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Model they start from

A new anisotropic continuum model for traffic flow

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Abstract

This paper develops a new continuum model based upon an improved car-following model by using a series expansion of the headway in terms of the density. The new model contains an additional speed gradient term (Anisotropic term) in comparison to the Berg's model [Berg et al., Phys. Rev. E 61(2) (2000) 1056]. This anisotropic term guarantees the property that the characteristic speeds are always less than or equal to the macroscopic flow speed. This new model also overcomes the problem of negative flows and negative speeds (i.e. wrong way travel) that exists in almost every higher order continuum models. It is known that diffusive wave could lead to a "wrong way travel" under certain circumstances [C. F. Daganzo, Trans. Res. B 29 (1995) 277]. In our general model, a traffic dependent anisotropic factor controls the non-isotropic character and diffusive influence in these special circumstances. The stability of traffic flow is analyzed and found out that this new continuum model obeys the same stability criterion as Berg's model for zero anisotropic parameter.

Keywords: Wrong way travel; Anisotropic; Stability; Continuum model

1. Introduction

Modeling of traffic flow has been a key tool to simulate the behavior of transportation systems. Fundamental of traffic flow modeling and control are the basic relationship between three traffic states: flow rate (q), speed (v) and concentration (ρ). Numerous experimental measurements revealed that traffic flow possesses qualitatively distinct dynamic states [1,2]. It is now widely believed that at least three distinct dynamic phases exist on highways [3], i.e. (i) free-flow (ii) synchronized traffic and (iii) wide moving jams. The free traffic flow which is analogous to the laminar flow in fluid systems, the traffic jam state where vehicles almost do not move, and the synchronized traffic flow which is characterized by complicated temporal variations of the vehicle density and velocity.

Paralleled with experiments, many physical models have been proposed [4–6]. At present traffic problems have been investigated by many models: car following models [7–11], the cellular automaton models [12], continuum models [13–22] and so on. Continuum models differ from car following models with regards to carry out simulations. With continuum models one has to deal with two coupled partial differential equations instead of a few hundred or even thousands of ordinary differential equations in the car following case. Since

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the seminal work of Lighthill and Whitham [13] on kinematic waves of vehicular traffic flow, a number of continuum models have been developed to model traffic flow. The study of continuum models began with the LWR model developed independently by Lighthill and Whitham [13] and Richards (1956). The LWR model is known as the simple continuum model in which the relationships among the three aggregate variables (q, v and ρ) are modeled. The LWR model employs the conservation equation in the following form:

$$\rho_t + q_x = 0, \tag{1}$$

and is supplemented by the following equation of traffic flow:

$$q = \rho v, \tag{2}$$

and a relationship between the mean speed and the traffic density under equilibrium conditions

$$v = V_e(\rho), \tag{3}$$

where $V_e(\rho)$ is the generalized equilibrium velocity, which is given by the steady state relationship between highway velocity and density (e.g. fundamental diagrams). Here x and t represent space and time respectively.

The above LWR model, the simplest among all the continuum models, is also the most understood. It is a scalar hyperbolic conservation law that exhibits a wide range of phenomena such as traffic sound waves, shocks and rarefaction waves. The consistency and the existence of weak solution of such conservation laws have been studied by Zhang [23] and Lax [24], respectively.

However, the LWR model has its deficiencies, the most fatal one is that the speed is solely determined by the equilibrium speed density relationship (3), no fluctuation of speed around the equilibrium values is allowed, thus, the model is not suitable for the description of non-equilibrium situations like stop-and-go traffic etc.

