SIAM J. NUMER. ANAL.
Vol. 21, No. 2, April 1984

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RIEMANN SOLVERS, THE ENTROPY CONDITION, AND DIFFERENCE APPROXIMATIONS*

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Abstract. A condition on the numerical flux for semidiscrete approximations to scalar, nonconvex conservation laws is introduced, and shown to guarantee convergence to the correct physical solution. An equality which can be used to impose an entropy inequality for approximations to systems of equations is obtained. Roe's scheme is modified to satisfy this inequality. These considerations also lead to a simple closed form expression for the solution to the Riemann problem for scalar, nonconvex conservation laws.

Introduction. Recently a number of new shock capturing finite difference approximations have been constructed and found to be useful in shock calculations, e.g. [4], [10], [24], [3]. In addition to conservation form, these schemes were usually constructed to have some of the following properties:

(1) Stable and sharp steady discrete shock solutions.

(2) Limit solutions which satisfy a geometric and/or analytic entropy condition.

(3) A bound on the variation of the approximate solutions, at least in the scalar, or linear systems, case.

In this paper we present a very simple property E for numerical flux functions, which implies properties (2) and (3) above. In the scalar case, all schemes known to satisfy (2) and (3), also satisfy property E (to the best of our knowledge). The property is presented in Definition 2.1.

Unfortunately property E implies that the scheme is at most first order accurate. In a parallel joint paper with S. Chakravarthy [22], we construct second order accurate schemes for which (3) is valid (following Harten [10], van Leer [18]). We also obtain a single entropy inequality which is satisfied by all limit solutions to the schemes we constructed in [22]. Hence we can conclude, in the scalar convex flux function case, that the approximate solutions converge to the unique entropy solution if the initial data and its variation are bounded. This follows from results contained in [28].

In earlier work with Majda [20], we modified the Lax–Wendroff scheme to obtain a single entropy inequality, while keeping its second order accuracy and three-point bandwidth. Also in [4], with Engquist, we obtained a second order accurate upwind scheme, again having a single entropy inequality.

At this point, there seems to be no higher order accurate scheme known to converge to the entropy solution for general nonconvex flux functions. Property (3) does not imply property (2). An important scheme satisfying (3) and not (2) is Murman's [21], which was generalized to systems by Roe [24]. In § 3 we present a correction to Roe's scheme for systems, so that property (2) is satisfied.

Schemes satisfying property (1) for scalar equations include Godunov's [9], Engquist–Osher [4]. Schemes which do have sharp discrete shocks for systems also include Godunov's [9], Osher's [23], Roe's [24], and Harten–Lax's [12]. Second order accurate schemes with this property are constructed in [22]. We also construct a scheme in [22] satisfying property (3) and having *no* steady discrete shocks.

* Received by the editors December 14, 1982. This research was supported by the National Science Foundation under grant MCS 82-00788, the U.S. Army Research Office under grant DAAG29-82-K-0090, and the National Aeronautics and Space Administration under grant NAG1-270.

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Godunov's scheme plays a significant role in this theory. Condition E for scalar equations is shown to be equivalent to the following statement:

$$\text{(0.1)}\quad \begin{aligned} &\text{(a) If } u_j < u_{j+1}, \text{ then } h_{j+1/2} \leqq h^G_{j+1/2}. \\ &\text{(b) If } u_j > u_{j+1}, \text{ then } h_{j+1/2} \geqq h^G_{j+1/2}. \end{aligned}$$

Here $h_{j+1/2}$ is the numerical flux at the right boundary of the jth cell, and $h^G_{j+1/2}$ is Godunov's flux at the same point.

These considerations also led us to a new and simple closed form expression for the solution to the analytic Riemann initial value problem for general nonconvex scalar flux functions. This is presented in Theorem 1.1.

The outline of the paper is as follows. In § 1 we review the theory of weak solutions and state and prove our result concerning the nonconvex Riemann problem. In § 2 we review the theory of approximate solutions, define E schemes, and state the convergence result for scalar problems. This result is proven in § 3, along with a condition which assures property (2) above for systems of equations. We then verify that property (2) is valid for our first order scheme for systems of equations (under slightly more general hypotheses than in [23]). We also modify Roe's scheme in a fairly general fashion, and then prove that these modifications also satisfy property (2). In § 4 we solve a nonconvex scalar Riemann problem using our new formula. We also compare Godunov's scheme with the Engquist–Osher scheme for nonconvex $f(u)$. We end by showing that this last scheme can be interpreted geometrically, as a multi-valued Riemann solver, using rarefaction and compression waves only, even for nonconvex $f(u)$.

1. Review of theory of weak solutions and solution to nonconvex Riemann problems. We shall consider numerical approximations to the initial value problem for nonlinear hyperbolic systems of conservation laws:

$$\text{(1.1)}\quad \begin{aligned} &\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) = 0, \qquad t > 0, \quad -\infty < x < \infty, \\ &w(x, 0) = w_0(x). \end{aligned}$$

Here $w(x, t)$ is an m-vector of unknowns, and the flux function, $f(w)$, is vector valued, having m components. The system is hyperbolic when the Jacobian matrix has real eigenvalues.

It is well known that solutions of (1.1) may develop discontinuities in finite time, even when the initial data are smooth. Because of this, we seek a weak solution of (1.1), i.e. a bounded measurable function w, such that for all $\varphi \in C_0^\infty(R \times R^+)$

$$\text{(1.2)}\quad \begin{aligned} &\text{(a)} \qquad \iint_{R\times R^+} (w\varphi_t + f(w)\varphi_x)\, dx\, dt = 0, \\ &\text{(b)} \qquad \lim_{t\to 0} \|w(x, t) - w_0(x)\|_{L^1} = 0. \end{aligned}$$

Solutions of (1.2) are not necessarily unique. For physical reasons, the limit solution of the viscous equation, as viscosity tends to zero, is sought. In the scalar case, this solution must satisfy for all $\varphi \in C_0^\infty(R \times R^+)$, $\varphi \geqq 0$, and for all real constants c,

$$\text{(1.3)}\quad \text{(a)} \qquad -\iint_{R\times R^+} (|w - c|\varphi_t + \operatorname{sgn}(w - c)(f(w) - f(c))\varphi_x)\, dx\, dt \leqq 0,$$

which is equivalent to the statement:

(1.3) (b) $$\frac{\partial}{\partial t}|w-c|+\frac{\partial}{\partial x}((f(w)-f(c))\,\mathrm{sgn}\,(w-c))\leqq 0,$$

in the sense of distributions.

Such solutions are called entropy solutions. Kruzkov has shown in [14], that two solutions of (1.3) satisfy

(1.4) $$\|w(x,t_1)-v(x,t_1)\|_{L_1}\leqq\|w(x,t_0)-v(x,t_0)\|_{L_1}$$

for all $t_1 \geqq t_0$. Hence, condition (1.3) guarantees the uniqueness of solutions to the scalar version of (1.2). Existence was also obtained ın [14].

For systems of equations, Lax has defined an entropy inequality [16], with the help of an entropy function $V(w)$ for (1.1) defined as follows:

(i) V satisfies

(1.5) $$V_w f_w = F_w$$

where F is some other function, called the entropy flux.

