Draft: Control of a Second Order Model of Traffic

Density-velocity system

Let ρ denotes the density, v the velocity of the cars. The LWR model is

$$\partial_t \rho + \partial_x(\rho v) = 0\,. \tag{1}$$

The idea of the second order model is that the variation of the velocity is linked to the variation of the density through a function of pressure $p(\rho)$:

$$(\partial_t + v\partial_x)v + (\partial_t + v\partial_x)p(\rho) = 0\,. \tag{2}$$

The above equation can be rewritten as

$$\partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = 0\,. \tag{3}$$

Let us denote $w = v + p(\rho)$. The Aw-Rascle (AR) model is written as follows

$$\partial_t \rho + \partial_x(\rho v) = 0\,, \tag{4}$$

$$\partial_t w + v\partial_x w = 0\,, \tag{5}$$

The meaning of the second equation in the AR model is that the w quantity is advected at the velocity v and is affecting also the velocity v. The AR model becomes the Aw-Rascle-Zhang (ARZ) model when $p(\rho) = -V(\rho)$, where V is the equilibrium velocity profile. Hence ARZ is the AR model with $w = v - V(\rho)$:

$$\partial_t \rho + \partial_x(\rho v) = 0\,, \tag{6}$$

$$\partial_t(v - V(\rho)) + v\partial_x(v - V(\rho)) = 0\,. \tag{7}$$

Those equations are rewritten as

$$\partial_t \begin{pmatrix} \rho \\ v \end{pmatrix} + \begin{pmatrix} v & \rho \\ 0 & v + \rho V'(\rho) \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ v \end{pmatrix} = 0\,. \tag{8}$$

Obviously the eigenvalues of the matrix of the system are $\lambda_1 = v$ and $\lambda_2 = v + \rho V'(\rho)$. The corresponding right eigenvectors are $r_1 = \binom{1}{0}$ and $r_2 = \binom{-1}{-V'(\rho)}$. Of course, the Riemann invariants remained the same as before.

One additional term may be added to the right hand side of the second equation of (5), the relaxation term $\frac{V(\rho)-v}{\tau}$. This term expressed that the drivers accelerate or break to reach the prefered velocity V function of the surrounding density.

Whenever $\rho = 0$, we have $w = v$. It is the velocity that a vehicle should take if the road was empty. It does not correspond to the velocity that the vehicle should take if it works as predicted by the V function. This velocity w is called the *empty velocity*.

Two remarks about the characteristic speed. First, one has $\lambda_1, \lambda_2 \leq v$, it means that the car is a anisotropic particle, hence it answers at one of the criticism of Daganzo. Second, λ_2 could take negative values. This aspect is not a drawback of the model but describe the displacement of gap in the velocity due to a congestion (form by a traffic light for example).

Linearization
The ARZ is linearized around a steady state $(\tilde{\rho}, \tilde{q})$. We assume that this steady state is in congestion to obtain velocities with opposite signs and apply the strong theory of stabilization of subcritical regime for open channels governing by the Saint-Venant equations. Obviously it is not the better way to proceed since the control of the ARZ model would be to pass from congestion flow to free flow. But from an application point of view it is interesting to apply the well known techniques of stabilization of open channel to the traffic management. Moreover, the stabilization of jam could be defend saying that we dont want the situation goes worst than what it is. We linearized the ARZ equations around an equilibrium steady-state defined by $(\tilde{\rho}, \tilde{q})$. The steady state is uniform with respect to the state variable, that is $\partial_x v^\star = 0$ and $\partial_x \rho^\star = 0$, then the deviations $(\tilde{v}, \tilde{\rho})$ satisfies the following equations:

$$\partial_t \begin{pmatrix} \tilde{\rho} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} v^\star & \rho^\star \\ 0 & v^\star + \rho^\star V'(\rho^\star)e \end{pmatrix} \partial_x \begin{pmatrix} \tilde{\rho} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{V'(\rho^\star)}{\tau} & -\frac{1}{\tau} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{v} \end{pmatrix}, \tag{9}$$

