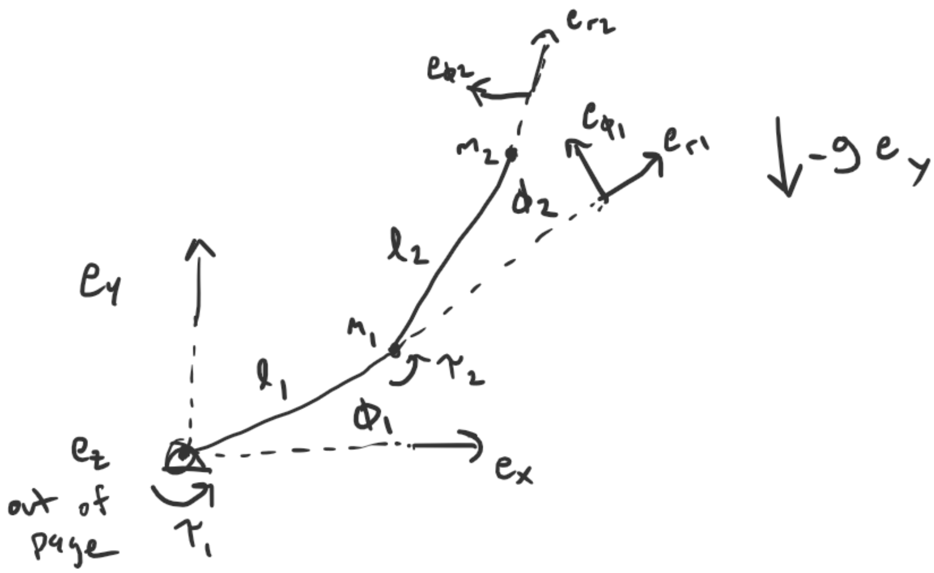
Two-Link Gradient Adaptive Controller



I. Dynamics

Consider the two-link dynamic system

$$M(\phi)\ddot{\phi} + C\left(\phi, \dot{\phi}\right) + G(\phi) = \tau$$

where $\phi(t) \triangleq \begin{bmatrix}\phi_1(t)\\ \phi_2(t)\end{bmatrix}, \dot{\phi}(t) \triangleq \begin{bmatrix}\dot{\phi}_1(t)\\ \dot{\phi}_2(t)\end{bmatrix}, \ddot{\phi}(t) \triangleq \begin{bmatrix}\ddot{\phi}_1(t)\\ \ddot{\phi}_2(t)\end{bmatrix}, \tau(t) \triangleq \begin{bmatrix}\tau_1(t)\\ \tau_2(t)\end{bmatrix} \in \mathbb{R}^2$ are the angle, angular velocity, angular acceleration, and input torque of the arm joints, the inertia terms $M(\phi)$, the Coriolis and centripetal terms $C\left(\phi, \dot{\phi}\right)$, and the gravity terms $G(\phi)$ are defined as

$$\begin{aligned}
M(\phi) &\triangleq \begin{bmatrix} m_1 l_1^2 + m_2\left(l_1^2 + 2 l_1 l_2 c_2 + l_2^2\right) & m_2\left(l_1 l_2 c_2 + l_2^2\right) \\ m_2\left(l_1 l_2 c_2 + l_2^2\right) & m_2 l_2^2 \end{bmatrix}, \\
C\left(\phi, \dot{\phi}\right) &\triangleq \begin{bmatrix} -2 m_2 l_1 l_2 s_2 \dot{\phi}_1 \dot{\phi}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\phi}_2^2 \\ m_2 l_1 l_2 s_2 \dot{\phi}_1^2 \end{bmatrix}, \\
G(\phi) &\triangleq \begin{bmatrix} (m_1 + m_2) g l_1 c_1 + m_2 g l_2 c_{12} \\ m_2 g l_2 c_{12} \end{bmatrix},
\end{aligned}$$

$m_i, l_i \in \mathbb{R}_{>0}$ are the **unknown** constant mass and length of link i, $g \in \mathbb{R}_{>0}$ is gravity,

$$\begin{aligned}
c_i &= \cos(\phi_i), \\
s_i &= \sin(\phi_i), \\
c_{12} &= \cos(\phi_1 + \phi_2), \\
s_{12} &= \sin(\phi_1 + \phi_2).
\end{aligned}$$

