Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

The intermediate Jacobian of the cubic threefold

Master's Thesis — DRAFT —

Marco Belli

November 28, 2025

Advisors: Prof. Dr. Rahul Pandharipande, Dr. Samir Canning

Department of Mathematics, ETH Zürich

Abstract

We begin by reviewing the history and modern theory of Abelian varieties, emphasizing their significance in complex geometry as invariants of more complicated algebraic varieties.

As a motivating example, we discuss the intermediate Jacobian of a smooth cubic threefold. Clemens and Griffiths showed that this principally polarized Abelian fivefold is naturally realized as the Albanese variety of the Fano surface of lines on the threefold. The interplay between the geometry of the Fano surface and the Hodge structure of the threefold yields strong birational information: in particular, the intermediate Jacobian furnishes a transcendental obstruction to rationality, leading to the classical result that the cubic threefold is not rational.

We then place this case study in a broader context by interpreting the intermediate Jacobian of the cubic threefold through the framework of Prym varieties. In this viewpoint, the resulting principally polarized Abelian variety can be described as the Prym variety of an étale double cover of a curve, and the Abel–Jacobi embedding of the Fano surface corresponds to the Abel–Prym map.

Time permitting, we conclude with a brief overview of related transcendental methods in algebraic geometry and their applications to questions of rationality and classification.

Contents

Chapter 1

Introduction

Motivation: Jacobians as birational invariants

The cubic threefold as a case study

Overview of results

Chapter 2

Abelian Varieties and Hodge Structures

2.1 Riemann forms on complex tori

2.2 Hodge structures of weight 1 and 3

2.3 Albanese Varieties and Jacobians of curves

2.3.1 Albanese functoriality

2.3.2 Abel-Jacobi maps and cycle theoretic interpretations

Chapter 3

The Geometry of the Cubic Threefold

3.1 Hodge structure

3.1.1 Hodge numbers and variation of Hodge structure

3.2 Fano Surface of Lines

3.3 The Albanese of the Fano Surface

Chapter 4

The Intermediate Jacobian

4.1 Construction of the Intermediate Jacobian $J^3(X)$

4.2 Abel-Jacobi map for the Fano Surface

4.3 The Clemens-Griffiths Isomorphism $\mathrm{Alb}(X) \cong J^3(X)$

4.4 Birational consequences

4.4.1 Birational invariance of $J^3(X)$

4.4.2 Comparison with Jacobians of curves

4.4.3 Obstruction to Rationality

Chapter 5

The Prym Interpretation

5.1 Étale Double Covers and Prym Varieties

5.2 Prym Variety associated with the Cubic Threefold

5.3 Abel-Jacobi vs Abel-Prym map

Appendix A

Cohomology and Line Bundles of Real and Complex Tori

In this appendix, we fix notation and recall the relationships between different cohomolgy theories on real and complex tori. Our exposition follows closely – but not exclusively – our reference on complex Abelian varieties [BL05, Chapters 1-3].

Real Tori

Let $\Lambda = \langle \lambda_i, \dots, \lambda_n \rangle$ be a free abelian group of rank n, $V = \Lambda \otimes \mathbb{R}$ the $\mathbb{R}$-vector space containing Λ as lattice and $X = V/\Lambda$ the n-dimensional torus. We denote

$$\lambda^i : \sum_j a_j \lambda_j \in V \mapsto a_i \in \mathbb{R}, \qquad \overline{\lambda}^i : \sum_j \overline{a_j} \lambda_j \in X \mapsto a_i \in [0,1)$$

the coordinate functions; λ^i will also be considered as a linear form in $\mathrm{Hom}(\Lambda, \mathbb{Z})$ or $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. The torus X has a basis of its singular cohomology consisting of the n cycles

$$\Lambda \xrightarrow{\sim} \mathrm{H}_1^{\mathrm{sin}}(X, \mathbb{Z}), \qquad \lambda_i \mapsto [\lambda_i^{\mathrm{sin}}], \qquad \left(\lambda_i^{\mathrm{sin}} : s \in [0,1] \mapsto \overline{s} \in X\right) \;\; \in \mathrm{C}_1^{\mathrm{sin}}(X, \mathbb{Z})\,.$$

The projection $\pi : V \twoheadrightarrow X$ is a universal covering, and its coordinate functions induce differential forms $d\lambda^i$ on X. These determine a basis of singular cohomology classes dual to the cycles above:

$$\mathrm{Hom}(\Lambda, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{sin}}(X, \mathbb{Z}), \quad \lambda^i \mapsto [\lambda^i_{\mathrm{sin}}], \quad \lambda^i_{\mathrm{sin}} = \left(\sigma \mapsto \int_\sigma d\lambda^i\right) \in \mathrm{C}^1_{\mathrm{sin}}(X, \mathbb{Z}), \;\; \lambda^i_{\mathrm{sin}}(\lambda_j^{\mathrm{sin}}) = \delta_{ij}\,.$$

