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WHAT DOES THE ENTROPY CONDITION MEAN IN TRAFFIC FLOW THEORY?

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Abstract—Mathematical models of freeway traffic flow do not include models of the *driver's ride impulse* (e.g. of the fact that drivers really start if a traffic light switches from red to green). Moreover, the occurrence of shocks leads to the necessity to deal with weak solutions of mathematical models, as far as models formulated in terms of conservation laws are concerned. But the transition from a classic conservation law differential equation to its weak formulation leads to a loss of uniqueness of the solution. This also happens in gas dynamics where an *entropy condition* was additionally introduced in order to pick out the physically true solution. In this paper, it will be pointed out that this entropy condition can also be used in traffic flow theory as a uniqueness criterion, and that it is—strangely enough—the missing mathematical model of the ride impulse. These ideas are exemplified in case of the Lighthill-Whitham model which so seems to become again an up-to-date model that can numerically be treated by means of very effective numerical procedures such as TVD methods.

1. INTRODUCTION

Continuum macroscopic freeway traffic models are described by the traffic variables

traffic flux	q	(vehicles per hour)
traffic density	ρ	(vehicles per km)
mean speed	v	(km per hour).

They are connected by

$$q = v\rho. \tag{1}$$

We assume the flow to be—hopefully—inviscid. For convenience, we restrict ourselves to the case of a long segment of the road without exits or entrances such that the segment can be modelled by the real axis $x \in IR$ (km) and that the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0,\ \forall x \in IR,\ \forall \geq 0 \tag{2}$$

(t: time variable) has to be fulfilled. If the initial data

$$\rho(x, 0) = \rho_0(x),\ x \in IR \tag{3}$$

are sufficiently smooth, a unique smooth† solution $\rho(x,t)$ exists at least locally (i.e. in a certain neighbourhood of the real axis); this holds because v is assumed to depend explicitly and smoothly only on ρ such that

$$q = q(\rho) \tag{4}$$

(cf. (1)) is also smooth. $q(\rho)$ is well known empirically (*Fundamental diagram*), and this empirical relation is strictly concave, i.e.

†"Smooth" always means continuously differentiable.

$$q''(\rho)<0 \quad \text{for} \quad 0<\rho<\rho^*$$

(cf. Fig. 1) ρ^* is assumed to be the maximum density, namely the traffic jam concentration.‡ $v=v(\rho)$ is a monotone decreasing function (cf. Fig. 2). v_f is the mean maximal speed of the cars§ which occurs if the road in front of the drivers is approximately empty.

The system that consists of (2) and (3) and of the fundamental diagram is called the *Lighthill-Whitham model* (Lighthill and Whitham, 1955) of freeway traffic flow (cf. also Drew, 1986). An extremely simple—hence not very precise—approximation of the empirical fundamental diagram is "Greenshields' Model"

$$q=v_{f\rho}\left(1-\frac{\rho}{\rho^*}\right).¶ \tag{5}$$

It will sometimes be used in this paper in order to illustrate some basic facts or examples or certain properties. Because of (4), the Lighthill-Whitham Model reduces to a scalar differential equation, namely to the scalar conservation law (2), and there is only one space variable.

Instationary flows of inviscid gases are also described mathematically by conservation laws but by systems of at least three equations (besides conservation of mass also conservation of momentum and conservation of energy) (e.g. Chorin and Marsden, 1984, p. 156). Nevertheless, many properties well known from fluid dynamics do also already occur in case of scalar equations. But—strangely enough—numerous important results from gas dynamics concerning theory and numerical treatment of conservation laws seem not yet to be realized in traffic flow engineering, though—vice versa—in mathematical gas dynamics often scalar models of the system model were studied at the beginning of a new theoretical or numerical approach (e.g. Burgers' equation).

