GETZLER-KAPRANOV GRAPH COMPLEX COHOMOLOGY COMPUTATIONS IN LOW EXCESSES

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Abstract. After describing a family of generators of the Getzler-Kapranov graph complex in weight 13, we list them with the aid of a python script using the excess $E(g,n) = 3(g-1)+2n$ as a measure of complexity and finally compute the cohomology. Following the approach of [4] for computing weight graded graded pieces of $H_c^*(\mathcal{M}_{g,n})$, we extend their results in weight 13 for the (g,n) pairs of excess 3.

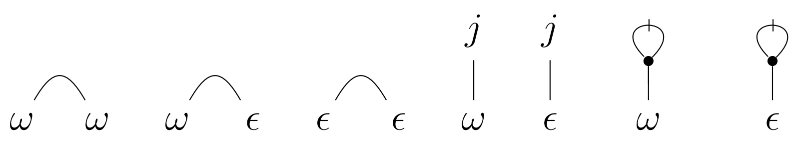
1. Computer program

We use Python to generate all possible blown-up representations of the relevant decorated graphs with a certain upper excess bound E_{max}. The script is in the format of a Jupyter Notebook. The graphs are generated using Sage's builtin interface for the Nauty library, and are later wrapped in Python objects that make available the graph features for easy access. We use Pandas to store the large amounts of python objects in dataframes along with many of their parameters (excess, edges, etc), which enables the user to take advantege of many useful data management tools like querying, sorting and grouping. Finally, we use Matplotlib to either show the graphs in the output cells or save them in a pdf file.

1.1. **Generate blown-up components.** These are all the connected graphs with trivalent vertices, hairs labeled by ϵ, ω or j such that $0 \leq 3(g-1)+3|\epsilon|+|\omega|+2|j| \leq E_{max}-22$, and falling under exactly one of the following cases:

(1) simple,
(2) simple and with a crossed ω hair,
(3) simple and with a crossed internal edge, possibly having maximum one multiple edge parallel to the crossed edge.

We also add manually the following seven bonus graphs:



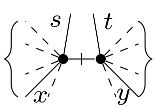
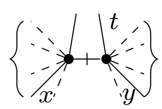
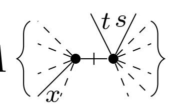
In this list lie also many graphs that will successively killed by relations, for example the lone hair with two ω labels, A_2 graphs with a loop at $\widetilde{v}$ or a a second ω hair incident to the crossed hair. Nonetheless, it is useful to have all these graphs in the list for completeness, and verifying that the algorithms run correctly in full generality.

First we generate graphs where ϵ and ω hairs are unlabeled, so called *unmarked blown-up components*, and then we mark them in every possible way. This is done in order to determine weight 11 relations later on, as explained below.

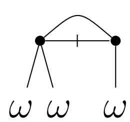
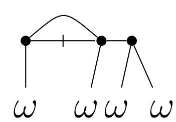
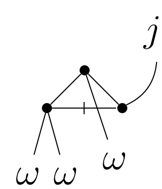
These graphs are wrapped in Python objects which compute at construction all graph parameters such as genus, valence at the special vertices, odd symmetries, Specht module contributions and plotting structures.

1.2. **Weight 2 and** $3b$**) relations.** Case B_1 graphs are grouped by the isomorphism class of their contraction at the crossed edge. This yields groups of blown-up components such that, any weight 2 cohomology relations or the $3b$) relation restrict between graphs in the same group. In low excess, for most B_1 graphs the valence at $\widetilde{v}$ is the minimal amount 4, and thus one can automatically pick any graph in the group to get a basis. But for higher valence, we manually go through each weight 2 relation group, determine a basis and hardcode it directly into the script.