We will refer to the horizonal compositions

$$
\begin{tikzcd}
\operatorname{Alt}^k(\Lambda,\mathbb{Z}) \arrow[r, "\sim"] \arrow[d, hook] & \bigwedge^k \operatorname{Hom}(\Lambda,\mathbb{Z}) \arrow[r, "\wedge_j \lambda^{i_j} \mapsto \wedge_j [\lambda^{i_j}_{\mathrm{sin}}]", "\sim"'] \arrow[d, hook] & \bigwedge^k \mathrm{H}^1_{\mathrm{sin}}(X,\mathbb{Z}) \arrow[r, "\smile", "\sim"'] & \mathrm{H}^k_{\mathrm{sin}}(X,\mathbb{Z}) \arrow[dr, hook] & \\
\operatorname{Alt}^k(\Lambda,\mathbb{R}) \arrow[r, "\sim"] & \bigwedge^k \operatorname{Hom}(\Lambda,\mathbb{R}) \arrow[r, "\wedge_j \lambda^{i_j} \mapsto \wedge_j [d\lambda^{i_j}]", "\sim"'] & \bigwedge^k \mathrm{H}^1_{\mathrm{dR}}(X,\underline{\mathbb{R}}) \arrow[r, "\wedge", "\sim"'] & \mathrm{H}^k_{\mathrm{dR}}(X,\underline{\mathbb{R}}) \arrow[r, "{[\omega] \mapsto [\int_\bullet \omega]}", "\sim"'] & \mathrm{H}^k_{\mathrm{sin}}(X,\mathbb{R})
\end{tikzcd}
$$

as the *Künneth* isomorphisms, where $\operatorname{Alt}^k(\Lambda,\mathbb{Z})$ is the group of k-alternating forms on Λ.

The constant automorphy factors of primitive functions of closed differential forms on V with respect to the deck transformation group $\operatorname{Deck}(\pi) \cong \pi_1(X) \cong \Lambda$ are equal to the periods of differential forms on X. This computation can be carried out at the level of singular cohomology, giving an isomorphism

$$
\mathrm{H}^1_{\mathrm{sin}}(X,\mathbb{Z}) \xrightarrow[\sim]{[\alpha] \mapsto [\lambda \mapsto \pi^*\alpha\,(0 \to \lambda)]} \mathrm{H}^1(\Lambda,\mathbb{Z}) ,
$$

where $(0 \to \lambda) : s \in [0,1] \mapsto s \cdot \lambda \in V$ is the singular 1-simplex on V starting in 0 and ending in λ. A generalization of this map, which we will call the *automorphy* map, is obtained through the corresponding cup products in singular and group cohomology:

$$
\begin{tikzcd}
\mathrm{H}^k_{\mathrm{sin}}(X,\mathbb{Z}) \arrow[rr, "{[\alpha] \mapsto [(l_1,\ldots,l_k) \mapsto \pi^*\alpha\,(0 \to l_1 \to l_1+l_2 \to \cdots \to \sum_j l_j)]}", "\sim"'] & & \mathrm{H}^k(\Lambda,\mathbb{Z}) & {[(l_1,\ldots,l_k) \mapsto F_{i_1}(l_1)\cdots F_{i_k}(l_k)]} \\
\bigwedge^k \mathrm{H}^1_{\mathrm{sin}}(X,\mathbb{Z}) \arrow[u, "\smile"', "\wr"] \arrow[rr, "{\wedge_j [\alpha_j] \mapsto \wedge_j [\lambda \mapsto \pi^*\alpha_j\,(0 \to \lambda)]}", "\sim"'] & & \bigwedge^k \mathrm{H}^1(\Lambda,\mathbb{Z}) \arrow[u, "\wr"] & \wedge_j [F_{i_j}] \arrow[u, mapsto] \\
& \bigwedge^k \operatorname{Hom}(\Lambda,\mathbb{Z}) \arrow[ul, "\sim"] \arrow[ur, "\sim"'] & \wedge_j [\lambda^{i_j}] & \\
& & \wedge_j \lambda^{i_j} \arrow[u, mapsto] &
\end{tikzcd}
$$

where $(0 \to l_1 \to l_1 + l_2 \to \cdots \to \sum_j l_j)$ is a singular k-simplex on V with vertices in order 0, l_1, …, $\sum_j l_j$. The inverse isomorphism from group cohomology to alternating forms is given by alternization:

