On "jamitons," self-sustained nonlinear waves in traffic flow

Morris R. Flynn (U. Alberta)
Aslan R. Kasimov (KAUST)
R. Ruben Rosales (MIT)
Benjamin Seibold (Temple U.)

KAUST
King Abdullah University of Science and Technology

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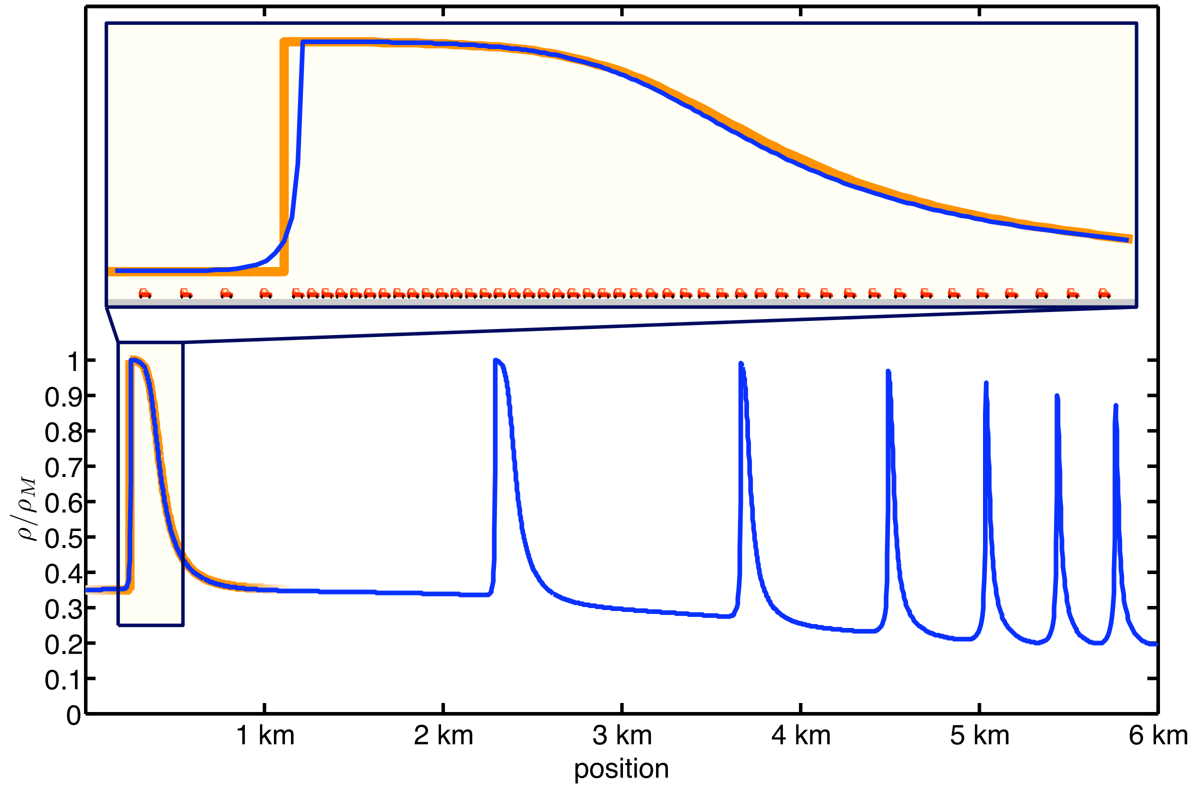

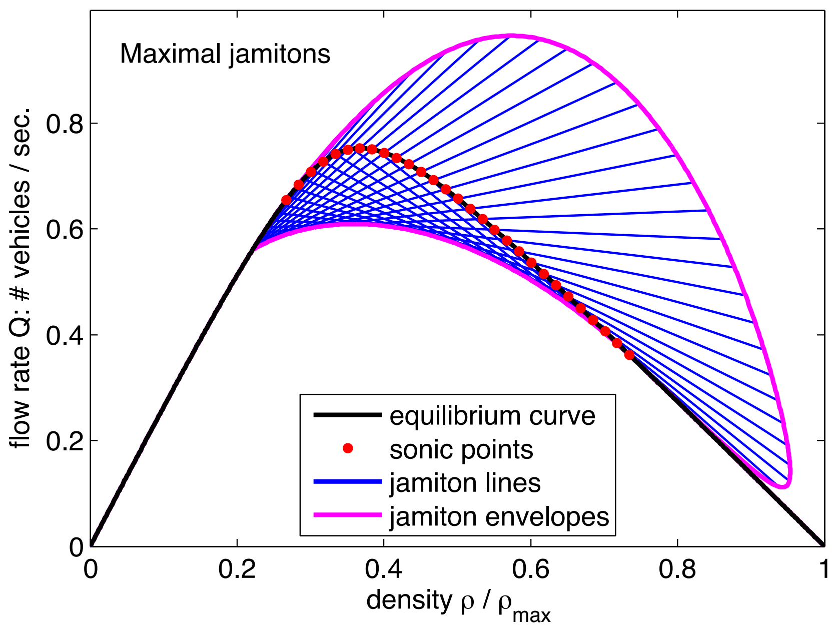

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Introduction

- Traffic jams are an aggravating feature of modern transportation systems that account for significant productivity losses
- Some traffic jams have as their root cause construction, lane closures, accidents, etc. while others ("phantom jams") arise for no apparent reason

Movie removed -- please see http://www.trafficforum.org/stopandgo

Phantom jams on a Cairo expressway. Movie credit: Dirk Helbing (http://www.trafficforum.org/stopandgo)

Observations from Helbing movie

- "Phantom jam" formation: characterized by inhomogeneities to initially uniform traffic that in turn evolve into full blown nonlinear waves

Properties:

- Vehicles run into a front, brake heavily then gradually speed up again
- Vehicles are always faster than the waves, which can either propagate upstream or downstream w.r.t. a stationary observer
- Waves are usually 5-15 vehicles wide, although this depends on the particulars of the road conditions

Q? Can we model phantom jams in a way that is physically illuminating?

Macroscopic traffic models

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Will address this question using a macroscopic traffic model with a heuristically-obtained equation of state

Road	$x \,\epsilon\, [0,\, \lambda]$
Traffic density	$\rho(x,\, t)$
Traffic speed	$u(x,\, t)$

Mass continuity equation "change equals inflow minus outflow":

$$\rho_t + (\text{flux})_x = 0 \qquad \text{flux} = \rho u$$

Momentum equation:

1. Algebraic relation: $u = u(\rho)$ — "First order" model ("Lighthill-Whitham-Richards" model)
2. Differential relation: $u_t + u u_x = \ldots$ — "Second order" model ("Payne-Whitham" or "Aw-Rascle-Zhang" model)

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Payne-Whitham (PW) model

- We start by examining the Payne-Whitham (PW) model, then discuss the Aw-Rascle-Zhang model; note that the order of presentation is not especially important because our method of construction is generic
- PW governing equations read

$$\rho_t + u\rho_x + \rho u_x = 0 \qquad (1)$$

$$u_t + uu_x + \frac{p_x}{\rho} = \frac{1}{\tau}(\tilde{u} - u) \qquad (2)$$

Traffic "pressure": $p \sim \rho^\gamma \quad \gamma \geq 1$

Equilibrium speed: $\tilde{u}$ a decreasing function of ρ

In contrast to Kerner & Konhauser (1993), Kurtze & Hong (1995), etc. our momentum equation does not include a (nebulous) viscous dissipation term proportional to u_{xx}. This facilitates finding traveling wave solutions...

Traveling wave solution to PW model

$$\rho_t + u\rho_x + \rho u_x = 0 \qquad (1)$$

$$u_t + uu_x + \frac{p_x}{\rho} = \frac{1}{\tau}(\tilde{u} - u) \qquad (2)$$

Traveling wave ansatz: $\eta = \dfrac{x - st}{\tau}$ s -- wave speed

Mass equation: $\{\rho(u-s)\}_\eta = 0 \quad \Rightarrow \quad \rho = \dfrac{m}{u-s}$ (3)

m -- mass flux of vehicles in the traveling frame of reference

(3)→(2): $$\frac{\mathrm{d}u}{\mathrm{d}\eta} = \frac{(u-s)(\tilde{u}-u)}{(u-s)^2 - c^2}$$

$c = (p_\rho)^{1/2}$ -- sound speed (Whitham 1974)

Traveling wave solution to PW model

$$\boxed{\frac{\mathrm{d}u}{\mathrm{d}\eta} = \frac{(u-s)(\tilde{u}-u)}{(u-s)^2 - c^2}}$$

- First order ODE
- Barring any pathological choices for $\tilde{u}$ and $c = (p_\rho)^{1/2}$ we expect to observe monotone solutions connected by shocks

- Rankine-Hugoniot conditions:

$$s[\rho] = [\rho u] \qquad \text{Mass continuity}$$
$$s[\rho u] = [p + \rho u^2] \qquad \text{Momentum continuity}$$

- Latter Rankine-Hugoniot condition can be defended in case of fluid flow but is incorrect, strictly speaking, when considering vehicles

- Lax entropy condition: in order to be dynamically stable, shocks must be compressive in the moving reference frame (LeVeque 1992)

- We focus on "left shocks" for which vehicles overtake the shock, not vice-versa (has mathematical and physical justification)

Left shock

- Left shock connects a supersonic to a subsonic state, i.e.

$$u^- - s > c^-, \qquad u^+ - s < c^+$$

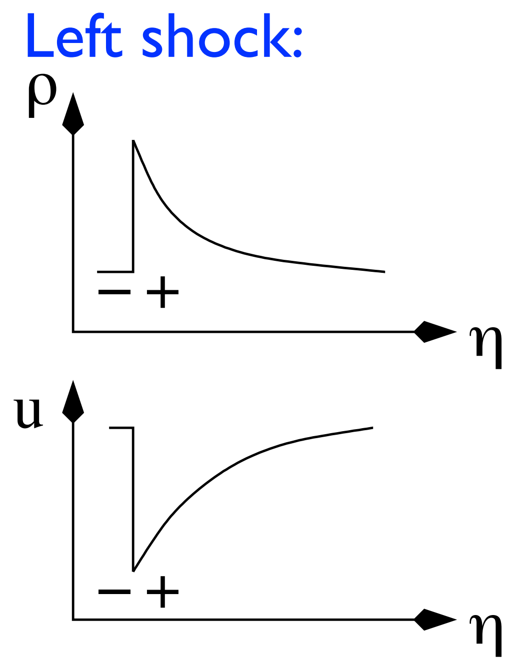
- ODE on u reads

$$\frac{\mathrm{d}u}{\mathrm{d}\eta} = \frac{(u-s)(\tilde{u}-u)}{(u-s)^2 - c^2}$$

so there is a unique point where the denominator vanishes

- In order to regularize the problem, we insist that the numerator exhibits a coincident root of equal order, i.e.

$$u = \tilde{u} \quad \text{when} \quad u - s = c$$

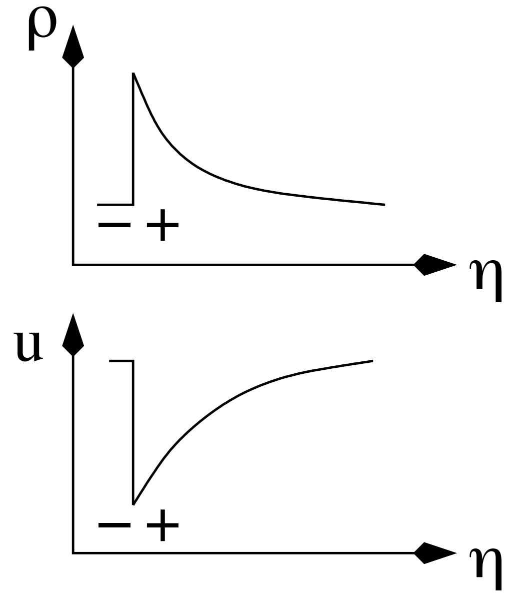


Chapman-Jouguet condition

$$\boxed{u = \tilde{u} \quad \text{when} \quad u - s = c}$$

- Equivalent to the Chapman-Jouguet condition from detonation theory (Fickett & Davis 1979) and is applied in determining the wave speed, s
- $u - s = c$ defines the sonic point, the information event horizon where the wave-adjusted traffic speed matches the speed of infinitesimal disturbances
- The traveling wave is self-sustained because the shock cannot "see" smooth disturbances that arise beyond the sonic point. This lack of foresight helps to explain why phantom jams are difficult to dissipate

Left shock:



Algorithm (periodic roadway)

- Guess a value for (ρ^-, u^-)

