MBELLI_Main

May 11, 2025

[57]:
```
from sage.all import graphs
import matplotlib.pyplot as plt
import pandas as pd
pd.options.display.max_columns = None
pd.set_option('display.max_rows', int(500))
pd.options.display.width = int(500)
pd.options.display.max_colwidth = int(500)
```

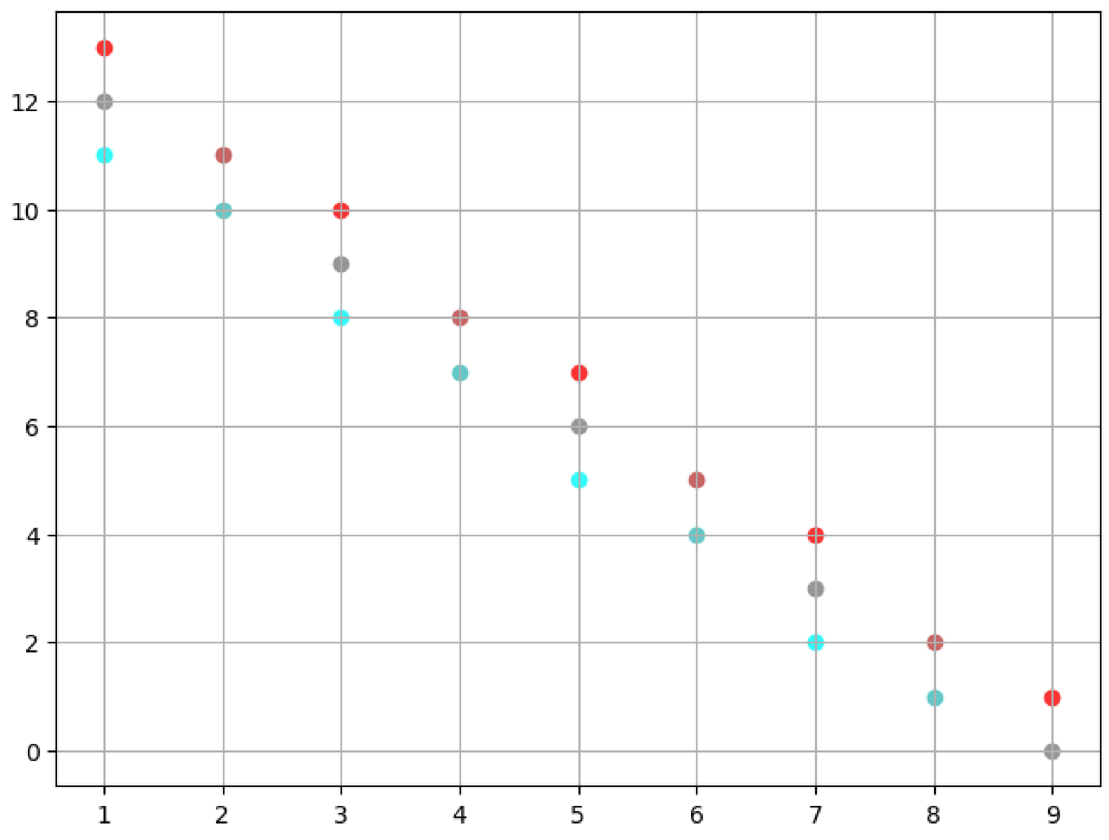
[58]:
```
# the whole file uses this variable as upper bound for the excess␣
↪classification of blowup representations
MAX_EXCESS = 4
print("MAX_EXCESS: ",MAX_EXCESS)

gn_table = [] # list of lists of tuples (g,n), indexed by excess
fig = plt.figure()
ax = fig.add_subplot()
plt.grid(visible=True)
for excess in range(MAX_EXCESS+1):
    ax.name = 'Excess '+str(excess)
    gn_table.append([])
    for g in range(1,floor((excess+22)/3+1)+1):
        if (excess+22-3*g+3)%2 == 0:
            n = (excess+22-3*g+3)/2
            gn_table[excess].append((g,n))
    ax.scatter(*zip(*gn_table[excess]),c='#'+str(hex(int(50+excess*200/
↪MAX_EXCESS))[2:])+str(hex(int(50+(MAX_EXCESS-excess)*200/MAX_EXCESS))[2:
↪])+str(hex(int(50+(MAX_EXCESS-excess)*200/MAX_EXCESS))[2:]))
plt.show()

# given a sequence of simple specht contributions (either Vn or V1^n), generate␣
↪the string of their box product
def specht_sequence_to_string(seq):
    if len(seq)>0:
        specht_string = r"$"
        first=True
        for specht in seq:
            if not first: specht_string+=r"\boxtimes "
            first = False
            specht_string+=r"V_{"+ (str(specht[0]) if specht[1]==1 else␣
↪(r"1^{"+str(specht[0])+r"}"))+r"}"
        specht_string+=r"$"
        return specht_string
    else: return ""

GRAPHVISUALS = True # controls wether to build graphics for the blowup objects
```

```
MAX_EXCESS:  4
```