$$
\begin{tikzcd}
\operatorname{Alt}^k(\Lambda,\mathbb{Z}) \arrow[r, "\sim"] & \bigwedge^k \operatorname{Hom}(\Lambda,\mathbb{Z}) \arrow[r, "\wedge_j \lambda^{i_j} \mapsto \wedge_j [\lambda^{i_j}]", "\sim"'] & \bigwedge^k \mathrm{H}^1(\Lambda,\mathbb{Z}) \arrow[r, "{\wedge_j [F_{i_j}] \mapsto [(l_1,\ldots,l_k) \mapsto F_{i_1}(l_1)\cdots F_{i_k}(l_k)]}", "\sim"'] & \mathrm{H}^k(\Lambda,\mathbb{Z}) \arrow[lll, bend left, "\sim", "{\sum_{\sigma \in S(\{1,\ldots,k\})} \operatorname{sgn}\sigma\, F \circ P_\sigma \,\mapsfrom\, [F]}"']
\end{tikzcd} ,
$$

where $P_\sigma(v_1 \ldots, v_k) = (v_{\sigma 1}, \ldots, v_{\sigma k})$ is the permutation of the arguments.

Relationship to Čech Cohomology

The universal δ-functor properties of sheaf cohomology relate by a unique isomorphism the Čech, de Rham and sheafified singular cohomology groups. Additionally, Čech cohomology of a sheaf of abelian groups $\mathcal{F}$ is related via the universal cover to group cohomology. Namely, if $\mathcal{U} = (U_i)_i$ is a good open cover of X, the universal cover $\pi : V \twoheadrightarrow X$ induces isomorphisms

$$\varphi^k_{\mathcal{U}} : \mathrm{H}^k(\Lambda, \Gamma(V, \pi^*\mathcal{F})) \xrightarrow{\sim} \check{\mathrm{H}}^k_{\mathcal{U}}(X, \mathcal{F}), \quad [F] \mapsto [(F(\lambda_{i_0 i_1}, \ldots, \lambda_{i_{k-1} i_k}))_{i_0 \cdots i_k}],$$

$$\lambda_{ij} = \pi_j^{-1} - \pi_i^{-1} \in \Lambda, \quad \pi_i : \widetilde{U_i} \xrightarrow{\sim} U_i,$$

where π_i are restrictions of π to a choice of connected components $\widetilde{U_i} \subseteq \pi^{-1}(U_i)$ for all i. The isomorphism is canonical, in the sense that it is indipendent of the choices made for $\widetilde{U_i}$.

Thus, there are two different maps relating sheaf cohomology theories to group cohomology. We will verify below that these actually coincide, up to an unfortunate sign difference.

$$\begin{tikzcd}
\mathrm{H}^k(\Lambda, \mathbb{Z}) \arrow[rr, "\cdot(-1)^{k+\frac{k(k+1)}{2}}", "\sim"'] & & \mathrm{H}^k(\Lambda, \mathbb{Z}) \arrow[d, "\varphi^k_{\mathcal{U}}", "\wr"'] \\
\mathrm{H}^k_{\mathrm{sin}}(X, \mathbb{Z}) \arrow[u, "\text{automorphy}", "\wr"'] \arrow[r, "\text{sheafification}", "\sim"'] \arrow[d, hook] & \mathrm{H}^k_{\mathrm{sin}}(X, \underline{\mathbb{Z}}) \arrow[r, "\text{sheaf cohomology}", "\sim"'] \arrow[d, hook] & \check{\mathrm{H}}^k_{\mathcal{U}}(X, \underline{\mathbb{Z}}) \arrow[d, hook] \\
\mathrm{H}^k_{\mathrm{sin}}(X, \mathbb{R}) \arrow[r, "\text{sheafification}", "\sim"'] & \mathrm{H}^k_{\mathrm{sin}}(X, \underline{\mathbb{R}}) \arrow[r, "\sim"] & \check{\mathrm{H}}^k_{\mathcal{U}}(X, \underline{\mathbb{R}}) \\
& \mathrm{H}^k_{\mathrm{dR}}(X, \underline{\mathbb{R}}) \arrow[ul, "\sim", "{[\int_\bullet \omega] \leftarrow\!\shortmid [\omega]}"'] \arrow[u, "\wr"] \arrow[ur, "\text{sheaf cohomology}", "\sim"'] &
\end{tikzcd}$$

As Čech cohomology groups computed using any particular good open cover relate to the limit $\varinjlim_{\mathcal{U}} \check{\mathrm{H}}^k_{\mathcal{U}}(X, \underline{\mathbb{Z}})$ in a natural way, it is sufficient to verify commutativity and compute these signs for a particular cover $\mathcal{U}$ and choices of restrictions $\pi_i : \widetilde{U_i} \xrightarrow{\sim} U_i$. To this end, we first describe a standard simplicial complex structure on X.