- Find wave speed, s, from Chapman-Jouguet condition:

$$u = \tilde{u} \quad \text{when} \quad u - s = c$$

- Find (ρ^+, u^+) from the Rankine-Hugoniot conditions:

$$s[\rho] = [\rho u]\,, \qquad s[\rho u] = [p + \rho u^2]$$

- Integrate the ODE

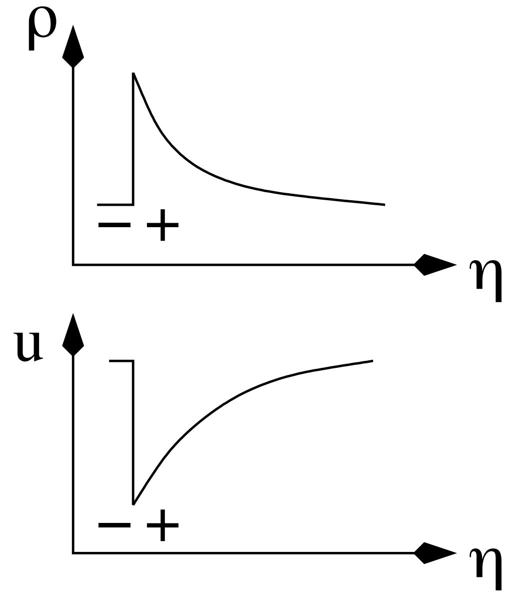
$$\frac{\mathrm{d}u}{\mathrm{d}\eta} = \frac{(u-s)(\tilde{u}-u)}{(u-s)^2 - c^2}$$

forward in η starting from u^+

- Once $u = u^-$ this defines the length of the periodic roadway. The total number of vehicles can then be computed from

$$\mathcal{N} = \int_0^\lambda \rho \,\mathrm{d}x$$

Left shock:



PW model example

- Let's consider solution behavior with:

$$\tilde{u} = \tilde{u}_0 \left(1 - \frac{\rho}{\rho_M}\right)$$ Equilibrium speed

$$\frac{p_x}{\rho} = \frac{\beta \rho_x}{\rho_M - \rho}, \qquad \beta > 0$$ Traffic pressure

- Traffic pressure: singularity keeps traffic density below maximum density; p an increasing function of ρ with $\rho \times p$ convex

- Will consider analytical* and numerical solutions

- Numerical details: SPH "mesh-free" algorithm where the number of particles is much larger than the number of vehicles (macroscopic, not a microscopic, description, albeit one with an intuitive link between the particle and vehicle densities) -- Monaghan (1988)

* (Implicit) solution is messy! See Flynn et al. (2009) for details

Model solutions

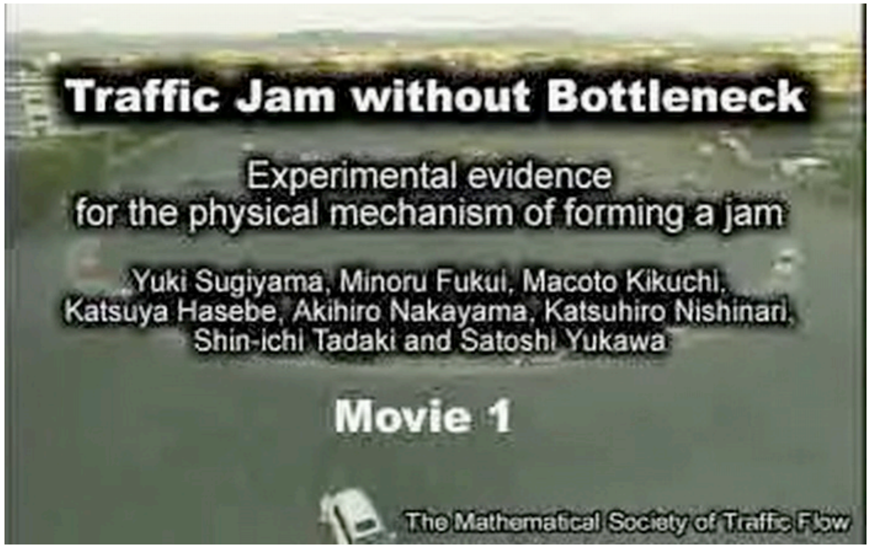
Movie removed -- please see http://math.mit.edu/projects/traffic

18 vehicles on a periodic road of length 230 m.

- Good agreement between theory and numerics
- $s = 0$ i.e. jam is stationary

Q? Can we make a more explicit comparison with observational data?

Experiment of Sugiyama et al. (2008)



Youtube: http://www.youtube.com/watch?v=Suugn-p5C1M

Model solutions

Movie removed -- please see http://math.mit.edu/projects/traffic

22 vehicles on a periodic road of length 230 m as in Japanese experiment.

In good agreement with the observations, we see a backward-propagating density wave about 6-8 vehicles thick

Q? Do shocks arise with fewer vehicles along the roadway?

Movie credit: J.-C. Nave (McGill U.)

Model solutions

Movie removed -- please see http://math.mit.edu/projects/traffic

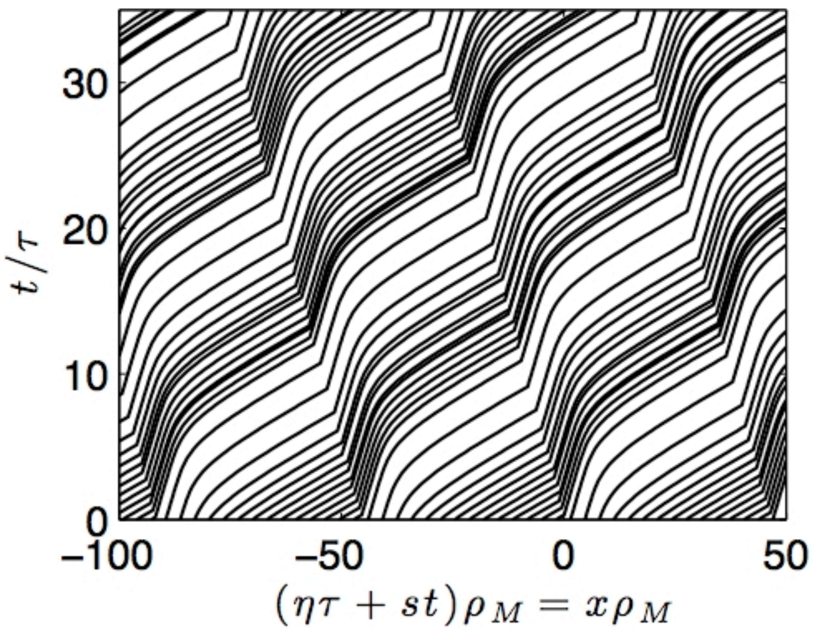
14 vehicles on a periodic road of length 230 m.

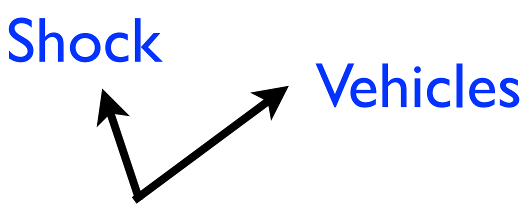
With a lower traffic density, the shock now travels in the same direction as individual vehicles

Movie credit: J.-C. Nave (McGill U.)

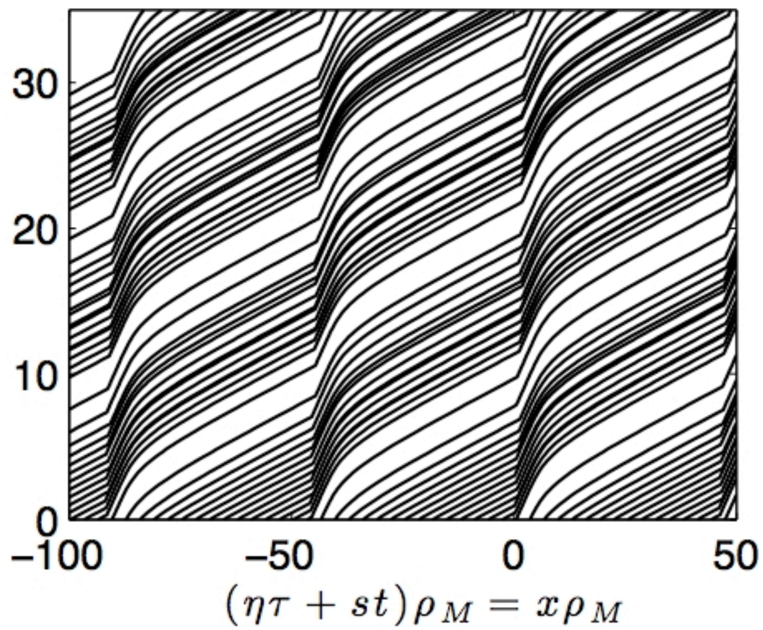
Vehicle trajectories

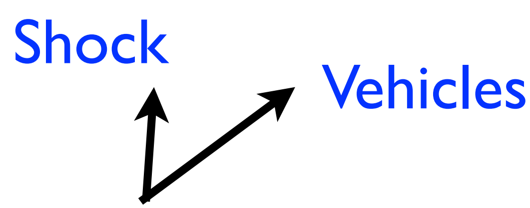
High traffic density (22 vehicles)





Lower traffic density (16 vehicles)





Infinite road

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Sequence of waves is observed c.f. roll waters in gutters on rainy days (Balmforth & Mandre 2004).

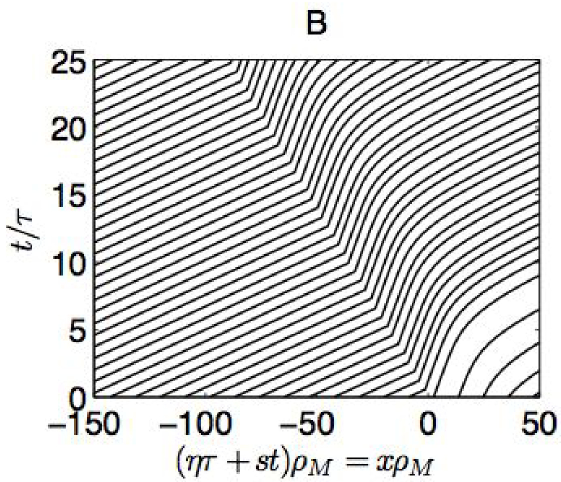
Q? What determines the spacing between adjacent density peaks? Is spacing bounded in long time limit?

Example -- infinite road

Movie removed -- please see http://math.mit.edu/projects/traffic

Movie credit: J.-C. Nave (McGill U.)

Vehicle trajectories (infinite road)

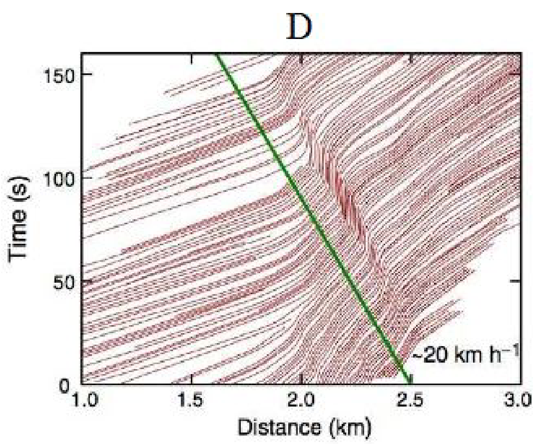


Theoretical vehicle trajectories for:

$$(\rho_-/\rho_M,\, u_-/\tilde{u}_0) = (0.35,\, 0.65)$$

Shock

Vehicles



Vehicle trajectories measured by aerial photographs of a congested freeway by J. Treiterer (1974)

Aw-Rascle-Zhang model

- Discussion till now has considered the Payne-Whitham model, but one may instead consider the Aw-Rascle-Zhang (ARZ) model whereby the momentum equation is modified, i.e.

$$u_t + uu_x + \frac{p_x}{\rho} = \frac{1}{\tau}(\tilde{u} - u)$$

Payne-Whitham model

$$(u+p)_t + u(u+p)_x = \frac{1}{\tau}(\tilde{u} - u)$$

Aw-Rascle-Zhang model

- Apply convective rather than spatial derivative to traffic "pressure," which now has units of speed, not vehicles/s^2

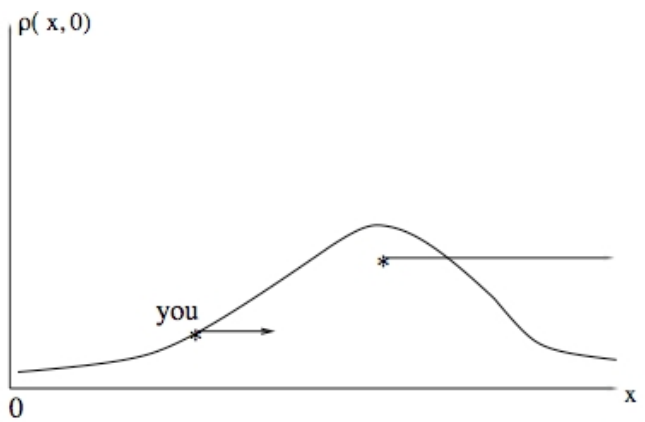


FIG. 1.1. *With such dense and fast traffic in front of you, would you brake, or accelerate?*

Payne-Whitham: "brake"

Aw-Rascle-Zhang: "accelerate"

Aw & Rascle (2001), figure 1.1

Aw-Rascle-Zhang model

- Notwithstanding this fundamental difference, the traveling wave solutions admitted by the two model types are similar to one another

Payne-Whitham (PW) model

ODE:

$$\frac{\mathrm{d}u}{\mathrm{d}\eta} = \frac{(u-s)(\tilde{u}-u)}{(u-s)^2 - c^2}$$

Rankine-Hugoniot condition #2:

$$s[\rho u] = [p + \rho u^2]$$

Aw-Rascle-Zhang model

ODE:

$$\frac{\mathrm{d}u}{\mathrm{d}\eta} = \frac{(u-s)(\tilde{u}-u)}{(u-s)^2 - mc^2}$$

Rankine-Hugoniot condition #2:

$$s[\rho(u+p)] = [\rho u p + \rho u^2]$$

- An advantage of Aw-Rascle-Zhang is that it avoids shocks overtaking individual vehicles. However, and although vehicles can overtake shocks in PW model, this odd behavior does not occur in context of PW traveling wave solutions

