Traffic flow on a road network using the Aw–Rascle model.

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Abstract

The paper deals with a fluid dynamic model for traffic flow on a road network. This consists of a hyperbolic system of two equations proposed by Aw and Rascle in [1]. A method to solve Riemann problems at junctions is given assigning rules on traffic distributions and maximizations of fluxes and other quantities. Then we discuss stability in L^∞ norm of such solutions. Finally, we prove existence of entropic solutions to the Cauchy problem when the road network has only one junction.

Key Words: Traffic flow, conservation laws, road network.

AMS subject classification: 90B20, 35L65.

1 Introduction.

Fluid dynamic models for traffic flow seem appropriate to describe phenomena as shock formation and propagation, see [1, 3, 6, 7, 9, 10, 11, 14]. Most results regard traffic on a single road, but in some recent papers [5, 6, 10, 11], also the case of a road network was considered. For example, [6] deals with the first order model proposed by Lighthill and Whitham [15] and Richards [17]. In this paper, traffic flow is described by the second order model proposed in 2000 by Aw and Rascle, see [1].

The first prototype of a second order traffic system was the so called Payne–Whitham model, see [16, 19]. Some other second order models were

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then proposed in the literature, see [11, 12, 13]. However, in [8], Daganzo showed that these second order systems are not satisfactory to describe the behavior of traffic flow, since car may have negative speed. Therefore Aw and Rascle in [1] proposed a correction to the Payne–Whitham equations overcoming the bad behavior of previous works.

We consider a network of roads connected by junctions and, on each road, the traffic flow is described by two equations, written in conservation form. The macroscopic variables are the density ρ of the cars and the "momentum" y (see [1]), which can be expressed as function of the density, of the speed of cars and of the "pressure".

The evolution on roads is thus defined by equations (2.1) below, while one has to define a solution in presence of junctions. In [6], the authors suppose that

(A) there are some prescribed preferences of drivers, i.e. the traffic from incoming roads is distributed on outgoing roads according to fixed coefficients;

(B) respecting (A), drivers behave so as to maximize fluxes.

The rule (A) is represented by a matrix A whose coefficients $\alpha_{i,j}$ are the percentage of flux from an incoming to a outgoing road. Following [6], we assume again that (A) and (B) hold. Unfortunately these rules are not sufficient to determine a unique solution for the Riemann problem at junctions, i.e. for Cauchy problems with initial data constant on each road. More precisely, the Riemann problem is still underdetermined in outgoing roads, hence the necessity to give an additional rule. We propose three different additional rules:

(AR-1) maximization of the speed v of cars;

(AR-2) maximization of the density of cars;

(AR-3) minimization of the total variation of the density along the solution.

The first two rules are motivated by modelling, while the third one is introduced only for mathematical reasons. We show that each of these rules permits to solve in unique way the Riemann problem at junctions.

The obtained solutions are then studied in detail. We define equilibria as constant solutions to a Riemann problem at junctions, while stability is meant with respect to small L^∞ perturbations. The study of stability is a key point to determine the existence of solutions to the Cauchy problem. We provide some stability results for equilibria with any number of incoming and outgoing roads. However, not all possible cases are considered: we completely solve only the generic cases for a junction with two incoming and two outgoing roads.

Using this analysis we prove the existence of solutions to a Cauchy problem when the road network has only one junction and the initial data are sufficiently close in total variation to a stable equilibrium configuration. The main technique is a modified Glimm functional containing a first order term for waves approaching the junction.

The paper is organized in the following way. In Section 2, we give the basic definitions and notations. In Section 3, we determine characteristic curves and speeds of the first and second family. In Section 4, we determine an invariant domain. In Section 5, we solve in details the Riemann problem at junctions and in Section 6 we analyze the stability of equilibria. Section 7 is devoted to prove the existence of a Cauchy problem on a road network with only one junction and finally an appendix, with a counterexample to conservation of flux variation, concludes the paper.

2 Basic definitions and notations.

We consider a network of roads, that is modeled by a finite collection of connected intervals $I_i = [a_i, b_i] \subseteq \mathbb{R}$, $i = 1, \ldots, N$, possibly with either $a_i = -\infty$ or $b_i = +\infty$, on which the dynamic is governed by the system:

$$\begin{cases} \partial_t \rho + \partial_x (y - \rho^{\gamma+1}) = 0 \\ \partial_t y + \partial_x (\frac{y^2}{\rho} - y\rho^\gamma) = 0 \end{cases} \tag{2.1}$$

where $\gamma > 0$, ρ is the density of the cars and $y = \rho v + \rho^{\gamma+1}$ is the momentum (v is the velocity of the cars). Thus, on each road, the datum is given by two functions $\rho_i, y_i : [0, +\infty[\times I_i \to \mathbb{R}$.

On each road I_i, we say that $U_i := (\rho_i, y_i) : [0, +\infty[\times I_i \to \mathbb{R}$ is a weak solution to (2.1) if, for every C^∞–function $\varphi : [0, +\infty[\times I_i \to \mathbb{R}^2$ with compact support in $]0, +\infty[\times]a_i, b_i[$,

$$\int_0^{+\infty} \int_{a_i}^{b_i} \left(U_i \cdot \frac{\partial \varphi}{\partial t} + f(U_i) \cdot \frac{\partial \varphi}{\partial x} \right) dx\, dt = 0 \tag{2.2}$$

where

$$f(U_i) = \begin{pmatrix} y_i - \rho_i^{\gamma+1} \\ \frac{y_i^2}{\rho_i} - y_i\rho_i^\gamma \end{pmatrix}, \tag{2.3}$$

is the flux of the system (2.1). For the definition of entropic solution, we refer to [4].

As in [6], we assume that the roads connect together at junctions and that each road could be an incoming road for at most one junction and an outgoing road for at most one junction, that is in each road cars can run in a unique direction.

With *flux of the density* we indicate the first component of the flux f and precisely the quantity $y - \rho^{\gamma+1}$, while *flux of momentum* stands for the second component of the flux, i.e. $y^2/\rho - y\rho^\gamma$.

3 Characteristic fields.

We observe that the system (2.1) is strictly hyperbolic when $\rho > 0$. In fact the Jacobian matrix of the flux of the system (2.1) is given by

$$\begin{pmatrix} -(\gamma+1)\rho^\gamma & 1 \\ -\frac{y^2}{\rho^2} - \gamma y \rho^{\gamma-1} & 2\frac{y}{\rho} - \rho^\gamma \end{pmatrix} \tag{3.4}$$

whose eigenvalues are

$$\lambda_1 = \frac{y}{\rho} - (\gamma+1)\rho^\gamma, \qquad \lambda_2 = \frac{y}{\rho} - \rho^\gamma. \tag{3.5}$$

Therefore, if $\rho > 0$, then $\lambda_1 < \lambda_2$ and the system is strictly hyperbolic. Notice that the second eigenvalue λ_2 is equal to the velocity v of the cars.

It is easy to see that the first characteristic field is genuinely nonlinear, while the second characteristic field is linearly degenerate, see [1, 4]. Moreover the rarefaction curves of the first family are lines passing through the origin. Since the rarefaction curves are lines, also the shock curves of the first family are lines and they have the same expression.

Instead, the curves of the second family through (ρ_0, y_0) are given by

$$y = \frac{y_0}{\rho_0}\rho + \rho^{\gamma+1} - \rho_0^\gamma \rho. \tag{3.6}$$

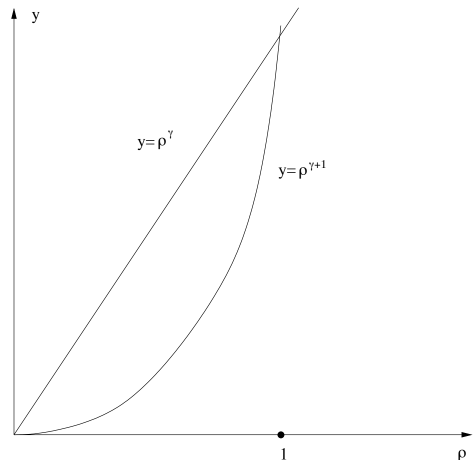


Figure 1: Domain of invariance.

4 Domains of invariance.

It is natural to assume that in each road the density ρ is positive and bounded by a constant ρ_{max}, which for simplicity we assume to be 1.

Also the velocity v of cars must be positive and bounded. In particular we suppose that the maximum velocity of cars is decreasing with respect to the density ρ and it has the following expression:

$$v_{max}(\rho) = 1 - \rho^\gamma.$$

Thus we obtain that $\rho^{\gamma+1} \leqslant y \leqslant \rho$; see Figure 1. Thus (ρ, y) takes value in the domain

$$\mathcal{D} = \left\{(\rho, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \rho^{\gamma+1} \leqslant y \leqslant \rho\right\}. \tag{4.7}$$

We show that the region $\mathcal{D}$ is invariant for the system (2.1). To this purpose, it is enough to show that the solution to every Riemann problem with data in $\mathcal{D}$, remains in $\mathcal{D}$. Consider a road I, modeled by $\mathbb{R}$ and the following Riemann problem:

$$\begin{cases} \partial_t \rho + \partial_x (y - \rho^{\gamma+1}) = 0, \\ \partial_t y + \partial_x (\frac{y^2}{\rho} - y\rho^\gamma) = 0, \\ (\rho(0,x), y(0,x)) = (\rho_-, y_-), & \text{if}\, x < 0, \\ (\rho(0,x), y(0,x)) = (\rho_+, y_+), & \text{if}\, x > 0. \end{cases} \tag{4.8}$$

As pointed out in [1], there are some different cases.

1. The points (ρ_-, y_-) and (ρ_+, y_+) belong either to a curve of the first family or to a curve of the second family. In this case the two points can be connected either by a wave of the first family or by a wave of the second family. Notice that (ρ_-, y_-) or (ρ_+, y_+) can be equal to $(0, 0)$.

2. $\rho_- > 0$, $\rho_+ > 0$ and the curve of the first family through (ρ_-, y_-) intersects the curve of the second family through (ρ_+, y_+) in a point of $\mathcal{D}$ different from $(0, 0)$. We call (ρ_0, y_0) this point.

   If $\rho_0 < \rho_-$ then $\lambda_1(\rho_-, y_-) < \lambda_1(\rho_0, y_0) < \lambda_2(\rho_0, y_0) = \lambda_2(\rho_+, y_+)$. So it is possible to connect (ρ_-, y_-) with (ρ_0, y_0) by a wave of the first family with maximum speed $\lambda_1(\rho_0, y_0)$ and then (ρ_0, y_0) with (ρ_+, y_+) by a wave of the second family with speed $\lambda_2(\rho_0, y_0)$.

   If instead $\rho_0 > \rho_-$, then it is possible to connect (ρ_-, y_-) with (ρ_0, y_0) by a shock wave of the first family with speed

$$\frac{(y_- - \rho_-^{\gamma+1}) - (y_0 - \rho_0^{\gamma+1})}{\rho_- - \rho_0}$$

   and then (ρ_0, y_0) with (ρ_+, y_+) by a wave of the second family with speed

$$\lambda_2(\rho_0, y_0) = \frac{y_0}{\rho_0} - \rho_0^{\gamma}.$$

   Clearly this process is admissible if and only if

$$\frac{(y_- - \rho_-^{\gamma+1}) - (y_0 - \rho_0^{\gamma+1})}{\rho_- - \rho_0} < \frac{y_0}{\rho_0} - \rho_0^{\gamma}. \tag{4.9}$$

   Since (ρ_-, y_-) and (ρ_0, y_0) belong to the same line $y = c\rho$ with $c > 0$, (4.9) is valid if and only if

$$\frac{c(\rho_- - \rho_0) - (\rho_-^{\gamma+1} - \rho_0^{\gamma+1})}{\rho_- - \rho_0} < c - \rho_0^{\gamma}$$

   which is equivalent to

$$\frac{\rho_-^{\gamma+1} - \rho_0^{\gamma+1}}{\rho_- - \rho_0} > \rho_0^{\gamma}.$$

   Multiplying by $(\rho_- - \rho_0)$ the last inequality, it results $\rho_-^{\gamma+1} - \rho_0^{\gamma+1} < \rho_0^{\gamma}\rho_- - \rho_0^{\gamma+1}$ and so (4.9) is equivalent to $\rho_-^{\gamma} < \rho_0^{\gamma}$ which is clearly true. Thus the analysis of this case is completed.