[59]:
```
class BlowUpComponent:
    def __init__(self):
        self.nameID='-1'
        self.ID='-1'
        self.crossed=False
        self.crossType='0'
        self.stability = 0

        self.marked=False # vars that only make sense after marking
        self.unmarkedComponent = None
        self.odd_component= None
        self.odd = None
        self.odd_symmetry = None
        self.nadd_permutation_matrix = None
        self.specht_sequence = None
        self.specht_string = None

        self.crossContractedComponent = None # in case B, the BlowUpComponent␣
↪object isomorphic to the contraction of the crossed edge in this graph
        self.crossValence = None # valence of tilde{v}, number of hairs of␣
↪half-edges incodent with the crossed edge
        self.crossValence_none = None
        self.crossValence_nadd = None
        self.crossValence_eps = None
        self.relation2groupID = None # index position of its group in the␣
↪sorted dataframe relations2groups. set externally
        self.in_relation2_basis = None # wether this component is a basis␣
↪element of its relation2 group

        self.crossTypeA2_blowupComponents = None # in case A2, the one or two␣
↪BlowUp component objects (unmarked or marked) after blowing up the crossed␣
↪edge

    def build(self,skeleton=None,␣
↪unmarked_valence=[],nadd_valence=[],doubleleg=None):
        if doubleleg==None:
            self.graph = skeleton.copy(immutable=False)  # non empty graph of␣
↪the internal vertices of the components
            self.vertices = skeleton.order() #internal, and counting the␣
↪potential crossed edge as two !
            if self.vertices!=len(unmarked_valence) or self.vertices!
↪=len(nadd_valence): raise ValueError('vertices!=length arrays')
            self.edges = skeleton.size() #internal, and counting the potential␣
↪crossed edge !
            self.unmarked_valence = unmarked_valence # list by index of the␣
↪internal vertex of how many unmarked hairs it has
            self.nadd_valence = nadd_valence # list by index of the internal␣
↪vertex of how many j hairs it has
            self.unmarked = sum(unmarked_valence)
            self.nadd = sum(nadd_valence)
            self.n = self.unmarked+self.nadd # total hairs of the components
            self.v_add = self.vertices #vertex contribution when glued to the␣
↪special vertex
            self.e_add = self.edges+self.unmarked #edges contribution when␣
↪glued to the special vertex
```

```
            ucount=ncount=0
            for v,u in enumerate(unmarked_valence):
                for x in range(u):
                    self.graph.add_vertex('u'+str(ucount))
                    self.graph.add_edge('u'+str(ucount),v,'u')
                    ucount += 1
            for v,n in enumerate(nadd_valence):
                for x in range(n):
                    self.graph.add_vertex('n'+str(ncount))
                    self.graph.add_edge('n'+str(ncount),v,'n')
                    ncount += 1

            for v in range(self.vertices):
                if self.graph.degree(v)<3: raise ValueError('Component not␣
↪stable at vertex ',v)
                self.stability += self.graph.degree(v)-2
            self.g = (self.stability-self.n+2)/2    # total genus of the hairy␣
↪graph
            self.g_add = self.g+self.unmarked-1   # genus contribution when␣
↪glued to the special vertex
        else:
            if doubleleg==0:
                self.graph = Graph({'u0':['u1'],'u1':[]})
                self.graph.set_edge_label('u0','u1','u')
                self.unmarked = 2
                self.nadd = 0
                self.g_add = 1
                self.e_add = 1
            else:
                self.graph = Graph({'u0':['n0'],'n0':[]})
                self.graph.set_edge_label('u0','n0','n')  # double leg edge␣
↪label is the one of its second vertex
                self.unmarked = 1
                self.nadd = 1
                self.g_add = 0
                self.e_add = 0
            self.vertices = 0
            self.edges = 0
            self.n = 2
            self.g = 0
            self.v_add = 0
        self.build_visuals()
        return self

    def cross(self,crosstype,vertex,vertex2=None):
        if self.vertices==0: raise ValueError("doubleleg can't be crossed")
        self.crossed = True
        self.crossType = crosstype
        self.v_add+= -1
        self.e_add+= -1
        # throughout the program it is assumed that self.crossedEdge[0] is the␣
↪internal vertex
        if crosstype=='A':
            if self.unmarked_valence[vertex]==0: raise ValueError('No unmarked␣
↪at crossed vertex',vertex)
            if self.marked:
                node_index = sum(self.ome_valence[:vertex])
                self.crossedEdge = [vertex,'o'+str(node_index)]
            else:
                node_index = sum(self.unmarked_valence[:vertex])
                self.crossedEdge = [vertex,'u'+str(node_index)]
            if self.graph.degree(vertex)==3: self.crossType='A2'
        if crosstype=='B':
            self.crossedEdge = [vertex,vertex2]
        if crosstype=='Birr':
            self.crossedEdge = [vertex,vertex]
        # if it has a multiple edge, I assume that its label has already been␣
↪set at construction of the graph
        if not self.graph.has_multiple_edges():
            self.graph.set_edge_label(*self.crossedEdge,'CROSS')

    def compute_crossContractedComponent(self):
        # compute the contraction of the B1 graph at the crossed vertex
        # this graph maintains all edge and vertex labelings, it might have␣
↪vertex labels which skip one integer
        if not self.crossType=='B': raise ValueError('component is not of␣
↪crossType B')
        G = copy(self.graph)
        G.allow_multiple_edges(new=True)
        G.allow_loops(new=True)
        G.contract_edge(self.crossedEdge)
        v = self.crossedEdge[0] if self.crossedEdge[0]<self.crossedEdge[1] else␣
↪self.crossedEdge[1]
        G.relabel({v:'CROSS'})
        # we add an extra edge, specially labeled, so that when we check␣
↪edge-label preserving isomorphism of the crossContracted graph we also fix␣
↪the crossed vertex
        G.add_vertex(name='CROSS2')
        G.add_edge(('CROSS','CROSS2','CROSS'))
        self.crossContractedComponent = G
        self.crossValence = -1   # correct for the added extra CROSS edge
        self.crossValence_nadd, self.crossValence_eps, self.crossValence_none =␣
↪0, 0, 0
```

```
        for edge in G.edges(vertices='CROSS',labels=True,sort=False):
            self.crossValence+=1
            if edge[0]==edge[1]:
                self.crossValence+=1
                self.crossValence_none+=2
            elif edge[2]==None: self.crossValence_none+=1
            elif edge[2][0]=='n': self.crossValence_nadd+=1
            elif edge[2][0]=='e': self.crossValence_eps+=1

        self.in_relation2_basis = False

    def compute_A2blowup(self,frame):
        # compute the splitting of the A2 graph at the crossed vertex, then␣
↪search in the given frame for the BlowUpComponent objects isomorphic to the␣
↪one or two resulting components.
        if not self.crossType=='A2': raise ValueError('component is not of␣
↪crossType A2')
        self.crossTypeA2_blowupComponents = []
        components = []
        if self.graph.has_loops():
            if self.marked:
                G = Graph({'o0':['o1'],'o1':[]})
                G.set_edge_label('o0','o1','o')
            else:
                G = Graph({'u0':['u1'],'u1':[]})
                G.set_edge_label('u0','u1','u')
            components = [G]
        else:
            G = copy(self.graph)
            G.delete_vertex(self.crossedEdge[1])
            edges = []
            for edge in G.edges(vertices=self.crossedEdge[0],␣
↪sort=False,labels=True):
                if edge[2]!='CROSS': edges.append((edge[0],edge[1],edge[2]) if␣
↪edge[0]==self.crossedEdge[0] else (edge[1],edge[0],edge[2]))
                if len(edges)!=2: raise ValueError('blowup at A2 went wrong',self.
↪__str__(True))
            G.delete_edge(edges[1])
            G.relabel({self.crossedEdge[0]:'w1'})
            G.add_vertex('w2')
            G.add_edge(edges[1][1],'w2',edges[1][2])
            if edges[0][2]==None: G.set_edge_label('w1',edges[0][1],'o' if self.
↪marked else 'u')
            if edges[1][2]==None: G.set_edge_label('w2',edges[1][1],'o' if self.
↪marked else 'u')
```

```
            if not G.is_connected(): components = G.
↪connected_components_subgraphs()
            else: components = [G]
        if self.marked:
            for C in components:
                eps = ome = nadd = vertices = 0
                for v in C.get_vertices():
                    if isinstance(v,str):
                        if v[0]=='o' or v[0]=='w': ome+=1
                        elif v[0]=='e': eps+=1
                        else: nadd+=1
                    else: vertices+=1
                for row in frame[(frame['ome']==ome) & (frame['eps']==eps) &␣
↪(frame['nadd']==nadd) & (frame['vertices']==vertices)].itertuples():
                    if row.graph.graph.is_isomorphic(C,edge_labels=True):
                        self.crossTypeA2_blowupComponents.append(row.graph)
                        break
        else:
            for C in components:
                unmarked = nadd = vertices = 0
                for v in C.get_vertices():
                    if isinstance(v,str):
                        if v[0]=='u' or v[0]=='w': unmarked+=1
                        else: nadd+=1
                    else: vertices+=1
                for row in frame[(frame['unmarked']==unmarked) &␣
↪(frame['nadd']==nadd) & (frame['vertices']==vertices)].itertuples():
                    if row.graph.graph.is_isomorphic(C,edge_labels=True):
                        self.crossTypeA2_blowupComponents.append(row.graph)
                        break

        if len(components) != len(self.crossTypeA2_blowupComponents):
            print("comps")
            for C in components: C.graphplot(edge_labels=True).show()
            print("found")
            for C in self.crossTypeA2_blowupComponents: C.show()
            raise ValueError("didn't find the isomorphic A2 blow ups in the␣
↪frame")

    def label_hairs_uniquely(self):
        for x in range(self.nadd):
            for edge in self.graph.edges(vertices=['n'+str(x)],sort=False):
                self.graph.set_edge_label(edge[0],edge[1],'n'+str(x))
    def unlabel_hairs(self):
        for x in range(self.nadd):
            for edge in self.graph.edges(vertices=['n'+str(x)],sort=False):
                self.graph.set_edge_label(edge[0],edge[1],'n')
    def mark(self,ome_valence,unmarkedComponent=None,doubleleg_ome=None):
        if self.marked: raise ValueError('component has already been marked')
        self.marked = True
        self.unmarkedComponent = unmarkedComponent # reference to a not marked␣
↪BlowUpComponent object
        self.ome_valence = ome_valence # number of omega hairs indexed by vertex
        self.ome = sum(ome_valence) if self.vertices!=0 else doubleleg_ome    #␣
↪ number of omega hairs
        if any(self.unmarked_valence[v]<ome_valence[v] for v in range(self.
↪vertices)): raise ValueError('ome_valence too high',self.unmarked_valence,␣
↪ome_valence)
        self.eps_valence = [self.unmarked_valence[v]-ome_valence[v] for v in␣
↪range(self.vertices)]
        self.eps = sum(self.eps_valence) if self.vertices!=0 else self.
↪unmarked-doubleleg_ome   #  number of epsilon hairs
        self.excess = 3*self.g-3+3*self.eps+self.ome+2*self.nadd   # excess of␣
↪blown up components

        if self.vertices==0:
            if self.unmarked==1: self.graph.relabel({'u0':'o0' if␣
↪doubleleg_ome==1 else 'e0'})
            elif doubleleg_ome==0: self.graph.relabel({'u0':'e0', 'u1':'e1'})
            else: self.graph.relabel({'u0':'o0' if doubleleg_ome==2 else 'e0' ,␣
↪'u1':'o0' if doubleleg_ome==1 else 'o1'})
            edge = self.graph.edges(sort=False)[0]
            if self.unmarked==1: self.graph.set_edge_label(edge[0],edge[1],'n')␣
↪ # nadd has priority over omega and epsilon for labeling the edge
            else: self.graph.set_edge_label(edge[0],edge[1],'o' if␣
↪doubleleg_ome==2 else 'e') # epsilon has priority over omega for labeling␣
↪the edge
            self.odd_component = (doubleleg_ome%2==0 and self.nadd==0)
            if doubleleg_ome==2:
                self.odd =True
                self.odd_symmetry = 'u see it'
            else: self.odd=False
        else:
            count = ecount = ocount = 0
            for v in range(self.vertices):
                for x in range(self.eps_valence[v]):
                    self.graph.relabel({'u'+str(count):
↪'e'+str(ecount)},inplace=True)
                    self.graph.set_edge_label('e'+str(ecount),v,'e')
                    count+=1
                    ecount+=1
                for o in range(self.ome_valence[v]):
                    self.graph.relabel({'u'+str(count):
↪'o'+str(ocount)},inplace=True)
                    self.graph.set_edge_label('o'+str(ocount),v,'o')
                    count+=1
                    ocount+=1
            if self.crossType=='A' or self.crossType=='A2':
                o = sum(ome_valence[:self.crossedEdge[0]])
                self.crossedEdge[1] = 'o'+str(o)
                self.graph.set_edge_label(*self.crossedEdge,'CROSS')

            self.odd_component = ((self.edges + self.eps)%2!=0 and self.nadd==0␣
↪and not self.crossed)
            # a (non empty) blow up component has an odd symmetry if and only␣
↪if there is a generator of the automorphism group with odd sign of␣
↪permutation on the internal edges + epsilon hairs
            #the symmetry automorphisms must fix the n_add hairs, so relabel␣
↪their edges from the anonymous 'n' to 'nx', where 1 x n_add is the index of␣
↪the hair
            self.label_hairs_uniquely()
            autom = self.graph.automorphism_group(edge_labels=True)
            self.odd = False
            self.odd_symmetry = None
            for f in autom.gens():
                sign_count = 0
                orbits = []
                for edge in self.graph.edges(sort=False,labels=True):
                    if (edge[2]==None or edge[2]=='e') and not␣
↪any(({edge[0],edge[1]} in orbit) for orbit in orbits):
                        orbits.append([{edge[0],edge[1]}])
                        edge2 = (edge[0],edge[1])
                        while {f(edge2[0]),f(edge2[1])}!={edge[0],edge[1]}:
                            edge2 = (f(edge2[0]),f(edge2[1]))
                            sign_count+=1
                            orbits[len(orbits)-1].append({edge2[0],edge2[1]})
                if sign_count%2!=0:
                    self.odd = True
                    self.odd_symmetry = f
                    break
            #relabel to original state the n_add hairs
            self.unlabel_hairs()

        self.build_visuals()

    def compute_specht(self):
        # if has odd symmetry, then the sign of an action on the nadd hairs is␣
↪not well defined
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```
        if self.odd: return

        # sequence of tuples (k,o) where k is the size of a group which can be␣
↪freely permuted by o-signed automorphisms
        self.specht_sequence = []
        if self.nadd==0: pass
        elif self.nadd==1:
            self.specht_sequence.append((1,1))  # this case covers also the two␣
↪double legs with a nadd hair
        else:
            # indexed by pairs of hairs, 0 if there's no transposition, +-1 if␣
↪there's a +-1 signed permutation
            self.nadd_transpositions_matrix = [([0]*self.nadd) for n in␣
↪range(self.nadd)]
            for i in range(self.nadd): self.nadd_transpositions_matrix[i][i]=1
            autom = self.graph.automorphism_group(edge_labels=True)
            for f in autom:
                transp = [(f('n'+str(i))!='n'+str(i)) for i in range(self.nadd)]
                if transp.count(True)!=2: continue # if f is not a pure␣
↪transposition on the hairs
                # find indices of potential transposition
                i1 = 0
                while not transp[i1]: i1+=1
                i2 = i1+1
                while not transp[i2]: i2+=1
                if self.nadd_transpositions_matrix[i1][i2]!=0: continue  # if␣
↪this transposition has already been reviews
                sign_count = 0
                orbits = []
                for edge in self.graph.edges(sort=False,labels=True):
                    if (edge[2]==None or edge[2]=='e') and not␣
↪any(({edge[0],edge[1]} in orbit) for orbit in orbits):
                        orbits.append([{edge[0],edge[1]}])
                        edge2 = (edge[0],edge[1])
                        while {f(edge2[0]),f(edge2[1])}!={edge[0],edge[1]}:
                            edge2 = (f(edge2[0]),f(edge2[1]))
                            sign_count+=1
                            orbits[len(orbits)-1].append({edge2[0],edge2[1]})
                sign = 1 if sign_count%2==0 else -1
                self.nadd_transpositions_matrix[i1][i2] = sign
                self.nadd_transpositions_matrix[i2][i1] = sign
            # group nadd hairs by existence of a pure transposition between any␣
↪two (a transitive relation)
            symmetric_subsets = []
            for x in range(self.nadd):
                found = False
                for y in range(x):
                    if self.nadd_transpositions_matrix[x][y]!=0:
                        for subset in symmetric_subsets:
                            if y in subset:
                                subset.append(x)
                                found = True
                                break
                        if found: break
                if not found: symmetric_subsets.append([x])
            if sum([len(subset) for subset in symmetric_subsets])!=self.nadd:
                print(self.nadd_transpositions_matrix)
                print(symmetric_subsets)
                self.show(name=True)
                raise ValueError("partition of nadd hairs is wrong")
            for subset in symmetric_subsets:
                if not all((x==y) or (self.
↪nadd_transpositions_matrix[x][y]==self.
↪nadd_transpositions_matrix[subset[0]][subset[1]]) for x in subset for y in␣
↪subset):
                    print(self.nadd_transpositions_matrix)
                    print(symmetric_subsets)
                    self.show(name=True)
                    raise ValueError("not all transpositions in partition of␣
↪nadd hairs have the same sign")
                if len(subset)==1: self.specht_sequence.append((1,1))
                else: self.specht_sequence.append((len(subset),self.
↪nadd_transpositions_matrix[subset[0]][subset[1]]))

        self.specht_string = specht_sequence_to_string(self.specht_sequence)



    def build_visuals(self):
        if not GRAPHVISUALS: return
        self.vertex_position = {}
        if self.marked:
            self.partition = {'vertices': [x for x in range(self.vertices)],␣
↪'eps':['e'+str(x) for x in range(self.eps)],'ome':['o'+str(x) for x in␣
↪range(self.ome)], 'nadd':['n'+str(x) for x in range(self.nadd)]}
            DX = 1/(self.eps+self.ome)
            X = DX/2
            Y = 0
            for i in range(self.ome):
                self.vertex_position['o'+str(i)] = [X,Y]
                X+=DX
            for i in range(self.eps):
                self.vertex_position['e'+str(i)] = [X,Y]
                X+=DX
        else:
            self.partition = {'vertices': [x for x in range(self.vertices)],␣
↪'unmarked':['u'+str(x) for x in range(self.unmarked)], 'nadd':['n'+str(x)␣
↪for x in range(self.nadd)]}
            DX = 1/(self.unmarked)
            X = DX/2
            Y = 0
            for i in range(self.unmarked):
                self.vertex_position['u'+str(i)] = [X,Y]
                X+=DX
        if self.nadd!=0:
            DX = 1/self.nadd
            X=DX/2
            Y=1
            for i in range(self.nadd):
                self.vertex_position['n'+str(i)] = [X,Y]
                X+=DX
        if self.vertices == 1 or self.vertices==2:
            DX = 1/self.vertices
            X=DX/2
            Y=0.5
            for i in range(self.vertices):
                self.vertex_position[i] = [X,Y]
                X+=DX
        elif self.vertices>=3:
            DT = 2*math.pi/self.vertices
            T = -math.pi/2-DT/2
            for i in range(self.vertices):
                self.vertex_position[i] = [0.5+cos(T)*0.45,0.5+sin(T)*0.3]
                T-=DT

    def __str__(self, long=False):
        if self.marked:
            if not long:
                name = ""
                if self.vertices==0:
                    if self.ome==1 and self.nadd==1: name="|"
                    elif self.ome==1 and self.eps==1: name=",_"
                    elif self.eps==1 and self.nadd==1: name="!"
                    elif self.eps==2: name="._"
                elif self.crossType=='Birr':
                    if self.ome==1: name = 'oIrr'
                    elif self.eps==1: name = 'eIrr'
                else:
                    for x in range(self.ome): name+='o'
                    if self.crossType=='A' or self.crossType=='A2': name+='CR'
                    for x in range(self.eps): name+='e'
                    for x in range(self.vertices): name+='v'
                    if self.crossType=='B': name+='CR'
                    for x in range(self.nadd): name+='j'
                    if self.vertices!=1: name += str(self.nameID)
                return name
            else:
                string = f"BlowUpComponent ID {self.ID}  name: {self.__str__()}␣
↪ excess: {str(self.excess)}  eps,ome,nadd: {str(self.eps)}, {str(self.ome)},␣
↪{str(self.nadd)}   vertices: {str(self.vertices)}   edges: {str(self.edges)}␣
↪ g: {str(self.g)}"
                if self.crossed: string+="  crossType: "+str(self.crossType)
                if self.crossType=='B': string+=" rel2basis: "+str(self.
↪in_relation2_basis)
                if self.odd_component: string+="  , odd edges"
                if self.odd: string+="  odd symmetry: "+str(self.odd_symmetry)
                return string
        else:
            if not long:
                name = ""
                if self.vertices==0:
                    if self.unmarked==1 and self.nadd==1: name="|"
                    elif self.unmarked==2: name="_"
                elif self.crossed and self.crossType=='Birr':
                    name = 'uIrr'
                else:
                    for x in range(self.unmarked): name+='u'
                    if self.crossType=='A' or self.crossType=='A2': name+='CR'
                    for x in range(self.vertices): name+='v'
                    if self.crossType=='B': name+='CR'
                    for x in range(self.nadd): name+='j'
                    if self.vertices!=1: name += str(self.nameID)
                return name
            else:
                string = f"UnmarkedBlowUpComponent ID: {self.ID}  name: {self.
↪__str__()}  unmarked,nadd: {str(self.unmarked)}, {str(self.nadd)}   vertices:
↪ {str(self.vertices)}   edges: {str(self.edges)}  g: {str(self.g)} ␣
↪stability: {self.stability}"
                if self.crossed: string+="  crossType: "+str(self.crossType)
                if self.crossType=='B': string+=" rel2basis: "+str(self.
↪in_relation2_basis)
                return string

    def show(self,name=False,title=False, specht=False,graphics=False):
        if name: print(self.__str__(True))
        graphplot_object = {}
        if self.marked:
            if not self.crossed:
                graphplot_object = self.graph.graphplot(edge_labels=True,␣
↪edge_labels_background='#FFFFFF90', pos=self.
↪vertex_position,vertex_colors={'darkgrey':self.partition['vertices'], 'gold':
↪self.partition['eps'],'dodgerblue':self.partition['ome'], 'lime':self.
↪partition['nadd']})
            else:
                graphplot_object = self.graph.graphplot(edge_labels=True,␣
↪edge_labels_background='#FFFFFF90', pos=self.vertex_position,␣
↪edge_colors={'red':[self.crossedEdge]},vertex_colors={'darkgrey':self.
↪partition['vertices'], 'gold':self.partition['eps'],'dodgerblue':self.
↪partition['ome'], 'lime':self.partition['nadd']})
        else:
            if not self.crossed:
                graphplot_object = self.graph.graphplot(edge_labels=True,␣
↪edge_labels_background='#FFFFFF90', pos=self.
↪vertex_position,vertex_colors={'darkgrey':self.partition['vertices'],␣
↪'sandybrown':self.partition['unmarked'], 'lime':self.partition['nadd']})
            else:
                graphplot_object = self.graph.graphplot(edge_labels=True,␣
↪edge_labels_background='#FFFFFF90', pos=self.vertex_position,␣
↪edge_colors={'red':[self.crossedEdge]},vertex_colors={'darkgrey':self.
↪partition['vertices'], 'sandybrown':self.partition['unmarked'], 'lime':self.
↪partition['nadd']})
        if title and not specht: graphplot_object = graphplot_object.
↪plot(axes=False, title=f'ID: {self.ID}    {self.__str__()}')
        elif title and specht: graphplot_object = graphplot_object.
↪plot(axes=False, title=f'ID: {self.ID}    {self.specht_string}')
        if not graphics: graphplot_object.show()
        else:  return graphplot_object
```