(ii) V is a convex function of w.

It follows from (1.1), upon multiplication by V_w, that every smooth solution of (1.1) also satisfies

(1.6) $$V_t + F_x = 0.$$

It was shown in [16], that if w is the bounded a.e. limit of solutions to the regularized equation, then the limit satisfies, in the weak sense, the following inequality:

(1.7) $$V_t + F_x \leqq 0.$$

Inequality (1.3)(b), for scalar equations, is just (1.7) with $V(w)=|w-c|$.

Inequalities (1.3) and (1.7) have important geometric consequences for piecewise continuous solutions. Suppose $w(x,t)$ is a piecewise continuous solution having a jump discontinuity, $w_L(t)$, $w_R(t)$, moving with speed $s(t)$. Then (1.2) implies the well-known jump conditions

(1.8) $$f(w_L)-f(w_R)=s(w_L-w_R).$$

In the scalar case, (1.3) is equivalent to Oleinik's condition E across the shock

(1.9) $$\frac{f(w)-f(w_R)}{w-w_R}\leqq\frac{f(w_L)-f(w_R)}{w_L-w_R}$$

for all w between w_L and w_R.

If f is convex, then (1.9) is equivalent to the statement that characteristics flow *into* the discontinuity as t increases.

For hyperbolic systems of equations, the Jacobian of f, denoted by ∂f, has real eigenvalues which are usually assumed to be distinct:

$$\lambda_1<\lambda_2<\cdots<\lambda_m$$

corresponding to right eigenvectors: $r_1, r_2, \cdots, r_m$. Lax [16] then defines the kth field to be linearly degenerate if

$$\nabla_w \lambda_k \cdot r_k \equiv 0$$

where r_k is the kth eigenvector of ∂f. He also defines the field to be genuinely nonlinear if

$$\nabla_w \lambda_k \cdot r_k \neq 0.$$

For genuinely nonlinear fields, a k shock moving with speed s is defined to be a discontinuity of w such that $m+1$ characteristics flow into the shock and

$$\lambda_k(w_L) - s > 0 > \lambda_k(w_R) - s.$$

This geometric condition is equivalent to (1.7) for weak shocks [16].

The Riemann problem for (1.1) is the solution to the initial value problem where the initial data is

$$w(x, 0) \equiv \begin{cases} w^L & \text{for } x < 0, \\ w^R & \text{for } x \geqq 0, \end{cases}$$

w^L, w^R constant vectors. Lax [16] constructed a solution for $|w^L - w^R|$ small enough, assuming that the system has only genuinely nonlinear, or linearly degenerate fields. Liu [19] extended this construction to include some other cases.

For the scalar case, a solution exists for w^L, w^R arbitrary. The solution for $t > 0$ is of the form

$$w = w(x/t) = w(\zeta).$$

It takes on values between w^L and w^R [15], may depend discontinuously on ζ, but has variation bounded by $|w^L - w^R|$.

For nonconvex $f(w)$ with $f'(w)$ having several zeros, the actual geometric construction of the entropy solution involves several different waves, and can be quite complicated. Oleinik's condition must be satisfied across each discontinuity.

Our first main result gives a closed form expression for the solution $w(\zeta)$. It is derived from a simple expression for the quantity $f(w(\zeta)) - \zeta w(\zeta)$. This expression is valid for general f and is the content of:

LEMMA 1.1. *For $w(x, t) = w(x/t) = w(\zeta)$, the solution to the scalar Riemann problem for* (1.1), *we have*:

if $w^L < w^R$, then

(1.10) (a) $$f(w(\zeta)) - \zeta w(\zeta) = \min_{u \in [w^L, w^R]} [f(u) - \zeta u];$$

or

if $w^L > w^R$, then

(1.10) (b) $$f(w(\xi)) - \zeta w(\zeta) = \max_{u \in [w^R, w^L]} [f(u) - \zeta u].$$

It is easy to see that the expressions on the right are uniformly Lipschitz continuous with Lipschitz constant max $(|w^R|, |w^L|)$, and are concave and convex respectively, as functions of ζ.

Taking the distribution derivatives of the above gives us an expression for $w(\zeta)$:

THEOREM 1.1.

(1.11) (a) *If $w^L < w^R$, then*

$$w(\zeta) = -\frac{d}{d\zeta}\left(\min_{u \in [w^L, w^R]} [f(u) - \zeta u]\right).$$

(1.11) (b) *If* $w^L > w^R$, *then*

$$w(\zeta) = -\frac{d}{d\zeta}\left(\max_{u\in[w^L, w^R]} [f(u) - \zeta u]\right).$$

Lemma 1.1 is of some interest, since it gives the numerical flux for Godunov's finite difference scheme, on a grid moving with speed ζ.

Analytic and numerical studies of problems involving nonconvex conservation laws have recently been performed by several authors, [8], [2], [13]. Such problems arise in various areas of chemistry and geophysics.

Lax [16] has given a closed form expression to the solution for the *convex* scalar case, for *general* initial data. For the convex, scalar, Riemann problem, his expression and (1.11) yield the same result, but are apparently quite different. Fleming[1] [6] obtains a solution, again to the convex, scalar problem for general initial data, via a different variational approach than in [16]. His variational principle involves (1.10)(b), but seems to depend heavily on convexity.

Proof of Lemma 1.1 *and Theorem* 1.1. Consider the initial value problem, (1.1), for scalar conservation laws. We change independent variables

$$y = x - st, \qquad \tau = t,$$

obtaining the initial value problem

$$\begin{aligned} &\frac{\partial}{\partial \tau} w + \frac{\partial}{\partial y} g(w) = 0, \qquad \tau > 0, \quad -\infty < y < \infty, \\ &w(y, 0) = w_0(y), \end{aligned} \tag{1.12}$$

where

$$g(w) = f(w) - sw.$$

We take $w_0(y)$ to be piecewise constant, say

$$w_0(y) \equiv \begin{cases} w^L, & y < -\frac{1}{2}, \\ w^M, & -\frac{1}{2} \leqq y < \frac{1}{2}, \\ w^R, & \frac{1}{2} \leqq y. \end{cases}$$

Solve (1.12), obtaining $w(y, t)$. For τ_0 small enough,

$$0 \leqq \tau_0 < \tfrac{1}{2} \max\left(|g'(w^L)|, |g'(w^M)|, |g'(w^R)|\right)$$

the solution consists of two noninteracting similarity solutions to two different Riemann problems. Next define

$$w^M(\tau_0) = \int_{-1/2}^{1/2} w(y, \tau_0)\, dy = w^M - \tau_0[h^G(w^R, w^M) - h^G(w^M, w^L)], \tag{1.13}$$

where, by the divergence theorem,

$$h^G(w^R, w^M) = g(w(\tfrac{1}{2}, \tau_0)), \qquad h^G(w^M, w^L) = g(w(-\tfrac{1}{2}, \tau_0)).$$

This is, of course, Godunov's construction for approximate solutions to (1.12). It is well known that the right side of (1.13) is a nondecreasing function of w^R, w^M, w^L. In particular (fact (1)) this means that $h^G(a, b)$ is nonincreasing in a, nondecreasing

[1] We are grateful to Peter Lax for mentioning this reference.

in b. We also know (fact (2)) from [15], that $w(\frac{1}{2}, \tau_0)$ lies between w^M and w^R, and $w(-\frac{1}{2}, \tau_0)$ lies between w^L and w^M.