3. $\rho_- > 0$, $\rho_+ > 0$ and the curve of the first family through (ρ_-, y_-) intersects in $\mathcal{D}$ the curve of the second family through (ρ_+, y_+) only at $(0,0)$. Let $y = c_1\rho$ be the curve of the first family through (ρ_-, y_-) and let $y = c_2\rho + \rho^{\gamma+1}$ be the curve of the second family through (ρ_+, y_+). In this case it is easy to see that $c_1 \leqslant c_2$. It is possible to connect (ρ_-, y_-) to $(0,0)$ by a wave of the first family whose maximum speed is
$$\lim_{\rho\to 0^+} \lambda_1(\rho, c_1\rho) = \lim_{\rho\to 0^+} c_1 - (\gamma+1)\rho^\gamma = c_1$$
and then $(0,0)$ to (ρ_+, y_+) by a wave of the second family with speed c_2. The conclusion follows from the fact that $c_2 \geqslant c_1$.

5 Riemann problems at junctions.

To construct solutions on the network, we need to define a solution to Riemann problems at junctions, that is a solution to the Cauchy problem with initial data constant on each road.

In the whole section, we consider a fixed junction J with n incoming roads (say $I_1, \ldots, I_n$) and m outgoing roads (say $I_{n+1}, \ldots, I_{n+m}$) and we assume that $((\rho_{1,0}, y_{1,0}), \ldots, (\rho_{n+m,0}, y_{n+m,0}))$ are the initial data on the roads. It is natural to impose to solutions to the Riemann problem at J the following rules:

(R-1) the waves produced must have negative speed in incoming roads and positive speed in outgoing roads;

(R-2) the first component of the flux (i.e. the flux of the density) must be conserved;

(R-3) there exist some fixed coefficients describing the preferences of the drivers. Each of them determines the percentage of the flux of the density which passes from an incoming road to an outgoing one;

(R-4) the sum of the first components of the flux in incoming roads is maximized.

The first rule means that the waves produced by solving a Riemann problem at a junction travel in the right direction in each road. The second one is conservation of car density, i.e. cars cannot be created or destroyed at junctions. The third one requires that each driver knows her destination and then she chooses the direction according to it. The last rule implies the maximization of the number of cars passing through the junction.

In next sections it is shown that these four rules are not sufficient to solve in a unique way the Riemann problem at junctions. More precisely, these rules are sufficient to isolate a unique solution to the Riemann problem only for incoming roads, but for outgoing roads there exist, in general, infinitely many solutions respecting rules (R-1)–(R-4). Therefore we need an additional rule and we propose three different ones:

(AR-1) maximize the velocity v of cars in outgoing roads;

(AR-2) maximize the density ρ of cars in outgoing roads;

(AR-3) minimize the total variation of ρ along the solution of the Riemann problem in outgoing roads.

Remark 1 *The solution of the Riemann problem at junctions for this model implies the conservation of the density of car, but does not imply the conservation of the momentum. This means that the solution of the Riemann problem at junctions is not a weak solution to (2.1), that is it is not a solution to (2.1) in integral sense.* ◁

Remark 2 *Rules (AR-1) and (AR-2) are given for model reason, assuming that drivers prefer to maximize ρ or v. On the other side rule (AR-3) is motivated mathematically to control the BV norm.*

To satisfy rule (R-3), we fix an $m \times n$ matrix

$$A = \begin{pmatrix} \alpha_{n+1,1} & \alpha_{n+1,2} & \cdots & \alpha_{n+1,n} \\ \alpha_{n+2,1} & \alpha_{n+2,2} & \cdots & \alpha_{n+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n+m,1} & \alpha_{n+m,2} & \cdots & \alpha_{n+m,n} \end{pmatrix} \tag{5.10}$$

where every $\alpha_{j,i}$ represents the percentage of flux of the density of the cars of the I_i incoming road which goes to the I_j outgoing road and

(A1) $0 < \alpha_{j,i} < 1$ for every $j \in \{n+1, \ldots, n+m\}$ and for every $i \in \{1, \ldots, n\}$;

(A2) $\sum_{j=n+1}^{n+m} \alpha_{j,i} = 1$ for every $i \in \{1, \ldots, n\}$;

(A3) denoting with $\{e_1, \dots, e_n\}$ the canonical basis of $\mathbb{R}^n$, with $H_i = \{e_i\}^\perp$ the orthogonal hyperplane to e_i, with H_j the orthogonal hyperplane to $\alpha_j = (\alpha_{j,1}, \dots, \alpha_{j,n})$ ($j \in \{n+1, \dots, n+m\}$), it holds

$$(1, \dots, 1) \notin H^\perp$$

for every H defined as the intersection of l distinct hyperplanes H_h, where $l \in \{1, \dots, n-1\}$ and $h \in \{1, \dots, n+m\}$.

Remark 3 *Condition (A3) is a technical condition, which allows us to determine in a unique way the values of the fluxes of the density satisfying conditions (R-1)–(R-4). From (A3) we immediately derive $m \geqslant n$. Otherwise, since by definition $(1, \dots, 1) = \sum_{j=n+1}^{n+m} \alpha_j$, we get $(1, \dots, 1) \in H^\perp$, where*

$$H = \cap_{j=n+1}^{n+m} H_j.$$

Moreover if $n \geqslant 2$, then (A3) implies that, for every $j \in \{n+1, \dots, n+m\}$ and for every distinct elements $i, i' \in \{1, \dots, n\}$, it holds $\alpha_{j,i} \neq \alpha_{j,i'}$. Otherwise, without loss of generalities, we may suppose that $\alpha_{n+1,1} = \alpha_{n+1,2}$. If we consider

$$H = (\cap_{2 < j \leqslant n} H_j) \cap H_{n+1},$$

then, by (A3), there exists an element $(x_1, x_2, 0, \dots, 0) \in H$ such that $x_1 + x_2 \neq 0$ and $\alpha_{n+1,1}(x_1 + x_2) = 0$.

In the case of a simple junction J with 2 incoming roads and 2 outgoing ones, the condition (A3) is completely equivalent to the fact that, for every $j \in \{3, 4\}$, $\alpha_{j,1} \neq \alpha_{j,2}$.

Remark 4 *Notice that the matrix A with properties (A1), (A2), (A3) could have identical lines. In fact the matrix A defined by*

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & \frac{3}{5} \end{pmatrix}$$

satisfies all the hypotheses (A1), (A2) and (A3) and the first two lines coincide.

We want to determine $((\hat{\rho}_1, \hat{y}_1), \dots, (\hat{\rho}_{n+m}, \hat{y}_{n+m}))$ such that,

- for every $i \in \{1, \ldots, n\}$, waves generated by $((\rho_{i,0}, y_{i,0}), (\hat{\rho}_i, \hat{y}_i))$ have negative velocity;

- for every $j \in \{n+1, \ldots, n+m\}$, waves obtained by $((\hat{\rho}_j, \hat{y}_j), (\rho_{j,0}, y_{j,0}))$ have positive velocity;

- for every $j \in \{n+1, \ldots, n+m\}$, we have $\hat{y}_j - \hat{\rho}_j^{\gamma+1} = \sum_{i=1}^n \alpha_{j,i}(\hat{y}_i - \hat{\rho}_i^{\gamma+1})$;

- the summation $\sum_{i=1}^n (\hat{y}_i - \hat{\rho}_i^{\gamma+1})$ is maximum subject to the previous constrains.

- one of (AR-1), (AR-2), (AR-3) holds.

First of all, we have to calculate all the admissible final states for incoming roads and for outgoing ones, that is we want to find all the final states $((\hat{\rho}_1, \hat{y}_1), \ldots, (\hat{\rho}_{n+m}, \hat{y}_{n+m}))$ such that the first two conditions hold.

In the following analysis, some curves in the domain $\mathcal{D}$ play a crucial role:

1. the curves of the first family;

2. the curves of the second family;

3. the curve $y = (\gamma+1)\rho^{\gamma+1}$.

We call the last one *curve of maxima*, since the first component of the flux restricted to a curve of the first family has the maximum point at the intersection with such curve. Moreover $y = (\gamma+1)\rho^{\gamma+1}$ divides the domain $\mathcal{D}$ into two subdomains $\mathcal{D}_1$ and $\mathcal{D}_2$:

$$\mathcal{D}_1 := \left\{(\rho, y) \in \mathcal{D} : \, y \geqslant (\gamma+1)\rho^{\gamma+1}\right\} \tag{5.11}$$

and

$$\mathcal{D}_2 := \left\{(\rho, y) \in \mathcal{D} : \, y \leqslant (\gamma+1)\rho^{\gamma+1}\right\}. \tag{5.12}$$

We use the symbols $\mathring{\mathcal{D}}_1$ and $\mathring{\mathcal{D}}_2$ to denote the sets:

$$\mathring{\mathcal{D}}_1 := \left\{(\rho, y) \in \mathcal{D}_1 : \, y > (\gamma+1)\rho^{\gamma+1}\right\} \tag{5.13}$$

and

$$\mathring{\mathcal{D}}_2 := \left\{(\rho, y) \in \mathcal{D}_2 : \, y < (\gamma+1)\rho^{\gamma+1}\right\}. \tag{5.14}$$

5.1 Admissible states in incoming roads.

Fix an incoming road I_i, with an initial state $(\rho_{i,0}, y_{i,0})$. We want to find all the possible states $(\hat{\rho}_i, \hat{y}_i)$ such that waves generated by the Riemann problem with data $(\rho_{i,0}, y_{i,0})$ and $(\hat{\rho}_i, \hat{y}_i)$ have negative speed.

Proposition 5.1 *Let $(\rho_{i,0}, y_{i,0}) \neq (0,0)$ be the initial value in an incoming road. The admissible states $(\hat{\rho}_i, \hat{y}_i)$ generated by the Riemann problem at the junction must belong to the curve of the first family through $(\rho_{i,0}, y_{i,0})$. More precisely, we have the following cases:*

1. *$(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1$. In this case, the two states are connected by a shock wave of the first family. There exists a unique point $(\bar{\rho}, \bar{y}) \in \mathcal{D}_2$ on the curve of the first family through $(\rho_{i,0}, y_{i,0})$ with the properties:*
   - *(a) $y_{i,0} - \rho_{i,0}^{\gamma+1} = \bar{y} - \bar{\rho}^{\gamma+1}$;*
   - *(b) $(\hat{\rho}_i, \hat{y}_i)$ is admissible if and only if $\hat{\rho}_i > \bar{\rho}$; see Figure 2.*
2. *$(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_2$. In this case all the admissible final states belong to $\mathcal{D}_2$; see Figure 3.*

If instead $(\rho_{i,0}, y_{i,0}) = (0,0)$ then the only admissible final state is the same point $(0,0)$.

Proof. If we connect two states with a wave of the second family, then the speed of the wave is greater or equal to 0. Therefore, to obtain waves with negative speed one has to restrict to waves of the first family. First, consider the case $(\rho_{i,0}, y_{i,0}) \neq (0,0)$.