[60]:
```
def generate_UnmarkedBlowUpComponents(unmarked,nadd,vertices,genus):
    #return a list of not marked blowup components objects which contains all␣
↪graphs with the following features:
    # connected, with given number of internal vertices and of total genus
    # with unmarked+nadd hairs partitioned in the two families by labeling␣
↪their edges as 'u' or 'n'
    # at least trivalent (internal) vertices (i.e. stable)
    # either:  simple
    #    or:   simple and has a crossed unmarked hair
    #    or:   has a crossed internal edge, is simple
    #    or:   has a crossed internal edge, is simple except for having exactly␣
↪one multiple edge parallel the crossed edge

    # The remaining 4 unmarked graphs of interest (2 double legs, A2 case with␣
↪loop at its end and Birr case) are added manually afterwards.
```

```
  pool_skeletons = graphs.nauty_geng(str(vertices)+" -c") # all connected␣
↪graphs with given vertices
  pool_UnmarkedComponents = []
  pool_UnmarkedCrossedInt = []
  pool_UnmarkedCrossedOme = []
  pool_UnmarkedCrossedIntMultiple = []

  n_hairs = Subsets(list(range(vertices))*nadd,nadd,submultiset=True) #all␣
↪multisubsets of the vertices with size n_add
  unm_hairs =␣
↪Subsets(list(range(vertices))*unmarked,unmarked,submultiset=True) #all␣
↪multisubsets of the vertices with size unmarked
  for skeleton in pool_skeletons:
      #skip skeletons with wrong genus (or not of g-1), as they will be␣
↪considered for another value of the excess
      g = sum([skeleton.degree(v)-2 for v in range(vertices)])/2+1
      if genus==g:
          pool_UnmarkedComponents.append([])
          pool_UnmarkedCrossedInt.append([])
          pool_UnmarkedCrossedOme.append([])
          pool_UnmarkedCrossedIntMultiple.append([])
          for f_unm in unm_hairs:
              for f_n in n_hairs:
                  unmarked_valence = [f_unm.count(v) for v in range(vertices)]
                  nadd_valence = [f_n.count(v) for v in range(vertices)]

                  # if stable at every vertex
                  if all((skeleton.
↪degree(v)+unmarked_valence[v]+nadd_valence[v]>=3) for v in range(vertices)):
                      GR = BlowUpComponent().
↪build(skeleton,unmarked_valence,nadd_valence)
                      #up to isomorphism of hairy graphs with labeled hairs.␣
↪It is sufficient to check isomorphism with previous graphs which have the␣
↪same skeleton S
                      if not any((G.graph.is_isomorphic(GR.graph,␣
↪edge_labels=True)) for G in pool_UnmarkedComponents[-1]):
                          pool_UnmarkedComponents[-1].append(GR)

                          #crossed at internal edge
                          for edge in skeleton.edges(sort=False):
                              GRcrossedB =  BlowUpComponent().
↪build(skeleton,unmarked_valence,nadd_valence)
                              GRcrossedB.cross('B',edge[0],edge[1])
                              #up to isomorphism of hairy graphs with labeled␣
↪hairs and crossed edge
```

```
                              if not any((G.graph.is_isomorphic(GRcrossedB.
↪graph, edge_labels=True)) for G in pool_UnmarkedCrossedInt[-1]):
                                  pool_UnmarkedCrossedInt[-1].
↪append(GRcrossedB)

                          #crossed at omega hair. Up to isomorphism, there is␣
↪only one way of crossing for every internal vertex with an omega hair
                          for v in range(vertices):
                              if unmarked_valence[v]!=0:
                                  GRcrossedA = BlowUpComponent().
↪build(skeleton,unmarked_valence,nadd_valence)
                                  GRcrossedA.cross('A',v)
                                  #up to isomorphism of hairy graphs with␣
↪labeled hairs and crossed edge
                                  if not any((G.graph.
↪is_isomorphic(GRcrossedA.graph, edge_labels=True)) for G in␣
↪pool_UnmarkedCrossedOme[-1]):
                                      pool_UnmarkedCrossedOme[-1].
↪append(GRcrossedA)
        elif genus-1==g:
            pool_UnmarkedComponents.append([])
            pool_UnmarkedCrossedInt.append([])
            pool_UnmarkedCrossedOme.append([])
            pool_UnmarkedCrossedIntMultiple.append([])
            for f_unm in unm_hairs:
                for f_n in n_hairs:
                    unmarked_valence = [f_unm.count(v) for v in range(vertices)]
                    nadd_valence = [f_n.count(v) for v in range(vertices)]
                    # try also adding a parallel edge in every possible way and␣
↪crossing the component
                    multiedged = {}
                    for edge in skeleton.edges(sort=False):
                        multiedged = skeleton.copy(immutable=False)
                        multiedged.allow_multiple_edges(new=True)
                        multiedged.add_edge(edge[0],edge[1],'CROSS') # have to␣
↪label as crossed already
                        # if it has become stable
                        if all((multiedged.
↪degree(v)+unmarked_valence[v]+nadd_valence[v]>=3) for v in range(vertices)):
                            GR = BlowUpComponent().
↪build(multiedged,unmarked_valence,nadd_valence)
                            GR.cross('B',edge[0],edge[1])
                            if not any((G.graph.is_isomorphic(GR.graph,␣
↪edge_labels=True)) for G in pool_UnmarkedCrossedIntMultiple[-1]):
                                pool_UnmarkedCrossedIntMultiple[-1].
↪append(GR)
```

```
    #unwrap list of lists
    return [G for pool in pool_UnmarkedComponents for G in pool]+[G for pool in␣
  ↪pool_UnmarkedCrossedInt for G in pool]+[G for pool in␣
  ↪pool_UnmarkedCrossedOme for G in pool]+[G for pool in␣
  ↪pool_UnmarkedCrossedIntMultiple for G in pool]
```

[61]:
```
unmarked_components = pd.
  ↪DataFrame(columns=['ID','name','unmarked','nadd','vertices','stability','g','crossed','cros
def add_UnmarkedBlownUpComponent(G):
    unmarked_components.loc[len(unmarked_components)] = {'name':G.
  ↪__str__(),'graph':G,'unmarked':G.unmarked,'nadd':G.nadd,'vertices':G.
  ↪vertices,'stability':G.stability,'g':G.g, 'crossed':G.crossed,'crossType':G.
  ↪crossType}

visited = []
for excess in range(0,MAX_EXCESS+1):
    for unmarked in range(1,excess+4):
        for n_add in range(floor((excess+3-unmarked)/2)+1):
            for genus in range(floor((excess+3-unmarked-2*n_add)/3)+1):
                n = unmarked+n_add
                stability = 2*genus-2+unmarked+n_add #the stability parameter␣
  ↪gives an upper bound for the number of vertices of stable graphs
                for vertices in range(1,stability+1):
                    if (unmarked,n_add,vertices,genus) in visited: continue
                    visited.append((unmarked,n_add,vertices,genus))

                    components =␣
  ↪generate_UnmarkedBlowUpComponents(unmarked,n_add,vertices,genus)
                    for G in components:
                        add_UnmarkedBlownUpComponent(G)
UTRIPOD =␣
  ↪unmarked_components[(unmarked_components['vertices']==1)&(unmarked_components['unmarked']==
  ↪iloc[int(0)].graph

UU_HAIR = BlowUpComponent().build(doubleleg=0)
add_UnmarkedBlownUpComponent(UU_HAIR)
UJ_HAIR = BlowUpComponent().build(doubleleg=1)
add_UnmarkedBlownUpComponent(UJ_HAIR)
UA2IRR = BlowUpComponent().build(Graph({0:[0]}),[1],[0])
UA2IRR.cross('A',0)
add_UnmarkedBlownUpComponent(UA2IRR)
UBIRR = BlowUpComponent().build(Graph({0:[0]}),[1],[0])
UBIRR.cross('Birr',0)
add_UnmarkedBlownUpComponent(UBIRR)

#set short name id of the components
```

```
namegrouped = unmarked_components.groupby('name')
for namegroup in namegrouped.groups.keys():
    sorted = namegrouped.get_group(namegroup).
  ↪sort_values(by=['crossType','vertices','nadd']).reset_index(drop=True)
    for row in sorted.itertuples():
        row.graph.nameID=str(row.Index)
        row.graph.graph.name(new=row.graph.__str__())
unmarked_components['name'] = unmarked_components['graph'].apply(lambda G: G.
  ↪__str__())
#verify that every nameID is unique
namegrouped = unmarked_components.groupby('name')
for namegroup in namegrouped.groups.keys():
    if len(namegrouped.get_group(namegroup))>int(1):
        for G in namegrouped.get_group(namegroup).itertuples(): G.graph.show()
        raise ValueError('nameID not unique',namegrouped.get_group(namegroup))

unmarked_components.
  ↪sort_values(by=['crossType','nadd','unmarked','vertices','stability'],inplace=True)
unmarked_components.reset_index(drop=True,inplace=True)
unmarked_components['ID'] = unmarked_components.index  # copying the index in␣
  ↪an extra column helps for bookkeeping when the k-fold cartesian product of␣
  ↪the dataframe is taken later
for row in unmarked_components.itertuples():
        row.graph.ID=int(row.ID)

print("unmarked_components shape:")
print(unmarked_components.groupby(['vertices']).size())
```

```
unmarked_components shape:
vertices
0      2
1     28
2     73
3    103
4     64
5     13
dtype: int64
```

[62]:
```
for G in unmarked_components[(unmarked_components['crossType']=='A2') &␣
  ↪(unmarked_components['vertices']<=2)].itertuples():
    break
    G.graph.show(True)

for row in unmarked_components[unmarked_components['crossType']=='A'].
  ↪itertuples():
    break
    if row.graph.graph.has_multiple_edges(): row.graph.show(True)
```

```
[63]: def generate_blowup_components(eps,ome,nadd,g):
          #return a list of marked blowup components objects which is obtained from␣
       ↪the list unmarked_components by dividing in every possible way the unmarked␣
       ↪hairs in two families eps and ome, such that:
          # the unmarked component has an internal vertex, genus g, eps+ome unmarked␣
       ↪hairs and nadd nadd hairs

          # the case A2 with a loop at its end and the case Birr with eps/ome hairs␣
       ↪are added by this function
          # The remaining graphs of interest (5 double legs) are added manually␣
       ↪afterwards.

          # In excess  3, every case B graph with a multiple edge parallel to the␣
       ↪crossed edge vanishes bc of odd symmetry, so we need not consider it.

          # At this level, the nadd hairs are not uniquely labeled because we care␣
       ↪about blow up components up to isomorphisms that are allowed to permute them,
          #   afterwards, we will relabel them uniquely in all possible ways that␣
       ↪create non isomorphic blow up representations

          pool_components = [] # list of lists of blow up components, for every␣
       ↪unmarked_component

          for unmarked_component in␣
       ↪unmarked_components[(unmarked_components['vertices']!=0) &␣
       ↪(unmarked_components['g']==g) & (unmarked_components['unmarked']==eps+ome) &␣
       ↪(unmarked_components['nadd']==nadd)].itertuples():
              unmarked_graph = unmarked_component.graph

              unmarked_valence_multiset = []
              for v,u in enumerate(unmarked_graph.unmarked_valence):
                  for x in range(u): unmarked_valence_multiset.append(v)

              ome_multisets = Subsets(unmarked_valence_multiset,ome,submultiset=True)␣
       ↪# all multisubsets of the vertices with size ome

              pool_components.append([])
              for f_ome in ome_multisets:
                  # case A requires at least one omega hair at the crossed edge
                  if (unmarked_graph.crossType=='A' or unmarked_graph.
       ↪crossType=='A2') and not unmarked_graph.crossedEdge[0] in f_ome: continue

                  ome_valence = [f_ome.count(v) for v in range(unmarked_graph.
       ↪vertices)]
                  marked_graph = deepcopy(unmarked_graph)
                  marked_graph.mark(ome_valence,unmarkedComponent=unmarked_graph)
```

```
            #up to isomorphism of hairy graphs with labeled hairs. It is␣
↪sufficient to check isomorphism with previous graphs which have the same␣
↪unmarked component
            if not any((G.graph.is_isomorphic(marked_graph.graph,␣
↪edge_labels=True)) for G in pool_components[-1]):
                pool_components[-1].append(marked_graph)
    #unwrap list of lists
    return [G for pool in pool_components for G in pool]
```

[64]:
```
blowup_components = pd.
↪DataFrame(columns=['ID','unmarkedID','name','excess','eps','ome','nadd','vertices','stabili
def add_BlownUpComponent(G):
    blowup_components.loc[len(blowup_components)] = {'graph':G,'excess':G.
↪excess,'eps':G.eps,'ome':G.ome,'nadd':G.nadd,'n':G.n,'vertices':G.
↪vertices,'stability':G.stability,'g':G.g,'crossed':G.crossed, 'crossType':G.
↪crossType,'odd':G.odd,'odd_component':G.odd_component,'name':G.
↪__str__(),'unmarkedID':G.unmarkedComponent.ID}

def unmarkedCompID_fromCompID(i): return blowup_components.iloc[i].graph.
↪unmarkedComponent.ID

for excess in range(0,MAX_EXCESS+1):
    for ome in range(excess+4):
        for eps in range(floor((excess+3-ome)/3)+1):
            if eps+ome>=1:
                for n_add in range(floor((excess+3-3*eps-ome)/2)+1):
                    n = eps+ome+n_add
                    if (excess+3-3*eps-ome-2*n_add)%3!=0: continue
                    genus = (excess+3-3*eps-ome-2*n_add)/3
                    components = generate_blowup_components(eps,ome,n_add,genus)
                    for G in components:
                        add_BlownUpComponent(G)

TRIPOD = blowup_components.loc[(blowup_components['vertices']==1) &␣
↪(blowup_components['eps']==0) & (blowup_components['ome']==3) &␣
↪(blowup_components['nadd']==0)& (blowup_components['crossed']==False)].
↪iloc[int(0)].graph
# TRIPOD.show()

#BONUS GRAPHS: the double legs o-j  e-j  o-e  e-e,  o--crossed--v--loop, ␣
↪irreducible e--v--crossedloop
OJ_HAIR= deepcopy(UJ_HAIR)
OJ_HAIR.mark([],unmarkedComponent=UJ_HAIR,doubleleg_ome=1)
add_BlownUpComponent(OJ_HAIR)
EJ_HAIR = deepcopy(UJ_HAIR)
EJ_HAIR.mark([],unmarkedComponent=UJ_HAIR,doubleleg_ome=0)
```

```
add_BlownUpComponent(EJ_HAIR)
OO_HAIR= deepcopy(UU_HAIR)
OO_HAIR.mark([],unmarkedComponent=UU_HAIR,doubleleg_ome=2)
add_BlownUpComponent(OO_HAIR)
EO_HAIR = deepcopy(UU_HAIR)
EO_HAIR.mark([],unmarkedComponent=UU_HAIR,doubleleg_ome=1)
add_BlownUpComponent(EO_HAIR)
EE_HAIR = deepcopy(UU_HAIR)
EE_HAIR.mark([],unmarkedComponent=UU_HAIR,doubleleg_ome=0)
add_BlownUpComponent(EE_HAIR)
# OA2IRR = deepcopy(UA2IRR) # these two are already added by the generating␣
↪function above
# OA2IRR.mark([1])
# add_BlownUpComponent(OA2IRR)
# EBIRR = deepcopy(UBIRR)
# EBIRR.mark([0])
# add_BlownUpComponent(EBIRR)

blowup_components.
↪sort_values(by=['crossType','excess','vertices','nadd','eps','odd'],inplace=True)
blowup_components.reset_index(drop=True,inplace=True)
blowup_components['ID'] = blowup_components.index  # copying the index in an␣
↪extra column helps for bookkeeping when the k-fold cartesian product of the␣
↪dataframe is taken later
for row in blowup_components.itertuples(): row.graph.ID=int(row.ID)

for row in blowup_components[blowup_components['crossType']=='A2'].itertuples():
↪ row.graph.compute_A2blowup(blowup_components)

print("blowup_components shape:")
print(blowup_components.groupby(['excess','g','odd']).size())
```

```
blowup_components shape:
excess  g  odd
-1      0  True       1
0       0  False      3
1       0  False      8
        1  False      2
2       0  False     19
           True       2
        1  False      1
3       0  False     55
           True       8
        1  False      4
           True       3
4       0  False    168
           True      24
        1  False     18
           True      12
        2  True       1
dtype: int64
```

[65]:
```
for row in blowup_components.itertuples():
    row.graph.compute_specht()
```

