

ARZ model

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1 Non linearized models

1.1 Saint-Venant vs ARZ

Compare Saint-Venant and ARZ, different coupling behaviors.

Both models rely on the fundamental relation $q(x, t) = v(x, t) \rho(x, t)$.

(ρ, v)	Saint-Venant	ARZ
First order	$\frac{\partial a}{\partial t} + \frac{\partial av}{\partial x} = 0$	$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$
Second order	$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{g}{T} \frac{\partial a}{\partial x} = g(S_b - S_f(x, t))$	$\frac{\partial v}{\partial t} + \left(v + \rho V'(\rho)\right) \frac{\partial v}{\partial x} = \frac{V(\rho) - v}{\tau}$

1.2 Different expressions for the ARZ model

ARZ (ρ, v)	First order	$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0$
	Second order	$\frac{\partial v}{\partial t} + \left(v + \rho V'(\rho)\right) \frac{\partial v}{\partial x} = 0$
ARZ (ρ, q)	First order	$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$
	Second order	$\frac{\partial q}{\partial t} - \frac{q}{\rho} \left(\frac{q}{\rho} + \rho V'(\rho)\right) \frac{\partial \rho}{\partial x} + \left(2\frac{q}{\rho} + \rho V'(\rho)\right) \frac{\partial q}{\partial x} = \frac{\rho V(\rho) - q}{\tau}$
ARZ (v, q)	Second order	$\frac{\partial v}{\partial t} + \left(v + \frac{q}{v} V'\left(\frac{q}{v}\right)\right) \frac{\partial v}{\partial x} = \frac{V\left(\frac{q}{v}\right) - v}{\tau}$
	Second order	$\frac{\partial q}{\partial t} + \frac{q}{v} \left(v + \frac{q}{v} V'\left(\frac{q}{v}\right)\right) \frac{\partial v}{\partial x} + v \frac{\partial q}{\partial x} = \frac{\frac{q}{v} V\left(\frac{q}{v}\right) - q}{\tau}$

2 Structure of solutions and relaxation time

One considers the (ρ, v) linearized system for example: $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ -\frac{1}{\tau} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$.

2.1 Solving for ξ_1 along its characteristic lines

Let $x \in \mathbb{R}$ and $F_x^1 : t \rightarrow \xi_1(t, x + \lambda_1 t)$.

$$\frac{dF_x^1}{dt}(t) = -\frac{1}{\tau} \xi_1(t, x + t\lambda_1) = -\frac{1}{\tau} F_x^1(t)$$

Therefore, $F_x^1(t) = K_x e^{-\frac{t}{\tau}}$ and

$$\xi_1(t, x + \lambda_1 t) = \xi_1(0, x) e^{-\frac{t}{\tau}}$$

2.2 Solving for ξ_2 along its characteristic lines

Let $x \in \mathbb{R}$ and $F_x^2 : t \rightarrow \xi_2(t, x + \lambda_2 t)$.

$$\frac{dF_x^2}{dt}(t) = -\frac{1}{\tau} \xi_1(t, x + t\lambda_2)$$

Therefore, $F_x^2(t) = -\frac{1}{\tau} \int_{u=0}^t \xi_1(u, x + u\lambda_2) du + F_x^2(0)$.

$$\xi_1(t, x + \lambda_2 t) = \xi_1(t, x - (\lambda_1 - \lambda_2)t + \lambda_1 t) = \xi_1(0, x - (\lambda_1 - \lambda_2)t) e^{-\frac{t}{\tau}}$$

$$\xi_2(t, x + \lambda_2 t) = -\frac{1}{\tau} \int_{u=0}^t \xi_1(0, x - (\lambda_1 - \lambda_2)u) e^{-\frac{u}{\tau}} du + \xi_2(0, x)$$

The general expression of the solution is $\begin{pmatrix} \xi_1(t, x + \lambda_1 t) \\ \xi_2(t, x + \lambda_2 t) \end{pmatrix} = \begin{pmatrix} \xi_1(0, x) e^{-\frac{t}{\tau}} \\ -\frac{1}{\tau} \int_{u=0}^t \xi_1(0, x - (\lambda_1 - \lambda_2)u) e^{-\frac{u}{\tau}} du + \xi_2(0, x) \end{pmatrix}$.

3 Linearized velocity flow system

The linearized velocity flow system is

$$\begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}_t + \underbrace{\begin{pmatrix} v^* + \frac{q^*}{v^*} V'(\frac{q^*}{v^*}) & 0 \\ \frac{q^*}{v^*} (v^* + \frac{q^*}{v^*} V'(\frac{q^*}{v^*})) & v^* \end{pmatrix}}_A \begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}_x = \underbrace{\begin{pmatrix} -\frac{(v^*)^2 + q^* V'(\frac{q^*}{v^*})}{(v^*)^2 \tau} & \frac{V'(\frac{q^*}{v^*})}{v^* \tau} \\ -\frac{q^* ((v^*)^2 + q^* V'(\frac{q^*}{v^*}))}{(v^*)^3 \tau} & \frac{q^* V'(\frac{q^*}{v^*})}{(v^*)^2} \end{pmatrix}}_B \begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}$$

3.1 Preliminary simplifications

So calculations are easier, matrices A and B will be expressed with ρ^* and v^* exclusively. Indeed, one has $q^* = \rho^* v^*$.