Return briefly to first order model

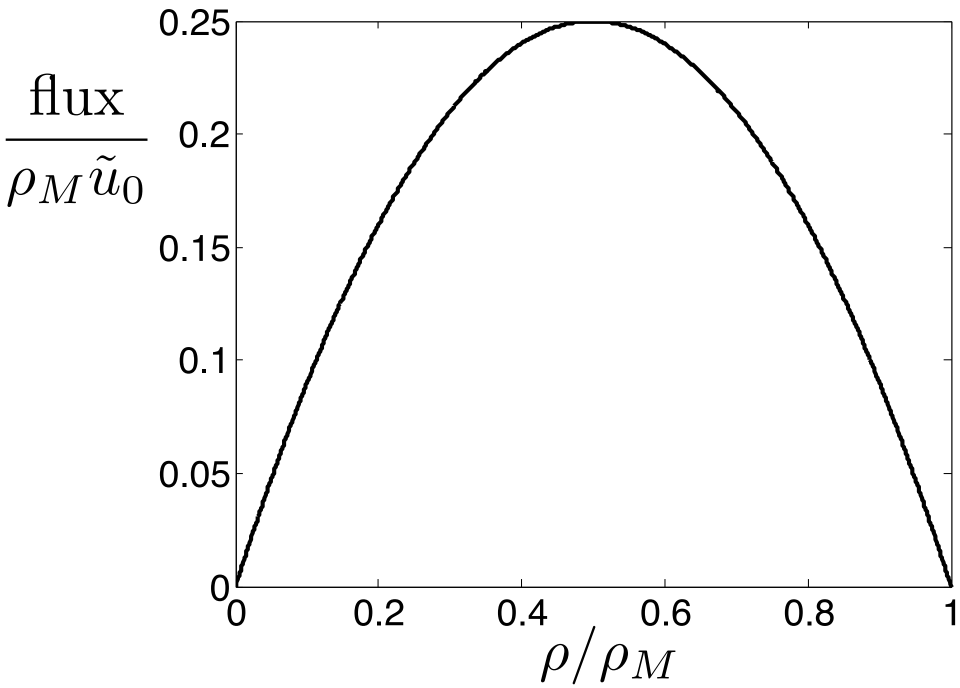
- Lighthill-Whitham-Richards model assumes an algebraic relationship between traffic speed and traffic density, i.e.

$$\rho_t + (\text{flux})_x = 0 \qquad \text{flux} = \rho u$$

$$u = u(\rho) \qquad \text{e.g.} \qquad u = \tilde{u}_0 \left(1 - \frac{\rho}{\rho_M}\right)$$

$$\Rightarrow \qquad \text{flux} = \rho \tilde{u}_0 \left(1 - \frac{\rho}{\rho_M}\right)$$

Parabolic flux function



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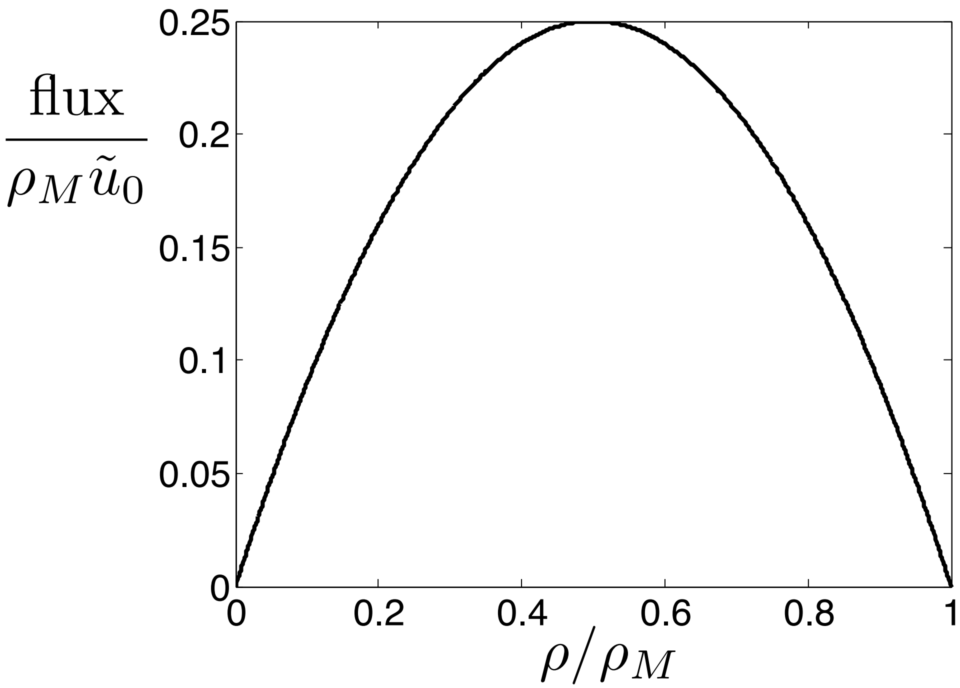


Is the shape of the *equilibrium curve* in this *fundamental diagram* reflected in measurements of real traffic flow?

Fundamental diagram

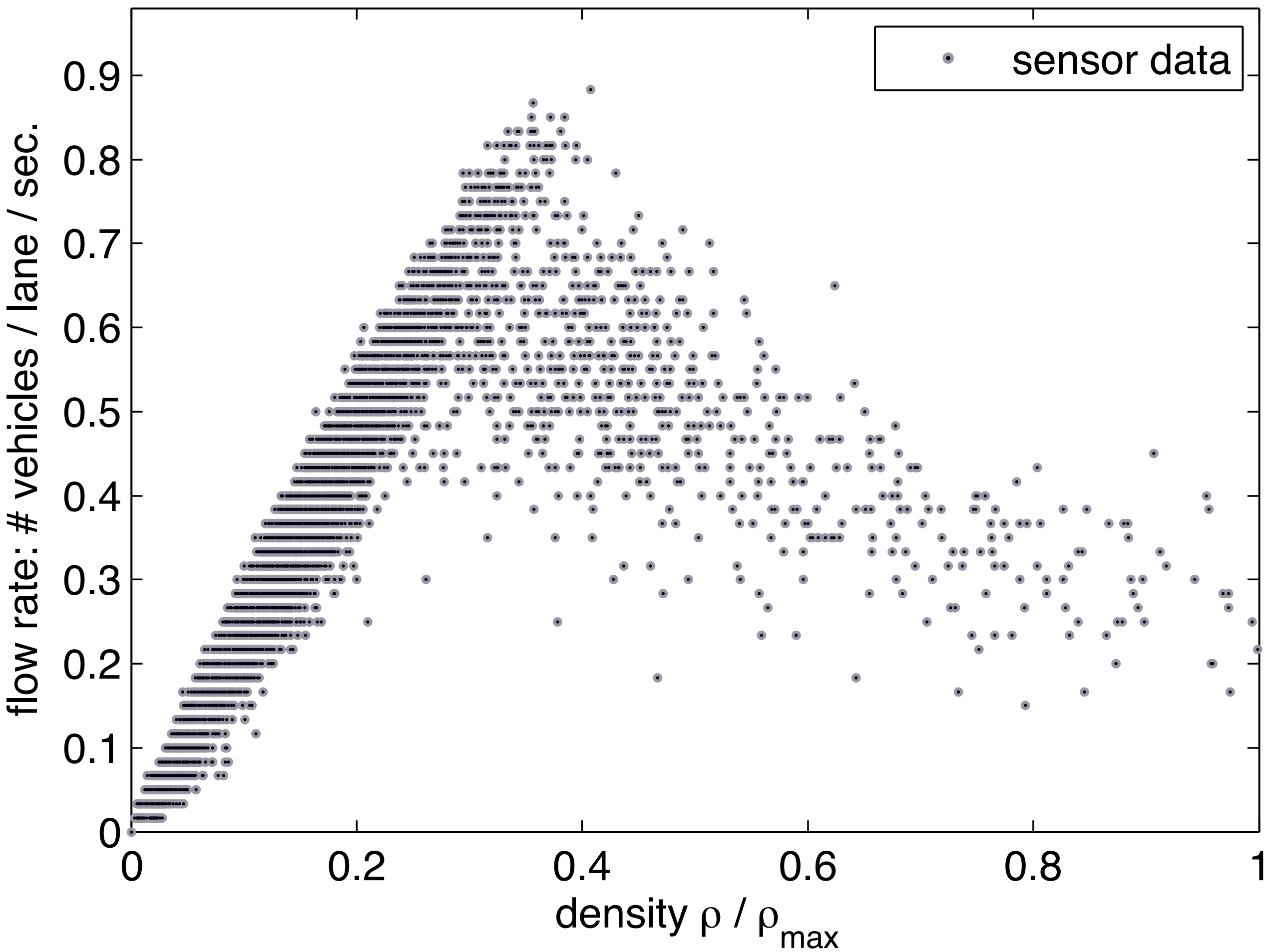
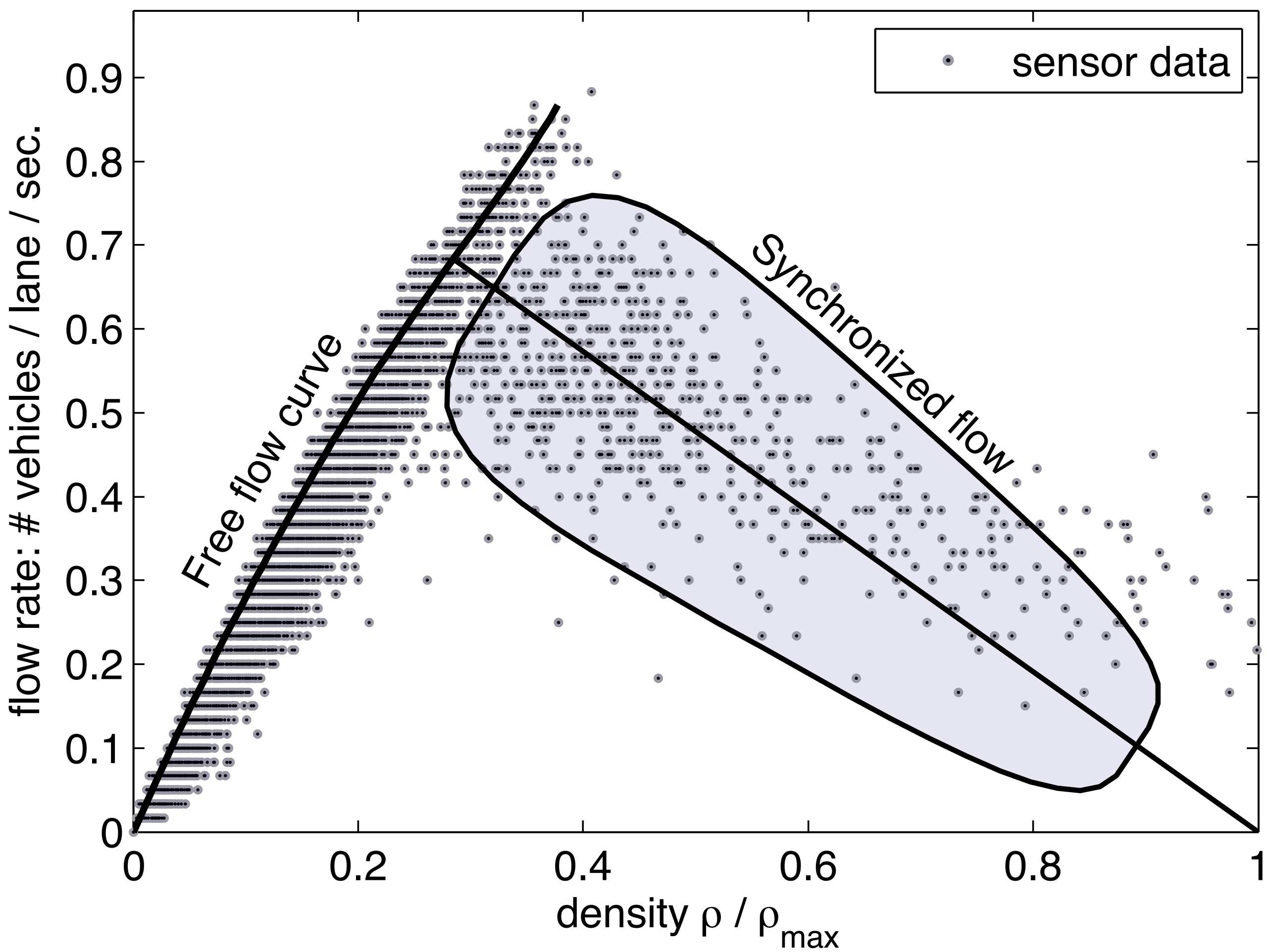