[66]:
```
relations2_groups = pd.
 ↪DataFrame(columns=['graphs_ID_set','excess','ome','eps','nadd','g','vertices','crossContrac
# graphs (and graphs_ID_set) is a list of BlowUpComponent objects in case B␣
 ↪whose contraction results in the same graph
# valence is the amount of hairs of halfedges incident with the crossed edge,␣
 ↪the 'valence' of tilde{v} in the paper
# basis (and basis_ID_set) is a sublist of graphs which spans a basis through␣
 ↪the weight 2 relations and 3b relation
# the other columns are invariants of each relation 2 group

for row in blowup_components[blowup_components['crossType']=='B'].itertuples():
    row.graph.compute_crossContractedComponent()
    for row2 in relations2_groups[(relations2_groups['g']==row.g) &␣
 ↪(relations2_groups['ome']==row.ome) & (relations2_groups['eps']==row.eps) &␣
 ↪(relations2_groups['nadd']==row.nadd) & (relations2_groups['vertices']==row.
 ↪vertices-1) & (relations2_groups['valence']==row.graph.crossValence)  &␣
 ↪(relations2_groups['valence_none']==row.graph.crossValence_none) &␣
 ↪(relations2_groups['valence_eps']==row.graph.crossValence_eps) &␣
 ↪(relations2_groups['valence_nadd']==row.graph.crossValence_nadd) ].
 ↪itertuples():
        if row2.crossContractedComponent.is_isomorphic(row.graph.
 ↪crossContractedComponent, edge_labels=True):
            row2.graphs.append(row.graph)
            break
    else:
        relations2_groups.loc[len(relations2_groups)] = {'excess': row.
 ↪excess,'eps': row.eps,'ome': row.ome,'nadd': row.nadd, 'g': row.g,␣
 ↪'vertices': row.vertices-1, 'crossContractedComponent': row.graph.
 ↪crossContractedComponent, 'valence':row.graph.crossValence, 'valence_none':
 ↪row.graph.crossValence_none, 'valence_nadd':row.graph.crossValence_nadd,␣
 ↪'valence_eps':row.graph.crossValence_eps,'graphs': [row.graph], 'basis':[]}
relations2_groups.sort_values(['excess','nadd','eps','vertices'],inplace=True)
relations2_groups.reset_index(drop=True,inplace=True)
relations2_groups['graphs_ID_set'] = relations2_groups['graphs'].apply(lambda␣
 ↪row: tuple([G.ID for G in row]))
for row in relations2_groups.itertuples():
    for G in row.graphs: G.relation2groupID=int(row.Index)
blowup_components['relation2groupID'] = blowup_components['graph'].apply(lambda␣
 ↪G: G.relation2groupID)
```

```
# compute basis after applying weight 2 relations and 3b relation
# DONE MANUALLY
def weight2basis_raise(e,v,graph=None):
    if graph:
        print(f"loops: {graph.crossContractedComponent.has_loops()} ,  multiple␣
 ↪edges: {graph.crossContractedComponent.has_multiple_edges()}")
        graph.crossContractedComponent.show()
        graph.show(True)
    raise ValueError(f"weight 2 basis computation: unhandled case   excess:␣
 ↪{e}, valence: {v}")
for row in relations2_groups.itertuples():
    # first easy case
    if row.valence==4:
        if any([G.odd for G in row.graphs]) or (row.crossContractedComponent.
 ↪has_loops() or row.crossContractedComponent.has_multiple_edges()): pass
        else: row.basis.append(row.graphs[0])
    elif len(row.graphs)==1:
        # second easy case
        if row.graphs[0].odd: pass
        # third easy case
        elif not (row.crossContractedComponent.has_loops() or row.
 ↪crossContractedComponent.has_multiple_edges()): row.basis.append(row.
 ↪graphs[0])
    # we have hardcoded manually the remaining cases
        elif row.excess==3: pass   # only one such
        elif row.excess==4: pass   # only one such
        else: weight2basis_raise(row.excess,row.valence,row.graphs[0])
    elif row.excess==1: weight2basis_raise(row.excess,row.valence)
    elif row.excess==2: weight2basis_raise(row.excess,row.valence, row.
 ↪graphs[0])
    elif row.excess==3: row.basis.extend(row.graphs) # there's 3 groups of␣
 ↪valence 5,5,6 of group length 2 and no relations
    elif row.excess==4:
        if row.valence==5:
            if row.crossContractedComponent.has_loops(): pass # there are two␣
 ↪groups, each with two graphs with loops and two indipendent equations
            elif row.valence_none==2: pass # there is one group with three␣
 ↪graphs, two are odd and they kill also the third
            elif row.valence_none==1 and row.valence_nadd==1: row.basis.
 ↪extend(row.graphs[:3]) # there is one group with four graphs and one equation
            elif row.valence_nadd==2: # there is one group with three graphs␣
 ↪with two nadd hairs and one equation between V2 specht contributions
                first=True
                for G in row.graphs:
                    if len(G.specht_sequence)==1 and first: # specht␣
 ↪contribution V2
                        row.basis.append(G)
                        first=False
                    elif len(G.specht_sequence)==2: row.basis.append(G) #␣
 ↪specht contribution V1^2 + V2
            else: row.basis.extend(row.graphs)
            # relation 3b imposes one equation with non zero coefficients on␣
 ↪the weight 2 relation group, so a basis is given by excluding any element
        elif row.valence==6:
            if row.crossContractedComponent.has_loops(): row.basis.append(row.
 ↪graphs[0]) # there's 1 group with two graphs with a loop, and one equation
            else: row.basis.extend(row.graphs)  # there's 2 groups with 3␣
 ↪graphs and no relations
        elif row.valence==7: row.basis.extend(row.graphs) # there's 1 group␣
 ↪with two graphs and no relations
        else: weight2basis_raise(row.excess,row.valence)
    else: weight2basis_raise(row.excess,row.valence)

relations2_groups['basis_ID_set'] =  relations2_groups['basis'].apply(lambda␣
 ↪row: tuple([G.ID for G in row]))
for row in relations2_groups.itertuples():
    for G in row.graphs: G.in_relation2_basis = G in row.basis

print("relations2_groups size: ")
print(relations2_groups.groupby(['excess','valence']).size())

for row in relations2_groups[relations2_groups['graphs'].apply(lambda graphs:
    len(graphs)>1 and graphs[0].excess==4 and graphs[0].crossValence==6
    and (True or graphs[0].crossContractedComponent.has_loops() or graphs[0].
 ↪crossContractedComponent.has_multiple_edges()) )].itertuples():
    break
    print(f'excess: {row.excess}, g: {row.g}, valence,none,nadd,eps: {row.
 ↪valence},{row.valence_none},{row.valence_nadd},{row.valence_eps}, graphs ID:␣
 ↪{row.graphs_ID_set}, basis ID: {row.basis_ID_set}')
    row.crossContractedComponent.graphplot(edge_labels=True).show()
    for G in row.graphs: G.show(title=True,specht=True)
```

```
relations2_groups size:
excess  valence
1       4           1
2       4           3
        5           1
3       4          10
        5           3
        6           1
4       4          35
```

```
        5          10
        6           3
        7           1
dtype: int64
```

[67]:
```
for row in blowup_components[blowup_components['graph'].apply(lambda G: G.
 ↪excess>0 and G.crossed and G.ome+G.eps==1 and G.g==0)].itertuples():
    break
    print(row)
    row.graph.show(True,title=True,specht=True)
```

[68]:
```
# Build collection of all components, up to isomorphism, up to odd symmetry,␣
 ↪and up to relations (at least the ones that can be checked on the individual␣
 ↪connected components).
# When a list of these components is glued to the special vertex, they may␣
 ↪still have odd symmetry (by permuting isomorphic components with odd edges)␣
 ↪or relations due to the special vertex decoration in weight 13.

# The dataframe of blowup_components that will be used later to construct all␣
 ↪different blowup representations of graphs
blowup_components_basis = blowup_components

# relations 2 and 3b:  keep only blowup components which are contained in the␣
 ↪basis of their respective relation2_group determined above
blowup_components_basis =␣
 ↪blowup_components_basis[blowup_components_basis['graph'].apply(lambda G: G.
 ↪crossType!='B' or G.in_relation2_basis)]

# A2 case with OO_HAIR: removed through weight 13 relations and 4ab relation
blowup_components_basis =␣
 ↪blowup_components_basis[blowup_components_basis['graph'].apply(lambda G: G.
 ↪crossType!='A2' or OO_HAIR not in G.crossTypeA2_blowupComponents)]

# odd symmetries are part of the relations created by coinvariants
blowup_components_basis =␣
 ↪blowup_components_basis[(blowup_components_basis['odd']==False)]

print('blowup_components_basis size: ')
print(blowup_components_basis.groupby(['excess','g']).size())

for G in blowup_components_basis[(blowup_components_basis['excess']<=2) &␣
 ↪(blowup_components_basis['odd']==True)].itertuples():
    break
    G.graph.show(True)
```

```
blowup_components_basis size:
excess  g
```

```
0        0      2
1        0      6
         1      1
2        0     15
3        0     44
         1      1
4        0    129
         1      5
dtype: int64
```

[69]:
```python
class BlowUpGraph:
    # wrapper object for lists of blown up components, i.e. what uniquely␣
↪determines the blown up representation of a graph
    def __init__(self,components):
        self.ID = -1
        if len(components)==0: raise ValueError("No components fed")
        self.components = components
        self.components.sort(key=lambda C: C.ID)  # list of marked␣
↪BlowUpComponent objects
        self.unmarked_components = [C.unmarkedComponent for C in components] #␣
↪list of unmarked BlowUpComponent objects
        self.crossedComponent=None    # BlowUpComponent object in the list␣
↪which is crossed
        for C in components:
            if C.crossed:
                self.crossedComponent = C
                break
        if self.crossedComponent==None: raise ValueError("no crossed␣
↪components")
        self.crossType = self.crossedComponent.crossType
        self.partition = [] # excess partition
        for C in components: self.partition.append(C.excess)
        self.partition = tuple(self.partition)

        self.eps_nadd_factor = 0 # a simple measure for heavily the eps hairs␣
↪are distributed on components with hairs, used in case B and Birr to sort␣
↪graphs in their relation11 group
        for C in self.components: self.eps_nadd_factor += (C.eps+int(C.
↪crossed))*C.nadd

        #parameters of blown up graph obtained through gluing the components to␣
↪the special vertex of genus 1
        self.n = sum(G.nadd for G in components)   # number of hairs
        self.unmarked = sum(G.unmarked for G in components) # number of␣
↪elements in the set A
        self.eps = sum(G.eps for G in components)   #number of elements in the␣
↪set A\B of the decoration
```

```
        self.ome = sum(G.ome for G in components)   # number of elements in the␣
↪set B of the decoration
        if self.ome>11: raise ValueError('ome value too high',self.ome)
        if self.unmarked!=self.ome+self.eps: raise ValueError('unmarked is not␣
↪ome + eps',self.unmarked)
        self.excess = sum(G.excess for G in components) # excess of the glued␣
↪graph
        self.g = 1+sum(G.g_add for G in components)  # total genus of the glued␣
↪graph
        if self.excess!=(3*self.g-3+2*self.n - 2*self.ome): raise␣
↪ValueError(f"excess formula does not match: {self.excess}!={(3*self.
↪g-3+2*self.n - 2*self.ome)}")
        self.vertices = 1+sum(G.v_add for G in components) #internal vertices␣
↪of the glued graph, identifying the crossed edge as one vertex
        self.edges = sum(G.e_add for G in components) #internal edges, not␣
↪counting the crossed edge

        self.virtual_gn_range = [] # the possible (g,n) tuples to which this␣
↪blowup can be completed
        for (g,n) in gn_table[self.excess]:
            if g>=self.g and n>=self.n and (g-self.g)%2==0:
                oj_hairs = n-self.n
                tripods = ((g-self.g)/2)
                if 11==3*tripods+self.ome+oj_hairs: self.virtual_gn_range.
↪append((g,n))
                else: raise ValueError("mismatch in omega hairs of possible␣
↪completion")
        self.virtual_gn_range=tuple(self.virtual_gn_range)

        self.build_visuals()

        self.specht_sequence = [] # list of tuples (k,o)
        visited_ID = []
        for C in self.components:
            if C.ID in visited_ID or C.nadd==0: continue
            visited_ID.append(C.ID)
            multiplicity = [F.ID for F in self.components].count(C.ID) # amount␣
↪of components isomotphic to C in this graph
            if multiplicity==1:
                self.specht_sequence.extend(C.specht_sequence)
            elif C.nadd==1: # the only case I need to handle, fortunately
                # the resulting action on these components has sign +-1␣
↪depending upon wether C has odd internal edges + epsilon hairs  (or is the␣
↪oj_hair and is not the eps_hair)
                if (C.vertices==0 and C.ome==1) or (C.vertices>0 and (C.edges+C.
↪eps)%2!=0):
```

```
                    self.specht_sequence.append((multiplicity,-1))
                else: self.specht_sequence.append((multiplicity,1))
            else:
                print(self.__str__(True))
                self.show(title=True,specht=True)
                raise ValueError("uncovered case in computing specht sequence␣
↪of BlowUpGraph object")
        self.specht_string=r""
        if len(self.specht_sequence)>0:
            self.specht_string = r"$"
            first=True
            for specht in self.specht_sequence:
                if not first: self.specht_string+=r"\boxtimes "
                first = False
                self.specht_string+=r"V_{"+ (str(specht[0]) if specht[1]==1␣
↪else (r"1^{"+str(specht[0])+r"}"))+r"}"
            self.specht_string+=r"$"

    def build_visuals(self):
        if not GRAPHVISUALS: return
        self.vertex_partition = {'vertices':[],'eps':[],'ome':[],'nadd':[]}
        for c,C in enumerate(self.components):
            self.vertex_partition['vertices'] = self.
↪vertex_partition['vertices']+[(str(c)+','+str(x)) for x in range(C.vertices)]
            self.vertex_partition['eps'] = self.
↪vertex_partition['eps']+[(str(c)+',e'+str(x)) for x in range(C.eps)]
            self.vertex_partition['ome'] = self.
↪vertex_partition['ome']+[(str(c)+',o'+str(x)) for x in range(C.ome)]
            self.vertex_partition['nadd'] = self.
↪vertex_partition['nadd']+[(str(c)+',n'+str(x)) for x in range(C.nadd)]

        self.vertex_position = {}
        node_width = 1/6
        max_nodes_per_row=[max([C.ome+C.eps,C.vertices,C.nadd]) for C in self.
↪components]
        width = 0.4+sum(max_nodes_per_row)*node_width
        component_widths=[width*max_nodes_per_row[c]/sum(max_nodes_per_row) for␣
↪c,C in enumerate(self.components)]
        height = 0.7
        XX = 0
        for c,C in enumerate(self.components):
            DX = component_widths[c]/(C.eps+C.ome)
            X = XX+DX/2
            Y = 0
            for i in range(C.ome):
                self.vertex_position[str(c)+',o'+str(i)] = [X,Y]
                X+=DX
            for i in range(C.eps):
                self.vertex_position[str(c)+',e'+str(i)] = [X,Y]
                X+=DX
            if C.nadd!=0:
                DX = component_widths[c]/C.nadd
                X=XX+DX/2
                if C.vertices==0 and C.ome!=0: Y=height*0.5 # oj_hairs
                elif C.vertices==0: Y=height*0.75 # ej_hairs
                else: Y = height
                for i in range(C.nadd):
                    self.vertex_position[str(c)+',n'+str(i)] = [X,Y]
                    X+=DX
            Y=height/2
            if C.vertices == 1 or C.vertices==2:
                DX = component_widths[c]/C.vertices
                X=XX+DX/2
                for i in range(C.vertices):
                    self.vertex_position[str(c)+','+str(i)] = [X,Y]
                    X+=DX
            elif C.vertices>=3:
                DT = 2*math.pi/C.vertices
                T = -math.pi/2-DT/2
                for i in range(C.vertices):
                    self.vertex_position[str(c)+','+str(i)] =␣
↪[XX+component_widths[c]/2+component_widths[c]*cos(T)*0.45,height/
↪2+sin(T)*height/4]
                    T-=DT
            XX+=component_widths[c]

        self.UnitedGraph = Graph({},loops=True, multiedges=True)
        self.crossedEdge = []
        self.UnitedGraph.add_vertices(self.vertex_partition['vertices']+self.
↪vertex_partition['eps']+self.vertex_partition['ome']+self.
↪vertex_partition['nadd'])
        for c,C in enumerate(self.components):
            for edge in C.graph.edges(sort=False,labels=True):
                self.UnitedGraph.
↪add_edge(str(c)+','+str(edge[0]),str(c)+','+str(edge[1]),edge[2])
                if C.crossed: self.crossedEdge = (str(c)+','+str(C.
↪crossedEdge[0]),str(c)+','+str(C.crossedEdge[1]))

    def __str__(self, long=False):
        if not long:
            name=""
            for c in self.components:
                name+=c.__str__()+' '
            return name
        return f"BlowUpGraph; Excess: {str(self.partition)}  g: {str(self.g)} ␣
↪n: {str(self.n)}   vertices: {str(self.vertices)}  edges: {str(self.edges)}␣
↪crossType: {str(self.crossType)}"

    def␣
↪show(self,graphics=False,name=False,title=False,gn_range=False,specht=False,append="",kws=N
↪
        if name: print(self.__str__(True))

        graphplot_object = self.UnitedGraph.graphplot(vertex_labels=False ,␣
↪vertex_size=30,vertex_colors={'darkgrey':self.vertex_partition['vertices'],␣
↪'gold':self.vertex_partition['eps'],'dodgerblue':self.
↪vertex_partition['ome'], 'lime':self.vertex_partition['nadd']}, pos=self.
↪vertex_position, edge_colors={'red':[self.crossedEdge]})
        if title:
            titlestring = f'ID: {self.ID}'
            if specht: titlestring += "    "+self.specht_string
            if gn_range:
                titlestring+= ('\n' if specht else "") + f'    n: '
                for (g,n) in self.virtual_gn_range:
                    titlestring+=f'{str(n)} '
                titlestring+= "    "+append
            else: titlestring+= "\n    "+append
            graphplot_object = graphplot_object.plot(axes=False,␣
↪title=titlestring, fontsize=11)
        else: graphplot_object = graphplot_object.plot(axes=False)
        if kws!=None: graphplot_object.plot(**kws)
        if not graphics: graphplot_object.show()
        else:  return graphplot_object
```