If $\hat{\rho}_i < \rho_{i,0}$ then there exists a rarefaction wave of the first family connecting $(\rho_{i,0}, y_{i,0})$ to $(\hat{\rho}_i, \hat{y}_i)$. The maximum speed of the wave is given by

$$\lambda_1(\hat{\rho}_i, \hat{y}_i) = \frac{\hat{y}_i}{\hat{\rho}_i} - (\gamma+1)\hat{\rho}_i^{\gamma}.$$

Since we need $\lambda_1(\hat{\rho}_i, \hat{y}_i) \leqslant 0$, then

$$\hat{\rho}_i^{\gamma+1} \leqslant \hat{y}_i \leqslant (\gamma+1)\hat{\rho}_i^{\gamma+1}.$$

If $\hat{\rho}_i > \rho_{i,0}$ then there exists a shock wave of the first family connecting $(\rho_{i,0}, y_{i,0})$ to $(\hat{\rho}_i, \hat{y}_i)$. Since the speed of the wave, given by the Rankine-Hugoniot condition, must be negative, it results

$$\hat{\rho}_i^{\gamma+1} - \frac{y_{i,0}}{\rho_{i,0}}\hat{\rho}_i + y_{i,0} - \rho_{i,0}^{\gamma+1} > 0.$$

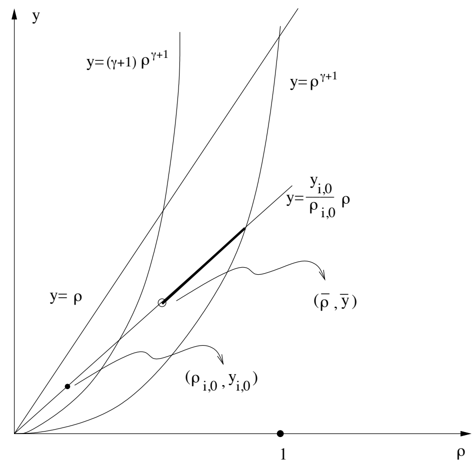


Figure 2: Admissible states in an incoming road I_i when $y_{i,0} > (\gamma+1)\rho_{i,0}^{\gamma+1}$. The final state either is $(\rho_{i,0}, y_{i,0})$ or belongs to the part in bold of the line $y = \frac{y_{i,0}}{\rho_{i,0}}\rho$.

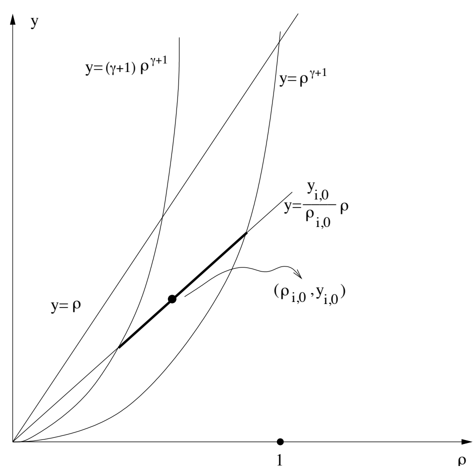


Figure 3: Admissible states in an incoming road I_i when $y_{i,0} < (\gamma+1)\rho_{i,0}^{\gamma+1}$. The final state belongs to the part in bold of the line $y = \frac{y_{i,0}}{\rho_{i,0}}\rho$.

The previous inequality can also be written in the form

$$\frac{y_{i,0}}{\rho_{i,0}} < \frac{\rho_{i,0}^{\gamma+1} - \hat{\rho}_i^{\gamma+1}}{\rho_{i,0} - \hat{\rho}_i}. \tag{5.15}$$

If $y_{i,0} \leqslant (\gamma+1)\rho_{i,0}^{\gamma+1}$, then all the points on the curve of the first family through $(\rho_{i,0}, y_{i,0})$ with $\hat{\rho}_i > \rho_{i,0}$ satisfy the last inequality. In fact $y_{i,0}/\rho_{i,0}$ is the slope of the curve of the first family through $(\rho_{i,0}, y_{i,0})$, while

$$\frac{\rho_{i,0}^{\gamma+1} - \hat{\rho}_i^{\gamma+1}}{\rho_{i,0} - \hat{\rho}_i}$$

is strictly greater than the minimum of the derivative of $\rho^{\gamma+1}$ when ρ belongs to the interval

$$\left[\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}} \left(\frac{y_{i,0}}{\rho_{i,0}}\right)^{\frac{1}{\gamma}}, \left(\frac{y_{i,0}}{\rho_{i,0}}\right)^{\frac{1}{\gamma}} \right],$$

which is exactly $y_{i,0}/\rho_{i,0}$.

Instead, if $y_{i,0} > (\gamma+1)\rho_{i,0}^{\gamma+1}$, then there exists a unique point $(\bar{\rho}, \bar{y})$ on the curve of the first family through $(\rho_{i,0}, y_{i,0})$ with $\bar{\rho} > \rho_{i,0}$ such that

$$\frac{y_{i,0}}{\rho_{i,0}} = \frac{\rho_{i,0}^{\gamma+1} - \bar{\rho}^{\gamma+1}}{\rho_{i,0} - \bar{\rho}}.$$

In fact, since the function $\rho \mapsto \rho^{\gamma+1}$ is convex, then the function

$$\rho \mapsto \frac{\rho_{i,0}^{\gamma+1} - \rho^{\gamma+1}}{\rho_{i,0} - \rho}$$

is strictly increasing when $\rho \geqslant \rho_{i,0}$; moreover

$$\lim_{\rho \to (\frac{1}{\gamma+1})^{1/\gamma} (\frac{y_{i,0}}{\rho_{i,0}})^{1/\gamma}} \frac{\rho_{i,0}^{\gamma+1} - \rho^{\gamma+1}}{\rho_{i,0} - \rho} < \frac{y_{i,0}}{\rho_{i,0}} \quad \text{and} \quad \frac{\rho_{i,0}^{\gamma+1} - \left(\frac{y_{i,0}}{\rho_{i,0}}\right)^{\frac{\gamma+1}{\gamma}}}{\rho_{i,0} - \left(\frac{y_{i,0}}{\rho_{i,0}}\right)^{\frac{1}{\gamma}}} > \frac{y_{i,0}}{\rho_{i,0}},$$

gives the existence of $(\bar{\rho}, \bar{y}) \in \mathring{\mathcal{D}}_2$. Notice that the points $(\rho_{i,0}, y_{i,0})$ and $(\bar{\rho}, \frac{y_{i,0}}{\rho_{i,0}}\bar{\rho})$ have the same first component of the flux.

Now, it remains the case $(\rho_{i,0}, y_{i,0}) = (0,0)$. In this case no point $(\hat{\rho}_i, \hat{y}_i)$ is admissible, since the speed of the wave of the first family connecting $(0,0)$ to $(\hat{\rho}_i, \hat{y}_i)$ is given by

$$\frac{\hat{y}_i - \hat{\rho}_i^{\gamma+1}}{\hat{\rho}_i},$$

which is clearly positive. Therefore the proof is finished. $\square$

By the previous proposition, the first component of the flux in an incoming road I_i, may take values in the set

$$\Omega_i = \begin{cases} \left[0, \gamma \left(\frac{1}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma}} \left(\frac{y_{i,0}}{\rho_{i,0}}\right)^{\frac{\gamma+1}{\gamma}}\right], & \text{if } (\rho_{i,0}, y_{i,0}) \in \mathcal{D}_2, \\ \left[0, y_{i,0} - \rho_{i,0}^{\gamma+1}\right], & \text{if } (\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1. \end{cases} \tag{5.16}$$

5.2 Admissible states in outgoing roads.

Consider an outgoing road I_j, with an initial state $(\rho_{j,0}, y_{j,0})$. We describe the solutions given by an intermediate state $(\bar{\rho}, \bar{y})$ and a final state $(\hat{\rho}_j, \hat{y}_j)$.

Proposition 5.2 *Any state $(\bar{\rho}, \bar{y})$ on a curve of the second family through the point $(\rho_{j,0}, y_{j,0})$ can be connected to $(\rho_{j,0}, y_{j,0})$ by a contact discontinuity wave of the second family with speed greater than or equal to 0.*

Proof. The proof follows from the fact that the second eigenvalue λ_2 is greater than or equal to 0 in $\mathcal{D}$. $\square$

Proposition 5.3 *A state $(\hat{\rho}_j, \hat{y}_j) \neq (0,0)$ is connectible to a given state $(\bar{\rho}, \bar{y})$ by a wave of the first family with strictly positive speed if and only if $\bar{y} = \frac{\hat{y}_j}{\hat{\rho}_j}\bar{\rho}$ and one of the followings holds:*

1. *$\bar{y} < (\gamma+1)\bar{\rho}^{\gamma+1}$. In this case there exists $\tilde{\rho} < \bar{\rho}$ such that all the possible final states $(\hat{\rho}_j, \hat{y}_j)$ are those with $\hat{\rho}_j < \tilde{\rho}$.*

2. *$\bar{y} \geqslant (\gamma+1)\bar{\rho}^{\gamma+1}$. In this case we have that*

$$0 \leqslant \hat{\rho}_j \leqslant \left(\frac{1}{\gamma+1}\right)^{1/\gamma} \left(\frac{\hat{y}_j}{\hat{\rho}_j}\right)^{1/\gamma}.$$

   If $\hat{\rho}_j < \bar{\rho}$, then the wave of the first family connecting $(\hat{\rho}_j, \hat{y}_j)$ to $(\bar{\rho}, \bar{y})$ is a shock wave, while, if $\hat{\rho}_j > \bar{\rho}$, then the wave of the first family connecting $(\hat{\rho}_j, \hat{y}_j)$ to $(\bar{\rho}, \bar{y})$ is a rarefaction wave.

Proof. First, we note that, if $(\hat{\rho}_j, \hat{y}_j)$ is connectible to $(\bar{\rho}, \bar{y})$ with a wave of the first family, then $\bar{y} = \frac{\hat{y}_j}{\hat{\rho}_j}\bar{\rho}$.

If $\bar{\rho} < \hat{\rho}_j$, then the minimum speed of the wave of the first family connecting $(\hat{\rho}_j, \hat{y}_j)$ to $(\bar{\rho}, \bar{y})$ is given by

$$\lambda_1(\hat{\rho}_j, \hat{y}_j) = \frac{\hat{y}_j}{\hat{\rho}_j} - (\gamma+1)\hat{\rho}_j^{\gamma}.$$

Therefore the speed is positive if and only if

$$\hat{y}_j \geqslant (\gamma+1)\hat{\rho}_j^{\gamma+1}.$$

Instead, if $\bar{\rho} > \bar{\rho}_j$, then the speed of the wave of the first family connecting $(\hat{\rho}_j, \hat{y}_j)$ to $(\bar{\rho}, \bar{y})$ is positive if and only if

$$\frac{(\hat{y}_j - \hat{\rho}_j^{\gamma+1}) - (\bar{y} - \bar{\rho}^{\gamma+1})}{\hat{\rho}_j - \bar{\rho}} > 0,$$

which is equivalent to

$$\frac{\bar{y}}{\bar{\rho}} > \frac{\bar{\rho}^{\gamma+1} - \hat{\rho}_j^{\gamma+1}}{\bar{\rho} - \hat{\rho}_j}. \tag{5.17}$$

If $\bar{y} \geqslant (\gamma+1)\bar{\rho}^{\gamma+1}$, then the supremum of the second member of (5.17) when $0 < \hat{\rho}_j < \bar{\rho}$ is equal to $(\gamma+1)\bar{\rho}^{\gamma}$, which is lower than or equal to $\bar{y}/\bar{\rho}$.

If instead $\bar{y} < (\gamma+1)\bar{\rho}^{\gamma+1}$, then as in the previous subsection there exists $\tilde{\rho} < \bar{\rho}$ with the desired property. □

Putting together the results of the last two propositions, we obtain the set of admissible states in an outgoing road; see Figures 4 and 5.

The possible values of the first component of the flux in an outgoing road I_j is given by

$$\Omega_j = \left[0, \gamma\left(\frac{1}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma}}\right] \tag{5.18}$$

if the curve of the second family through $(\rho_{j,0}, y_{j,0})$ is completely inside $\mathcal{D}_1$, while

$$\Omega_j = \left[0, \frac{1}{\rho_{j,0}}(y_{j,0} - \rho_{j,0}^{\gamma+1})\left(1 + \rho_{j,0}^{\gamma} - \frac{y_{j,0}}{\rho_{j,0}}\right)^{\frac{1}{\gamma}}\right] \tag{5.19}$$

in the other case.