Figure from Seibold et al. (2012); data from Minnesota DOT

Fundamental diagram



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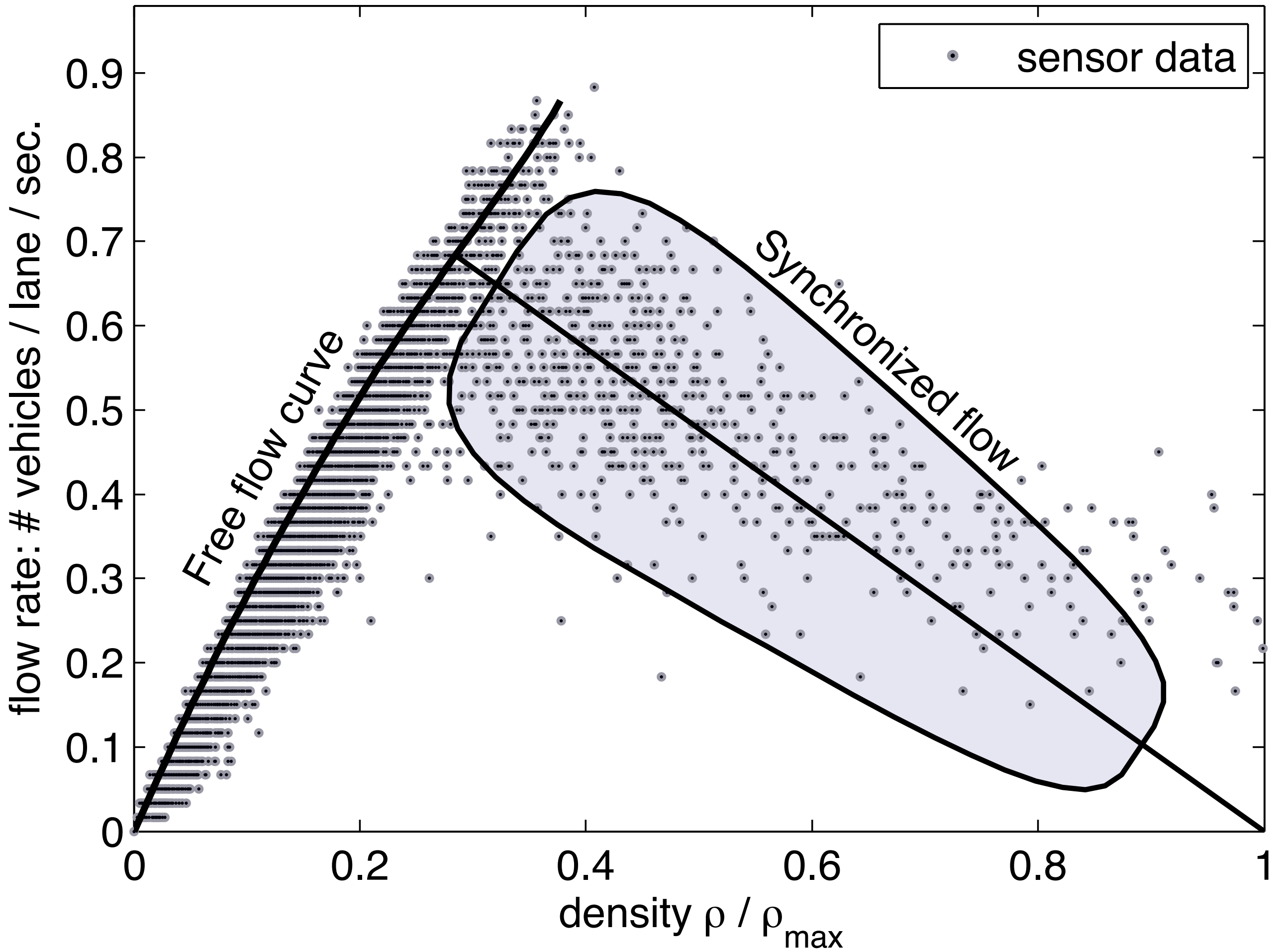
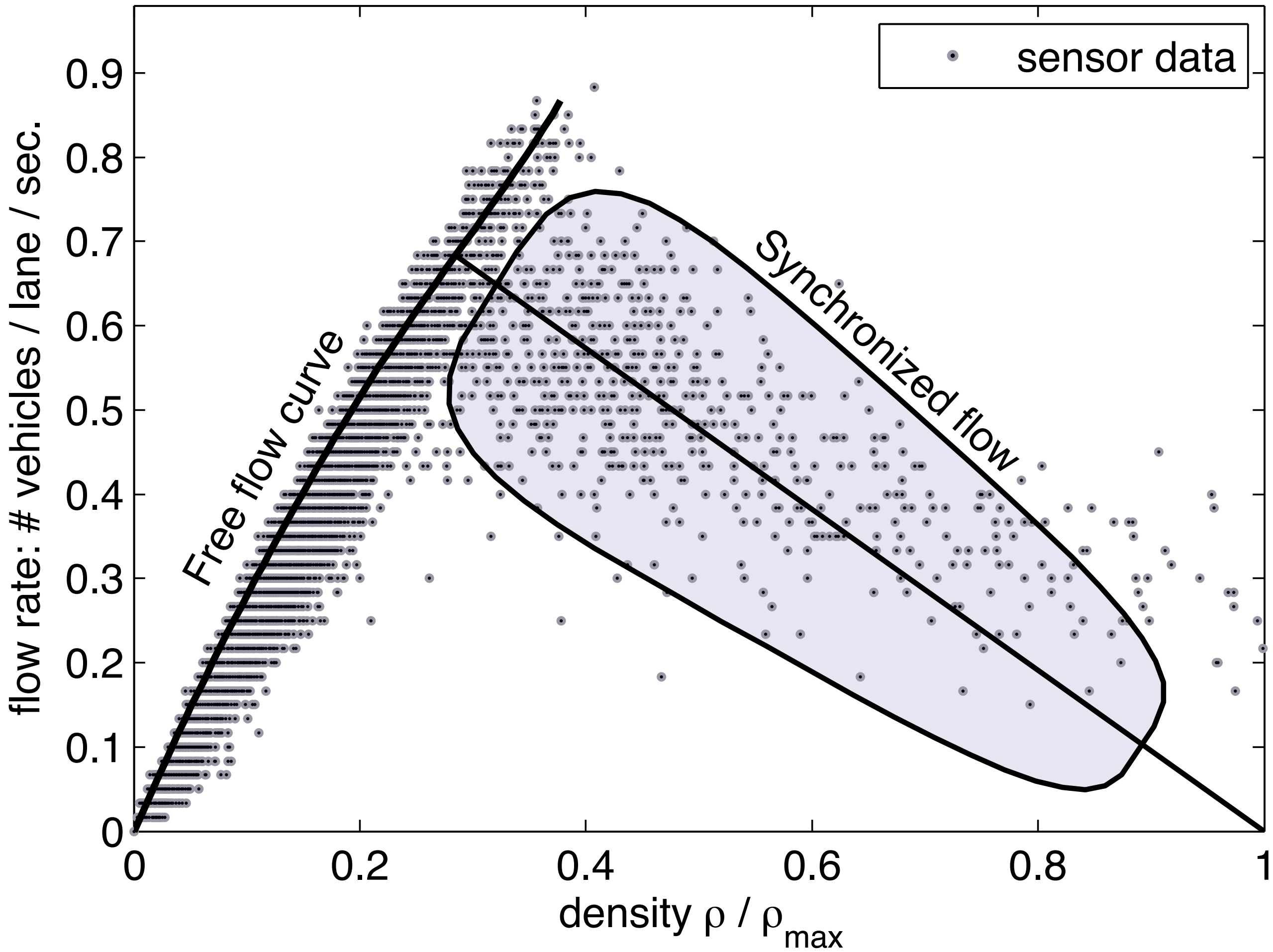


Do the previously studied "jamitons" i.e. self-sustained traveling wave solutions help to explain the scatter in the synchronized flow regime?

Figure from Seibold et al. (2012); data from Minnesota DOT

Fundamental diagram



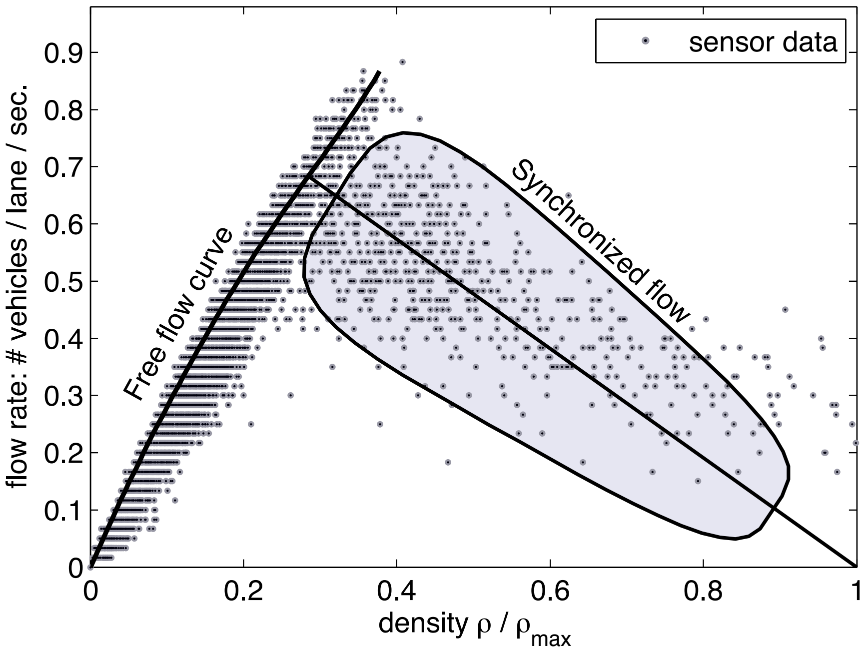
Do the previously studied "jamitons" i.e. self-sustained traveling wave solutions help to explain the scatter in the synchronized flow regime?

Answer: Yes!

Figure from Seibold et al. (2012); data from Minnesota DOT

Scatter in fundamental diagram

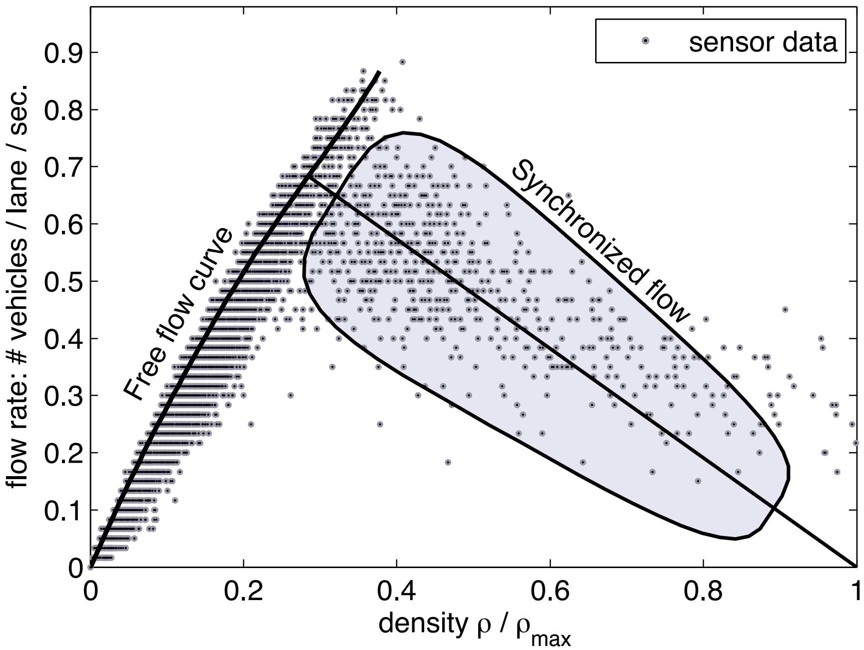
Qualitative rationale:

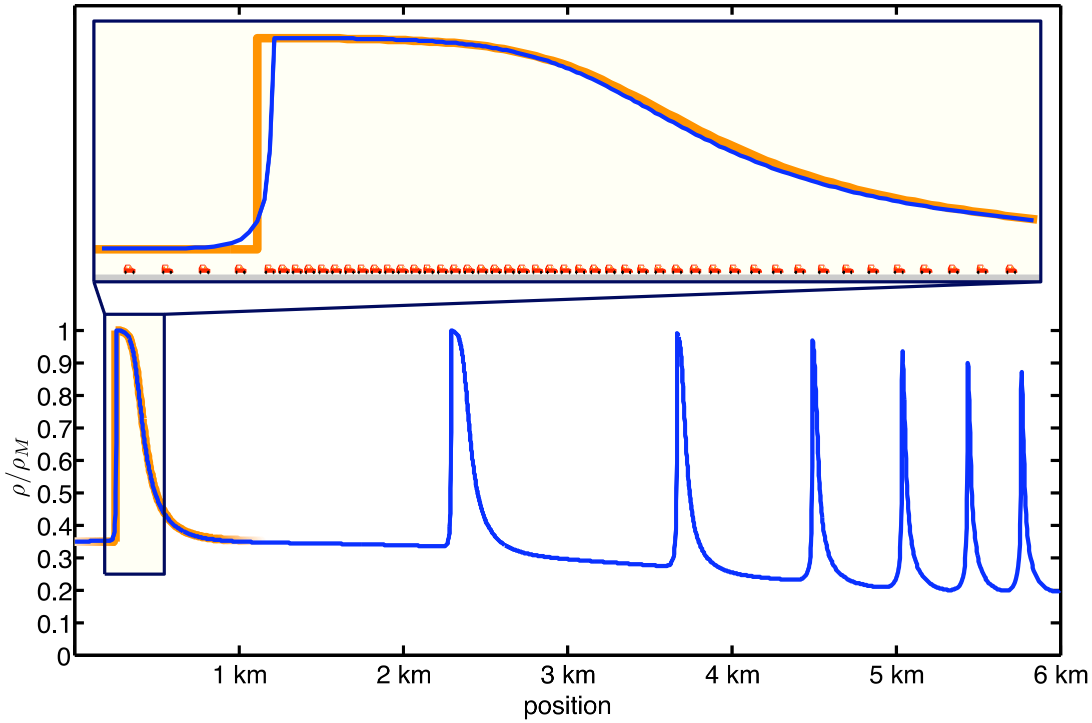


Flow is unstable for intermediate traffic densities.