Standard Simplicial Complex Structure on the Torus

As 0-simplices we take the 3^n points $(x_i)_i$ given as the image of the composition $\frac{1}{3}\Lambda \hookrightarrow V \twoheadrightarrow X$, ordered lexicographically by the rule

$$i < j :\iff \overline{\lambda}^k(x_i) < \overline{\lambda}^k(x_j), \text{ for } k = \max\{1 \leq l \leq n \mid \overline{\lambda}^l(x_i) \neq \overline{\lambda}^l(x_j)\}.$$

As 1-simplices we take all oriented segments $(x_i \to x_j)$ from x_i to x_j such that $i < j$ and

$$\left(\forall 1 \leq k \leq n : 0 \leq \lambda^k(x_j) - \lambda^k(x_i) \leq \frac{1}{3}\right) \text{ or } \left(\forall 1 \leq k \leq n : -\frac{1}{3} \leq \lambda^k(x_j) - \lambda^k(x_i) \leq 0\right).$$

This just means that either $(x_i \to x_j)$ or its opposite stays within one of the 3^n subcubes and points positively in all directions. The rest of the skeleton is uniquely determined by adding all possible higher dimensional simplices that stay withing one of subcubes. We denote a k-simplex $(x_{i_0} \to \cdots \to x_{i_k})$ by its ordered sequence of 0-simplices. Consider the following basis for the Δ-complex (co)homology of X.

$$
\begin{aligned}
\Lambda &\xrightarrow{\sim} \mathrm{H}_1^{\Delta}(X,\mathbb{Z}) \\
\lambda_k &\mapsto [\lambda_k^{\Delta}]
\end{aligned}
\qquad
\lambda_k^{\Delta} = \left(0 \to \frac{\lambda_k}{3}\right) + \left(\frac{\lambda_k}{3} \to \frac{2\lambda_k}{3}\right) - \left(0 \to \frac{2\lambda_k}{3}\right) \in C_1^{\Delta}(X,\mathbb{Z})
$$

$$
\begin{aligned}
\mathrm{Hom}(\Lambda,\mathbb{Z}) &\xrightarrow{\sim} \mathrm{H}^1_{\Delta}(X,\mathbb{Z}) \\
\lambda^k &\mapsto [\lambda^k_{\Delta}]
\end{aligned}
\qquad
\lambda^k_{\Delta} : (x_i \to x_j) \in C_1^{\Delta}(X,\mathbb{Z}) \mapsto \begin{cases} 1 & \text{if } \overline{\lambda}^k(x_i) = 0,\, \overline{\lambda}^k(x_j) = \frac{1}{3} \\ -1 & \text{if } \overline{\lambda}^k(x_i) = \frac{1}{3},\, \overline{\lambda}^k(x_j) = 0 \\ 0 & \text{else} \end{cases}
$$

The standard simplicial complex structure is realized in a natural way as the Čech nerve $N\mathcal{U}$ of a good open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X with 3^n open sets, each U_i corresponding to $x_i \in X$.

Fix connected components $\widetilde{U}_i \subset \pi^{-1}(U_i)$ such that

$$
\widetilde{U}_i \cap \widetilde{U}_j = \emptyset \iff \exists 1 \le k \le n : \overline{\lambda}^k(x_i) = 0,\, \overline{\lambda}^k(x_j) = \frac{1}{3}, \quad \text{for all } i < j.
$$

Then, the group elements measuring the difference between the restrictions of $\pi_i : \widetilde{U}_i \xrightarrow{\sim} U_i$ are

$$
\Lambda \ni \lambda_{ij} = \pi_j^{-1} - \pi_i^{-1} = \sum_k \begin{cases} -\lambda_k & \text{if } \overline{\lambda}^k(x_i) = 0,\, \overline{\lambda}^k(x_j) = \frac{1}{3} \\ \lambda_k & \text{if } \overline{\lambda}^k(x_i) = \frac{1}{3},\, \overline{\lambda}^k(x_j) = 0 \\ 0 & \text{else} \end{cases}, \quad \text{for all } i < j,
$$

letting us compute explicitely

$$
\lambda^l(\lambda_{ij}) = \begin{cases} -1 & \text{if } \overline{\lambda}^l(x_i) = 0,\, \overline{\lambda}^l(x_j) = \frac{1}{3} \\ 1 & \text{if } \overline{\lambda}^l(x_i) = \frac{1}{3},\, \overline{\lambda}^l(x_j) = 0 \\ 0 & \text{else} \end{cases} = -\lambda^l_{\Delta}(x_i \to x_j), \quad \text{for all } l,\, i < j,
$$

giving us commutativity of the following diagram.

$$
\begin{tikzcd}
\mathrm{H}^1_{\Delta}(X,\mathbb{Z}) & & \mathrm{H}^1(\Lambda,\mathbb{Z}) \arrow[ll, "{[(x_i \to x_j) \mapsto F(\lambda_{ij})] \,\mapsfrom\, [F]}"', "\sim"] \\
\mathrm{Hom}(\Lambda,\mathbb{Z}) \arrow[u, "{[\lambda^i_{\Delta}] \,\text{\rotatebox{90}{$\mapsto$}}\, \lambda^i}"', "\wr"] \arrow[rr, "\cdot(-1)", "\sim"'] & & \mathrm{Hom}(\Lambda,\mathbb{Z}) \arrow[u, "{[\lambda^i] \,\text{\rotatebox{90}{$\mapsto$}}\, \lambda^i}"', "\wr"]
\end{tikzcd}
$$