A. Controller Design

1) Error Development: Assume we want to track the desired angles, angular velocities, and angular accelerations $\phi_d(t), \dot{\phi}_d(t), \ddot{\phi}_d(t) \in \mathbb{R}^2$. Then let the tracking error be defined as

$$\begin{aligned}
e &\triangleq \phi_d - \phi \\
\dot{e} &= \dot{\phi}_d - \dot{\phi} \\
\ddot{e} &= \ddot{\phi}_d - \ddot{\phi}
\end{aligned}$$

and let the filtered tracking be defined as

$$r = \dot{e} + \alpha e$$
$$\dot{r} = \ddot{e} + \alpha \dot{e}$$

where $\alpha \in \mathbb{R}^{2\times 2}$ is a positive definite design gain matrix (usually a diagonal). Now we want to substitute our dynamics so multiply both sides by $M(\phi)$

$$\begin{aligned}
M(\phi)\dot{r} &= M(\phi)\ddot{e} + M(\phi)\alpha\dot{e} \\
M(\phi)\dot{r} &= M(\phi)\left(\ddot{\phi}_d - \ddot{\phi}\right) + M(\phi)\alpha\dot{e} \\
M(\phi)\dot{r} &= M(\phi)\ddot{\phi}_d - M(\phi)\ddot{\phi} + M(\phi)\alpha\dot{e}
\end{aligned}$$

where since

$$M(\phi)\ddot{\phi} + C\left(\phi,\dot{\phi}\right) + G(\phi) = \tau$$
$$\implies M(\phi)\ddot{\phi} = -C\left(\phi,\dot{\phi}\right) - G(\phi) + \tau$$

we get

$$\begin{aligned}
M(\phi)\dot{r} &= M(\phi)\ddot{\phi}_d - \left(-C\left(\phi,\dot{\phi}\right) - G(\phi) + \tau\right) + M(\phi)\alpha\dot{e} \\
M(\phi)\dot{r} &= M(\phi)\ddot{\phi}_d + C\left(\phi,\dot{\phi}\right) + G(\phi) - \tau + M(\phi)\alpha\dot{e} \\
M(\phi)\dot{r} &= M(\phi)\left(\ddot{\phi}_d + \alpha\dot{e}\right) + C\left(\phi,\dot{\phi}\right) + G(\phi) - \tau
\end{aligned}$$

Since the masses and lengths are unknowns we will have an approximation error of the unknowns $\theta \in \mathbb{R}^p$, $\widetilde{\theta}(t) \in \mathbb{R}^p$

$$\widetilde{\theta}(t) \triangleq \theta - \widehat{\theta}(t)$$
$$\dot{\widetilde{\theta}}(t) \triangleq -\dot{\widehat{\theta}}(t)$$

where $\widehat{\theta}(t) \in \mathbb{R}^p$ is the estimate and $\dot{\widehat{\theta}}(t) \in \mathbb{R}^p$ is the adaptive update law.

2) Analysis: Let the stacked error be defined as

$$\zeta(t) \triangleq \begin{bmatrix} e(t) \\ r(t) \\ \widetilde{\theta}(t) \end{bmatrix}$$

and choose a Lyapunov candidate where $M(\phi)$ is positive definite

$$\begin{aligned}
V(\zeta(t),t) &\triangleq \frac{1}{2}e^\top e + \frac{1}{2}r^\top M(\phi) r + \frac{1}{2}\widetilde{\theta}^\top \Gamma^{-1}\widetilde{\theta} \\
\dot{V}(\zeta(t),t) &= e^\top \dot{e} + \frac{1}{2}r^\top \dot{M}(\phi) r + r^\top M(\phi)\dot{r} + \widetilde{\theta}^\top \Gamma^{-1}\dot{\widetilde{\theta}}
\end{aligned}$$

Since

$$r = \dot{e} + \alpha e$$
$$\implies \dot{e} = r - \alpha e$$

$$\begin{aligned}
\dot{M}(\phi) &= \frac{d}{dt}(M(\phi)) = \frac{d}{dt}\left(\begin{bmatrix} m_1 l_1^2 + m_2\left(l_1^2 + 2l_1 l_2 c_2 + l_2^2\right) & m_2\left(l_1 l_2 c_2 + l_2^2\right) \\ m_2\left(l_1 l_2 c_2 + l_2^2\right) & m_2 l_2^2 \end{bmatrix}\right) \\
\dot{M}(\phi) &= \begin{bmatrix} m_2\left(-2l_1 l_2 s_2 \dot{\phi}_2\right) & m_2\left(-l_1 l_2 s_2 \dot{\phi}_2\right) \\ m_2\left(-l_1 l_2 s_2 \dot{\phi}_2\right) & 0 \end{bmatrix} \\
\dot{M}(\phi) &= \begin{bmatrix} -2m_2 l_1 l_2 s_2 \dot{\phi}_2 & -m_2 l_1 l_2 s_2 \dot{\phi}_2 \\ -m_2 l_1 l_2 s_2 \dot{\phi}_2 & 0 \end{bmatrix}
\end{aligned}$$

$$M(\phi)\dot{r} = M(\phi)\left(\ddot{\phi}_d + \alpha\dot{e}\right) + C\left(\phi,\dot{\phi}\right) + G(\phi) - \tau$$