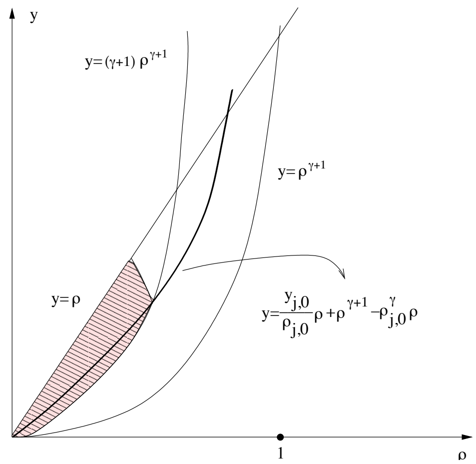


Figure 4: Admissible states in an outgoing road I_j when the curve of the second family through $(\rho_{j,0}, y_{j,0})$ (in bold) is not completely in $\mathcal{D}_1$. The admissible final states $(\hat{\rho}_j, \hat{y}_j)$ belongs to that curve or to the drawn region.

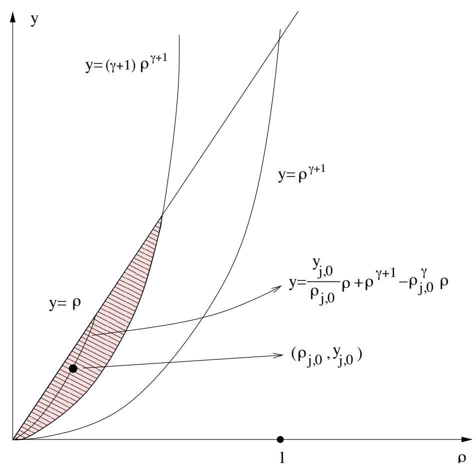


Figure 5: Admissible states in an outgoing road I_j when the curve of the second family through $(\rho_{j,0}, y_{j,0})$ is completely in $\mathcal{D}_1$. The admissible final states $(\hat{\rho}_j, \hat{y}_j)$ belongs to the drawn region.

Remark 5 *Notice that if* $(\rho_{j,0}, y_{j,0}) \neq (0,0)$ *satisfies* $y_{j,0} = \rho_{j,0}^{\gamma+1}$*, then the final state* $(\hat{\rho}_j, \hat{y}_j)$ *must be equal to* $(\rho_{j,0}, y_{j,0})$*. In fact, if* $(\hat{\rho}_j, \hat{y}_j)$ *belongs to the curve of the second family through* $(\rho_{j,0}, y_{j,0})$*, then the wave connecting the two states has zero speed and so it is not admissible, while if* $(\hat{\rho}_j, \hat{y}_j)$ *belongs to the curve of the first family through* $(\rho_{j,0}, y_{j,0})$*, then the speed of the wave is negative.*

5.3 Riemann problem with rules (R-1)–(R-4).

By the analysis of the previous subsections, (R-1) gives the possible density fluxes in each road of J, (R-3) individuates the set

$$\Omega := \{(\delta_1, \ldots, \delta_n) \in \Omega_1 \times \cdots \times \Omega_n | A \cdot (\delta_1, \ldots, \delta_n) \in \Omega_{n+1} \times \cdots \times \Omega_{n+m}\}, \tag{5.20}$$

(R-4) prescribes the maximization of the function

$$E : (\delta_1, \ldots, \delta_n) \mapsto \sum_{i=1}^{n} \delta_i, \tag{5.21}$$

while (R-2) is granted once (R-3) is satisfied. The set Ω is closed, convex and non empty. Moreover, by (A3), ∇E is not orthogonal to any nontrivial subspace contained in a supporting hyperplane to Ω, hence there exists a unique vector $\hat{\delta} = (\hat{\delta}_1, \ldots, \hat{\delta}_n) \in \Omega$ such that

$$E(\hat{\delta}_1, \ldots, \hat{\delta}_n) = \max_{(\delta_1, \ldots, \delta_n) \in \Omega} E(\delta_1, \ldots, \delta_n). \tag{5.22}$$

With this procedure we find uniquely the values of the fluxes of density of the solution of the Riemann problem at the junction J. More precisely, $\hat{\delta}_i$ gives the value of density fluxes in incoming roads, while density fluxes in outgoing roads are defined by

$$(\hat{\delta}_{n+1}, \ldots, \hat{\delta}_{n+m})^T = A \cdot (\hat{\delta}_1, \ldots, \hat{\delta}_n)^T.$$

For every $i \in \{1, \ldots, n\}$, we have to choose an element $(\hat{\rho}_i, \hat{y}_i)$, which is an admissible state as discussed in Subsection 5.1 and such that the flux $\hat{y}_i - \hat{\rho}_i^{\gamma+1}$ is equal to $\hat{\delta}_i$. In order to do this, we need to solve the system

$$\begin{cases} y = \frac{y_{i,0}}{\rho_{i,0}} \rho, \\ y = \rho^{\gamma+1} + \hat{\delta}_i. \end{cases} \tag{5.23}$$

This system in general admits two solutions in $\mathcal{D}$, but only one is admissible. So we take

$$(\hat{\rho}_i, \hat{y}_i) = (\rho_{i,0}, y_{i,0}) \tag{5.24}$$

if $y_{i,0} = \rho_{i,0}^{\gamma+1} + \hat{\delta}_i$, otherwise $(\hat{\rho}_i, \hat{y}_i)$ is the unique solution in $\mathcal{D}_2$ of the system (5.23).

The situation is more complicated in outgoing roads. In fact, by the analysis of subsection 5.2, it is evident that, in general, there are infinitely many solutions satisfying rules (R-1)–(R-4), since the intersection between the level curve of the flux $y = \rho^{\gamma+1} + \hat{\delta}_j$ with the region of admissible states is an one dimensional manifold.

5.4 (AR-1): maximize the speed.

Proposition 5.4 *The rule (AR-1) determines a unique solution of the Riemann problem at the junction J. The final state $(\hat{\rho}_j, \hat{y}_j)$ belongs to the line $y = \rho$.*

Proof. First of all, we recall that the second characteristic field is linearly degenerate and the second eigenvalue λ_2 is equal to the velocity v of the cars. So the curves of the second family are the level sets for the speed v of the cars. On the contrary, the speed v is monotone decreasing in ρ on level curves of density flux. Therefore, the solution is given by

$$\begin{cases} y = \rho^{\gamma+1} + \hat{\delta}_j, \\ y = \rho. \end{cases} \tag{5.25}$$

There are some different cases.

1. $\hat{\delta}_j < \sup \Omega_j$. In this case $(\hat{\rho}_j, \hat{y}_j)$ is the solution to the system (5.25), that is in $\mathcal{D}_1$. In general, to connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$ we use a wave of the first family with positive speed and a wave of the second family; see Figure 6.

Remark 6 *If we recalculate Ω_j using $(\hat{\rho}_j, \hat{y}_j)$ instead of $(\rho_{j,0}, y_{j,0})$, then the obtained set $\hat{\Omega}_j$ may be bigger than Ω_j. Thus it seems that the solution can be found in two steps, but this is not the case, since $\hat{\delta}_j < \sup(\Omega_j)$ implies that the maximization problem (5.22) has the same solution.*

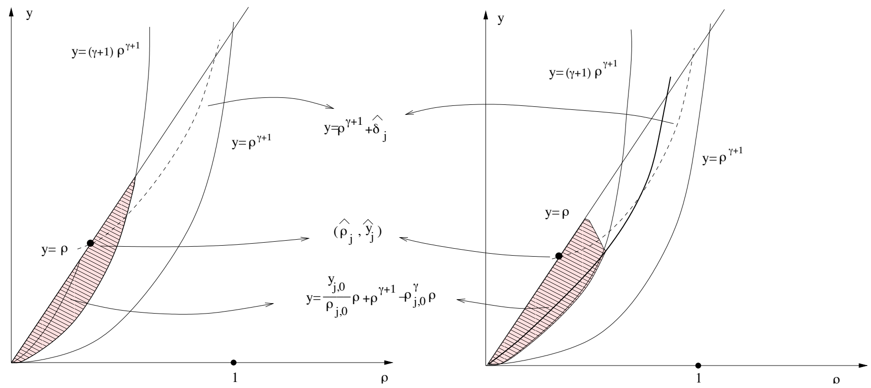


Figure 6: Solution $(\hat{\rho}_j, \hat{y}_j)$ to the Riemann problem on an outgoing road I_j in the case 1 with the additional rule (AR-1). In the first picture it is drawn the case in which the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ does not intersect in $\mathcal{D}$ the curve of the second family through $(\rho_{j,0}, y_{j,0})$, while in the second picture the other case.

2. $\hat{\delta}_j = \sup(\Omega_j)$, $y_{j,0} < \rho_{j,0}$, and the curve of the second family through $(\rho_{j,0}, y_{j,0})$ lies completely in the region $\mathcal{D}_1$; see Figure 7. By the analysis of subsection 5.2, the set Ω_j is given by (5.18) and so it is the maximum possible. Hence there exists only one point in $\mathcal{D}$ with the first component of the flux equal to $\hat{\delta}_j$ and this point is precisely given by the intersection between the line $y = \rho$ and the curve of maxima. Thus

$$(\hat{\rho}_j, \hat{y}_j) = \left(\left(\frac{1}{\gamma+1} \right)^{\frac{1}{\gamma}}, \left(\frac{1}{\gamma+1} \right)^{\frac{1}{\gamma}} \right),$$

and to connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$ we may use a wave of the first family with positive speed and a wave of the second family.

3. $\hat{\delta}_j = \sup(\Omega_j)$, $y_{j,0} < \rho_{j,0}$ and the curve of the second family through $(\rho_{j,0}, y_{j,0})$ is not completely contained in the region $\mathcal{D}_1$. In this case $(\hat{\rho}_j, \hat{y}_j)$ is given by the solution of the system

$$\begin{cases} y = \rho \\ y = \rho^{\gamma+1} + \hat{\delta}_j \\ (\rho, y) \in \mathcal{D}_2. \end{cases}$$

since as in the previous case the intersection between the region of admissible

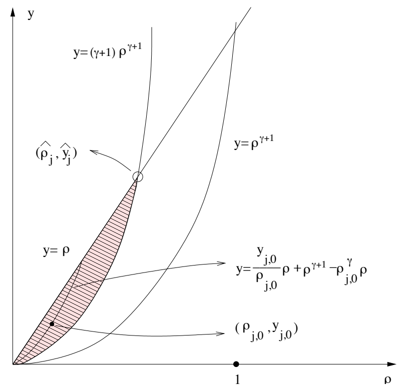


Figure 7: Solution $(\hat{\rho}_j, \hat{y}_j)$ to the Riemann problem on an outgoing road I_j in the case 2 with the additional rule (AR-1).

final states and the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ consists of a single point; see Figure 8. To connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$ we use only a wave of the second family.

4. $\hat{\delta}_j = \sup(\Omega_j)$ and $y_{j,0} = \rho_{j,0}$. If $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_1$, then as in case 2 the set Ω_j is the maximum possible and so the solution is given by

$$(\hat{\rho}_j, \hat{y}_j) = \left(\left(\frac{1}{\gamma+1} \right)^{\frac{1}{\gamma}}, \left(\frac{1}{\gamma+1} \right)^{\frac{1}{\gamma}} \right),$$

and to connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$ we use only a wave of the first family with positive speed. If instead $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_2$, then, as in case 3, the solution is

$$(\hat{\rho}_j, \hat{y}_j) = (\rho_{j,0}, y_{j,0}),$$

and no wave is produced.

So the proof is finished. $\square$

Remark 7 *Notice that the problem of finding the solution to the Riemann problem with the additional rule (AR-1) is completely equivalent to using the additional rule: minimize the density of the cars in outgoing roads.*

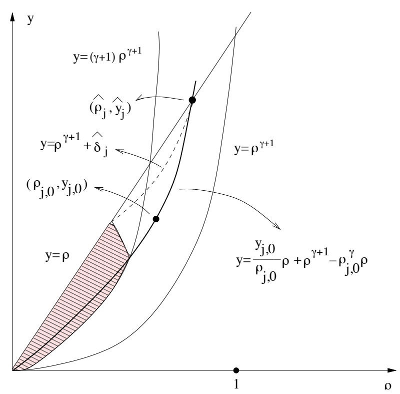


Figure 8: Solution $(\hat{\rho}_j, \hat{y}_j)$ to the Riemann problem on an outgoing road I_j in the case 3 with the additional rule (AR-1).