In the past decades, many efforts were devoted to improving the LWR model through developing high-order models, which use dynamic equation for the speed v to replace the equilibrium relationship (3). Perhaps the most well known result of these efforts is the higher-order model developed by Payne [14]. Payne's theory comprises a system of quasi-linear hyperbolic equations: one equation describing conservation of mass such as (1) and the other evolution of traffic speed given by

$$v_t + v v_x = -\frac{1}{\tau}\left\{ v - V_e(\rho) - \left(\frac{V_e'(\rho)}{2\rho}\right)\rho_x \right\}, \tag{4}$$

where v is the traffic speed, $V_e(\rho)$ is the equilibrium speed, τ is the relaxation time constant and $V_e'(\rho) = \mathrm{d}V_e(\rho)/\mathrm{d}\rho$.

Later several authors have suggested a considerable number of modifications to Payne's model. On reformulating the models given by Phillips [15], Kerner and Konhäuser [16] and Zhang [17], we can classify all these models into the following general form:

$$\rho_t + q_x = 0. \tag{5}$$

$$v_t + v v_x = -\left(\frac{1}{\rho}\right)\zeta_x + \frac{1}{\tau}(V_e(\rho) - v), \tag{6}$$

where ζ is the so-called traffic pressure and τ is the so-called relaxation time, which is the time constant of the regulating traffic velocity (v) to the equilibrium velocity $V_e(\rho)$.

For various macroscopic models, the traffic pressure $\zeta(\rho, v)$, relaxation time τ and the generalized equilibrium velocity $V_e(\rho, v)$ are different.

(1) $\zeta = \rho\theta$ with $\theta = \theta_0(1 - (\rho/\rho_{\max}))$ results in the model given in Phillips [15].
(2) If $\zeta = \rho\theta_0 - \eta v_x$, where η is the so-called viscosity coefficient, we obtain the model given in Kerner and Konhäuser [16].
(3) $\zeta = \frac{1}{3}\rho^3 V_e'^2(\rho)$, we obtain the model given in Zhang [17].

Berg et al. [18] proposed a continuum version of the car following OV model [7] by using a series expansion of the headway in terms of the density.

However, as pointed out by Daganzo [25], it was shown that the Payne model (and the other listed non-equilibrium models) had two families of characteristic, along which traffic information is transmitted: one is slower and the other is faster than the speed of the traffic stream that carries them. The faster characteristic leads to a gas like behavior—vehicle from behind can force vehicles in front to speed up (Zhang [26]), and diffusion causes 'Wrong way' travel (Daganzo [25]). One fundamental principal of the traffic flow is that vehicles are anisotropic and respond only to the frontal stimuli. Treiber et al. [27] in his gas kinetic traffic model (GKT) has solved the issue of wrong way travel problem. Recently, Zhang [19,20] has proposed new non-equilibrium models and shown that the new models overcome the backward travel problem.

In this paper, we develop a new macroscopic continuum model by introducing a new speed gradient term as the anticipation term in the equation of motion. It will solve the characteristic speed problem that exists in the previous high-order models and therefore can describe the traffic flow dynamics more realistically.

In Section 2, we present an improved continuum model based upon the car following model given by Jiang et al. [21] and using the series expansion between headway and density given by Berg et al. [18]. In Section 3, we analyze some qualitative features of the model and in Section 4, we study its stability properties and finally in Section 5, we provide a discussion on our results.

2. A new continuum model

The car-following model was developed to model the motion of vehicles following each other on a single lane without overtaking. It is believed that the dynamic state of the following car depends upon the speeds of the leading car and the following car itself, the distance between the two cars, the road conditions, the capability of the car, the personality of the drivers and so on. The classical car following theory given by Gazis [8] is represented by the following equation:

$$\frac{\mathrm{d}v_n(t+\Delta t)}{\mathrm{d}t} = \lambda\,\Delta v, \tag{7}$$

where $\Delta v = v_{n-1}(t) - v_n(t)$, with v_{n-1} and v_n being the speed of leading and following car, respectively. λ is the sensitivity and is given by

