# GETZLER-KAPRANOV GRAPH COMPLEX COHOMOLOGY COMPUTATIONS IN LOW EXCESSES

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Abstract. After describing a family of generators of the Getzler-Kapranov graph complex in weight 13, we list them with the aid of a python script using the excess $E(g,n) = 3g+2n$ as a measure of complexity and finally compute the cohomology. Following the approach of [4] for computing weight graded graded pieces of $H_c^*(\mathcal{M}_{g,n})$, we extend their results in weight 13 for the $(g,n)$ pairs of excess 28.

## 1. Introduction

The authors of [4] study the weight graded pieces of $H_c^*(\mathcal{M}_{g,n})$ using graph complexes. The associated graded of the weight filtration is identified with the cohomology of the modular cooperad whose $(g,n)$ part is $H^*(\overline{\mathcal{M}}_{g,n})$, it is known as the Getzler-Kapranov graph complex:

$$H^*(\mathsf{GK}^k_{g,n}) \cong \operatorname{gr}^W_k H_c^*(\mathcal{M}_{g,n}) := W_k H_c^*(\mathcal{M}_{g,n})/W_{k-1} H_c^*(\mathcal{M}_{g,n}) \tag{1.1}$$

They obtain two following results in weight 13.

**Proposition 1.1.** *If $3g+2n \le 25$ then $\operatorname{gr}^W_{13} H_c^*(\mathcal{M}_{g,n}) = 0$.*

**Theorem 1.2.** *Suppose $3g+2n \in \{26, 27\}$. Then $\operatorname{gr}^W_{13} H_c^*(\mathcal{M}_{g,n})$ is nonzero only in degree*

$$k(g,n) = 3g+n-2-\delta_{0,n},$$

*and there is an $\mathbb{S}_n$-equivariant isomorphism $\operatorname{gr}^W_{13} H_c^{k(g,n)}(\mathcal{M}_{g,n}) \cong Z_{g,n} \otimes \mathsf{LS}_{12}$, where*

$$\begin{array}{lll} Z_{1,12} \cong V_{21^{10}} & Z_{2,10} \cong V_{1^{10}} & Z_{3,9} \cong V_{1^9} \\ Z_{4,7} \cong V_{1^7} & Z_{5,6} \cong V_{1^6} \oplus V_{21^4}^{\oplus 2} & Z_{6,4} \cong V_{1^4} \\ Z_{7,3} \cong V_{1^3} \oplus V_{21}^{\oplus 2} & Z_{8,1} \cong \mathbb{Q} & Z_{9,0} \cong \mathbb{Q} \end{array}$$

With our computations we extend to the following case.

**Theorem 1.3.** *Suppose $3g+2n = 28$. Then $\operatorname{gr}^W_{13} H_c^*(\mathcal{M}_{g,n})$ vanishes outside the degrees*

$$k_1(g,n) = 3g+n-2 \quad \text{and} \quad k_2(g,n) = 3g+n-3$$

*and there are $\mathbb{S}_n$-equivariant isomorphisms*

$$\operatorname{gr}^W_{13} H_c^{k_1(g,n)}(\mathcal{M}_{g,n}) \cong Z_{g,n} \otimes \mathsf{LS}_{12} \qquad \operatorname{gr}^W_{13} H_c^{k_2(g,n)}(\mathcal{M}_{g,n}) \cong W_{g,n} \otimes \mathsf{LS}_{12},$$

*where*

$$Z_{2,11} \cong V_{21^9} \oplus V_{221^7} \oplus V_{31^8}^{\oplus 2} \qquad Z_{4,8} \cong V_{21^6} \oplus V_{221^4} \oplus V_{31^5}^{\oplus 3} \qquad Z_{6,5} \cong V_{21^3} \oplus V_{221} \oplus V_{31^2}^{\oplus 3}$$

$$Z_{8,2} = 0 \qquad W_{2,11} = V_{1^{11}}^{\oplus 2} \qquad W_{4,8} = V_{1^8}^{\oplus 2} \qquad W_{6,5} = V_{1^5}^{\oplus 2} \qquad W_{8,2} = V_{1^2}$$

## 2. Graph Complexes in weight 13

In this sections, we recall the definition of the Getzler-Kapranov graph complex and its simplified version obtained as a quasi-isomorphic quotient in [4]. Then we model combinatorially it's weight 13 graded piece using the graphical depictions of generators of the cohomology groups $H^{12,1}(\overline{\mathcal{M}}_{1,n_{\overline{v}}})$, $H^{11,0}(\overline{\mathcal{M}}_{1,n})$, $H^{1,1}(\overline{\mathcal{M}}_{g,n})$.

2.1. **The simplified Getzler-Kapranov graph complex in weight 13.** In the following we will consider stable graphs $\Gamma \in \Gamma((g,n))$ equipped with a decoration $(\gamma_v)_{v\in V(\Gamma)}$ of their vertices by cohomology classes $\gamma_v \in H^*(\overline{\mathcal{M}}_{g_v,n_v})$ which behaves tensorially in $v$, and with an alternating ordering of the internal edges $(o_e)_{e\in E(\Gamma)}$; we will refer to such graphs $(\Gamma, \gamma\otimes o)$ shortly as *decorated graphs*. Isomorphisms $\phi : \Gamma \xrightarrow{\sim} \Gamma'$ of stable graphs act on the decorated graphs with underlying graph $\Gamma$ by permutation:

$$\phi_*(\Gamma, (\gamma_v)_v \otimes (o_e)_e) = (\Gamma', (\gamma_v)_{\phi(v)} \otimes (o_e)_{\phi(e)}).$$

The Getzler-Kapranov graph complex has a presentation by the space of coinvariants of decorated graphs under isomorphisms:

$$\begin{aligned}(2.1)\quad \mathsf{GK}_{g,n} &= \left( \bigoplus_{\Gamma\in\Gamma((g,n))} \bigotimes_{v\in V(\Gamma)} H^*(\overline{\mathcal{M}}_{g_v,n_v}) \otimes \mathbb{Q}[-1]^{\otimes |E(\Gamma)|} \right) \Big/ \sim \\ &= \bigoplus_{[\Gamma]\in[\Gamma((g,n))]} \left( \bigotimes_{v\in V(\Gamma)} H^*(\overline{\mathcal{M}}_{g_v,n_v}) \otimes \mathbb{Q}[-1]^{\otimes |E(\Gamma)|} \right) \Big/ \mathrm{Aut}_\Gamma,\end{aligned}$$

where the $\Gamma((g,n))$ are stable graphs and $[\Gamma((g,n))]$ their isomorphism classes, of which there are only finitely for every pair $(g,n)$. Automorphisms of stable graphs are understood to fix the $n$ labeled hairs and to act by the sign of their induced permutation on the internal edges $E(\Gamma)$.

The *weight* of a decorated graph $(\Gamma, \gamma\otimes o)$ with all $\gamma_v$ of pure cohomological degree $k_v$ is $\kappa = \sum_v k_v$. For each $\kappa \in \mathbb{N}$, the homogeneous slice of weight $\kappa$ of $\mathsf{GK}_{g,n}$ is then given by

$$(2.2)\qquad \mathsf{GK}^{\kappa}_{g,n} = \bigoplus_{[\Gamma]\in[\Gamma((g,n))]} \bigoplus_{\substack{\lambda \text{ partition}\\ \text{of } \kappa}} \left( \bigoplus_{\substack{k:V(\Gamma)\to\mathbb{N}\\ k \text{ respects } \lambda}} \bigotimes_{v\in V(\Gamma)} H^{k_v}(\overline{\mathcal{M}}_{g_v,n_v}) \otimes \mathbb{Q}[-1]^{\otimes |E(\Gamma)|} \right) \Big/ \mathrm{Aut}_\Gamma,$$

The grading of the complex $\mathsf{GK}_{g,n}$ is given by the weight plus the number of internal edges of the stable graphs. For the definition of the differential we refer to [4]. In 2.3 we will describe its action on the graphical depictions of the generators.