5.5 (AR-2): maximize the density.

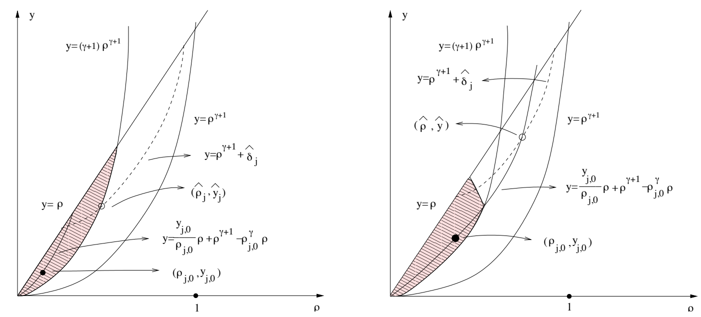
In this case we have to find the admissible point $(\hat{\rho}_j, \hat{y}_j)$ belonging to the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ with the maximum ρ.

Proposition 5.5 *The solution of the Riemann problem at the junction J with the additional rule (AR-2) is unique and the final state $(\hat{\rho}_j, \hat{y}_j)$ belongs to the region $\mathcal{D}_2$. Moreover, $(\hat{\rho}_j, \hat{y}_j)$ belongs to the curve of the second family through $(\rho_{j,0}, y_{j,0})$ or $(\hat{\rho}_j, \hat{y}_j)$ belongs to the curve of maxima.*

Proof. We have two different possibilities.
1. The curve of the second family through $(\rho_{j,0}, y_{j,0})$ is completely in the region $\mathcal{D}_1$. In this case the admissible final states are exactly all the points of $\mathcal{D}_1$ and so the solution $(\hat{\rho}_j, \hat{y}_j)$ belongs to the part of the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ which lies in $\mathcal{D}_1$. If we want to maximize the density we have to choose the point $(\hat{\rho}_j, \hat{y}_j)$ given by

$$\begin{cases} y = \rho^{\gamma+1} + \hat{\delta}_j, \\ y = (\gamma+1)\rho^{\gamma+1}, \end{cases}$$

as we clearly see in figure 9.a. To connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$, we have to use in general a wave of the first family with positive speed and a wave of the second family.
2. The curve of the second family through $(\rho_{j,0}, y_{j,0})$ is not completely in the region $\mathcal{D}_1$. If the curve of the second family through $(\rho_{j,0}, y_{j,0})$ intersects the



(a) (b)

Figure 9: Solution $(\hat{\rho}_j, \hat{y}_j)$ to the Riemann problem on an outgoing road I_j with the additional rule (AR-3). The first figure shows the case where the curve of the second family through $(\rho_{j,0}, y_{j,0})$ is completely in the region $\mathcal{D}_1$, while the second figure shows the other case.

curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ only in the region $\mathcal{D}_1$, then the solution $(\hat{\rho}_j, \hat{y}_j)$ is given as in the previous case by the system

$$\begin{cases} y = \rho^{\gamma+1} + \hat{\delta}_j, \\ y = (\gamma+1)\rho^{\gamma+1}, \end{cases}$$

and to connect the two states we use a wave of the first family with positive speed and a wave of the second family. Otherwise $(\hat{\rho}_j, \hat{y}_j)$ is given solving

$$\begin{cases} y = \rho^{\gamma+1} + \hat{\delta}_j, \\ y = \frac{y_{j,0}}{\rho_{j,0}}\rho + \rho^{\gamma+1} - \rho_{j,0}^{\gamma}\rho, \end{cases}$$

as we see in figure 9.b. To connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$, we use only a wave of the second family.

Thus the proof is finished. □

5.6 (AR-3): minimize the total variation.

We start by proving the following lemmata.

Lemma 5.1 *If $(\rho_{j,0}, y_{j,0}) = (0, 0)$ then the point $(\hat{\rho}_j, \hat{y}_j)$ belongs to the line $y = \rho$ and to the region $\mathcal{D}_1$.*

Proof. Here the set of admissible states is the whole $\mathcal{D}$ and in this case minimizing the total variation of ρ along a solution is equivalent to choose the point of the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ with minimum ρ. $\square$

From Remark 5, we have immediately

Lemma 5.2 *Let $(\rho_{j,0}, y_{j,0}) \neq (0,0)$ and $y_{j,0} = \rho_{j,0}^{\gamma+1}$. In this case the solution $(\hat{\rho}_j, \hat{y}_j)$ is equal to $(\rho_{j,0}, y_{j,0})$.*

Lemma 5.3 *Assume $y_{j,0} > \rho_{j,0}^{\gamma+1}$. If the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ intersects the curve of the second family through $(\rho_{j,0}, y_{j,0})$, then the solution $(\hat{\rho}_j, \hat{y}_j)$ is given by the unique intersection of those curves.*

Proof. It is easy to see that the intersection between the curve of the second family through $(\rho_{j,0}, y_{j,0})$ and the curve $y = \rho^{\gamma+1} + \hat{\rho}_j$ consists of at most one point. Then $(\hat{\rho}_j, \hat{y}_j)$ is such intersection and the total variation of the density ρ along the solution is simply given by $|\hat{\rho}_j - \rho_{j,0}|$. In order to prove that the solution attains the minimum of variation in ρ, let us consider an other admissible point $(\bar{\rho}, \bar{y})$ such that

$$\begin{cases} \bar{y} = \bar{\rho}^{\gamma+1} + \hat{\delta}_j, \\ \bar{\rho} \neq \hat{\rho}_j. \end{cases}$$

We must have $\min\{\hat{\rho}_j, \rho_{j,0}\} \leqslant \bar{\rho} \leqslant \max\{\hat{\rho}_j, \rho_{j,0}\}$. If such a point $(\bar{\rho}, \bar{y})$ exists, then to connect $(\bar{\rho}, \bar{y})$ with $(\rho_{j,0}, y_{j,0})$ we need to use first a wave of the first family until a point $(\tilde{\rho}, \tilde{y})$ and then a wave of the second family. Thus the total variation of the density ρ along this solution is given by $|\bar{\rho} - \tilde{\rho}| + |\tilde{\rho} - \rho_{j,0}|$, where $\tilde{\rho}$ satisfies either $\tilde{\rho} < \min\{\hat{\rho}_j, \rho_{j,0}\}$ or $\tilde{\rho} > \max\{\hat{\rho}_j, \rho_{j,0}\}$ and so the proof is finished. $\square$

Lemma 5.4 *Assume $y_{j,0} > \rho_{j,0}^{\gamma+1}$. If the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ does not intersect the curve of the second family through $(\rho_{j,0}, y_{j,0})$, then the solution $(\hat{\rho}_j, \hat{y}_j)$ is given by*

$$\begin{cases} y = \rho^{\gamma+1} + \hat{\delta}_j, \\ y = \rho, \\ (\rho, y) \in \mathcal{D}_1. \end{cases} \tag{5.26}$$

Proof. The only possibility is that the curve of the second family through $(\rho_{j,0}, y_{j,0})$ is completely inside the region $\mathcal{D}_1$. Let us call $(\hat{\rho}_j, \hat{y}_j)$ the solution to (5.26). To connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$ we have to use first a rarefaction wave of the first family until a state $(\bar{\rho}, \bar{y})$ with $\rho_{j,0} < \bar{\rho} < \hat{\rho}_j$ and then a wave of the second family. The total variation of the density ρ along this solution is equal to $|\hat{\rho}_j - \rho_{j,0}|$. Any other point of $y = \rho^{\gamma+1} + \hat{\delta}_j$ generates a variation in ρ strictly bigger than $|\hat{\rho}_j - \rho_{j,0}|$ and so the lemma is proved. $\square$

6 Stability of solutions to Riemann problems at junctions.

The aim of this section is to investigate stability of constant (on each road) solutions to Riemann problem, called equilibria. Stability simply means that small perturbations of the data in L^∞ norm, that may be produced by waves arriving at junctions, produce small variations of the equilibrium in L^∞ norm. As in the previous section, we have to consider different cases according to the additional rules (AR-1), (AR-2) or (AR-3).

In the whole section, we consider a fixed junction J with n incoming roads (say $I_1, \ldots, I_n$) and m outgoing roads (say $I_{n+1}, \ldots, I_{n+m}$) and we assume that $((\rho_{1,0}, y_{1,0}), \ldots, (\rho_{n+m,0}, y_{n+m,0}))$ is an equilibrium at J.

We want to remark that waves of the second family have always positive speed. Moreover waves of the first family connecting two states in the region $\mathcal{D}_1$ have positive speed, while waves of the first family connecting two states in the region $\mathcal{D}_2$ have negative speed. The consequences of this fact are the followings.

Claim 1. In an outgoing road only waves of the first family can reach the junction. Therefore if $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_1$, then it can not be perturbed by waves connecting $(\rho_{j,0}, y_{j,0})$ with an other state $(\bar{\rho}, \bar{y}) \in \mathring{\mathcal{D}}_1$; in fact, in this case, also waves of the first family have positive speed.

Claim 2. Assume $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$. If a wave on a road different from I_i produces a variation of the solution of the Riemann problem at the junction, then the new solution $(\hat{\rho}_i, \hat{y}_i)$ in the incoming road I_i either is equal to $(\rho_{i,0}, y_{i,0})$ or $(\hat{\rho}_i, \hat{y}_i)$ belongs to $\mathring{\mathcal{D}}_2$. In the latter case the distance between $(\rho_{i,0}, y_{i,0})$ and $(\hat{\rho}_i, \hat{y}_i)$ is proportional to the distance between $(\rho_{i,0}, y_{i,0})$ and the curve of maxima. Thus, such configuration is unstable.

6.1 (AR-1): maximize the speed.

Recall that, by Proposition 5.4, all equilibria for outgoing roads must belong to the line $y = \rho$. The analysis of all equilibria is very complicated, hence we prefer to treat in detail only some significant cases. We also consider all the general case when $n = m = 2$.

We have some different possibilities.