The characteristics of (2) are the straight lines†

$$x-x_0=q'(\rho_0(x_0))t,\ x_0\in IR, \tag{6}$$

and ρ is constant along each of these lines, namely

$$\rho(x,t)=\rho_0(x_0).$$

Hence, it can happen that characteristics cross, and at a point where they cross (2) predicts two different values of ρ (i.e. a discontinuity). This leads to the introduction of the concept of *weak solutions*‡ but this concept was not yet consistently used in traffic flow engineering. Let $\Omega=\{(x,t)|x\in IR,\ t\in IR_0^+\}$ be the upper half-plane of the (x,t)-plane including the

‡e.g.: $\rho^*=200$ vehicles/km.

§e.g. $v_f=130$ km/h.

¶In this model, $v=v(\rho)$ (cf. Fig. 2) becomes a straight line.

†q' is assumed to be smooth for $0<\rho<\rho^*$

‡Though the problem is nonlinear, it can easily be replaced by a weak solution model because of the particular form of conservation laws.

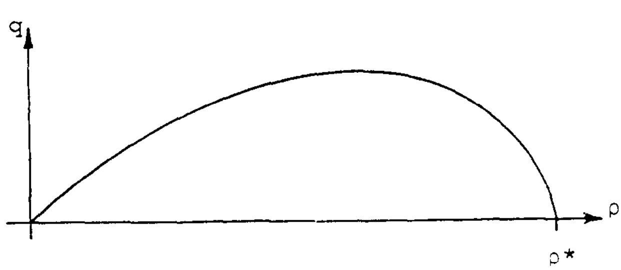

Fig. 1. Fundamental diagram.

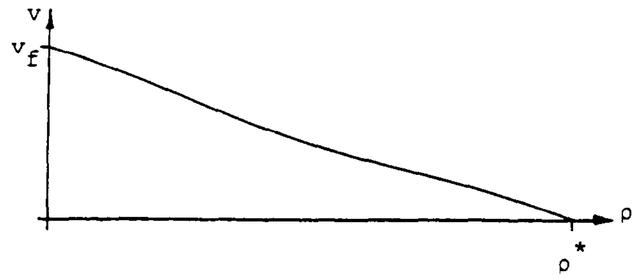

Fig. 2. Relation betwen v and ρ.

x=axis, and let $C_0^1(\Omega)$ be the set of test functions (i.e. of functions continuously differentiable on Ω having compact support). Thus, every test function ϕ vanishes on the boundary of its compact support as far as this boundary does not include a part of the x-axis; on such a part ϕ does not necessarily vanish. The weak formulation of the Lighthill-Whitham model, therefore, reads as follows:

find $\rho \in L_1^{loc}(\Omega)$ such that

$$-\int_{-\infty}^{+\infty}\int_0^{\infty}[\rho\phi_t+q\phi_x]dtdx-\int_{-\infty}^{+\infty}\phi(x,0)\rho_0(x)dx=0,\ \forall\phi\in C_0^1(\Omega). \tag{7}$$

Every smooth solution also is a weak solution, and if a weak solution is somewhere smooth, it also solves there the problem in its classical formulation.

Let us assume that the sets $\{(x,t)|(x,t)$ is point of a discontinuity of a weak solution$\}$ form continuous curves. Such a curve Γ is called a *shock*.

Let

$$\begin{aligned} f_l(x_s,t) &:= \lim_{\epsilon\to 0} f(x_s-\epsilon,t) \\ & \qquad\qquad , \epsilon>0,\ (x_s,t)\in\Gamma \\ f_r(x_s,t) &:= \lim_{\epsilon\to 0} f(x_s+\epsilon,t) \end{aligned}$$

$$[\rho]:=\rho_l-\rho_r,\quad [q]:=q_l-q_r=q(\rho_l)-q(\rho_r) \tag{8}$$

and assume the weak solution ρ to be smooth outside Γ.

Then, (7) is equivalent with the validity of (2), (3) outside the shocks connected with the jump relation

$$[\rho]\dot{x}_s=[q] \tag{9}$$

where $x_s=x_s(t)$ describes Γ such that $\dot{x}_s$ gives the *congestion velocity* the discontinuity is running with along the road.* (9) is well known in traffic flow engineering.§ Of course, it is also well known from gas dynamics where it is called the *Rankine-Hugoniot condition*.