Since $H^k(\overline{\mathcal{M}}_{g,n}) = 0$ for odd $k < 11$ [1], the only weight labelings $k : V(\Gamma) \to \mathbb{N}$ with $13 = \sum_v k_v$ giving rise to non trivial decorated graphs are the ones with zero weight on every vertex except either $(A)$ $k_{\overline{v}} = 13$ or $(B)$ $k_{\overline{v}} = 11$, $k_{\widetilde{v}} = 2$ for exactly one special vertex $\overline{v}$ and in case $(B)$ a second vertex $\widetilde{v}$. We will denote by $N_{\overline{v}}$ and $N_{\widetilde{v}}$ the hairs and half-edges at $\overline{v}$ and $\widetilde{v}$. We can reduce further the coinvariants by summing over all possible choices of $\overline{v}$ in case $(A)$ and $\overline{v}, \widetilde{v}$ in case $(B)$, and requiring that automorphisms fix these vertices:

$$\mathsf{GK}^{13}_{g,n} = \bigoplus_{\substack{[\Gamma]\in[\Gamma((g,n))]\\ \overline{v}\in V(\Gamma)}} \left(H^{13}(\overline{\mathcal{M}}_{g_{\overline{v}},n_{\overline{v}}})\otimes\mathbb{Q}[-1]^{\otimes|E(\Gamma)|}\right)/\mathrm{Aut}_{\Gamma,\overline{v}} \tag{2.3}$$
$$\oplus \bigoplus_{\substack{[\Gamma]\in[\Gamma((g,n))]\\ \overline{v}\neq\widetilde{v}\in V(\Gamma)}} \left(H^{11}(\overline{\mathcal{M}}_{g_{\overline{v}},n_{\overline{v}}})\otimes H^{2}(\overline{\mathcal{M}}_{g_{\widetilde{v}},n_{\widetilde{v}}})\otimes\mathbb{Q}[-1]^{\otimes|E(\Gamma)|}\right)/\mathrm{Aut}_{\Gamma,\overline{v},\widetilde{v}}.$$

As the cohomology groups come equipped with a $\mathbb{Q}$-Hodge decomposition 5.1, we obtain accordingly a decomposition $\mathsf{GK}^{13}_{g,n}\otimes\mathbb{C} = \mathsf{GK}^{12,1}_{g,n}\oplus\mathsf{GK}^{1,12}_{g,n}$, where $\mathsf{GK}^{12,1}_{g,n}$ is obtained as in 2.3 by replacing $H^{13}, H^{11}$ and $H^2$ by $H^{12,1}, H^{11,0}$ and $H^{1,1}$ respectively. After quotienting by an appropriate subspace closed under the differential, one obtain a quasi-isomorphic complex $\mathsf{GK}^{13}_{g,n}\twoheadrightarrow\overline{\mathsf{GK}}^{13}_{g,n}$ generated by a much smaller set of decorated graphs [4, Section 2.2]. This quasi-isomorphism preserves the above $\mathbb{Q}$-Hodge decomposition: $\overline{\mathsf{GK}}^{13}_{g,n}\otimes\mathbb{C} = \overline{\mathsf{GK}}^{12,1}_{g,n}\oplus\overline{\mathsf{GK}}^{1,12}_{g,n}$. From now on we will focus on the $\overline{\mathsf{GK}}^{12,1}_{g,n}$ part, which is generated by the subset of decorated graphs satisfying the following properties:

1) Weight zero vertices have genus zero, valence at least 3 and don't have loops.
2) In both cases $(A)$ and $(B)$ the special vertex $\overline{v}$ has $g_{\overline{v}}=1$.
2b) In case $(B)$ the special vertex $\widetilde{v}$ has either $g_{\widetilde{v}}=0$, or $g_{\widetilde{v}}=1$ and $n_{\widetilde{v}}=1$.

Using the presentation of the cohomology groups described in 5, we consider the subspace of $\mathsf{GK}^{12,1}_{g,n}$ spanned by generators with propreties $1), 2), 2b)$ and we exchange the order of the cohomology relations and the coinvariants relations to obtain

$$\overline{\mathsf{GK}}^{12,1}_{g,n} \twoheadleftarrow \bigoplus_{\substack{[\Gamma],\overline{v},\,B\subseteq A\subseteq N_{\overline{v}}\\ |B|=10,|A^c|\geq 3}} \left(\langle Z_{B\subseteq A}\rangle\otimes\mathbb{Q}[-1]^{\otimes|E(\Gamma)|}\right)/\mathrm{Aut}^{B,A}_{\Gamma,\overline{v}} \tag{2.4}$$
$$\oplus \bigoplus_{\substack{[\Gamma],\overline{v},\,E\subseteq N_{\overline{v}}\\ |E|=12}} \left(\bigoplus_{\substack{B\subseteq A\subseteq N_{\overline{v}}\\ B\sqcup A^c=E}} \left(\langle Z_{B\subseteq A}\rangle\otimes\mathbb{Q}[-1]^{\otimes|E(\Gamma)|}\right)/\mathrm{Aut}^{B,A}_{\Gamma,\overline{v}}\right)\Big/\left\{\begin{smallmatrix}\text{weight 13}\\ \text{relations}\end{smallmatrix}\right\}$$
$$\oplus \bigoplus_{[\Gamma],\overline{v},\widetilde{v}} \left(\bigoplus_{\substack{B\subseteq N_{\widetilde{v}},|B|=11\\ A\sqcup A'=N_{\widetilde{v}},|A|,|A'|\geq 2}} \left(\langle\omega_B\otimes\delta\{\begin{smallmatrix}A\\ A'\end{smallmatrix}\}\rangle\otimes\mathbb{Q}[-1]^{\otimes|E(\Gamma)|}\right)/\mathrm{Aut}^{B,\{A,A'\}}_{\Gamma,\overline{v},\widetilde{v}}\right)\Big/\left\{\begin{smallmatrix}\text{weight 11 and 2}\\ \text{relations}\end{smallmatrix}\right\}$$
$$\oplus \bigoplus_{[\Gamma],\overline{v},\widetilde{v}} \left(\bigoplus_{\substack{B\subseteq N_{\widetilde{v}}\\ |B|=11}} \left(\langle\omega_B\otimes\delta_{irr}\rangle\otimes\mathbb{Q}[-1]^{\otimes|E(\Gamma)|}\right)/\mathrm{Aut}^{B}_{\Gamma,\overline{v},\widetilde{v}}\right)\Big/\left\{\begin{smallmatrix}\text{weight 11}\\ \text{relations}\end{smallmatrix}\right\},$$

where, for example in the third term, $\mathrm{Aut}^{B,\{A,A'\}}_{\Gamma,\overline{v},\widetilde{v}}$ is the group of automorphisms leaving the set $B$ and the partition $A\sqcup A'$ invariant. These are precisely the symmetries of the decorated graphs $(\Gamma, Z_{B\subseteq A}\otimes o)$, $(\Gamma,\omega_B\otimes\delta\{\begin{smallmatrix}A\\ A'\end{smallmatrix}\}\otimes o)$ and $(\Gamma,\omega_B\otimes\delta_{irr}\otimes o)$ respectively, which act by the sign of the permutation of the half-edges of $\overline{v}$ in $B$ multiplied by the sign of the permutation on the internal edges $E(\Gamma)$. In each case, the coinvariants relations restrict to killing the decorated graphs with odd symmetry. In the following we will consider the familiy of generators indexed by all the direct sums in 2.4 whose automorphism group does not have odd symmetries. In particular we can impose the further restriction on the generators:

1.2) There are no multiple edges, except possibly for edges incident at $\overline{v}$ or $\widetilde{v}$.

The relations introduced by the quotient manifest as follows between the generators:

Case $(A)$:

3a) Graphs with a loop $(s,t)$ at $\overline{v}$ and decorated by a $Z_{B\subseteq A}$ with $|A^c| \geq 3$, $s,t \in A^c$ are set to zero.

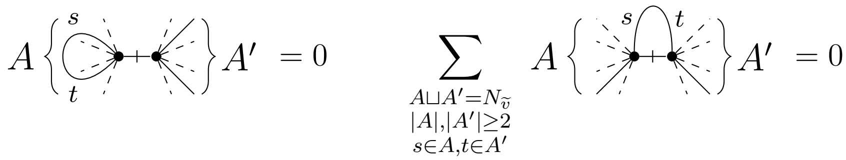
Case $(B)$:

3b) Graphs with $g_{\widetilde{v}} = 0$ and a loop at $\widetilde{v}$ decorated by a class in the image of the pullback $H^2(\overline{\mathcal{M}}_{1,n_{\widetilde{v}}-2}) \xrightarrow{\xi_*} H^2(\overline{\mathcal{M}}_{0,n_{\widetilde{v}}})$ are set to zero.