$$A = \begin{pmatrix} v^* + \rho^* V'(\rho^*) & 0 \\ \rho^* (v^* + \rho^* V'(\rho^*)) & v^* \end{pmatrix}$$

$$B = \begin{pmatrix} -\frac{v^* + \rho^* V'(\rho^*)}{v^* \tau} & \frac{V'(\rho^*)}{v^* \tau} \\ -\frac{\rho^* (v^* + \rho^* V'(\rho^*))}{v^* \tau} & \frac{\rho^* V'(\rho^*)}{v^* \tau} \end{pmatrix} = \frac{1}{v^* \tau} \begin{pmatrix} -(v^* + \rho^* V'(\rho^*)) & V'(\rho^*) \\ -\rho^* (v^* + \rho^* V'(\rho^*)) & \rho^* V'(\rho^*) \end{pmatrix}$$

Let $\lambda_1 = v^*$ and $\lambda_2 = v^* + \rho^* V'(\rho^*) = v^* + \frac{q^*}{v^*} V'(\frac{q^*}{v^*})$. $V'(\rho^*) = \lambda_2 - \lambda_1 / \rho^*$

The system simplifies as

$$\begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}_t + \begin{pmatrix} \lambda_2 & 0 \\ \rho^* \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}_x = \frac{1}{v^* \tau} \begin{pmatrix} -\lambda_2 & \frac{\lambda_2 - \lambda_1}{\rho^*} \\ -\rho^* \lambda_2 & \lambda_2 - \lambda_1 \end{pmatrix} \begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix}$$

$$A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$$

$$\text{With } P = \begin{pmatrix} 0 & \lambda_2 - \lambda_1 \\ 1 & \rho^* \lambda_2 \end{pmatrix}.$$

$$\text{As } P^{-1} B P = \frac{1}{v^* \tau} \begin{pmatrix} \frac{\rho^* \lambda_2}{\lambda_1 - \lambda_2} & 1 \\ -\frac{1}{\lambda_1 - \lambda_2} & 0 \end{pmatrix} \begin{pmatrix} -\lambda_2 & \frac{\lambda_2 - \lambda_1}{\rho^*} \\ -\rho^* \lambda_2 & \lambda_2 - \lambda_1 \end{pmatrix} \begin{pmatrix} 0 & \lambda_2 - \lambda_1 \\ 1 & \rho^* \lambda_2 \end{pmatrix} = \frac{1}{v^* \tau} \begin{pmatrix} -\lambda_1 & 0 \\ \frac{1}{\rho^*} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ \frac{1}{\tau \rho^*} & 0 \end{pmatrix}$$

the system can be rewritten in the form

$$\begin{pmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ \frac{1}{\tau q^*} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{pmatrix}$$

with $\begin{pmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{pmatrix} = P^{-1} \begin{pmatrix} \tilde{v} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} -\frac{(v^* + \rho^* V'(\rho^*))}{V'(\rho^*)} \tilde{v} + \tilde{q} \\ \frac{1}{\rho^* V'(\rho^*)} \tilde{v} \end{pmatrix}.$

Therefore, with $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \tilde{\xi}_1 \\ -q^* \tilde{\xi}_2 \end{pmatrix} = \begin{pmatrix} -\frac{(v^* + \frac{q^*}{v^*} V'(\frac{q^*}{v^*}))}{V'(\frac{q^*}{v^*})} \tilde{v} + \tilde{q} \\ -\frac{v^*}{V'(\frac{q^*}{v^*})} \tilde{v} \end{pmatrix}.$

The system becomes

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ -\frac{1}{\tau} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

We find the same relaxation time, as expected.

3.2 Preliminary study of the (v, q) system:

3.2.1 Diagonalized system:

One obtains $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ -\frac{1}{\tau} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$

With $\begin{cases} \xi_1 = -\frac{(v^* + \frac{q^*}{v^*} V'(\frac{q^*}{v^*}))}{V'(\frac{q^*}{v^*})} \tilde{v} + \tilde{q} \\ \xi_2 = -\frac{v^*}{V'(\frac{q^*}{v^*})} \tilde{v} \end{cases}.$

Once again the eigen values are $\begin{cases} \lambda_1 = v^* \\ \lambda_2 = v^* + \frac{q^*}{v^*} V'(\frac{q^*}{v^*}) \end{cases}.$

3.2.2 Froude number:

$$F = \frac{q^* V'(\rho^*)}{(v^*)^2}$$

$$\lambda_2 > 0 \Leftrightarrow F < 1$$

$$\lambda_2 < 0 \Leftrightarrow F > 1$$

3.2.3 Characteristics:

See figure

3.2.4 Laplace transform:

$$\xi_t + A \xi_x = B \xi \text{ with } A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Frequency domain:

$$\text{Let } \mathcal{A}(s) = A^{-1} (B - sI) = \begin{pmatrix} -\frac{\frac{1}{\tau} + s}{\lambda_1} & 0 \\ -\frac{1}{\lambda_2 \tau} & -\frac{s}{\lambda_2} \end{pmatrix}$$

$$\mathcal{B} = A^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix}$$

$$\frac{\partial \widehat{\xi}}{\partial x}(x, s) = \mathcal{A}(s) \widehat{\xi}(x, s) + \mathcal{B}\xi(x, t=0)$$

$$\mathcal{A}(s) = P(s) \begin{pmatrix} \nu_1(s) & 0 \\ 0 & \nu_2(s) \end{pmatrix} P^{-1}(s)$$

with

$$\nu_1(s) = -\frac{1+s\tau}{\lambda_1\tau}$$

$$\nu_2(s) = -\frac{s}{\lambda_2}$$

and

$$P(s) = \begin{bmatrix} 0 & -\frac{s\lambda_1\tau - \lambda_2(1+s\tau)}{\lambda_1} \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\widehat{\xi}(x, s) = \Phi(x, s) \left(\widehat{\xi}(0, s) + \int_0^x \Phi^{-1}(v, s) \mathcal{B}\xi(v, 0) dv \right)$$