The uniqueness of smooth solutions is lost in case of really weak solutions, and also this fact is known in traffic flow theory though weak solutions in the sense of (7) are normally not discussed in traffic flow literature. As an example, consider the dissolution of a traffic jam, i.e.

$$\rho_0(x)=\begin{cases}\rho^* \text{ for } x<0 \\ 0 \text{ for } x\geq 0\end{cases}.¶ \tag{10}$$

*cf. (Osher and Chakravarty, 1984), pp. 247–248

§cf. (Braun, Coleman and Drew, 1978), p. 213, and (Newell, 1988)

If Greenshields' model is used, we find the following two solutions:

$$\rho_1(x,t)=\begin{cases}\rho^* & \text{for} \quad x<-v_f t,\ t\geq 0\\ \rho^*\dfrac{v_f t-x}{2v_f t} & \text{for} \quad -v_f t\leq x< v_f t,\ t>0\\ 0 & \text{for} \quad x\geq v_f t,\ t\geq 0\end{cases} \tag{11}$$

and

$$\rho_2(x,t)=\rho_0(x),\ \forall t\geq 0. \tag{12}$$

Within the frame of the simplifications of Greenshield's model, (11) describes the reality correctly whereas (12) is unrealistic: ρ_2 describes a situation where the car drivers at the front of a traffic jam do not start though there is a green traffic light.

Hence, the Lighthill-Whitham model must be completed by a uniqueness condition that models mathematically something like the *driver's ride impulse* and that selects out of the set of weak solutions the *physically relevant* solution.

Of course, a corresponding situation is well known from gas dynamics, and Lax (1971) therefore recommended a completion of the theory by the 2nd Main Theorem of thermodynamics, i.e. by adding the entropy condition

$$dS\geq 0 \tag{13}$$

where S represents the physical entropy, and to generalize this condition properly to general systems of conservations laws. This concept does not work in every case: If the system consists of more than two differential equations, an entropy function in the sense of Lax exists only if the system of conservation laws shows a suitable structure.† But even if an entropy function exists, uniqueness of the weak solution has not been proven in general.‡

For the scalar case—hence, also for the traffic flow model under consideration—Oleinik (1957) showed uniqueness under a slightly different formulation of an entropy condition, and also existence of an entropy function in the sense of Lax and uniqueness of the *entropy solution* (i.e. the solution that fulfills not only (7) but also the properly generalized inequality (13)) is ensured.

But though thermodynamics can hardly explain the behaviour of car drivers, we may ask the question whether or not the entropy solution of (7) is at the same time also the physically relevant solution from the point of view of traffic flow. And if this seems to be true, the question of a specific traffic flow interpretation of the entropy condition arises.

2. ENTROPY CONDITION AND DRIVER'S RIDE IMPULSE

A (weak) solution ρ of the problem (2), (3) is called an entropy solution, if there are a strictly convex smooth *entropy function* $V=V(\rho)$ and a smooth *entropy flux* function $F=F(\rho)$ with $F(0)=0$ such that

$$V_t+\frac{\partial}{\partial x}F(\rho)=0 \tag{14}$$

¶Problems of this type are called *Riemann problems.*

†Of course, in the case of gas dynamics, the physical entropy also yields an entropy function in the sense of the general theory.

‡But there are no counterexamples.

holds automatically on regions where ρ is smooth§, and such that ρ fulfills the *entropy condition*

$$V_t + \frac{\partial}{\partial x} F(\rho) \leq 0. \tag{15}$$

Inequality (15) must be considered as an inequality in its weak form¶, i.e.

$$\begin{aligned} &-\int_{-\infty}^{+\infty}\int_0^{\infty} [V(\rho(x,t))\phi_t(x,t) + F(\rho(x,t))\phi_x(x,t)]dtdx \\ &-\int_{-\infty}^{+\infty} V(\rho_0(x))\phi(x,0)dx \leq 0, \quad \forall \phi \in C_0^1(\Omega) \text{ with } \phi \geq 0. \end{aligned} \tag{16}$$