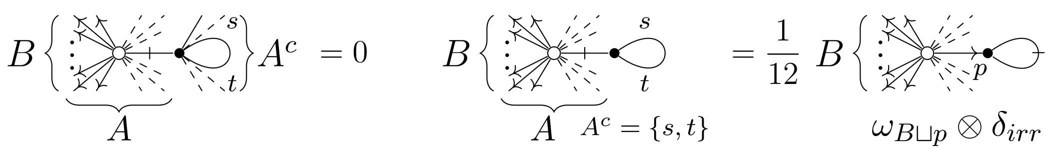
4) Case $(A)$ graphs with a loop $(s,t)$ at $\overline{v}$ and decorated by a $Z_{B\subseteq A}$ with $A^c = \{s,t\}$ are identified with $\frac{1}{12}$ times the case $(B)$ graph obtained by adding a genus 1 vertex $\widetilde{v}$, connecting it only to $\overline{v}$ through an edge $(p',p)$, redecorating $\overline{v}$ with the weight 11 class $\omega_{B\sqcup p}$ and decorating $\widetilde{v}$ with the weight 2 class $\delta_{irr}$.

We represent these relations using the graphical depictions of cohomology classes in 5. When talking about the depiction, the genus one vertex and the two ends of the crossed edge will be referenced by the keys $\overline{v}$ and $\widetilde{v}$ in case $(A)$ and $\overline{v}$,$\widetilde{v}_1$ and $\widetilde{v}_2$ in case $(B)$.

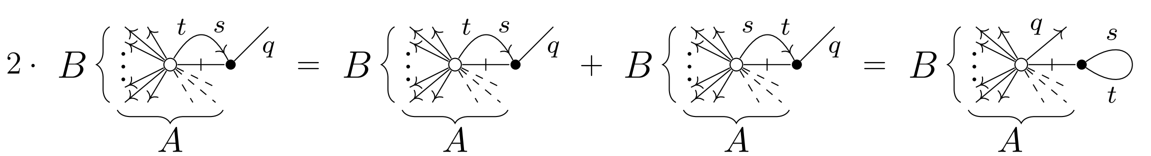
3b):




3a) and 4):




With condition 1) and relations $3a), 3b), 4)$ we can consider only the generators whose graphical depiction has no loops except possibly at $\overline{v}$, but including the special case of exactly one loop at $\widetilde{v}$ for the case $(B)$ with $\omega_B \otimes \delta_{irr}$. In addition, we can ignore case $(A)$ generators with $|A^c| = 2$ and a multiple edge parallel to the crossed edge with an endpoint in the set $B$ using coinvariants and the weight 13 relation:




Considering now the graphical depictions, we obtain four families of $(g,n)$-stable graphs with features, which we call $A_3$, $A_2$, $B_1$ and $B_{irr}$; denote graphs with features in these families as $G^{\widetilde{v}}_{\overline{v},B\subseteq A}$, $G^{\widetilde{v},\{A,A'\}}_{\overline{v},B}$, $G^{\widetilde{v},irr}_{\overline{v},B}$. The variables range in each case over the following objects:

all: Any $(g,n)$-stable graph $G$ with exactly one genus 1 vertex $\overline{v}$ and every other vertex having genus 0.

$A_3, A_2$: Any choice of a crossed internal edge $(\overline{v},\widetilde{v})$ adjacent to $\overline{v}$. In this case, with $N_{\overline{v}}$ we will always mean the hairs and half-edges adjacent to $\overline{v}$ or to $\widetilde{v}$ excluding the two crossed half-edges, and we fix a canonical ordering of the set $N_{\overline{v}}$. In addition, there's any choice of subsets $B \subseteq A \subseteq N_{\overline{v}}$ with $|B| = 10$ and either $|A^c| \geq 3$ or $|A^c| = 2$, called $A_3$ and $A_2$ cases respectively. $G$ is required to be simple, except possibly with loops at $\overline{v}$ or multiple edges parallel to the crossed edge.

$B_1, B_{irr}$: Any choice of subset $B \subseteq N_{\overline{v}}$ with $|B| = 11$. We fix a canonical ordering on $N_{\overline{v}}$.

$B_1$: Any choice of a crossed internal edge $\widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2)$ not adjacent to $\overline{v}$, and any partition $A \sqcup A' = N_{\widetilde{v}}$ with $|A|, |A'| \geq 2$; in this case we denote by $N_{\widetilde{v}}$ the hairs and half-edges adjacent to $\widetilde{v}_1$ or $\widetilde{v}_2$ excluding the two crossed half-edges. $G$ is required to be simple, except possibly with loops at $\overline{v}$ or multiple edges parallel to the crossed edge.

$B_{irr}$: $G$ is required to be simple, except possibly with loops at $\overline{v}$, and with exactly one vertex $\widetilde{v} \neq \overline{v}$ with unique neighbour $\overline{v}$ and one loop, which we call the crossed edge.

all: A choice of alternating order of the internal edges of $G$ which are not crossed.

all: The resulting graph with features must have no odd symmetries. Here a symmetry is a graph automorphism that preserves all the choices made above.

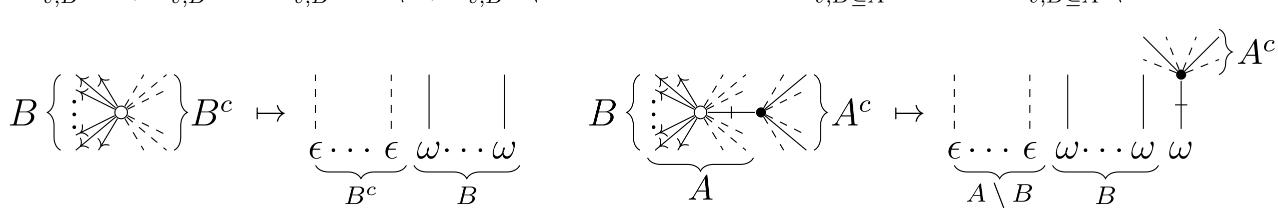
The only relations amongst these four families of generators are the relations of the cohomology classes $Z_{B\subseteq A}$ in case $A_2$ and $\omega_B$ in case $B_1, B_{irr}$ and the remaining quotient relation 3b) in case $B_1$, which can be seen as just an extension of the weight 2 relations in case $B_1$. Using these four families of generators, we obtain

$$\overline{\mathsf{GK}}^{12,1}_{g,n} = \bigoplus_{\left[\substack{G,\overline{v},\widetilde{v}\\ B,A,|A^c|\geq 3}\right]} \left\langle G^{\widetilde{v}}_{\overline{v},B\subseteq A}\right\rangle \oplus \left( \bigoplus_{\left[\substack{G,\overline{v},\widetilde{v}\\ B,A,|A^c|=2}\right]} \left\langle G^{\widetilde{v}}_{\overline{v},B\subseteq A}\right\rangle \right) \Big/ \left\{\substack{\text{weight 13}\\ \text{relations}}\right\}$$
$$\oplus \left( \bigoplus_{\left[\substack{G,\overline{v},\widetilde{v}\\ B,A}\right]} \left\langle G^{\widetilde{v},\{A,A'\}}_{\overline{v},B}\right\rangle \right) \Big/ \left\{\substack{\text{weight 11, 2, and}\\ 3b)\text{ relations}}\right\} \oplus \left( \bigoplus_{[G,\overline{v},\widetilde{v},B]} \left\langle G^{\widetilde{v},irr}_{\overline{v},B}\right\rangle \right) \Big/ \left\{\substack{\text{weight 11}\\ \text{relations}}\right\}, \tag{2.5}$$

where the square brackets indicate that we range over all the admissible choices listed above, up to isomorphisms of graphs with these features.

2.2. **Blown-up representation of the generators.** In order to determine $\overline{\mathsf{GK}}^{12,1}_{g,n}$, we need to classify the four families of graphs with features above up to isomorphism. To this end, we look at their connected components after deleting the special vertex but keeping it's half-edges and hairs; this is called the blown-up representation. We keep track of the features at the special vertex by labeling the hairs of the blown-up components either $\epsilon$ or $\omega$; in each case there are precisely 11 $\omega$ labels. In the pictures below, one has to imagine the rest of the ambient graph to exist unchanged, potentially grouping the $\epsilon$ and $\omega$ hairs into connected components.

$G^{\widetilde{v},\{A,A'\}}_{\overline{v},B}, G^{\widetilde{v},irr}_{\overline{v},B} \mapsto G^{\widetilde{v},\{A,A'\}}_{\overline{v},B} \setminus \overline{v}, G^{\widetilde{v},irr}_{\overline{v},B} \setminus \overline{v}$ $\qquad$ $G^{\widetilde{v}}_{\overline{v},B\subseteq A} \mapsto G^{\widetilde{v}}_{\overline{v},B\subseteq A} \setminus \overline{v}$




The crucial observation is that, if $C_1, .., C_k$ are the blown-up components of $G$, graph automorphisms $\phi : G \to G$ fixing $\overline{v}$ precisely correlate to permutations $\sigma \in \mathbb{S}_k$ and graph isomorphisms $\phi_i : C_i \to C_{\sigma(i)}$. Thus, a generator $G$ is uniquely determined by its list of blown-up components up to reordering and isomorphism of the components preserving the features $\epsilon, \omega$, the original $n$ hairs and the crossed edge. In addition, a generator has an odd symmetry if and only if it has either a blown-up component with an odd symmetry (when restricted to internal edges and $\epsilon$ hairs, as the sign on $\omega$ cancels out by the action on the cohomology class) or at least two isomorphic blown-up components wich have an odd number of internal edges plus $\epsilon$ hairs (as the action on $\omega$ hairs cancels out).

