2.1) consists of two rarefaction waves and an intermediate state $(\rho_*^\varepsilon, u_*^\varepsilon)$ besides two constant states $(\rho_\pm, u_\pm)$ for $0 < \varepsilon < \varepsilon_0$ when $u_- < u_+$. Furthermore, we have the following result.

Theorem 4.2. *In the case $u_- < u_+$, the limit of the Riemann solution for the PAR model* (1.8) *with the Riemann initial data* (2.1) *as $\varepsilon \to 0$ is two contact discontinuities connecting the constant states $(\rho_\pm, u_\pm)$ and the intermediate vacuum state as follows*:

$$(\rho, u) = \begin{cases} (\rho_-, u_-), & -\infty < \xi \leqslant u_-, \\ (0, \xi), & u_- \leqslant \xi \leqslant u_+, \\ (\rho_+, u_+), & u_+ \leqslant \xi < \infty, \end{cases} \tag{4.5}$$

which is exactly the corresponding Riemann solution to the PGD model (1.7) *with the same Riemann initial data.*

Proof. It follows from Lemma 4.1 that the Riemann solution of (1.8) and (2.1) consists of two rarefaction waves R_1, R_2 and an intermediate state $(\rho_*^\varepsilon, u_*^\varepsilon)$ besides two constant states $(\rho_\pm, u_\pm)$ for $0 < \varepsilon < \varepsilon_0$ when $u_- < u_+$. They satisfy

$$R_1\colon \quad \begin{cases} \lambda_1 = \xi = u - \sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1} u}, \quad \rho_*^\varepsilon \leqslant \rho \leqslant \rho_-, \\ \sqrt{u} = \sqrt{u_-} - \sqrt{\dfrac{\varepsilon}{\gamma-1}\rho^{\gamma-1}} + \sqrt{\dfrac{\varepsilon}{\gamma-1}\rho_-^{\gamma-1}}, \end{cases} \tag{4.6}$$

and

$$R_2\colon \quad \begin{cases} \lambda_2 = \xi = u + \sqrt{\varepsilon(\gamma-1)\rho^{\gamma-1} u}, \quad \rho_*^\varepsilon \leqslant \rho \leqslant \rho_+, \\ \sqrt{u} = \sqrt{u_+} - \sqrt{\dfrac{\varepsilon}{\gamma-1}\rho_+^{\gamma-1}} + \sqrt{\dfrac{\varepsilon}{\gamma-1}\rho^{\gamma-1}}. \end{cases} \tag{4.7}$$

Then, there exists an $\varepsilon_1 > 0$ such that a vacuum state appears for $\varepsilon \leqslant \varepsilon_1$. It can be derived from (4.6) and (4.7) that the critical value ε_1 is determined by

$$\sqrt{u_-} + \sqrt{\frac{\varepsilon_1}{\gamma-1}\rho_-^{\gamma-1}} = \sqrt{u_+} - \sqrt{\frac{\varepsilon_1}{\gamma-1}\rho_+^{\gamma-1}}, \tag{4.8}$$

i.e., ε_1 can be given by

$$\varepsilon_1 = (\gamma-1)\left(\frac{\sqrt{u_+} - \sqrt{u_-}}{\sqrt{\rho_+^{\gamma-1}} + \sqrt{\rho_-^{\gamma-1}}}\right)^2. \tag{4.9}$$

If $\varepsilon_1 < \varepsilon < \varepsilon_0$, then $(\rho_+, u_+) \in \mathrm{I}(\rho_-, u_-)$ and there is no vacuum in the Riemann solution, which implies that no vacuum occurs for a fluid with strong pressure in the Riemann solution of (1.8) and (2.1).

If $\varepsilon < \varepsilon_1$, then $(\rho_+, u_+) \in \mathrm{V}(\rho_-, u_-)$ and the intermediate state $(\rho_*^\varepsilon, u_*^\varepsilon)$ becomes a vacuum state with $\rho_*^\varepsilon = 0$ and $u_1^\varepsilon \leqslant u_*^\varepsilon \leqslant u_2^\varepsilon$, where

$$\sqrt{u_1^\varepsilon} = \sqrt{u_-} + \sqrt{\frac{\varepsilon}{\gamma-1}\rho_-^{\gamma-1}}, \qquad \sqrt{u_2^\varepsilon} = \sqrt{u_+} - \sqrt{\frac{\varepsilon}{\gamma-1}\rho_+^{\gamma-1}}. \tag{4.10}$$

From above, we can get $\lim_{\varepsilon\to 0} u_1^\varepsilon = u_-$ and $\lim_{\varepsilon\to 0} u_2^\varepsilon = u_+$. Thus λ_1 converges to u_- and λ_2 converges to u_+ as $\varepsilon \to 0$, i.e., the left boundary of 1-rarefaction wave and the right boundary of 2-rarefaction wave become two contact discontinuities of the PGD model (1.7) with the same Riemann initial data and the vacuum state fills up the region between the two contact discontinuities (see Fig. 4(b)). □

5. The vanishing pressure limit of (1.8) for $u_- = u_+$

In this section, we study the formation of contact discontinuity in the Riemann problem for the PAR model (1.8) with the Riemann initial data (2.1) when $u_- = u_+$ as $\varepsilon \to 0$, which can be fully explained in the following theorem.

Theorem 5.1. *In the case $u_- = u_+$, the limit of the Riemann solution for the PAR model* (1.8) *with the Riemann initial data* (2.1) *as $\varepsilon \to 0$ is a contact discontinuity connecting the constant states $(\rho_\pm, u_\pm)$ as follows*:

$$\rho(\xi) = \rho_- + [\rho]H(\xi - u_-), \qquad u(\xi) = u_- = u_+, \tag{5.1}$$

which is the Riemann solution to the PGD model (1.7) *with the same Riemann initial data.*

Proof. 1. For $u_- = u_+$, two constant states $(\rho_\pm, u_\pm)$ can be connected by 1-shock wave S_1, an intermediate state $(\rho_*^\varepsilon, u_*^\varepsilon)$ and 2-rarefaction wave R_2 for $\rho_+ > \rho_-$ or by 1-rarefaction wave R_1, an intermediate state $(\rho_*^\varepsilon, u_*^\varepsilon)$ and 2-shock wave S_2 for $\rho_+ < \rho_-$. In particular, (ρ, u) is a constant state (ρ_-, u_-) for $\rho_- = \rho_+$.