$$\lambda = a\frac{[v_n(t+\Delta t)]^m}{(b_n)^l}. \tag{8}$$

In Eq. (8), l and m are non-negative parameters; $b_n = x_{n-1}(t) - x_n(t)$, where x_{n-1} and x_n are the positions of the leading vehicle and the following vehicle, respectively; and a is the positive constant of proportionality. This model does not work in realistic situations, as the distance between successive vehicles can be arbitrarily close when the leading and following vehicle has identical speed. To overcome such type of difficulties, Bando et al. [7] proposed a car following model called the optimal velocity model (OV model). In the OV model, the OV function increases monotonically to its maximal value and has a turning point (i.e. critical point), which corresponds to the safety distance. In the OV model, the acceleration of every car is determined by its velocity v_n and optimal speed $V(b_n)$ depending on the headway b_n as follows:

$$\frac{\mathrm{d}v_n}{\mathrm{d}t} = a[V(b_n) - v_n], \tag{9}$$

where $V(b_n) = \tanh(b_n - h_c) + \tanh(h_c)$ is the optimal velocity function, h_c being the safety distance and a is the driver's sensitivity, which equals the inverse of the driver's reaction time.

Treiber et al. [27] explained that there exists a common driver's behavior that none of the existing car-following model can explain. He argued that the driver of the following vehicle may not decelerate if the distance between the two vehicles is shorter than the safe distance and the preceding vehicle travels faster than the following vehicle. Such type of sensitivity of drivers with respect to velocity differences is already taken into account by Treiber et al. [28]. Jiang et al. [21] observed that the phenomenon does exist and the OV model given by Bando et al. [7] can't explain this phenomenon. To remove this deficiency, Jiang et al. [21] proposed an improved car following model in which he explicitly considered that the relative speed between the leading and the following vehicles, has an impact on the behavior of the following driver. Jiang's model has the following form:

$$\frac{\mathrm{d}v_n(t)}{\mathrm{d}t} = a[V(b_n) - v_n(t)] + \alpha\lambda\,\Delta v. \tag{10}$$

Therefore, both the classical car following model and the OV model are the special case of the above model.

In order to develop the macroscopic continuum version of the improved car following model, we transform the discrete variable of individual vehicles into the continuous flow variables. The equation of motion thus obtained is given by

$$v_t + vv_x = a(V(b) - v) - (2\beta c(\rho))v_x. \tag{11}$$

The proof of the above equation can be found in the Appendix A of this paper. It is generally believed that the relation between headway and density is of great importance. When there are long-range fluctuations in the headway or the density along the road, the usual definition of the density in terms of the headway (i.e. $\rho = b^{-1}$) is only an approximation. The headway b can be written as a perturbation series [18].

Following this, the conservation equation can be written as

$$\rho_t + (\rho v)_x = 0, \tag{12}$$

where v is the speed of the traffic flow and the traffic dynamics equation can be obtained by inserting the approximate expression of headway as a perturbation series [18] into Eq. (11) and is given by

$$v_t + vv_x = a[\bar{V}(\rho) - v] + a\bar{V}'(\rho)\left[\frac{\rho_x}{2\rho} + \frac{\rho_{xx}}{6\rho^2} - \frac{\rho_x^2}{2\rho^3}\right] - 2\beta c(\rho)v_x, \tag{13}$$

where $\bar{V}'(\rho) = \mathrm{d}\bar{V}(\rho)/\mathrm{d}\rho$.

Now we hypothesize that traffic sound speed does not change from equilibrium flow to non-equilibrium flow and depends upon traffic density and relaxation time. For light traffic (i.e. $\bar{V}'(\rho) = 0$), traffic sound speed becomes zero (Zhang [20]).

Traffic sound speed is given by

$$c^2(\rho) = -\frac{a\bar{V}'(\rho)}{2}, \tag{14}$$

where $\bar{V}(\rho) = V(1/\rho) = \tanh((1/\rho) - h_c) + \tanh(h_c)$. Note that if $\beta = 0$, then the model reduces to the Berg's model [18] and after leaving the term of ρ_x^2, the model converts to the model given by Zhou et al. [22]. Eq. (13) is now analogous to the Zhang model [20]. However, an important difference between that model and the new model lies in the coefficient of higher order terms.

