Two-Link Integral Concurrent Learning Control

I. Two-Link Dynamics

Again consider the two-link dynamic system

$$M(\phi)\ddot{\phi} + C\left(\phi, \dot{\phi}\right) + G(\phi) = \tau$$

where $\phi(t) \triangleq \begin{bmatrix}\phi_1(t)\\ \phi_2(t)\end{bmatrix}, \dot{\phi}(t) \triangleq \begin{bmatrix}\dot{\phi}_1(t)\\ \dot{\phi}_2(t)\end{bmatrix}, \ddot{\phi}(t) \triangleq \begin{bmatrix}\ddot{\phi}_1(t)\\ \ddot{\phi}_2(t)\end{bmatrix}, \tau(t) \triangleq \begin{bmatrix}\tau_1(t)\\ \tau_2(t)\end{bmatrix} \in \mathbb{R}^2$ are the angle, angular velocity, angular acceleration, and input torque of the arm joints, the inertia terms $M(\phi)$, the Coriolis and centripetal terms $C\left(\phi, \dot{\phi}\right)$, and the gravity terms $G(\phi)$ are defined as

$$\begin{aligned}
M(\phi) &\triangleq \begin{bmatrix} m_1 l_1^2 + m_2\left(l_1^2 + 2 l_1 l_2 c_2 + l_2^2\right) & m_2\left(l_1 l_2 c_2 + l_2^2\right) \\ m_2\left(l_1 l_2 c_2 + l_2^2\right) & m_2 l_2^2 \end{bmatrix}, \\
C\left(\phi, \dot{\phi}\right) &\triangleq \begin{bmatrix} -2 m_2 l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\phi}_2^2 \\ m_2 l_1 l_2 s_2 \dot{\phi}_1^2 \end{bmatrix}, \\
G(\phi) &\triangleq \begin{bmatrix} (m_1 + m_2) g l_1 c_1 + m_2 g l_2 c_{12} \\ m_2 g l_2 c_{12} \end{bmatrix},
\end{aligned}$$

$m_i, l_i \in \mathbb{R}_{>0}$ are the **unknown** constant mass and length of link i, $g \in \mathbb{R}_{>0}$ is gravity,

$$\begin{aligned}
c_i &= \cos(\phi_i), \\
s_i &= \sin(\phi_i), \\
c_{12} &= \cos(\phi_1 + \phi_2), \\
s_{12} &= \sin(\phi_1 + \phi_2).
\end{aligned}$$

As shown in the previous section, the above dynamics are linear in the unknown parameters and we were able to develop a gradient update law for the unknown parameters

$$\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{bmatrix} = \begin{bmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 \\ m_2 l_1 l_2 \\ m_2 l_2^2 \\ (m_1 + m_2) l_1 \\ m_2 l_2 \end{bmatrix}$$

and we were able to develop an update law of the form

$$\dot{\hat{\theta}} \triangleq \Gamma Y^\top r$$

from the analysis since we showed that

$$Y\theta = M(\phi)\varphi + C\left(\phi, \dot{\phi}\right) + G(\phi) + \frac{1}{2}\dot{M}\left(\phi, \dot{\phi}\right) r$$