The first fact implies

$$\operatorname{sgn}(w^R - w^M)[g(w(\tfrac{1}{2}, \tau_0)) - g(u)] \leqq 0 \tag{1.14}$$

for all u between w^M and w^R, because we may replace (1.14) by

$$\begin{aligned} &\operatorname{sgn}(w^R - u)(hG(w^R, w^M) - hG(u, w^M)) \\ &\quad + \operatorname{sgn}(u - w^M)(hG(u, w^M) - hG(u, u)), \end{aligned} \tag{1.15}$$

and each of these expressions is nonpositive. The second fact thus implies that

(a) if $w^M < w^R$, then

$$g(w(\tfrac{1}{2}, \tau_0)) = \min_{u \in [w^M, w^R]} g(u),$$

(b) if $w^M > w^R$, then

$$g(w(\tfrac{1}{2}, \tau_0)) = \max_{u \in [w^R, w^M]} g(u).$$

Letting $s = \zeta$, and translating the coordinate system, gives us Lemma 1.1.

To prove Theorem 1.1, we merely differentiate $f(w(\zeta)) - \zeta w(\zeta)$ in the sense of distributions. The result follows from the following.

CLAIM.

$$\frac{d}{d\zeta}(f(w(\zeta)) - \zeta w(\zeta)) = -w(\zeta). \tag{1.16}$$

Proof of (1.16). For any $\varphi \in C_0^\infty(R \times R^+)$, and any weak solution w, we have

$$0 = \iint (w\varphi_t + f(w)\varphi_x)\, dx\, dt,$$

(letting $\tau = t$, $\zeta = x/t$, gives us):

$$\begin{aligned} &= \iint (w(\varphi_\zeta(-\zeta) + \varphi_\tau \tau) + f(w)\varphi_\zeta)\, d\zeta\, d\tau \\ &= \iint \left(w\left(\frac{\partial}{\partial \zeta}(-\zeta\varphi) + \frac{\partial}{\partial \tau}(\tau\varphi) \right) + f(w)\varphi_\zeta \right) d\zeta\, d\tau \\ &= \iint \left(w \cdot \frac{\partial}{\partial \zeta}(-\zeta\varphi) + f(w)\varphi_\zeta \right) d\zeta\, d\tau \end{aligned}$$

(if w depends only on ζ). Thus, in the sense of distributions:

$$0 = -\zeta w_\zeta + \frac{d}{d\zeta} f(w(\zeta)) = \frac{d}{d\zeta}(f(w(\zeta)) - \zeta w(\zeta)) + w(\zeta). \qquad \square$$

2. Review of theory of approximate solutions and statement of convergence theorem. For simplicity, we consider a semidiscrete, method of lines, approximation to (1.1). We partition

$$R^1 = \cup\, \Omega_j,$$

where

$$\Omega_j = [x_{j-1/2}, x_{j+1/2}),$$

with

$$x_j = \tfrac{1}{2}(x_{j+1/2} + x_{j-1/2}).$$

As a measure of refinement, we call

$$\Delta = \max_j \Delta x_j = \max_j (x_{j+1/2} - x_{j-1/2}).$$

Define the step function for each $t > 0$, as

$$v_\Delta(x, t) \equiv u_j(t)$$

for $x \in \Omega_j$.

The initial data of (1.1) is discretized via the averaging operator, T_Δ,

$$T_\Delta w_0(x) = \frac{1}{\Delta x_j} \int_{\Omega_j} w_0(s)\, ds = u_j(0) \tag{2.1}$$

when $x \in \Omega_j$. Throughout this paper, we assume $w_0(x) \in L^1 \cap L^\infty \cap BV$ as in [25].

For any step function, we define the difference operators

$$\Delta_\pm u_j = \pm(u_{j\pm1} - u_j), \qquad D_\pm u_j = \frac{1}{\Delta x_j}(\Delta_\pm u_j).$$

A method of lines, conservation form, discretization of (1.1), is a system of differential equations:

$$\begin{aligned} &\frac{\partial}{\partial t} u_j + D_+ h_{j-1/2} = 0, \qquad j = 0, \pm 1, \cdots, \\ &v_\Delta(x, 0) = T_\Delta w_0(x), \quad \text{for } x \in \Omega_j. \end{aligned} \tag{2.2}$$

We use the notation:

$$D_+ h_{j-1/2} = D_- h_{j+1/2} = \frac{1}{\Delta x_j}(h_{j+1/2} - h_{j-1/2})$$

throughout this work. Here

$$h_{j-1/2} = h(u_{j+k-1}, \cdots, u_{j-k}), \tag{2.3}$$

for $k \geqq 1$, is a Lipschitz continuous function of its arguments satisfying the consistency condition

$$h(w, w, \cdots, w) = f(w).$$

It is well known that bounded a.e. limits, as $\Delta \to 0$, of approximate solutions converge to weak solutions of (1.1), i.e. (1.2)(a) is satisfied. However, this does not also imply that the limit solution will satisfy any of the entropy conditions mentioned above, e.g. [11], [20]. Some restrictions on h are required.

A simple class of flux functions h, for which (2.2) converges to the unique entropy solution in $L^\infty(L^1(R); [0, T])$ as $\Delta \to 0$, for any $T > 0$, are three-point scalar monotone approximations. Such schemes have

$$h_{j-1/2} = h(u_j, u_{j-1}),$$

with h nonincreasing in its first argument, nondecreasing in its second [25].

It follows from results of [11] that these approximations are, at most, first order accurate. Some of these schemes do treat steady, or close to steady, shock solutions very well.

Together with an entropy inequality, a key estimate involved many convergence proof, is a bound on the variation. For completeness, we present Sanders' proof [25] of that bound there.

For any fixed $t>0$, the x variation of $v_\Delta(x,t)$ is

$$B(v)=\sum_j |\Delta_+ u_j(t)|.$$

Let

$$\chi_{j+1/2}=\begin{cases} 1 & \text{if } \Delta_+ u_j \geqq 0, \\ -1 & \text{if } \Delta_+ u_j < 0. \end{cases}$$

Then

$$\begin{aligned} \frac{d}{dt}B(v) &= \sum_j \frac{d}{dt}(\Delta_+ u_j \chi_{j+1/2}) \\ &= \sum_j \chi_{j+1/2}\frac{d}{dt}\Delta_+ u_j \\ &= -\sum_j \chi_{j+1/2}\Delta_+\left(\frac{1}{\Delta x_j}\Delta_+ h_{j-1/2}\right) \qquad (2.4) \\ &= \sum_j (\chi_{j+1/2}-\chi_{j-1/2})\frac{\Delta_+ h_{j-1/2}}{\Delta x_j} \\ &= \sum_j \frac{(\chi_{j+1/2}-\chi_{j-1/2})}{\Delta x_j}([h(u_{j+1},u_j)-h(u_j,u_j)]+[h(u_j,u_j)-h(u_j,u_{j-1})]) \leqq 0. \end{aligned}$$

The last inequality follows as a consequence of monotonicity, i.e., for any j

$$(2.5)\quad \begin{aligned} &\text{(a)} \qquad \chi_{j+1/2}[h(u_{j+1},u_j)-h(u_j,u_j)] \leqq 0, \\ &\text{(b)} \qquad -\chi_{j+1/2}[h(u_{j+1},u_{j+1})-h(u_{j+1},u_j)] \leqq 0. \end{aligned}$$

See [1, Thm. 2.1], for the fully discrete argument.

