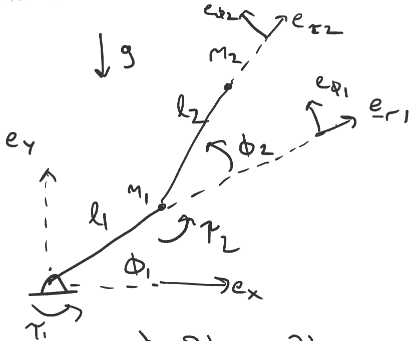
FBD

Thursday, January 6, 2022 6:04 PM

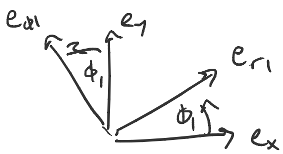



$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \quad \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\tau = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\mathcal{L} = \overset{\text{kinetic}}{K} - \overset{\text{potential}}{P}$$

$$\tau_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} - \frac{\partial \mathcal{L}}{\partial \phi_i}$$

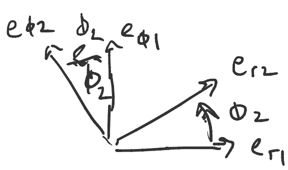



$$e_{r_1} = \cos\phi_1 e_x + \sin\phi_1 e_y$$
$$e_{\phi_1} = -\sin\phi_1 e_x + \cos\phi_1 e_y$$

find the velocities

$$p_1 = l_1 e_{r_1} = l_1 \cos\phi_1 \, e_x + l_1 \sin\phi_1 e_y = l_1 \begin{bmatrix} \cos\phi_1 \\ \sin\phi_1 \end{bmatrix}$$

$$\Rightarrow v_1 = l_1 \begin{bmatrix} -\sin\phi_1 \\ \cos\phi_1 \end{bmatrix} \dot{\phi}_1 = \underbrace{\begin{bmatrix} -l_1 \sin\phi_1 & 0 \\ l_1 \cos\phi_1 & 0 \end{bmatrix}}_{B_1} \underbrace{\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}}_{\dot{\phi}} = B_1(\phi)\dot{\phi}$$




$$e_{r_2} = \cos\phi_2 e_{r_1} + \sin\phi_2 e_{\phi 1}$$
$$e_{\phi 2} = -\sin\phi_2 e_{r_1} + \cos\phi_2 e_{\phi 1}$$

$$P_2 = P_1 + P_{12}$$

$$P_{12} = l_2 e_{r_2} = l_2 \cos\phi_2 \, e_{r_1} + l_2 \sin\phi_2 \, e_{\phi 2}$$
$$= l_2 \cos\phi_2 (\cos\phi_1 e_x + \sin\phi_1 e_y) + l_2 \sin\phi_2 (-\sin\phi_1 e_x + \cos\phi_1 e_y)$$
$$= l_2 \cos\phi_1 \cos\phi_2 e_x + l_2 \sin\phi_1 \cos\phi_2 \, e_y - l_2 \sin\phi_1 \sin\phi_2 \, e_x + l_2 \cos\phi_1 \sin\phi_2 \, e_y$$

$$= l_2 \cos\phi_1 \cos\phi_2 e_x + l_2 \sin\phi_1 \cos\phi_2 e_y - l_2 \sin\phi_1 \sin\phi_2 e_x + l_2 \cos\phi_1 \sin\phi_2 e_y$$

$$= l_2 \underbrace{(\cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2)}_{\cos(\phi_1+\phi_2)} e_x + l_2 \underbrace{(\sin\phi_1 \cos\phi_2 + \cos\phi_1 \sin\phi_2)}_{\sin(\phi_1+\phi_2)} e_y$$

$$= l_2 \cos(\phi_1+\phi_2) e_x + l_2 \sin(\phi_1+\phi_2) e_y = l_2 \begin{bmatrix} \cos(\phi_1+\phi_2) \\ \sin(\phi_1+\phi_2) \end{bmatrix}$$