where $\varphi \triangleq \left(\ddot{\phi}_d + \alpha \dot{e}\right)$

$$\begin{aligned}
Y &\triangleq Y_M(\phi, \varphi) + Y_C\left(\phi, \dot{\phi}\right) + Y_G(\phi) + \frac{1}{2} Y_{\dot{M}}(\phi, r), \\
Y_M(\phi, \varphi) &\triangleq \begin{bmatrix} \varphi_1 & (2 c_2 \varphi_1 + c_2 \varphi_2) & \varphi_2 & 0 & 0 \\ 0 & c_2 \varphi_1 & (\varphi_1 + \varphi_2) & 0 & 0 \end{bmatrix}, \\
Y_C\left(\phi, \dot{\phi}\right) &\triangleq \begin{bmatrix} 0 & -\left(2 s_2 \dot{\phi}_1 \dot{\phi}_2 + s_2 \dot{\phi}_2^2\right) & 0 & 0 & 0 \\ 0 & s_2 \dot{\phi}_1^2 & 0 & 0 & 0 \end{bmatrix}, \\
Y_G(\phi) &\triangleq \begin{bmatrix} 0 & 0 & 0 & g c_1 & g c_{12} \\ 0 & 0 & 0 & 0 & g c_{12} \end{bmatrix}, \\
Y_{\dot{M}}(\phi) &\triangleq \begin{bmatrix} 0 & -\left(2 s_2 \dot{\phi}_2 r_1 + s_2 \dot{\phi}_2 r_2\right) & 0 & 0 & 0 \\ 0 & -s_2 \dot{\phi}_2 r_1 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Now we wish to examine how to use data based methods to determine better estimates of the parameters.

A. Integral Concurrent Learning

Just like with CL, we will form a data-based learning regressor using the dynamics. Recall that for the above system we would do a CL update as

$$\mathcal{Y}_{CL}\theta = M(\phi)\ddot{\phi} + C\left(\phi, \dot{\phi}\right) + G(\phi).$$

Now if we compare this relationship with $Y\theta$, we see that the only differences between the above and $Y\theta$ are that $Y_M(\phi, \varphi)$ is a function of φ which can be replaced by $\ddot{\phi}$ since they are mapped the same way and that $\frac{1}{2}Y_{\dot{M}}(\phi, r)$ is added at the end. This implies that

$$\mathcal{Y}_{CL}(t) \triangleq Y_M\left(\phi, \ddot{\phi}\right) + Y_C\left(\phi, \dot{\phi}\right) + Y_G(\phi)$$

where

$$Y_M\left(\phi, \ddot{\phi}\right) \triangleq \begin{bmatrix} \ddot{\phi}_1 & \left(2c_2\ddot{\phi}_1 + c_2\ddot{\phi}_2\right) & \ddot{\phi}_2 & 0 & 0 \\ 0 & c_2\ddot{\phi}_1 & \left(\ddot{\phi}_1 + \ddot{\phi}_2\right) & 0 & 0 \end{bmatrix}.$$

Now using this we can develop a CL approach using

$$\mathcal{Y}_{IC}\theta = \tau.$$

Now instead, ICL takes the integral of the dynamics over a finite window and uses that for learning instead. Specifically we integrate as

$$\underbrace{\int_{t-\Delta t}^{t} \mathcal{Y}_{CL}(\iota)\, d\iota}_{\mathcal{Y}_{ICL}(t)}\theta = \underbrace{\int_{t-\Delta t}^{t} \tau(\iota)\, d\iota}_{\mathcal{U}_{ICL}(t)}$$
$$\mathcal{Y}_{ICL}\theta = \mathcal{U}_{ICL}$$

where Δt is the size of the integration window. Now with this integral we essentially are filtering the concurrent learning data and in cases where we may not be able to measure the highest order derivative or it is noisy we could do integration by parts however we will just assume it is measurable. From here, we can design an update law just like before with the CL update to learn θ. Now lets work on the design for this system. Now let

$$\mathcal{Y} \triangleq \mathcal{Y}_{ICL}$$
$$\mathcal{U} \triangleq \mathcal{U}_{ICL}$$

and we will take a sum over a set of this data collected online. Specifically, we can determine how to augment the update law so we can use the above relationship. Here we multiply both sides by $\mathcal{Y}^\top$to yield

$$\mathcal{Y}^\top\mathcal{Y}\theta = \mathcal{Y}^\top\mathcal{U}$$

which is true for any time t_j

$$\mathcal{Y}^\top(t_j)\,\mathcal{Y}(t_j)\,\theta = \mathcal{Y}^\top(t_j)\,\mathcal{U}(t_j).$$