2. In the case $\rho_+ > \rho_-$, the intermediate state $(\rho_*^\varepsilon, u_*^\varepsilon)$ between S_1 and R_2 can be directly obtained from (3.22) and (3.9) as follows:

$$u_*^\varepsilon - u_- = -\sqrt{\frac{\varepsilon}{\gamma}\left((\gamma-1)\left(\frac{1}{\rho_-} - \frac{1}{\rho_*^\varepsilon}\right)\left(\left(\rho_*^\varepsilon\right)^\gamma u_*^\varepsilon - \rho_-^\gamma u_-\right) + \left(u_- - u_*^\varepsilon\right)\left(\left(\rho_*^\varepsilon\right)^{\gamma-1} - \rho_-^{\gamma-1}\right)\right)}, \tag{5.2}$$

$$\sqrt{u_+} - \sqrt{u_*^\varepsilon} = \sqrt{\frac{\varepsilon}{\gamma-1}\rho_+^{\gamma-1}} - \sqrt{\frac{\varepsilon}{\gamma-1}\left(\rho_*^\varepsilon\right)^{\gamma-1}}, \tag{5.3}$$

where $\rho_- < \rho_*^\varepsilon < \rho_+$.

Therefore, the Riemann solution of (1.8) and (2.1) can be given by

$$\left(\rho^\varepsilon(\xi), u^\varepsilon(\xi)\right) = \begin{cases} (\rho_-, u_-), & -\infty < \xi < \sigma_1^\varepsilon, \\ (\rho_*^\varepsilon, u_*^\varepsilon), & \sigma_1^\varepsilon < \xi \leqslant \lambda_2(\rho_*^\varepsilon, u_*^\varepsilon), \\ R_2, & \lambda_2(\rho_*^\varepsilon, u_*^\varepsilon) \leqslant \xi \leqslant \lambda_2(\rho_+, u_+), \\ (\rho_+, u_+), & \lambda_2(\rho_+, u_+) \leqslant \xi < +\infty, \end{cases} \tag{5.4}$$

where $\sigma_1^\varepsilon = \frac{\rho_*^\varepsilon u_*^\varepsilon - \rho_- u_-}{\rho_*^\varepsilon - \rho_-}$ is the propagating speed of S_1 and R_2 consists of (ρ, u) which satisfies (3.9) under the condition $\lambda_2(\rho_*^\varepsilon, u_*^\varepsilon) \leqslant \xi \leqslant \lambda_2(\rho_+, u_+)$.

Noticing that ρ_*^ε is bounded and letting $\varepsilon \to 0$ in (5.3), it immediately leads to $\lim_{\varepsilon\to 0} u_*^\varepsilon = u_+$. Noting $u_+ = u_-$ and substituting (5.3) into (5.2), it follows that

$$\begin{aligned} &\left(\sqrt{\rho_+^{\gamma-1}} - \sqrt{\left(\rho_*^\varepsilon\right)^{\gamma-1}}\right)\left(\sqrt{u_-} + \sqrt{u_*^\varepsilon}\right) \\ &\quad = \sqrt{\frac{\gamma-1}{\gamma}\left((\gamma-1)\left(\frac{1}{\rho_-} - \frac{1}{\rho_*^\varepsilon}\right)\left(\left(\rho_*^\varepsilon\right)^\gamma u_*^\varepsilon - \rho_-^\gamma u_-\right) + \left(u_- - u_*^\varepsilon\right)\left(\left(\rho_*^\varepsilon\right)^{\gamma-1} - \rho_-^{\gamma-1}\right)\right)}. \end{aligned} \tag{5.5}$$

Letting $\varepsilon \to 0$ in (5.5), we have

$$2\lim_{\varepsilon\to 0}\left(\sqrt{\rho_+^{\gamma-1}} - \sqrt{\left(\rho_*^\varepsilon\right)^{\gamma-1}}\right) = (\gamma-1)\lim_{\varepsilon\to 0}\sqrt{\frac{1}{\gamma}\left(\left(\frac{1}{\rho_-} - \frac{1}{\rho_*^\varepsilon}\right)\left(\left(\rho_*^\varepsilon\right)^\gamma - \rho_-^\gamma\right)\right)}, \tag{5.6}$$

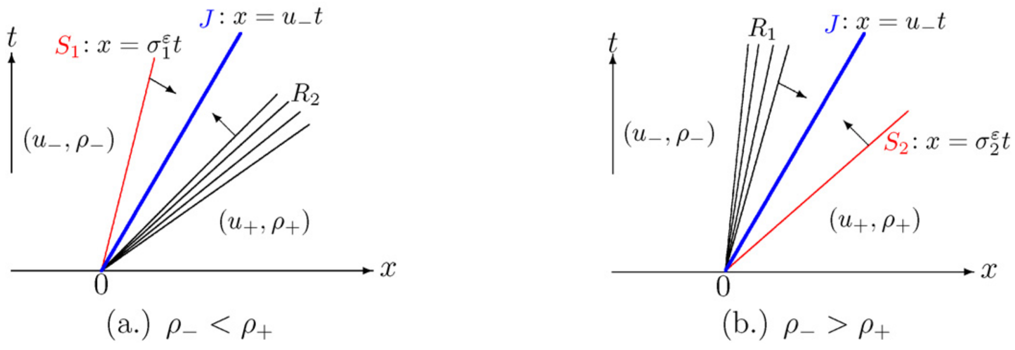

Fig. 5.

which implies that $\lim_{\varepsilon\to 0}\rho_*^\varepsilon \neq \rho_-$. Thus, we get

$$\lim_{\varepsilon\to 0}\sigma_1^\varepsilon = \lim_{\varepsilon\to 0}\frac{\rho_*^\varepsilon u_*^\varepsilon - \rho_- u_-}{\rho_*^\varepsilon - \rho_-} = \lim_{\varepsilon\to 0}\left(u_*^\varepsilon + \frac{\rho_-(u_*^\varepsilon - u_-)}{\rho_*^\varepsilon - \rho_-}\right) = u_-. \tag{5.7}$$

On the other hand, we have

$$\lim_{\varepsilon\to 0}\lambda_2(\rho_*^\varepsilon, u_*^\varepsilon) = \lim_{\varepsilon\to 0}\left(u_*^\varepsilon + \sqrt{\varepsilon(\gamma-1)(\rho_*^\varepsilon)^{\gamma-1}u_*^\varepsilon}\right) = u_-, \tag{5.8}$$

$$\lim_{\varepsilon\to 0}\lambda_2(\rho_+, u_+) = \lim_{\varepsilon\to 0}\left(u_+ + \sqrt{\varepsilon(\gamma-1)\rho_+^{\gamma-1}u_+}\right) = u_+ = u_-. \tag{5.9}$$

According to (5.7), (5.8) and (5.9), it follows that the Riemann solution (5.4) converges to (5.1) as $\varepsilon \to 0$ (see Fig. 5(a)).