with

$$\Phi(x, s) = P(s) \begin{pmatrix} \exp(\nu_1(s)x) & 0 \\ 0 & \exp(\nu_2(s)x) \end{pmatrix} P^{-1}(s)$$

therefore

$$\Phi(x, s) = \begin{bmatrix} e^{-\frac{sx}{\lambda_1}} e^{-\frac{x}{\lambda_1\tau}} & 0 \\ \lambda_1 \left(e^{-\frac{sx}{\lambda_1}} e^{-\frac{x}{\lambda_1\tau}} - e^{-\frac{sx}{\lambda_2}} \right) & e^{-\frac{sx}{\lambda_2}} \end{bmatrix} = \begin{bmatrix} \phi_{11}(x, s) & \phi_{12}(x, s) \\ \phi_{21}(x, s) & \phi_{22}(x, s) \end{bmatrix}$$

3.3 Input-output and time domain response with $\lambda_2 > 0$:

3.3.1 Frequency domain:

with zero initial conditions the transition equation writes

$$\begin{pmatrix} \widehat{\xi}_1(x, s) \\ \widehat{\xi}_2(x, s) \end{pmatrix} = \Phi(x, s) \begin{pmatrix} \widehat{\xi}_1(0, s) \\ \widehat{\xi}_2(0, s) \end{pmatrix}$$

Let

$$Q = \begin{bmatrix} \frac{\rho^* \lambda_2}{\lambda_1 - \lambda_2} & 1 \\ \frac{\rho^* \lambda_1}{\lambda_1 - \lambda_2} & 0 \end{bmatrix}$$

One has

$$\begin{pmatrix} \widehat{v}(x, s) \\ \widehat{q}(x, s) \end{pmatrix} = \underbrace{Q^{-1} \Phi(x, s) Q}_{\Psi(x, s)} \begin{pmatrix} \widehat{v}(0, s) \\ \widehat{q}(0, s) \end{pmatrix}$$

with

- $\psi_{11}(x, s) = -\frac{e^{-\frac{(1+s\tau)x}{\lambda_1\tau}}\lambda_2 - e^{-\frac{sx}{\lambda_2}}s\tau(\lambda_1 - \lambda_2)}{s\tau(\lambda_1 - \lambda_2) - \lambda_2}$
- $\psi_{12}(x, s) = -\frac{(\lambda_1 - \lambda_2)\left(e^{-\frac{(1+s\tau)x}{\lambda_1\tau}} - e^{-\frac{sx}{\lambda_2}}\right)}{\rho^*(s\tau(\lambda_1 - \lambda_2) - \lambda_2)}$
- $\psi_{21}(x, s) = \frac{\rho^*s\tau\lambda_2\left(e^{-\frac{(1+s\tau)x}{\lambda_1\tau}} - e^{-\frac{sx}{\lambda_2}}\right)}{s\tau(\lambda_1 - \lambda_2) - \lambda_2}$
- $\psi_{22}(x, s) = -\frac{e^{-\frac{sx}{\lambda_2}}\lambda_2 - e^{-\frac{(1+s\tau)x}{\lambda_1\tau}}s\tau(\lambda_1 - \lambda_2)}{s\tau(\lambda_1 - \lambda_2) - \lambda_2}$

Let $\alpha = -\frac{\lambda_2}{\tau(\lambda_1 - \lambda_2)} = \frac{v^* + \frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right)}{\tau\frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right)}$

- $\psi_{11}(x, s) = \left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{sx}{\lambda_2}}\right)\frac{\alpha}{s+\alpha} + e^{-\frac{sx}{\lambda_2}}$
- $\psi_{12}(x, s) = \frac{(\lambda_1 - \lambda_2)}{\lambda_2\rho^*}\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{sx}{\lambda_2}}\right)\frac{\alpha}{s+\alpha}$
- $\psi_{21}(x, s) = \frac{\rho^*\lambda_2}{(\lambda_1 - \lambda_2)}s\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{sx}{\lambda_2}}\right)\frac{1}{s+\alpha}$
- $\psi_{22}(x, s) = -\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{sx}{\lambda_2}}\right)\frac{\alpha}{s+\alpha} + e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}}$

3.3.2 Constants

- Constants:
 - $\lambda_2 = v^* + \frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right) > 0$
 - $\lambda_1 = v^* > 0$
 - $\lambda_1 - \lambda_2 = -\frac{q^*}{v^*}V'\left(\frac{q^*}{v^*}\right) > 0$
 - $\alpha = -\frac{\lambda_2}{\tau(\lambda_1 - \lambda_2)} < 0$