Helbing et al. [29] proposed a new way of obtaining macroscopic models from microscopic models, which are different from gas kinetic approach, and allow us to carry out simultaneous micro-and macro- simulations of neighboring freeway sections.

3. Qualitative properties of the model

The system (12) and (13) has a similar structure to Berg's model [18] and Zhou's models [22] but is more general than these models. On comparing, we get that our model has an additional term $2\beta c(\rho)v_x$. Moreover the term $c(\rho)$ varies with OV function. These differences, however, are not the structural differences. So one would expect that the new model behaves roughly the same as Berg's model and Zhou's model and perhaps gives a more accurate description of traffic flow owing to its greater generality.

We rewrite the system (12) and (13) as follows:

$$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v & \rho \\ \frac{-a\bar{V}'(\rho)}{2\rho} & v + 2\beta c(\rho) \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ a(\bar{V}(\rho) - v + a\bar{V}'\left(\frac{\rho_{xx}}{6\rho^2} - \frac{\rho_x^2}{2\rho^3}\right) \end{pmatrix}. \tag{15}$$

The corresponding homogeneous system of (15) can be written as

$$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v & \rho \\ \frac{-a\bar{V}'(\rho)}{2\rho} & v + 2\beta c(\rho) \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = 0. \tag{16}$$

or $(\partial U/\partial t) + [A](\partial U/\partial x) = 0$, is strictly hyperbolic, where

$$U = \begin{pmatrix} \rho \\ v \end{pmatrix} \text{ and } [A] = \begin{pmatrix} v & \rho \\ \frac{-a\bar{V}'(\rho)}{2\rho} & v + 2\beta c(\rho) \end{pmatrix}. \tag{17}$$

The eigen values, λ, of the matrix A are found by setting

$$|[A] - \lambda[I]| = 0, \tag{18}$$

where I is the identity matrix

$$\begin{vmatrix} v - \lambda & \rho \\ \frac{-a\bar{V}'(\rho)}{2\rho} & v + 2\beta c(\rho) - \lambda \end{vmatrix} = 0. \tag{19}$$

Thus,

$$\lambda_1 = v + \left(\beta + \sqrt{1 + \beta^2}\right) c(\rho), \tag{20}$$

and

$$\lambda_2 = v + \left(\beta - \sqrt{1 + \beta^2}\right) c(\rho). \tag{21}$$

These are the characteristic speeds for our new model, i.e.

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_1 = v + \left(\beta + \sqrt{1 + \beta^2}\right) c(\rho), \tag{22}$$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_2 = v + \left(\beta - \sqrt{1 + \beta^2}\right) c(\rho), \tag{23}$$

and the right eigen vectors

$$r_1(\rho, v) = \left(1, \left(\beta + \sqrt{1 + \beta^2}\right) \frac{c(\rho)}{\rho}\right)^t, \tag{24}$$

$$r_2(\rho, v) = \left(1, \left(\beta - \sqrt{1 + \beta^2}\right) \frac{c(\rho)}{\rho}\right)^t. \tag{25}$$

The homogeneous system (16) has two families of traffic sound waves, shock and rarefaction waves, one family for each characteristic field. For the first characteristic field the properties of these waves are quantitatively identical to those of the LWR model because of $\lambda_1 \leqslant v$. But for the second characteristic, the waves behave quite differently as they travel faster than traffic ($\lambda_2 \geqslant v$ and $\nabla\lambda_2(\rho, v) r_2(\rho, v) > 0$). This means that the future conditions of the traffic flow will be affected by the traffic conditions behind the flow. Such type of behavior, however, can be controlled by the factor β in our model. We call it the anisotropic factor. Note that $\beta \gg 1$, the second characteristic approaches v, the velocity of the traffic. Thus, information can never reach vehicles from behind. The term having this anisotropic factor in the system (13) is known as anisotropic term.