Passing to the exterior product, we obtain diagram (A.1), where the sign $(-1)^{\frac{k(k+1)}{2}}$ is needed to ensure commutativity with the singular cohomology groups; we derive this result in [Bel25].

$$
\begin{tikzcd}
\mathrm{H}^k_{\mathrm{sin}}(X,\underline{\mathbb{Z}}) \arrow[r, "\substack{\text{sheaf}\\ \text{cohomology}}", "\sim"'] & \check{\mathrm{H}}^k_{\mathcal{U}}(X,\underline{\mathbb{Z}}) & \mathrm{H}^k(\Lambda,\mathbb{Z}) \arrow[l, "{[F(\lambda_{i_0 i_1},\dots,\lambda_{i_{k-1} i_k})_{i_0\cdots i_k}] \mapsfrom [F]}"', "\sim"] \\
\mathrm{H}^k_{\mathrm{sin}}(X,\mathbb{Z}) \arrow[u, "\wr"', "\text{sheafification}"] \arrow[r, "\text{restriction}", "\sim"'] & \mathrm{H}^k_{\Delta}(X,\mathbb{Z}) \arrow[u, "\wr"', "\substack{[(\alpha(x_{i_0}\to\cdots\to x_{i_k}))_{i_0\cdots i_k}]\cdot(-1)^{\frac{k(k+1)}{2}} \\ \text{\rotatebox{90}{$\mapsto$}} \\ {[\alpha]}}"] & \mathrm{H}^k(\Lambda,\mathbb{Z}) \arrow[u, "\wr"', "\cdot(-1)^{\frac{k(k+1)}{2}}"] \arrow[l, "{[(x_{i_0}\to\cdots\to x_{i_k})\mapsto F(\lambda_{i_0 i_1},\dots,\lambda_{i_{k-1} i_k})] \mapsfrom [F]}"', "\sim"] \\
\bigwedge^k \mathrm{H}^1_{\mathrm{sin}}(X,\mathbb{Z}) \arrow[u, "\wr"', "\smile"] \arrow[r, "\text{restriction}", "\sim"'] & \bigwedge^k \mathrm{H}^1_{\Delta}(X,\mathbb{Z}) \arrow[u, "\wr"', "\smile"] & \bigwedge^k \mathrm{H}^1(\Lambda,\mathbb{Z}) \arrow[u, "\wr"', "\substack{[(l_1,\dots,l_k)\mapsto F_{i_1}(l_1)\cdots F_{i_k}(l_k)] \\ \text{\rotatebox{90}{$\mapsto$}} \\ \wedge_j [F_{i_j}]}"] \arrow[l, "{\wedge_j [(x_\mu\to x_\nu)\mapsto F_{i_j}(\lambda_{\mu\nu})] \mapsfrom \wedge_j [F_{i_j}]}"', "\sim"] \\
 & \bigwedge^k \mathrm{Hom}(\Lambda,\mathbb{Z}) \arrow[ul, "\sim"', "{\wedge_j [\lambda^{i_j}_{\mathrm{sin}}] \mapsfrom \wedge_j \lambda^{i_j}}"] \arrow[u, "\wr"', "\substack{\wedge_j [\lambda^{i_j}_{\Delta}] \\ \text{\rotatebox{90}{$\mapsto$}} \\ \wedge_j \lambda^{i_j}}"] \arrow[r, "\cdot(-1)^k", "\sim"'] & \bigwedge^k \mathrm{Hom}(\Lambda,\mathbb{Z}) \arrow[u, "\wr"', "\substack{\wedge_j [\lambda^{i_j}] \\ \text{\rotatebox{90}{$\mapsto$}} \\ \wedge_j \lambda^{i_j}}"]
\end{tikzcd}
\tag{A.1}
$$

Passing to field coefficients, commutativity between de Rham and Δ-complex cohomology is verified by integrating $\int_{\lambda_i^\Delta} d\,\lambda^j = \delta_{ij} = \lambda^j_\Delta(\lambda_i^\Delta)$, giving us diagram (A.2).