3.3.3 Fundamental responses in diagonal form:

- Expression of Φ :
 - $\phi_{11}(x, s) = e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}}$
 - $\phi_{12}(x, s) = 0$
 - $\phi_{11}(x, s) = -\alpha\frac{\lambda_1}{\lambda_2}\left(e^{-\frac{x}{\lambda_1\tau}}e^{-\frac{sx}{\lambda_1}} - e^{-\frac{sx}{\lambda_2}}\right)\frac{1}{s+\alpha}$
 - $\phi_{12}(x, s) = e^{-\frac{sx}{\lambda_2}}$
- $\begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} H(t) \\ 0 \end{pmatrix}$:
 - $\tilde{\xi}_1(0, t) = e^{-\frac{x}{\lambda_1\tau}}H\left(t - \frac{x}{\lambda_1}\right)$

$$\begin{aligned}
& - \tilde{\xi}_2(0, t) = -\frac{\lambda_1}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} \left(1 - e^{-\alpha \left(t - \frac{x}{\lambda_1} \right)} \right) H \left(t - \frac{x}{\lambda_1} \right) - \left(1 - e^{-\alpha \left(t - \frac{x}{\lambda_2} \right)} \right) H \left(t - \frac{x}{\lambda_2} \right) \right) \\
& \bullet \begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} 0 \\ H(t) \end{pmatrix}: \\
& \quad - \tilde{\xi}_1(0, t) = 0 \\
& \quad - \tilde{\xi}_2(0, t) = H \left(t - \frac{x}{\lambda_2} \right) \\
& \bullet \begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} \sin(\omega t) \\ 0 \end{pmatrix}: \\
& \quad - \tilde{\xi}_1(0, t) = e^{-\frac{x}{\lambda_1 \tau}} \sin \left(\omega \left(t - \frac{x}{\lambda_1} \right) \right) H \left(t - \frac{x}{\lambda_1} \right) \\
& \quad - \tilde{\xi}_2(0, t) = -\frac{\lambda_1 \alpha}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa_{\alpha, \omega}^{\sin} \left(t - \frac{x}{\lambda_1} \right) - \kappa_{\alpha, \omega}^{\sin} \left(t - \frac{x}{\lambda_2} \right) \right) \\
& \bullet \begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} 0 \\ \sin(\omega t) \end{pmatrix}: \\
& \quad - \tilde{\xi}_1(0, t) = 0 \\
& \quad - \tilde{\xi}_2(0, t) = \sin \left(\omega \left(t - \frac{x}{\lambda_2} \right) \right) H \left(t - \frac{x}{\lambda_2} \right) \\
& \bullet \begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ 0 \end{pmatrix}: \\
& \quad - \tilde{\xi}_1(0, t) = e^{-\frac{x}{\lambda_1 \tau}} \cos \left(\omega \left(t - \frac{x}{\lambda_1} \right) \right) H \left(t - \frac{x}{\lambda_1} \right) \\
& \quad - \tilde{\xi}_2(0, t) = -\frac{\lambda_1 \alpha}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa_{\alpha, \omega}^{\cos} \left(t - \frac{x}{\lambda_1} \right) - \kappa_{\alpha, \omega}^{\cos} \left(t - \frac{x}{\lambda_2} \right) \right) \\
& \bullet \begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} 0 \\ \cos(\omega t) \end{pmatrix}: \\
& \quad - \tilde{\xi}_1(0, t) = 0 \\
& \quad - \tilde{\xi}_2(0, t) = \cos \left(\omega \left(t - \frac{x}{\lambda_2} \right) \right) H \left(t - \frac{x}{\lambda_2} \right)
\end{aligned}$$

3.3.4 Fundamental responses to step and sinusoidal stimulations in (v, q) domain

$$\begin{aligned}
& \bullet \begin{pmatrix} \tilde{v}(0, t) \\ \tilde{q}(0, t) \end{pmatrix} = \begin{pmatrix} H(t) \\ 0 \end{pmatrix}: \\
& \quad - \tilde{v}(x, t) = \left(1 - e^{-\alpha \left(t - \frac{x}{\lambda_1} \right)} \right) e^{-\frac{x}{\lambda_1 \tau}} H \left(t - \frac{x}{\lambda_1} \right) + e^{-\alpha \left(t - \frac{x}{\lambda_2} \right)} H \left(t - \frac{x}{\lambda_2} \right) \\
& \quad - \tilde{q}(x, t) = \frac{\rho \lambda_2}{(\lambda_1 - \lambda_2)} \left(e^{-\alpha \left(t - \frac{x}{\lambda_1} \right)} e^{-\frac{x}{\lambda_1 \tau}} H \left(t - \frac{x}{\lambda_1} \right) - e^{-\alpha \left(t - \frac{x}{\lambda_2} \right)} H \left(t - \frac{x}{\lambda_2} \right) \right) \\
& \bullet \begin{pmatrix} \tilde{v}(0, t) \\ \tilde{q}(0, t) \end{pmatrix} = \begin{pmatrix} 0 \\ H(t) \end{pmatrix}:
\end{aligned}$$