The behavior of the inhomogeneous system (15) is qualitatively similar to that of its corresponding homogeneous system (16). The additional relaxation and viscosity terms, however, modify the behavior of the system as viscosity smoothes shocks and relaxation drives the system to the equilibrium state exponentially fast. Thus Eq. (15) admits smooth traveling waves rather than shocks and behaves much like the viscous equilibrium model in large time.

$$\rho_t + f_*(\rho)_x = \tau(\upsilon(\rho)\rho_x)_x, \quad f''_*(\rho) < 0, \tag{26}$$

where τ and υ are referred to as the relaxation time and viscosity coefficient, respectively. The traveling waves of (15) are expected to approach the corresponding shocks of (16), when the relaxation time approaches to zero. The viscosity term smoothes variations in traffic density and has a stabilizing effect on traffic flow, which counteracts the pressure term. So one would expect that diffusive wave arising from (15) are nonlinearly stable (Zhang [20]). This, however, needs further analysis.

4. Linear stability analysis

Assuming ρ_0 and $v_0 = \bar{V}(\rho_0)$ are the steady-state solution of Eqs. (12) and (13). The analogous criterion for the continuum model may be found by linearizing the model (12 and 13) around some initial values ρ_0 and v_0.

$$\rho = \rho_0 + \hat{\rho}. \tag{27}$$

$$v = v_0 + \hat{v}. \tag{28}$$

After taking Taylor series expansions of the perturbed equations at ρ_0 and v_0 leads to the perturbation equations

$$\hat{\rho}_t + \rho_0 \hat{v}_x + v_0 \hat{\rho}_x = 0 \tag{29}$$

and

$$\hat{v}_t + v_0 \hat{v}_x = a[\bar{V}(\rho_0)\hat{\rho} - \hat{v}] + a\bar{V}'\left[\frac{\hat{\rho}_x}{2\rho_0} + \frac{\hat{\rho}_{xx}}{6\rho_0^2}\right] - 2\beta c_0 \hat{v}_x, \tag{30}$$

where $c_0 = c(\rho_0)$ and $\bar{V}' = \bar{V}'(\rho_0)$.

The linear stability of the system (12 and 13) can be determined by examining the sinusoidal solution of the perturbed Eqs. (27) and (28). It is found that the system is stable, when

$$\rho_0^2 - \frac{2\beta c_0 \rho_0}{\bar{V}'} < -\left(\frac{a(1+\beta^2)}{2\bar{V}'}\right). \tag{31}$$

The proof of the linear stability analysis can be found in Appendix B of this paper. This shows that the model is stable against all infinitesimal perturbations for the inequality (31). There is an intermediate range of density, $0 \leqslant \rho_{c_1} \leqslant \rho \leqslant \rho_{c_2}$, in which $\bar{V}(\rho)$ is so sensitive to change in ρ that homogeneous flow is unstable. As it follows from inequality (31), one can find the critical values (ρ_{c_1} and ρ_{c_2}) from the equation

$$\rho_0^2 - \frac{2\beta c_0 \rho_0}{\bar{V}'} = -\left(\frac{a(1+\beta^2)}{2\bar{V}'}\right). \tag{32}$$

Note that for $\beta = 0$; we get

$$-\left(\frac{a}{2\rho_0^2 \bar{V}'}\right) > 1, \tag{33}$$

which is exactly the stability criterion found by Bando et al. [7].

Due to the presence of anisotropic parameter, the intermediate range of instability in our model is different from the Berg's model. This explains the basic difference between these two models. These results are crucial in explaining the appearance of a "Phantom traffic jam", which is observed in real traffic flow. In this regime traffic flow breaks down and forms the well-known stop-and-go pattern of traffic jam [30].

It is easy to find out that the critical disturbances travels with a speed

$$\bar{C}(\rho_0) = \bar{V}(\rho_0) + \bar{V}'(\rho_0)\left(\rho_0 + \frac{a\beta}{2c_0}\right), \tag{34}$$

which is slower than the steady state traffic speed $v_0 = \bar{V}(\rho_0)$, since $\bar{V}'$ is negative.