The ICL approach then takes a sum over a set of these terms to get

$$\sum_{j=1}^{N}\mathcal{Y}^\top(t_j)\,\mathcal{Y}(t_j)\,\theta = \sum_{j=1}^{N}\mathcal{Y}^\top(t_j)\,\mathcal{U}(t_j)$$

where $N \in \mathbb{Z}_{>0}$ is the number of data points required that resulted in

$$\lambda_{\min}\left\{\sum_{j=1}^{N}\mathcal{Y}^\top(t_j)\,\mathcal{Y}(t_j)\right\} > \lambda_{CL},\, t \geq T_{CL} > \Delta t,$$

$\lambda_{CL} \in \mathbb{R}_{>0}$ is a user selected minimum eigenvalue condition, and $T_{CL} \in \mathbb{R}_{>0}$ is the time the condition was satisfied. The above relationship can now be used to determine a relationship for $\widetilde{\theta}$ since

$$\widetilde{\theta}(t) = \theta - \widehat{\theta}(t),$$

we can multiple both sides by the relationship $\sum_{j=1}^{N}\mathcal{Y}^\top(t_j)\,\mathcal{Y}(t_j)$ to yield

$$\sum_{j=1}^{N}\mathcal{Y}^\top(t_j)\,\mathcal{Y}(t_j)\,\widetilde{\theta}(t) = \sum_{j=1}^{N}\mathcal{Y}^\top(t_j)\,\mathcal{Y}(t_j)\,\theta - \sum_{j=1}^{N}\mathcal{Y}^\top(t_j)\,\mathcal{Y}(t_j)\,\widehat{\theta}(t).$$

Now using the above relationship we know $\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{Y}(t_j)\theta=\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{U}(t_j)$ so

$$\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t)=\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{U}(t_j)-\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{Y}(t_j)\widehat{\theta}(t).$$

Since the right side of this relationship only contains measurable quantities, we can use them to determine and once enough data is collected, the eigenvalue condition holds and we can use the data to improve the update law.

Remark 1. Notice that once we get the dynamics into the regressor form, it follows essentially the same as the mass-spring-damper example.

Since the right side of this relationship only contains measurable quantities, we can use them to determine an update law. Once enough data is collected, the eigenvalue condition holds and we can use the data to improve the update law as

$$\dot{\widehat{\theta}}(t)\triangleq\Gamma Y^{\top}(t)r(t)+\Gamma k_{CL}\left(\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{U}(t_j)-\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{Y}(t_j)\widehat{\theta}(t)\right)$$

which is implementable and as shown above is equivalent to

$$\dot{\widehat{\theta}}(t)\triangleq\Gamma Y^{\top}(t)r(t)+\Gamma k_{CL}\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t)$$

which we can use for analysis.

1) Analysis: Using this changed update law, the analysis follows identically to the traditional concurrent learning method. Lets examine starting off where the input and update law were defined in the previous section and $\dot{V}(\zeta(t),t)$ became

$$\dot{V}(\zeta(t),t)=-e^{\top}\alpha e-r^{\top}\beta r+\widetilde{\theta}^{\top}Y^{\top}r-\widetilde{\theta}^{\top}\Gamma^{-1}\dot{\widehat{\theta}}$$

now we can use the analysis form of the above design of $\dot{\widehat{\theta}}$ in $\dot{V}(\zeta(t),t)$ to yield

$$\dot{V}(\zeta(t),t)=-e^{\top}\alpha e-r^{\top}\beta r+\widetilde{\theta}^{\top}Y^{\top}r-\widetilde{\theta}^{\top}\Gamma^{-1}\left(\Gamma Y^{\top}(t)r(t)+\Gamma k_{CL}\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t)\right)$$

$$\dot{V}(\zeta(t),t)=-e^{\top}\alpha e-r^{\top}\beta r+\widetilde{\theta}^{\top}Y^{\top}r-\widetilde{\theta}^{\top}Y^{\top}r-\widetilde{\theta}^{\top}k_{CL}\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t)$$