In any case, a B_1 graph with a multiple edge and valence 4 are equivalent to a B_1 graph with a loop, and thus vanish because of relation $3b$). More generally, we note that if there exist two hairs or half-edges $s, t \in N_{\widetilde{v}}$ in a graph $G_{\widetilde{v},B}^{\widetilde{v},\{A,A'\}}$ with the property that it vanishes whenever the partitioin $A \sqcup A'$ doesn't separate s and t, then, for every other $x, y \in N_{\widetilde{v}}$ distinct from s, z, the following relation holds:

$$\sum_{\substack{A\sqcup A'=N_{\widetilde{v}}\\ s,x\in A,t,y\in A'}} A$$  $$A' = \sum_{\substack{A\sqcup A'=N_{\widetilde{v}}\\ |A|,|A'|\geq 2\\ x\in A,t,y\in A'}} A$$  $$A' = \sum_{\substack{A\sqcup A'=N_{\widetilde{v}}\\ |A|,|A'|\geq 2\\ x\in A,t,s\in A'}} A$$  $$A' = 0$$

This observation can be employed to kill some graphs in excess 3 and 4 which have multiple edges or form a triangle:

$3 \cdot$  $= 0$ $\quad 3 \cdot$  $= 0$ $\quad 3 \cdot$  $= 0$

The cohomological degree of the graphs $\Gamma_{irr}^{(0)}$ and $\Gamma_{\delta}^{(0)}$ above is $k = 24 - n$, while that of $\Gamma_s^{(0)}$ is $k = 23 - n$. Using (**??**), one checks that the differential sends $\Gamma_s^{(0)}$ to a non-zero multiple of $\Gamma_{\delta}^{(0)}$ plus a multiple of $\Gamma_{irr}^{(0)}$. Note that of the six terms in (**??**) only the the third and the fifth term contribute, with the third term producing a multiple of $\Gamma_{irr}^{(0)}$ (after using the relation (**??**)), and the fifth term producing $-\Gamma_{\delta}^{(0)}$ (after replacing the ψ class with a boundary class). In particular, the first term in the second line of (**??**) is zero in this case because there cannot be a weight 13 vertex of genus 1 and valence 11. In the end, we retain only one-dimensional cohomology, generated by $\Gamma_{irr}^{(0)}$. We hence obtain:

$$H^k(\overline{\mathsf{GK}}_{2,10}^{12,1}) = \begin{cases} V_{1^{10}} & \text{for } k = 14 \\ 0 & \text{otherwise} \end{cases} \qquad H^k(\overline{\mathsf{GK}}_{4,7}^{12,1}) = \begin{cases} V_{1^7} & \text{for } k = 17 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(\overline{\mathsf{GK}}_{6,4}^{12,1}) = \begin{cases} V_{1^4} & \text{for } k = 20 \\ 0 & \text{otherwise} \end{cases} \qquad H^k(\overline{\mathsf{GK}}_{8,1}^{12,1}) = \begin{cases} V_1 & \text{for } k = 23 \\ 0 & \text{otherwise.} \end{cases}$$

We can now compute the cohomology of $\overline{\mathsf{GK}}_{g,n}^{12,1}$ for $3g + 2n = 27$. The complex is concentrated in degrees $23 - n$, $24 - n$, $25 - n$. Note that some generators exist only for higher genus or higher n. There is no cocycle of degree $23-n$, since the image of the differential is of full dimension. This is seen already by looking only at the leading terms $\Gamma^{(2)}_{b\bar{4}}$, $\Gamma^{(2)}_{\epsilon\bar{b}}$, $\Gamma^{(2)}_{\overline{B}'}$. Note that these generators only exist for g large enough, i.e., $g \geq 4$ or $g \geq 7$ respectively. But if they do not (because g is too small), the same holds for the corresponding generators of degree $23-n$. Hence we conclude that $H^{23-n}(\overline{\mathsf{GK}}^{12,1}_{g,n}) = 0$ for all g, n considered.

