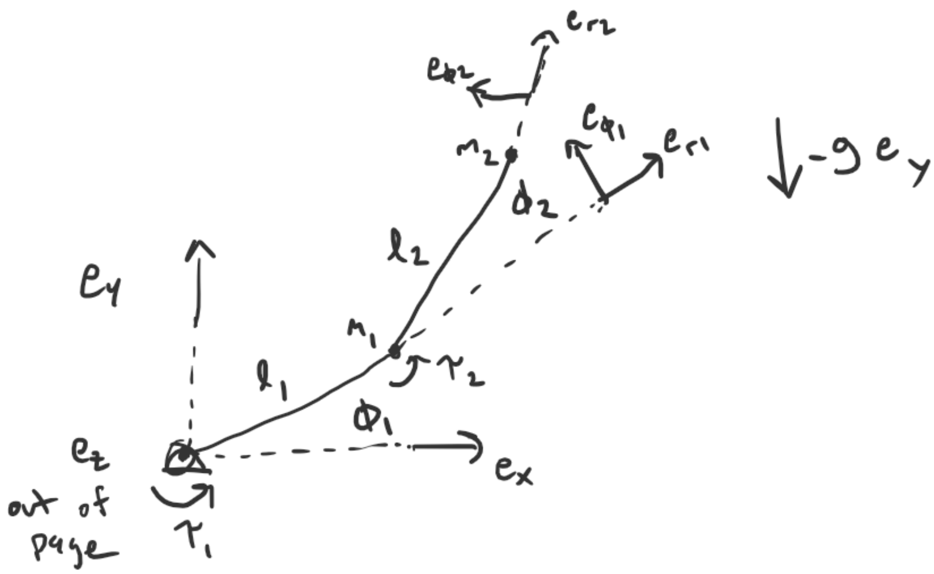
Two-Link Concurrent Learning Control



I. Dynamics

Again consider the two-link dynamic system

$$M(\phi)\ddot{\phi} + C\left(\phi,\dot{\phi}\right) + G(\phi) = \tau$$

where $\phi(t) \triangleq \begin{bmatrix}\phi_1(t)\\ \phi_2(t)\end{bmatrix}, \dot{\phi}(t) \triangleq \begin{bmatrix}\dot{\phi}_1(t)\\ \dot{\phi}_2(t)\end{bmatrix}, \ddot{\phi}(t) \triangleq \begin{bmatrix}\ddot{\phi}_1(t)\\ \ddot{\phi}_2(t)\end{bmatrix}, \tau(t) \triangleq \begin{bmatrix}\tau_1(t)\\ \tau_2(t)\end{bmatrix} \in \mathbb{R}^2$ are the angle, angular velocity, angular acceleration, and input torque of the arm joints, the inertia terms $M(\phi)$, the Coriolis and centripetal terms $C\left(\phi,\dot{\phi}\right)$, and the gravity terms $G(\phi)$ are defined as

$$\begin{aligned}
M(\phi) &\triangleq \begin{bmatrix} m_1 l_1^2 + m_2\left(l_1^2 + 2l_1l_2c_2 + l_2^2\right) & m_2\left(l_1l_2c_2 + l_2^2\right) \\ m_2\left(l_1l_2c_2 + l_2^2\right) & m_2l_2^2 \end{bmatrix},\\
C\left(\phi,\dot{\phi}\right) &\triangleq \begin{bmatrix} -2m_2l_1l_2s_2\dot{\phi}_1\dot{\phi}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -m_2l_1l_2s_2\dot{\phi}_2^2 \\ m_2l_1l_2s_2\dot{\phi}_1^2 \end{bmatrix},\\
G(\phi) &\triangleq \begin{bmatrix} (m_1+m_2)\,gl_1c_1 + m_2gl_2c_{12} \\ m_2gl_2c_{12} \end{bmatrix},
\end{aligned}$$

$m_i, l_i \in \mathbb{R}_{>0}$ are the **unknown** constant mass and length of link i, $g \in \mathbb{R}_{>0}$ is gravity,

$$\begin{aligned}
c_i &= \cos(\phi_i),\\
s_i &= \sin(\phi_i),\\
c_{12} &= \cos(\phi_1+\phi_2),\\
s_{12} &= \sin(\phi_1+\phi_2).
\end{aligned}$$