$$P_2 = l_1 \begin{bmatrix} \cos\phi_1 \\ \sin\phi_1 \end{bmatrix} + l_2 \begin{bmatrix} \cos(\phi_1+\phi_2) \\ \sin(\phi_1+\phi_2) \end{bmatrix}$$

$$\Rightarrow V_2 = l_1 \begin{bmatrix} -\sin\phi_1 \\ \cos\phi_1 \end{bmatrix} \dot{\phi}_1 + l_2 \begin{bmatrix} -\sin(\phi_1+\phi_2) \\ \cos(\phi_1+\phi_2) \end{bmatrix} (\dot{\phi}_1 + \dot{\phi}_2)$$

$$V_2 = \underbrace{\begin{bmatrix} -l_1 \sin\phi_1 - l_2 \sin(\phi_1+\phi_2) & -l_2 \sin(\phi_1+\phi_2) \\ l_1 \cos\phi_1 + l_2 \cos(\phi_1+\phi_2) & l_2 \cos(\phi_1+\phi_2) \end{bmatrix}}_{B_2(\phi)} \underbrace{\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}}_{\dot{\phi}} = B_2(\phi)\dot{\phi}$$

Now we find the kinetic energy

$$K_1 = \frac{1}{2} m_1 V_1^T V_1$$

$$K_1 = \frac{1}{2} m_1 V_1^T V_1 = \frac{1}{2} m_1 (B_1 \dot{\phi})^T B_1 \dot{\phi} = \frac{1}{2} m_1 \dot{\phi}^T B_1^T B_1 \dot{\phi}$$

$$B_1^T B_1 = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \end{bmatrix}^T \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 s_1 & l_1 c_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (-l_1 s_1)(-l_1 s_1) + (l_1 c_1)(l_1 c_1) & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2 s_1^2 + l_1^2 c_1^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} l_1^2 (s_1^2 + c_1^2) & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow K_1 = \frac{1}{2} m_1 \dot{\phi}^T \begin{bmatrix} l_1^2 & 0 \\ 0 & 0 \end{bmatrix} \dot{\phi} = \frac{1}{2} m_1 \begin{bmatrix} \dot{\phi}_1 & \dot{\phi}_2 \end{bmatrix} \begin{bmatrix} l_1^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

$$= \frac{1}{2} m_1 \begin{bmatrix} \dot{\phi}_1 l_1^2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2$$

$$K_2 = \frac{1}{2} m_2 V_2^T V_2 = \frac{1}{2} m_2 (B_2 \dot{\phi})^T (B_2 \dot{\phi}) = \dot{\phi}^T B_2^T B_2 \dot{\phi}$$

$$B_2^T B_2 = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}^T \\ \begin{array}{cc} c & d \end{array} \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array}$$

$$= \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix}$$

$$\dot{\phi}^T B_2^T B_2 \dot{\phi} = \begin{bmatrix} \dot{\phi}_1 & \dot{\phi}_2 \end{bmatrix} \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

$$= \begin{bmatrix} (a^2+c^2)\dot{\phi}_1 + (ab+cd)\dot{\phi}_2 & (ab+cd)\dot{\phi}_1 + (b^2+d^2)\dot{\phi}_2 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

$$= (a^2+c^2)\dot{\phi}_1^2 + (ab+cd)\dot{\phi}_1\dot{\phi}_2 + (ab\,cd)\dot{\phi}_1\dot{\phi}_2 + (b^2+d^2)\dot{\phi}_2^2$$

$$= (a^2+c^2)\dot{\phi}_1^2 + 2(ab+cd)\dot{\phi}_1\dot{\phi}_2 + (b^2+d^2)\dot{\phi}_2^2$$

now determine each one

$$a^2+c^2 = (-l_1 s_1 - l_2 s_{12})^2 + (l_1 c_1 + l_2 c_{12})^2$$

$$= (-l_1 s_1 - l_2 s_{12})(-l_1 s_1 - l_2 s_{12}) + (l_1 c_1 + l_2 c_{12})(l_1 c_1 + l_2 c_{12})$$