It is clear that (2.5)(a) and (b) themselves imply the variation bound. For explicit time discretized approximations, Harten [10] constructed a class of schemes with a five-point bandwidth, which are second order accurate except near critical points and for which the discrete analogue of (2.4) is proven (on a uniform grid). In [17], van Leer modified some classical second order accurate scheme in order to achieve this property. The idea used by both authors reduces, in the semidiscrete context, to requiring:

$$(2.6)\quad \text{(a)} \qquad \Delta_+ h_{j-1/2} = C_{j+1/2}\Delta_+ u_j - D_{j-1/2}\Delta_- u_j,$$

with

$$\text{(b)} \qquad C_{j+1/2} = C(u_{j+2}, u_{j+1}, u_j, u_{j-1}) \leqq 0,$$

$$\text{(c)} \qquad D_{j-1/2} = D(u_{j+1}, u_j, u_{j-1}, u_{j-2}) \leqq 0.$$

A glance at the fourth line of (2.4) shows that $dB(v)/dt \leqq 0$, for such schemes, called TVD schemes after Harten [10].

It was shown in [22] that at critical points of $f(w)$ such schemes are at most first order accurate. However, their overall second order accuracy does not seem to be degraded by this fact.

A well-known TVD scheme which has stable entropy condition violating solutions [21] is due to Murman. Here

$$h_{j-1/2}=\tfrac{1}{2}[f(u_j)+f(u_{j-1})-a_{j-1/2}(u_j-u_{j-1})] \tag{2.7 (a)}$$

with

$$a_{j-1/2}=\left|\frac{\Delta_- f(u_j)}{\Delta_- u_j}\right|. \tag{b}$$

Various fixes have been suggested for this scheme, and for Roe's vector valued generalization [10], [17].

We now define a class of numerical flux functions which generate TVD, and entropy condition satisfying, solutions.

DEFINITION 2.1. A consistent scheme whose numerical flux satisfies

$$\operatorname{sgn}(u_{j+1}-u_j)[h_{j+1/2}-f(u)]\leqq 0, \tag{2.8}$$

for all u between u_j and u_{j+1}, is said to be an E scheme. (These schemes may depend on more than three points.)

Setting

$$C_{j+1/2}=\frac{h_{j+1/2}-f(u_j)}{\Delta_+ u_j}, \qquad D_{j+1/2}=\frac{h_{j+1/2}-f(u_{j+1})}{\Delta_+ u_j}$$

makes it evident that an E scheme is TVD. Moreover, setting $f(u)=h(u,u)$ in (2.8) makes it obvious that a three-point monotone scheme is an E scheme.

Unfortunately, E schemes share with monotone schemes the following property.

LEMMA 2.1. *E schemes are at most first order accurate.*

The proof will be given at the end of this section.

We now state:

THEOREM 2.1. *The solutions to* (2.2) *for E schemes converge in* $L^\infty(L^1(R),[0,T])$, *as* $\Delta\to 0$, *to the unique entropy solution of* (1.1).

We shall prove this in the next section.

The monotone scheme devised by Godunov [9] plays a special role in this theory. His scheme reduces in this semidiscrete scalar case to (2.2), with

$$h^G_{j-1/2}=h^G(u_j,u_{j-1})=f(w(x_{j-1/2},t^+)), \tag{2.9}$$

where $w(x_{j-1/2},s)$ for $s>t$ is the unique entropy solution to the Riemann problem for (1.1) with initial data:

$$w(x,t)\equiv\begin{cases}u_{j-1} & \text{if } x<x_{j-1/2},\\ u_j & \text{if } x\geqq x_{j-1/2},\end{cases}$$

and

$$w(x_{j-1/2},t^+)=\lim_{s\downarrow t} w(x_{j-1/2},s).$$

This scheme is monotone (as mentioned in the proof of Lemma 1.1), but thought to be complicated for nonconvex $f(u)$. However, we have shown above that the scalar Riemann problem is rather easily solved, and moreover we do not even need its complete solution. The one item of information we need is:

(2.10)

(a) $$h^G_{j-1/2} = \min_{u_{j-1} \leqq u \leqq u_j} f(u) \quad \text{if } u_{j-1} < u_j,$$

(b) $$h^G_{j-1/2} = \max_{u_{j-1} \geqq u \geqq u_j} f(u) \quad \text{if } u_{j-1} > u_j.$$

Thus the scalar version of Godunov's scheme is not only simple, it is the limiting case of E schemes in the sense that E schemes are precisely those for which

(2.11)

(a) $$h_{j-1/2} \leqq h^G_{j-1/2} \quad \text{if } u_{j-1} < u_j,$$

(b) $$h_{j-1/2} \geqq h^G_{j-1/2} \quad \text{if } u_j > u_{j-1}.$$

The following restriction is equivalent to schemes of type (2.7)(a), being E schemes.

For all u between u_j and u_{j+1}

$$\frac{f(u_{j+1}) + f(u_j) - 2f(u)}{\Delta_+ u_j} \leqq a_{j+1/2}. \tag{2.12}$$

Suppose f is convex, with $\bar{u}$, the unique (sonic) point at which

$$f'(\bar{u}) = 0.$$

Then we may define $a_{j+1/2}$ as in (2.7)(b) unless $u_j < \bar{u} < u_{j+1}$. In this case, any definition which keeps $h_{j+1/2}$ Lipschitz continuous, and for which

$$h_{j+1/2} \leqq f(\bar{u}) = h^G_{j+1/2} \tag{2.13}$$

serves to make this an E scheme. Roe suggested such a modification in [18] and Sweby has made a similar suggestion in [26]. Our result gives a different proof of its correctness.

Proof of Lemma 2.1. For E schemes, we have

$$\frac{h(u_{j+k}, \cdots, u_{j+1}, u_j, \cdots, u_{j-k+1}) - h(u, u, \cdots, u)}{\Delta_+ u_j} \leqq 0 \tag{2.14}$$

for all u between u_j and u_{j+1}. Let

$$u_{j+\gamma} = u_j + \alpha_\gamma \Delta_+ u_j,$$

and consider various cases. Define $h_\gamma = (\partial/\partial u_{j+\gamma}) h(u_{j+k}, \cdots, u_{j+1}, u_j, \cdots, u_{j-k+1})$.