1. $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_1$ for every $j \in \{n+1, \ldots, n+m\}$. Therefore, the maximization problem (5.22) implies that $(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1$ for every $i \in \{1, \ldots, n\}$. In this case, by (5.16) and (5.18), we deduce that: for incoming roads

$$\Omega_i = \left[0, y_{i,0} - \rho_{i,0}^{\gamma+1}\right],$$

while for outgoing ones

$$\Omega_j = \left[0, \gamma \left(\frac{1}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma}}\right].$$

If we denote by $\delta_{i,0} := y_{i,0} - \rho_{i,0}^{\gamma+1}$ for every $i \in \{1, \ldots, n\}$ and by $\delta_{j,0} := y_{j,0} - \rho_{j,0}^{\gamma+1}$ for every $j \in \{n+1, \ldots, n+m\}$, then clearly $(\delta_{1,0}, \ldots, \delta_{n,0})$ is the solution of the maximization problem (5.22) and

$$(\delta_{n+1,0}, \ldots, \delta_{n+m,0})^T = A \cdot (\delta_{1,0}, \ldots, \delta_{n,0})^T.$$

The hypothesis $y_{j,0} > (\gamma+1)\rho_{j,0}^{\gamma+1}$ for every $j \in \{n+1, \ldots, n+m\}$ has the following two consequences. Firstly, $\delta_{j,0} < \sup \Omega_j$ and hence the outgoing roads give no constraint for the maximization problem (5.22). Secondly, by claim 1, the outgoing roads cannot be perturbed by waves with negative speed. Consider a perturbation produced by a wave of the first or second family from an incoming road I_i connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$. The possible density fluxes are in the set

$$\widetilde{\Omega}_i = \left[0, \tilde{y}_i - \tilde{\rho}_i^{\gamma+1}\right]$$

if $(\tilde{\rho}_i, \tilde{y}_i) \in \mathcal{D}_1$, while

$$\widetilde{\Omega}_i = \left[0, \gamma \left(\frac{1}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma}} \left(\frac{\tilde{y}_i}{\tilde{\rho}_i}\right)^{\frac{\gamma+1}{\gamma}}\right]$$

in the other case. Since the outgoing roads are not constraints for the maximization problem (5.22), we may suppose the following, provided the perturbation is sufficiently small:

(a) the new maximum point for (5.22) is

$$(\hat{\delta}_1, \dots, \hat{\delta}_n) := \left(y_{1,0} - \rho_{1,0}^{\gamma+1}, \dots, \tilde{y}_i - \tilde{\rho}_i^{\gamma+1}, \dots, y_{n,0} - \rho_{n,0}^{\gamma+1}\right)$$

if $(\tilde{\rho}_i, \tilde{y}_i) \in \mathcal{D}_1$, while

$$(\hat{\delta}_1, \dots, \hat{\delta}_n) := \left(y_{1,0} - \rho_{1,0}^{\gamma+1}, \dots, \gamma \left(\frac{1}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma}} \left(\frac{\tilde{y}_i}{\tilde{\rho}_i}\right)^{\frac{\gamma+1}{\gamma}}, \dots, y_{n,0} - \rho_{n,0}^{\gamma+1} \right).$$

in the other case;

(b) the solution $(\hat{\delta}_{n+1}, \dots, \hat{\delta}_{n+m})$ defined by

$$(\hat{\delta}_{n+1}, \dots, \hat{\delta}_{n+m})^T = A \cdot (\hat{\delta}_1, \dots, \hat{\delta}_n)^T$$

satisfies

$$\hat{\delta}_j < \sup \widetilde{\Omega}_j$$

for every $j \in \{n+1, \dots, n+m\}$ (the outgoing roads do not become constraints for the maximization problem (5.22));

(c) there exists a positive constant C such that

$$|(\hat{\rho}_i, \hat{y}_i) - (\tilde{\rho}_i, \tilde{y}_i)| + \sum_{j=n+1}^{n+m} |(\hat{\rho}_j, \hat{y}_j) - (\rho_{j,0}, y_{j,0})| < C\,|(\rho_i, y_i) - (\tilde{\rho}_i, \tilde{y}_i)|\,.$$

Moreover in outgoing roads waves of the first family are produced, while in incoming roads no waves are produced except in the I_i road.

The conclusion is that this kind of equilibrium is stable under small perturbations.

2. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$ for every $i \in \{1, \dots, n\}$ and $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_2$ for some $j \in \{n+1, \dots, n+m\}$. This is an unstable equilibrium. In fact, let I_{j_1} be the outgoing road with the property $y_{j_1,0} \leqslant (\gamma+1)\rho_{j_1,0}^{\gamma+1}$. It is possible to consider a perturbation generated by a wave of the first family connecting $(\rho_{j_1,0}, y_{j_1,0})$ with $(\tilde{\rho}_{j_1}, \tilde{y}_{j_1})$ such that

$$\sup \widetilde{\Omega}_{j_1} < \sup \Omega_{j_1},$$

where $\widetilde{\Omega}_{j_1}$ is defined as in (5.19) for the state $(\tilde{\rho}_{j_1}, \tilde{y}_{j_1})$. In this case the maximization problem (5.22) produces a flux in an incoming road I_i, which is strictly lower than $\sup \Omega_i$, hence the final state jumps into the region $\mathring{\mathcal{D}}_2$.

3. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \dots, n\}$ and $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_2$ for every $j \in \{n+1, \dots, n+m\}$. The fact that $y_{i,0} < (\gamma+1)\rho_{i,0}^{\gamma+1}$ for every $i \in \{1, \dots, n\}$ implies that $\delta_{i,0} := y_{i,0} - \rho_{i,0}^{\gamma+1} < \sup \Omega_i$ for every $i \in \{1, \dots, n\}$ and hence the incoming roads are not constraints for the maximization problem (5.22). Therefore we have stability for perturbations by waves from incoming roads.

Instead the perturbation of an outgoing road in general produces a variation of the maximization problem (5.22), since by hypotheses $\delta_{j,0} := y_{j,0} - \rho_{j,0}^{\gamma+1} = \sup \Omega_j$ for every $j \in \{n+1, \dots, n+m\}$ (all the outgoing roads are constraints for the maximization problem (5.22)).

First of all, let us consider the case $m > n$. The maximum for (5.22) is determined only by n constraints. Consider a wave of the first family in an outgoing road I_j connecting $(\rho_{j,0}, y_{j,0})$ with $(\tilde{\rho}_j, \tilde{y}_j) \in \mathring{\mathcal{D}}_2$. We denote by $\widetilde{\Omega}_j$ the set defined by (5.19) where $(\tilde{\rho}_j, \tilde{y}_j)$ is the initial state. If $\sup \widetilde{\Omega}_j > \sup \Omega_j$, then the maximum for (5.22) does not vary, but the I_j road is no more an active constraint since $\delta_j < \sup \widetilde{\Omega}_j$. Then the final state $(\hat{\rho}_j, \hat{y}_j) \in \mathring{\mathcal{D}}_1$ and the equilibrium is unstable.

Now let us consider the case $m = n$. We consider a perturbation in an outgoing road I_j by a wave of the first family connecting $(\rho_{j,0}, y_{j,0})$ with $(\tilde{\rho}_j, \tilde{y}_j)$. If the perturbation is sufficiently small, then we may suppose the following:

(a) the new solution $(\hat{\delta}_1, \dots, \hat{\delta}_n)$ of the maximization problem (5.22) satisfies $\hat{\delta}_i < \sup \Omega_i$ for every $i \in \{1, \dots, n\}$ (the incoming roads are not constraints for the maximization problem (5.22)) and the final states $(\hat{\rho}_i, \hat{y}_i)$ in the incoming roads belong to $\mathring{\mathcal{D}}_2$;

(b) $\hat{\delta}_j = \tilde{y}_j - \tilde{\rho}_j^{\gamma+1}$, the fluxes for the other outgoing roads remain the same and the final state $(\hat{\rho}_j, \hat{y}_j)$ in the I_j outgoing road coincides with $(\tilde{\rho}_j, \tilde{y}_j)$;

(c) there exists a positive constant C such that

$$\sum_{i=1}^{n} \left| (\rho_{i,0}, y_{i,0}) - (\hat{\rho}_i, \hat{y}_i) \right| < C \left| (\rho_{j,0}, y_{j,0}) - (\tilde{\rho}_j, \tilde{y}_j) \right| .$$

Therefore the equilibrium is stable.

We may summarize all these results in the following.

Theorem 6.1 *If $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_1$ for every $j \in \{n+1, \ldots, n+m\}$, then the equilibrium is stable.*

If $m = n$, $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \ldots, n\}$ and $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_2$ for every $j \in \{n+1, \ldots, 2n\}$, then the equilibrium is stable.

Consider now the generic case for $m = n = 2$. For generic we mean that the active constraints are given exactly by two roads and the states belong to $\mathring{\mathcal{D}}_1$ and $\mathring{\mathcal{D}}_2$.

1. $(\rho_{1,0}, y_{1,0}) \in \mathring{\mathcal{D}}_1$, $(\rho_{2,0}, y_{2,0}) \in \mathring{\mathcal{D}}_1$, $(\rho_{3,0}, y_{3,0}) \in \mathring{\mathcal{D}}_1$, $(\rho_{4,0}, y_{4,0}) \in \mathring{\mathcal{D}}_1$. This case is covered by the previous theorem.

2. $(\rho_{1,0}, y_{1,0}) \in \mathring{\mathcal{D}}_1$, $(\rho_{2,0}, y_{2,0}) \in \mathring{\mathcal{D}}_2$, $(\rho_{3,0}, y_{3,0}) \in \mathring{\mathcal{D}}_2$, $(\rho_{4,0}, y_{4,0}) \in \mathring{\mathcal{D}}_1$. In this case the active constraints are given by the roads I_1 and I_3. By claim 1, we know that the datum $(\rho_{4,0}, y_{4,0})$ can not be perturbed. Consider a perturbation produced by a wave of the second family connecting $(\tilde{\rho}_2, \tilde{y}_2)$ with $(\rho_{2,0}, y_{2,0})$. If the strength of the wave is sufficiently small, then the maximization problem (5.22) admits the same maximum point. Therefore no change happens in road I_1 and I_3, and

$$\left|(\tilde{\rho}_2, \tilde{y}_2) - (\hat{\rho}_2, \hat{y}_2)\right| + \left|(\rho_{4,0}, y_{4,0}) - (\hat{\rho}_4, \hat{y}_4)\right| \leqslant C \left|(\tilde{\rho}_2, \tilde{y}_2) - (\rho_{2,0}, y_{2,0})\right|,$$

where C is a positive constant and $(\hat{\rho}_2, \hat{y}_2)$ and $(\hat{\rho}_4, \hat{y}_4)$ are the final states respectively in roads I_2 and I_4.

Consider now a perturbation produced by a wave connecting $(\tilde{\rho}_1, \tilde{y}_1)$ with $(\rho_{1,0}, y_{1,0})$. We may suppose the followings, provided the perturbation is small:

(a) the active constraints remain the roads I_1 and I_3;

(b) the final state in I_3 is $(\hat{\rho}_3, \hat{y}_3) = (\rho_{3,0}, y_{3,0})$, while in I_1 is $(\hat{\rho}_1, \hat{y}_1) = (\tilde{\rho}_1, \tilde{y}_1)$;

(c) for some $C > 0$ we have

$$\left|(\hat{\rho}_2, \hat{y}_2) - (\rho_{2,0}, y_{2,0})\right| + \left|(\rho_{4,0}, y_{4,0}) - (\hat{\rho}_4, \hat{y}_4)\right| \leqslant C \left|(\tilde{\rho}_1, \tilde{y}_1) - (\rho_{1,0}, y_{1,0})\right|,$$

where $(\hat{\rho}_2, \hat{y}_2)$ and $(\hat{\rho}_4, \hat{y}_4)$ are the final states respectively in roads I_2 and I_4.

The case of a perturbation in I_3 is completely similar. Therefore this equilibrium is stable. The other cases, in which the active constraints are given by an incoming road and an outgoing road, are similar to this one and so stable.

3. $(\rho_{1,0}, y_{1,0}) \in \mathring{\mathcal{D}}_2$, $(\rho_{2,0}, y_{2,0}) \in \mathring{\mathcal{D}}_2$, $(\rho_{3,0}, y_{3,0}) \in \mathring{\mathcal{D}}_2$, $(\rho_{4,0}, y_{4,0}) \in \mathring{\mathcal{D}}_2$. This case is covered by the previous theorem.

We conclude with the following.

Theorem 6.2 *Let J be a junction with 2 incoming and 2 outgoing roads. A generic equilibrium is stable.*

6.2 (AR-2): maximize the density.

By Proposition 5.5, we know that all equilibria in outgoing roads must be in the region $\mathcal{D}_2$. We notice that the instability for the equilibrium for the Riemann problem at J happens when there is a jump in incoming roads from the region $\mathring{\mathcal{D}}_1$ to the region $\mathring{\mathcal{D}}_2$.

We have some possibilities.

1. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$ for every $i \in \{1, \ldots, n\}$ and $y_{j,0} = \rho_{j,0}$ for some $j \in \{n+1, \ldots, n+m\}$. This implies that $\delta_{i,0} := y_{i,0} - \rho_{i,0}^{\gamma+1} = \sup \Omega_i$ for every $i \in \{1, \ldots, n\}$ and $\delta_{j,0} := y_{j,0} - \rho_{j,0}^{\gamma+1} \leqslant \sup \Omega_j$ for every $j \in \{n+1, \ldots, n+m\}$. Moreover there exists $j_1 \in \{n+1, \ldots, n+m\}$ such that $\delta_{j_1,0} = \sup \Omega_{j_1}$. This means that all the incoming roads and at least one outgoing road give a constraint for the maximization problem (5.22). This fact implies that the equilibrium is unstable. Indeed consider an incoming road I_i and a wave of the first family connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$ such that the set $\widetilde{\Omega}_i$, defined as in (5.16) for the state $(\tilde{\rho}_i, \tilde{y}_i)$, strictly contains Ω_i. There are at least n active constraints, so the point of maximum does not change and, if the perturbation is sufficiently small, then we produce a jump on the road I_i.

2. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$ for every $i \in \{1, \ldots, n\}$ and $y_{j,0} < \rho_{j,0}$ for every $j \in \{n+1, \ldots, n+m\}$. Define η as

$$\eta := \min_{j \in \{n+1, \ldots, n+m\}} \left\{ \sup \Omega_j - \left(y_j - \rho_j^{\gamma+1} \right) \right\},$$

then, by hypotheses we have that $\eta > 0$. Assume that a wave of the first family on an outgoing road I_j connecting $(\rho_{j,0}, y_{j,0})$ with $(\tilde{\rho}_j, \tilde{y}_j)$ arrives to

J. If the perturbation is sufficiently small, then the new set $\tilde{\Omega}_j$ defined as in (5.19) with the new state $(\tilde{\rho}_j, \tilde{y}_j)$ satisfies

$$\left|\sup \tilde{\Omega}_j - \sup \Omega_j\right| \leqslant \frac{\eta}{2}$$

and this implies that the maximization problem (5.22) remains unchanged. Then only a wave of the second family on I_j connecting $(\hat{\rho}_j, \hat{y}_j)$ with $(\tilde{\rho}_j, \tilde{y}_j)$ is created. Moreover if the perturbation is sufficiently small, then

$$|(\hat{\rho}_j, \hat{y}_j) - (\tilde{\rho}_j, \tilde{y}_j)| \leqslant C\, |(\rho_{j,0}, y_{j,0}) - (\tilde{\rho}_j, \tilde{y}_j)|\,,$$

where C is a positive constant. Now, suppose that a wave connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$ arrives at J. Assume first $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$. If the perturbation is sufficiently small, then:

(a) the new solution of the maximization problem (5.22) is given by

$$(\delta_{1,0} := y_{1,0} - \rho_{1,0}^{\gamma+1}, \ldots, \hat{\delta}_i, \ldots, \delta_{n,0} := y_{n,0} - \rho_{n,0}^{\gamma+1})$$

with $\hat{\delta}_i := \tilde{y}_i - \tilde{\rho}_i^{\gamma+1}$ and the final state $(\hat{\rho}_i, \hat{y}_i)$ is equal to $(\tilde{\rho}_i, \tilde{y}_i)$;

(b) the solution

$$(\hat{\delta}_{n+1}, \ldots, \hat{\delta}_{n+m})^T = A \cdot (\delta_{1,0}, \ldots, \hat{\delta}_i, \ldots, \delta_{n,0})^T$$

satisfies $\hat{\delta}_j < \sup \Omega_j$ for every $j \in \{n+1, \ldots, n+m\}$, the final states $(\hat{\rho}_j, \hat{y}_j)$ are such that $\hat{y}_j < \hat{\rho}_j$ for every $j \in \{n+1, \ldots, n+m\}$ (the outgoing roads are not constraints for the maximization problem (5.22)) and

$$\sum_{j=n+1}^{n+m} |(\hat{\rho}_j, \hat{y}_j) - (\rho_{j,0}, y_{j,0})| < C\, |(\tilde{\rho}_i, \tilde{y}_i) - (\rho_{i,0}, y_{i,0})|$$

for some C positive constant.

If, on the contrary, $(\rho_{i,0}, y_{i,0})$ is on the curve of maxima, then $\left|\hat{\delta}_i - \delta_{i,0}\right|$ is proportional to the incoming wave, $(\hat{\rho}_i, \hat{y}_i)$ is on the curve of maxima, (b) holds and we conclude similarly. So the equilibrium is stable.

3. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \ldots, n\}$, and $y_{j,0} = \rho_{j,0}$ for at least n indices $j \in \{n+1, \ldots, n+m\}$. Define

$$\eta := \min_{i \in \{1, \ldots, n\}} \left\{ \sup \Omega_i - \left(y_i - \rho_i^{\gamma+1}\right) \right\}.$$

If from an incoming road I_i a wave connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$ arrives at J, then the new set $\tilde{\Omega}_i$, defined as in (5.16) for the state $(\tilde{\rho}_i, \tilde{y}_i)$, satisfies

$$\left|\sup \tilde{\Omega}_i - \sup \Omega_i\right| \leqslant \frac{\eta}{2}$$

provided that the perturbation is sufficiently small. Thus the maximization problem (5.22) remains unchanged and only a wave of the first family connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\hat{\rho}_i, \hat{y}_i)$ is created. Moreover,

$$|(\hat{\rho}_i, \hat{y}_i) - (\tilde{\rho}_i, \tilde{y}_i)| \leqslant C\, |(\rho_i, y_i) - (\tilde{\rho}_i, \tilde{y}_i)|\,,$$

where C is a positive constant.

A similar case happens if the perturbation is on an outgoing road I_j with $y_{j,0} < \rho_{j,0}$.

Now, consider a wave connecting $(\rho_{j,0}, y_{j,0})$ with $(\tilde{\rho}_j, \tilde{y}_j)$ on an outgoing road I_j with $y_{j,0} = \rho_{j,0}$. For the maximization problem (5.22), the active constraints remain the same. Waves are produced only in incoming roads and on outgoing roads that give no active constraints. Then

$$\sum_{i=1}^{n+m} |(\hat{\rho}_i, \hat{y}_i) - (\rho_{i,0}, y_{i,0})| \leqslant C\, |(\hat{\rho}_j, \hat{y}_j) - (\rho_{j,0}, y_{j,0})|\,,$$

where C is a positive constant. Thus the equilibrium is stable.

Putting together all the previous results, we obtain the following.

Theorem 6.3 *If $(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1$ for every $i \in \{1, \ldots, n\}$ and $y_{j,0} < \rho_{j,0}$ for every $j \in \{n+1, \ldots, n+m\}$, then the equilibrium is stable.*

If $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \ldots, n\}$ and $y_{j,0} = \rho_{j,0}$ for at least n outgoing roads, then the equilibrium is stable.

Consider now the generic case when $m = n = 2$. Generically the states in outgoing roads belong to the region $y < \rho$, hence the outgoing roads are not constraints. Therefore there is only one generic case: the incoming roads are constraints for the maximization problem (5.22). So this is a stable equilibrium by Theorem 6.3. We have the following.

Theorem 6.4 *Let J be a junction with 2 incoming and 2 outgoing roads. A generic equilibrium is stable.*

6.3 (AR-3): minimize the total variation.

Notice that, in this case, the instability for the equilibrium happens when there is a jump in incoming roads from the region $\mathring{\mathcal{D}}_1$ to the region $\mathring{\mathcal{D}}_2$. We have some possibilities.

1. For every index $j \in \{n+1, \ldots, n+m\}$, $(\rho_{j,0}, y_{j,0}) \in \mathcal{D} \setminus \{(\rho, y) : \rho = y, \rho \geqslant (\frac{1}{\gamma+1})^{1/\gamma}\}$. Then $(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1$, $y_{i,0} - \rho_{i,0}^{\gamma+1} = \sup \Omega_i$ for every $i \in \{1, \ldots, n\}$ and $y_{j,0} - \rho_{j,0}^{\gamma+1} < \sup \Omega_j$ for every $j \in \{n+1, \ldots, n+m\}$. Define

$$\eta := \min_{j \in \{n+1, \ldots, n+m\}} \left\{ \sup \Omega_j - \left(y_{j,0} - \rho_{j,0}^{\gamma+1} \right) \right\}.$$

Assume that a wave connecting $(\rho_{j,0}, y_{j,0})$ with $(\tilde{\rho}_j, \tilde{y}_j)$ reaches J. This may happen only if $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_2$. If the wave is sufficiently small, then

$$\left| \sup \widetilde{\Omega}_j - \sup \Omega_j \right| \leqslant \frac{\eta}{2}$$

which implies that the maximization problem (5.22) remains unchanged. Only a wave connecting $(\hat{\rho}_j, \hat{y}_j)$ with $(\tilde{\rho}_j, \tilde{y}_j)$ is created. Moreover, if the perturbation is small, then

$$|(\hat{\rho}_j, \hat{y}_j) - (\tilde{\rho}_j, \tilde{y}_j)| \leqslant C \left| (\rho_{j,0}, y_{j,0}) - (\tilde{\rho}_j, \tilde{y}_j) \right|,$$

with C positive constant. Now, consider a wave connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$ on the incoming road I_i. If the perturbation is sufficiently small, then the maximization problem (5.22) has the following solution:

$$(\delta_{1,0}, \ldots, \tilde{\delta}_i, \ldots, \delta_{n,0}),$$

with $\tilde{\delta}_i := \sup \widetilde{\Omega}_i$ and $\delta_{l,0} := y_{l,0} - \rho_{l,0}^{\gamma+1}$. Moreover the fluxes of the density in the outgoing roads change in a continuous way with respect the strength of the perturbation. Thus this equilibrium is stable.

2. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$ for every $i \in \{1, \ldots, n\}$ and $y_{j,0} = \rho_{j,0}$, $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_2$ for some $j \in \{n+1, \ldots, n+m\}$. This is an unstable case. In fact, if a wave on an incoming road I_i reaches J, in such a way the set Ω_i increases, then the maximization problem (5.22) admits the same point of maximum (at least one outgoing road is an active constraint) and a jump happens in the incoming road I_i.

3. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \ldots, n\}$ and $y_{j,0} = \rho_{j,0}$, $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_2$ for at least n indices $j \in \{n+1, \ldots, n+m\}$. The active constraints are given by the outgoing roads. For small perturbations these are again the only active constraints. Thus the equilibrium is stable.

Putting together all the previous results we have:

Theorem 6.5 *If $(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1$ for every $i \in \{1, \dots, n\}$ and if, for every $j \in \{n+1, \dots, n+m\}$, $(\rho_{j,0}, y_{j,0}) \in \mathcal{D} \setminus \{(\rho, y) : \rho = y, \rho \geqslant (\frac{1}{\gamma+1})^{1/\gamma}\}$, then the equilibrium is stable.*

If $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \dots, n\}$ and $y_{j,0} = \rho_{j,0}$, $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_2$ for at least n indices $j \in \{n+1, \dots, n+m\}$, then the equilibrium is stable.

Consider now the generic case when $m = n = 2$. As in the previous subsection, the outgoing roads are not constraints. Therefore there is only one generic case: the incoming roads are constraints for the maximization problem (5.22). So this is a stable equilibrium by Theorem 6.5. We have the following.

Theorem 6.6 *Let J be a junction with 2 incoming and 2 outgoing roads. A generic equilibrium is stable.*

7 Existence of solution at a junction.

Fix a road network with only one junction J with n incoming and m outgoing roads and fix $((\rho_{1,0}, y_{1,0}), \dots, (\rho_{n+m,0}, y_{n+m,0}))$, a stable equilibrium for the Riemann problem at J for one of the additional rules (AR-1),(AR-2) or (AR-3). Assume the following hypothesis:

(H) there exist $k_1, k_2 \in \{1, 2\}$ such that $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_{k_1}$ for every $i \in \{1, \dots, n\}$ and $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_{k_2}$ for every $j \in \{n+1, \dots, n+m\}$.

By the analysis of the previous section, the following proposition holds.