$$
\begin{tikzcd}
\mathrm{H}^k_{\mathrm{sin}}(X,\underline{\mathbb{R}}) \arrow[r, "\sim"'] & \check{\mathrm{H}}^k_{\mathcal{U}}(X,\underline{\mathbb{R}}) & \mathrm{H}^k(\Lambda,\mathbb{R}) \arrow[l, "{[F(\lambda_{i_0 i_1},\dots,\lambda_{i_{k-1} i_k})_{i_0\cdots i_k}] \mapsfrom [F]}"', "\sim"] \\
\mathrm{H}^k_{\mathrm{dR}}(X,\underline{\mathbb{R}}) \arrow[u, "\wr"', "\substack{[\int_\bullet \omega] \\ \text{\rotatebox{90}{$\mapsto$}} \\ {[\omega]}}"] \arrow[ur, "\substack{\text{sheaf}\\ \text{cohomology}}", "\sim"'] \arrow[r, "{[\omega]\mapsto[\int_\bullet \omega]}", "\sim"'] & \mathrm{H}^k_{\Delta}(X,\mathbb{R}) \arrow[u, "\wr"', "\substack{[(\alpha(x_{i_0}\to\cdots\to x_{i_k}))_{i_0\cdots i_k}]\cdot(-1)^{\frac{k(k+1)}{2}} \\ \text{\rotatebox{90}{$\mapsto$}} \\ {[\alpha]}}"] & \\
\bigwedge^k \mathrm{H}^1_{\mathrm{dR}}(X,\underline{\mathbb{R}}) \arrow[u, "\wr"', "\wedge"] \arrow[r, "{\wedge_j [\omega_j]\mapsto \wedge_j [\int_\bullet \omega_j]}", "\sim"'] & \bigwedge^k \mathrm{H}^1_{\Delta}(X,\mathbb{Z}) \arrow[u, "\wr"', "\smile"] & \bigwedge^k \mathrm{H}^1(\Lambda,\mathbb{R}) \arrow[uu, "\wr"', "\substack{[(l_1,\dots,l_k)\mapsto F_{i_1}(l_1)\cdots F_{i_k}(l_k)] \\ \text{\rotatebox{90}{$\mapsto$}} \\ \wedge_j [F_{i_j}]}"] \\
 & \bigwedge^k \mathrm{Hom}(\Lambda,\mathbb{R}) \arrow[ul, "\sim"', "{\wedge_j [d\lambda^{i_j}] \mapsfrom \wedge_j \lambda^{i_j}}"] \arrow[u, "\wr"', "\substack{\wedge_j [\lambda^{i_j}_{\Delta}] \\ \text{\rotatebox{90}{$\mapsto$}} \\ \wedge_j \lambda^{i_j}}"] \arrow[r, "\cdot(-1)^k\cdot(-1)^{\frac{k(k+1)}{2}}", "\sim"'] & \bigwedge^k \mathrm{Hom}(\Lambda,\mathbb{R}) \arrow[u, "\wr"', "\substack{\wedge_j [\lambda^{i_j}] \\ \text{\rotatebox{90}{$\mapsto$}} \\ \wedge_j \lambda^{i_j}}"]
\end{tikzcd}
\tag{A.2}
$$

Complex Tori

From now on, we assume that the real vector space V is given a complex structure, making it necessarily of even dimension $2g$. The quotient V/Λ by a lattice $\Lambda \subseteq V$ is then a complex torus.

The complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ decomposes in two complex vector spaces $V^{1,0} \oplus V^{0,1}$ with $V \cong V^{1,0} \cong \overline{V^{0,1}}$ over $\mathbb{C}$. The groups of alternating forms on $V_{\mathbb{C}}$ decompose accordingly as

$$\begin{tikzcd}
\mathrm{Alt}^k(\Lambda, \mathbb{Z}) \arrow[d, hook] & \bigwedge^k \mathrm{Hom}(\Lambda, \mathbb{Z}) \arrow[l, "\sim"'] \arrow[d, hook] & & \\
\mathrm{Alt}^k(\Lambda, \mathbb{C}) \arrow[d, "\wr"] & \bigwedge^k \mathrm{Hom}(\Lambda, \mathbb{C}) \arrow[l, "\sim"'] \arrow[d, "\wr"] & & \\
\mathrm{Alt}^k_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) & \bigwedge^k \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) \arrow[l, "\sim"'] & \bigwedge^k \Omega \oplus \overline{\Omega} \arrow[l, "\sim"'] & \bigoplus_{p+q=k} \Omega^p \otimes \overline{\Omega}^q \arrow[l, "\sim"']
\end{tikzcd},$$

where $\Omega = \mathrm{Hom}_{\mathbb{C}}(V^{1,0}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\overline{\Omega} = \mathrm{Hom}_{\mathbb{C}}(V^{0,1}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(\overline{V}, \mathbb{C})$. It is the content of the Hodge theorem that the maps extending $(p.q)$-alternating forms to (p,q)-differential forms on X induce isomorphisms $\Omega^p \otimes \overline{\Omega}^q \xrightarrow{\sim} \mathrm{H}^q_{\mathrm{Dol}}(X, \Omega^p_X)$ to (p,q)-Dolbeault cohomolgy, where Ω^p_X is the sheaf of holomorphic p-forms on X. Thus, we obtain the Hodge decomposition.