Characteristic form

$$\begin{cases} \partial_t \zeta_1(t,x) + \lambda_1 \partial_x \zeta_1(t,x) = -\frac{1}{\tau}\zeta_1(t,x)\,, \\ \partial_t \zeta_2(t,x) + \lambda_2 \partial_x \zeta_2(t,x) = -\frac{1}{\tau}\zeta_1(t,x)\,,, \end{cases} \tag{10}$$

with the characteristic coordinates are defined as follows

$$\begin{matrix} \zeta_1(t,x) = \tilde{v}(t,x) - V'(\rho^\star)\tilde{\rho}(t,x)\,, \\ \zeta_2(t,x) = \tilde{v}(t,x)\,, \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} \tilde{\rho}(t,x) = -\frac{1}{V'(\rho^\star)}\left(\zeta_1(t,x) - \zeta_2(t,x)\right)\,, \\ \tilde{v}(t,x) = \zeta_2(t,x)\,. \end{matrix}$$

We consider system (10) with $t \in [0, +\infty)$ and $x \in [0, L]$, under linear boundary conditions, as in the work of Bastin and Coron, of the form

$$\begin{pmatrix} \zeta_1(t,0) \\ \zeta_2(t,L) \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \zeta_1(t,L) \\ \zeta_2(t,0) \end{pmatrix} \begin{pmatrix} \zeta_1(t,L) \\ \zeta_2(t,0) \end{pmatrix}, \quad t \in [0, +\infty) \tag{11}$$

and an initial condition

$$\left[\zeta_1(0,x) = \zeta^0(x), \zeta_2(0,x) = \zeta_2^0(x)\right] \in L^2((0,L); \mathbb{R}^2)\,. \tag{12}$$

The exactly same Lyapunov analysis as in [1] may be lead. Let us consider the following candidate Lyapunov function:

$$V = \int_0^L \left(\zeta_1^2 p_1 e^{-\mu x} + \zeta_2^2 p_2 e^{\mu x}\right)\,, \quad p_1, p_2, \mu > 0\,. \tag{13}$$

Density-flow system

Now one could work with a PDE for the flux rather directly the velocity. The computation of such PDE is following:

$$\begin{aligned} \partial_t(\rho v) &= v\partial_t \rho + \rho \partial_t v \\ &= v(-\partial_x(\rho v)) + \rho\left(-(v + \rho V'(\rho))\partial_x v + \frac{V(\rho) - v}{\tau}\right) \\ &= -v\partial_x(\rho v) + v(v + \rho V'(\rho))\partial_x \rho - (v + \rho V'(\rho))\partial_x(\rho v) + \frac{\rho V(\rho) - \rho v}{\tau} \end{aligned}$$

Therefore, one gets the following two equations for (ρ, q) where q denotes the flow $q = \rho v$

$$\partial_t \begin{pmatrix} \rho \\ q \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 1 \\ \alpha\beta & \alpha - \beta \end{pmatrix}}_{A} \partial_x \begin{pmatrix} \rho \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\rho V(\rho) - q}{\tau} \end{pmatrix}, \tag{14}$$

where $\alpha(\rho, q) = \frac{q}{\rho}$ and $\beta(\rho, q) = -\frac{q}{\rho} - \rho V'(\rho)$. The eigenvalues of A are

$$\lambda_1 = \frac{q}{\rho} = \alpha\,, \tag{15}$$

$$\lambda_2 = \frac{q}{\rho} + \rho V'(\rho) = -\beta\,. \tag{16}$$

The corresponding right eigenvectors are $r_1 = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$ and $r_2 = \begin{pmatrix} 1 \\ -\beta \end{pmatrix}$. Hence corresponding left eigenvectors are $l_1 = (\beta, 1)$ and $l_2 = (-\alpha, 1)$.

Riemann Invariants form

The Riemann Invariants are $J_1 = \frac{q}{\rho} - V(\rho)$ and $J_2 = \frac{q}{\rho}$. In the Riemann Invariants coordinates the non-linear system. In this coordinates one gets

$$\partial_t J_1 + \alpha \partial_x J_1 = -\frac{1}{\tau} J_1\,, \tag{17}$$

$$\partial_t J_2 - \beta \partial_x J_2 = -\frac{1}{\tau} J_1\,. \tag{18}$$

Hence we obtain the following expressions:

$$\frac{dJ_1}{dt}(t, x_1) = h(x_1, J_1(t, x_1))\,, \tag{19}$$

$$\frac{dx_1}{dt} = \alpha(t, x_1)\,, \tag{20}$$

$$\frac{dJ_2}{dt}(t, x_2) = h(x_2, J_1(t, x_2))\,, \tag{21}$$

$$\frac{dx_2}{dt} = -\beta(t, x_2)\,, \tag{22}$$

with $h(x, J_1(t, x)) = -\frac{1}{\tau} J_1(t, x)$.