$$\dot{V}(\zeta(t),t)=-e^{\top}\alpha e-r^{\top}\beta r-\widetilde{\theta}^{\top}k_{CL}\sum_{j=1}^{N}\mathcal{Y}^{\top}(t_j)\mathcal{Y}(t_j)\widetilde{\theta}(t)$$

$$\dot{V}(\zeta(t),t)\leq\begin{cases}-\underline{\alpha}\|e\|^2-\underline{\beta}\|r\|^2 & \forall t<T_{CL}\\ -\underline{\alpha}\|e\|^2-\underline{\beta}\|r\|^2-k_{CL}\lambda_{CL}\|\widetilde{\theta}\|^2, & \forall t\geq T_{CL}\end{cases}$$

$$\dot{V}(\zeta(t),t)\leq\begin{cases}-\underline{\alpha}\|e\|^2-\underline{\beta}\|r\|^2 & \forall t<T_{CL}\\ -\min\left\{\underline{\alpha},\underline{\beta},k_{CL}\lambda_{CL}\right\}\|\zeta\|^2, & \forall t\geq T_{CL}\end{cases}$$

We can use the Lyapunov candidate bounds to show

$$\frac{1}{2}\min\left\{1,\lambda_{\min}\left\{M(\phi)\right\},\lambda_{\min}\left\{\Gamma^{-1}\right\}\right\}\|\zeta\|^2\leq V(\zeta(t),t)\leq\frac{1}{2}\max\left\{1,\lambda_{\max}\left\{M(\phi)\right\},\lambda_{\max}\left\{\Gamma^{-1}\right\}\right\}\|\zeta\|^2$$

$$-2V(\zeta(t),t)\geq-\max\left\{1,\lambda_{\max}\left\{M(\phi)\right\},\lambda_{\max}\left\{\Gamma^{-1}\right\}\right\}\|\zeta\|^2$$

$$-\frac{2}{\max\left\{1,\lambda_{\max}\left\{M(\phi)\right\},\lambda_{\max}\left\{\Gamma^{-1}\right\}\right\}}V(\zeta(t),t)\geq-\|\zeta\|^2$$

which is substituted into the above relationship yielding

$$\dot{V}(\zeta(t),t)\leq\begin{cases}-\underline{\alpha}\|e\|^2-\underline{\beta}\|r\|^2 & \forall t<T_{CL}\\ -\frac{2\min\left\{\underline{\alpha},\underline{\beta},k_{CL}\lambda_{CL}\right\}}{\max\left\{1,\lambda_{\max}\left\{M(\phi)\right\},\lambda_{\max}\left\{\Gamma^{-1}\right\}\right\}}V(\zeta(t),t), & \forall t\geq T_{CL}\end{cases}$$

and let

$$\beta_v\triangleq\frac{2\min\left\{\underline{\alpha},\underline{\beta},k_{CL}\lambda_{CL}\right\}}{\max\left\{1,\lambda_{\max}\left\{M(\phi)\right\},\lambda_{\max}\left\{\Gamma^{-1}\right\}\right\}}$$

$$\dot{V}\left(\zeta\left(t\right),t\right)\leq\begin{cases}0 & \forall t<T_{CL}\\ -\beta_{v}V\left(\zeta\left(t\right),t\right), & \forall t\geq T_{CL}\end{cases}$$

$$V\left(\zeta\left(t\right),t\right)\leq\begin{cases}V\left(\zeta\left(0\right),0\right) & \forall t<T_{CL}\\ V\left(\zeta\left(T_{CL}\right),T_{CL}\right)\exp\left(-\beta_{v}\left(t-T_{CL}\right)\right), & \forall t\geq T_{CL}\end{cases}$$

which implies

$$V\left(\zeta\left(t\right),t\right)\leq V\left(\zeta\left(0\right),0\right)\exp\left(\beta_{v}\right)\exp\left(-\beta_{v}t\right).$$