$$\begin{tikzcd}
\mathrm{Alt}^k_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) \arrow[d, "\wr"] & \bigoplus_{p+q=k} \Omega^p \otimes \overline{\Omega}^q \arrow[l, "\sim"'] \arrow[d, "\wr"] \\
\mathrm{H}^k_{\mathrm{dR}}(X, \underline{\mathbb{C}}) & \bigoplus_{p+q=k} \mathrm{H}^q_{\mathrm{Dol}}(X, \Omega^p_X) \arrow[l, "\sim"']
\end{tikzcd}$$

Similarly as for constanst coefficients, computing automorphy factors of primitives of $(0,1)$-forms on V with respect to the deck transformation group of the universal cover gives an isomorphism to group cohomology

$$\begin{tikzcd}
\mathrm{H}^1_{\mathrm{dR}}(X, \underline{\mathbb{C}}) \arrow[r, "{[\omega] \mapsto [\lambda \mapsto \int_{(0 \to \lambda)} \pi^*\omega] = \left[\lambda \ \mapsto \underset{f \in C^\infty_V(V) \text{ with } df = \pi^*\omega}{f \circ \sigma_\lambda - f}\right]}", "\sim"'] \arrow[d, two heads, "\substack{\text{inclusion} \\ \| \\ \text{hodge} \\ \text{projection}}"] & \mathrm{H}^1(\Lambda, \mathbb{C}) \arrow[d, two heads, "\text{inclusion}"] \\
\mathrm{H}^1_{\mathrm{Dol}}(X, \mathcal{O}_X) \arrow[r, "{[\omega] \mapsto \left[\lambda \mapsto \underset{f \in C^\infty_V(V) \text{ with } \overline{\partial} f = \pi^*\omega}{f \circ \sigma_\lambda - f}\right]}", "\sim"'] & \mathrm{H}^1(\Lambda, \mathcal{O}_V(V))
\end{tikzcd}$$

where, for each $\lambda \in \Lambda \cong \pi_1(X)$, σ_λ is the induced deck transformation on $\pi : V \to X$.

Line Bundles on Complex Tori

The Picard group

$$\mathrm{Pic}(X) = \{L \mid L \text{ holomorphic line bundle on } X\}/\text{isomorphism}$$

is descibed using Čech cohomology and the exponential sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp(2\pi i \bullet)} \mathcal{O}_X^* \to 0 \, .$$

The Čech cocycles of holomorphic transition functions can be described in terms of factors of automorphy, i.e. elements of group cohomology $\mathrm{H}^1(\Lambda, \mathcal{O}_V^*(V))$. In turn, these factors can be described by integral hermitian forms and semicharacters on the multiplicative group $\mathbb{C}_1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Namely, if $\mathcal{P}(X)$ is the set of all pairs (H, χ), where H is a hermitian form on V with $H(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and $\chi : \Lambda \to \mathbb{C}_1$ is a semicharacter for H, under the group operation $(H_1, \chi_1) \circ (H_2, \chi_2) = (H_1 + H_2, \chi_1 \cdot \chi_2)$, then there is the isomorphism

$$\mathcal{P}(X) \xrightarrow{\sim} \mathrm{H}^1(\Lambda, \mathcal{O}_V^*(V)), \quad (H, \chi) \mapsto \left[(\lambda, v) \in \Lambda \times V \mapsto \chi(\lambda) e^{\pi H(v,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)}\right]$$

that associates to each pair (H, χ) its so called *canonical factor* $a_{(H,\chi)}$. The Appel-Humbert theorem [BL05, 2.2.3] then states that the kernel of the first Chern class $\mathrm{Pic}(X) \overset{c_1}{\twoheadrightarrow} \mathrm{NS}(X)$ corresponds to the group of characters $\mathrm{Hom}(\Lambda, \mathbb{C}_1) \hookrightarrow \mathcal{P}(X)$. Thus, the Néron-Severi group $\mathrm{NS}(X)$ is isomorphic to the group $\mathcal{H}(X)$ of integral hermitian forms on V. Everything fits together in diagram (A.3), where it is understood that each horizontal sequence of homomorphisms is exact, as well as $\check{\mathrm{H}}^1_{\mathcal{U}}(X, \underline{\mathbb{Z}}) \hookrightarrow \check{\mathrm{H}}^1_{\mathcal{U}}(X, \mathcal{O}_X) \twoheadrightarrow \mathrm{Pic}^0(X)$. However, note that some of the abelian groups in the lower part of the diagram are not $\mathbb{C}$-vector spaces in an obvious way, unless one transports that structure through the vertical isomorphisms.