Obviously, in the scalar case under consideration, V and F can be chosen as

$$V = \frac{\rho^2}{2}, \quad F(\rho) = \int_0^{\rho} \alpha q'(\alpha) d\alpha \tag{17}$$

because of $\frac{d^2 V}{d\rho^2} = 1 > 0$ (convexity) and because

$$V_t + \frac{\partial}{\partial x} F(\rho) = \rho \cdot \rho_t + \rho \cdot q'(\rho)\rho_x = \rho\{\rho_t + \frac{\partial}{\partial x} q(\rho)\} = 0$$

holds automatically for smooth solutions ρ.

In case of the particular example of Greenshields' model, (17) yields

$$F(\rho) = v_f \frac{\rho^2}{2}\left(1 - \frac{4}{3}\frac{\rho}{\rho^*}\right)$$

such that (16), (17) lead to the entropy condition

$$-\int_{-\infty}^{+\infty}\int_0^{\infty} \frac{\rho^2}{2}\left[\phi_t + v_f\left(1 - \frac{4}{3}\frac{\rho}{\rho^*}\right)\phi_x\right] dtdx - \int_{-\infty}^{+\infty} \frac{\rho_0^2(x)}{2}\phi(x,0)dx \leq 0, \tag{18}$$

$$\forall \phi \in C_0^1(\Omega) \text{ with } \phi \geq 0.$$

If applied to the example of dissolution of a traffic jam, (i.e. to ρ_2) (cf. (12)), (18) yields on its left hand side the value of

$$\frac{v_f \rho^{*2}}{6}\int_0^{\infty} \phi(0,t)dt$$

§This condition establishes a relation between V and F, namely

$$F(\rho) = \int_0^{\rho} V'(\alpha) q'(\alpha) d\alpha.$$

¶In gas dynamics, (15) is equivalent to (13) if $V = -s$ (s: entropy/volume)

$F = \frac{\kappa}{\rho} s$ (κ: momentum/volume).

which – obviously – is positive for certain non-negative test functions ϕ. Hence, (18) is not fulfilled in case of $\rho=\rho_2$.

For $\rho=\rho_1$ (cf. (11)), the left hand side of (18) equals zero for all test functions ϕ; hence, (18) is fulfilled in this case. In other words: The entropy condition motivated from thermodynamics also works in case of this example of traffic flow by selecting the physically true solution.

Let us consider another example – for convenience, again in combination with Greenshields' model – namely the propagation of the back part of a traffic jam opposite to the direction of the flow. More precisely, let us assume that there is maximal flux for $t=0$ if $x\leq 0$, stagnant traffic jam for $x\geq 1$ and a linear and continuous transition of density between these two situations, i.e.

$$\rho_0(x)=\begin{cases}\dfrac{\rho^*}{2} & \text{for} \quad x<0\\ \dfrac{\rho^*}{2}\,(1+x) & \text{for} \quad 0\leq x\leq 1\\ \rho^* & \text{for} \quad x<1.\end{cases} \tag{19}$$

The characteristics of (cf. (6)) are

$$\begin{aligned} x&= x_0 && \text{for} \quad x_0\leq 0\\ t&= \frac{1}{v_f}(1-\frac{x}{x_0}) && \text{for} \quad 0<x_0\leq 1\\ t&=-\frac{1}{v_f}(x-x_0) && \text{for} \quad x_0>1. \end{aligned} \tag{20}$$

Figure 3 shows that the desired unknown ρ keeps its initial continuity for a certain time, namely up to $t=\dfrac{1}{v_f}$. Then suddenly a discontinuity occurs† – reason for many crashes – which proceeds for $t>\dfrac{1}{v_f}$ along a shock Γ. Obviously, Γ can only be a curve within the upper left half plane such that along this curve we find $\rho_\ell=\dfrac{\rho^*}{2}$, $\rho_r=\rho^*$. Hence, the Rankine Hugoniot condition (cf. (9)) reads as

$$\dot{x}_s=\frac{v_f}{2}. \tag{21}$$

Together with the initial value $x_s\left(\dfrac{1}{v_f}\right)=0$ (cf. Fig. 3), (21) yields

$$x_s(t)=\frac{1-v_f t}{2}, \text{ i.e. } t=\frac{1}{v_f}(1-2x) \text{ for } x\leq 0,$$

such that Γ is a straight line (as already plotted in Fig. 3).