[70]:
```
blowups = pd.
↪DataFrame(columns=['excess','partition','comp_ID_set','g','n','vertices','edges','graph','c
# Generate all (unordered) collections of components in blowup_components_basis␣
↪(of excess >=1) which have:
# -- exactly one crossed component
# -- excesses summing up to MAX_EXCESS
# -- at maximum one not crossed component with the odd_component attribute (to␣
↪avoid odd symmetries of the glued graph)

def partitions(n, I=1):
    yield (n,)
    for i in range(I, n//2 + 1):
        for p in partitions(n-i, i):
            yield (i,) + p
```

```
grouped_components = blowup_components_basis.groupby(['excess','crossed'])
for E in range(1,MAX_EXCESS+1):
    for partition in partitions(E):
        #there has to be exactly one crossed component, which has to be chosen␣
 ↪without repetition nor order
        crossed_visited = []
        for crossedchoice in range(len(partition)):
            if partition[crossedchoice] in crossed_visited: continue
            crossed_visited.append(partition[crossedchoice])

            #form the cartesian product of selected subsets of␣
 ↪blowup_components_basis
            components_array = grouped_components.
 ↪get_group((partition[0],0==crossedchoice))
            components_array = components_array[['ID','graph']].add_suffix('_0')
            id_columns = ['ID_0']
            graph_columns = ['graph_0']
            for i in range(1,len(partition)):
                components_array = components_array.merge(grouped_components.
 ↪get_group((partition[i],i==crossedchoice))[['ID','graph']].
 ↪add_suffix('_'+str(i)), how='cross')
                id_columns.append('ID_'+str(i))
                graph_columns.append('graph_'+str(i))
            #remove duplicates up to reordering of the components in cartesian␣
 ↪product
            #components_array = components_array.assign(comp_ID_set=lambda row:␣
 ↪row[id_columns].apply(lambda l: sorted(list(l)),axis=1) ) # does not work␣
 ↪for the love of god
            def sort_inplace(row): row.sort()
            components_array = components_array.assign(comp_ID_set=lambda row:␣
 ↪row[id_columns].values.tolist() )
            components_array['comp_ID_set'].apply(sort_inplace)
            components_array.
 ↪drop_duplicates(subset=['comp_ID_set'],inplace=True)

            #if at least two elements are not crossed, remove entries with at␣
 ↪least two components with the odd_component attribute
            if len(partition)>2:
                def double_odd_check(row):
                    odd_components=[]
                    for C in list(row):
                        if C.odd_component:
                            if any(C_ is C for C_ in odd_components): return␣
 ↪True
                            odd_components.append(C)
                    return False
                components_array = components_array.
↪assign(has_double_odd=lambda row: row[graph_columns].
↪apply(double_odd_check,axis=1))
                components_array =␣
↪components_array[components_array['has_double_odd']==False]

            # construct the BlowUpGraph objects and store them in the dataframe
            for row in components_array[graph_columns+['comp_ID_set']].
↪itertuples():
                components = list(row[1:-1])
                if sum([C.ome for C in components])>11: continue
                G = BlowUpGraph(components)
                comp_ID_set = [C.ID for C in components]
                comp_ID_set.sort()
                comp_ID_set = tuple(comp_ID_set)
                unmarkedComp_ID_set = [C.unmarkedComponent.ID for C in␣
↪components]
                unmarkedComp_ID_set.sort()
                unmarkedComp_ID_set = tuple(unmarkedComp_ID_set)
                A2Comp_ID_set=None
                if G.crossType=='A2':
                    A2Comp_ID_set = [C.ID for C in components if not C.crossed]␣
↪+ [C.ID for C in G.crossedComponent.crossTypeA2_blowupComponents]
                    A2Comp_ID_set.sort()
                    A2Comp_ID_set = tuple(A2Comp_ID_set)
                blowups.loc[len(blowups)] = {'graph':G,'comp_ID_set':
↪comp_ID_set,'excess':G.excess,'partition':partition,'g':G.g,'n':G.
↪n,'vertices':G.vertices,'edges':G.edges,'crossType':G.crossType,␣
↪'unmarkedComp_ID_set':unmarkedComp_ID_set,'A2Comp_ID_set':A2Comp_ID_set,␣
↪'relation2groupID':G.crossedComponent.relation2groupID}
print("blowups shape:")
print(blowups.groupby(['excess','partition']).size())
```

```
blowups shape:
excess  partition
1       (1,)              3
2       (1, 1)           12
        (2,)              8
3       (1, 1, 1)        23
        (1, 2)           53
        (3,)             30
4       (1, 1, 1, 1)     30
        (1, 1, 2)       140
        (1, 3)          165
        (2, 2)           56
        (4,)             99
dtype: int64
```

```
[71]: def␣
      ↪get_blowups(excess=None,partition=None,vertices=None,unmarkedComp_ID_set=None,crossType=Non
      ↪g=None, n=None):
          frame = blowups
          conditions = frame['g']>=0
          if crossType!=None:
              if crossType=='BB': conditions = conditions & (frame['crossType']=='B')␣
      ↪| (frame['crossType']=='Birr')
              else: conditions = conditions & (frame['crossType']==crossType)
          if excess!=None: conditions = conditions & (frame['excess']==excess)
          if g!=None: conditions = conditions & (frame['g']==g)
          if n!=None: conditions = conditions & (frame['n']==n)
          if partition!=None: conditions = conditions &␣
      ↪(frame['partition']==partition)
          if vertices!=None: conditions = conditions & (frame['vertices']==vertices)
          if unmarkedComp_ID_set!=None: conditions = conditions &␣
      ↪(frame['unmarkedComp_ID_set']==unmarkedComp_ID_set)
          return frame[conditions]

      #print(blowups[blowups['graph'].apply(lambda G: G.partition == [1,1,1] )].
      ↪groupby(['vertices']).size())
      for row in blowups[blowups['graph'].apply(lambda G: G.excess==4 and G.n==2 and␣
      ↪len(G.specht_sequence)==1)].itertuples():
          break
          row.graph.show(title=True,specht=True)

      for row in get_blowups(excess=2, crossType='BB').itertuples():
          break
          row.graph.show()
```

```
[72]: blowups_virtual = pd.
      ↪DataFrame(columns=['excess','partition','gn_range','comp_ID_set','crossType','edges_plus_n'
      # wrapper dataframe for the blowups dataframe to characterize them up to␣
      ↪completition by oj_hairs and tripods
      # given the excess, the parameter n determines g,vertices, edges, number of␣
      ↪oj_hairs and number of tripods of the completed blow up representation in␣
      ↪the following way:
      #  oj_hairs = n - n_virtual    tripods = (11-ome_virtual-oj_hairs)/3 =
      #  g = g_virtual+2*tripods
      #  vertices = vertices_virtual+tripods
      #  edges = edges_virtual 3*tripods = edges_virtual + (11-ome_virtual+n_virtual)␣
      ↪- n
      #  we define   edges_plus_n = edges_virtual + 11-ome_virtual + n_virtual

      # this task is grouped by excess and the possible (g,n) pairs corresponding to␣
      ↪the excess
```

```
# the pair (g,n) determines uniquely which and how many excess 0 components␣
↪must be appended to obtain a (g,n)-graph when they are glued to the special␣
↪vertex
for blowup in get_blowups().itertuples():
        blowup.graph.specht_string = r"$V_{1^{n"+("-"+str(blowup.graph.n) if␣
↪blowup.graph.n!=0 else "")+r"}}$"+(r"$\boxtimes $" if len(blowup.graph.
↪specht_sequence)>0 else "")+specht_sequence_to_string(blowup.graph.
↪specht_sequence) # ugly, modify directly the attribute of each BlowUpGraph␣
↪object
        components = list(blowup.graph.components)
        comp_ID_set = [C.ID for C in components]
        comp_ID_set.sort()
        comp_ID_set = tuple(comp_ID_set)
        unmarkedComp_ID_set = [C.unmarkedComponent.ID for C in components]
        while UJ_HAIR.ID in unmarkedComp_ID_set: unmarkedComp_ID_set.
↪remove(UJ_HAIR.ID)
        while UTRIPOD.ID in unmarkedComp_ID_set: unmarkedComp_ID_set.
↪remove(UTRIPOD.ID)
        unmarkedComp_ID_set.sort()
        unmarkedComp_ID_set = tuple(unmarkedComp_ID_set)
        A2Comp_ID_set=None
        if blowup.crossType=='A2':
            A2Comp_ID_set = [C.ID for C in components if not C.crossed] + [C.ID␣
↪for C in blowup.graph.crossedComponent.crossTypeA2_blowupComponents]
            while OJ_HAIR.ID in A2Comp_ID_set: A2Comp_ID_set.remove(OJ_HAIR.ID)
            while TRIPOD.ID in A2Comp_ID_set: A2Comp_ID_set.remove(TRIPOD.ID)
            A2Comp_ID_set.sort()
            A2Comp_ID_set = tuple(A2Comp_ID_set)
        blowups_virtual.loc[len(blowups_virtual)] = {'graph':blowup.
↪graph,'comp_ID_set':comp_ID_set,'crossType':blowup.crossType,'excess':blowup.
↪excess,'partition':blowup.partition,'edges_plus_n': blowup.edges+11-blowup.
↪graph.ome+blowup.n, 'gn_range':tuple(blowup.graph.virtual_gn_range),␣
↪'unmarkedComp_ID_set':unmarkedComp_ID_set,'A2Comp_ID_set':A2Comp_ID_set,␣
↪'relation2groupID':blowup.relation2groupID, 'specht_sequence': tuple(blowup.
↪graph.specht_sequence)}

blowups_virtual.
↪sort_values(by=['excess','edges_plus_n','crossType','gn_range','relation2groupID','unmarked
↪inplace=True)
blowups_virtual.reset_index(drop=True, inplace=True)
for row in blowups_virtual.itertuples(): row.graph.ID = int(row.Index)

print("blowups_virtual size: ")
print(blowups_virtual.groupby(['excess','edges_plus_n','crossType']).size())
```

```
blowups_virtual size:
excess  edges_plus_n  crossType
```

```
1       10           A            1
        11           B            1
                     Birr         1
2       10           A            3
        11           A            4
                     B            3
                     Birr         2
        12           A2           2
                     B            4
                     Birr         2
3       10           A            6
        11           A           17
                     B            7
                     Birr         4
        12           A           12
                     A2           8
                     B           19
                     Birr         9
        13           A2           8
                     B           12
                     Birr         4
4       10           A           10
        11           A           45
                     B           13
                     Birr         6
        12           A           66
                     A2          19
                     B           58
                     Birr        22
        13           A           34
                     A2          45
                     B           75
                     Birr        24
        14           A2          30
                     B           34
                     Birr         9
dtype: int64
```

[73]:
```
def␣
↪get_blowups_virtual(excess=None,partition=None,vertices=None,unmarkedComp_ID_set=None,cross
↪gn=None,edges_plus_n=None):
    frame = blowups_virtual
    conditions = frame['excess']>=0
    if crossType!=None:
        if crossType=='BB': conditions = conditions & (frame['crossType']=='B')␣
↪| (frame['crossType']=='Birr')
        else: conditions = conditions & (frame['crossType']==crossType)
    if excess!=None: conditions = conditions & (frame['excess']==excess)
    if gn!=None: conditions = conditions & frame['gn_range'].apply(lambda␣
 ↪gn_range: gn in gn_range)
    if partition!=None: conditions = conditions &␣
 ↪(frame['partition']==partition)
    if vertices!=None: conditions = conditions & (frame['vertices']==vertices)
    if unmarkedComp_ID_set!=None: conditions = conditions &␣
 ↪(frame['unmarkedComp_ID_set']==unmarkedComp_ID_set)
    if edges_plus_n!=None: conditions = conditions &␣
 ↪(frame['edges_plus_n']==edges_plus_n)
    #if modified!=None: conditions = conditions & (frame)
    return frame[conditions]

for row in get_blowups_virtual(excess=4, partition=(4,), crossType='B',␣
 ↪edges_plus_n=13).itertuples():
    break
    print(f"gn_range: {row.gn_range}   unmarkedComps: {row.unmarkedComp_ID_set}␣
 ↪  A2Comps: {row.A2Comp_ID_set}")
    row.graph.show(title=True, specht=True, gn_range=True)
    #for C in row.graph.crossedComponent.crossTypeA2_blowupComponents: C.show()
for (id, group) in blowups_virtual.groupby('relation2groupID'):
    break
    if len(relations2_groups.loc[id].basis)>1:
        print(f"rel2 group ID:  {id}")
        for row in group.itertuples():
            row.graph.show(title=True, specht=True, gn_range=True)

print(get_blowups_virtual(excess=3).groupby(['gn_range','edges_plus_n']).size())
```

```
gn_range                            edges_plus_n
((2, 11), (4, 8), (6, 5))           13               1
((2, 11), (4, 8), (6, 5), (8, 2))   12               5
                                    13               3
((4, 8), (6, 5), (8, 2))            11               5
                                    12              16
                                    13              10
((6, 5), (8, 2))                    10               3
                                    11              14
                                    12              19
                                    13               8
((8, 2),)                           10               3
                                    11               9
                                    12               8
                                    13               2
dtype: int64
```