$$
\begin{tikzcd}
 & \mathrm{H}^1_{\mathrm{dR}}(X,\underline{\mathbb{C}}) \arrow[d, "\wr"', "\text{sheaf cohomology}"] \arrow[dr, two heads, "\text{Hodge projection}"] & & \mathrm{Pic}^0(X) \arrow[r, hook] & \mathrm{Pic}(X) \arrow[r, two heads, "c_1"] \arrow[dd, "\wr", "\text{transition functions}"'] & \mathrm{NS}(X) \arrow[ddr, hook] & & \\
 & \check{\mathrm{H}}^1_{\mathcal{U}}(X,\underline{\mathbb{C}}) \arrow[dr, two heads] & \mathrm{H}^1_{\mathrm{Dol}}(X,\mathcal{O}_X) \arrow[d, "\wr"] & & & & & \check{\mathrm{H}}^2_{\mathcal{U}}(X,\underline{\mathbb{C}}) \\
\check{\mathrm{H}}^1_{\mathcal{U}}(X,\underline{\mathbb{Z}}) \arrow[ur, hook] \arrow[rr, hook] & & \check{\mathrm{H}}^1_{\mathcal{U}}(X,\mathcal{O}_X) \arrow[uur] \arrow[rr, "\exp(2\pi i\bullet)"] & & \check{\mathrm{H}}^1_{\mathcal{U}}(X,\mathcal{O}_X^*) \arrow[rr] & & \check{\mathrm{H}}^2_{\mathcal{U}}(X,\underline{\mathbb{Z}}) \arrow[ur, hook] & \\
 & \mathrm{H}^1(\Lambda,\mathbb{C}) \arrow[uu, "\wr", "\varphi^1_{\mathcal{U}}"'] \arrow[dr, two heads] & & & & & & \mathrm{H}^2(\Lambda,\mathbb{C}) \arrow[uu, "\wr", "\varphi^2_{\mathcal{U}}"'] \\
\mathrm{H}^1(\Lambda,\mathbb{Z}) \arrow[uu, "\wr", "\varphi^1_{\mathcal{U}}"'] \arrow[ur, hook] \arrow[rr, hook] & & \mathrm{H}^1(\Lambda,\mathcal{O}_V(V)) \arrow[uu, "\wr", "\varphi^1_{\mathcal{U}}"'] \arrow[rr, "\exp(2\pi i\bullet)"] & & \mathrm{H}^1(\Lambda,\mathcal{O}_V^*(V)) \arrow[uu, "\wr", "\varphi^1_{\mathcal{U}}"'] \arrow[rr] & & \mathrm{H}^2(\Lambda,\mathbb{Z}) \arrow[uu, "\wr", "\varphi^2_{\mathcal{U}}"'] \arrow[ur, hook] & \\
 & & & & & & & \mathrm{Alt}^2(\Lambda,\mathbb{C}) \arrow[uu, "\wr"] \\
 & \mathrm{Hom}(\Lambda,\mathbb{C}) \arrow[uuu, "\wr"] \arrow[dr, two heads, "F \mapsto \substack{\mathrm{Re}F(\bullet)\\ -\mathrm{Im}F(i\bullet)}"] & & \mathrm{Hom}(\Lambda,\mathbb{C}_1) \arrow[uuuuuu, "\wr"] \arrow[r, hook, "\chi\mapsto(0,\chi)"] & \mathcal{P}(X) \arrow[uu, "\wr", "a_{(H,\chi)} \mapsfrom (H,\chi)"'] \arrow[r, two heads, "(H,\chi)\mapsto H"] & \mathcal{H}(X) \arrow[uuuuuu, "\wr"] \arrow[dr, "\sim", "\substack{H\mapsto Im\, H\\ E(i\bullet,\bullet)+iE(\bullet,\bullet)\mapsfrom E}"'] & \mathrm{Alt}^2(\Lambda,\mathbb{Z}) \arrow[uu, "\wr"] \arrow[ur, hook] & \Omega\otimes\overline{\Omega} \arrow[u, hook] \\
\mathrm{Hom}(\Lambda,\mathbb{Z}) \arrow[uuu, "\wr"] \arrow[ur, hook] \arrow[rr, hook] & & \mathrm{Hom}(\Lambda,\mathbb{R}) \arrow[uuu, "\wr"] \arrow[ur, two heads, "\exp(2\pi i\bullet)"'] & & & & \mathrm{Alt}^2(\Lambda,\mathbb{Z})\cap\Omega\otimes\overline{\Omega} \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}
\tag{A.3}
$$

Bibliography

[Bel25] Marco Belli. The isomorphism to Čech cohomology as evaluation on the Čech nerve. 2025. arXiv/2510.21486.

[BL05] Christina Birkenhake and Herbert Lange. *Complex Abelian Varieties*. Springer Berlin, Heidelberg, 2nd edition, 2005. springer/978-3-662-06307-1.

[CG72] C. Herbert Clemens and Phillip A. Griffiths. The intermediate jacobian of the cubic threefold. *Annals of Mathematics, Second Series*, (Vol. 95, No. 2), 1972. jstor/1970801.