$$= l_1^2 s_1^2 + 2 l_1 l_2 s_1 s_{12} + l_2^2 s_{12}^2 + l_1^2 c_1^2 + 2 l_1 l_2 c_1 c_{12} + l_2 c_{12}^2$$

$$= l_1^2 + l_2^2 + 2 l_1 l_2 (s_1 s_{12} + c_1 c_{12})$$

$$s_1 s_{12} + c_1 c_{12} = s_1(s_1 c_2 + c_1 s_2) + c_1(c_1 c_2 - s_1 s_2)$$

$$= s_1^2 c_2 + \cancel{s_1 c_1 s_2} + c_1^2 c_2 - \cancel{s_1 c_1 s_2} = (s_1^2 + c_1^2) c_2$$

$$= c_2$$

$$\Rightarrow = l_1^2 + l_2^2 + 2 l_1 l_2 c_2$$

$$ab+cd = (-l_1 s_1 - l_2 s_{12})(-l_2 s_{12}) + (l_1 c_1 + l_2 c_{12})(l_2 c_{12})$$

$$= l_1 l_2 s_1 s_{12} + l_2^2 s_{12}^2 + l_1 l_2 c_1 c_{12} + l_2^2 c_{12}^2$$

$$= l_2^2 (s_{12}^2 + c_{12}^2) + l_1 l_2 (s_1 s_{12} + c_1 c_{12})$$

$$= l_2^2 + l_1 l_2 c_2$$

$$b^2+d^2 = (-l_2 s_{12})^2 + (l_2 c_{12})^2$$

$$= l_2^2 s_{12}^2 + l_2^2 c_{12}^2 = l_2^2 (s_{12}^2 + c_{12}^2)$$

$$= l_2^2$$

$$\Rightarrow K_2 = \frac{1}{2} m_2 \left( \left(l_1^2 + l_2^2 + 2 l_1 l_2 c_2\right)\dot{\phi}_1^2 + 2\left(l_2^2 + l_1 l_2 c_2\right)\dot{\phi}_1 \dot{\phi}_2 + l_2^2 \dot{\phi}_2^2 \right)$$

now calculate the potential energy from gravity

gravity force

$$f_g = -m_i g e_y = -m_i g \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P_1 = -f_g^T P_1 = -\left(-m_1 g \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^T \left( l_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} \right)$$

$$P_1 = m_1 g l_1 s_1$$

$$P_2 = -f_g^T P_2 = -\left(-m_2 g \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^T \left( l_1 \begin{bmatrix} c_1 \\ s_1 \end{bmatrix} + l_2 \begin{bmatrix} c_{12} \\ s_{12} \end{bmatrix} \right)$$

$$P_2 = m_2 g l_1 s_1 + m_2 g l_2 s_{12}$$

$$P_2 = m_2 g \left(l_1 s_1 + l_2 s_{12}\right)$$

now we can calculate the Lagrangian

$$\mathcal{L} = \sum_i K_i - P_i$$

$$\mathcal{L} = K_1 - P_1 + K_2 - P_2$$

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2$$

$$P_1 = m_1 g l_1 s_1$$

$$K_2 = \frac{1}{2} m_2 \left( \left(l_1^2 + l_2^2 + 2 l_1 l_2 c_2\right)\dot{\phi}_1^2 + 2\left(l_2^2 + l_1 l_2 c_2\right)\dot{\phi}_1 \dot{\phi}_2 + l_2^2 \dot{\phi}_2^2 \right)$$

$$P_2 = m_2 g \left(l_1 s_1 + l_2 s_{12}\right)$$

$$\tau_i = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} - \frac{\partial \mathcal{L}}{\partial \phi_i}$$

$$\tau_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} - \frac{\partial \mathcal{L}}{\partial \phi_i}$$