From this, we find immediately the (weak) solution

$$\rho(x,t)=\begin{cases}\dfrac{\rho^*}{2} & \text{for} \quad 0\leq t<\dfrac{1}{v_f}(1-2x), -\infty<x\leq 0\\ \dfrac{\rho^*}{2}(1+\dfrac{x}{1-v_f t}) & \text{for} \quad 0\leq t<\dfrac{1-x}{v_f}, 0\leq x<1.\\ \rho^* & \end{cases} \tag{22}$$

†This corresponds to results of Newell's discrete model (Newell, 1961), namely that stable shocks only occur for vehicle deceleration (not acceleration).

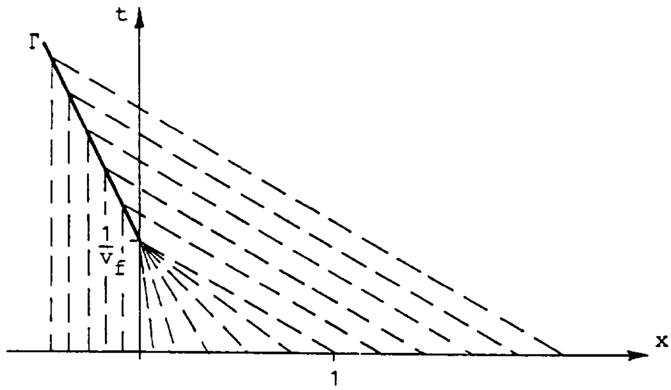

Fig. 3. Characteristics if the back part of a traffic jam propagates opposite to the traffic flow direction.

Within the frame of Greenshields' model, the back part of the stagnant traffic jam originally ($t=0$) situation at $x=1$, propagates opposite to the flow direction with velocity $-v_f$ until it reaches $x=0$. At this very moment, the discontinuity occurs and propagates the opposite to the flow direction with speed $-\frac{v_f}{2}$. Neglecting the errors following from simplification of Greenshields' model, (22) coincides with empirical results. Hence, (22) is the physically relevant solution. And again, this physically relevant solution also fullfills the entropy condition: The left hand side of (16) applied to (22) yields

$$-\frac{\rho^{*2}}{24}\int_{-\infty}^{0}\phi(x,\frac{1-2x}{v_f})dx,$$

and this term is never positive because only non-negative test functions ϕ are admitted.

Hence, the question formulated at the end of Section 1 of this paper should be answered: Does there exist a specific traffic flow interpretation of the entropy condition such that this condition can be used as a completion of the Lighthill-Whitham traffic flow model? In order to find the answer, it should be mentioned again that eqn (7) was equivalent with the validity of (2), (3) outside the shocks connected with the Rankine-Hugoniot condition

$$[\rho]\dot{x}_s=[q]$$

(cf. (9)). In the same way and by the same proof it can easily be shown that the weak inequality (15) (i.e. inequality (16)), is equivalent with the validity of (2), (3) outside the shocks in connection with

$$[V]\dot{x}_s\leq[F] \tag{23}$$

($[U]:=U_\ell-U_r=U(\rho_\ell)-U(\rho_r)$; $U=V,F$). Both relations together yield

$$[V]\frac{[q]}{[\rho]}\leq[F] \tag{24}$$

along every shock. With the functions V, F from (17), (24) reads as

$$\frac{1}{2}(\rho_\ell^2-\rho_r^2)\frac{q_\ell-q_r}{\rho_\ell-\rho_r}\leq\int_{\rho_r}^{\rho_\ell}\alpha q'(\alpha)d\alpha=\rho_\ell q_\ell-\rho_r q_r-\int_{\rho_r}^{\rho_\ell}q(\alpha)d\alpha,$$

i.e.