3. The case $\rho_+ < \rho_-$ can be done in a similar way. The intermediate state $(\rho_*^\varepsilon, u_*^\varepsilon)$ between R_1 and S_2 can be directly obtained from (3.8) and (3.23) as follows:

$$\sqrt{u_*^\varepsilon} - \sqrt{u_-} = -\sqrt{\frac{\varepsilon}{\gamma-1}(\rho_*^\varepsilon)^{\gamma-1}} + \sqrt{\frac{\varepsilon}{\gamma-1}\rho_-^{\gamma-1}}, \tag{5.10}$$

$$u_+ - u_*^\varepsilon = -\sqrt{\frac{\varepsilon}{\gamma}\left((\gamma-1)\left(\frac{1}{\rho_*^\varepsilon} - \frac{1}{\rho_+}\right)\left(\rho_+^\gamma u_+ - (\rho_*^\varepsilon)^\gamma u_*^\varepsilon\right) + (u_*^\varepsilon - u_+)\left(\rho_+^{\gamma-1} - (\rho_*^\varepsilon)^{\gamma-1}\right)\right)}, \tag{5.11}$$

in which $\rho_+ < \rho_*^\varepsilon < \rho_-$.

Therefore, the Riemann solution of (1.8) and (2.1) can be given by

$$\left(\rho^\varepsilon(\xi), u^\varepsilon(\xi)\right) = \begin{cases} (\rho_-, u_-), & -\infty < \xi \leqslant \lambda_1(\rho_-, u_-), \\ R_1, & \lambda_1(\rho_-, u_-) \leqslant \xi \leqslant \lambda_1(\rho_*^\varepsilon, u_*^\varepsilon), \\ (\rho_*^\varepsilon, u_*^\varepsilon), & \lambda_1(\rho_*^\varepsilon, u_*^\varepsilon) \leqslant \xi < \sigma_2^\varepsilon, \\ (\rho_+, u_+), & \sigma_2^\varepsilon < \xi < \infty, \end{cases} \tag{5.12}$$

where $\sigma_2^\varepsilon = \frac{\rho_+ u_+ - \rho_*^\varepsilon u_*^\varepsilon}{\rho_+ - \rho_*^\varepsilon}$ is the propagating speed of S_2 and R_1 consists of (ρ, u) which satisfies (3.8) under the condition $\lambda_1(\rho_-, u_-) \leqslant \xi \leqslant \lambda_1(\rho_*^\varepsilon, u_*^\varepsilon)$.

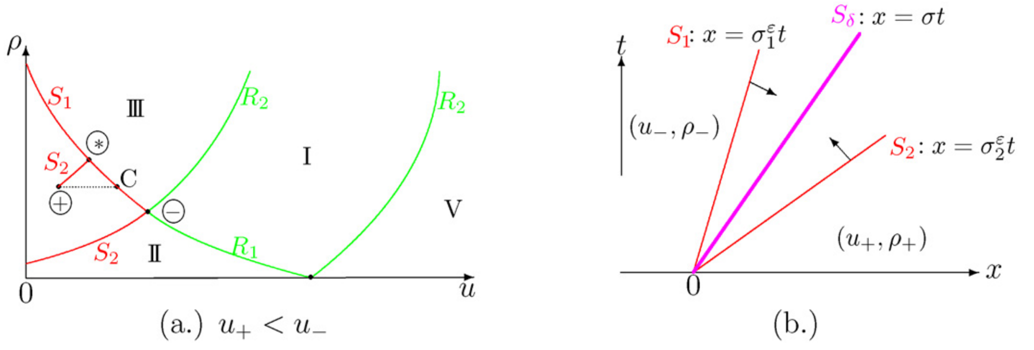

Fig. 6.

Similarly, the limit relations can be obtained as follows:

$$\lim_{\varepsilon\to 0}\lambda_1(\rho_-,u_-)=\lim_{\varepsilon\to 0}\lambda_1\big(\rho_*^\varepsilon,u_*^\varepsilon\big)=\lim_{\varepsilon\to 0}\sigma_2^\varepsilon=u_-. \tag{5.13}$$

From above, it is easily seen that the Riemann solution (5.12) converges to (5.1) as $\varepsilon\to 0$ (see Fig. 5(b)). □

6. The vanishing pressure limit of (1.8) for $u_- > u_+$

In this section, we study the formation of δ-shock in the Riemann problem (1.8) and (2.1) when $u_- > u_+$ as $\varepsilon\to 0$.

Lemma 6.1. *If $u_- > u_+$, then there exists an $\varepsilon_0 > 0$ such that $(\rho_+,u_+)\in \mathrm{IV}(\rho_-,u_-)$ for $0<\varepsilon<\varepsilon_0$.*

Proof. It is easy to see that $S_1(\rho_-,u_-)$ and $S_2(\rho_-,u_-)$ have the same expression from (3.22) and (3.23). Let us denote

$$u_C=u_--\sqrt{\frac{\varepsilon}{\gamma}\bigg((\gamma-1)\bigg(\frac{1}{\rho_-}-\frac{1}{\rho_+}\bigg)\big(\rho_+^\gamma u_C-\rho_-^\gamma u_-\big)+(u_--u_C)\big(\rho_+^{\gamma-1}-\rho_-^{\gamma-1}\big)\bigg)}. \tag{6.1}$$

We can see that (ρ_+,u_C) lies in $S_1(\rho_-,u_-)$ in the phase plane if $\rho_+>\rho_-$, otherwise (ρ_+,u_C) lies in $S_2(\rho_-,u_-)$ in the phase plane if $\rho_+<\rho_-$ (see Fig. 6(a)). If $u_+<u_C$, then we have $(\rho_+,u_+)\in \mathrm{IV}(\rho_-,u_-)$. The conclusion is obviously true for $\rho_-=\rho_+$ due to the fact that $u_C=u_->u_+$. When $\rho_-\neq\rho_+$, the conclusion can be drawn by taking

$$\varepsilon_0=\frac{\gamma(u_+-u_-)^2}{(\gamma-1)(\frac{1}{\rho_-}-\frac{1}{\rho_+})(\rho_+^\gamma u_+-\rho_-^\gamma u_-)+(u_--u_+)(\rho_+^{\gamma-1}-\rho_-^{\gamma-1})}. \tag{6.2}$$

In the above, ε_0 can be obtained by taking $u_C=u_+$ in (6.1), namely $(\rho_+,u_+)\in S_1(\rho_-,u_-)$ for $\rho_+>\rho_-$ or $(\rho_+,u_+)\in S_2(\rho_-,u_-)$ for $\rho_+<\rho_-$ if $\varepsilon=\varepsilon_0$ is taken. □