Next we consider cocycles x of degree $24-n$. The general cocycle is a linear combination of the 9 generators in degree $24-n$. By adding an exact term to x we may however assume that this linear combination does not involve $\Gamma^{(2)}_{b\bar{4}}$, $\Gamma^{(2)}_{\epsilon\bar{b}}$, $\Gamma^{(2)}_{\overline{B}'}$. Assume first that $n \geq 1$. Then we claim that no linear combination of the remaining generators can be closed. This is so because the image of the generators under the differential is already of full rank if projected to the subspace spanned by the "leading terms" $\Gamma^{(2)}_{birr}$, $\Gamma^{(2)}_{b\bar{b}}$, $\Gamma^{(2)}_{j\bar{b}}$, $\Gamma^{(2)}_{j,irr}$, $\Gamma^{(2)}_{\bar{b}\bar{b}}$, $\Gamma^{(2)}_{\bar{b}i}$. Hence we have $H^{24-n}(\overline{\mathsf{GK}}^{12,1}_{g,n}) = 0$ for $n \geq 1$. In the special case $n = 0$, i.e., $g = 9$, the generator $\Gamma^{(2)}_{j,irr}$ of degree 25 does not exist, while $\Gamma^{(2)}_{\epsilon,irr}$ does exist. Hence a linear combination of $\Gamma^{(2)}_{\epsilon,irr}$ and $\Gamma^{(2)}_{4irr}$ is a cocycle, so that $H^{24}(\overline{\mathsf{GK}}^{12,1}_{9,0})$ is one-dimensional.

Finally, we consider degree $25-n$. Any element x of that degree is a cocyle. By adding exact terms we may ensure that x does not involve the generators $\Gamma^{(2)}_{birr}$, $\Gamma^{(2)}_{b\bar{b}}$, $\Gamma^{(2)}_{j\bar{b}}$, $\Gamma^{(2)}_{\bar{b}\bar{b}}$, $\Gamma^{(2)}_{\bar{b}i}$, and in addition we have to mod out the linear combination $\sum_j \pm\Gamma^{(2)}_{j,irr}$ (for $g > 1$, $n > 0$), that accounts for one copy of the sign representation V_{1^n} of $\mathbb{S}_n$. Our x can hence be a linear combination of the generators $\Gamma^{(2)}_{i}$, $\Gamma^{(2)}_{j,irr}$ and $\Gamma^{(2)}_{ij}$. The generators $\Gamma^{(2)}_{i}$ exist for $g \geq 3$ and $n \geq 1$ and contribute the representation $V_{1^n} \oplus V_{21^{n-2}}$ of $\mathbb{S}_n$. As explained above the generators $\Gamma^{(2)}_{j,irr}$ and $\Gamma^{(2)}_{ij}$ in total contribute an $\mathbb{S}_n$ representation $V_{21^{n-2}}$ if $g = 1$, and a representation $V_{1^n} \oplus V_{21^{n-2}}$ for $g > 1$, $n > 0$. As mentioned above, in the case $g > 1$ we have to remove the V_{1^n} again (since it is in the image of the differential). Hence we arrive at the following cohomology table:

$$H^k(\overline{\mathsf{GK}}^{12,1}_{1,12}) = \begin{cases} V_{21^{10}} & \text{for } k = 13 \\ 0 & \text{otherwise} \end{cases} \qquad H^k(\overline{\mathsf{GK}}^{12,1}_{3,9}) = \begin{cases} V_{1^9} \oplus V_{21^7} \oplus V_{21^7} & \text{for } k = 16 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(\overline{\mathsf{GK}}^{12,1}_{5,6}) = \begin{cases} V_{1^6} \oplus V_{21^4} \oplus V_{21^4} & \text{for } k = 19 \\ 0 & \text{otherwise} \end{cases} \qquad H^k(\overline{\mathsf{GK}}^{12,1}_{7,3}) = \begin{cases} V_{1^3} \oplus V_{21} \oplus V_{21} & \text{for } k = 22 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(\overline{\mathsf{GK}}^{12,1}_{9,0}) = \begin{cases} \mathbb{C} & \text{for } k = 24 \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of Theorem **??**.