This is already a concrete description that could be implemented in a computer algebra system. However, it would be very computationally wasteful to generate for each $(g, n)$ all isomorphism classes of $(g, n)$-stable graphs because the most efficient graph generating algorithms run based on the number of vertices.

Notice that the $(g, n)$ type of a graph $G$ can be be computed from the $(g, n)$ type and the number of $\epsilon$ and $\omega$ labels of each blown-up component. This means that we could first classify a set of blown-up components based on these parameters and afterwards generate all blown-up representations with these components. If we knew that the blow-up of every graph in some $(g, n)$ classes is so representable we would obtain the full $\overline{\mathsf{GK}}^{12,1}_{g,n}$ for multiple $(g, n)$ at a time, thus minimizing computational redundancy.

The authors of [4, Section 3.2] have chosen to classify blown-up representations by their excess value $E(g, n) = 3g+2n$, which fits together as follows with the parameters of blown-up components:

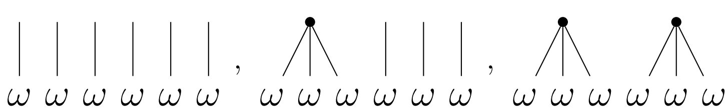
$$n_{\overline{v}} = \sum_i \epsilon_i + \omega_i \geq 11 \qquad n = \sum_i n_i \qquad g = 1 + \sum_i (g_i + \epsilon_i + \omega_i - 1)$$

$$2(g-1) + n = n_{\overline{v}} + \sum_i 2(g_i - 1) + \epsilon_i + \omega_i + n_i \qquad \forall i : 3(g_i - 1) + 3\epsilon_i + \omega_i + 2n_i \geq 0$$

$$3(g-1) + 2n = n_{\overline{v}} + \sum_i 3(g_i - 1) + 2(\epsilon_i + \omega_i + n_i) = 22 + \sum_i 3(g_i - 1) + 3\epsilon_i + \omega_i + 2n_i,$$

where $g_i, \epsilon_i, \omega_i, n_i$ are the genus, number of $\epsilon$ labels, number of $\omega$ labels and number of original hairs of the $i$-th blown-up component. The excess is greater or equal to 25 and additive over the non-negative paramenter $e(C_i) = 3(g_i - 1) + 3\epsilon_i + \omega_i + 2n_i$ of blown-up components, meaning that excess $E$ graphs can only be obtained from a list of blown-up components with $e$ summing up to $E - 25$.

One reason for considering this function as a measure of complexity is its invariance under replacing three hairs at the special vertex by a 'tripod', meaning that in the excess classification we deal simultaneously with every combination of these two types of graphs, which are arguably the simplest stable subgraphs connected to the special vertex.



all have $3(g-1) + 2n = 12$

2.3. **Action of the differential on the generators.** In the simplified graph complex $\overline{\mathsf{GK}}^{12,1}_{g,n}$, the differential $d$ acts by summing over every vertex and every way of splitting that vertex [4, Section 2.6]. More precisely, for every vertex $v$ of a decorated graph $(\Gamma, \gamma \otimes o)$, we consider the one-vertex decorated graph $(*_v, \gamma_v)$ whose hairs correspond to the hairs and neighbours of $v$ in $\Gamma$. The differential on one-vertex graphs decorated by the relevant cohomology classes is described in 5.6. We denote by $d_v(\Gamma, \gamma \otimes o)$ the decorated graph obtained by replacing the vertex $v$ and it's decoration $\gamma_v$ by $d(*_{g,n}, \gamma_v)$, gluing it's hairs to the hairs and neighbours of $v$ in $\Gamma$; the newly created edge is understood to be placed last in the alternating ordering $o$. Thus, the total differential takes the form

$$\text{case (A):}\quad (\Gamma,\gamma\otimes o) \xrightarrow{d} \; d_{\overline{v}}(\Gamma,\gamma\otimes o) \;+ \sum_{\overline{v},\neq v\in V(G)} d_v(\Gamma,\gamma\otimes o)$$

$$\text{case (B):}\quad (\Gamma,\gamma\otimes o) \xrightarrow{d} \; d_{\overline{v}}(\Gamma,\gamma\otimes o) \;+ d_{\widetilde{v}}(\Gamma,\gamma\otimes o) + \sum_{\overline{v},\widetilde{v}\neq v\in V(G)} d_v(\Gamma,\gamma\otimes o).$$

## 3. Computer program

We use Python to generate all possible blown-up representations of the relevant decorated graphs with a certain upper excess bound $E_{max}$. The script is in the format of a Jupyter Notebook. The graphs are generated using Sage's builtin interface for the Nauty library, and are later wrapped in Python objects that store many graph features. We use Pandas to store the large amounts of python objects in dataframes along with many of their parameters (excess, edges, etc), which enables the user to take advantege of many useful data management tools like querying, sorting and grouping. Finally, we use Matplotlib to either show the graphs in the output cells or save them in a pdf file.

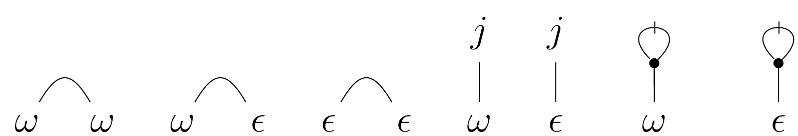
The project is saved in this GitHub repository.

### 3.1. **Generate blown-up components.**

These are all the connected graphs with trivalent vertices, hairs labeled by $\epsilon, \omega$ or $j$ such that $0 \leq 3(g-1)+3|\epsilon|+|\omega|+2|j| \leq E_{max}-25$, and falling under exactly one of the following cases:

1. simple,
2. simple and with a crossed $\omega$ hair,
3. simple and with a crossed internal edge, possibly having maximum one multiple edge parallel to the crossed edge.

We also add manually the following seven bonus graphs:




In this list of blown-up graphs also lie many that will be later killed by relations, for example the lone hair with two $\omega$ labels, $A_2$ graphs with a loop at $\widetilde{v}$ or a a second $\omega$ hair incident to the crossed hair. Nonetheless, it is useful to have all these graphs in the list for completeness, and verifying that the algorithms run correctly in full generality.

These graphs are wrapped in Python objects which compute at construction all graph parameters such as genus, valence at the special vertices, odd symmetries, Specht module contributions and plotting structures.

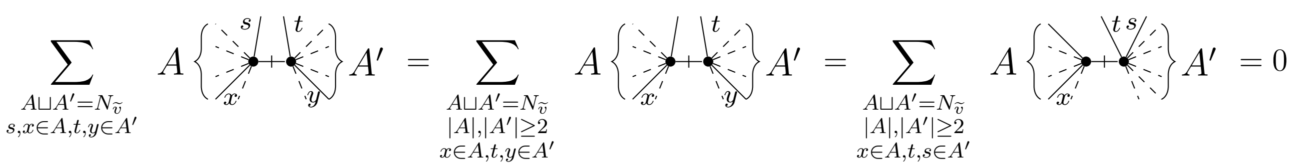
### 3.2. **Weight 2 and $3b$) relations.**

Case $B_1$ graphs are grouped by the isomorphism class of their contraction at the crossed edge. This yields groups of blown-up components such that, any weight 2 cohomology relations or the $3b$) relation restrict between graphs in the same group.

It is important to keep in mind that the $B_1$ graphs we generated in 3.1 do not represent all decorated graphs with $\delta\{{A \atop A'}\}$ classes. We have ignored the ones with loops or multiple edges incident to just one end of the crossed edge. Because they are killed by the $3b$ relation or odd symmetries, they might also influence the relation group without appearing in our list.