Case 1. $\alpha_1 = 1$, all other $\alpha_\gamma = 0$, and $u = u_j$. We let $u_{j+1} \to u_j$, and arrive at

$$h_1(u, u, \cdots, u) \leqq 0.$$

Case 2. $\alpha_0 = 0$, all other $\alpha_\gamma = 1$, and $u = u_{j+1}$. We let $u_{j+1} \to u_j$, and arrive at

$$h_0(u, u, \cdots, u) \geqq 0.$$

Case 3. $\alpha_1 = 1, \alpha_0 = 0$, all other $\alpha_\gamma = 0$ except α_{γ_0}, and $u = u_j$. Let $u_{j+1} \to u_j$, we have

$$\alpha_{\gamma_0} h_{\gamma_0}(u, u, \cdots, u) + h_1(u, u, \cdots, u) \leqq 0$$

for any α_{γ_0}. This implies $h_\gamma(u, u, \cdots, u) = 0$ if $\gamma \neq 0$ and $\gamma \neq 1$.

This means that if we discretize (2.2) explicitly in time, writing

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \Delta_- h(u^n_{j+k}, \cdots, u^n_{j-k+1}) = H(u^n_{j+k}, \cdots, u^n_{j-k}),$$

then for each integer γ between $-k$ and k, we have

$$\frac{\partial H}{\partial u_{j+\gamma}}(u, u, \cdots, u) \geqq 0,$$

if the simple CFL condition

$$1 \geqq \frac{\Delta t}{\Delta x}(h_1(u, u, \cdots, u) - h_0(u, u, \cdots, u))$$

is met.

Following the proof that monotone schemes are at most first order accurate in [11], we see that this weakened hypothesis also implies the same result for this explicit case. The semi-discrete case follows from a simple limiting procedure.

3. An approximate entropy inequality and systems of equations. We approximate (1.1) for systems by (2.2). Let $V(w)$ be a convex entropy, with entropy flux $F(w)$. We multiply (2.2) by $\Delta x_j V_w(u_j)$. This gives us

$$\Delta x_j \frac{d}{dt} V(u_j) = -V_w(u_j) \cdot \Delta_+ h_{j-1/2}. \tag{3.1}$$

Next, we add to both sides of (3.1) the quantity

$$\Delta_+(V_w(u_j) \cdot [h_{j-1/2} - f(u_j)]) + \Delta_+ F(u_j) = \Delta_+ \tilde{F}(u_j). \tag{3.2}$$

We now have

$$\Delta x_j \left(\frac{d}{dt} V(u_j) + D_+ \tilde{F}(u_j) \right) = (\Delta_+ V_w(u_j)) \cdot h_{j+1/2} - \Delta_+(V_w(u_j) \cdot f(u_j) - F(u_j)). \tag{3.3}$$

Let $\Gamma_{j+1/2}$ be any continuous, piecewise differentiable curve connecting u_j to u_{j+1} in a phase space. Then

$$\begin{aligned} \Delta_+ F(u_j) &= \int_{\Gamma_{j+1/2}} F_w \cdot dw = \int_{\Gamma_{j+1/2}} V_w(w) \cdot \frac{\partial f}{\partial w}(w) \cdot dw \\ &= \Delta_+(V_w(u_j) \cdot f(u_j)) - \int_{\Gamma_{j+1/2}} dw \cdot V_{ww} \cdot f. \end{aligned} \tag{3.4}$$

Also

$$\Delta_+ V_w(u_j) \cdot h_{j+1/2} = \int_{\Gamma_{j+1/2}} dw \cdot V_{ww} \cdot h_{j+1/2}. \tag{3.5}$$

Substituting (3.4), (3.5) into (3.3), gives us the equality

$$\Delta x_j \left(\frac{d}{dt} V(u_j) + D_+ \tilde{F}(u_j) \right) = \int_{\Gamma_{j+1/2}} dw \cdot V_{ww}(w) \cdot [h_{j+1/2} - f(w)]. \tag{3.6}$$

(The quantity on the right of (3.6) is path independent.)

We have the following:

LEMMA 3.1. *Suppose the right side of* (3.6) *is nonpositive for each j and for all $t > 0$, and the approximate solutions, $v_\Delta(x, t)$, converge boundedly a.e. to a function $w(x, t)$. Then the entropy inequality,* (1.7), *is valid for the limit solution.*

The proof follows from multiplying (3.6) by $\varphi(x_j, t)$, for $0 \leqq \varphi \in C_0^\infty(R \times R^+)$, then summing and integrating by parts.

In the scalar case, the right side of (3.6) can be written

$$\int_{u_j}^{u_{j+1}} V''(w)[h_{j+1/2}-f(w)]\,dw.$$

A necessary and sufficient condition that this be nonpositive for *all* convex V is that (2.2) be an E scheme. Theorem 2.1 now follows, as in [25, Thm. 1].

For systems, Godunov's scheme is of the form (2.2) with

$$h^G_{j-1/2}=h(u_j,u_{j-1})=f(u(x_{j-1/2},t^+)) \tag{3.7}$$

as in the scalar case, where $u(x_{j-1/2},s)$ is Lax's entropy satisfying similarity solution to the Riemann problem, with initial data

$$u(x,t)\equiv\begin{cases}u_{j-1} & \text{if } x<x_{j-1/2},\\ u_j & \text{if } x\geqq x_{j-1/2},\end{cases}$$

and $u(x_{j-1/2},t^+)=\lim_{s\downarrow t}u(x_{j-1/2},s)$.

Using Jensen's inequality, Godunov showed [9] (cf. also [12]) that a discrete entropy inequality is satisfied for explicit schemes. Letting $\Delta t\downarrow 0$, it easily follows that

$$\frac{\partial}{\partial t}V(u_j,t)+\frac{1}{\Delta x_j}\Delta_+F(u(x_{j-1/2},t))\leqq 0. \tag{3.8}$$

Thus limit solutions satisfy (1.7).

Various simplifications of this scheme have been developed recently. In [12], Harten and Lax constructed an approximate Riemann solver for which the analysis in [9] is modified to give an approximate entropy inequality.

In [23], the author has constructed a scheme which has a relatively simple form for Euler's equations and other physical systems. An entropy inequality was proven in [23]. We reprove it here (with slightly more general hypotheses).

The numerical flux for this scheme is defined as follows. First, define a piecewise smooth continuous path in phase space connecting u_j to u_{j+1} made up of subpaths, along which

$$\frac{du}{ds}=r_k(u(s))$$

(with r_k the kth eigenvalue of $\partial f(u)$). On such a path, the $(n-1)$ Riemann invariants corresponding to field k are constants. We begin at u, with $k=m$, stop at some end point, use $k=m-1$, etc., arriving at u_{j+1} with the subpath corresponding to $k=1$. Call each of these subpaths $\Gamma^\gamma_{j+1/2}$, $\gamma=m,m-1,\cdots,1$. There exists one such path, $\Gamma^\gamma_{j+1/2}=\bigcup^1_{\gamma=m}\Gamma^\gamma_{j+1/2}$, for $|u_{j+1}-u_j|$ small. Then define

$$h^0_{j+1/2}=\frac{1}{2}\left[f(u_{j+1})+f(u_j)-\int_{\Gamma_{j+1/2}}|\partial f(w)|\,dw\right]. \tag{3.9}$$

Here $|\cdot|$ denotes the absolute value of a diagonalizable matrix.