$$\begin{aligned}
- \tilde{v}(x, t) &= \frac{(\lambda_1 - \lambda_2)}{\lambda_2 \rho} \left(\left(1 - e^{-\alpha \left(t - \frac{x}{\lambda_1} \right)} \right) e^{-\frac{x}{\lambda_1 \tau}} H \left(t - \frac{x}{\lambda_1} \right) - \left(1 - e^{-\alpha \left(t - \frac{x}{\lambda_2} \right)} \right) H \left(t - \frac{x}{\lambda_2} \right) \right) \\
- \tilde{q}(x, t) &= \left(1 - e^{-\alpha \left(t - \frac{x}{\lambda_2} \right)} \right) H \left(t - \frac{x}{\lambda_2} \right) + e^{-\frac{x}{\lambda_1 \tau}} e^{-\alpha \left(t - \frac{x}{\lambda_1} \right)} H \left(t - \frac{x}{\lambda_1} \right) \\
\bullet \begin{pmatrix} \tilde{v}(0, t) \\ \tilde{q}(0, t) \end{pmatrix} &= \begin{pmatrix} \sin(\omega t) \\ 0 \end{pmatrix} : \\
- \tilde{v}(x, t) &= \alpha \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa_{\alpha, w}^{\sin} \left(t - \frac{x}{\lambda_1} \right) - \kappa_{\alpha, w}^{\sin} \left(t - \frac{x}{\lambda_2} \right) \right) + \sin \left(\omega \left(t - \frac{x}{\lambda_2} \right) \right) \\
- \tilde{q}(x, t) &= \frac{\rho^* \lambda_2}{(\lambda_1 - \lambda_2)} \left(e^{-\frac{x}{\lambda_1 \tau}} \iota_{\alpha, \omega}^{\sin} \left(t - \frac{x}{\lambda_1} \right) - \iota_{\alpha, \omega}^{\sin} \left(t - \frac{x}{\lambda_2} \right) \right) \\
\bullet \begin{pmatrix} \tilde{v}(0, t) \\ \tilde{q}(0, t) \end{pmatrix} &= \begin{pmatrix} 0 \\ \sin(\omega t) \end{pmatrix} : \\
- \tilde{v}(x, t) &= \frac{(\lambda_1 - \lambda_2)}{\lambda_2 \rho^*} \alpha \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa_{\alpha, w}^{\sin} \left(t - \frac{x}{\lambda_1} \right) - \kappa_{\alpha, w}^{\sin} \left(t - \frac{x}{\lambda_2} \right) \right) \\
- \tilde{q}(x, t) &= -\alpha \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa_{\alpha, w}^{\sin} \left(t - \frac{x}{\lambda_1} \right) - \kappa_{\alpha, w}^{\sin} \left(t - \frac{x}{\lambda_2} \right) \right) + e^{-\frac{x}{\lambda_1 \tau}} \sin \left(\omega \left(t - \frac{x}{\lambda_1} \right) \right) \\
\bullet \begin{pmatrix} \tilde{v}(0, t) \\ \tilde{q}(0, t) \end{pmatrix} &= \begin{pmatrix} \cos(\omega t) \\ 0 \end{pmatrix} : \\
- \tilde{v}(x, t) &= \alpha \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa_{\alpha, w}^{\cos} \left(t - \frac{x}{\lambda_1} \right) - \kappa_{\alpha, w}^{\cos} \left(t - \frac{x}{\lambda_2} \right) \right) + \cos \left(\omega \left(t - \frac{x}{\lambda_2} \right) \right) \\
- \tilde{q}(x, t) &= \frac{\rho^* \lambda_2}{(\lambda_1 - \lambda_2)} \left(e^{-\frac{x}{\lambda_1 \tau}} \iota_{\alpha, \omega}^{\cos} \left(t - \frac{x}{\lambda_1} \right) - \iota_{\alpha, \omega}^{\cos} \left(t - \frac{x}{\lambda_2} \right) \right) \\
\bullet \begin{pmatrix} \tilde{v}(0, t) \\ \tilde{q}(0, t) \end{pmatrix} &= \begin{pmatrix} 0 \\ \cos(\omega t) \end{pmatrix} : \\
- \tilde{v}(x, t) &= \frac{(\lambda_1 - \lambda_2)}{\lambda_2 \rho^*} \alpha \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa_{\alpha, w}^{\cos} \left(t - \frac{x}{\lambda_1} \right) - \kappa_{\alpha, w}^{\cos} \left(t - \frac{x}{\lambda_2} \right) \right) \\
- \tilde{q}(x, t) &= -\alpha \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa_{\alpha, w}^{\cos} \left(t - \frac{x}{\lambda_1} \right) - \kappa_{\alpha, w}^{\cos} \left(t - \frac{x}{\lambda_2} \right) \right) + e^{-\frac{x}{\lambda_1 \tau}} \cos \left(\omega \left(t - \frac{x}{\lambda_1} \right) \right)
\end{aligned}$$

3.4 Time domain response with $\lambda_2 < 0$:

$$\begin{pmatrix} \hat{\xi}_1(x, s) \\ \hat{\xi}_2(x, s) \end{pmatrix} = \Phi(x, s) \begin{pmatrix} \hat{\xi}_1(0, s) \\ \hat{\xi}_2(0, s) \end{pmatrix}$$

therefore

$$\begin{pmatrix} \hat{\xi}_1(x, s) \\ \hat{\xi}_2(x, s) \end{pmatrix} = \Phi(x, s) \begin{pmatrix} 1 & 0 \\ -\frac{\phi_{21}(L, s)}{\phi_{22}(L, s)} & \frac{1}{\phi_{22}(L, s)} \end{pmatrix} \begin{pmatrix} \hat{\xi}_1(0, s) \\ \hat{\xi}_2(0, s) \end{pmatrix}$$

let

$$\Gamma(x, s) = \begin{pmatrix} e^{-\frac{x}{\lambda_1 \tau}} e^{-\frac{s x}{\lambda_1}} & 0 \\ \alpha \frac{\lambda_1}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} e^{-\frac{s x}{\lambda_1}} - e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2} \left(x - L \frac{\lambda_1 - \lambda_2}{\lambda_1} \right)} \right) & \frac{1}{s + \alpha} e^{-\frac{s(x-L)}{\lambda_2}} \end{pmatrix}$$

$$\bullet \gamma_{11}(x, s) = e^{-\frac{x}{\lambda_1 \tau}} e^{-\frac{s x}{\lambda_1}}$$