Scatter in fundamental diagram

Qualitative rationale:





Flow is unstable for intermediate traffic densities. Jamitons can arise anywhere that linear stability condition fails

Different segments of the jamiton admit different relationships between flux and density

Jamitons and the fundamental diagram

Quantitative details:

Mass continuity: $\rho_t + u\rho_x + \rho u_x = 0$ $\Rightarrow$ $u = \dfrac{m}{\rho} + s$

$\Rightarrow$ $\text{flux}, Q = m + \rho s$

Jamitons and the fundamental diagram

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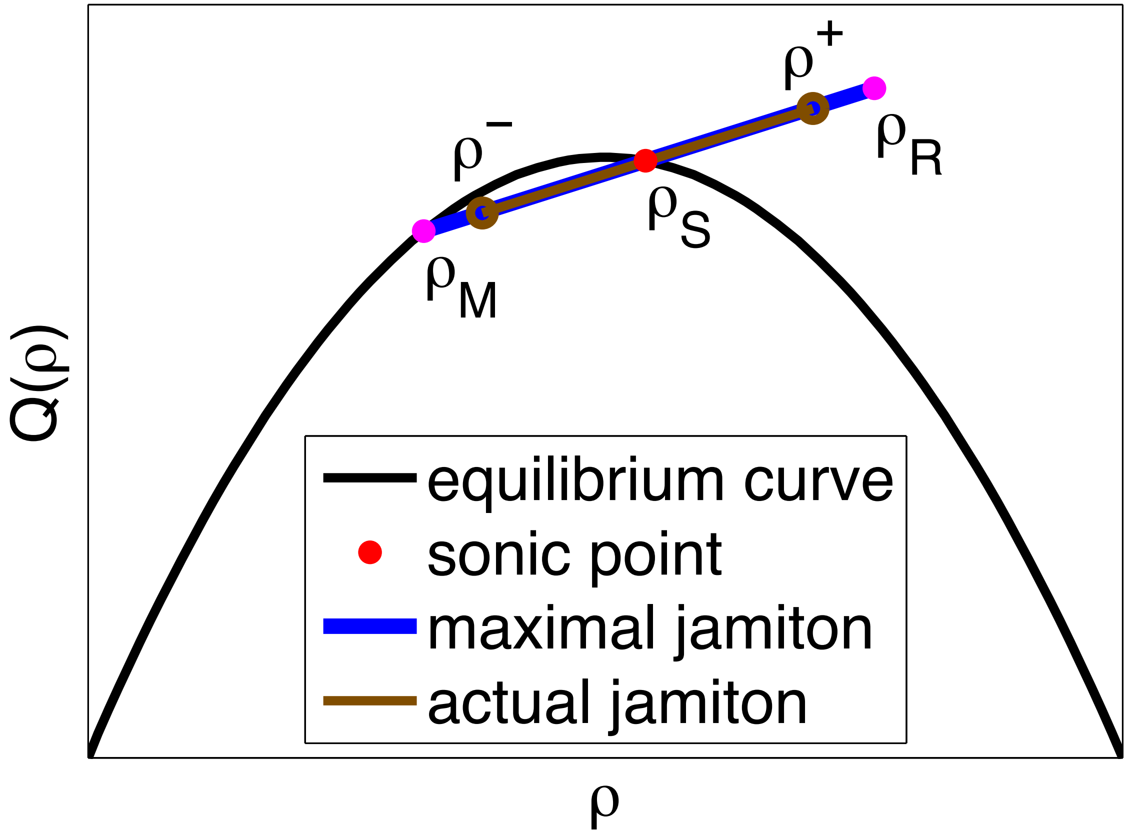

Boxed equation is a straight line in the plane of the fundamental diagram with slope s and intercept m

m -- mass flux (veh./s)

s -- shock speed (m/s)

Chapman-Jouget condition requires that line intersect equilibrium curve at the sonic point

Jamitons and the fundamental diagram

Quantitative details:

Mass continuity: $\rho_t + u\rho_x + \rho u_x = 0 \quad \Rightarrow \quad u = \dfrac{m}{\rho} + s$

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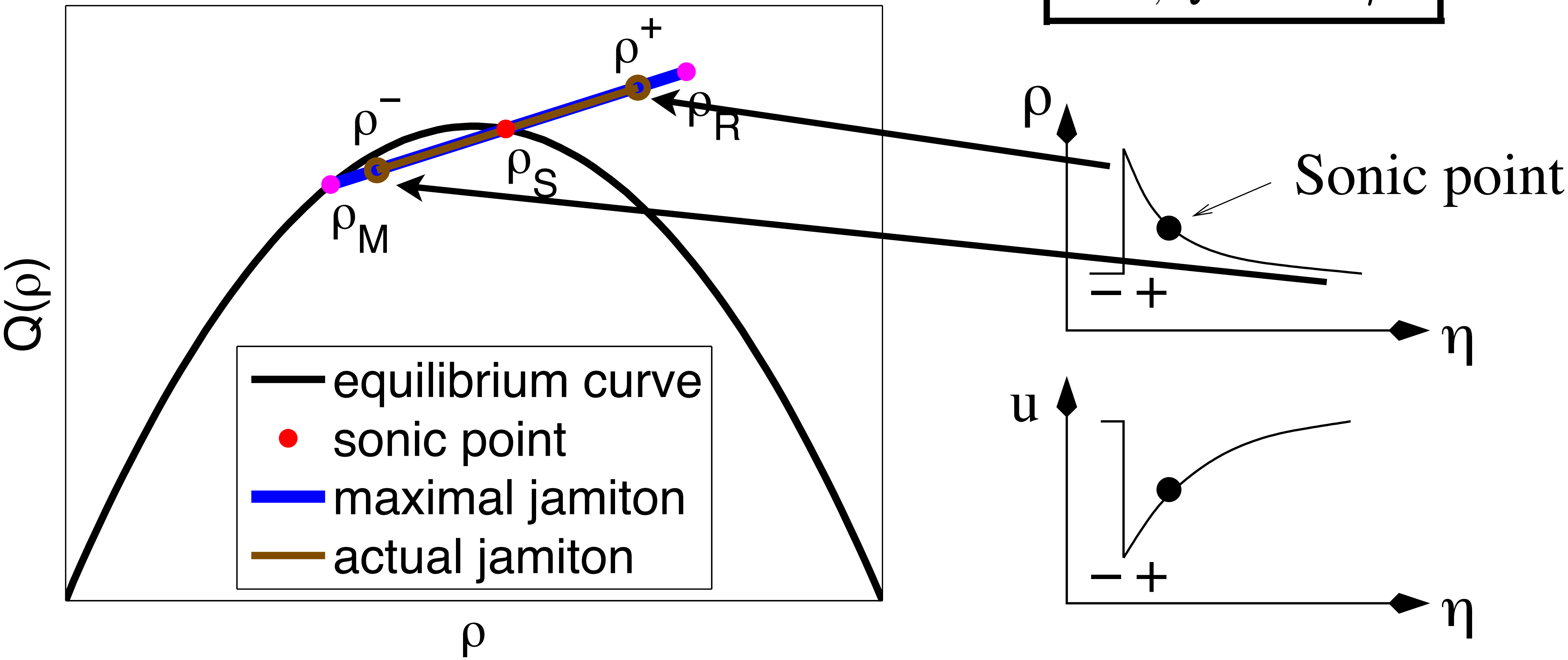


Jamitons and the fundamental diagram

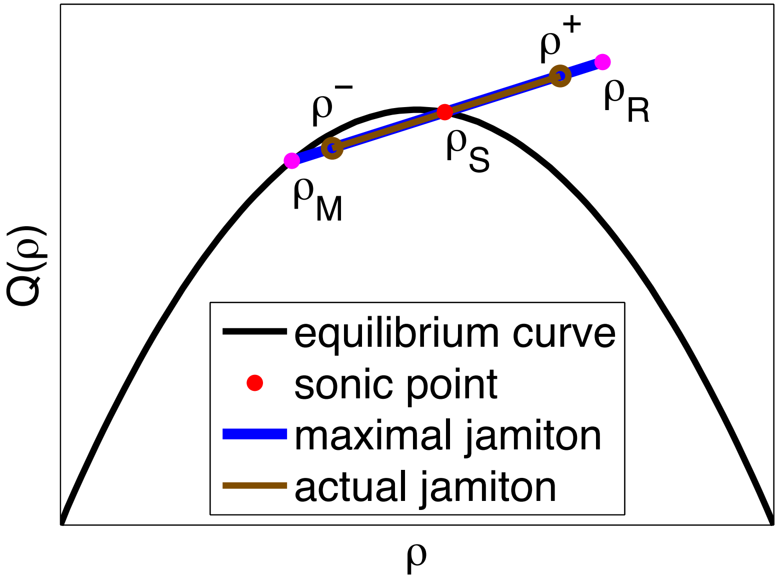
$$\boxed{\text{flux}, Q = m + \rho s}$$

- Mass flux, m, is determined from the sonic point density, ρ_S, and the definition of the sonic point
- Shock speed, s, is determined from m and the Chapman-Jouget conditions

1. m and s are not independent parameters
2. Lines are parameterized by ρ_S i.e. as sonic density changes, so too will the slope and intercept of the lines defined by boxed equation

Jamitons and the fundamental diagram

- Considering all possible values of the sonic point density in the linearly unstable regime, we replace



Jamitons and the fundamental diagram

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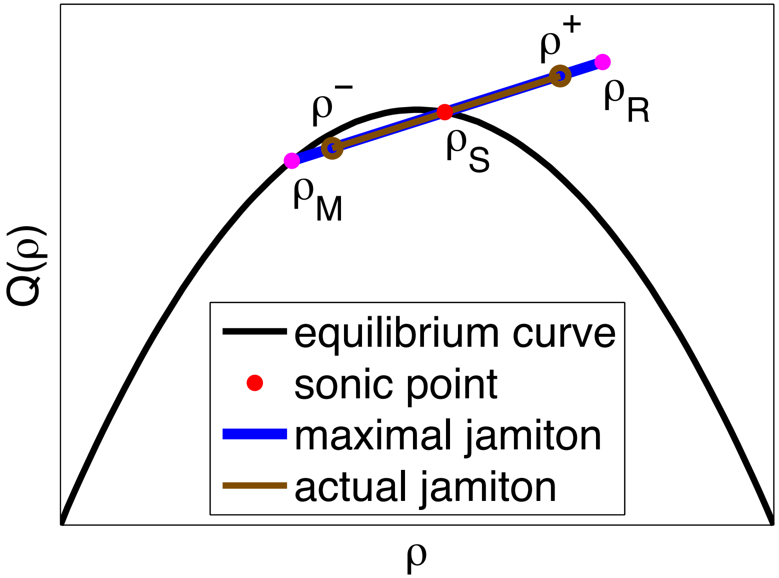
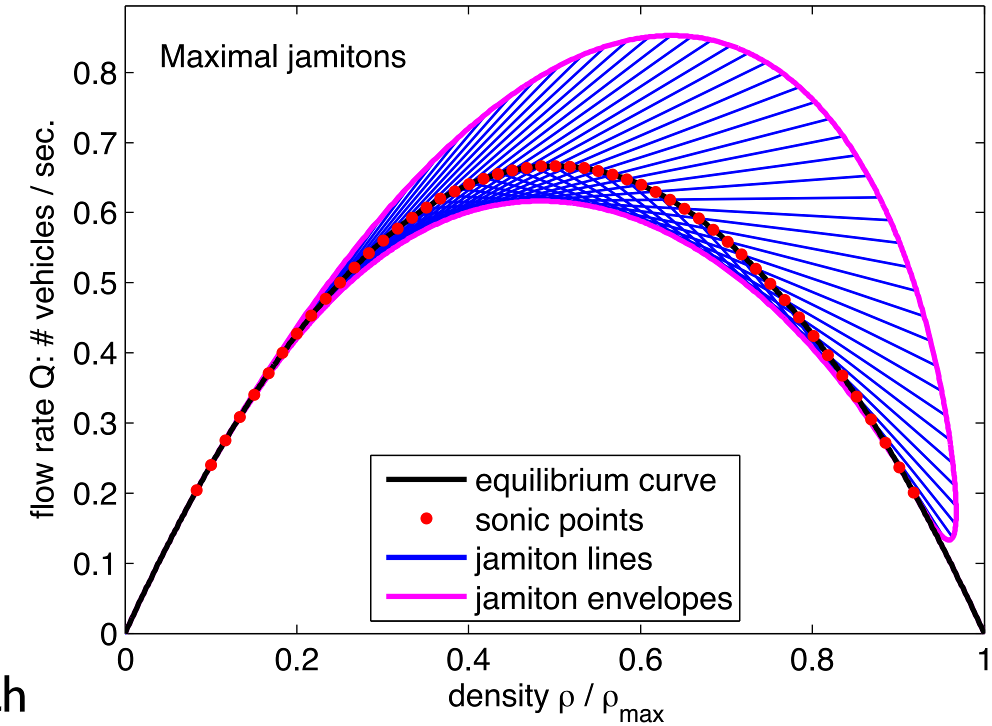
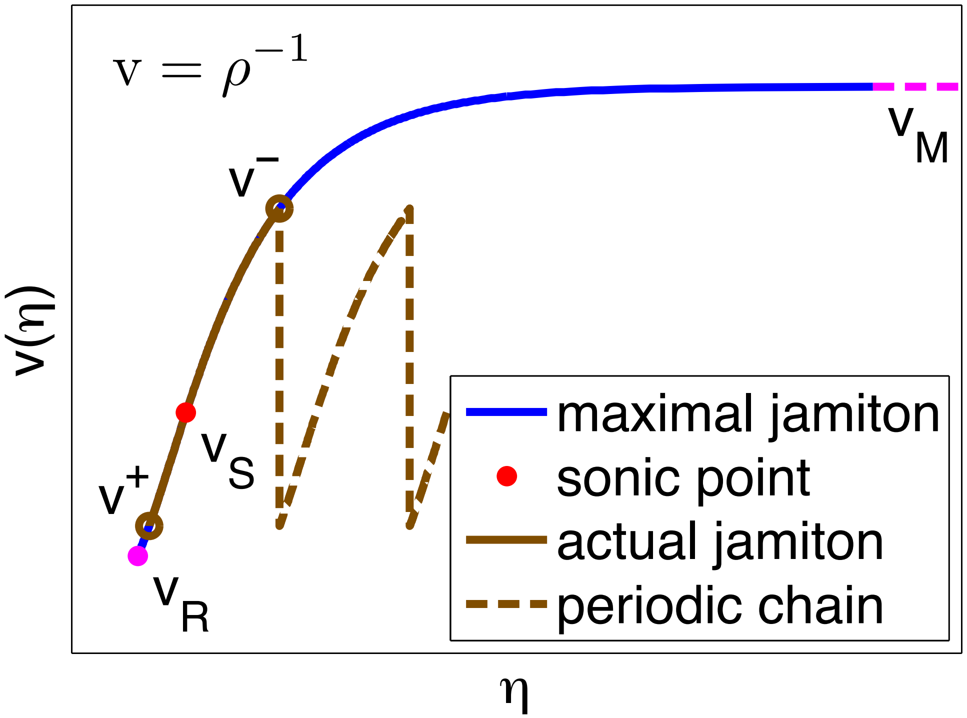