This algorithm yields a closed formula for $h^0_{j+1/2}$ for various physical systems [23], [3], because their Riemann invariants can be tabulated, and also because all fields are either linearly degenerate or genuinely nonlinear. This property of the fields means that the integral in (3.9) involves only $\pm f(u)$ at endpoints of the subpaths, and perhaps at a sonic point $\bar{u}$, for which $\lambda_k(\bar{u})=0$, on a genuinely nonlinear subpath.

This algorithm can be interpreted as solving the incoming Riemann problem in phase space, using only rarefactions, compression, or contact waves, then averaging the resulting multivalued solution as in Godunov's method.

To verify that the right side of (3.6) is nonpositive, we may write

$$\begin{aligned} h_{j+1/2}-f(u) &= \frac{1}{2}\left[f(u_{j+1})-f(u)+f(u_j)-f(u)-\int_{\Gamma_{j+1/2}} |\partial f(u)|\, du \right] \\ &= \left[\int_u^{u_{j+1}} (\partial f(v))^-\, dv - \int_{u_j}^{u} (\partial f(v))^+\, dv \right], \end{aligned} \tag{3.10}$$

where u is connected to u_{j+1} using the part of $\Gamma_{j+1/2}$ which begins at u, u_j is connected to u by the remaining part of $\Gamma_{j+1/2}$, and

$$(\partial f)^- = \tfrac{1}{2}[(\partial f)-|\partial f|], \qquad (\partial f)^+ = \tfrac{1}{2}[(\partial f)+|\partial f|].$$

Then we consider (3.6) using $\Gamma_{j+1/2} = \bigcup_{\gamma=m}^{1} \Gamma^{\gamma}_{j+1/2}$.

Since V_{ww} symmetrizes ∂f, we have

$$r_k^T V_{ww}\, \partial f = r_k^T (\partial f)^T V_{ww} = \lambda_k r_k^T V_{ww}. \tag{3.11}$$

Hence $V_{ww} r_k = l_k$ is a left eigenvector of ∂f, corresponding to eigenvalue λ_k. Thus, the integral in (3.6) can be written

$$\begin{aligned} \sum_{\gamma=1}^{m} \int_0^{s_\gamma} ds\, l_\gamma(u(s)) \cdot \Bigg[& \int_s^{s_\gamma} (\lambda_k(u(t)))^- r_k(u(t))\, dt \\ & + \int_0^{s} (-(\lambda_k(u(t)))^+) r_k(u(t))\, dt \Bigg] + R, \end{aligned} \tag{3.12}$$

where $\lambda_k^- = \min(\lambda_k, 0)$, $\lambda_k^+ = \max(\lambda_k, 0)$ and

$$|R| \leqq C_1 \sum_{\mu} \sum_{\gamma \neq \mu} |s_\mu|^2 |s_\gamma|.$$

(Here, and below, each C_i is a positive universal constant.)

The first quantity in (3.12) is bounded above by

$$-(\inf |\lambda_\mu|)\, C_0 \sum_{\gamma} |s_\gamma|^2.$$

Thus, if all λ_k are bounded away from zero, the right side of (3.6) is nonpositive for sufficiently small oscillation.

If one λ_γ is within $O(s_\gamma)$ of zero, we require that it correspond to a genuinely nonlinear field. Then, genuine nonlinearity implies that λ_γ is strictly monotone on $\Gamma^k_{\gamma+1/2}$, and the corresponding contribution is bounded below by

$$-C_2^\gamma |s_\gamma|^3,$$

while the other contributions are as above (if the variation is small enough). It remains to show that

$$0 \geqq C_1 \sum_{\mu \neq \gamma} |s_\mu|^2 s_\gamma - C_3 \sum_{k \neq \gamma} |s_k|^2 - C_2^\gamma |s_\gamma|^3$$

for sufficiently small oscillation. This elementary estimate was obtained in [23].

Thus we have:

THEOREM 3.2. *Suppose the approximate solutions to* (2.2) *for* $h_{j-1/2} = h^0_{j-1/2}$ *have sufficiently small oscillation. Moreover, suppose all eigenvalues of* ∂f *for the approximate solutions are bounded away from zero, except perhaps the genuinely nonlinear ones. Then bounded a.e. limits, as* $\Delta \to 0$, *of this scheme satisfy the entropy condition.*

Remark 3.1. The same result follows if we reorder the subpaths $\Gamma^{\gamma}_{j+1/2}$. We chose the ordering above so that the sharp, discrete, steady shock property, mentioned in the introduction, is valid.

We now turn to a class of schemes based on Roe's decomposition [24]. Roe defined a matrix $A_{j+1/2}$, so that

$$f(u_{j+1})-f(u_j)=A_{j+1/2}(u_{j+1}-u_j) \tag{3.13}$$

where (a) $A_{j+1/2}$ has real eigenvalues, (b) $A_{j+1/2}$ is Lipschitz continuous, (c) $\lim_{u_{j+1}\to u_j} A_{j+1/2}=\partial f(u_j)$.

Such a matrix has been shown to exist [27] if the underlying hyperbolic system has a convex entropy. Roe's original scheme has numerical flux function

$$h^R_{j+1/2}=\tfrac{1}{2}(f(u_{j+1})+f(u_j)-|A_{j+1/2}|(u_{j+1}-u_j))$$

(which agrees with Murman's (2.7) in the scalar convex case).

Unfortunately, it possesses stable, steady, expansion shock solutions. Various modifications have been suggested, e.g. [10]. The framework is as follows. Let the eigenvalues, right, and left eigenvectors of $A_{j+1/2}$ be $\lambda_k^{j+1/2}$, $r_k^{j+1/2}$, $l_k^{j+1/2}$, respectively, for $k=1,\cdots,n$. Then

$$h^R=\frac{1}{2}\left[f(u_{j+1})+f(u_j)-\sum_k|\lambda_k^{j+1/2}|\alpha_k^{j+1/2}r_k^{j+1/2}\right],$$

where

$$\alpha_k^{j+1/2}=l_k^{j+1/2}(u_{j+1}-u_j).$$

Here $l_k^{j+1/2}\cdot r_k^{j+1/2}=1$.

One then defines a modified Roe flux as

$$h^{MR}_{j+1/2}=\frac{1}{2}\left[f(u_{j+1})+f(u_j)-\sum_k \alpha_k^{j+1/2}\beta_k^{j+1/2}r_k^{j+1/2}\right], \tag{3.14}$$

where the $\beta_k^{j+1/2}$ are chosen so as to give consistency with the entropy inequality.

We now make the following restrictions on the $\beta_k^{j+1/2}$:

$$\begin{aligned}&\text{(a) If both } |\lambda_k(u_{j+1})|, |\lambda_k(u_j)|>D(\textstyle\sum_k|\alpha_k^{j+1/2}|), \text{ let } \beta_k^{j+1/2}\geqq|\lambda_k^{j+1/2}|.\\ &\text{(b) If either } |\lambda_k(u_{j+1})|, |\lambda_k(u_j)|\leqq D(\textstyle\sum_k|\alpha_k^{j+1/2}|), \text{ let } \beta_k^{j+1/2}\geqq|\lambda_k^{j+1/2}|+C|\alpha_k^{j+1/2}|.\end{aligned} \tag{3.15}$$

Here C and D are sufficiently large positive constants.