Proposition 7.1 *There exists a positive constant C such that, if a wave in an incoming road I_i connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$ arrives at J and if the wave has sufficiently small total variation, then the solution to the Riemann problem at J $((\hat{\rho}_1, \hat{y}_1), \dots, (\hat{\rho}_{n+m}, \hat{y}_{n+m}))$ has the following properties:*

1. *if $(\rho_{l,0}, y_{l,0}) \in \mathring{\mathcal{D}}_i$ for some $i \in \{1, 2\}$ and $l \in \{1, \dots, n+m\}$, then $(\hat{\rho}_l, \hat{y}_l) \in \mathring{\mathcal{D}}_1$;*

2. *we have:*

$$\sum_{l=1,l\neq i}^{n+m} |(\hat{\rho}_l,\hat{y}_l)-(\rho_{l,0},y_{l,0})|+|(\hat{\rho}_i,\hat{y}_i)-(\tilde{\rho}_i,\tilde{y}_i)| \leqslant C\,|(\tilde{\rho}_i,\tilde{y}_i)-(\rho_{i,0},y_{i,0})|\,.$$

The same holds for a perturbation on an outgoing road.

Under assumption (H) we can prove the following.

Theorem 7.1 *Assume (H), then there exists $\varepsilon > 0$ such that the following holds. For every initial datum $((\rho_{1,0}(x), y_{1,0}(x)), \ldots, (\rho_{n+m,0}(x), y_{n+m,0}(x)))$ with*

$$\|(\rho_{l,0}(x), y_{l,0}(x))\|_{BV} \leqslant \varepsilon$$

and

$$\sup_{x\in(a_l,b_l)} |\rho_{l,0}(x)-\rho_{l,0}| + \sup_{x\in(a_l,b_l)} |y_{l,0}(x)-y_{l,0}| \leqslant \varepsilon$$

for every $l \in \{1,\ldots,n+m\}$, there exists a solution

$$((\rho_1(t,x), y_1(t,x)), \ldots, (\rho_{n+m}(t,x), y_{n+m}(t,x))),$$

defined for every $t \geqslant 0$, such that

1. *$(\rho_l(0,x), y_l(0,x)) = (\rho_{l,0}(x), y_{l,0}(x))$ for a.e. $x \in I_l$ and for every $l \in \{1,\ldots,n+m\}$;*
2. *$(\rho_l(t,x), y_l(t,x))$ is an entropic solution to (2.1) on each road I_l;*
3. *for a.e. $t > 0$,*

$$((\rho_1(t,b_1-), y_1(t,b_1-)), \ldots, (\rho_{n+m}(t,a_{n+m}+), y_{n+m}(t,a_{n+m}+)))$$

provides an equilibrium at J.

Proof. We consider a wave front tracking approximate solution, see [4]. For every $t > 0$, we denote by (x_k^i, σ_k^i) and (z_l^i, θ_l^i) the positions and strengths in the road I_i of all waves respectively of the first family and of the second family, where k and l belong to some finite sets of indices. For every road I_i, we consider as in [4] the two functionals

$$V_i(t) := \sum_k \left|\sigma_k^i\right| + \sum_l \left|\theta_l^i\right|$$

and

$$Q_i(t) := \sum_{z_l^i < x_k^i} \left|\sigma_k^i \theta_l^i\right| + \sum_{\sigma_k^i < 0} \left|\sigma_k^i \sigma_{k'}^i\right|,$$

which are the classical components of the Glimm functional (notice that the second family is linearly degenerate, hence in the functional Q_i interactions between waves of the second family do not appear). We introduce also a functional $\widetilde{V}$ measuring the strength of waves approaching J. If $i \in \{1, \ldots, n\}$, then define

$$\widetilde{V}_i(t) := \begin{cases} V_i(t), & \text{if } (\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1, \\ \sum_l \left|\theta_l^i\right|, & \text{if } (\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2. \end{cases}$$

For an outgoing road I_j, we put

$$\widetilde{V}_j(t) := \begin{cases} 0, & \text{if } (\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_1, \\ \sum_k \left|\sigma_k^j\right|, & \text{if } (\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_2. \end{cases}$$

Define $V(t) := \sum_{i=1}^{n+m} V_i(t)$, $Q(t) := \sum_{i=1}^{n+m} Q_i(t)$ and $\widetilde{V}(t) := \sum_{i=1}^{n+m} \widetilde{V}_i(t)$. We claim that there exist two positive constants C_1 and C_2 such that the functional

$$\Upsilon(t) := V(t) + C_1 \widetilde{V}(t) + C_2 Q(t)$$

is decreasing in time.

Assuming this, for every $t > 0$,

$$\begin{aligned} \Upsilon(t) \leqslant \Upsilon(0) &= V(0) + C_1 \widetilde{V}(0) + C_2 Q(0) \\ &\leqslant V(0) + C_1 V(0) + C_2 V^2(0) \end{aligned}$$

and, since Υ is equivalent to the total variation as norm, then the total variation of the approximate wave front tracking solution remains bounded for every $t > 0$, hence we have the conclusion by standard compactness arguments.

We prove now that Υ is decreasing in time. Clearly Υ changes only at times where two waves interact or a wave approaches J. If at a time $\tau > 0$ two waves interact in a road I_i, then, by standard estimates (see [4]), we have

$$\begin{aligned} \Delta V_i(\tau) &\leqslant C \cdot \text{ product of strength of waves}, \\ \Delta \widetilde{V}_i(\tau) &\leqslant C \cdot \text{ product of strength of waves}, \\ \Delta Q_i(\tau) &\leqslant -\frac{\text{product of strength of waves}}{2}, \end{aligned} \tag{7.27}$$

for some $C > 0$, provided that V is sufficiently small. If

$$\frac{C_2}{2} \geqslant C(1 + C_1), \tag{7.28}$$

then $\Delta\Upsilon \leqslant 0$ when waves interact in the roads. Consider now an interaction of a wave with J. For simplicity we assume that a wave of the second family (z_l^1, θ_l^1) arrives at J from the incoming road I_1 at time τ. The other cases are completely similar. By Proposition 7.1 , we have that:

$$\Delta V(\tau) \leqslant C \left|\theta_l^1\right|, \quad \Delta Q(\tau) \leqslant C \left|\theta_l^1\right| V(\tau-),$$

and

$$\Delta\widetilde{V}_1(\tau) = -\left|\theta_k^1\right|, \quad \Delta\widetilde{V}_i(\tau) = 0 \quad \text{for } i \neq 1.$$

Therefore $\Delta\widetilde{V}(\tau) = -\left|\theta_k^1\right|$ and

$$\Delta\Upsilon(\tau) \leqslant C\left|\theta_k^1\right| - C_1\left|\theta_k^1\right| + C_2 C\left|\theta_k^1\right| V(\tau-).$$

If

$$C_1 \geqslant C + CC_2 V, \tag{7.29}$$

then $\Delta\Upsilon \leqslant 0$ when a wave interacts with J.

Fix $C_1 \geqslant C$. Then it is possible to take C_2 satisfying (7.28). There exists $\delta > 0$ depending on C_1 and C_2 such that, if $V < \delta$, then (7.29) and (7.27) hold. As long as $V < \delta$, then Υ is decreasing, thus

$$\begin{aligned} V(t) \leqslant \Upsilon(t) \leqslant \Upsilon(0) &= V(0) + C_1\widetilde{V}(0) + C_2 Q(0) \\ &\leqslant (1 + C_1)V(0) + C_2 V^2(0) \leqslant C_3 \cdot V(0), \end{aligned}$$

for some constant $C_3 > 1$. Choosing $\varepsilon = \frac{\delta}{(n+m)C_3}$, we have that $V(0) \leqslant \frac{\delta}{C_3}$ thus $V(t) \leqslant \delta$ for every $t > 0$. So we conclude the proof. □

Appendix: total variation of the flux.

In the case of a road network where the Lighthill–Whitham–Richards scalar model is considered in each road, if every junction has exactly 2 incoming and 2 outgoing roads, then an increment of the total variation of the flux can happen only when a wave on an outgoing road interacts with the junction;

see [6]. Here the situation is different since there are cases in which the total variation of the flux of the density strictly increases after an interaction of a wave from an incoming road, even if we are considering a junction with 2 incoming and 2 outgoing roads. In fact, consider a junction J with I_1 and I_2 incoming roads and I_3 and I_4 outgoing roads. Moreover suppose $\gamma = 1$ and the matrix A defined by

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}.$$

Consider the point

$$(\delta_{1,0}, \delta_{2,0}, \delta_{3,0}, \delta_{4,0}) = (1/8, 1/8, 5/48, 7/48).$$

It is clear that

$$\begin{pmatrix} \delta_{3,0} \\ \delta_{4,0} \end{pmatrix} = A \cdot \begin{pmatrix} \delta_{1,0} \\ \delta_{2,0} \end{pmatrix}.$$

We show that there exists an equilibrium configuration with $\delta_{i,0}$ as density fluxes. In I_1 we consider the point on the curve of maxima

$$(\rho_{1,0}, y_{1,0}) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{4}\right)$$

so that $\Omega_1 = [0, 1/8]$ and $y_{1,0} - \rho_{1,0}^2 = 1/8$. In road I_2 we consider a point $(\rho_{2,0}, y_{2,0})$ such that $y_{2,0} - \rho_{2,0}^2 = 1/8$, $y_{2,0} < 2\rho_{2,0}^2$ and $\frac{1}{8} < \sup \Omega_2$. In road I_3 we consider the point

$$(\rho_{3,0}, y_{3,0}) = \left(\frac{1 + \sqrt{\frac{7}{12}}}{2}, \frac{1 + \sqrt{\frac{7}{12}}}{2}\right)$$

and so $\frac{5}{48} = y_{3,0} - \rho_{3,0}^2 = \sup \Omega_3$. Finally in I_4 we consider a point $(\rho_{4,0}, y_{4,0})$ such that $y_{4,0} - \rho_{4,0}^2 = \frac{7}{48} < \sup \Omega_4$. Notice that for every additional rule, it is possible to choose $(\rho_{4,0}, y_{4,0})$ such that $((\rho_{1,0}, y_{1,0}), \dots, (\rho_{4,0}, y_{4,0}))$ is an equilibrium for the Riemann problem at J. For this equilibrium, the active constraints are given by roads I_1 and I_3.

We perturb the equilibrium with a wave of the second family connecting $(\tilde{\rho}_1, \tilde{y}_1)$ with $(\rho_{1,0}, y_{1,0})$ such that the set $\widetilde{\Omega}_1$, defined as in (5.16) for the state $(\tilde{\rho}_1, \tilde{y}_1)$, is equal to $[0, 1/8 + \varepsilon]$, where ε is a small positive parameter. This is possible by taking

$$(\tilde{\rho}_1, \tilde{y}_1) = \left(\frac{\frac{1}{4} + 2\varepsilon}{\sqrt{\frac{1}{8} + \varepsilon}} - \frac{\sqrt{2}}{4}, \frac{\left(\frac{1}{4} + 2\varepsilon\right)^2}{\frac{1}{8} + \varepsilon} - \frac{\sqrt{2}}{4} \cdot \frac{\frac{1}{4} + 2\varepsilon}{\sqrt{\frac{1}{8} + \varepsilon}} \right) \in \mathring{\mathcal{D}}_2.$$

The new solution of (5.22) is given by

$$(\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4) = (1/8 + \varepsilon, 1/8 - 2\varepsilon/3, 5/48, 7/48 + \varepsilon/3).$$

Therefore the total variation of the first component of the flux after the interaction is given by

$$\begin{aligned} &\left|\hat{\delta}_1 - \delta_1\right| + \left|\hat{\delta}_2 - \delta_{2,0}\right| + \left|\hat{\delta}_3 - \delta_{3,0}\right| + \left|\hat{\delta}_4 - \delta_{4,0}\right| = \\ &= \hat{\delta}_1 - \delta_1 + \frac{2}{3}\varepsilon + \frac{\varepsilon}{3} = \frac{1}{8} - \delta_1 + 2\varepsilon, \end{aligned}$$

where $\delta_1 = y_1 - \rho_1^2 < \hat{\delta}_1$. Instead, the total variation of the first component of the flux before the interaction is given by

$$|\delta_1 - \delta_{1,0}| = \delta_1 - \frac{1}{8},$$

and so an increment of the total variation of the density flux happens, since $\delta_1 < \hat{\delta}_1 = 1/8 + \varepsilon$.

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