As shown in the previous section, the above dynamics are linear in the unknown parameters and we were able to develop a gradient update law for the unknown parameters

$$\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{bmatrix} = \begin{bmatrix} m_1l_1^2 + m_2l_1^2 + m_2l_2^2 \\ m_2l_1l_2 \\ m_2l_2^2 \\ (m_1+m_2)\,l_1 \\ m_2l_2 \end{bmatrix}$$

and we were able to develop an update law of the form

$$\dot{\hat{\theta}} \triangleq \Gamma Y^{\top} r$$

from the analysis since we showed that

$$Y\theta = M(\phi)\varphi + C\left(\phi,\dot{\phi}\right) + G(\phi) + \frac{1}{2}\dot{M}\left(\phi,\dot{\phi}\right) r$$

where $\varphi \triangleq \left(\ddot{\phi}_d + \alpha\dot{e}\right)$

$$Y \triangleq Y_M(\phi,\varphi) + Y_C\left(\phi,\dot{\phi}\right) + Y_G(\phi) + \frac{1}{2}Y_{\dot{M}}(\phi,r),$$

$$Y_M(\phi,\varphi) \triangleq \begin{bmatrix} \varphi_1 & (2c_2\varphi_1 + c_2\varphi_2) & \varphi_2 & 0 & 0 \\ 0 & c_2\varphi_1 & (\varphi_1+\varphi_2) & 0 & 0 \end{bmatrix},$$

$$Y_C\left(\phi,\dot{\phi}\right) \triangleq \begin{bmatrix} 0 & -\left(2s_2\dot{\phi}_1\dot{\phi}_2 + s_2\dot{\phi}_2^2\right) & 0 & 0 & 0 \\ 0 & s_2\dot{\phi}_1^2 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_G(\phi) \triangleq \begin{bmatrix} 0 & 0 & 0 & gc_1 & gc_{12} \\ 0 & 0 & 0 & 0 & gc_{12} \end{bmatrix},$$

$$Y_{\dot{M}}(\phi) \triangleq \begin{bmatrix} 0 & -\left(2s_2\dot{\phi}_2 r_1 + s_2\dot{\phi}_2 r_2\right) & 0 & 0 & 0 \\ 0 & -s_2\dot{\phi}_2 r_1 & 0 & 0 & 0 \end{bmatrix}.$$

Now we wish to examine how to use data based methods to determine better estimates of the parameters.

A. Traditional Concurrent Learning

1) Regressor Development: Using the standard approach, we must first find a regressor for the dynamics such that

$$\mathcal{Y}\theta = M(\phi)\ddot{\phi} + C\left(\phi,\dot{\phi}\right) + G(\phi).$$

Now if we compare this relationship with $Y\theta$, we see that the only differences between the above and $Y\theta$ are that $Y_M(\phi,\varphi)$ is a function of φ which can be replaced by $\ddot{\phi}$ since they are mapped the same way and that $\frac{1}{2}Y_{\dot{M}}(\phi,r)$ is added at the end. This implies that

$$\mathcal{Y}(t) \triangleq Y_M\left(\phi,\ddot{\phi}\right) + Y_C\left(\phi,\dot{\phi}\right) + Y_G(\phi)$$

where

$$Y_M\left(\phi,\ddot{\phi}\right) \triangleq \begin{bmatrix} \ddot{\phi}_1 & \left(2c_2\ddot{\phi}_1 + c_2\ddot{\phi}_2\right) & \ddot{\phi}_2 & 0 & 0 \\ 0 & c_2\ddot{\phi}_1 & \left(\ddot{\phi}_1 + \ddot{\phi}_2\right) & 0 & 0 \end{bmatrix}.$$

Now using this we can develop a CL approach using

$$\mathcal{Y}\theta = \tau.$$

2) Improved Update Law using CL: Now we can determine how to augment the update law so we can use the above relationship. Here we multiply both sides by $\mathcal{Y}^{\top}$to yield

$$\mathcal{Y}^{\top}\mathcal{Y}\theta = \mathcal{Y}^{\top}\tau$$

which is true for any time t_j

$$\mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\,\theta = \mathcal{Y}^{\top}(t_j)\,\tau(t_j).$$

The CL approach then takes a sum over a set of these terms to get

$$\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\,\theta = \sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\,\tau(t_j)$$

where $N \in \mathbb{Z}_{>0}$ is the number of data points required that resulted in

$$\lambda_{\min}\left\{\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\right\} > \lambda_{CL},\, t \geq T_{CL},$$

$\lambda_{CL} \in \mathbb{R}_{>0}$ is a user selected minimum eigenvalue condition, and $T_{CL} \in \mathbb{R}_{>0}$ is the time the condition was satisfied. The above relationship can now be used to determine a relationship for $\widetilde{\theta}$ since

$$\widetilde{\theta}(t) = \theta - \widehat{\theta}(t),$$

we can multiple both sides by the relationship $\sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)$ to yield

$$\sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t) = \sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\theta - \sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\widehat{\theta}(t).$$

Now using the above relationship we know $\sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\theta = \sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\tau(t_j)$ so

$$\sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t) = \sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\tau(t_j) - \sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\widehat{\theta}(t).$$

Since the right side of this relationship only contains measurable quantities, we can use them to determine

Once enough data is collected, the eigenvalue condition holds and we can use the data to improve the update law as

$$\dot{\widehat{\theta}}(t) \triangleq \Gamma Y^\top(t)r(t) + \Gamma k_{CL}\left(\sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\tau(t_j) - \sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\widehat{\theta}(t)\right)$$

which is implementable and as shown above is equivalent to

$$\dot{\widehat{\theta}}(t) \triangleq \Gamma Y^\top(t)r(t) + \Gamma k_{CL}\sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t)$$

which we can use for analysis.