[74]:
```
relations11_groups = pd.
  ↪DataFrame(columns=['excess','edges_plus_n','unmarkedComp_ID_set','graphs','group_length','c
# unmarkedComp_ID_set is the sorted tuple of IDs of each unmarked␣
  ↪BlowUpComponent object that determines the relation11 group
# graphs is the list of BlowUpGraph objects in case B or Birr that have been␣
  ↪generated from the unmarked components with IDs in unmarkedComp_ID_set. It␣
  ↪is sorted descendingly w.r.t. the eps_nadd_factor of each BlowUpGraph object

for (excess,edges_plus_n,unmarkedComp_ID_set),group in␣
  ↪get_blowups_virtual(crossType='BB').
  ↪groupby(['excess','edges_plus_n','unmarkedComp_ID_set']):
    if not (group['edges_plus_n']==group.iloc[int(0)].edges_plus_n).all():␣
  ↪raise ValueError('not all blowups in the relation 13 group have same␣
  ↪properties')
    graphs = group['graph'].tolist()
    graphs.sort(key=lambda G: G.virtual_gn_range)
    graphs.sort(key=lambda G: G.eps_nadd_factor, reverse=True)
    relations11_groups.loc[len(relations11_groups)] = {'excess':
  ↪excess,'edges_plus_n':edges_plus_n,'unmarkedComp_ID_set':␣
  ↪unmarkedComp_ID_set, 'graphs':graphs, 'crossType':graphs[0].crossType}
relations11_groups['group_length'] = relations11_groups['graphs'].apply(lambda␣
  ↪graphs:len(graphs))
relations11_groups['gn_range'] = relations11_groups['graphs'].apply(lambda␣
  ↪graphs:graphs[0].virtual_gn_range)
relations11_groups.
  ↪sort_values(['excess','edges_plus_n','crossType','gn_range','group_length','unmarkedComp_ID

def get_relations11_groups(excess=None, edges_plus_n=None, gn=None,␣
  ↪group_length=None):
    frame = relations11_groups
    conditions = frame['excess']>=0
    if excess!=None: conditions = conditions & (frame['excess']==excess)
    if gn!=None: conditions = conditions & frame['graphs'].apply(lambda graphs:␣
  ↪any(gn in G.virtual_gn_range for G in graphs))
    if edges_plus_n!=None: conditions = conditions &␣
  ↪(frame['edges_plus_n']==edges_plus_n)
    if group_length!=None:
        if group_length==-1: conditions = conditions & (frame['group_length']>1)
        else: conditions = conditions & (frame['group_length']==group_length)
    return frame[conditions]

print("relations11_groups size: ")
print(relations11_groups.groupby(['excess','edges_plus_n','group_length']).
  ↪size())
```

```
for row in relations11_groups[relations11_groups['graphs'].apply(lambda graphs:␣
 ↪len(graphs)>0 and graphs[0].excess==2)].itertuples():
    break
    print("unmarkedComp_ID_set: ", row.unmarkedComp_ID_set)
    for G in row.graphs: G.show()
```

```
relations11_groups size:
excess  edges_plus_n  group_length
1       11            1               2
2       11            1               5
        12            1               6
3       11            1              11
        12            1              22
                      3               2
        13            1              16
4       11            1              19
        12            1              60
                      4               5
        13            1              76
                      3               1
                      4               5
        14            1              43
dtype: int64
```

[75]:
```
relations13_groups = pd.
 ↪DataFrame(columns=['excess','edges_plus_n','A2Comp_ID_set','graphs'])
# A2Comp_ID_set is the sorted tuple of IDs of each marked BlowUpComponent␣
 ↪object that determines the relation13 group
# graphs is the list of BlowUpGraph objects in case A2 whose further blowup at␣
 ↪the special vertex results in the marked components with IDs in␣
 ↪A2Comp_ID_set.  It is sorted descendingly w.r.t. the eps_nadd_factor of each␣
 ↪BlowUpGraph object, and ascendingly w.r.t. the amount o hairs on the crossed␣
 ↪component.

for (excess,edges_plus_n,A2Comp_ID_set),group in␣
 ↪get_blowups_virtual(crossType='A2').
 ↪groupby(['excess','edges_plus_n','A2Comp_ID_set']):
    if not (group['edges_plus_n']==group.iloc[int(0)].edges_plus_n).all():␣
 ↪raise ValueError('not all blowups in the relation 13 group have same␣
 ↪properties')
    graphs = group['graph'].tolist()
    graphs.sort(key=lambda G: G.virtual_gn_range)
    graphs.sort(key=lambda G: G.eps_nadd_factor, reverse=True)
    graphs.sort(key=lambda G: G.crossedComponent.nadd)
    graphs.sort(key=lambda G: len(G.components))
    relations13_groups.loc[len(relations13_groups)] = {'graphs':graphs,␣
 ↪'excess':excess, 'edges_plus_n':edges_plus_n, 'A2Comp_ID_set':A2Comp_ID_set}
```

```
relations13_groups['group_length'] = relations13_groups['graphs'].apply(lambda␣
↪graphs:len(graphs))
relations13_groups.
↪sort_values(['excess','edges_plus_n','group_length','A2Comp_ID_set'],inplace=True)

def get_relations13_groups(excess=None, edges_plus_n=None, gn=None):
    frame = relations13_groups
    conditions = frame['excess']>=0
    if excess!=None: conditions = conditions & (frame['excess']==excess)
    if gn!=None: conditions = conditions & frame['graphs'].apply(lambda graphs:␣
↪any(gn in G.virtual_gn_range for G in graphs))
    if edges_plus_n!=None: conditions = conditions &␣
↪(frame['edges_plus_n']==edges_plus_n)
    return frame[conditions]

print("relations13_groups size: ")
print(relations13_groups.groupby(['excess','edges_plus_n','group_length']).
↪size())

for row in relations13_groups[relations13_groups['graphs'].apply(lambda graphs:␣
↪len(graphs)>0 and graphs[0].excess==3)].itertuples():
    break
    print("A2Comp_ID_set: ", row.A2Comp_ID_set)
    for G in row.graphs: G.show()
```

```
relations13_groups size:
excess  edges_plus_n  group_length
2       12            2               1
3       12            4               2
        13            4               2
4       12            4               3
                      7               1
        13            2               1
                      4               2
                      7               5
        14            3               1
                      4               2
                      5               1
                      7               2
dtype: int64
```

[76]:
```
# we choose manually which graphs to discard from each relation group in order␣
↪to form a basis
relation_discard_ID = [
    71,69,75,72,79,96, # excess 3, edges 12-n
    108,105,111,107, # excess 3, edges 13-n
    273, 269, 286, 279, 284, 272, 281, 275, 280, 271, 288,  297, 353, 318, 347,␣
 ↪# excess 4, edges 12-n
    426, 408, 418, 406, 420, 404, 413, 403, 434, 419, 433, 412, 436, 415, 432,␣
 ↪414, 445, 435, 444, 431, 442,427, 439, 416, 497, # excess 4, edges 13-n, A2
    451, 526, 464, 468, 532, # excess 4, edges 13-n, B,Birr
    575, 570, 551, 546, 564, 573, 567, 559, 548, 571, 558, 569,557, 568, 554,␣
 ↪562, 550, # excess 4, edges 14-n, A2
]

if len(relation_discard_ID)!=len(set(relation_discard_ID)): raise␣
 ↪ValueError("relation discard list has duplicates")

# for most virtual blowups we choose (manually, after having analyzed all) a␣
 ↪blowup whose coefficient in the image under the differential is not zero
# of course, one has to be sure of the fact that the IDs of the graphs didn't␣
 ↪change (for example due to reordering) from the time I hardcoded this␣
 ↪dictionary
differential_mapping_ID = {
    0:1, # excess 1
    3:10, 4:8, 5:9, # excess 2, edges 10-n
    6:17, 7:18, 11:19, 12:20, 14:22, # excess 2, edges 11-n
    23:46, 24:34, 25:39, 26:41, 27:44, 28:45, # excess 3, edges 10-n
    29:78, 30:73, 31:77, 32:61, 33:82, 35:62, 36:64, 37:65, 38:84, 40:66, 42:
 ↪67, 43:68, # excess 3, edges 11-n, A3
    47:86, 48:89, 49:88, 50:91, 52:94, 51:93, 53:100, 54:101, 55:103, 56:104, #␣
 ↪excess 3, edges 11-n, B,Birr
    57:113, 58:115, 59:114, 60:110, 63:118, 76:112, 74:109, 80:116, 81:117, 87:
 ↪121, 85:120, 83:119, 90:122, 92:123, 95:124, 99:127, 102:128, # excess 3,␣
 ↪edges 12-n
    # EXCESS 4 BY EXACT N RANGE.
    # n range 1
    133:167, 134:170, 135:171, 136:178, 137:181, 138:183, # excess 4, n range␣
 ↪1, edges 10-n
    168:255, 169:256, 172:257, 173:261, 174:262, 175:263, 176:264, 177:265, 179:
 ↪266, 180:267, 182:268, 190:327, 193:335, 191:329, 195:341, 196:342, 194:337,␣
 ↪192:333, 201:364, 202:366, # excess 4, n range: 1, edges 11-n
    258:399, 259:400, 260:401, 287:446, 286:444, 328:507, 338:514, 340:517, 336:
 ↪513, 330:509, 343:518, 345:521, 344:519, 332:511, 339:516, 334:512, 365:544,␣
 ↪367:545, 331:510, # excess 4, n range: 1, edges 12-n
    445:575, 515:608, 508:607, 520:609, # excess 4, n range: 1, edges 13-n
    # n range 4
    227:476, 228:387, 229:388, 230:390, 231:391, 279:431, 389:596, 432:569, 480:
 ↪599, 477:595, 479:598, 478:597, 537:617,
    # n range 4 1
    129:152, 130:148, 131:153, 132:166, # excess 4, n range: 4 1, edges 10-n
    147:243, 149:232, 150:233, 151:234, 154:235, 155:238, 156:239, 157:240, 158:
↪241, 159:242, 160:245, 161:247, 162:249, 163:250, 164:253, 165:251, 184:303,␣
↪185:310, 186:316, 187:317, 188:323, 189:322, 198:356, 199:358, 200:363, #␣
↪excess 4, n range: 4 1, edges 11-n
    236:392, 237:393, 244:394, 246:395, 248:396, 252:398, 254:397, 283:438, 285:
↪441, 284:439, 281:434, 282:437, 280:433, 301:484, 311:491, 307:488, 302:481,␣
↪315:496, 319:498, 320:499, 312:492, 313:494, 314:493, 308:487, 306:486, 321:
↪504, 324:505, 325:506, 326:503, 357:538, 359:539, 361:542, 362:543, 309:490,␣
↪305:483, 304:482, 318:497, 360:541, # excess 4, n range: 4 1, edges 12-n
    440:572, 436:571, 443:574, 435:570, 442:573, 500:603, 501:604, 495:602, 489:
↪601, 485:600, 502:605, 540:618, # excess 4, n range 4 1, edges 13-n
    # n range 7 4
    207:456, 208:452, 209:374, 210:453, 211:378, 212:379, 213:380, 214:381, 272:
↪416, 271:412,  # excess 4, n range: 7 4, edges 12-n
    375:586, 376:582, 377:561, 382:584, 418:562, 417:560, 415:558, 414:557, 460:
↪590, 458:588, 459:589, 457:587, 455:585, 454:583, 528:613, 529:615, # excess␣
↪4, n range: 7 4, edges 13-n
    # n range 7 4 1
    139:291, 140:219, 141:292, 142:215, 143:224, 144:220, 145:221, 146:225, 197:
↪351, # excess 4, n range 7 4 1, edges 11-n
    216:465, 217:384, 218:422, 222:429, 223:386, 226:385, 274:425, 278:430, 277:
↪424, 276:421, 293:469, 294:471, 295:423, 298:472, 299:474, 300:473, 349:535,␣
↪350:530, 355:536, 289:466, 290:462, 296:467, 354:533, 352:531, # excess 4, n␣
↪range 7 4 1, edges 12-n
    383:565, 428:566, 470:591, 475:592, 463:593, 461:594, 534:616, # excess 4,␣
↪n range 7 4 1, edges 13-n
    # n range 10 7 4
    203:448, 204:372, 205:373, 269:408, 369:578, 370:577, 371:553, 409:552, 449:
↪581, 450:580, 522:611, 524:612, 447:579,
    # n range 10 7 4 1
    206:410, 270:525, 411:556,
    # n range 13 10 7 4
    368:576,
}
differential_mapping_ID_inv = {v:k for k,v in differential_mapping_ID.items()}

if len(differential_mapping_ID.keys())!=len(set(v for v in␣
↪differential_mapping_ID.items())): raise ValueError(f"images of differential␣
↪have duplicates")
for (key,val) in differential_mapping_ID.items():
    if val in differential_mapping_ID.keys():
        blowups_virtual.loc[int(key)].graph.
↪show(name=True,title=True,gn_range=True,specht=True)
        blowups_virtual.loc[int(val)].graph.
↪show(name=True,title=True,gn_range=True,specht=True)
```

```
        blowups_virtual.loc[differential_mapping_ID[val]].graph.
↪show(name=True,title=True,gn_range=True,specht=True)
        raise ValueError(f"differential is defined for an ID in its image, this␣
↪is not necessary")
for key,val in differential_mapping_ID.items():
    if blowups_virtual.loc[int(key)].excess!=blowups_virtual.loc[int(val)].
↪excess or blowups_virtual.loc[int(key)].edges_plus_n + 1 != blowups_virtual.
↪loc[int(val)].edges_plus_n:
        raise ValueError("differential argument and image have different excess␣
↪or consecutive edges")
    if blowups_virtual.loc[int(key)].gn_range != blowups_virtual.loc[int(val)].
↪gn_range or blowups_virtual.loc[int(key)].specht_sequence != blowups_virtual.
↪loc[int(val)].specht_sequence :
        if key==int(368): continue # there is one graph which is actually␣
↪compatible with its image, but the specht module computed is wrong bc of␣
↪weight 2 relations
        blowups_virtual.loc[int(key)].graph.
↪show(name=True,title=True,gn_range=True,specht=True)
        blowups_virtual.loc[int(val)].graph.
↪show(name=True,title=True,gn_range=True,specht=True)
        raise ValueError("differential argument and image have different␣
↪gn_range or specht sequence")
    if key in relation_discard_ID and not val in relation_discard_ID:
        blowups_virtual.loc[int(key)].graph.
↪show(name=True,title=True,gn_range=True,specht=True)
        blowups_virtual.loc[int(val)].graph.
↪show(name=True,title=True,gn_range=True,specht=True)
        raise ValueError("differential discarded argument maps to non discarded␣
↪element")
    elif not key in relation_discard_ID and val in relation_discard_ID:
        blowups_virtual.loc[int(key)].graph.
↪show(name=True,title=True,gn_range=True,specht=True)
        blowups_virtual.loc[int(val)].graph.
↪show(name=True,title=True,gn_range=True,specht=True)
        raise ValueError("differential argument maps to discarded element")

differential_mapping_string = {}
for key in differential_mapping_ID.keys(): differential_mapping_string[key] =␣
↪r"$\mapsto $ID "+str(differential_mapping_ID[key])
for key in differential_mapping_ID_inv.keys(): differential_mapping_string[key]␣
↪= r"$\mapsfrom $ID "+str(differential_mapping_ID_inv[key])

for id in relation_discard_ID:
    #if id in differential_mapping_string.keys():␣
↪differential_mapping_string[id] += " RED"
    #else:
```

```
    differential_mapping_string[id] = "Redundant"    # even if we computed the␣
    ↪elimination, if its discarded bc of weight relations, overwrite its string

reduced_blowups_virtual = blowups_virtual[blowups_virtual['graph'].apply(lambda␣
  ↪G: not (G.ID in relation_discard_ID))]
print(f"blowups_virtual length: {len(blowups_virtual)}, redundant:␣
  ↪{len(relation_discard_ID)}, remaining: {len(reduced_blowups_virtual)}")
print(f"differential mapping arguments:  {len(differential_mapping_ID.keys())},␣
  ↪ images:  {len(differential_mapping_ID.values())}")
print(f"total eliminated: {len(differential_mapping_string.keys())}")

print("blowups_virtual in excess 3 without redundant:")
print(reduced_blowups_virtual[reduced_blowups_virtual['excess']==3].
  ↪groupby(['edges_plus_n','crossType']).size())
```

```
blowups_virtual length: 619, redundant: 72, remaining: 547
differential mapping arguments:  278,  images:  278
total eliminated: 594
blowups_virtual in excess 3 without redundant:
edges_plus_n  crossType
10            A             6
11            A            17
              B             7
              Birr          4
12            A            12
              A2            4
              B            18
              Birr          8
13            A2            4
              B            12
              Birr          4
dtype: int64
```