2. Appendix

The $\mathbb{Q}$-Hodge structure on the cohomology of the moduli space of curves delivers the following decompositions.

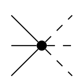
$$
\begin{aligned}
H^{13}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{C} &= H^{12,1}(\overline{\mathcal{M}}_{g,n}) \oplus H^{1,12}(\overline{\mathcal{M}}_{g,n}) \\
H^{11}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{C} &= H^{11,0}(\overline{\mathcal{M}}_{g,n}) \oplus H^{0,11}(\overline{\mathcal{M}}_{g,n}) \\
H^{2}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{C} &= H^{1,1}(\overline{\mathcal{M}}_{g,n})
\end{aligned}
\tag{2.1}
$$

In this paper we are interested in cohomology classes $\gamma_v \in H^k(\overline{\mathcal{M}}_{g,n})$ when viewed as decorations of a vertex of genus g and valence n in some graph Γ. Since it is cumbersome to bring along γ whenever we want to reference a specific decorated graph (Γ, γ), we will encode this datum in symbols drawn onto the vertex being decorated. The goal is to have graphical depictions that determine uniquely all relevant cohomology classes, so that just by the drawing it is possible to unambiguosly determine what is the decoration at each vertex. We will give these graphical depictions onto the one vertex graph $*_{g,n}$ of genus g and with n hairs, labeled by a set N; these depictions are understood to transfer onto the vertex being decorated of a general ambient graph.

To understand automorphisms of decorated graphs we will have to keep track of the $\mathbb{S}_n$-action on each cohomolgy group induced by the permutation of N, which in some cases involves a sign representation.

In the following depictions, solid lines represent the minimum amount of edges *necessary* for the considered class to exist, whereas dashed lines represent a *potential* existance of edges.

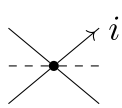
2.1. **The case $k = 0$, $g = 0$, $n \geq 3$.** We have $H^{0,0}(\overline{\mathcal{M}}_{0,n}) = \mathbb{C}$ and thus we can draw weight 0 vertices without any graphical depiction.

 $:= (*_{0,n}, 1)$

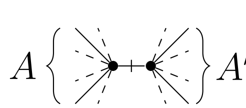
2.2. **The case $k = (1,1)$, $g = 1$,$n = 1$.** $H^{1,1}(\overline{\mathcal{M}}_{1,1})$ is one dimensional, spanned the class we call δ_{irr}. Since this is the only case where a non special vertex might have genus 1, we will introduce a symbolic loop with a crossed edge and draw the node black as if it were a genus 0 vertex. As $n = 1$, there is no $\mathbb{S}_n$-action on δ_irr to talk about.

 $:= (*_{1,1}, \delta_{irr})$

2.3. **The case $k = (1,1)$, $g = 0$.** For this case we refer to [5, Section 3]. In $g = 0$, $H^{1,1}(\overline{\mathcal{M}}_{0,n})$ is non zero only for $n \geq 4$; so we operate under this assumption. The group is generated by classes ψ_i for every $1 \leq i \leq n$ and $\delta\{{}^{A}_{A'}\}$ for every partition $A \sqcup A' = N$ with $|A|, |A'| \geq 2$. To depict $\delta\{{}^{A}_{A'}\}$ we split symbolically the vertex in two parts connected by a crossed edge and draw on one side the subset of hairs A and on the other A'. The notation $\{{}^{A}_{A'}\}$ is chosen to express the fact that swapping A and A' doesn't change the class, which graphically means it doesn't matter on which sides the two sets of hairs are chosen to be drawn. The $\mathbb{S}_n$-action is given by $\sigma\,\psi_i = \psi_{\sigma i}$ and $\sigma\,\delta\{{}^{A}_{A'}\} = \delta\{{}^{\sigma A}_{\sigma A'}\}$.