so lets calculate all the partials

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \frac{\partial K_1}{\partial \phi_1} - \frac{\partial P_1}{\partial \phi_1} + \frac{\partial K_2}{\partial \phi_1} - \frac{\partial P_2}{\partial \phi_1} \quad \text{same for 2}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = \frac{d}{dt}\left(\frac{\partial K_1}{\partial \dot{\phi}_1} - \frac{\partial P_1}{\partial \dot{\phi}_1} + \frac{\partial K_2}{\partial \dot{\phi}_1} - \frac{\partial P_2}{\partial \dot{\phi}_1}\right) \quad \text{same for 2}$$

start with $K_1$

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2$$

$$\frac{\partial K_1}{\partial \phi_1} = 0$$

$$\frac{\partial K_1}{\partial \phi_2} = 0$$

$$\frac{d}{dt}\frac{\partial K_1}{\partial \dot{\phi}_1} = \frac{d}{dt} 2\left(\frac{1}{2} m_1 l_1^2 \dot{\phi}_1\right) = m_1 l_1^2 \ddot{\phi}_1$$

$$\frac{d}{dt}\frac{\partial K_1}{\partial \dot{\phi}_2} = 0$$

$$P_1 = m g l_1 s_1$$

$$-\frac{\partial P_1}{\partial \phi_1} = -m_1 g l_1 c_1$$

$$-\frac{\partial P_2}{\partial \phi_2} = 0$$

$$-\frac{d}{dt}\frac{\partial P_1}{\partial \dot{\phi}_1} = 0$$

$$-\frac{d}{dt}\frac{\partial P_1}{\partial \dot{\phi}_2} = 0$$

$$K_2 = \frac{1}{2} m_2 \left(\left(l_1^2 + l_2^2 + 2 l_1 l_2 c_2\right)\dot{\phi}_1^2 + 2\left(l_2^2 + l_1 l_2 c_2\right)\dot{\phi}_1\dot{\phi}_2 + l_2^2 \dot{\phi}_2^2\right)$$

$$\frac{\partial K_2}{\partial \phi_1} = 0$$

$$\frac{\quad}{\partial \phi_1} = 0$$

$$\frac{\partial K_2}{\partial \phi_2} = \cancel{\tfrac{1}{2}} m_2\left((-\cancel{2} l_1 l_2 s_2)\dot{\phi}_1^2 + \cancel{2}(-l_1 l_2 s_2)\dot{\phi}_1\dot{\phi}_2\right) = -m_2 l_1 l_2 s_2(\dot{\phi}_1^2 + \dot{\phi}_1\dot{\phi}_2)$$

$$\frac{d}{dt}\frac{\partial K_2}{\partial \dot{\phi}_1} = \frac{d}{dt}\left(\cancel{\tfrac{1}{2}} m_2\left(\cancel{2}(l_1^2 + l_2^2 + 2 l_1 l_2 c_2)\dot{\phi}_1 + \cancel{2}(l_2^2 + l_1 l_2 c_2)\dot{\phi}_2\right)\right)$$

$$= m_2\left(-2 l_1 l_2 s_2 \dot{\phi}_2 \dot{\phi}_1 + (l_1^2 + l_2^2 + 2 l_1 l_2 c_2)\ddot{\phi}_1\right)$$
$$+ m_2\left(-l_1 l_2 s_2 \dot{\phi}_2^2 + (l_2^2 + l_1 l_2 c_2)\ddot{\phi}_2\right)$$

$$\frac{d}{dt}\frac{\partial K_2}{\partial \dot{\phi}_2} = \frac{d}{dt}\left(\cancel{\tfrac{1}{2}} m_2\left(\cancel{2}(l_2^2 + l_1 l_2 c_2)\dot{\phi}_1 + \cancel{2} l_2^2 \dot{\phi}_2\right)\right)$$

$$= m_2\left(-l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2 + (l_2^2 + l_1 l_2 c_2)\ddot{\phi}_1 + l_2^2 \ddot{\phi}_2\right)$$

$$P_2 = m_2 g(l_1 s_1 + l_2 s_{12}) = m_2 g\left(l_1 s_1 + l_2(s_1 c_2 + c_1 s_2)\right)$$