[ ]:
```
from matplotlib.backends.backend_pdf import PdfPages
plt.rcParams['figure.constrained_layout.use'] = True

window_width = 8.5      # pdf sizes in inches
blowup_per_row = 4      # how many blowups to print per row
blowup_height = 1.3     # height to calculate for each blowup
window_height_buffer = 0.6 # window height will be determined as␣
  ↪blowup_height*rows+window_height_buffer
w_pad = 0.3             # padding between the subplots of each blowup␣
  ↪representations in the matplotlib figure
h_pad = 0.2
wspace = 0
```

```
# show in the output or save to pdf file the list of virtual blowup␣
↪representations of a given excess
def plot_VirtualBlowups(excess, exact_gn_range=None, show_differential=False,␣
↪show_eliminated=True, show_redundant=False, force_show_relations=False,␣
↪save_pdf=False):
    pdf = {}
    if save_pdf:
        if exact_gn_range==None:
            if show_eliminated: pdf = PdfPages(f'Results/Excess{excess}/
↪VirtualBlowups_excess{excess}.pdf')
            else: pdf = PdfPages(f'Results/Excess{excess}/
↪VirtualBlowups_excess{excess}_eliminated.pdf')
        else: pdf = PdfPages(f'Results/Excess{excess}/
↪VirtualBlowups_gn_range{exact_gn_range}.pdf')


    gn_string=""
    for gn in gn_table[excess]: gn_string+= ("" if len(gn_string)==0 else ", ")␣
↪+ str(gn)
    excess_group = get_blowups_virtual(excess=excess)
    eliminated = excess_group['graph'].apply(lambda G: G.ID in␣
↪differential_mapping_string.keys()).sum()

    excess_title = f'----  excess: {excess}      (g,n): {gn_string}      graphs:
↪ {len(excess_group)}    eliminated+redundant: {eliminated}  ----\n'
    first_group = True

    for edges_plus_n,edge_group in excess_group.groupby('edges_plus_n'):
        edge_title = (excess_title if first_group else "") + f'---------    ␣
↪edges: {edges_plus_n}-n       graphs: {len(edge_group)}      ---------\n'
        print(edge_title)
        first_group=False
        first_edge_group = True

        graphics = []
        count=0
        for row in edge_group[edge_group['crossType']=='A'].itertuples():
            if exact_gn_range!=None and row.graph.virtual_gn_range!
↪=exact_gn_range: continue
            if show_eliminated!=True and (row.graph.ID in␣
↪differential_mapping_ID.keys() or row.graph.ID in differential_mapping_ID.
↪values()): continue
            if count%blowup_per_row==0: graphics.append([])
            graphics[-1].append(row.graph.
↪show(title=True,gn_range=True,specht=True,graphics=True,␣
↪append=differential_mapping_string.get(row.graph.ID,"") if show_differential␣
↪else ""))
            count+=1
        while count%blowup_per_row!=0:
            graphics[-1].append(text(' ',(0,0),axes=False))
            count+=1
        if len(graphics)!=0:
            title = (edge_title if first_edge_group else "") + f"---  A3 case  ␣
↪   graphs: {(edge_group['crossType']=='A').sum()}  ---"
            fig = plt.figure()
            fig.suptitle(title,fontsize=10)
            fig.
↪set_size_inches(window_width,blowup_height*len(graphics)+window_height_buffer*(2␣
↪if first_edge_group else 1))
            fig.get_layout_engine().set(w_pad=w_pad, h_pad=h_pad,wspace=wspace)
            first_edge_group = False
            graphics_array(graphics).matplotlib(figure=fig)
            if save_pdf: pdf.savefig(fig)
            else: plt.show()
            plt.close()

        graphics = []
        count=0
        for rel13group in␣
↪relations13_groups[(relations13_groups['excess']==excess) &␣
↪(relations13_groups['edges_plus_n']==edges_plus_n)].itertuples():
            #if count!=0: graphics.append([text('␣
↪',(0,0),axes=False)]*blowup_per_row) # buffer space between relation groups
            for G in rel13group.graphs:
                if exact_gn_range!=None and G.virtual_gn_range!=exact_gn_range:␣
↪continue
                if show_redundant==False and G.ID in relation_discard_ID:␣
↪continue
                if force_show_relations==False and show_eliminated!=True and (G.
↪ID in differential_mapping_ID.keys() or G.ID in differential_mapping_ID.
↪values()): continue
                if count%blowup_per_row==0: graphics.append([])
                graphics[-1].append(G.
↪show(title=True,gn_range=True,specht=True,graphics=True,␣
↪append=differential_mapping_string.get(G.ID,"") if show_differential else␣
↪""))
                count+=1
            while count%blowup_per_row!=0:
                graphics[-1].append(text(' ',(0,0),axes=False))
                count+=1
        if len(graphics)!=0:
            title = (edge_title if first_edge_group else "") + f"--- A2 case␣
↪with weight 13 relations      relation groups:␣
↪{len(get_relations13_groups(excess=excess,edges_plus_n=edges_plus_n))} ---"
            fig = plt.figure()
            fig.suptitle(title,fontsize=10)
            fig.
↪set_size_inches(window_width,blowup_height*len(graphics)+window_height_buffer*(2␣
↪if first_edge_group else 1))
            fig.get_layout_engine().set(w_pad=w_pad, h_pad=h_pad,wspace=wspace)
            first_edge_group = False
            graphics_array(graphics).matplotlib(figure=fig)
            if save_pdf: pdf.savefig(fig)
            else: plt.show()
            plt.close()

        graphics = []
        count=0
        for rel11group in␣
↪get_relations11_groups(excess=excess,edges_plus_n=edges_plus_n,group_length=1).
↪itertuples():
            G = rel11group.graphs[0]
            if exact_gn_range!=None and G.virtual_gn_range!=exact_gn_range:␣
↪continue
            if show_eliminated!=True and (G.ID in differential_mapping_ID.
↪keys() or G.ID in differential_mapping_ID.values()): continue
            if count%blowup_per_row==0: graphics.append([])
            graphics[-1].append(G.
↪show(title=True,gn_range=True,specht=True,graphics=True,␣
↪append=differential_mapping_string.get(G.ID,"") if show_differential else␣
↪""))
            count+=1
        while count%blowup_per_row!=0:
            graphics[-1].append(text(' ',(0,0),axes=False))
            count+=1
        if len(graphics)!=0:
            title = (edge_title if first_edge_group else "") + f"--- B,Birr␣
↪cases without weight 11 relations      graphs:␣
↪{len(get_relations11_groups(excess=excess,edges_plus_n=edges_plus_n,group_length=1))}␣
↪---"
            fig = plt.figure()
            fig.suptitle(title,fontsize=10)
            fig.
↪set_size_inches(window_width,blowup_height*len(graphics)+window_height_buffer*(2␣
↪if first_edge_group else 1))
```

```
            fig.get_layout_engine().set(w_pad=w_pad, h_pad=h_pad,wspace=wspace)
            first_edge_group = False
            graphics_array(graphics).matplotlib(figure=fig)
            if save_pdf: pdf.savefig(fig)
            else: plt.show()
            plt.close()

        graphics = []
        count=0
        for rel11group in␣
↪get_relations11_groups(excess=excess,edges_plus_n=edges_plus_n,group_length=-1).
↪itertuples():
            #if count!=0: graphics.append([text('␣
↪',(0,0),axes=False)]*blowup_per_row) # buffer space between relation groups
            for G in rel11group.graphs:
                if exact_gn_range!=None and G.virtual_gn_range!=exact_gn_range:␣
↪continue
                if show_redundant==False and G.ID in relation_discard_ID:␣
↪continue
                if force_show_relations==False and show_eliminated!=True and (G.
↪ID in differential_mapping_ID.keys() or G.ID in differential_mapping_ID.
↪values()): continue
                if count%blowup_per_row==0: graphics.append([])
                graphics[-1].append(G.
↪show(title=True,gn_range=True,specht=True,graphics=True,␣
↪append=differential_mapping_string.get(G.ID,"") if show_differential else␣
↪""))
                count+=1
            while count%blowup_per_row!=0:
                graphics[-1].append(text(' ',(0,0),axes=False))
                count+=1
        if len(graphics)!=0:
            title = (edge_title if first_edge_group else "") + f"--- B,Birr␣
↪cases with weight 11 relations     relation groups:␣
↪{len(get_relations11_groups(excess=excess,edges_plus_n=edges_plus_n,␣
↪group_length=-1))} ---"
            fig = plt.figure()
            fig.suptitle(title,fontsize=10)
            fig.
↪set_size_inches(window_width,blowup_height*len(graphics)+window_height_buffer*(2␣
↪if first_edge_group else 1))
            fig.get_layout_engine().set(w_pad=w_pad, h_pad=h_pad,wspace=wspace)
            first_edge_group = False
            graphics_array(graphics).matplotlib(figure=fig)
            if save_pdf: pdf.savefig(fig)
            else: plt.show()
            plt.close()

        # if no graphs where added from this edge group, print a blank page
        if first_edge_group:
            fig = plt.figure()
            fig.suptitle(edge_title, fontsize=10)
            fig.set_size_inches(window_width,window_height_buffer*2)
            first_edge_group = False
            fig.add_subplot(1,1,1).set_axis_off()
            if save_pdf: pdf.savefig(fig)
            else: plt.show()
            plt.close()

    if save_pdf: pdf.close()

#for e in range(1,MAX_EXCESS+1): plot_VirtualBlowups(e, show_differential=True,␣
 ↪show_eliminated=False, save_pdf=True)
# plot_VirtualBlowups(4, save_pdf=True, show_differential=True,␣
 ↪show_eliminated=True, exact_gn_range=((1,13),(3,10),(5,7),(7,4),))
plot_VirtualBlowups(3, save_pdf=False, show_differential=True,␣
 ↪show_eliminated=True, force_show_relations=False, show_redundant=True)
```

```
----  excess: 3      (g,n): (2, 11), (4, 8), (6, 5), (8, 2)      graphs: 106
eliminated+redundant: 100  ----
---------     edges: 10-n       graphs: 6      ---------

---------     edges: 11-n       graphs: 28      ---------

---------     edges: 12-n       graphs: 48      ---------

---------     edges: 13-n       graphs: 24      ---------
```

[78]:
```
blowups_complete = pd.
 ↪DataFrame(columns=['excess','partition','g','n','crossType','vertices','edges','graph','spe
# Completions of virtual blowups to an actual blowup representation by adding␣
 ↪oj_hair and tripod components.
# Creates a blowupgraph object for every possible g,n tuple in the gn_range of␣
 ↪the virtual blowup, so potentially uses a lot of memory.

for blowup in blowups_virtual.itertuples():
    blowup.graph.gn_completions = {}
    for (g,n) in blowup.gn_range:
        oj_hairs = n-blowup.graph.n
        tripods = ((g-blowup.graph.g)/2)
        if 11!=3*tripods+blowup.graph.ome+oj_hairs: raise ValueError("mismatch␣
 ↪in number of omega hairs of completion")
```

```
        components = [OJ_HAIR for x in range(oj_hairs)] + [TRIPOD for x in␣
↪range(tripods)]
        components = components + list(blowup.graph.components)
        G = BlowUpGraph(components)
        if G.edges!=blowup.edges_plus_n-n: raise ValueError("mismatch in number␣
↪of edges of completion")
        G.ID = blowup.graph.ID # inherit the ID of the virtual blowup, as it is␣
↪unique for every (g,n) pair
        blowup.graph.gn_completions[g]=G
        blowups_complete.loc[len(blowups_complete)] = {'graph':G,'crossType':G.
↪crossType,'excess':blowup.excess,'partition':G.partition,'g':g,'n':
↪n,'vertices':G.vertices,'edges':G.edges,'oj_hairs':oj_hairs,'tripods':
↪tripods,'specht_sequence':G.specht_sequence}
blowups_complete.
↪sort_values(by=['excess','g','n','edges','crossType','vertices'],␣
↪inplace=True)
print("blowups_complete size: ")
print(blowups_complete.groupby(['excess','g','n']).size())
```

```
blowups_complete size:
excess  g  n
1       2  10      1
        4  7       3
        6  4       3
        8  1       3
2       1  12      1
        3  9       6
        5  6      16
        7  3      20
        9  0      13
3       2  11      9
        4  8      40
        6  5      84
        8  2     105
4       1  13      5
        3  10     55
        5  7     207
        7  4     392
        9  1     364
dtype: int64
```