If $0<\varepsilon<\varepsilon_0$, then $(\rho_+,u_+)\in \mathrm{IV}(\rho_-,u_-)$, and the Riemann solution consists of two shock waves S_1, S_2 and an intermediate state $(\rho_*^\varepsilon,u_*^\varepsilon)$ besides two constant states $(\rho_\pm,u_\pm)$. From (3.22) and (3.23), it can be derived that $(\rho_*^\varepsilon,u_*^\varepsilon)$ is determined by

$$u_*^\varepsilon - u_- = -\sqrt{\frac{\varepsilon}{\gamma}\left((\gamma-1)\left(\frac{1}{\rho_-}-\frac{1}{\rho_*^\varepsilon}\right)\left((\rho_*^\varepsilon)^\gamma u_*^\varepsilon - \rho_-^\gamma u_-\right) + \left(u_- - u_*^\varepsilon\right)\left((\rho_*^\varepsilon)^{\gamma-1} - \rho_-^{\gamma-1}\right)\right)} \tag{6.3}$$

for $\rho_*^\varepsilon > \rho_-$ and

$$u_+ - u_*^\varepsilon = -\sqrt{\frac{\varepsilon}{\gamma}\left((\gamma-1)\left(\frac{1}{\rho_*^\varepsilon}-\frac{1}{\rho_+}\right)\left(\rho_+^\gamma u_+ - (\rho_*^\varepsilon)^\gamma u_*^\varepsilon\right) + \left(u_*^\varepsilon - u_+\right)\left(\rho_+^{\gamma-1} - (\rho_*^\varepsilon)^{\gamma-1}\right)\right)} \tag{6.4}$$

for $\rho_*^\varepsilon > \rho_+$.

Introduce

$$f(\rho_1,\rho_2,u_1,u_2) = (\gamma-1)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\left(\rho_2^\gamma u_2 - \rho_1^\gamma u_1\right) + (u_1-u_2)\left(\rho_2^{\gamma-1} - \rho_1^{\gamma-1}\right) \tag{6.5}$$

for $\rho_1,\rho_2,u_1,u_2 > 0$.

The addition of (6.3) and (6.4) gives

$$u_+ - u_- = -\sqrt{\frac{\varepsilon}{\gamma}}\left(\sqrt{f\left(\rho_-,\rho_*^\varepsilon,u_-,u_*^\varepsilon\right)} + \sqrt{f\left(\rho_*^\varepsilon,\rho_+,u_*^\varepsilon,u_+\right)}\right) < 0. \tag{6.6}$$

Taking the limit in (6.6), it follows that

$$\lim_{\varepsilon\to 0} f\left(\rho_-,\rho_*^\varepsilon,u_-,u_*^\varepsilon\right) = \lim_{\varepsilon\to 0} f\left(\rho_*^\varepsilon,\rho_+,u_*^\varepsilon,u_+\right) = +\infty, \tag{6.7}$$

which means that $\lim_{\varepsilon\to 0}\rho_*^\varepsilon = +\infty$.

With $u_+ < u_*^\varepsilon < u_-$ and $\rho_\pm \ll \rho_*^\varepsilon$ as $\varepsilon\to 0$ in mind, let us consider the limit in (6.6) again, then it follows that

$$u_+ - u_- = -\lim_{\varepsilon\to 0}\sqrt{\frac{\varepsilon}{\gamma}}\left(\sqrt{(\gamma-1)\frac{1}{\rho_-}\left(\rho_*^\varepsilon\right)^\gamma u_*^\varepsilon} + \sqrt{(\gamma-1)\frac{1}{\rho_+}\left(\rho_*^\varepsilon\right)^\gamma u_*^\varepsilon}\right), \tag{6.8}$$

which implies that

$$\lim_{\varepsilon\to 0}\sqrt{\varepsilon\left(\rho_*^\varepsilon\right)^\gamma u_*^\varepsilon} = \frac{\sqrt{\gamma\rho_-\rho_+}(u_- - u_+)}{\sqrt{\gamma-1}(\sqrt{\rho_-}+\sqrt{\rho_+})}. \tag{6.9}$$

Let u_- and u_*^ε be connected by 1-shock S_1 with speed σ_1^ε, while u_*^ε and u_+ are connected by 2-shock S_2 with speed σ_2^ε. From (3.13), σ_1^ε and σ_2^ε can be calculated by

$$\sigma_1^\varepsilon = \frac{\rho_*^\varepsilon u_*^\varepsilon - \rho_- u_-}{\rho_*^\varepsilon - \rho_-}, \qquad \sigma_2^\varepsilon = \frac{\rho_+ u_+ - \rho_*^\varepsilon u_*^\varepsilon}{\rho_+ - \rho_*^\varepsilon}. \tag{6.10}$$

Therefore, we have the following result.

Lemma 6.2. *Set* $\sigma = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}}$. *Then, we have the following relations*:

$$\lim_{\varepsilon\to 0} u_*^\varepsilon = \lim_{\varepsilon\to 0} \sigma_1^\varepsilon = \lim_{\varepsilon\to 0} \sigma_2^\varepsilon = \sigma, \tag{6.11}$$

$$\lim_{\varepsilon\to 0} \varepsilon\big(\rho_*^\varepsilon\big)^\gamma = \frac{\gamma\rho_-\rho_+(u_- - u_+)^2}{(\gamma-1)(\sqrt{\rho_-}+\sqrt{\rho_+})(\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+)}, \tag{6.12}$$

$$\lim_{\varepsilon\to 0} \int\limits_{\sigma_1^\varepsilon t}^{\sigma_2^\varepsilon t} \rho_*^\varepsilon\,dx = \big(\sigma[\rho] - [\rho u]\big)t. \tag{6.13}$$

Proof. Letting $\varepsilon \to 0$ in (6.3), we have

$$\lim_{\varepsilon\to 0} u_*^\varepsilon = u_- - \lim_{\varepsilon\to 0}\sqrt{\frac{\varepsilon}{\gamma}(\gamma-1)\frac{1}{\rho_-}\big(\rho_*^\varepsilon\big)^\gamma u_*^\varepsilon}. \tag{6.14}$$

Substituting (6.9) into (6.14), one can easily see that

$$\lim_{\varepsilon\to 0} u_*^\varepsilon = u_- - \frac{\sqrt{\rho_+}(u_- - u_+)}{\sqrt{\rho_-}+\sqrt{\rho_+}} = \sigma.$$

Letting $\varepsilon \to 0$ in (6.10) and noting $\lim_{\varepsilon\to 0}\rho_*^\varepsilon = +\infty$, $\lim_{\varepsilon\to 0}\sigma_1^\varepsilon = \lim_{\varepsilon\to 0}\sigma_2^\varepsilon = \sigma$ can be drawn. And (6.12) can be directly obtained by substituting $\lim_{\varepsilon\to 0} u_*^\varepsilon = \sigma$ into (6.9), which means that the intermediate pressure $\varepsilon(\rho_*^\varepsilon)^\gamma$ is always bounded and this result will be used later.

From the result as above, we can conclude that the two shocks coincide as $\varepsilon \to 0$ (see Fig. 6(b)). The velocity σ is the weighted average of initial velocities u_- and u_+, which is identical with the velocity of the δ-shock of the PGD model (1.7) with the same Riemann data $(\rho_\pm, u_\pm)$.