$$\dot{\widetilde{\theta}} = -\dot{\widehat{\theta}}$$

$$\dot{V}(\zeta(t),t) \triangleq e^\top \dot{e} + \frac{1}{2} r^\top \dot{M}(\phi) r + r^\top M(\phi)\dot{r} + \widetilde{\theta}^\top \Gamma^{-1} \dot{\widetilde{\theta}}$$
$$\dot{V}(\zeta(t),t) = e^\top r - e^\top \alpha e + \frac{1}{2} r^\top \dot{M}(\phi) r + r^\top \left(M(\phi)\left(\ddot{\phi}_d + \alpha\dot{e}\right) + C\left(\phi,\dot{\phi}\right) + G(\phi) - \tau \right) - \widetilde{\theta}^\top \Gamma^{-1} \dot{\widehat{\theta}}$$

pull the $\dot{M}$ in

$$\dot{V}(\zeta(t),t) = -e^\top \alpha e + e^\top r + r^\top \left(M(\phi)\left(\ddot{\phi}_d + \alpha\dot{e}\right) + C\left(\phi,\dot{\phi}\right) + G(\phi) + \frac{1}{2}\dot{M}(\phi) r - \tau \right) - \widetilde{\theta}^\top \Gamma^{-1} \dot{\widehat{\theta}}$$

now we need to show that

$$Y\theta = M(\phi)\underbrace{\left(\ddot{\phi}_d + \alpha\dot{e}\right)}_{\varphi} + C\left(\phi,\dot{\phi}\right) + G(\phi) + \frac{1}{2}\dot{M}(\phi) r$$

is linear in the parameters so lets simplify each one a bit to find the minimal realization

$$M(\phi)\varphi = \begin{bmatrix} m_1 l_1^2 + m_2\left(l_1^2 + 2l_1 l_2 c_2 + l_2^2\right) & m_2\left(l_1 l_2 c_2 + l_2^2\right) \\ m_2\left(l_1 l_2 c_2 + l_2^2\right) & m_2 l_2^2 \end{bmatrix}\varphi$$
$$M(\phi)\varphi = \begin{bmatrix} m_1 l_1^2 + m_2 l_1^2 + 2m_2 l_1 l_2 c_2 + m_2 l_2^2 & m_2 l_1 l_2 c_2 + m_2 l_2^2 \\ m_2 l_1 l_2 c_2 + m_2 l_2^2 & m_2 l_2^2 \end{bmatrix}\varphi$$
$$M(\phi)\varphi = \begin{bmatrix} \overbrace{m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2}^{\theta_1} + 2\overbrace{m_2 l_1 l_2}^{\theta_2} c_2 & \overbrace{m_2 l_1 l_2}^{\theta_2} c_2 + \overbrace{m_2 l_2^2}^{\theta_3} \\ \overbrace{m_2 l_1 l_2}^{\theta_2} c_2 + \overbrace{m_2 l_2^2}^{\theta_3} & \overbrace{m_2 l_2^2}^{\theta_3} \end{bmatrix}\varphi$$
$$M(\phi)\varphi = \begin{bmatrix} \theta_1 + 2\theta_2 c_2 & \theta_3 + \theta_2 c_2 \\ \theta_3 + \theta_2 c_2 & \theta_3 \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$
$$M(\phi)\varphi = \begin{bmatrix} (\theta_1 + 2\theta_2 c_2)\varphi_1 + (\theta_3 + \theta_2 c_2)\varphi_2 \\ (\theta_3 + \theta_2 c_2)\varphi_1 + \theta_3\varphi_2 \end{bmatrix}$$
$$M(\phi)\varphi = \begin{bmatrix} \varphi_1\theta_1 + (2c_2\varphi_1 + c_2\varphi_2)\theta_2 + \varphi_2\theta_3 \\ c_2\varphi_1\theta_2 + (\varphi_1 + \varphi_2)\theta_3 \end{bmatrix}$$
$$M(\phi)\varphi = \underbrace{\begin{bmatrix} \varphi_1 & (2c_2\varphi_1 + c_2\varphi_2) & \varphi_2 & 0 & 0 \\ 0 & c_2\varphi_1 & (\varphi_1 + \varphi_2) & 0 & 0 \end{bmatrix}}_{Y_M(\varphi)} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}}_{\theta}$$
$$M(\phi)\varphi = Y_M(\phi,\varphi)\theta$$