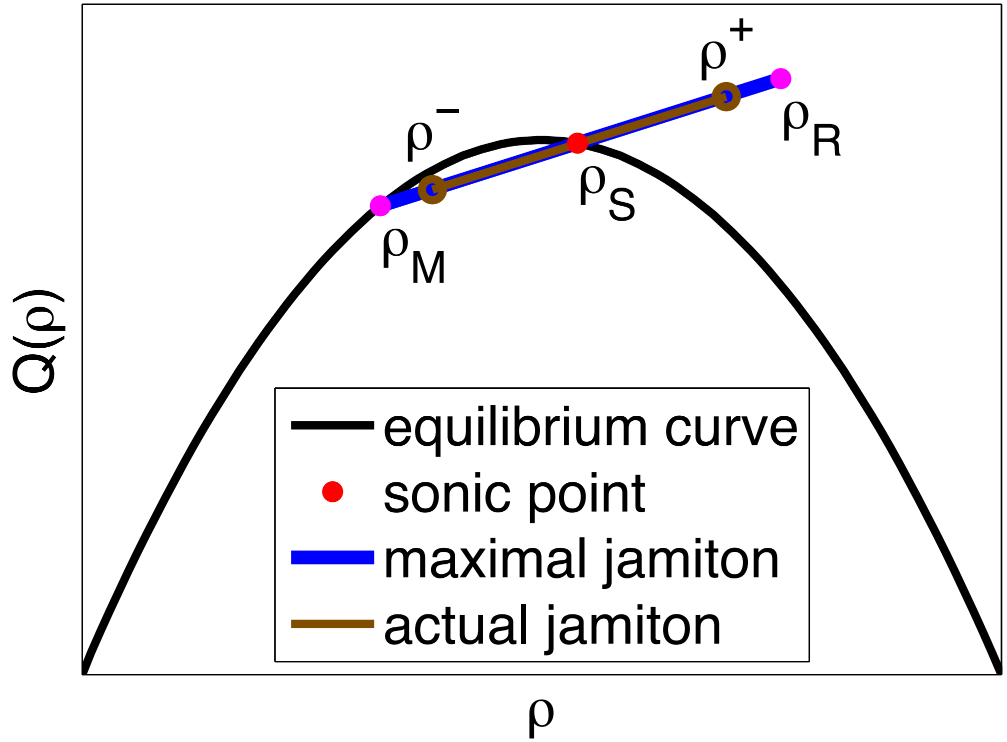
with



Calculation details: Payne-Whitham model with "usual" equilibrium speed and singular traffic pressure -- see Seibold et al. (2012)

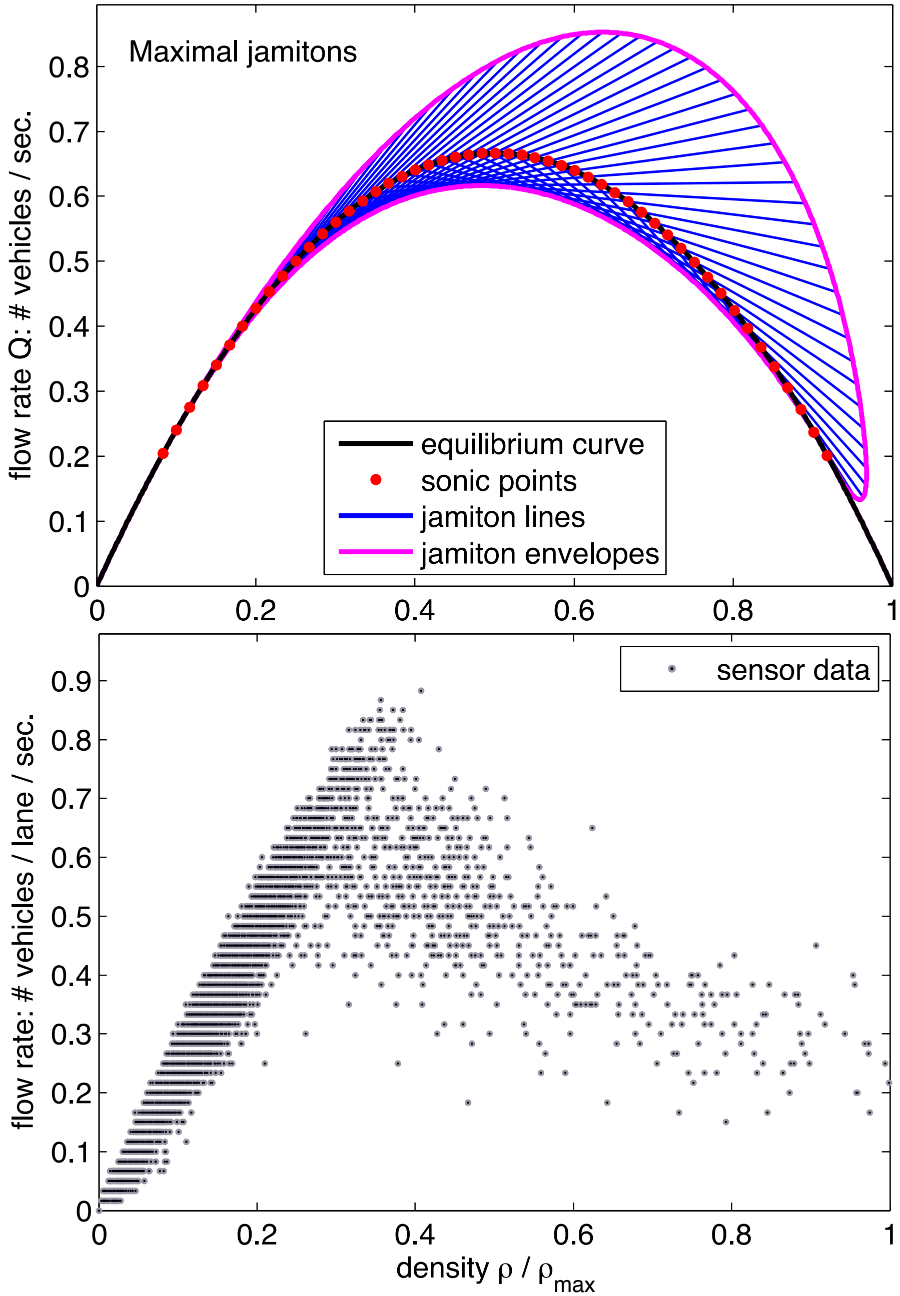
Maximal vs. non-maximal jamitons





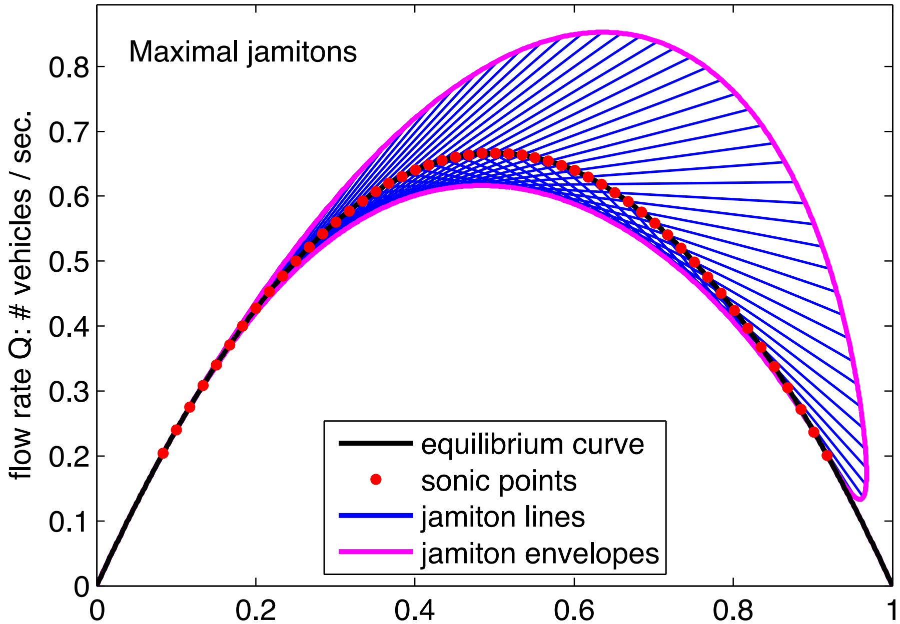
- Blue curves correspond to jamitons that have infinitely long tails
- Correspondingly, blue line in fundamental diagram coincides with equilibrium curve at *two points*
- Real jamitons (brown curves) don't, of course, extend this far and so only coincide with the equilibrium curve once, i.e. at the sonic point

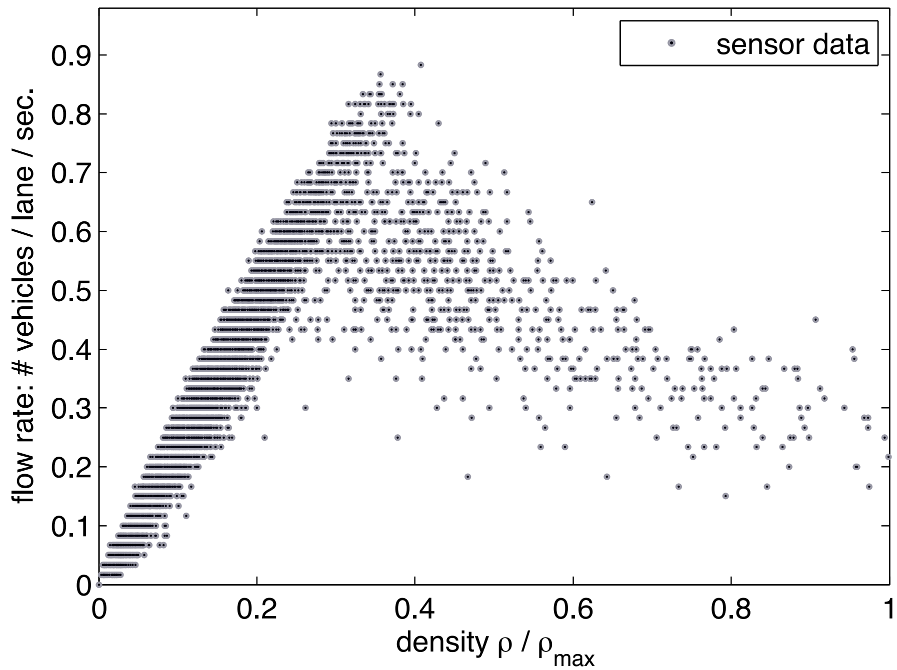
Comparison with measurement 1



- Data points can fall anywhere between the magenta curves
- Envelope defined by magenta curves is broadest for intermediate ρ, vanishingly small when traffic density is small or large
- However, comparison with measured data shows only qualitative agreement