- $\gamma_{12}(x, s) = 0$
- $\gamma_{21}(x, s) = \alpha \frac{\lambda_1}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} e^{-\frac{s x}{\lambda_1}} - e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2} \left(x - L \frac{\lambda_1 - \lambda_2}{\lambda_1} \right)} \right) \frac{1}{s + \alpha}$
- $\gamma_{22}(x, s) = e^{-\frac{s(x-L)}{\lambda_2}}$

and as

$$\begin{pmatrix} \widehat{\xi}_1(0, s) \\ \widehat{\xi}_2(L, s) \end{pmatrix} = \begin{pmatrix} \frac{\rho \lambda_2}{\lambda_1 - \lambda_2} \widehat{v}(0, s) + \widehat{q}(0, s) \\ \frac{\rho \lambda_1}{\lambda_1 - \lambda_2} \widehat{v}(L, s) \end{pmatrix}$$

one obtains

$$\begin{pmatrix} \widehat{v}(x, s) \\ \widehat{q}(x, s) \end{pmatrix} = Q^{-1} \Phi(x, s) \begin{pmatrix} 1 & 0 \\ -\frac{\phi_{21}(L, s)}{\phi_{22}(L, s)} & \frac{1}{\phi_{22}(L, s)} \end{pmatrix} \begin{pmatrix} \frac{\rho \lambda_2}{\lambda_1 - \lambda_2} & 1 & 0 \\ 0 & 0 & \frac{\rho \lambda_1}{\lambda_1 - \lambda_2} \end{pmatrix} \begin{pmatrix} \widehat{v}(0, s) \\ \widehat{q}(0, s) \\ \widehat{v}(L, s) \end{pmatrix}$$

which finally gives

$$\begin{pmatrix} \widehat{v}(x, s) \\ \widehat{q}(x, s) \end{pmatrix} = \Upsilon(x, s) \begin{pmatrix} \widehat{v}(0, s) \\ \widehat{q}(0, s) \\ \widehat{v}(L, s) \end{pmatrix}$$

where

- $\Upsilon_{11}(x, s) = -\frac{\lambda_2 \left(e^{-\frac{x}{\lambda_1 \tau}} e^{-\frac{s x}{\lambda_1}} - e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2} \left(x - L \frac{\lambda_1 - \lambda_2}{\lambda_1} \right)} \right)}{s \tau (\lambda_1 - \lambda_2) - \lambda_2}$
- $\Upsilon_{12}(x, s) = -\frac{(\lambda_1 - \lambda_2)}{\rho} \frac{\left(e^{-\frac{x}{\lambda_1 \tau}} e^{-\frac{s x}{\lambda_1}} - e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2} \left(x - L \frac{\lambda_1 - \lambda_2}{\lambda_1} \right)} \right)}{s \tau (\lambda_1 - \lambda_2) - \lambda_2}$
- $\Upsilon_{13}(x, s) = e^{-\frac{s(x-L)}{\lambda_2}}$
- $\Upsilon_{21}(x, s) = \frac{\rho \lambda_2}{(\lambda_1 - \lambda_2)} \frac{\left(s \tau (\lambda_1 - \lambda_2) e^{-\frac{-x}{\lambda_1 \tau}} e^{-\frac{-s x}{\lambda_1}} - \lambda_2 e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2} \left(x - L \frac{\lambda_1 - \lambda_2}{\lambda_1} \right)} \right)}{s \tau (\lambda_1 - \lambda_2) - \lambda_2}$
- $\Upsilon_{22}(x, s) = \frac{s \tau (\lambda_1 - \lambda_2) e^{-\frac{-x}{\lambda_1 \tau}} e^{-\frac{-s x}{\lambda_1}} - \lambda_2 e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2} \left(x - L \frac{\lambda_1 - \lambda_2}{\lambda_1} \right)}}{s \tau (\lambda_1 - \lambda_2) - \lambda_2}$
- $\Upsilon_{23}(x, s) = -\frac{\rho \lambda_2}{(\lambda_1 - \lambda_2)} e^{-\frac{s(x-L)}{\lambda_2}}$

A simplified version of these expressions is

- $\Upsilon_{11}(x, s) = \alpha \left(e^{-\frac{x}{\lambda_1 \tau}} e^{-\frac{s x}{\lambda_1}} - e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2} \left(x - L \frac{\lambda_1 - \lambda_2}{\lambda_1} \right)} \right) \frac{1}{s + \alpha}$
- $\Upsilon_{12}(x, s) = -\frac{1}{\rho \tau} \left(e^{-\frac{x}{\lambda_1 \tau}} e^{-\frac{s x}{\lambda_1}} - e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2} \left(x - L \frac{\lambda_1 - \lambda_2}{\lambda_1} \right)} \right) \frac{1}{s + \alpha}$
- $\Upsilon_{13}(x, s) = e^{-\frac{s(x-L)}{\lambda_2}}$
- $\Upsilon_{21}(x, s) = \frac{\rho \lambda_2}{\lambda_1 - \lambda_2} \left(s \cdot e^{-\frac{-x}{\lambda_1 \tau}} e^{-\frac{-s x}{\lambda_1}} + \alpha \cdot e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2} \left(x - L \frac{\lambda_1 - \lambda_2}{\lambda_1} \right)} \right) \frac{1}{s + \alpha}$

- $\Upsilon_{22}(x, s) = \left(s \cdot e^{-\frac{x}{\lambda_1 \tau}} e^{-\frac{s x}{\lambda_1}} + \alpha \cdot e^{-\frac{L}{\lambda_1 \tau}} e^{-\frac{s}{\lambda_2}} \left(x - L \frac{\lambda_1 - \lambda_2}{\lambda_1} \right) \right) \frac{1}{s + \alpha}$