$:= (*_{0,n}, \psi_i)$



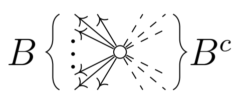
$:= (*_{0,n}, \delta\{{}^{A}_{A'}\})$

There are two equivalent families of relations between these generators. For any three pairwise distinct $i, x, y \in N$, or for any $i \neq j \in N$, it holds:

$$\psi_i = \sum_{\substack{A \sqcup A' = N, |A|, |A'| \geq 2 \\ i \in A, x, y \in A'}} \delta\{{}^{A}_{A'}\} \qquad\qquad \psi_i + \psi_j = \sum_{\substack{A \sqcup A' = N, |A|, |A'| \geq 2 \\ i \in A, j \in A'}} \delta\{{}^{A}_{A'}\}. \tag{2.2}$$

So the ψ_i classes are actually superfluous in the case $g = 0$, but algebraically they can be more convenient to work with. The dimension of $H^{1,1}(\overline{\mathcal{M}}_{0,n})$ turns out to be $2^{n-1} - \binom{n}{2} - 1$, in particular when $n = 4$ every single class forms a basis.

2.4. **The case** $k = (11, 0)$**,** $g = 1$**.** This case is studied in [2, Section 2]. $H^{11,0}(\overline{\mathcal{M}}_{1,n})$ is non zero only for $n \geq 11$; so we operate under this assumption. The group is generated by classes ω_B for every *alternatingly ordered* $B \subseteq N$ with $|B| = 11$. This means that the underlying set determines ω_B up to sign, and if we choose a canonical labeling $N = \{1, ..., n\}$ we can stipulate that every subset comes equipped with the increasing ordering. We draw arrows onto the hairs contained in B to depict the ω_B decoration.



$$:= (*_{1,n}, \omega_B)$$

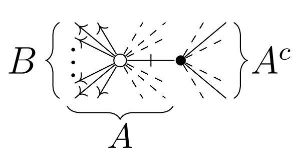
The only relations are amongst the classes ω_B whose set B is contained in the same size 12 subset. Namely, if we choose a canonical labeling on N, then for every $E = \{e_1, ..., e_{12}\} \subseteq N$ with e_i increasing there is the relation

$$\sum_{i=1}^{12} (-1)^i \omega_{E \setminus e_i} = 0 \tag{2.3}$$

Therefore, choosing a distinguished hair $e \in N$ (for example $e = 1$), the classes ω_B with $e \in B$ form a basis of $H^{11,0}(\overline{\mathcal{M}}_{1,n})$.

The $\mathbb{S}_n$-action on ω_B is given by $\sigma\, \omega_B = \omega_{\sigma B}$, which is equal to $\operatorname{sgn} \sigma\, \omega_B$ if σ preserves B setwise.

2.5. **The case** $k = (12, 1)$**,** $g = 1$**.** This case is studied in [3, Section 4.2]. $H^{12,1}(\overline{\mathcal{M}}_{1,n})$ is non zero only for $n \geq 12$; so we operate under this assumption. The group is generated by classes $Z_{B \subseteq A}$ for every subset $A \subseteq N$ with $|A^c| \geq 2$ and *alternatingly ordered* subset $B \subseteq A$ with $|B| = 10$; the underlying set B determines $Z_{B \subseteq A}$ up to sign. We draw arrows onto the hairs contained in B, and we split symbolically the vertex in a genus 1 vertex, where we attach the hairs in A, and a genus 0 vertex, where we attach the hairs in A^c.



$$:= (*_{1,n}, Z_{B \subseteq A})$$

The only relations are amongst the classes $Z_{B \subseteq A}$ having $|A^c| = 2$ and same set $B \sqcup A^c$. Namely, if we choose a canonical labeling on N, then for every $E = \{e_1, ..., e_{12}\} \subseteq N$ with e_i increasing and every $1 \leq i < j < k \leq 12$ we have the relation