Rankine–Hugoniot conditions (3.13) for both shocks S_1 and S_2 imply

$$\begin{cases} \sigma_1^\varepsilon\big(\rho_*^\varepsilon - \rho_-\big) = \rho_*^\varepsilon u_*^\varepsilon - \rho_- u_-, \\ \sigma_2^\varepsilon\big(\rho_+ - \rho_*^\varepsilon\big) = \rho_+ u_+ - \rho_*^\varepsilon u_*^\varepsilon. \end{cases} \tag{6.15}$$

Thus,

$$\big(\sigma_1^\varepsilon - \sigma_2^\varepsilon\big)\rho_*^\varepsilon = \rho_+ u_+ - \rho_- u_- + \sigma_1^\varepsilon\rho_- - \sigma_2^\varepsilon\rho_+. \tag{6.16}$$

Letting $\varepsilon \to 0$ in (6.16), we have

$$\lim_{\varepsilon\to 0}\big(\sigma_1^\varepsilon - \sigma_2^\varepsilon\big)\rho_*^\varepsilon = [\rho u] - \sigma[\rho]. \tag{6.17}$$

Hence, (6.13) can be derived directly from (6.17). It can be seen that the density becomes a singular measure as $\varepsilon \to 0$, which is a linear function of t and is also consistent with the density of the δ-shock of the PGD model (1.7) with the same Riemann data $(\rho_\pm, u_\pm)$. Therefore, the solution is no longer a self-similar solution even though Eqs. (1.8) and initial data (2.1) are invariant under the self-similar transform. □

We now show the following theorem characterizing the vanishing pressure limit of the PAR model (1.8) with the Riemann data (2.1) in the case $u_- > u_+$. And the following theorem is similar to Theorem 3.1 in [11] for the isentropic Euler equations.

Theorem 6.3. *In the case $u_- > u_+$, assume that $(\rho^\varepsilon(\xi), u^\varepsilon(\xi))$ is the Riemann solution of the PAR model* (1.8) *with Riemann initial data* (2.1) *for $0 < \varepsilon < \varepsilon_0$, then the limit of the Riemann solution $(\rho^\varepsilon(\xi), u^\varepsilon(\xi))$ as $\varepsilon \to 0$ is a delta shock wave connecting the constant states $(\rho_\pm, u_\pm)$ as follows*:

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_-), & x < \sigma t, \\ (w_1(t)\delta(x-\sigma t), u_\delta(t)), & x = \sigma t, \\ (\rho_+, u_+), & x > \sigma t, \end{cases} \tag{6.18}$$

where $w_1(t) = \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho u])$ and $u_\delta(t) = \sigma$, which is the solution of the Riemann problem to the PGD model (1.7) *with the same Riemann initial data $(\rho_\pm, u_\pm)$.*

Furthermore, we have the limit of $\rho^\varepsilon u^\varepsilon$ as follows:

$$\lim_{\varepsilon\to 0} \rho^\varepsilon u^\varepsilon = \rho_- u_- + [\rho u] H(x - \sigma t) + w_2(t)\delta(x - \sigma t), \tag{6.19}$$

where $w_2(t) = \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho u] - [\rho u^2])$ and $H(x - \sigma t)$ is the Heaviside function.

Proof. 1. For $0 < \varepsilon < \varepsilon_0$, the Riemann solution of the PAR model (1.8) with Riemann initial data (2.1) can be expressed as

$$\big(\rho^\varepsilon(\xi), u^\varepsilon(\xi)\big) = \begin{cases} (\rho_-, u_-), & -\infty < \xi < \sigma_1^\varepsilon, \\ (\rho_*^\varepsilon, u_*^\varepsilon), & \sigma_1^\varepsilon < \xi < \sigma_2^\varepsilon, \\ (\rho_+, u_+), & \sigma_2^\varepsilon < \xi < +\infty, \end{cases} \tag{6.20}$$

which satisfies the following weak equalities:

$$\big\langle -\xi\big(\rho^\varepsilon(\xi)\big)_\xi + \big(\rho^\varepsilon(\xi)u^\varepsilon(\xi)\big)_\xi, \psi(\xi)\big\rangle = 0, \tag{6.21}$$

$$\left\langle -\xi\left(\rho^\varepsilon(\xi)u^\varepsilon(\xi) + \frac{\varepsilon}{\gamma}\big(\rho^\varepsilon(\xi)\big)^\gamma\right)_\xi + \big(\rho^\varepsilon(\xi)\big(u^\varepsilon(\xi)\big)^2 + \varepsilon\big(\rho^\varepsilon(\xi)\big)^\gamma u^\varepsilon(\xi)\big)_\xi, \psi(\xi)\right\rangle = 0, \tag{6.22}$$

for any $\psi(\xi) \in C_0^\infty(R)$.

Which can be expressed in the integral formulations as follows:

$$\int_{-\infty}^{\infty} \big(\xi - u^\varepsilon(\xi)\big)\rho^\varepsilon(\xi)\psi'(\xi)\,d\xi + \int_{-\infty}^{\infty} \rho^\varepsilon(\xi)\psi(\xi)\,d\xi = 0, \tag{6.23}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\xi\rho^\varepsilon(\xi)u^\varepsilon(\xi) + \frac{\varepsilon}{\gamma}\xi\big(\rho^\varepsilon(\xi)\big)^\gamma - \rho^\varepsilon(\xi)\big(u^\varepsilon(\xi)\big)^2 - \varepsilon\big(\rho^\varepsilon(\xi)\big)^\gamma u^\varepsilon(\xi)\right)\psi'(\xi)\,d\xi \\ &\quad + \int_{-\infty}^{\infty} \left(\rho^\varepsilon(\xi)u^\varepsilon(\xi) + \frac{\varepsilon}{\gamma}\big(\rho^\varepsilon(\xi)\big)^\gamma\right)\psi(\xi)\,d\xi = 0. \end{aligned} \tag{6.24}$$

2. The first integral in (6.23) can be decomposed into

$$\left\{\int_{-\infty}^{\sigma_1^\varepsilon} + \int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon} + \int_{\sigma_2^\varepsilon}^{\infty}\right\}\big(\xi - u^\varepsilon(\xi)\big)\rho^\varepsilon(\xi)\psi'(\xi)\,d\xi. \tag{6.25}$$

By applying integration by parts, the first and the last terms of (6.25) can be computed by

$$\int_{-\infty}^{\sigma_1^\varepsilon} (\xi - u_-)\rho_-\psi'(\xi)\,d\xi = \left(\sigma_1^\varepsilon - u_-\right)\rho_-\psi\left(\sigma_1^\varepsilon\right) - \int_{-\infty}^{\sigma_1^\varepsilon} \rho_-\psi(\xi)\,d\xi,$$

$$\int_{\sigma_2^\varepsilon}^{\infty} (\xi - u_+)\rho_+\psi'(\xi)\,d\xi = -\left(\sigma_2^\varepsilon - u_+\right)\rho_+\psi\left(\sigma_2^\varepsilon\right) - \int_{\sigma_2^\varepsilon}^{\infty} \rho_+\psi(\xi)\,d\xi.$$