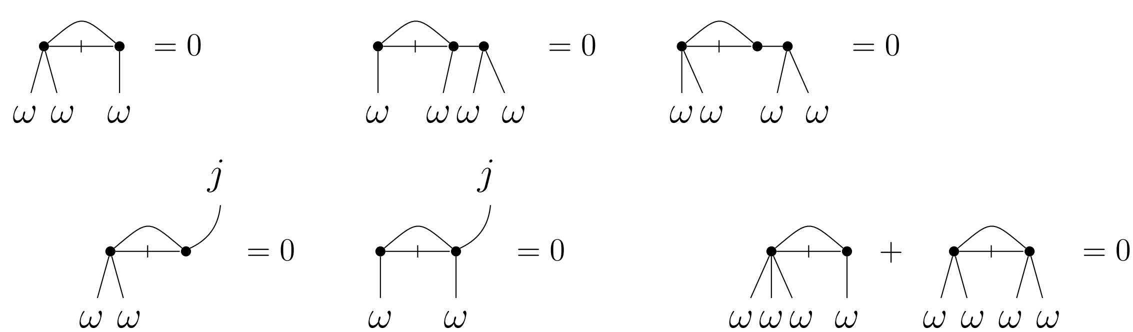
There are three easy cases to handle automatically. If a weight 2 relation group has size one with an odd graph, then it vanishes. If a group has size one and the contraction at the crossed edge doesn't have loops nor multiple edges, then it vanishes if and only if the single graph in the group has an odd symmetry. If a group is of minimal valence $n_{\widetilde{v}} = 4$, then it vanishes if and only if any of its graphs have odd symmetries or its contraction at the crossed edge has loops or multiple edges. For higher group sizes or higher valence, we manually check each weight 2 relation group and hardcode a basis into the script.

**Remark 3.1.** A $B_1$ graph with a multiple edge and valence 4 is equivalent to a $B_1$ graph with a loop, and thus vanishes because of relation 3*b*). More generally, we note that if there exist two hairs or half-edges $s, t \in N_{\widetilde{v}}$ in a graph $G_{\widetilde{v},B}^{\widetilde{v},\{A,A'\}}$ with the property that it vanishes whenever the partition $A \sqcup A'$ doesn't separate $s$ and $t$, then, for every other $x, y \in N_{\widetilde{v}}$ distinct from $s, t$, the following relation holds:



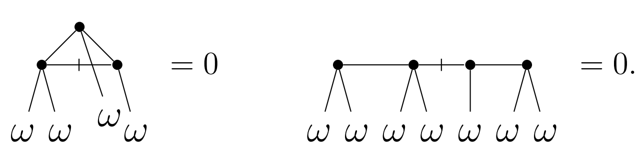
This observation can be employed to kill all $B_1$ graphs in excess 3 and 4 which have multiple edges and valence $n_{\widetilde{v}} \geq 5$, except one. These are shown in 3.1, and they come in three different weight 2 relation groups. They all vanish, except for the last ones, which merely have a relation between them.

(3.1)

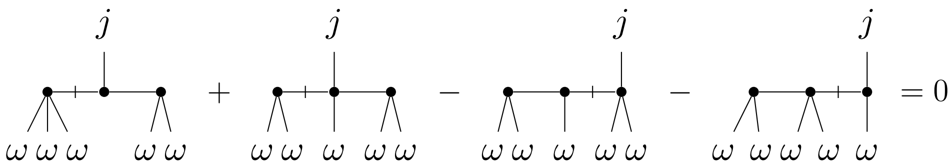


Remark 3.1 can also be used to kill other graphs, such as ones where the crossed edge is part of a triangle or where two half-edges being in at the same endpoint would create an odd symmetry:

(3.2)



3.3 is another example in excess 4 of a weight 2 relation group. It contains the four graphs which appear as terms in the equation, and one checks that all the relations turn out equivalent to the one displayed. Thus, a basis for this group is determined by any three graphs.

(3.3) 


3.3. **Generating virtual blown-up graphs.** First we remove odd symmetries, weight 2 and 3*b*) relations, and $A_2$ graphs which are related to $B_{irr}$ graphs through relation 4. Then, we build so called *virtual blown-up graphs*. These are all unordered lists of the remaining blown-up components of excess $\geq 1$ which have exactly one crossed component, total number of $\omega$ hairs less than or equal to 11, excesses summing up to less than or equal to $E_{max} - 22$, and which don't contain more than once components that create an odd symmetry when exchanged by an automorphism. We explain this last condition.

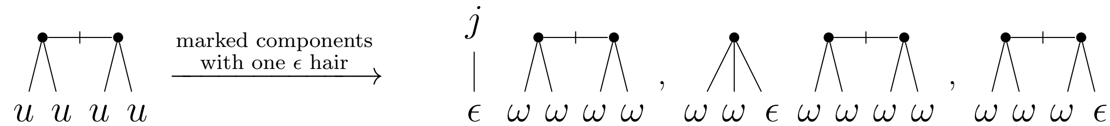
Two isomorphic blown-up components (which don't individually have an odd symmetry already) create an odd symmetry if and only if they have no $j$ hairs (because automorphisms don't exchange the $j$ hairs) and have an odd number of internal edges plus $\epsilon$ hairs. Also the lone hair with two $\epsilon$ labels is of this type.

Virtual blown-up graphs are likewise wrapped in a Python object and stored in a dataframe, along with aggregate quantities from the list of blown-up components. Note that these lists have an excess value but don't yet determine a unique generator, as we haven't pinned down which $(g, n)$ pair of the excess class we are considering, hence the name *virtual*. The possible $(g, n)$ pairs are determined by how many lone $j$ hairs vs tripods it is possible to append to the list of components, they are called the *(g,n)-range* of the virtual blown-up graph.

3.4. **Weight 11 relations.** The first step of the program is actually to generate so called *unmarked blown-up components*, which have a unique label $u$ for the half-edges that where incident to $\overline{v}$. Only afterwards, blown-up components with $\omega$ and $\epsilon$ hairs are generated by marking the $u$ hairs in every possible way. The unmarked component which was used as a template for the marking is stored in each respective blown-up component. In so doing, a list of blown-up components can be obtained from another by permuting its $\omega$ and $\epsilon$ hairs if and only if they have the same list of unmarked components and same number of total $\epsilon$ hairs, which can be instantly checked instead of testing for isomorphism.

Thus, we group by the list of unmarked components and number of $\epsilon$ hairs to determine weight 11 relations groups of $B$ and $B_{irr}$ blown-up graphs. In addition, this can be done at the virtual level without pinning down a $(g, n)$ pair in the excess class by ignoring components obtained as a marking of a lone $j$ hair or a tripod.

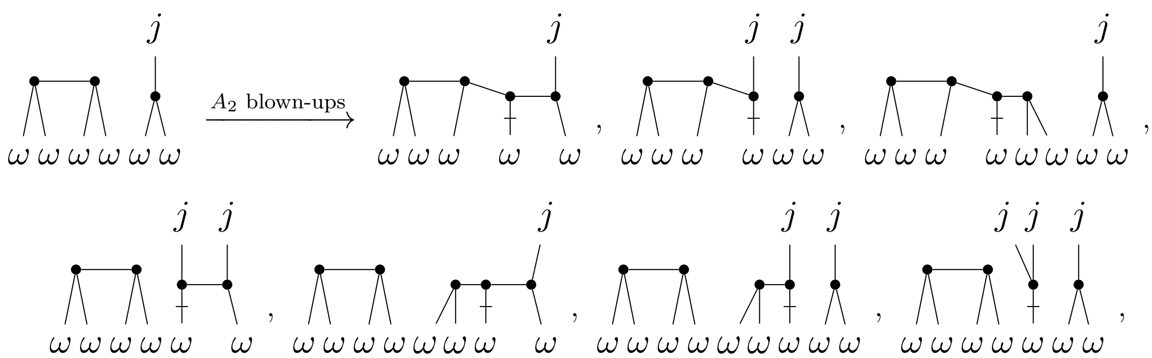
For example, the unmarked component in 3.4 determines in excess 3 (the excess determines the number of $\epsilon$ hairs) a weight 11 relation group of three virtual blown-up graphs. Note that not all graphs in the same relation group must have the same $(g, n)$-range: the first and third exist for $(4, 8), (6, 5), (8, 2)$, whereas the middle one only for $(6, 5), (8, 2)$. The corresponding weight 11 relation is an integer-weighted alternating sum of the three or two graphs, depending upon the $(g, n)$ pair. Thus, discarding the third graph leaves us with a basis of the relation group in for all three $(g, n)$ pairs.

(3.4) 


3.5. **Weight 13 relations.** The process is similar to weight 11, only that now we have to determine virtual blown-up graphs up to permutation of the $\omega$ hairs with the two hairs or half-edges at $\widetilde{v}$. This is done again by blowing up at $\widetilde{v}$: the crossed edge and $\widetilde{v}$ are discarded, the two newly created hairs are marked $\omega$, and we end up with a list of uncrossed blown-up components with a total of 12 $\omega$ hairs.

Thus, we group by the list of so called $A_2$ *components* to determine weight 13 relation groups of $A_2$ blown-up graphs. Again, this can be done at the virtual level by ignoring the lone $\omega - j$ hairs and the tripods with $\omega$ labels.