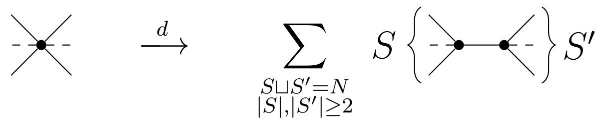
$$(-1)^{i+j} Z_{E \setminus e_i, e_j \subseteq N \setminus e_i, e_j} - (-1)^{i+k} Z_{E \setminus e_i, e_k \subseteq N \setminus e_i, e_k} + (-1)^{j+k} Z_{E \setminus e_j, e_k \subseteq N \setminus e_j, e_k} = 0 \tag{2.4}$$

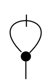
For this subset $E \subseteq N$, choosing a distinguished element $e \in E$ (for example $e = e_1$), the subspace PB_E spanned by the classes $Z_{B\subseteq A}$ with $B \sqcup A^c = E$ has basis the ones with $e \in A^c$, of which there are 11. So $H^{12,1}(\overline{\mathcal{M}}_{1,n})$ decomposes into a direct sum $PB_3 \oplus \bigoplus_{|E|=12} PB_E$, where PB_3 has basis the classes with $|A^c| \geq 3$. The $\mathbb{S}_n$-action on $Z_{B\subseteq A}$ is given by $\sigma\, Z_{B\subseteq A} = Z_{\sigma B\subseteq \sigma A}$, which is equal to $\operatorname{sgn}\sigma\, Z_{B\subseteq \sigma A}$ if σ preserves B setwise.

2.6. **Action of the differential on cohomology classes.** In this section we describe the action of the differential operator that splits one-vertex decorated graphs. A splitting of a decorated graph $(*_{g,n}, \gamma)$ is of the form $(*' - *'', \gamma' \otimes \gamma'')$, where $*' - *''$ is the connection of two vertices $*'_{g',n'}, *''_{g'',n''}$ with $g = g' + g''$, $n = n' + n'' + 2$, and γ', γ'' are their respective decorations. The hairs of $*'$ and $*''$ form a partition of N. If q' and q'' are the two half-edges connecting $*'$ and $*''$, then γ' is obtained from γ by pullback along the map $\overline{\mathcal{M}}_{g',n'+1} \to \overline{\mathcal{M}}_{g,n}$ determined by the subset of hairs ending up on $*'$. For the computation of the pullbacks of cohomology classes we refer to [5], [2] and [3] for the weight 2, weight 11 and weight 13 cases respectively, in this paper we limit ourselves to translating those computations in graphical form.

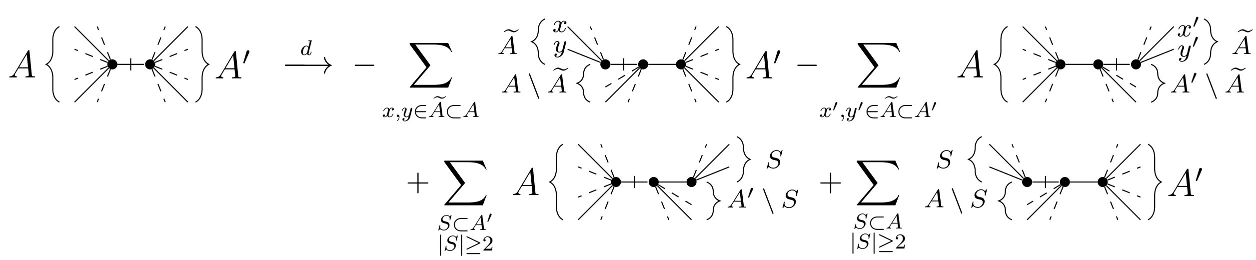
The image under the differential is given by summing over these two-vertex decorated graphs for all possible splittings. In the case of interest, we always have either $g' = g'' = g = 0$ or $g' = g = 1, g'' = 0$, so the only determining datum of the splitting is the partition of the hairs of $*_{g,n}$.