$$\int_{\rho_r}^{\rho_\ell} q(\alpha)d\alpha \le (\rho_\ell - \rho_r)\frac{q_\ell + q_r}{2}. \tag{25}$$

Hence, inequality (25)—together with the validity of (2), (3) outside the shocks—is equivalent with the entropy condition. The left hand side of this inequality is an integral, the right hand side is the approximation of this integral by the trapezoidal rule. But because $q(\alpha)$ is a concave function, this approximation can satisfy inequality (25) if and only if

$$\rho_\ell \le \rho_r. \tag{26}$$

Thus, the completion of the Lighthill-Whitham model of traffic flow by adding the Lax entropy condition reduces to the addition of inequality (26) along every shock. But do motorists on an average really behave as described by (26)?

Inequality (26) does not necessarily imply that motorists always and everywhere try to increase the traffic density or to keep it constant, but it says that they try to smooth a discontinuous situation to a continuous one ($\rho_\ell = \rho_r$) (as it turned out to characterize the physically true solution of the traffic jam dissolution example) or not to decrease the density if they cross a discontinuity (as seen in the congestion propagation example: The drivers do not stop if they notice that there is a traffic jam in front of them a certain distance ahead but they ride into this jam which leads to an increase of density).

Thus, this interpretation of the entropy condition in the traffic flow model under consideration seems to coincide with real life behaviour of motorists, and because of this observation, relation (26) should be called *driver's ride impulse* and should become an additional part of the Lighthill-Whitham model.

It should be noticed that (26) is equivalent with a geometric property of the characteristic straight lines (cf. (6)), and it is this geometric property which can more easily be discretely imitated by numerical methods than (26).† In order to find this equivalent geometric property, let us apply the mean value theorem to the Rankine-Hugoniot condition, i.e. to

$$\dot{x}_s = \frac{q(\rho_\ell) - q(\rho_\ell)}{\rho_\ell - \rho_r}$$

(cf. (9)). This leads to

$$\dot{x}_s = q'(\rho_\ell + \vartheta(\rho_r - \rho_\ell)),\ 0 < \vartheta < 1.$$

Because of the concave structure of the function $q(\rho)$, (26) yields

$$q'(\rho_r) \le \dot{x}_s \le q'(\rho_\ell).\ddagger \tag{27}$$

Moreover, because $q(\rho)$ is not only concave but strictly concave (cf. (4)), (27)—vice versa—implies (26). Hence, (27) and (26) are really eqivalent in our model. But (27) now has a simple geometric meaning: Because q' represents the ascent of the characteristics in the $t-x$-plane (cf. (6)), (27) means that the entropy solution is that particular weak solution for which the characteristics run *into* the shocks for increasing time (cf. Fig. 3).

†Such an imitation helps to ensure that the numerical solution really approximates the entropy solution instead of an other weak solution.

‡(27) is just a particular case of a much more general property of systems of conservation laws (e.g. (Smoller, 1983), p. 327).

3. COMPARISON WITH OTHER MODELS

There is a paper in the literature on traffic flow theory where the entropy plays a certain role, namely the paper of Beylich (1978). Beylich developed a discrete traffic flow theory with formation of clusters. This theory corresponds to the kinetic theory of gases, and this also holds with respect to the entropy as part of the classic Boltzmann statistics. The entropy condition was not yet introduced into continuous models.

Leutzbach and Schwerdtfeger (1981) noted that the Lighthill-Whitham model is able to explain the formation of shocks—analogously to our second example—but that it fails to describe the dissolution of traffic jams. As demonstrated by our first example, this seems not to be true if the model is completed by adding the Lax entropy condition.