Remark 3.2. The choice of $\beta_k^{j+1/2}$ in (3.15)(b), while enforcing an entropy inequality, will destroy a desirable property of Roe's original scheme. Steady, discrete shocks are no longer resolved exactly on the grid. This may not be too serious, since this entropy fix keeps the scheme upwind away from sonic points.

Remark 3.3. The numerical implementation of this modification is via

$$h^{MR}_{j+1/2}=h^R_{j+1/2}-\tfrac{1}{2}\alpha_{k_0}^{j+1/2}\,|\alpha_{k_0}^{j+1/2}|C_{k_0}^{j+1/2}r_{k_0}^{j+1/2},$$

if field k_0 satisfies the hypothesis of (3.15)(b). Thus, one needs to monitor the eigenvalues of $A_{j+1/2}$, and to modify, if necessary, by adding a scalar multiple of the appropriate eigenfunction.

We have the following:

THEOREM 3.3. *Suppose the approximate solutions to* (2.2), (3.14), *satisfying* (3.15), *have sufficiently small oscillation. Then bounded a.e. limits, as* $\Delta\to 0$, *satisfy the entropy condition.*

Proof. We choose the path, $\Gamma_{j+1/2}$, in (3.6), to be the straight line

$$\Gamma_{j+1/2}=u(s)=u_j+s(u_{j+1}-u_j)\quad \text{for } 0\leqq s\leqq 1.$$

Then the right side of (3.6) becomes

$$\begin{aligned}
&\frac{1}{2}\sum_k\int_0^1 ds\,\alpha_k^{j+1/2}(r_k^{j+1/2})^T V_{ww}(u(s))\\
&\qquad\cdot\Big[f(u_{j+1})+f(u_j)-2f(u(s))-\sum_\gamma \alpha_\gamma^{j+1/2}\beta_\gamma^{j+1/2}r_\gamma^{j+1/2}\Big]\\
(3.16)\qquad &\quad=\frac{1}{2}(1+O(|u_{j+1}-u_j|))\\
&\qquad\cdot\sum_k\int_0^1 \alpha_k^{j+1/2}(\tilde l_k^{j+1/2})^T[f(u_{j+1})+f(u_j)-2f(u(s))-\alpha_k^{j+1/2}\beta_k^{j+1/2}r_k^{j+1/2}]\,ds,
\end{aligned}$$

where we defined $|u_{j+1}-u_j|=\sum_k|\alpha_k^{j+1/2}|$. We wish to show this is nonpositive.

Here, each $\tilde l_k^{j+1/2}$ is a positive multiple of a normalized left eigenvector $l_k^{j+1/2}$,

$$\tilde l_k^{j+1/2}=C_k^{j+1/2}l_k^{j+1/2},\quad \text{with } C_k^{j+1/2}\geqq\delta>0,$$

for δ some universal constant.

When $s=0$, each of the integrands above is nonpositive if $\beta_k^{j+1/2}\geqq\pm\lambda_k^{j+1/2}$. Thus (3.15)(a) suffices.

Next, we take the derivative with respect to s, of each integrand for which both

$$\lambda_k(u_{j+1}),\ \lambda_k(u_j)>D|u_{j+1}-u_j|.$$

We have:

$$\begin{aligned}
&-2\alpha_k^{j+1/2}(\tilde l_k^{j+1/2})^T A(u(s))\Big(\sum_{\gamma=1}^m \alpha_\gamma^{j+1/2}\cdot r_\gamma^{j+1/2}\Big)\\
(3.17)\qquad &\quad=-2(\alpha_k^{j+1/2})^2\lambda_k^{j+1/2}(\tilde l_k^{j+1/2}\cdot r_k^{j+1/2})+O(|u_{j+1}-u_j|)^3.
\end{aligned}$$

If both

$$\lambda_k(u_{j+1}),\ \lambda_k(u_j)<-D|u_{j+1}-u_j|,$$

we let $s\to 1-s$ in the kth integral, and then take the s derivative of the new integrand. We then have

$$\begin{aligned}
&2\alpha_k^{j+1/2}(\tilde l_k^{j+1/2})^T A(u(1-s))\Big(\sum_{\gamma=1}^m \alpha_\gamma^{j+1/2}\Omega_\gamma^{j+1/2}\Big)\\
(3.18)\qquad &\quad=2(\alpha_k^{j+1/2})^2\lambda_k^{j+1/2}(\tilde l_k^{j+1/2}\cdot r_k^{j+1/2})+O(|u_{j+1}-u_j|)^3.
\end{aligned}$$

If all k are of either type, then the sum over k of (3.17), (3.18) is negative, and hence (3.15)(a) suffices to show the nonpositivity of (3.16) (assuming that the oscillation is small enough).

Finally we allow the case when $\lambda_{k_0}^{j+1/2}=O(|u_{j+1}-u_j|)$. Taking the derivative gives us (3.17), for $k=k_0$.

We next sum over $k \neq k_0$. The sum of these derivatives is

$$(3.19)\qquad -\sum_{k\neq k_0} 2(\alpha_k^{j+1/2})^2|\lambda_k^{j+1/2}|(l_k^{j+1/2})\cdot r_k^{j+1/2} + O(|u_{j+1}-u_j|^3).$$

We add (3.17) for $k=k_0$ to (3.19). It is clear that choosing $\beta_{k_0}^{j+1/2}$ as in (3.15)(b), for C large enough, will imply the nonpositivity of (3.16). □

4. Scalar, nonconvex, conservation laws. We begin by applying our formula (1.11) to an illustrative example[2] of Gelfand [7] for which $f(u)$ is given in Fig. 1. Here, u^L is the left state for our Riemann data, $f''(u_2)=0$, below. We consider various values of u^R.

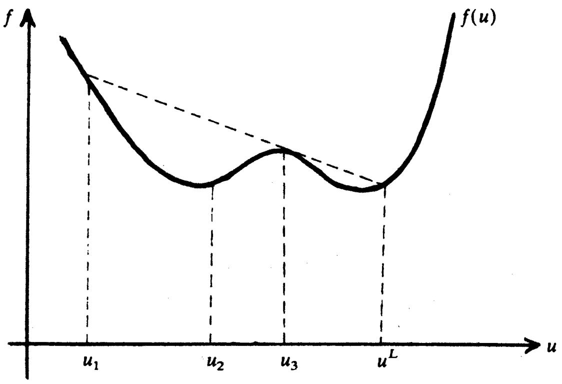

FIG. 1

If $u^R < u_1$, then a glance at Fig. 1 shows us that

$$f(u^R)-\zeta u^R > f(u)-\zeta u$$

for $u\in[u^R, u^L]$ and $\zeta > (f(u^R)-f(u^L))/u^R-u^L = s_{LR}$,

$$f(u^L)-\zeta u^L \geqq f(u)-\zeta u$$

for $u\in[u^R, u^L]$ and $\zeta \leqq (f(u^R)-f(u^L))/u^R-u^L = s_{LR}$.

Thus, (1.11) gives us a single shock solution, moving with speed s_{LR}.

If $u_3 \leqq u^R \leqq u^L$, then a similar analysis give us a single shock solution, again moving with speed s_{LR}.