5. Conclusions

Traffic flow has many surprisingly rich and varied phenomena. So the models which attempts to describe the flow are numerous and having a role in explaining certain aspects of the system. The car-following model is one of the basic traffic models in which each driver can respond to the surrounding traffic conditions. Berg et al. [18] and Zhou et al. [22] have developed the continuum version of the OV model given by Bando et al. [7]. Due to the terms ρ_x and ρ_{xx} appears in the nonlinear continuous model, the density fluctuations in traffic flow can be described by this model. This paper has developed a traffic flow model that practically includes some of the well-known non-equilibrium models as a special case. However, an important difference between the new model and the previously developed model is that our model is isotropic. But the isotropic behavior can be controlled by the anisotropic factor β at the limit $\beta \to \infty$, the isotropic behavior dies out and the model becomes fully anisotropic. This model is applicable only for single lane flow with no overtaking for single species. Further research will focus on the study of synchronized flow, traveling wave solutions, numerical approximations, cluster effects and experimental validation of this general traffic flow anisotropic continuum model.

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Appendix A

To develop the macroscopic continuum model corresponding to the car-following model given by Eq. (10), we assume that the nth vehicle at position x represents the average traffic condition at $(x - (\Delta/2), x + (\Delta/2))$, which is determined by the average traffic condition in the preceding region $(x + (\Delta/2), x + (3\Delta/2))$. Here Δ corresponds to b_n in car following theory and it varies with different inter-vehicle spaces between different successive vehicles. So one can transform the discrete variables of individual vehicles into the continuous flow variables as follows (Jiang et al. [21]):

$$v_n(t) \to v(x,t), \quad v_{n-1}(t) \to v(x+\Delta,t), \quad V(b_n) \to V(b)$$
$$\text{and } \lambda \to \frac{1}{\tau}, \tag{A.1}$$

where b is the headway in continuum theory and τ is the time needed for the backward propagated disturbance to travel a distance b_n.

Appling Eq. (A.1) to the Eq. (10), we obtained

$$\frac{\mathrm{d}v(x,t)}{\mathrm{d}t} = a(V(b) - v) + \alpha\left[\frac{v(x+\Delta,t) - v(x,t)}{\tau}\right]. \tag{A.2}$$

Expanding the right-hand side of Eq. (A.2) and neglecting the higher-order terms, we get

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a(V(b) - v) + \left(\frac{\alpha\Delta}{\tau}\right)v_x. \tag{A.3}$$

To complete the model, we introduce a non-negative dimensionless parameter β such that

$$\alpha\Delta = 2\beta\delta, \quad \beta \geqslant 0. \tag{A.4}$$

Moreover the quantity δ/τ measures the speed at which a small change in traffic density propagates back along a line of vehicles. We call it traffic sound speed and hypothesize that it is a function of local density (Zhang [17,20]):

$$\frac{\delta}{\tau} = -c(\rho) \geqslant 0. \tag{A.5}$$

We know that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v_t + vv_x. \tag{A.6}$$

By substituting Eqs. (A.4), (A.5) and (A.6) into (A.3), we get

$$v_t + vv_x = a(V(b) - v) - (2\beta c(\rho))v_x. \tag{A.7}$$

Appendix B

To determine the stability condition of the system (15), we calculate the eigen value $\omega(k)$ of a harmonic disturbance:

$$\bar{f}(x,t) = \begin{pmatrix} \hat{\rho}(x,t) \\ \hat{v}(x,t) \end{pmatrix} = \begin{pmatrix} \hat{\rho}_0 \\ \hat{v}_0 \end{pmatrix} \exp\{\mathrm{i}[kx - \omega(k)t]\}, \tag{B.1}$$