For example, the two blown-up components in 3.5 determine in excess 4 a weight 13 relation group of seven virtual blown-up graphs. One can check that they all have the same list of blown-up components after removing $\widetilde{v}$ (up to lone hairs and tripods). The graph with two tripods glued to $\widetilde{v}$ (or with two hairs of the same tripod) are not listed because they would have an odd symmetry. Note again that not all blown-up graphs have the same $(g, n)$-range.

(3.5) 


3.6. **Completing virtual blown-up graphs.** The script also offers the possibility to focus on a particular $(g, n)$ pair by *completing* each blown-up graph in an excess class by adding the right amount of $\omega - j$ hairs and tripods to the list of blown-up components.

Having at disposal the Specht module contributions of each generator of the complex, we can compute the Euler Characteristic for each $(g, n)$ pair individually, but with a caveat: we haven't resolved the weight 11 and 13 relations. We have checked that, after removing the redundant contributions, the Euler Characteristic in excesses smaller or equal to 3 matches the one computed in [3, Figure 1].

3.7. **Reading the data.** $\omega, \epsilon$ and $j$ hairs are colored blue, yellow and green respectively. The crossed edge is colored red.

The virtual or completed blown-up graphs can be printed by excess or by the $(g, n)$ pair respectively. They are further grouped by the amount of internal edges (which for virtual blown-ups can be written dependently on $n$) and the cases $A_3, A_2, B_1, B_{irr}$.

Over each blown-up graph is written a unique identifier, the Specht module contribution of the graph and, for virtual blown-ups, the values of $n$ for which the graph can be completed to a $(g, n)$ graph of the given excess. Note that the actual Specht module contribution might not be accurate for graphs in non-trivial relation groups; it has to be computed manually for each group as a whole.

In addition, over most graphs we write the ID of the chosen term for the gaussian elimination process; this is explained in 4.

The start and endpoint of the list of components of each blown-up graph is distinguished by the crossed component, which is always displayed rightmost within its list. In the cases $A_2, B_1$ and $B_{irr}$, graphs in the same relation group are written side by side and on multiple consecutive rows if necessary. When a relation group ends, the next one starts on the row below. Since there are many weight 11 relation groups of size one, which thus form an indipendent set, we display them all in the same rows without line breaking, as we do for graphs in case $A_3$.

## 4. Computations

Our methodology is the following. Focusing on graphs of a specific excess, one edge group at a time, we compute the image of every graph under the differential and carry out gaussian elimination to determine simpler subspaces isomorphic to the kernel and the image. After the simple subspaces have been factored out, we compute the cohomology.

Specifically, for every graph in an edge group we try to choose one non-zero term in its image such that the linear map defined by these choices has full rank. This means that the image of the differential is isomorphic through a change of basis to the subspace in the next edge group spanned by the chosen terms. Thus, to compute the cohomology, one can focus on the list of graphs with the chosen terms removed.

The choices of these leading terms have been hardcoded in the Python script as a dictionary. The IDs of the pairs of graphs in this dictionary are displayed next to each other, with an arrow indicating which is the argument and which the image. Note that the ID is dependent upon the order in which the graphs have been generated and sorted throughout the script, so one has to be careful not to change those processes in order to use the same dictionary.

To avoid losing oneself in countless case distinctions, it is crucial that the choices of leading terms have matching $(g, n)$ range, so that for every $(g, n)$ pair they either both cancel out or both don't exist. It is also important to exploit the gaussian elimination process as much as possible by choosing to kill the graphs with the most complicated differential; these are the $A_3$ graphs and the ones with $\epsilon$ hairs.

*— show here example of a differential of a complicated graph —*

4.1. **Previous results in excess 25,26,27.** These cases have been analyzed in [4, Section 3]. In excess smaller or equal to 25 we have $H^*(\overline{\mathsf{GK}}^{12,1}_{g,n}) = 0$. In excess 26 and 27 the cohomology is concentrated in the top degree of the complex, except in $(g, n) = (9, 0)$ where it is in one degree lower.

$$H^k(\overline{\mathsf{GK}}^{12,1}_{2,10}) = \begin{cases} V_{1^{10}} & \text{for } k = 14 \\ 0 & \text{otherwise} \end{cases} \qquad H^k(\overline{\mathsf{GK}}^{12,1}_{4,7}) = \begin{cases} V_{1^7} & \text{for } k = 17 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(\overline{\mathsf{GK}}^{12,1}_{6,4}) = \begin{cases} V_{1^4} & \text{for } k = 20 \\ 0 & \text{otherwise} \end{cases} \qquad H^k(\overline{\mathsf{GK}}^{12,1}_{8,1}) = \begin{cases} V_{1} & \text{for } k = 23 \\ 0 & \text{otherwise.} \end{cases}$$

$$H^k(\overline{\mathsf{GK}}^{12,1}_{1,12}) = \begin{cases} V_{21^{10}} & \text{for } k = 13 \\ 0 & \text{otherwise} \end{cases} \qquad H^k(\overline{\mathsf{GK}}^{12,1}_{3,9}) = \begin{cases} V_{1^9} \oplus V_{21^7}^{\oplus 2} & \text{for } k = 16 \\ 0 & \text{otherwise} \end{cases}$$

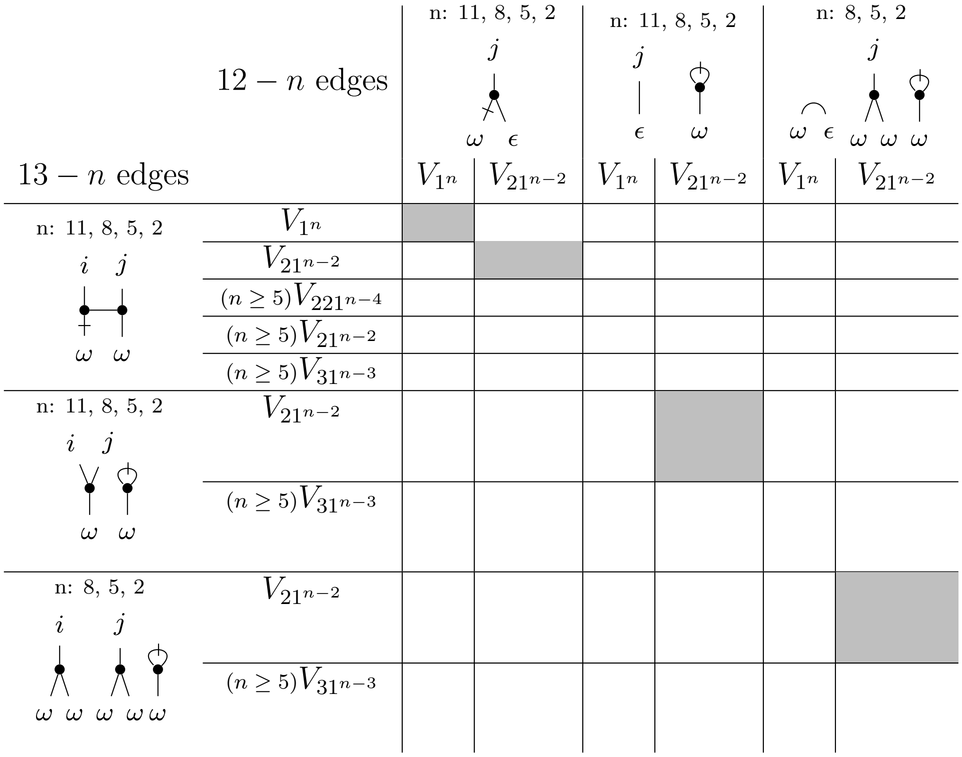
$$H^k(\overline{\mathsf{GK}}^{12,1}_{5,6}) = \begin{cases} V_{1^6} \oplus V_{21^4}^{\oplus 2} & \text{for } k = 19 \\ 0 & \text{otherwise} \end{cases} \qquad H^k(\overline{\mathsf{GK}}^{12,1}_{7,3}) = \begin{cases} V_{1^3} \oplus V_{21}^{\oplus 2} & \text{for } k = 22 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(\overline{\mathsf{GK}}^{12,1}_{9,0}) = \begin{cases} \mathbb{C} & \text{for } k = 24 \\ 0 & \text{otherwise.} \end{cases}$$

4.2. **Excess 28.** We are looking at the $(g, n)$ pairs $(2, 11), (4, 8), (6, 5)$ and $(8, 2)$. There are 106 virtual blown-up graphs of excess 28 to consider, spanning four degree classes.

We have determined bases for each relation group and indicated in the file which graphs are redundant. The choices of these bases have been made a posteriori to suit well the gaussian elimination process.