3) Analysis: Lets examine starting off where the input and update law were defined in the previous section and $\dot{V}(\zeta(t), t)$ became

$$\dot{V}(\zeta(t), t) = -e^\top\alpha e - r^\top\beta r + \widetilde{\theta}^\top Y^\top r - \widetilde{\theta}^\top\Gamma^{-1}\dot{\widehat{\theta}}$$

now we can use the analysis form of the above design of $\dot{\widehat{\theta}}$ in $\dot{V}(\zeta(t), t)$ to yield

$$\dot{V}(\zeta(t), t) = -e^\top\alpha e - r^\top\beta r + \widetilde{\theta}^\top Y^\top r - \widetilde{\theta}^\top\Gamma^{-1}\left(\Gamma Y^\top(t)r(t) + \Gamma k_{CL}\sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t)\right)$$

$$\dot{V}(\zeta(t), t) = -e^\top\alpha e - r^\top\beta r + \widetilde{\theta}^\top Y^\top r - \widetilde{\theta}^\top Y^\top r - \widetilde{\theta}^\top k_{CL}\sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t)$$

$$\dot{V}(\zeta(t), t) = -e^\top\alpha e - r^\top\beta r - \widetilde{\theta}^\top k_{CL}\sum_{j=1}^{N} \mathcal{Y}^\top(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t)$$

$$\dot{V}(\zeta(t), t) \leq \begin{cases} -\underline{\alpha}\|e\|^2 - \underline{\beta}\|r\|^2 & \forall t < T_{CL} \\ -\underline{\alpha}\|e\|^2 - \underline{\beta}\|r\|^2 - k_{CL}\lambda_{CL}\|\widetilde{\theta}\|^2, & \forall t \geq T_{CL} \end{cases}$$

$$\dot{V}(\zeta(t), t) \leq \begin{cases} -\underline{\alpha}\|e\|^2 - \underline{\beta}\|r\|^2 & \forall t < T_{CL} \\ -\min\left\{\underline{\alpha}, \underline{\beta}, k_{CL}\lambda_{CL}\right\}\|\zeta\|^2, & \forall t \geq T_{CL} \end{cases}$$

We can use the Lyapunov candidate bounds to show

$$\frac{1}{2}\min\left\{1, \lambda_{\min}\{M(\phi)\}, \lambda_{\min}\left\{\Gamma^{-1}\right\}\right\}\|\zeta\|^2 \leq V(\zeta(t), t) \leq \frac{1}{2}\max\left\{1, \lambda_{\max}\{M(\phi)\}, \lambda_{\max}\left\{\Gamma^{-1}\right\}\right\}\|\zeta\|^2$$

$$-2V(\zeta(t), t) \geq -\max\left\{1, \lambda_{\max}\{M(\phi)\}, \lambda_{\max}\left\{\Gamma^{-1}\right\}\right\}\|\zeta\|^2$$

$$-\frac{2}{\max\{1, \lambda_{\max}\{M(\phi)\}, \lambda_{\max}\{\Gamma^{-1}\}\}}V(\zeta(t), t) \geq -\|\zeta\|^2$$

which is substituted into the above relationship yielding

$$\dot{V}(\zeta(t), t) \leq \begin{cases} -\underline{\alpha}\|e\|^2 - \underline{\beta}\|r\|^2 & \forall t < T_{CL} \\ -\frac{2\min\left\{\underline{\alpha}, \underline{\beta}, k_{CL}\lambda_{CL}\right\}}{\max\{1, \lambda_{\max}\{M(\phi)\}, \lambda_{\max}\{\Gamma^{-1}\}\}}V(\zeta(t), t), & \forall t \geq T_{CL} \end{cases}$$

and let

$$\beta_v \triangleq \frac{2\min\left\{\underline{\alpha}, \underline{\beta}, k_{CL}\lambda_{CL}\right\}}{\max\left\{1, \lambda_{\max}\left\{M\left(\phi\right)\right\}, \lambda_{\max}\left\{\Gamma^{-1}\right\}\right\}}$$

$$\dot{V}\left(\zeta\left(t\right), t\right) \leq \begin{cases} 0 & \forall t < T_{CL} \\ -\beta_v V\left(\zeta\left(t\right), t\right), & \forall t \geq T_{CL} \end{cases}$$

$$V\left(\zeta\left(t\right), t\right) \leq \begin{cases} V\left(\zeta\left(0\right), 0\right) & \forall t < T_{CL} \\ V\left(\zeta\left(T_{CL}\right), T_{CL}\right)\exp\left(-\beta_v\left(t - T_{CL}\right)\right), & \forall t \geq T_{CL} \end{cases}$$

which implies

$$V\left(\zeta\left(t\right), t\right) \leq V\left(\zeta\left(0\right), 0\right)\exp\left(\beta_v\right)\exp\left(-\beta_v t\right).$$