- $\Upsilon_{23}(x, s) = \tau \rho \alpha \cdot e^{-\frac{s(x-L)}{\lambda_2}}$

3.4.1 Fundamental responses in diagonal form:

- $\begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} H(t) \\ 0 \end{pmatrix}:$

- $\tilde{\xi}_1(0, t) = e^{-\frac{x}{\lambda_1 \tau}} H\left(t - \frac{x}{\lambda_1}\right)$

- $\tilde{\xi}_2(0, t) = \frac{\lambda_1}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} \left(1 - e^{-\alpha \left(t - \frac{x}{\lambda_1} \right)} \right) H\left(t - \frac{x}{\lambda_1}\right) - e^{-\frac{L}{\lambda_1 \tau}} \left(1 - e^{-\alpha \left(t - \frac{x-L}{\lambda_1} \right)} \right) H\left(t - \frac{x-L}{\lambda_1}\right) \right)$

- $\begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} 0 \\ H(t) \end{pmatrix}:$

- $\tilde{\xi}_1(0, t) = 0$

- $\tilde{\xi}_2(0, t) = H\left(t - \frac{x-L}{\lambda_2}\right)$

- $\begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} \sin(\omega t) \\ 0 \end{pmatrix}:$

- $\tilde{\xi}_1(0, t) = e^{-\frac{x}{\lambda_1 \tau}} \sin\left(\omega \left(t - \frac{x}{\lambda_1}\right)\right) H\left(t - \frac{x}{\lambda_1}\right)$

- $\tilde{\xi}_2(0, t) = \frac{\lambda_1 \alpha}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa_{\alpha, \omega}^{\sin}\left(t - \frac{x}{\lambda_1}\right) - e^{-\frac{L}{\lambda_1 \tau}} \kappa_{\alpha, \omega}^{\sin}\left(t - \frac{x-L}{\lambda_1}\right) \right)$

- $\begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} 0 \\ \sin(\omega t) \end{pmatrix}:$

- $\tilde{\xi}_1(0, t) = 0$

- $\tilde{\xi}_2(0, t) = \sin\left(\omega \left(t - \frac{x-L}{\lambda_2}\right)\right) H\left(t - \frac{x-L}{\lambda_2}\right)$

- $\begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ 0 \end{pmatrix}:$

- $\tilde{\xi}_1(0, t) = e^{-\frac{x}{\lambda_1 \tau}} \cos\left(\omega \left(t - \frac{x}{\lambda_1}\right)\right) H\left(t - \frac{x}{\lambda_1}\right)$

- $\tilde{\xi}_2(0, t) = \frac{\lambda_1 \alpha}{\lambda_2} \left(e^{-\frac{x}{\lambda_1 \tau}} \kappa_{\alpha, \omega}^{\cos}\left(t - \frac{x}{\lambda_1}\right) - e^{-\frac{L}{\lambda_1 \tau}} \kappa_{\alpha, \omega}^{\cos}\left(t - \frac{x-L}{\lambda_1}\right) \right)$

- $\begin{pmatrix} \tilde{\xi}_1(0, t) \\ \tilde{\xi}_2(0, t) \end{pmatrix} = \begin{pmatrix} 0 \\ \cos(\omega t) \end{pmatrix}:$

- $\tilde{\xi}_1(0, t) = 0$

- $\tilde{\xi}_2(0, t) = \cos\left(\omega \left(t - \frac{x-L}{\lambda_2}\right)\right) H\left(t - \frac{x-L}{\lambda_2}\right)$

4 Bode plots

4.1 Spatial transforms

Initial domain: (v, q) .

$$\text{Diagonalization basis: } \begin{cases} \xi_1 &= \frac{\rho^* \lambda_2}{\lambda_1 - \lambda_2} v + q \\ \xi_2 &= \frac{\rho^* \lambda_1}{\lambda_1 - \lambda_2} v \end{cases}, \begin{cases} v &= \frac{\lambda_1 - \lambda_2}{\rho^* \lambda_1} \xi_2 \\ q &= \lambda_1 \xi_1 - \lambda_2 \xi_2 \end{cases}$$

4.2 Eigen values from fundamental diagram

$$\lambda_1 = \frac{q(\rho^*)}{\rho^*}, \lambda_2 = q'(\rho^*)$$

4.3 Values for parameters

$$L = 100 \text{ (meters)}$$

4.3.1 Greenshields fundamental diagram

$$q(\rho) = 4 \frac{q_m}{\rho_m^2} \rho (\rho_m - \rho)$$

$$q'(\rho) = 4 \frac{q_m}{\rho_m} \left(1 - 2 \frac{\rho}{\rho_m}\right)$$

$$\rho_m = 0.1 \text{ veh/m}$$

$$q_m = 0.36 \text{ veh/s}$$

$$\tau = 15\text{s}$$

Annex A:

Useful frequency to time domain conversions

In the transfer functions that have been computed, several components are recurrent:

- $s \rightarrow \frac{1}{s+\alpha}$
- $s \rightarrow \frac{s}{s+\alpha}$
- $s \rightarrow e^{-\theta s}$

The last item is an usual θ delay.

The inputs we will use are:

- $s \rightarrow \frac{1}{s}$ (step function)
- $s \rightarrow \frac{\omega}{s^2 + \omega^2}$ ($\sin(\omega t) H(t)$ in the time domain)
- $s \rightarrow \frac{s}{s^2 + \omega^2}$ ($\cos(\omega t) H(t)$ in the time domain)

Therefore, we need to compute the inverse transforms of the following functionals in the frequency domain: $s \rightarrow \frac{1}{s(s+\alpha)}$, $s \rightarrow \frac{1}{s+\alpha}$, $s \rightarrow \frac{1}{s+\alpha} \frac{\omega}{s^2 + \omega^2}$, $s \rightarrow \frac{s}{s+\alpha} \frac{\omega}{s^2 + \omega^2}$, $s \rightarrow \frac{1}{s+\alpha} \frac{s}{s^2 + \omega^2}$ and $s \rightarrow \frac{s}{s+\alpha} \frac{s}{s^2 + \omega^2}$.