One checks that our choices of leading terms in the differential of each graph are well-posed and indipendent from one another. Only the six following graphs remain after elimination, distributed in the last two edge groups $12 - n$ and $13 - n$; we draw the matrix of the differential between these two degrees.




A filled box in the above matrix means that the differential of the column has the row as a non-zero term. The graph in the second column actually has only one term in its differential, whereas the other two also have other terms, which have been removed through gaussian elimination.

We obtain the following cohomology groups. In particular, they are no longer concentrated in top degree, and as in excess 2, the pair of maximal genus $(8, 2)$ vanishes in top degree.

$$H^k(\overline{\mathsf{GK}}^{12,1}_{2,11}) = \begin{cases} V_{1^{11}} & \text{for } k = 14 \\ V_{21^9} \oplus V_{221^7} \oplus V_{31^8}^{\oplus 2} & \text{for } k = 15 \\ 0 & \text{otherwise} \end{cases} \quad H^k(\overline{\mathsf{GK}}^{12,1}_{4,8}) = \begin{cases} V_{1^8}^{\oplus 2} & \text{for } k = 17 \\ V_{21^6} \oplus V_{221^4} \oplus V_{31^5}^{\oplus 3} & \text{for } k = 18 \\ 0 & \text{otherwise} \end{cases}$$

$$H^k(\overline{\mathsf{GK}}^{12,1}_{6,5}) = \begin{cases} V_{1^5}^{\oplus 2} & \text{for } k = 20 \\ V_{21^3} \oplus V_{221} \oplus V_{31^2}^{\oplus 3} & \text{for } k = 21 \\ 0 & \text{otherwise} \end{cases} \quad H^k(\overline{\mathsf{GK}}^{12,1}_{8,2}) = \begin{cases} V_{1^2}^{\oplus 2} & \text{for } k = 23 \\ 0 & \text{otherwise} \end{cases}$$

# 5. Appendix

The $\mathbb{Q}$-Hodge structure on the cohomology of the moduli space of curves delivers the following decompositions.

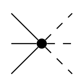
$$
\begin{aligned}
H^{13}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{C} &= H^{12,1}(\overline{\mathcal{M}}_{g,n}) \oplus H^{1,12}(\overline{\mathcal{M}}_{g,n}) \\
H^{11}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{C} &= H^{11,0}(\overline{\mathcal{M}}_{g,n}) \oplus H^{0,11}(\overline{\mathcal{M}}_{g,n}) \\
H^{2}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{C} &= H^{1,1}(\overline{\mathcal{M}}_{g,n})
\end{aligned}
\tag{5.1}
$$

In this paper we are interested in cohomology classes $\gamma_v \in H^k(\overline{\mathcal{M}}_{g,n})$ when viewed as decorations of a vertex of genus $g$ and valence $n$ in some graph $\Gamma$. Since it is cumbersome to bring along $\gamma$ whenever we want to reference a specific decorated graph $(\Gamma, \gamma)$, we will encode this datum in symbols drawn onto the vertex being decorated. The goal is to have graphical depictions that determine uniquely all relevant cohomology classes, so that just by the drawing it is possible to unambiguosly determine what is the decoration at each vertex. We will give these graphical depictions onto the one vertex graph $*_{g,n}$ of genus $g$ and with $n$ hairs, labeled by a set $N$; these depictions are understood to transfer onto the vertex being decorated of a general ambient graph.

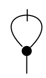
To understand automorphisms of decorated graphs we will have to keep track of the $\mathbb{S}_n$-action on each cohomolgy group induced by the permutation of $N$, which in some cases involves a sign representation.

In the following depictions, solid lines represent the minimum amount of edges *necessary* for the considered class to exist, whereas dashed lines represent a *potential* existance of edges.

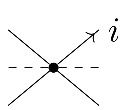
5.1. **The case $k = 0$, $g = 0$, $n \geq 3$.** We have $H^{0,0}(\overline{\mathcal{M}}_{0,n}) = \mathbb{C}$ and thus we can draw weight 0 vertices without any graphical depiction.

 $:= (*_{0,n}, 1)$

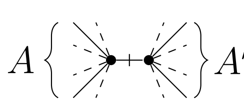
5.2. **The case $k = (1,1)$, $g = 1$,$n = 1$.** $H^{1,1}(\overline{\mathcal{M}}_{1,1})$ is one dimensional, spanned the class we call $\delta_{irr}$. Since this is the only case where a non special vertex might have genus 1, we will introduce a symbolic loop with a crossed edge and draw the node black as if it were a genus 0 vertex. As $n = 1$, there is no $\mathbb{S}_n$-action on $\delta_irr$ to talk about.

 $:= (*_{1,1}, \delta_{irr})$

5.3. **The case $k = (1,1)$, $g = 0$.** For this case we refer to [5, Section 3]. In $g = 0$, $H^{1,1}(\overline{\mathcal{M}}_{0,n})$ is non zero only for $n \geq 4$; so we operate under this assumption. The group is generated by classes $\psi_i$ for every $1 \leq i \leq n$ and $\delta\{{}^{A}_{A'}\}$ for every partition $A \sqcup A' = N$ with $|A|, |A'| \geq 2$. To depict $\delta\{{}^{A}_{A'}\}$ we split symbolically the vertex in two parts connected by a crossed edge and draw on one side the subset of hairs $A$ and on the other $A'$. The notation $\{{}^{A}_{A'}\}$ is chosen to express the fact that swapping $A$ and $A'$ doesn't change the class, which graphically means it doesn't matter on which sides the two sets of hairs are chosen to be drawn. The $\mathbb{S}_n$-action is given by $\sigma\,\psi_i = \psi_{\sigma i}$ and $\sigma\,\delta\{{}^{A}_{A'}\} = \delta\{{}^{\sigma A}_{\sigma A'}\}$.



$:= (*_{0,n}, \psi_i)$



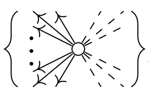
$:= (*_{0,n}, \delta\{{}^{A}_{A'}\})$

There are two equivalent families of relations between these generators. For any three pairwise distinct $i, x, y \in N$, or for any $i \neq j \in N$, it holds:

$$\psi_i = \sum_{\substack{A \sqcup A' = N, |A|, |A'| \geq 2 \\ i \in A, x, y \in A'}} \delta\{{}^{A}_{A'}\} \qquad \psi_i + \psi_j = \sum_{\substack{A \sqcup A' = N, |A|, |A'| \geq 2 \\ i \in A, j \in A'}} \delta\{{}^{A}_{A'}\}. \tag{5.2}$$

So the $\psi_i$ classes are actually superfluous in the case $g = 0$, but algebraically they can be more convenient to work with. The dimension of $H^{1,1}(\overline{\mathcal{M}}_{0,n})$ turns out to be $2^{n-1} - \binom{n}{2} - 1$, in particular when $n = 4$ every single class forms a basis.

5.4. **The case $k = (11, 0)$, $g = 1$.** This case is studied in [2, Section 2]. $H^{11,0}(\overline{\mathcal{M}}_{1,n})$ is non zero only for $n \geq 11$; so we operate under this assumption. The group is generated by classes $\omega_B$ for every *alternatingly ordered* $B \subseteq N$ with $|B| = 11$. This means that the underlying set determines $\omega_B$ up to sign, and if we choose a canonical labeling $N = \{1, ..., n\}$ we can stipulate that every subset comes equipped with the increasing ordering. We draw arrows onto the hairs contained in $B$ to depict the $\omega_B$ decoration.

$B\{$  $\}B^c \;:= (*_{1,n}, \omega_B)$

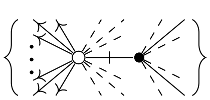
The only relations are amongst the classes $\omega_B$ whose set $B$ is contained in the same size 12 subset. Namely, if we choose a canonical labeling on $N$, then for every $E = \{e_1, ..., e_{12}\} \subseteq N$ with $e_i$ increasing there is the relation

$$\sum_{i=1}^{12} (-1)^i \omega_{E \setminus e_i} = 0 \tag{5.3}$$

Therefore, choosing a distinguished hair $e \in N$ (for example $e = 1$), the classes $\omega_B$ with $e \in B$ form a basis of $H^{11,0}(\overline{\mathcal{M}}_{1,n})$.

The $\mathbb{S}_n$-action on $\omega_B$ is given by $\sigma\, \omega_B = \omega_{\sigma B}$, which is equal to $\operatorname{sgn} \sigma\, \omega_B$ if $\sigma$ preserves $B$ setwise.