Now, we define $\rho_0(\xi) = \rho_- + [\rho]H(\xi)$, then the limit of the sum of the first and the last terms of (6.25) is

$$\lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{\sigma_1^\varepsilon} + \int_{\sigma_2^\varepsilon}^{\infty}\right\}\left(\xi - u^\varepsilon(\xi)\right)\rho^\varepsilon(\xi)\psi'(\xi)\,d\xi = \left([\rho u] - \sigma[\rho]\right)\psi(\sigma) - \int_{-\infty}^{\infty} \rho_0(\xi - \sigma)\psi(\xi)\,d\xi. \tag{6.26}$$

The second term of (6.25) can be calculated by

$$\begin{aligned}
&\int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon} \left(\xi - u^\varepsilon(\xi)\right)\rho^\varepsilon(\xi)\psi'(\xi)\,d\xi \\
&= \int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon} \left(\xi - u_*^\varepsilon\right)\rho_*^\varepsilon\psi'(\xi)\,d\xi \\
&= \rho_*^\varepsilon\left(\sigma_2^\varepsilon - u_*^\varepsilon\right)\psi\left(\sigma_2^\varepsilon\right) - \rho_*^\varepsilon\left(\sigma_1^\varepsilon - u_*^\varepsilon\right)\psi\left(\sigma_1^\varepsilon\right) - \int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon} \rho_*^\varepsilon\psi(\xi)\,d\xi \\
&= \rho_*^\varepsilon\left(\sigma_2^\varepsilon - \sigma_1^\varepsilon\right)\left(\frac{\sigma_2^\varepsilon\psi(\sigma_2^\varepsilon) - \sigma_1^\varepsilon\psi(\sigma_1^\varepsilon)}{\sigma_2^\varepsilon - \sigma_1^\varepsilon} - u_*^\varepsilon\frac{\psi(\sigma_2^\varepsilon) - \psi(\sigma_1^\varepsilon)}{\sigma_2^\varepsilon - \sigma_1^\varepsilon} - \frac{1}{\sigma_2^\varepsilon - \sigma_1^\varepsilon}\int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon}\psi(\xi)\,d\xi\right).
\end{aligned} \tag{6.27}$$

Noticing that $\lim_{\varepsilon\to 0}\sigma_1^\varepsilon = \lim_{\varepsilon\to 0}\sigma_2^\varepsilon = \sigma$, we have

$$\lim_{\varepsilon\to 0}\frac{\psi(\sigma_2^\varepsilon) - \psi(\sigma_1^\varepsilon)}{\sigma_2^\varepsilon - \sigma_1^\varepsilon} = \psi'(\sigma), \tag{6.28}$$

$$\lim_{\varepsilon\to 0}\frac{\sigma_2^\varepsilon\psi(\sigma_2^\varepsilon) - \sigma_1^\varepsilon\psi(\sigma_1^\varepsilon)}{\sigma_2^\varepsilon - \sigma_1^\varepsilon} = \lim_{\varepsilon\to 0}\left(\sigma_2^\varepsilon\frac{\psi(\sigma_2^\varepsilon) - \psi(\sigma_1^\varepsilon)}{\sigma_2^\varepsilon - \sigma_1^\varepsilon} + \psi\left(\sigma_1^\varepsilon\right)\right) = \sigma\psi'(\sigma) + \psi(\sigma). \tag{6.29}$$

In view of (6.11), (6.17), (6.28) and (6.29), the limit of the second term of (6.25) can be given by

$$\lim_{\varepsilon\to 0}\int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon}\big(\xi-u^\varepsilon(\xi)\big)\rho^\varepsilon(\xi)\psi'(\xi)\,d\xi=0. \tag{6.30}$$

Summarizing (6.26) and (6.30) leads to

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\big(\rho^\varepsilon(\xi)-\rho_0(\xi-\sigma)\big)\psi(\xi)\,d\xi=\big(\sigma[\rho]-[\rho u]\big)\psi(\sigma), \tag{6.31}$$

which is true for any $\psi(\xi)\in C_0^\infty(R)$.

3. Now, we compute the limit $\varepsilon\to 0$ of $\rho^\varepsilon(\xi)u^\varepsilon(\xi)$ by using the weak formulation (6.24). Like as before, we decompose the first integral in (6.24) into three parts:

$$\left\{\int_{-\infty}^{\sigma_1^\varepsilon}+\int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon}+\int_{\sigma_2^\varepsilon}^{\infty}\right\}\left(\xi\rho^\varepsilon(\xi)u^\varepsilon(\xi)+\frac{\varepsilon}{\gamma}\xi\big(\rho^\varepsilon(\xi)\big)^\gamma-\rho^\varepsilon(\xi)\big(u^\varepsilon(\xi)\big)^2-\varepsilon\big(\rho^\varepsilon(\xi)\big)^\gamma u^\varepsilon(\xi)\right)\psi'(\xi)\,d\xi. \tag{6.32}$$

The first term of (6.32) can be calculated by

$$\begin{aligned}
&\int_{-\infty}^{\sigma_1^\varepsilon}\left(\xi\rho^\varepsilon(\xi)u^\varepsilon(\xi)+\frac{\varepsilon}{\gamma}\xi\big(\rho^\varepsilon(\xi)\big)^\gamma-\rho^\varepsilon(\xi)\big(u^\varepsilon(\xi)\big)^2-\varepsilon\big(\rho^\varepsilon(\xi)\big)^\gamma u^\varepsilon(\xi)\right)\psi'(\xi)\,d\xi\\
&=\int_{-\infty}^{\sigma_1^\varepsilon}\left(\xi\rho_-u_-+\frac{\varepsilon}{\gamma}\xi\rho_-^\gamma-\rho_-u_-^2-\varepsilon\rho_-^\gamma u_-\right)\psi'(\xi)\,d\xi\\
&=\left(\rho_-u_-\sigma_1^\varepsilon+\frac{\varepsilon}{\gamma}\rho_-^\gamma\sigma_1^\varepsilon-\rho_-u_-^2-\varepsilon\rho_-^\gamma u_-\right)\psi\big(\sigma_1^\varepsilon\big)-\int_{-\infty}^{\sigma_1^\varepsilon}\left(\rho_-u_-+\frac{\varepsilon}{\gamma}\rho_-^\gamma\right)\psi(\xi)\,d\xi.
\end{aligned} \tag{6.33}$$

