OPTIMAL INTRANSITIVE DICE SETS FOR REGULAR TOURNAMENTS.

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Abstract. This note explores optimal non-transitive random variable tournaments, focusing on the case of 5 and 7 random variables.

1. Introduction

Consider a set of independent random variables X_1, X_2, X_3, such that the probability of each variable being larger than the next is more than 50%, that is $P(X_1 > X_2) > 0.5$ and $P(X_2 > X_3) > 0.5$. Trybuła [1] demonstrated a paradox that random variables need not exhibit transitivity in terms of probability of one value being larger than another, meaning it is possible that $P(X_1 > X_3) < 0.5$. One natural way to conceptualize random variables is dice. Thus, a classical example of this intransitive dice is Efron's dice, a set of four regular 6-sided dice $\{D_1, D_2, D_3, D_4\}$ with the following configurations of values on the faces:

Faces	D_1	D_2	D_3	D_4
1 and 2	4	3	6	5
3	4	3	2	5
4	4	3	2	1
5 and 6	0	3	2	1

In this example, $P(D_1 > D_2) = P(D_2 > D_3) = P(D_3 > D_4) = P(D_4 > D_1) = 2/3$. As it is shown independantly by Trybuła [2] and Usiskin [3], the maximal probablity exhibited by a cycle of four intransitive random variables is exactly $2/3$, rendering Efron's dice optimal in this sense. The optimal probabilities for cycles of other numbers of intransitive random variables are irrational. For three variables, the optimal probability is ϕ^{-1}, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio, the positive root of $x^2 - x - 1$. Using the connection between the golden ratio and the Fibonacci numbers, it is possible to construct regular dice sets that approximate this optimal probability very well. Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ and consider the set of three F_{2n}-sided dice (for $n > 2$):

Faces	D_1	D_2	D_3
$1 \ldots F_{2n-1} - 1$	2	8	5
F_{2n-1}	4	6	3
$F_{2n-1} + 1 \ldots F_{2n} - 1$	9	1	5
F_{2n}	9	1	7

Considering Cassini's identity $F_{2n-2}F_{2n}+1=F_{2n-1}^2$, we observe

$$P(D_1>D_2)=P(D_2>D_3)=P(D_3>D_1)=\frac{F_{2n-1}F_{2n}-1}{F_{2n}^2}\xrightarrow[n\to\infty]{}\phi^{-1}.$$

Physically, this can be emulated well with pairs of dodecahedra with $12^2=144$ outcomes each, which is luckily the twelfth Fibonacci number, so the winning probability is $\frac{89\cdot 144-1}{144^2}=0.618007$ only different from ϕ^{-1} beyond the fifth digit. In the continuous limit, we can consider a set of three 'golden' unfair coins $\{C_1,C_2,C_3\}$, each having $P(\text{Heads})=\phi^{-1}$ and $P(\text{Tails})=1-\phi^{-1}=\phi^{-2}$:

Side	C_1	C_2	C_3
Heads	2	4	3
Tails	5	1	3

2. Regular intransitive tournament with 5 variables

For a larger number of random variables, it is also interesting what the probabilities are for non-consecutive ones. Akin and Saccamano [4] showed the construction of an arbitrary tournament with a given number of dice and faces on each die. In particular, arbitrary tournament of $2n+1$ dice can be constructed with 3-sided dice and numbers $\{1\dots 3(2n+1)\}$ on the faces. This means that each probability $P(D_i>D_j)=5/9$. But it can be higher. For example, with 5-sided dice we can construct the set

Side	D_1	D_2	D_3	D_4	D_5
1	2	3	1	7	6
2	5	4	10	12	8
3	11	16	14	13	9
4	23	20	19	15	17
5	24	22	21	18	25

where $P(D_i>D_j)=0.56>5/9$ for $j-i\equiv 1$ or $2 \pmod 5$. It can be emulated with regular icosahedra, where each value is mapped to four faces. The natural question is how high can the winning probability be in a regular tournament. [1]

It seems that the maximum common probability for a 5-tournament can be obtained by generalizing the case of the 'golden' coin. In fact, let ψ be the positive real root of $x^4-x^3-1=0$, and note that $\psi^{-3}+\psi^{-4}+\psi^{-5}+\psi^{-6}=1$. Then let us consider a set of 5 four-sided dice with corresponding

[1] I also provide set of 11-sided dice tournament with more than 57% winning probabilities on my website `https://belliavesha.github.io/second.html#dice`

probabilities for each side $\{\psi^{-3}, \psi^{-4}, \psi^{-5}, \psi^{-6}\}$:

Side prob.	D_1	D_2	D_3	D_4	D_5
ψ^{-3}	19	18	17	14	8
ψ^{-4}	3	2	1	5	9
ψ^{-5}	4	6	10	12	20
ψ^{-6}	11	15	16	13	7

For this set,

$$P(D_i > D_j) = \frac{\psi}{1+\psi} \approx 0.57988 \quad j - i \equiv 1 \text{ or } 2 \pmod 5$$

However, it is not straightforward to generate regular n-sided dice set that approximates this probability as it was with Fibonacci numbers, because the (1, 4)-bonacci numbers (such that $f_{n+1} = f_n + f_{n-3}$) do not seem to have enough identities similar to Cassini's.

3. The case with 7 variables and beyond

For tournaments of size $2n+1 > 7$, similar constructions will not yield optimal variables, because for each positive real root ψ_{2n+1} of $x^{2n} - x^{2n-1} - 1 = 0$, the value

$$\frac{\psi_{2n+1}}{1+\psi_{2n+1}} < \frac{5}{9}.$$

That leaves the last number of dice, 7, for which this pattern would make sense to try:

$$\frac{\psi_7}{1+\psi_7} = 0.5624\cdots > \frac{5}{9},$$

where $\psi_7 = 0.7780895986786\ldots$, the root of $x^6 - x^5 - 1$. However, such a set proved to be quite difficult to find, but it is finally given in the following:

Side prob.	D_1	D_2	D_3	D_4	D_5	D_6	D_7
ψ_7^{-5}	17	14	11	8	27	31	20
ψ_7^{-6}	24	28	32	38	3	2	37
ψ_7^{-7}	18	15	12	9	41	22	1
ψ_7^{-8}	25	29	33	39	6	40	21
ψ_7^{-9}	19	16	13	10	42	5	35
ψ_7^{-10}	26	30	34	36	7	23	4

In this set.

$$P(D_i > D_j) = \frac{\psi_7}{1+\psi_7} \quad j - i \equiv 1,\ 2,\ \text{or } 3 \pmod 7$$

There are two more regular tournaments on 7 vertices, and, they are most probably also constructable with this scheme.

The hypothesis is that the aforementioned sets of random values are optimal, that is, achieving the maximum winning probability in tournaments of size 3, 5, and 7. The optimal sets for bigger regular tournaments are likely the ones emulated with 3-sided dice and the maximum probability is 5/9.

References

[1] S. Trybuła, *On the paradox of three random variables*, Applicationes Mathematicae **5** (1961), no. 4, 321-332 (eng).

[2] ———, *On the paradox of n random variables*, Applicationes Mathematicae **8** (1965), no. 2, 143-156 (eng).

[3] Zalman Usiskin, *Max-Min Probabilities in the Voting Paradox*, The Annals of Mathematical Statistics **35** (1964), DOI 10.1214/aoms/1177703585.

[4] Ethan Akin and Julia Saccamano, *Generalized Intransitive Dice II: Partition Constructions*, 2019.

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