Steady Flow Solution

Let us consider the steady flow for the ARZ model. A steady-flow is given by the condition $q^\star(x) = \rho^\star(x) V(\rho^\star(x))$. Let us compute it. Replacing the term ∂_t in (14) by 0 one hqs

$$\frac{dq^\star(x)}{dx} = 0\,, \tag{23}$$

$$\left(V(\rho^\star(x))^2 + \rho^\star(x) V(\rho^\star(x)) V'(\rho^\star(x))\right) \frac{d\rho^\star(x)}{dx} = 0\,. \tag{24}$$

The equation (24) gives

$$V(\rho^\star(x))^2 + \rho^\star(x) V(\rho^\star(x)) V'(\rho^\star(x)) = 0 \text{ and/or } \frac{d\rho^\star(x)}{dx} = 0\,.$$

Now we note that $V(\rho^\star(x))^2 + \rho^\star(x)V(\rho^\star(x))V'(\rho^\star(x))$ is the second eigenvalue which is assumed to be non-zero, hence $\frac{d\rho^\star(x)}{dx} = 0$. Finally, the steady state is uniform over the road.

Linearization around a steady-state

Let us denote $(\tilde{\rho}(t,x), \tilde{q}(t,x))$ the deviations of the state with respect to the steady state previously defined. One gets

$$\partial_t \tilde{\rho} + \partial_x \tilde{q} = 0\,, \tag{25}$$

$$\partial_t + (\alpha^\star - \beta^\star)\partial_x(x)\tilde{q} - \alpha^\star\beta^\star\partial_x\tilde{\rho} = \sigma\tilde{q} + \delta\tilde{\rho}\,, \tag{26}$$

with

$$\alpha^\star = \frac{q^\star}{\rho^\star}\,, \tag{27}$$

$$\beta^\star = -\frac{q^\star}{\rho^\star} - \rho^\star V'(\rho^\star)\,, \tag{28}$$

$$\delta^\star = \left(\frac{V(\rho^\star) + \rho^\star V'(\rho^\star)}{\tau}\right)\,, \tag{29}$$

$$\sigma = -\frac{1}{\tau}\,. \tag{30}$$

Hence the system is rewritten as

$$\partial_t \xi(t,x) + A(x)\partial_x\xi(t,x) = B(x)\xi(t,x)\,, \tag{31}$$

where $\xi(t,x) = (\tilde{\rho}(t,x), \tilde{q}(t,x))^\top$ and $A(x) = \begin{pmatrix} 0 & 1 \\ -\alpha^\star\beta^\star & \alpha^\star - \beta^\star \end{pmatrix}$, $B(x) = \begin{pmatrix} 0 & 0 \\ \delta^\star & \sigma \end{pmatrix}$. The Riemann invariants are $J_1 = \frac{\tilde{q}}{\tilde{\rho}}$ and $J_2 = \frac{\tilde{q}}{\tilde{\rho}} - V(\tilde{\rho})$.

Characteristic form

We are looking at the congestion flow. It means $\lambda_2 < 0$. The characteristic coordinates are defined as follows:

$$\begin{aligned} \zeta_1(t,x) &= \beta^\star\tilde{\rho}(t,x) + \tilde{q}(t,x)\,, \\ \zeta_2(t,x) &= -\alpha^\star\tilde{\rho}(t,x) + \tilde{q}(t,x)\,, \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \tilde{\rho}(t,x) &= \tfrac{1}{\alpha^\star+\beta^\star}\left(\zeta_1(t,x) - \zeta_2(t,x)\right)\,, \\ \tilde{q}(t,x) &= \tfrac{1}{\alpha^\star+\beta^\star}\left(\alpha^\star\zeta_1(t,x) + \beta^\star\zeta_2(t,x)\right)\,. \end{aligned}$$

The structure of the linearized ARZ model is the same one as for the Saint-Venant equations. It means that the formal expressions of the transfer functions are the same for the two systems.

References

[1] Georges Bastin, Jean-Michel Coron, and Brigitte d'Andréa Novel. On lyapunov stability of linearised saint-venant equations for a sloping channel. *NHM*, 4(2):177–187, 2009.