Similarly, we can get the last term of (6.32) as

$$\begin{aligned}
&\int_{\sigma_2^\varepsilon}^{\infty}\left(\xi\rho^\varepsilon(\xi)u^\varepsilon(\xi)+\frac{\varepsilon}{\gamma}\xi\big(\rho^\varepsilon(\xi)\big)^\gamma-\rho^\varepsilon(\xi)\big(u^\varepsilon(\xi)\big)^2-\varepsilon\big(\rho^\varepsilon(\xi)\big)^\gamma u^\varepsilon(\xi)\right)\psi'(\xi)\,d\xi\\
&=-\left(\rho_+u_+\sigma_2^\varepsilon+\frac{\varepsilon}{\gamma}\rho_+^\gamma\sigma_2^\varepsilon-\rho_+u_+^2-\varepsilon\rho_+^\gamma u_+\right)\psi\big(\sigma_2^\varepsilon\big)-\int_{\sigma_2^\varepsilon}^{\infty}\left(\rho_+u_++\frac{\varepsilon}{\gamma}\rho_+^\gamma\right)\psi(\xi)\,d\xi.
\end{aligned} \tag{6.34}$$

Thus, the limit of the sum of (6.33) and (6.34) can be obtained by

$$\lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{\sigma_1^\varepsilon}+\int_{\sigma_2^\varepsilon}^{\infty}\right\}\left(\xi\rho^\varepsilon(\xi)u^\varepsilon(\xi)+\frac{\varepsilon}{\gamma}\xi\big(\rho^\varepsilon(\xi)\big)^\gamma-\rho^\varepsilon(\xi)\big(u^\varepsilon(\xi)\big)^2-\varepsilon\big(\rho^\varepsilon(\xi)\big)^\gamma u^\varepsilon(\xi)\right)\psi'(\xi)\,d\xi$$

$$= \big(\rho_- u_- \sigma - \rho_- u_-^2\big)\psi(\sigma) - \int_{-\infty}^{\sigma} \rho_- u_- \psi(\xi)\,d\xi - \big(\rho_+ u_+ \sigma - \rho_+ u_+^2\big)\psi(\sigma) - \int_{\sigma}^{\infty} \rho_+ u_+ \psi(\xi)\,d\xi$$

$$= \big(\big[\rho u^2\big] - \sigma[\rho u]\big)\psi(\sigma) - \int_{-\infty}^{+\infty} (\rho_0 u_0)(\xi - \sigma)\cdot \psi(\xi)\,d\xi, \tag{6.35}$$

where $(\rho_0 u_0)(\xi - \sigma) = \rho_- u_- + [\rho u]H(\xi - \sigma)$.

For the second term of (6.32), integrating by parts again, we deduce that

$$\int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon} \bigg(\xi \rho^\varepsilon(\xi) u^\varepsilon(\xi) + \frac{\varepsilon}{\gamma}\xi\big(\rho^\varepsilon(\xi)\big)^\gamma - \rho^\varepsilon(\xi)\big(u^\varepsilon(\xi)\big)^2 - \varepsilon\big(\rho^\varepsilon(\xi)\big)^\gamma u^\varepsilon(\xi)\bigg)\psi'(\xi)\,d\xi$$

$$= \int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon} \bigg(\xi \rho_*^\varepsilon u_*^\varepsilon + \frac{\varepsilon}{\gamma}\xi\big(\rho_*^\varepsilon\big)^\gamma - \rho_*^\varepsilon\big(u_*^\varepsilon\big)^2 - \varepsilon\big(\rho_*^\varepsilon\big)^\gamma u_*^\varepsilon\bigg)\psi'(\xi)\,d\xi$$

$$= \big(\sigma_2^\varepsilon \psi\big(\sigma_2^\varepsilon\big) - \sigma_1^\varepsilon \psi\big(\sigma_1^\varepsilon\big)\big)\bigg(\rho_*^\varepsilon u_*^\varepsilon + \frac{\varepsilon}{\gamma}\big(\rho_*^\varepsilon\big)^\gamma\bigg) - \big(\psi\big(\sigma_2^\varepsilon\big) - \psi\big(\sigma_1^\varepsilon\big)\big)\big(\rho_*^\varepsilon\big(u_*^\varepsilon\big)^2 + \varepsilon\big(\rho_*^\varepsilon\big)^\gamma u_*^\varepsilon\big)$$

$$- \int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon} \bigg(\rho_*^\varepsilon u_*^\varepsilon + \frac{\varepsilon}{\gamma}\big(\rho_*^\varepsilon\big)^\gamma\bigg)\psi(\xi)\,d\xi$$

$$= \frac{\sigma_2^\varepsilon \psi(\sigma_2^\varepsilon) - \sigma_1^\varepsilon \psi(\sigma_1^\varepsilon)}{\sigma_2^\varepsilon - \sigma_1^\varepsilon}\bigg(\big(\sigma_2^\varepsilon - \sigma_1^\varepsilon\big)\rho_*^\varepsilon u_*^\varepsilon + \big(\sigma_2^\varepsilon - \sigma_1^\varepsilon\big)\frac{\varepsilon}{\gamma}\big(\rho_*^\varepsilon\big)^\gamma\bigg)$$

$$- \frac{\psi(\sigma_2^\varepsilon) - \psi(\sigma_1^\varepsilon)}{\sigma_2^\varepsilon - \sigma_1^\varepsilon}\big(\big(\sigma_2^\varepsilon - \sigma_1^\varepsilon\big)\rho_*^\varepsilon\big(u_*^\varepsilon\big)^2 + \big(\sigma_2^\varepsilon - \sigma_1^\varepsilon\big)\varepsilon\big(\rho_*^\varepsilon\big)^\gamma u_*^\varepsilon\big)$$

$$- \frac{1}{\sigma_2^\varepsilon - \sigma_1^\varepsilon}\int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon} \bigg(\big(\sigma_2^\varepsilon - \sigma_1^\varepsilon\big)\rho_*^\varepsilon u_*^\varepsilon + \big(\sigma_2^\varepsilon - \sigma_1^\varepsilon\big)\frac{\varepsilon}{\gamma}\big(\rho_*^\varepsilon\big)^\gamma\bigg)\psi(\xi)\,d\xi. \tag{6.36}$$

Combining (6.11), (6.17), (6.28) and (6.29) and noting that $\varepsilon(\rho_*^\varepsilon)^\gamma$ is bounded from (6.12), the limit of the second term of (6.32) can be obtained by

$$\lim_{\varepsilon\to 0}\int_{\sigma_1^\varepsilon}^{\sigma_2^\varepsilon} \bigg(\xi \rho^\varepsilon(\xi) u^\varepsilon(\xi) + \frac{\varepsilon}{\gamma}\xi\big(\rho^\varepsilon(\xi)\big)^\gamma - \rho^\varepsilon(\xi)\big(u^\varepsilon(\xi)\big)^2 - \varepsilon\big(\rho^\varepsilon(\xi)\big)^\gamma u^\varepsilon(\xi)\bigg)\psi'(\xi)\,d\xi$$

$$= \big(\sigma[\rho] - [\rho u]\big)\big(\sigma\big(\sigma\psi'(\sigma) + \psi(\sigma)\big) - \sigma^2\psi'(\sigma) - \sigma\psi(\sigma)\big) = 0. \tag{6.37}$$

Taking the limit in (6.24), in view of (6.35) and (6.37), we have

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \big(\big(\rho^\varepsilon u^\varepsilon\big)(\xi) - (\rho_0 u_0)(\xi - \sigma)\big)\psi(\xi)\,d\xi = \big(\sigma[\rho u] - \big[\rho u^2\big]\big)\psi(\sigma), \tag{6.38}$$

which is true for any $\psi(\xi) \in C_0^\infty(R)$.