Comparison with measurement 1



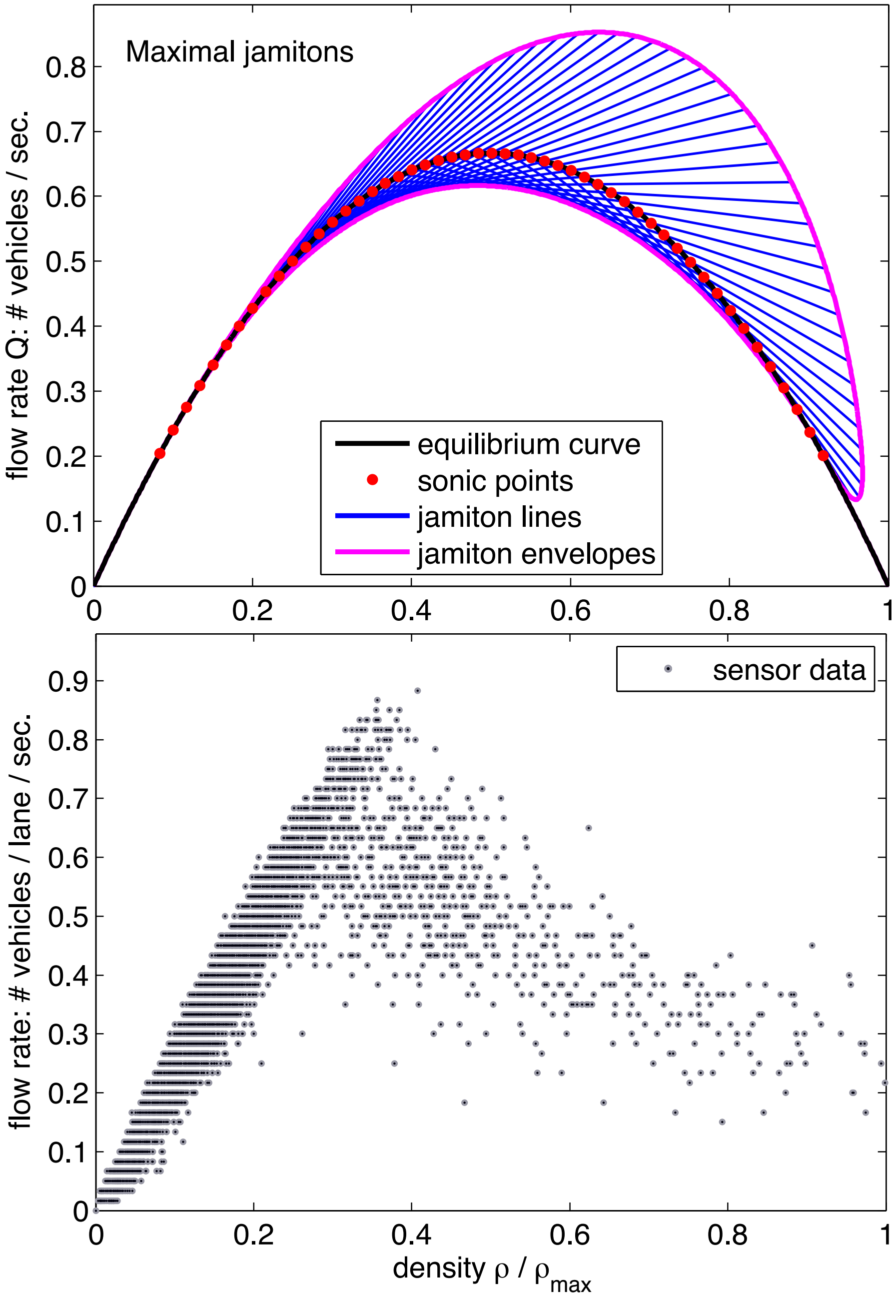


- However, comparison with measured data shows only qualitative agreement

1. Sample theoretical data in a manner consistent with the way real measurements are made

2. Use more realistic expression for equilibrium speed, $\tilde{u}$

Comparison with measurement I



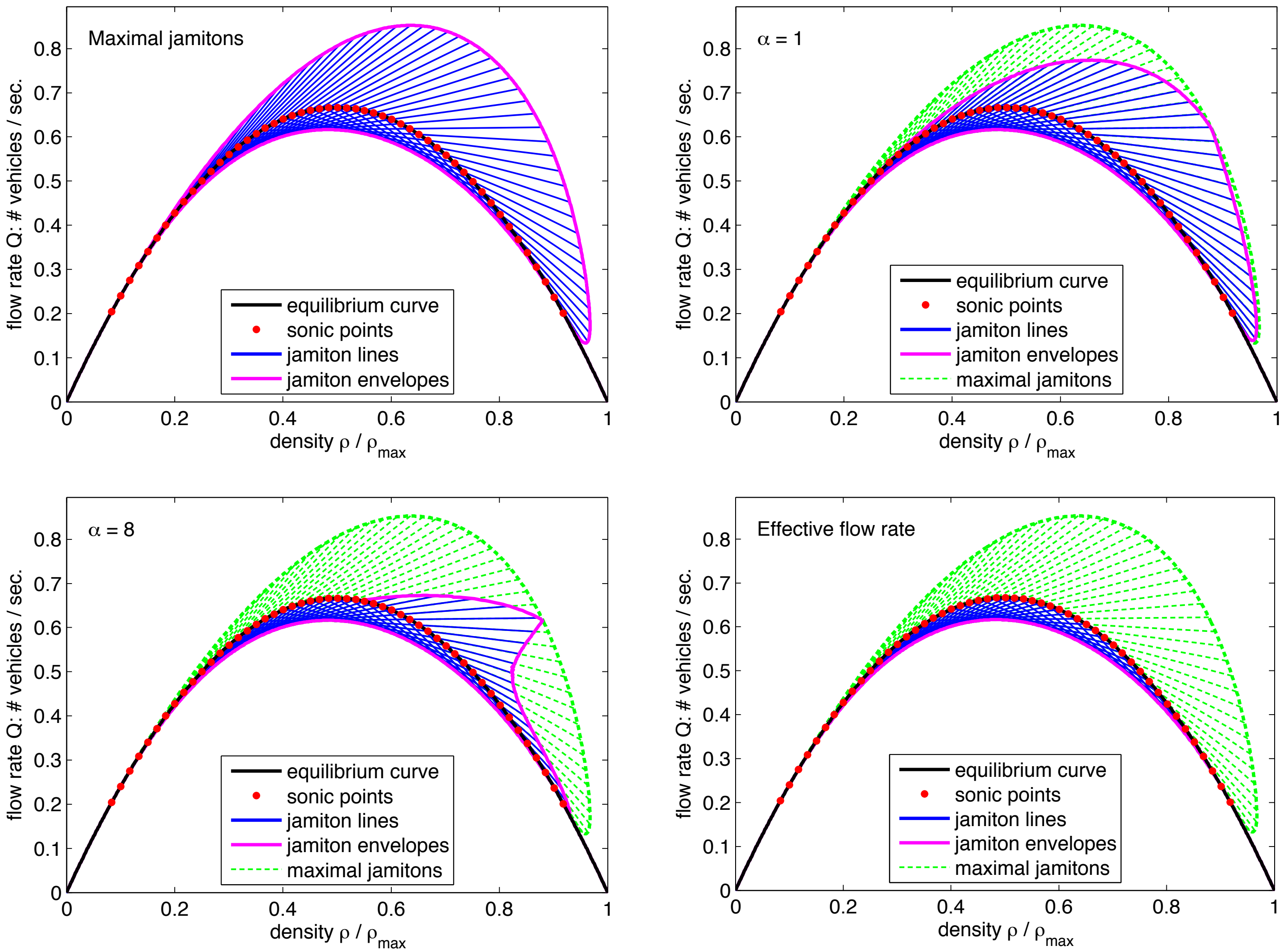
- However, comparison with measured data shows only qualitative agreement

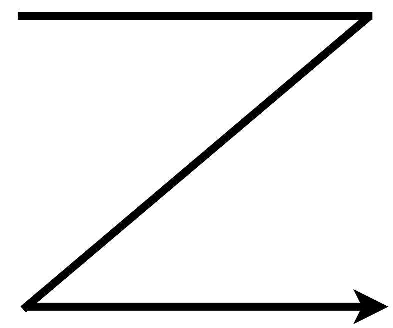
  1. Sample theoretical data in a manner consistent with the way real measurements are made

  2. Use more realistic expression for equilibrium speed, $\tilde{u}$

Measurement methodology

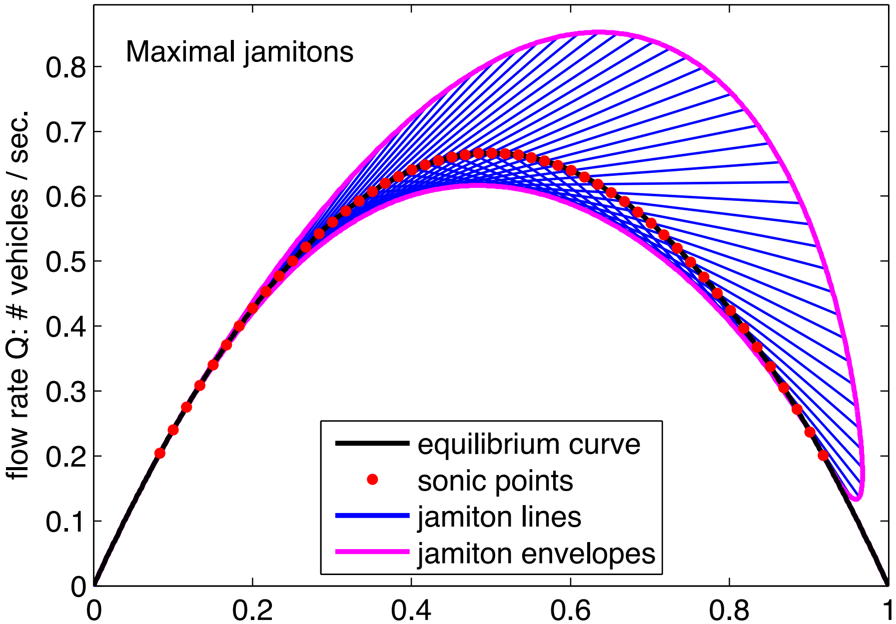
- We consider jamitons of various lengths (periodically-continued *ad infinitum*) and mimic a stationary sensor that makes estimates of the traffic density and flux in a finite time interval, Δt

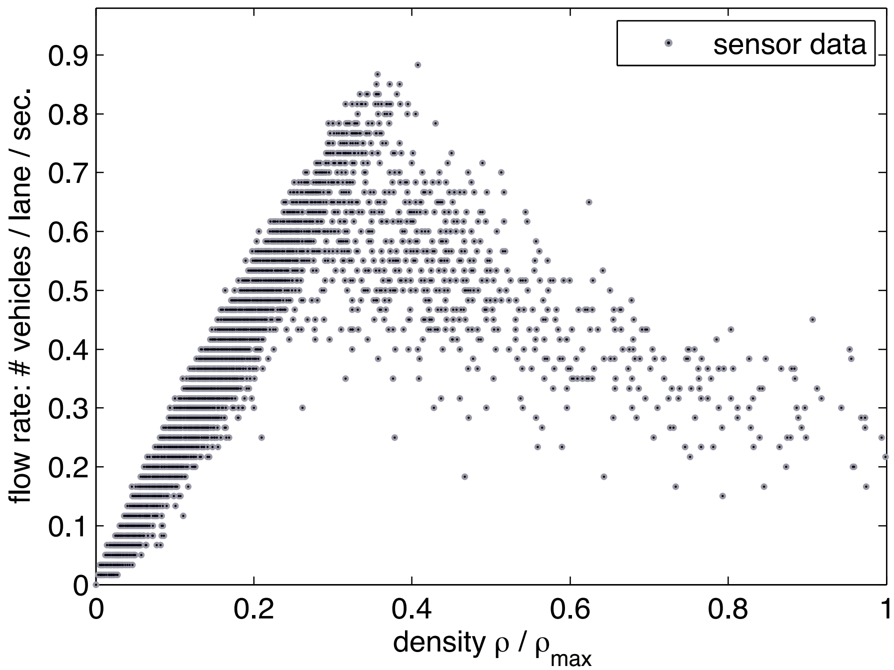




Δt increases from 0 to infinity

Comparison with measurement I



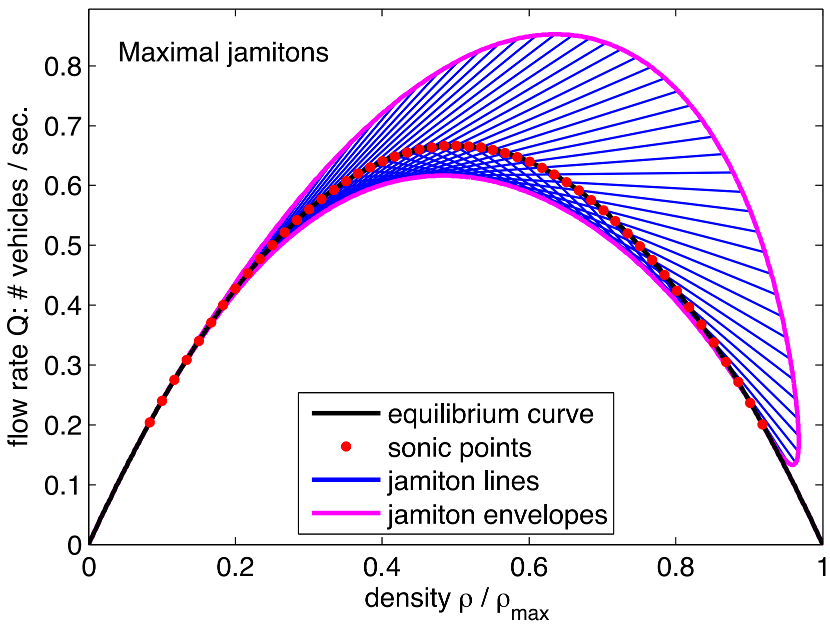


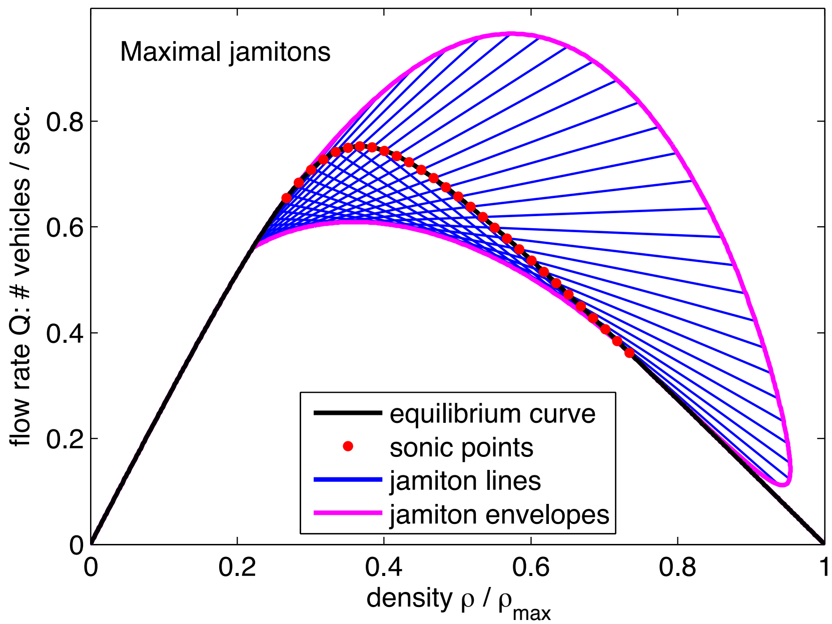
- However, comparison with measured data shows only qualitative agreement