so that we can rewrite the Eqs. (28) and (29) in the form as

$$\begin{pmatrix} \mathrm{i}(kv_0 - \omega) & \mathrm{i}k\rho_0 \\ -a\bar{V}' - \frac{a\mathrm{i}k\bar{V}'}{2\rho_0} + \frac{a\bar{V}'k^2}{6\rho_0^2} & \mathrm{i}(kv_0 - \omega) + a + 2\beta c_0 \mathrm{i}k \end{pmatrix} \times \begin{pmatrix} \hat{\rho}_0 \\ \hat{v}_0 \end{pmatrix} \exp\{\mathrm{i}[kx - \omega(k)t]\} = 0. \tag{B.2}$$

For nontrivial solution

$$\begin{vmatrix} \mathrm{i}(kv_0 - \omega) & \mathrm{i}k\rho_0 \\ -a\bar{V}' - \frac{a\mathrm{i}k\bar{V}'}{2\rho_0} + \frac{a\bar{V}'k^2}{6\rho_0^2} & \mathrm{i}(kv_0 - \omega) + a + 2\beta c_0 \mathrm{i}k \end{vmatrix} = 0. \tag{B.3}$$

On solving, we get

$$\omega_{1,2}(k) = k(v_0 + \beta c_0) - \frac{a\mathrm{i}}{2}\left[1 \pm \sqrt{1 + \frac{2\bar{V}'}{a}\left(k^2(\beta^2+1) - \frac{a\mathrm{i}\beta k}{c_0} - 2\mathrm{i}k\rho_0 + \frac{\mathrm{i}k^3}{3\rho_0}\right)}\right]. \tag{B.4}$$

The traffic flow will remain stable as long as the imaginary part of ω is negative.

Define

$$\Omega(k) = \mathrm{Re}\left[1 + \frac{2\bar{V}'}{a}\left(k^2(\beta^2+1) - \frac{a\mathrm{i}\beta k}{c_0} - 2\mathrm{i}k\rho_0 + \frac{\mathrm{i}k^3}{3\rho_0}\right)\right]^{1/2}. \tag{B.5}$$

The criterion is equivalent to $|\Omega(k)| < 1$, and

$$|\Omega(k)| = \left[\left(1 + \frac{2\bar{V}'}{a}k^2(\beta^2+1)\right)^2 + \left(\frac{4\beta k c_0}{a} - \frac{4k\rho_0\bar{V}'}{a} + \frac{2k^3\bar{V}'}{3a\rho_0}\right)^2\right]^{1/4}\sqrt{\frac{1+\cos\phi}{2}}, \tag{B.6}$$

where

$$\phi = \arg\left[1 + \frac{2\bar{V}'}{a}\left(k^2(\beta^2+1) - \frac{a\mathrm{i}\beta k}{c_0} - 2\mathrm{i}k\rho_0 + \frac{\mathrm{i}k^3}{3\rho_0}\right)\right]. \tag{B.7}$$

Solving $|\Omega(k)| = 1$ leads to three solutions.

$$k_0 = 0, \quad k_\pm = \sqrt{6\left(\rho_0^2 - \frac{\beta c_0 \rho_0}{\bar{V}'}\right) \pm \frac{3a\rho_0}{\bar{V}'}\sqrt{\frac{-2\bar{V}'}{a}(1+2\beta^2)}}. \tag{B.8}$$

We know that $\bar{V}'<0$ and k_- is always real whereas k_+ might be either real or complex. As pointed out by Berg et al. [18] the system is stable when $k_+ \notin R$, which means

$$6\left(\rho_0^2 - \frac{\beta c_0 \rho_0}{\bar{V}'}\right) + \frac{3a\rho_0}{\bar{V}'}\sqrt{\frac{-2\bar{V}'}{a}(1+2\beta^2)}<0 \tag{B.9}$$

or

$$\rho_0^2 - \frac{2\beta c_0 \rho_0}{\bar{V}'} < -\left(\frac{a(1+\beta^2)}{2\bar{V}'}\right). \tag{B.10}$$

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