[93]:
```
def␣
↪get_blowups_complete(excess=None,partition=None,vertices=None,crossType=None,␣
↪gn=None):
    frame = blowups_complete
    conditions = frame['excess']>=0
    if crossType!=None:
        if crossType=='BB': conditions = conditions & (frame['crossType']=='B')␣
 ↪| (frame['crossType']=='Birr')
        else: conditions = conditions & (frame['crossType']==crossType)
    if excess!=None: conditions = conditions & (frame['excess']==excess)
    if gn!=None: conditions = conditions & (frame['g']==gn[0]) &␣
 ↪(frame['n']==gn[1])
    if partition!=None: conditions = conditions &␣
 ↪(frame['partition']==partition)
    if vertices!=None: conditions = conditions & (frame['vertices']==vertices)
    return frame[conditions]

# compute euler characteristic, WITHOUT TAKING CARE OF WEIGHT 13 AND 11␣
 ↪RELATIONS
# ONLY TAKES INTO ACCOUNT SPECHT MODULES WITH n SMALLER OR EQUAL TO 3
def EulerCharacteristic(g,n):
    excess_group = get_blowups_virtual(gn=(g,n))
    contributions = pd.
 ↪DataFrame(columns=['edges','V0','V1','V2','V11','V1xV1','V1xV1xV1','V1xV2','V1xV11','V3','V
    for row in excess_group.itertuples():
        l = len(row.specht_sequence)
        V1 = row.specht_sequence.count((1,1))
        V2 = row.specht_sequence.count((2,1))
        V11 = row.specht_sequence.count((2,-1))
        V3 = row.specht_sequence.count((3,1))
        V111 = row.specht_sequence.count((3,-1))
        if V1+V2+V11+V3+V111!=l:
            print(row.specht_sequence)
            raise ValueError("unhandeled specht contribution")

        contributions.loc[len(contributions)] = {'edges':row.edges_plus_n-n,␣
 ↪'V0':1 if l==0 else 0, 'V1':V1 if l==1 else 0, 'V2':V2 if l==1 else 0, 'V11':
 ↪V11 if l==1 else 0, 'V1xV1':1 if (V1==2 and l==2) else 0, 'V1xV1xV1':1 if␣
 ↪(V1==3 and l==3) else 0, 'V1xV2': 1 if (V1==1 and V2==1 and l==2) else 0,␣
 ↪'V1xV11': 1 if (V1==1 and V11==1 and l==2) else 0, 'V3': V3 if l==1 else 0,␣
 ↪'V111': V111 if l==1 else 0 }

    return contributions

    # alternating sum of all the specht module contributions. sign -1 when the␣
 ↪degree 13 + edges = 13 + edges_plus_n - n is odd
    for edges_plus_n,edge_group in excess_group.groupby('edges_plus_n'):
        contributions[contributions['edges']==edges_plus_n-n] =␣
 ↪contributions[contributions['edges']==edges_plus_n-n].mul( 1 if␣
 ↪(edges_plus_n-n)%2!=0 else -1 )
```

```
    summed = ␣
↪contributions[['V0','V1','V2','V11','V1xV1','V1xV1xV1','V1xV2','V1xV11','V3','V111']].
↪sum(axis=0)


    specht_string=""
    if summed.V0!=0: specht_string += r"$"+str(summed.
↪V0)+r"$"+specht_sequence_to_string([(n,-1 if n>1 else 1)])
    if summed.V1!=0: specht_string += r"$"+("+" if len(specht_string)!=0 and␣
↪summed.V1>=0 else "")+str(summed.V1)+r"$"+specht_sequence_to_string([(n-1,-1␣
↪if n-1>1 else 1),(1,1)])
    if summed.V2!=0: specht_string += r"$"+("+" if len(specht_string)!=0 and␣
↪summed.V2>=0 else "")+str(summed.V2)+r"$"+specht_sequence_to_string([(n-2,-1␣
↪if n-2>1 else 1),(2,1)])
    if summed.V11!=0: specht_string += r"$"+("+" if len(specht_string)!=0 and␣
↪summed.V11>=0 else "")+str(summed.
↪V11)+r"$"+specht_sequence_to_string([(n-2,-1 if n-2>1 else 1),(2,-1)])
    if summed.V1xV1!=0: specht_string += r"$"+("+" if len(specht_string)!=0 and␣
↪summed.V1xV1>=0 else "")+str(summed.
↪V1xV1)+r"$"+specht_sequence_to_string([(n-2,-1 if n-2>1 else 1),(1,1),(1,1)])
    if summed.V3!=0: specht_string += r"$"+("+" if len(specht_string)!=0 and␣
↪summed.V3>=0 else "")+str(summed.V3)+r"$"+specht_sequence_to_string([(n-3,-1␣
↪if n-3>1 else 1),(3,1)])
    if summed.V111!=0: specht_string += r"$"+("+" if len(specht_string)!=0 and␣
↪summed.V111>=0 else "")+str(summed.
↪V111)+r"$"+specht_sequence_to_string([(n-3,-1 if n-3>1 else 1),(3,-1)])
    if summed.V1xV2!=0: specht_string += r"$"+("+" if len(specht_string)!=0 and␣
↪summed.V1xV2>=0 else "")+str(summed.
↪V1xV2)+r"$"+specht_sequence_to_string([(n-3,-1 if n-3>1 else 1),(1,1),(2,1)])
    if summed.V1xV11!=0: specht_string += r"$"+("+" if len(specht_string)!=0␣
↪and summed.V1xV11>=0 else "")+str(summed.
↪V1xV11)+r"$"+specht_sequence_to_string([(n-3,-1 if n-3>1 else␣
↪1),(1,1),(2,-1)])
    if summed.V1xV1xV1!=0: specht_string += r"$"+("+" if len(specht_string)!=0␣
↪and summed.V1xV1xV1>=0 else "")+str(summed.
↪V1xV1xV1)+r"$"+specht_sequence_to_string([(n-3,-1 if n-3>1 else␣
↪1),(1,1),(1,1),(1,1)])
    return specht_string

(g,n)=(4,8)
for row in get_blowups_virtual(gn=(g,n), crossType='A2').itertuples():
    break
    row.graph.gn_completions[g].show(title=True, specht=True)

contributions = EulerCharacteristic(g,n)
#print(contributions)
```

```
for edges,edge_group in contributions.groupby('edges'):
    print(g,n,edges)
    print(edge_group.sum())
```

```
4 8 3
edges        15
V0            2
V1            3
V2            0
V11           0
V1xV1         0
V1xV1xV1      0
V1xV2         0
V1xV11        0
V3            0
V111          0
dtype: object
4 8 4
edges        84
V0            4
V1           12
V2            4
V11           0
V1xV1         1
V1xV1xV1      0
V1xV2         0
V1xV11        0
V3            0
V111          0
dtype: object
4 8 5
edges        70
V0            0
V1            5
V2            5
V11           0
V1xV1         3
V1xV1xV1      0
V1xV2         1
V1xV11        0
V3            0
V111          0
dtype: object
```

[81]:
```
from matplotlib.backends.backend_pdf import PdfPages
plt.rcParams['figure.constrained_layout.use'] = True
```

```
window_width = 8.5      # pdf sizes in inches
blowup_per_row = 4      # how many blowups to print per row
blowup_height = 1.3     # height to calculate for each blowup
window_height_buffer = 0.6 # window height will be determined as␣
 ↪blowup_height*rows+window_height_buffer
w_pad = 0.15             # padding between the subplots of each blowup␣
 ↪representations in the matplotlib figure
h_pad = 0.2
wspace = 0

# show in the output or save to pdf file the list of virtual blowup␣
 ↪representations of a given excess
def plot_CompleteBlowups(g,n, save_pdf=False, show_differential=False,␣
 ↪show_eliminated=True, force_show_relations=False, show_redundant=False):
    excess = 3*g-3+2*n-22
    pdf = {}
    if save_pdf:
        if show_eliminated: pdf = PdfPages(f'Results/Excess{excess}/
 ↪CompletedBlowups_({g},{n}).pdf')
        else:  pdf = PdfPages(f'Results/Excess{excess}/
 ↪CompletedBlowups_({g},{n})_eliminared.pdf')

    gn_group = get_blowups_complete(gn=(g,n))
    EulerChar = EulerCharacteristic(g,n)

    eliminated = gn_group['graph'].apply(lambda G: G.ID in␣
 ↪differential_mapping_string.keys()).sum()
    excess_title = f'-------    g,n: {g},{n}    graphs: {len(gn_group)}   ␣
 ↪eliminated+redundant: {eliminated}  -------'
    excess_title+= f"\n  Euler Characteristic (without relations):  ␣
 ↪{EulerChar}\n"
    first_group = True


    for edges,edge_group in gn_group.groupby('edges'):
        edge_title= (excess_title if first_group else "") + f'---------    ␣
 ↪edges: {edges}      graphs: {len(edge_group)}      ---------\n'
        print(edge_title)
        first_group=False
        first_edge_group = True

        graphics = []
        count=0
        for row in edge_group[edge_group['crossType']=='A'].itertuples():
            if show_redundant==False and row.graph.ID in relation_discard_ID:␣
 ↪continue
```

```
            if show_eliminated!=True and (row.graph.ID in␣
↪differential_mapping_ID.keys() or row.graph.ID in differential_mapping_ID.
↪values()): continue
            if count%blowup_per_row==0: graphics.append([])
            graphics[-1].append(row.graph.
↪show(title=True,specht=True,graphics=True,␣
↪append=differential_mapping_string.get(row.graph.ID,"") if show_differential␣
↪else ""))
            count+=1
        while count%blowup_per_row!=0:
            graphics[-1].append(text(' ',(0,0),axes=False))
            count+=1
        if len(graphics)!=0:
            title = (edge_title if first_edge_group else "") + f"-------  A3␣
↪case       graphs: {(edge_group['crossType']=='A').sum()}  -------"
            fig = plt.figure()
            fig.suptitle(title, fontsize=10)
            fig.
↪set_size_inches(window_width,blowup_height*len(graphics)+window_height_buffer*(2␣
↪if first_edge_group else 1))
            fig.get_layout_engine().set(w_pad=w_pad, h_pad=h_pad,wspace=wspace)
            first_edge_group = False
            graphics_array(graphics).matplotlib(figure=fig)
            if save_pdf: pdf.savefig(fig)
            else: plt.show()
            plt.close()

        graphics = []
        count=0
        for rel13group in get_relations13_groups(excess,edges+n,(g,n)).
↪itertuples():
            #if count!=0: graphics.append([text('␣
↪',(0,0),axes=False)]*blowup_per_row) # buffer space between relation groups
            for G in rel13group.graphs:
                if (g,n) in G.virtual_gn_range:
                    if show_redundant==False and G.ID in relation_discard_ID:␣
↪continue
                    if force_show_relations==False and show_eliminated!=True␣
↪and (G.ID in differential_mapping_ID.keys() or G.ID in␣
↪differential_mapping_ID.values()): continue
                    if count%blowup_per_row==0: graphics.append([])
                    graphics[-1].append(G.gn_completions[g].
↪show(title=True,specht=True,graphics=True,␣
↪append=differential_mapping_string.get(G.ID,"") if show_differential else␣
↪""))
                    count+=1
            while count%blowup_per_row!=0:
                graphics[-1].append(text(' ',(0,0),axes=False))
                count+=1
        if len(graphics)!=0:
            title = (edge_title if first_edge_group else "") + f"------- A2␣
↪case with weight 13 relations       relation groups:␣
↪{len(get_relations13_groups(excess,edges+n,(g,n)))} -------"
            fig = plt.figure()
            fig.suptitle(title, fontsize=10)
            fig.
↪set_size_inches(window_width,blowup_height*len(graphics)+window_height_buffer*(2␣
↪if first_edge_group else 1))
            fig.get_layout_engine().set(w_pad=w_pad, h_pad=h_pad,wspace=wspace)
            first_edge_group = False
            graphics_array(graphics).matplotlib(figure=fig)
            if save_pdf: pdf.savefig(fig)
            else: plt.show()
            plt.close()

        graphics = []
        count=0
        for rel11group in get_relations11_groups(excess,edges+n,(g,n),1).
↪itertuples():
            G = rel11group.graphs[0]
            if (g,n) in rel11group.graphs[0].virtual_gn_range:
                if show_redundant==False and G.ID in relation_discard_ID:␣
↪continue
                if show_eliminated!=True and (G.ID in differential_mapping_ID.
↪keys() or G.ID in differential_mapping_ID.values()): continue
                if count%blowup_per_row==0: graphics.append([])
                graphics[-1].append(G.gn_completions[g].
↪show(title=True,specht=True,graphics=True,␣
↪append=differential_mapping_string.get(G.ID,"") if show_differential else␣
↪""))
                count+=1
        while count%blowup_per_row!=0:
            graphics[-1].append(text(' ',(0,0),axes=False))
            count+=1
        if len(graphics)!=0:
            title = (edge_title if first_edge_group else "") + f"------- ␣
↪B,Birr cases without weight 11 relations       graphs:␣
↪{len(get_relations11_groups(excess,edges+n,(g,n),1))}  -------"
            fig = plt.figure()
            fig.suptitle(title, fontsize=10)
            fig.
↪set_size_inches(window_width,blowup_height*len(graphics)+window_height_buffer*(2␣
↪if first_edge_group else 1))
            fig.get_layout_engine().set(w_pad=w_pad, h_pad=h_pad,wspace=wspace)
            first_edge_group = False
            graphics_array(graphics).matplotlib(figure=fig)
            if save_pdf: pdf.savefig(fig)
            else: plt.show()
            plt.close()

        graphics = []
        count=0
        for rel11group in␣
↪get_relations11_groups(excess,edges+n,(g,n),group_length=-1).itertuples():
            #if count!=0: graphics.append([text('␣
↪',(0,0),axes=False)]*blowup_per_row) # buffer space between relation groups
            for G in rel11group.graphs:
                if (g,n) in G.virtual_gn_range:
                    if show_redundant==False and G.ID in relation_discard_ID:␣
↪continue
                    if force_show_relations==False and show_eliminated!=True␣
↪and (G.ID in differential_mapping_ID.keys() or G.ID in␣
↪differential_mapping_ID.values()): continue
                    if count%blowup_per_row==0: graphics.append([])
                    graphics[-1].append(G.gn_completions[g].
↪show(title=True,specht=True,graphics=True,␣
↪append=differential_mapping_string.get(G.ID,"") if show_differential else␣
↪""))
                    count+=1
            while count%blowup_per_row!=0:
                graphics[-1].append(text(' ',(0,0),axes=False))
                count+=1
        if len(graphics)!=0:
            title = (edge_title if first_edge_group else "") + f"------- ␣
↪B,Birr cases with weight 11 relations        relation groups:␣
↪{len(get_relations11_groups(excess,edges+n,(g,n),group_length=-1))}  -------"
            fig = plt.figure()
            fig.suptitle(title, fontsize=10)
            fig.
↪set_size_inches(window_width,blowup_height*len(graphics)+window_height_buffer*(2␣
↪if first_edge_group else 1))
            fig.get_layout_engine().set(w_pad=w_pad, h_pad=h_pad,wspace=wspace)
            first_edge_group = False
            graphics_array(graphics).matplotlib(figure=fig)
            if save_pdf: pdf.savefig(fig)
            else: plt.show()
            plt.close()

        # if no graphs where added from this edge group, print a blank page
        if first_edge_group:
            fig = plt.figure()
            fig.suptitle(edge_title, fontsize=10)
            fig.set_size_inches(window_width,window_height_buffer*2)
            first_edge_group = False
            fig.add_subplot(1,1,1).set_axis_off()
            if save_pdf: pdf.savefig(fig)
            else: plt.show()
            plt.close()

    if save_pdf: pdf.close()

for e in range(1,MAX_EXCESS):
    for (g,n) in gn_table[4]:
        break
        plot_CompleteBlowups(g,n, show_differential=True, save_pdf=True,␣
 ↪show_eliminated=False,force_show_relations=False, show_redundant=False)
plot_CompleteBlowups(2,11, save_pdf=False, show_differential=True,␣
 ↪show_eliminated=True, force_show_relations=False, show_redundant=True)
```

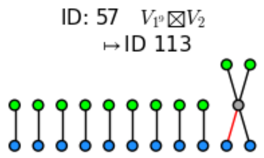
```
-------    g,n: 2,11    graphs: 9    eliminated+redundant: 5  -------
  Euler Characteristic (without relations):
$1$$V_{1^{11}}$$+2$$V_{1^{10}}\boxtimes V_{1}$$-1$$V_{1^{9}}\boxtimes
V_{1}\boxtimes V_{1}$$-1$$V_{1^{8}}\boxtimes V_{1}\boxtimes V_{2}$
---------     edges: 1      graphs: 5      ---------
```

------- g,n: 2,11 graphs: 9 eliminated+redundant: 5 -------

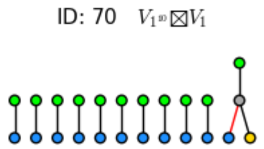
Euler Characteristic (without relations): $1V_{1^{11}}+2V_{1^{10}}\boxtimes V_1-1V_{1^9}\boxtimes V_1\boxtimes V_1-1V_{1^8}\boxtimes V_1\boxtimes V_2$

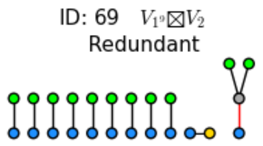
--------- edges: 1 graphs: 5 ---------

------- A3 case graphs: 1 -------

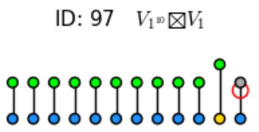


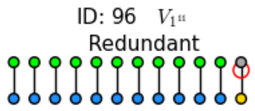
------- A2 case with weight 13 relations relations groups: 1 -------





------- B,Birr cases with weight 11 relations relation groups: 1 -------

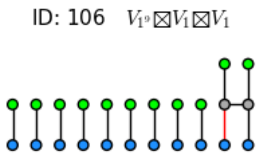


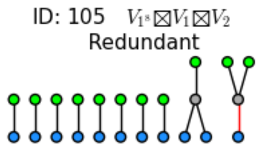


`---------      edges: 2      graphs: 4      ---------`

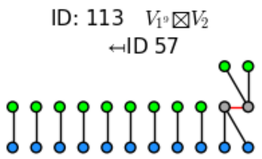
--------- edges: 2 graphs: 4 ---------

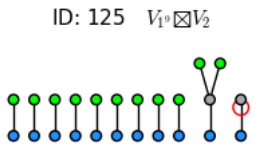
------- A2 case with weight 13 relations relation groups: 1 -------





------- B,Birr cases without weight 11 relations graphs: 2 -------


ID: 113 $V_{1^9} \boxtimes V_2$
↤ID 57


ID: 125 $V_{1^9} \boxtimes V_2$