5.5. **The case $k = (12, 1)$, $g = 1$.** This case is studied in [3, Section 4.2]. $H^{12,1}(\overline{\mathcal{M}}_{1,n})$ is non zero only for $n \geq 12$; so we operate under this assumption. The group is generated by classes $Z_{B \subseteq A}$ for every subset $A \subseteq N$ with $|A^c| \geq 2$ and *alternatingly ordered* subset $B \subseteq A$ with $|B| = 10$; the underlying set $B$ determines $Z_{B \subseteq A}$ up to sign. We draw arrows onto the hairs contained in $B$, and we split symbolically the vertex in a genus 1 vertex, where we attach the hairs in $A$, and a genus 0 vertex, where we attach the hairs in $A^c$.

$B\{$  $\}A^c \;:= (*_{1,n}, Z_{B \subseteq A})$

$A$

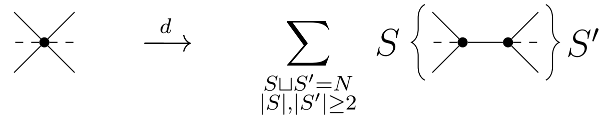
The only relations are amongst the classes $Z_{B \subseteq A}$ having $|A^c| = 2$ and same set $B \sqcup A^c$. Namely, if we choose a canonical labeling on $N$, then for every $E = \{e_1, ..., e_{12}\} \subseteq N$ with $e_i$ increasing and every $1 \leq i < j < k \leq 12$ we have the relation

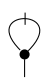
$$(-1)^{i+j} Z_{E \setminus e_i, e_j \subseteq N \setminus e_i, e_j} - (-1)^{i+k} Z_{E \setminus e_i, e_k \subseteq N \setminus e_i, e_k} + (-1)^{j+k} Z_{E \setminus e_j, e_k \subseteq N \setminus e_j, e_k} = 0 \tag{5.4}$$

For this subset $E \subseteq N$, choosing a distinguished element $e \in E$ (for example $e = e_1$), the subspace $PB_E$ spanned by the classes $Z_{B\subseteq A}$ with $B \sqcup A^c = E$ has basis the ones with $e \in A^c$, of which there are 11. So $H^{12,1}(\overline{\mathcal{M}}_{1,n})$ decomposes into a direct sum $PB_3 \oplus \bigoplus_{|E|=12} PB_E$, where $PB_3$ has basis the classes with $|A^c| \geq 3$. The $\mathbb{S}_n$-action on $Z_{B\subseteq A}$ is given by $\sigma\, Z_{B\subseteq A} = Z_{\sigma B\subseteq \sigma A}$, which is equal to $\operatorname{sgn}\sigma\, Z_{B\subseteq \sigma A}$ if $\sigma$ preserves $B$ setwise.

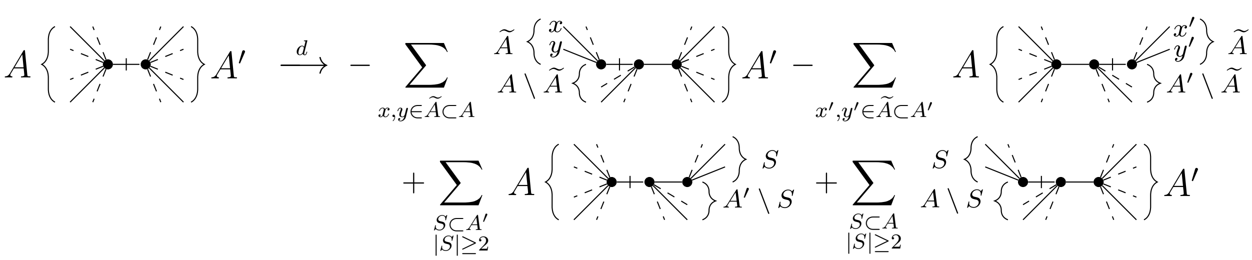
5.6. **Action of the differential on cohomology classes.** In this section we describe the action of the differential operator that splits one-vertex decorated graphs. A splitting of a decorated graph $(*_{g,n}, \gamma)$ is of the form $(*' - *'', \gamma' \otimes \gamma'')$, where $*' - *''$ is the connection of two vertices $*'_{g',n'}, *''_{g'',n''}$ with $g = g' + g''$, $n = n' + n'' + 2$, and $\gamma', \gamma''$ are their respective decorations. The hairs of $*'$ and $*''$ form a partition of $N$. If $q'$ and $q''$ are the two half-edges connecting $*'$ and $*''$, then $\gamma'$ is obtained from $\gamma$ by pullback along the map $\overline{\mathcal{M}}_{g',n'+1} \to \overline{\mathcal{M}}_{g,n}$ determined by the subset of hairs ending up on $*'$. For the computation of the pullbacks of cohomology classes we refer to [5], [2] and [3] for the weight 2, weight 11 and weight 13 cases respectively, in this paper we limit ourselves to translating those computations in graphical form.

The image under the differential is given by summing over these two-vertex decorated graphs for all possible splittings. In the case of interest, we always have either $g' = g'' = g = 0$ or $g' = g = 1, g'' = 0$, so the only determining datum of the splitting is the partition of the hairs of $*_{g,n}$.

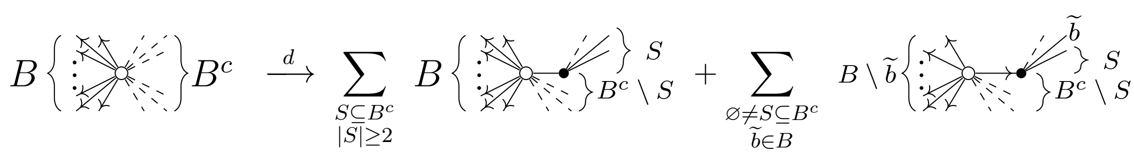
(5.5) 


(5.6)  $\xrightarrow{d}$ 0 because the valence of $*_{1,1}$ is less than 2

(5.7) For any choice of $x, y \in A$ and $x', y' \in A'$, the image can be expressed as follows:

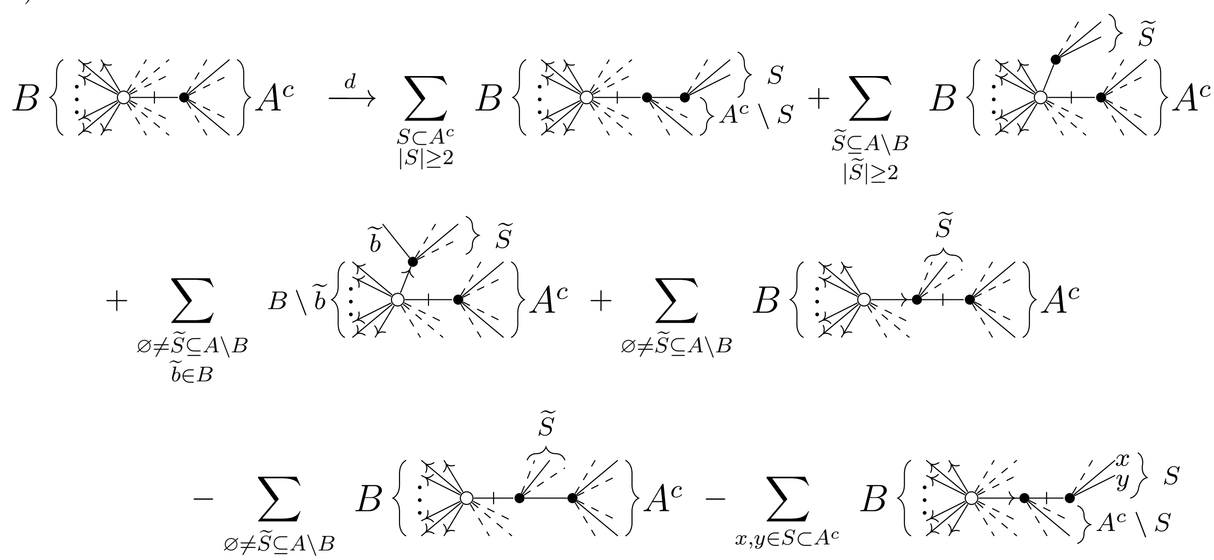



In the second term, the weight 11 decoration $\omega_B$ becomes $\omega_{B\setminus\widetilde{b}\sqcup q}$, where $q$ is the newly added half-edge to the genus 1 vertex and takes the place of $\widetilde{b}$ in the ordering of $B$.

(5.8) 

For any fixed choice of $x, y \in A^c$, the image can be expressed as in 5.9. In the two terms that create weight 11 and 2 vertices, the newly created weight 11 decoration $\omega_{B \sqcup p}$, where $p$ is the half-edge at the genus 1 vertex, is understood to have the ordering inherited from $B$ with $p$ appended at the end.

(5.9)

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