Some authors guess that the deficiencies of the Lighthill-Whitham model are due to the fact that this model works with a steady state connection between density and velocity (i.e. with the fundamental diagram $q=q(\rho)$) or equivalently with

$$v=\hat{v}(\rho) \tag{28}$$

(cf. Fig. 2), instead of a dynamic relation. And they also guess that the anticipation effect coming from the reaction of drivers to events occurring downstream is not taken into account early enough. Payne (1979) therefore constructed a model consisting of the continuity equation, the fundamental diagram as a model of the undisturbed steady state relation $\hat{v}(\rho)$ and of an additional differential equation, namely

$$\frac{dv}{dt}=v_t+vv_x=\frac{1}{\tau}\{\hat{v}(\rho)-v\}-c_0{}^2\frac{\rho_x}{\rho}. \tag{29}$$

Unfortunately, this acceleration model does not keep the conservation law character of the original Lighthill-Whitham model but it contains a relaxation towards the steady state velocity as well as an anticipation term which takes into account traffic conditions downstream. Cremer (1979), developed discrete models from (29), and his numerical results coincide very well with real life values. On the other hand, Hauer and Hurdle (1979) pointed out that (29) fails to model sharp shocks. Nevertheless Kühne (1987) introduced an additional viscosity term into (29) in order to smear out strong variations and to "avoid a complicated adaptive integration step width control for numerical calculations." But smoothing the shock leads to a loss of information.

Obviously, there are two main reasons in modern traffic flow engineering literature for the rejection of the Lighthill-Whitham model or its modification, leading to the destruction of its conservation law structure, namely:

(i) failure in the description of traffic jam dissolutions and in sudden shocks
(ii) numerical problems with respect to the calculation of strong variations.

The first objection will be eliminated by the addition of the entropy condition and by the renewed reference to the Rankine-Hugoniot condition, no doubt. Hence, there remains the question how to eliminate the numerical objection. If this question can be answered, the reason for the rejection of the Lighthill-Whitham model (i.e. of the conservation law structure), vanishes. And it is precisely this conservation law character of the model that helps to construct excellent and efficient shock preserving numerical procedures.

4. NUMERICAL METHODS

Some questions discussed in recent years in gas dynamics were:

(i) how to approximate weak solutions of conservation laws
(ii) how to construct methods just approximating the entropy solution

(iii) how to approximate real shocks sharply but to avoid the danger of bringing in artificial shocks arising from numerical errors and to avoid oscillations in the neighbourhood of shocks (cf. Engquist and Osher, 1980, 1981; Harten, 1984; Harten, Lax and van Leer, 1983; Einfeldt, 1987; Henshaw, 1987; Newell, 1988.

This discussion based on fundamental earlier papers such as Courant, Friedrichs and Lewy (1928), Glimm (1965), Godunov (1959), Lax and Wendroff (1960). Often—simply in order to make life easier—authors particularly treated as a first approach only scalar problems with one space variable instead of the full system of conservation laws. Such a scalar model of the mathematical model of gas dynamics is—for example—Burgers' equation.† But because we are just concerned with a scalar problem, also these attempts (e.g. Harten and Hyman 1983) are worth taking into account from the point of view of macroscopic freeway traffic flow models.

The well known Lax-Wendroff scheme approximates shocks fairly well but yields oscillations in the neighbourhood of real shocks. Oscillations do not occur if modern TVD schemes‡ are used. The TVD property can simultaneously be employed for the discrete imitation of the entropy condition in its geometric realization (27), and the consistency of suitable multi point formulas imitates the conservation law structure discretely.

Though the TVD property can be understood as introduction of an artificial numerical viscosity, this viscosity can be constructed in such a way that it vanishes exactly at the real shocks. This idea is due to Harten (1984) and guarantees that the shocks are approximated extremely sharply despite of the presence of viscosity close to the shock.

It cannot be the task of this paper to repeat in detail what has already been fully described in the literature referred to, but these recent papers justify the statement that also the numerical objections mentioned at the end of the last chapter are no longer tenable.

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†Also the Lighthill-Whitham model can easily be transformed into a Burgers' equation problem.

‡*t*otal *v*ariation *d*iminishing

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