1. Sample theoretical data in a manner consistent with the way real measurements are made

2. Use more realistic expression for equilibrium speed, $\tilde{u}$

Equilibrium speed, $\tilde{u}$





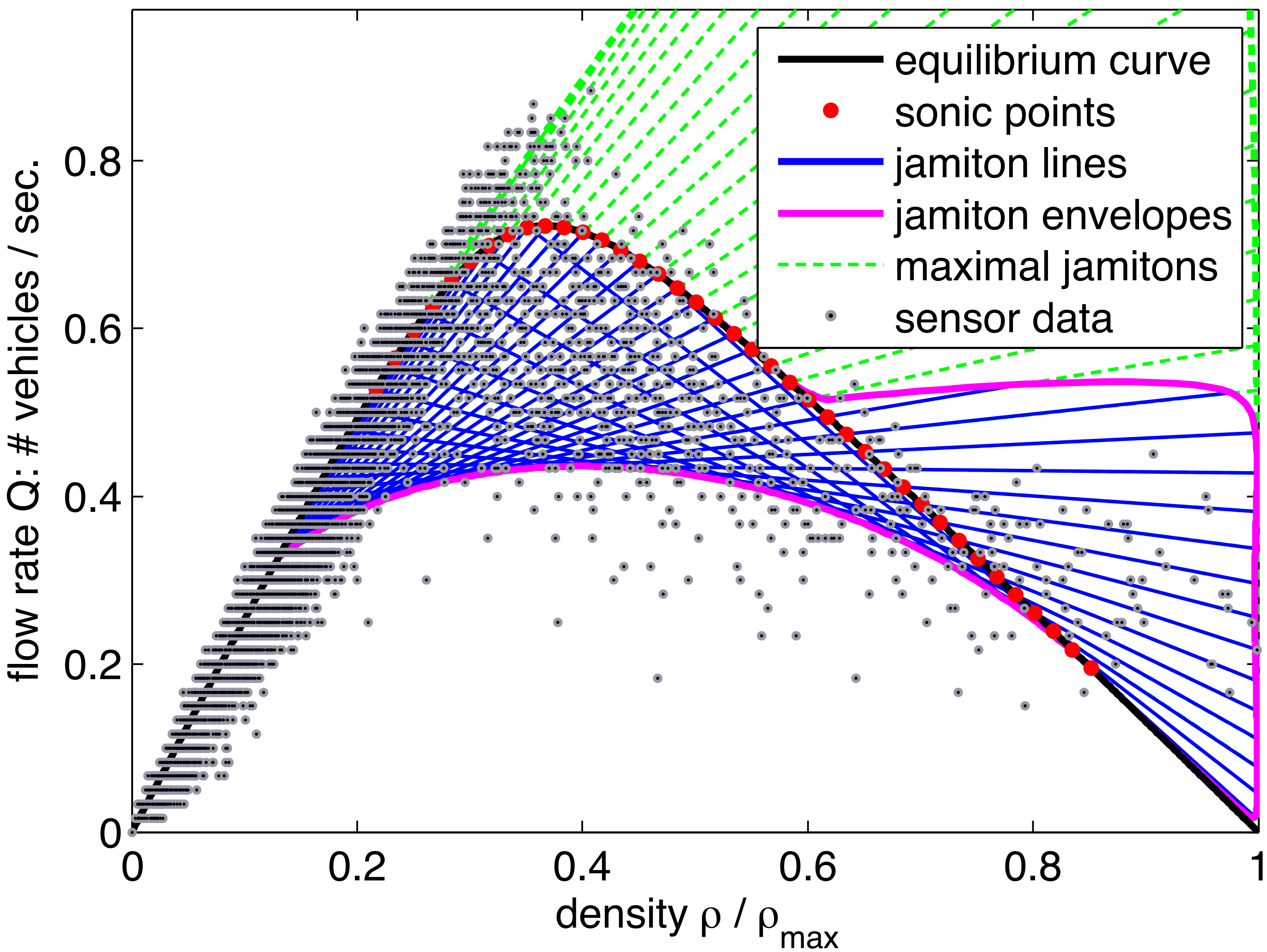
$\tilde{u}$ a linear function of ρ

$\tilde{u}$ a "kinky" function of ρ

Further details in Seibold et al. (2012)

Comparison with measurement 2

- Combining these considerations and performing some additional "fine tuning" yields a promising agreement between theory and measurement, as shown below



Calculation details: Aw-Rascle-Zhang model with "kinky" equilibrium speed and singular traffic pressure -- see Seibold et al. (2012)

Summary

- Jamitons are defined as self-sustained, nonlinear traveling waves in traffic; these are a natural feature of second order models e.g. Payne-Whitham (PW) and Aw-Rascle-Zhang

- Jamiton construction follows from ideas borrowed from gas-dynamics (sonic point, Chapman-Jouget and Lax-entropy conditions, etc.)

- Viewed from this jamiton-perspective, the PW and Aw-Rascle-Zhang models exhibit similar behavior

- By examining the details of the jamiton solutions, we can, in turn, gain insights into the multivalued region of fundamental diagrams, which show significant "scatter" at intermediate traffic densities

- Comparing measured and predicted data in fundamental diagrams may allow us to accept or reject particular choices for e.g. the traffic pressure as being either reasonable or unreasonable (*on-going research*)

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OLD SLIDES HEREAFTER

Phase plane analysis (PW model)

$$\boxed{\frac{\mathrm{d}u}{\mathrm{d}\eta} = \frac{(u-s)(\tilde{u}-u)}{(u-s)^2 - c^2}}$$

Introduce the phase plane variable ξ

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = (u-s)(\tilde{u}-u) = F_1(u)$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = (u-s)^2 - c^2 = F_2(u)$$

Jacobian: $\mathcal{J} = \begin{bmatrix} F_{1,u} & 0 \\ F_{2,u} & 0 \end{bmatrix}$

Sonic point is a critical point of the dynamical system

Sonic point eigenvalues:

$$r_1 = 0$$

$$r_2 = F_{1,u} = \frac{m|\tilde{u}_\rho|}{c} - c$$

(Degenerate system)

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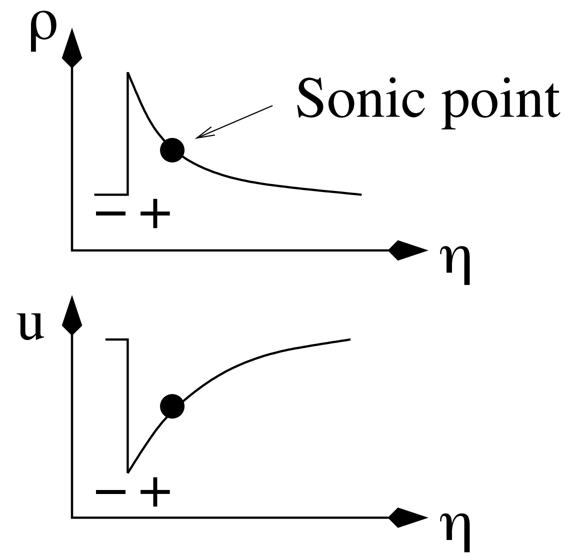

However, $r_2 > 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}\eta} > 0$

$$r_2 < 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}\eta} < 0$$

From above figure, we expect

$$\frac{\mathrm{d}u}{\mathrm{d}\eta} > 0 \quad \Leftrightarrow \quad m|\tilde{u}_\rho| > c^2 \quad \Leftrightarrow \quad |\tilde{u}_\rho| > \frac{c}{\rho}$$

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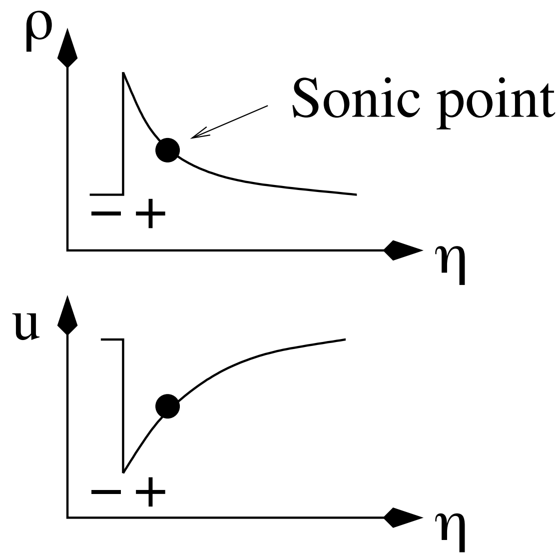

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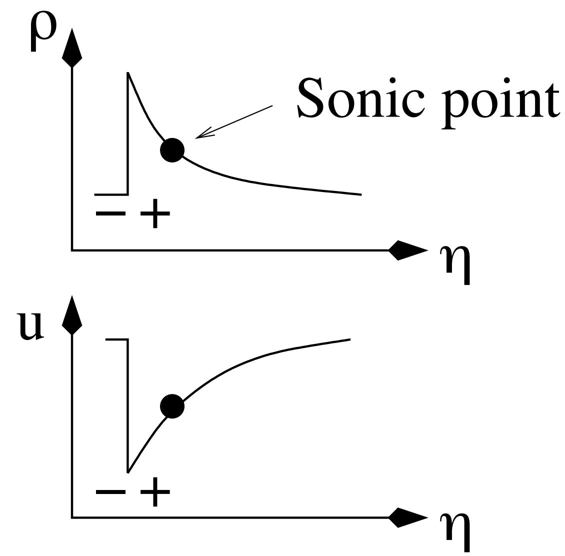

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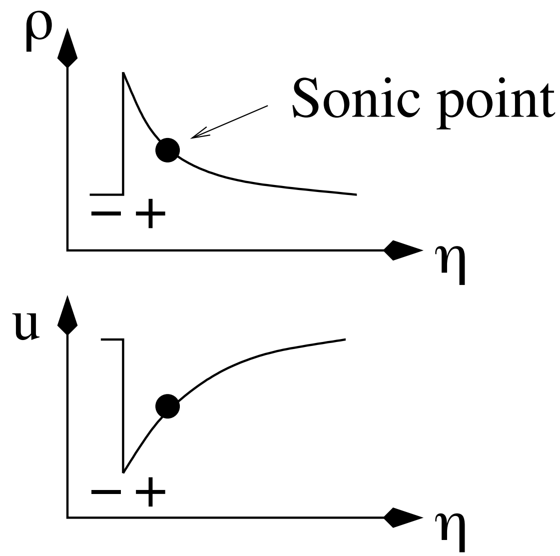

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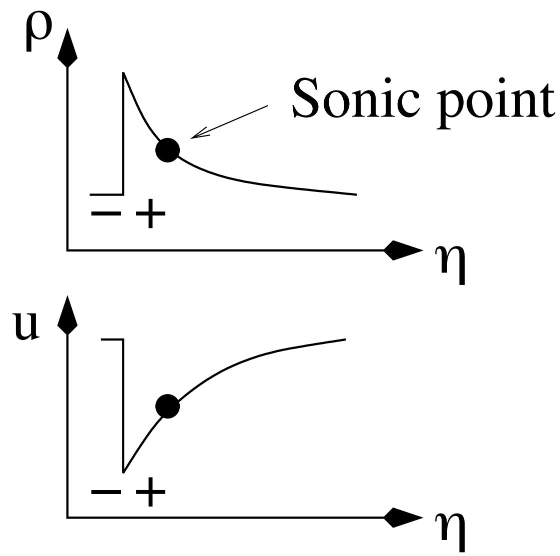

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When boxed inequality is satisfied...

(i) Sub-characteristic condition is violated

(ii) PW equations are unstable to infinitesimal perturbations

(iii) Self-sustained traveling wave solutions (i.e. jamitons) can exist