Classically one has the inverse transforms for the following functionals:

- $s \rightarrow \frac{1}{s(s+\alpha)} \xrightarrow{\text{time domain}} t \rightarrow \frac{1}{\alpha} (1 - e^{-\alpha t}) H(t)$
- $s \rightarrow \frac{1}{s+\alpha} \xrightarrow{\text{time domain}} t \rightarrow e^{-\alpha t} H(t)$

For the other functionals, one can compute the convolution products (denoted \star) in the time domain.

First sinusoidal:

$$s \rightarrow \frac{1}{s+\alpha} \frac{\omega}{s^2+\omega^2} \xrightarrow{\text{time domain}} t \rightarrow (e^{-\alpha \cdot} H(\cdot) \star \sin(\omega \cdot) H(\cdot))(t)$$

One has $(e^{-\alpha \cdot} H(\cdot) \star \sin(\omega \cdot) H(\cdot))(t) = \int_{u=-\infty}^{+\infty} e^{-\alpha u} H(u) \sin(\omega(t-u)) H(t-u) du = \int_{u=0}^t e^{-\alpha u} \sin(\omega(t-u)) du$

$$\text{and } \int_{u=0}^t e^{-\alpha u} \sin(\omega(t-u)) du = \frac{\omega e^{-\alpha t} - \omega \cos(\omega t) + \alpha \sin(\omega t)}{\alpha^2 + \omega^2}$$

Therefore

$$s \rightarrow \frac{1}{s+\alpha} \frac{\omega}{s^2+\omega^2} \xrightarrow{\text{time domain}} t \rightarrow \frac{\omega e^{-\alpha t} - \omega \cos(\omega t) + \alpha \sin(\omega t)}{\alpha^2 + \omega^2} = \kappa_{\alpha, \omega}^{\sin}(t)$$

Second sinusoidal:

$$s \rightarrow \frac{s}{s+\alpha} \frac{\omega}{s^2+\omega^2} \xrightarrow{\text{time domain}} t \rightarrow \frac{d}{dt} \left(\frac{\omega e^{-\alpha t} - \omega \cos(\omega t) + \alpha \sin(\omega t)}{\alpha^2 + \omega^2} \right) (t) + \frac{\omega e^{-\alpha t} - \omega \cos(\omega t) + \alpha \sin(\omega t)}{\alpha^2 + \omega^2} \Big|_{t=0-}$$

As, $\frac{\omega e^{-\alpha t} - \omega \cos(\omega t) + \alpha \sin(\omega t)}{\alpha^2 + \omega^2} \Big|_{t=0-} = 0$, we obtain

$$s \rightarrow \frac{s}{s+\alpha} \frac{\omega}{s^2+\omega^2} \xrightarrow{\text{time domain}} t \rightarrow \frac{-\alpha \omega e^{-\alpha t} + \omega^2 \sin(\omega t) + \alpha \omega \cos(\omega t)}{\alpha^2 + \omega^2} = \iota_{\alpha, \omega}^{\sin}(t)$$

First cosinusoidal:

$$s \rightarrow \frac{1}{s+\alpha} \frac{s}{s^2+\omega^2} \xrightarrow{\text{time domain}} t \rightarrow (e^{-\alpha \cdot} H(\cdot) \star \cos(\omega \cdot) H(\cdot))(t)$$

Therefore

$$s \rightarrow \frac{1}{s+\alpha} \frac{s}{s^2+\omega^2} \xrightarrow{\text{time domain}} t \rightarrow -\frac{\alpha e^{-\alpha t} - \alpha \cos(\omega t) - \omega \sin(\omega t)}{\alpha^2 + \omega^2} = \kappa_{\alpha, \omega}^{\cos}(t)$$

Second cosinusoidal:

$$s \rightarrow \frac{s}{s+\alpha} \frac{s}{s^2+\omega^2} \xrightarrow{\text{time domain}} t \rightarrow \frac{d}{dt} \left(-\frac{\alpha e^{-\alpha t} - \alpha \cos(\omega t) - \omega \sin(\omega t)}{\alpha^2 + \omega^2} \right) (t) - \frac{\alpha e^{-\alpha t} - \alpha \cos(\omega t) - \omega \sin(\omega t)}{\alpha^2 + \omega^2} \Big|_{t=0-}$$

As, $\frac{\alpha e^{-\alpha t} - \alpha \cos(\omega t) - \omega \sin(\omega t)}{\alpha^2 + \omega^2} \Big|_{t=0-} = 0$, we obtain

$$s \rightarrow \frac{s}{s+\alpha} \frac{s}{s^2+\omega^2} \xrightarrow{\text{time domain}} t \rightarrow -\frac{-\alpha^2 e^{-\alpha t} + \alpha \omega \sin(\omega t) - \omega^2 \cos(\omega t)}{\alpha^2 + \omega^2} = \iota_{\alpha, \omega}^{\cos}(t)$$

5 Practical values for parameters

5.1 Relaxation time

Values of $\tau \in [10, 30] s$

5.2 Typical length

Values of $L \in [100, 1000] m$

5.3 Triangular diagram

$$\rho_{max} =$$