4. Finally, we consider the limit of ρ^ε and $\rho^\varepsilon u^\varepsilon$. Since the solution is no longer a self-similar solution when $\varepsilon \to 0$, thus we seek the solution depending on the time.

Let $\phi(x,t) \in C_0^\infty(R \times R^+)$, then we have

$$\lim_{\varepsilon\to 0}\int_0^\infty\int_{-\infty}^\infty \rho^\varepsilon\left(\frac{x}{t}\right)\phi(x,t)\,dx\,dt = \lim_{\varepsilon\to 0}\int_0^\infty t\left(\int_{-\infty}^\infty \rho^\varepsilon(\xi)\phi(\xi t,t)\,d\xi\right)dt. \tag{6.39}$$

Regarding t as a parameter and applying (6.31), one can easily see that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^\infty \rho^\varepsilon(\xi)\phi(\xi t,t)\,d\xi = \int_{-\infty}^\infty \rho_0(\xi-\sigma)\phi(\xi t,t)\,d\xi + \bigl(\sigma[\rho]-[\rho u]\bigr)\phi(\sigma t,t). \tag{6.40}$$

Substituting $\xi = \frac{x}{t}$ into (6.40) and noting $\rho_0(\frac{x}{t}-\sigma) = \rho_0(x-\sigma t)$, we transform (6.39) into the form

$$\lim_{\varepsilon\to 0}\int_0^\infty\int_{-\infty}^\infty \left(\rho^\varepsilon\left(\frac{x}{t}\right) - \rho_0(x-\sigma t)\right)\phi(x,t)\,dx\,dt = \int_0^\infty t\bigl(\sigma[\rho]-[\rho u]\bigr)\phi(\sigma t,t)\,dt. \tag{6.41}$$

With the same reason as before, we arrive at

$$\lim_{\varepsilon\to 0}\int_0^\infty\int_{-\infty}^\infty \left(\left(\rho^\varepsilon u^\varepsilon\right)\left(\frac{x}{t}\right) - (\rho_0 u_0)(x-\sigma t)\right)\phi(x,t)\,dx\,dt = \int_0^\infty t\bigl(\sigma[\rho u]-\bigl[\rho u^2\bigr]\bigr)\phi(\sigma t,t)\,dt. \tag{6.42}$$

From (2.10), it can be concluded that the strengths of the Dirac delta functions are $w_1(t) = \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho]-[\rho u])$ and $w_2(t) = \frac{t}{\sqrt{1+\sigma^2}}(\sigma[\rho u]-[\rho u^2])$, respectively. It is not difficult to obtain $w_2(t) = \sigma w_1(t)$ by a straightforward computation. If we define the product of ρ and u by the point-wise product, then $\rho u = \lim_{\varepsilon\to 0}\rho^\varepsilon u^\varepsilon$ can be achieved. □

7. Discussions

Due to the fact that the Riemann solutions of (1.6) do not converge to the classical results on those of the PGD model (1.7) as $\varepsilon \to 0$, the PAR model (1.8) is proposed in order to solve this problem. In fact, the adjoint PAR model

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \bigl(\rho u + \varepsilon p(\rho)\bigr)_t + \bigl(\rho u^2 + \varepsilon\gamma u p(\rho)\bigr)_x = 0 \end{cases} \tag{7.1}$$

can also be used to solve this problem and the process is similar.

Furthermore, we can consider the general situation

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \bigl(\rho u + \varepsilon_1 p(\rho)\bigr)_t + \bigl(\rho u^2 + \varepsilon_2 u p(\rho)\bigr)_x = 0, \end{cases} \tag{7.2}$$

here $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. However the Riemann problem for (7.2) is not easy to solve and the discussion should be divided into many situations based on the relation between the convergent speeds of $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

The non-local macroscopic traffic model proposed in [23], assuming that repulsive vehicle interactions that depend on the vehicle distance and vehicle speed but not on the relative velocity, can be written in the following nonconservative form:

$$\begin{cases} \rho_t + u\rho_x = -\rho u_x, \\ u_t + uu_x = -\dfrac{1}{\rho}\dfrac{\partial P_1}{\partial \rho}\rho_x - \dfrac{1}{\rho}\dfrac{\partial P_2}{\partial u}u_x + \dfrac{u_0(\rho,u)-u}{\tau}, \end{cases} \tag{7.3}$$

in which $P_1(\rho,u)$ and $P_2(\rho,u)$ are contributions to the traffic pressure and $u_0(\rho,u)$ is the "optimal velocity" function.

The system (7.2) corresponds to (7.3) with $\tau \to \infty$ by taking $\frac{\partial P_1}{\partial \rho} = (\varepsilon_2 - \varepsilon_1)\gamma\rho^{\gamma-1}u$ and $\frac{\partial P_2}{\partial u} = (\varepsilon_2 - \varepsilon_1\gamma)\rho^{\gamma}$ when $p(\rho) = \rho^{\gamma}$ in (7.2). In particular, the PAR model (1.8) corresponds to (7.3) with $\tau \to \infty$ by taking $\frac{\partial P_1}{\partial \rho} = \varepsilon(\gamma-1)\rho^{\gamma-1}u$ and $\frac{\partial P_2}{\partial u} = 0$ when $p(\rho) = \rho^{\gamma}$ in (1.8) and the PGD model (1.7) corresponds to (7.3) with $\tau \to \infty$ by taking $\frac{\partial P_1}{\partial \rho} = \frac{\partial P_2}{\partial u} = 0$.

The results in this paper show that the limit behaviors of the AR traffic model (1.1) as the traffic pressure vanishes like as those of the sticky particle model proposed by Shandarin and Zeldovich [40] if we cancel the constraint condition $\rho \leqslant \rho_*$ in [4] (or $\rho \leqslant \rho_*(u)$ in [5]), namely cars will collide and coalesce into the infinite density. Thus the sticky particle model (1.7) can also be regarded as the singular limit of the traffic flow model in the extreme situation to describe the movement of the free particles on a line. On the other hand, the limit behaviors of the AR traffic model (1.1) are identical with those of the PGD model (1.7) in the free flow.

The results in this paper also show that if we need to approximate the PGD model (1.7) through perturbing (1.7), then the δ-entropy condition should be proposed suitably for the corresponding small perturbation of the PGD model (1.7), otherwise the results might be spurious.

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