(2.5)



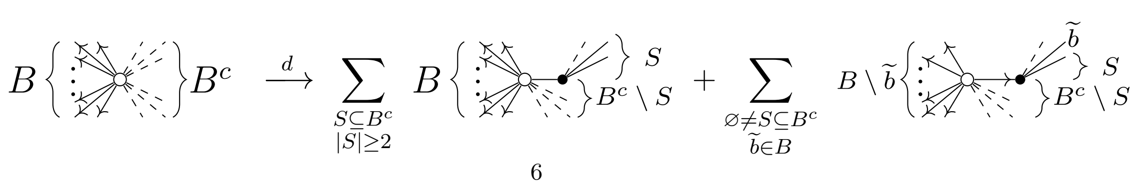
(2.6)  $\xrightarrow{d}$ 0 because the valence of $*_{1,1}$ is less than 2

(2.7) For any choice of $x, y \in A$ and $x', y' \in A'$, the image can be expressed as follows:



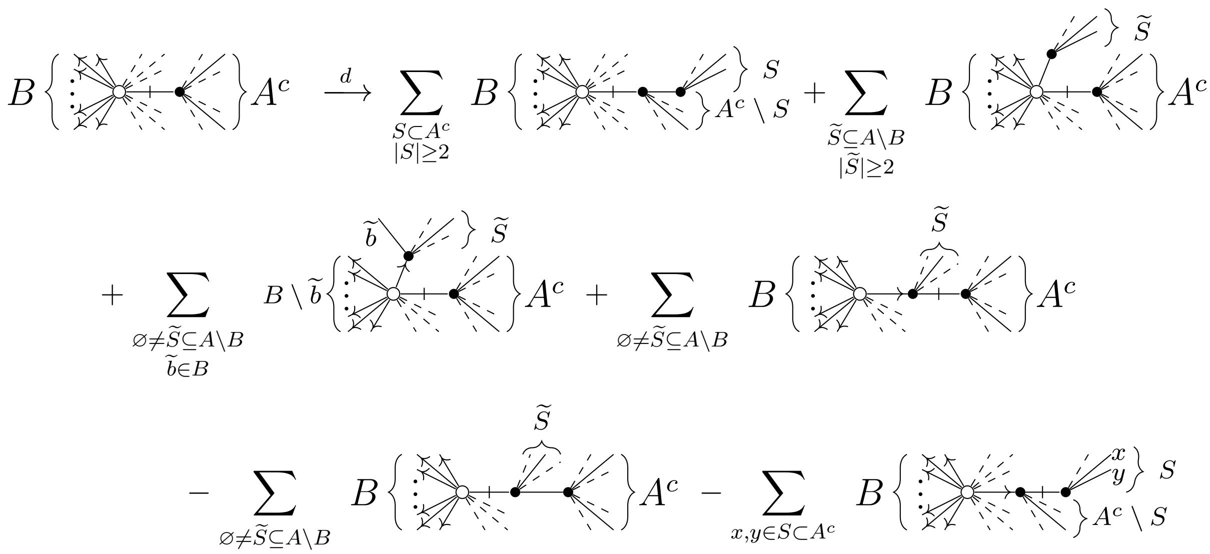
In the second term, the weight 11 decoration ω_B becomes $\omega_{B\setminus \widetilde{b} \sqcup q}$, where q is the newly added half-edge to the genus 1 vertex and takes the place of $\widetilde{b}$ in the ordering of B.

(2.8)



For any fixed choice of $x, y \in A^c$, the image can be expressed as in 2.9. In the two terms that create weight 11 and 2 vertices, the newly created weight 11 decoration $\omega_{B \sqcup p}$, where p is the half-edge at the genus 1 vertex, is understood to have the ordering inherited from B with p appended at the end.

(2.9)



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