$$-\frac{\partial P_2}{\partial \phi_1} = -\left(m_2 g l_1 c_1 + m_2 g l_2(c_1 c_2 - s_1 s_2)\right) = -m_2 g(l_1 c_1 + l_2 c_{12})$$

$$-\frac{\partial P_2}{\partial \phi_2} = -m_2 g l_2(-s_1 s_2 + c_1 c_2) = -m_2 g l_2 c_{12}$$

$$-\frac{d}{dt}\frac{\partial P_2}{\partial \dot{\phi}_1} = 0$$

$$-\frac{d}{dt}\frac{\partial P_2}{\partial \dot{\phi}_2} = 0$$

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_1} - \frac{\partial L}{\partial \phi_1}, \quad L = K_1 - P_1 + K_2 - P_2$$

$$= m_1 l_1^2 \ddot{\phi}_1 + m_2\left(-2 l_1 l_2 s_2 \dot{\phi}_2 \dot{\phi}_1 + (l_1^2 + l_2^2 + 2 l_1 l_2 c_2)\ddot{\phi}_1\right)$$
$$+ m_2\left(-l_1 l_2 s_2 \dot{\phi}_2^2 + (l_2^2 + l_1 l_2 c_2)\ddot{\phi}_2\right)$$
$$-(-m_1 g l_1 c_1) - \left(-m_2 g(l_1 c_1 + l_2 c_{12})\right)$$

$$\tau_1 = (m_1 l_1^2 + m_2(l_1^2 + l_2^2 + 2 l_1 l_2 c_2)\ddot{\phi}_1$$
$$+ m_2(l_2^2 + l_1 l_2 c_2)\ddot{\phi}_2$$
$$- 2 m_2 l_1 l_2 s_2(\dot{\phi}_1 \dot{\phi}_2 + \dot{\phi}_2^2)$$

$$+(m_1+m_2) g l c_1 + m_2 g l_2 c_{12}$$

$$\tau_2 = m_2\left(-l_1 l_2 s_2 \dot\phi_1 \dot\phi_2 + (l_2^2 + l_1 l_2 c_2)\ddot\phi_1 + l_2^2 \ddot\phi_2\right)$$
$$-\left(-m_2 l_1 l_2 s_2 (\dot\phi_1^2 + \dot\phi_1 \dot\phi_2)\right)$$
$$-\left(-m_2 g l_2 c_{12}\right)$$

$$\tau_2 = m_2 (l_2^2 + l_1 l_2 c_2)\ddot\phi_1 + m_2 l_2^2 \ddot\phi_2$$
$$+ m_2 l_1 l_2 s_2 \dot\phi_1^2$$
$$+ m_2 g l_2 c_{12}$$

alright now we can write in a form everyone may be familiar with

$$M(\phi)\ddot\phi + C(\phi,\dot\phi) + G(\phi) = \tau$$

$$M(\phi) = \begin{bmatrix} m_1 l_1^2 + m_2(l_1^2 + l_2^2 + 2 l_1 l_2 c_2) & m_2(l_2^2 + l_1 l_2 c_2) \\ m_2(l_2^2 + l_1 l_2 c_2) & m_2 l_2^2 \end{bmatrix}$$

$$C(\phi,\dot\phi) = \begin{bmatrix} -2 m_2 l_1 l_2 s_2 (\dot\phi_1 \dot\phi_2 + \dot\phi_2^2) \\ m_2 l_1 l_2 s_2 \dot\phi_1^2 \end{bmatrix}$$

the $\dot\phi_1^2$ & $\dot\phi_2^2$ terms are called Centripetal terms

the $\dot\phi_1 \dot\phi_2$ terms are called the Coriolis terms

$$G(\phi) = \begin{bmatrix} (m_1+m_2) g l c_1 + m_2 g l_2 c_{12} \\ m_2 g l_2 c_{12} \end{bmatrix}$$