For $u_2 < u^R < u_3$, it is easy to see

$$\max_{u^R\leqq u\leqq u^L}[f(u)-\zeta u] = \begin{cases} f(u^L)-\zeta u^L & \text{if } \zeta < \dfrac{f(u_3)-f(u^L)}{u_3-u_L}, \\[2ex] f((f')^{-1}(\zeta))-\zeta(f')^{-1}(\zeta) & \text{if } \dfrac{f(u_3)-f(u^L)}{u_3-u^L} \leqq \zeta < (f')(u^R), \\[2ex] f(u^R)-\zeta u^R & \text{if } (f')(u^R)\leqq \xi. \end{cases}$$

Thus (1.11) implies that we have a shock (u^L, u_3) followed by a rarefaction beginning with $u=u_3$ and ending at $u=u^R$.

[2] We thank Burton Wendroff for bringing this paper to our attention.

Finally we consider the most complicated case, which has u^R between u_1 and u_2. We have

$$\max_{u^R \leqq u \leqq u^L} [f(u)-\zeta u] = \begin{cases} f(u^L)-\zeta u^L & \text{if } \zeta < \dfrac{f(u_3)-f(u^L)}{u_3-u^L}, \\ f((f')^{-1}(\zeta))-\zeta (f')^{-1}(\zeta) & \text{if } \dfrac{f(u_3)-f(u^L)}{u_3-u^L} \leqq \zeta < (f')(u^*), \\ f(u^R)-\zeta u^R & \text{if } (f')(u^*) \leqq \zeta. \end{cases}$$

Here u^* is chosen uniquely so that

$$f'(u^*) = \frac{f(u^R)-u^*}{u^R-u^*}, \qquad u_2 < u^* < u_3.$$

Finally, (1.11) gives us a shock (u^L, u_3), followed by a rarefaction beginning with $u = u_3$ and ending at u^*, and finally a shock, with states (u^*, u^R).

We next consider some computational aspects of (1.10). Suppose we solve (1.1) for the previous $f(u)$, using Godunov's method on a fixed grid. Then formula (2.10) gives us the numerical flux, which we rewrite here

$$h^G(u_{j+1}, u_j) = \begin{cases} \displaystyle\min_{u_j \leqq u \leqq u_{j+1}} f(u) & \text{if } u_j < u_{j+1}, \\ \displaystyle\max_{u_{j+1} \leqq u \leqq u_j} f(u) & \text{if } u_j > u_{j+1}. \end{cases} \tag{4.1}$$

Thus one needs the following programming logic. If f is monotone between u_j and u_{j+1}, then choose the appropriate end point value. Otherwise scan the interval, find the value of $f(u)$ at critical points, then choose the appropriate critical, or end point, value.

We compare this with a scalar, monotone scheme developed by Engquist and the author [4], [5]. The scheme's numerical flux may be written

$$h^{E0}(u_{j+1}, u_j) = \frac{1}{2}\left[f(u_{j+1}) + f(u_j) - \int_{u_j}^{u_{j+1}} |f'(s)|\, ds \right] = f_-(u_{j+1}) + f_+(u_j), \tag{4.2}$$

where

$$\begin{aligned} f_-(u) &= \int_a^u \min(f'(s), 0)\, ds, \\ f_+(u) &= \int_a^u \max(f'(s), 0)\, ds, \end{aligned} \tag{4.3}$$

with a trivial normalization of $f(a) = 0$.

Again, if f is monotone between u_j, u_{j+1}, we choose the appropriate end point value. (In fact, $h^{E0} = h^G$, in this case.) Otherwise, scan the interval, record the value of f at critical points, and take the appropriate weighted sum.

In [17], van Leer gave an interesting geometric interpretation to the flux defined by (4.2), in the special case of convex f. We shall now generalize it to nonconvex f, e.g. the f in Fig. 2. We wish to compute $h^{E0}(u^R, u^L)$.

Suppose $u^R < u^3$. Then formula (4.2) gives

$$h^{E0}(u^R, u^L) = f(u^L) - f(u^1) + f(u^2) - f(u^3) + f(u^R). \tag{4.4}$$

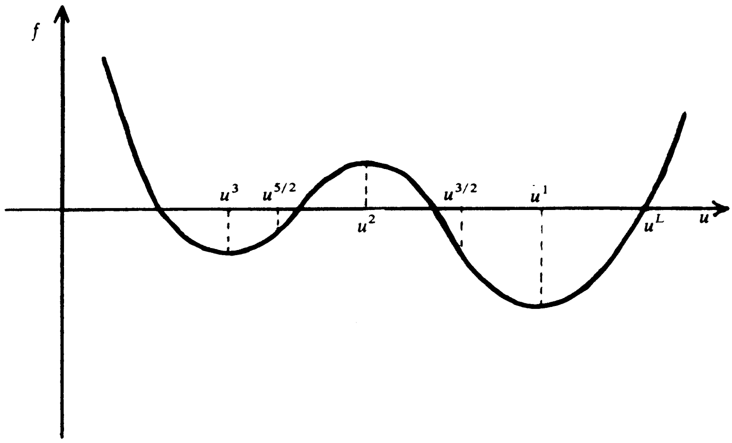

FIG. 2

We can also arrive at this formula from geometric considerations. We do this using multivalued solutions made of rarefaction and overturned compression waves. These are solutions, $w(x/t) = w(\zeta)$, of

$$\zeta = f'(w(\zeta)), \tag{4.5}$$

which may not be single valued. We assume f'' has only isolated zeros between u^L and u^R. (In the present case there is exactly one between each of the critical points, i.e. $u^{3/2}$ and $u^{5/2}$.)

Starting at

$$\zeta^L = f'(u^L), \qquad u(\zeta^L) = u^L,$$

we solve (4.5) and move ζ so that u decreases, i.e. goes from u^L to u^R. From (4.5), we see that

$$u'(\zeta) = \frac{1}{f''}(u(\zeta)),$$

so ζ decreases from $f'(u^L)$ to $f'(u^{3/2})$. We think of the resulting solution as multivalued. Starting with $u = u^L$ for $\zeta \geqq f'(u^L)$, it crosses $\zeta = 0$ at $u = u^1$. Since it does so in a right to left direction, we assign $-f(u^1)$ as the contribution to the numerical flux over this point. We add this to $f(u^L)$ which is the contribution from the constant state at $\zeta = 0$ (since $f'(u^L) > 0$). From $u^{3/2}$ to $u^{5/2}$, ζ increases and crosses 0 at $u = u^2$, so we add $f(u^2)$ to the numerical flux. Then at $u^{5/2}$, ζ begins to decrease, so we subtract $f(u^3)$ from our accumulating numerical flux. We finally connect to $u = u^R$, with $f'(u^R) < 0$, so we add $f(u^R)$ arriving at (4.4). The sum of contributions from this multivalued solution manifold is given by (4.4).

This illustrates the fact that the E–O scheme can be obtained by averaging a solution to the Riemann problem over a multivalued solution manifold which connects u^L to u^R, via compression and rarefaction waves only.

Acknowledgment. The author would like to thank Philip Roe for his constructive comments on a first draft of this paper.

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