$$C\left(\phi,\dot{\phi}\right) = \begin{bmatrix} -2m_2 l_1 l_2 s_2 \dot{\phi}_1\dot{\phi}_2 - m_2 l_1 l_2 s_2 \dot{\phi}_2^2 \\ m_2 l_1 l_2 s_2 \dot{\phi}_1^2 \end{bmatrix}$$
$$C\left(\phi,\dot{\phi}\right) = \begin{bmatrix} -2\overbrace{m_2 l_1 l_2}^{\theta_2} s_2 \dot{\phi}_1\dot{\phi}_2 - \overbrace{m_2 l_1 l_2}^{\theta_2} s_2 \dot{\phi}_2^2 \\ \overbrace{m_2 l_1 l_2}^{\theta_2} s_2 \dot{\phi}_1^2 \end{bmatrix}$$
$$C\left(\phi,\dot{\phi}\right) = \begin{bmatrix} -2\theta_2 s_2 \dot{\phi}_1\dot{\phi}_2 - \theta_2 s_2 \dot{\phi}_2^2 \\ \theta_2 s_2 \dot{\phi}_1^2 \end{bmatrix}$$
$$C\left(\phi,\dot{\phi}\right) = \begin{bmatrix} -\left(2s_2\dot{\phi}_1\dot{\phi}_2 + s_2\dot{\phi}_2^2\right)\theta_2 \\ s_2\dot{\phi}_1^2\theta_2 \end{bmatrix}$$
$$C\left(\phi,\dot{\phi}\right) = \underbrace{\begin{bmatrix} 0 & -\left(2s_2\dot{\phi}_1\dot{\phi}_2 + s_2\dot{\phi}_2^2\right) & 0 & 0 & 0 \\ 0 & s_2\dot{\phi}_1^2 & 0 & 0 & 0 \end{bmatrix}}_{Y_C\left(\phi,\dot{\phi}\right)} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}}_{\theta}$$
$$C\left(\phi,\dot{\phi}\right) = Y_C\left(\phi,\dot{\phi}\right)\theta$$

$$
\begin{aligned}
G(\phi) &= \begin{bmatrix} (m_1+m_2)\, g l_1 c_1 + m_2 g l_2 c_{12} \\ m_2 g l_2 c_{12} \end{bmatrix} \\
G(\phi) &= \begin{bmatrix} \overbrace{(m_1+m_2)\, g l_1}^{\theta_4} c_1 + \overbrace{m_2 g l_2}^{\theta_5} c_{12} \\ \overbrace{m_2 g l_2}^{\theta_5} c_{12} \end{bmatrix} \\
G(\phi) &= \begin{bmatrix} \theta_4 c_1 + \theta_5 c_{12} \\ \theta_5 c_{12} \end{bmatrix} \\
G(\phi) &= \underbrace{\begin{bmatrix} 0 & 0 & 0 & c_1 & c_{12} \\ 0 & 0 & 0 & 0 & c_{12} \end{bmatrix}}_{Y_G(\phi)} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}}_{\theta} \\
G(\phi) &= Y_G(\phi)\, \theta
\end{aligned}
$$

$$
\begin{aligned}
\dot{M}(\phi)\, r &= \begin{bmatrix} -2 m_2 l_1 l_2 s_2 \dot{\phi}_2 & -m_2 l_1 l_2 s_2 \dot{\phi}_2 \\ -m_2 l_1 l_2 s_2 \dot{\phi}_2 & 0 \end{bmatrix} r \\
\dot{M}(\phi)\, r &= \begin{bmatrix} -2 \overbrace{m_2 l_1 l_2}^{\theta_2} s_2 \dot{\phi}_2 & -\overbrace{m_2 l_1 l_2}^{\theta_2} s_2 \dot{\phi}_2 \\ -\overbrace{m_2 l_1 l_2}^{\theta_2} s_2 \dot{\phi}_2 & 0 \end{bmatrix} r \\
\dot{M}(\phi)\, r &= \begin{bmatrix} -2 \theta_2 s_2 \dot{\phi}_2 & -\theta_2 s_2 \dot{\phi}_2 \\ -\theta_2 s_2 \dot{\phi}_2 & 0 \end{bmatrix} r \\
\dot{M}(\phi)\, r &= \begin{bmatrix} -2 \theta_2 s_2 \dot{\phi}_2 & -\theta_2 s_2 \dot{\phi}_2 \\ -\theta_2 s_2 \dot{\phi}_2 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \\
\dot{M}(\phi)\, r &= \begin{bmatrix} -2 \theta_2 s_2 \dot{\phi}_2 r_1 - \theta_2 s_2 \dot{\phi}_2 r_2 \\ -\theta_2 s_2 \dot{\phi}_2 r_1 \end{bmatrix} \\
\dot{M}(\phi)\, r &= \begin{bmatrix} -\left(2 s_2 \dot{\phi}_2 r_1 + s_2 \dot{\phi}_2 r_2\right) \theta_2 \\ -s_2 \dot{\phi}_2 r_1 \theta_2 \end{bmatrix} \\
\dot{M}(\phi)\, r &= \underbrace{\begin{bmatrix} 0 & -\left(2 s_2 \dot{\phi}_2 r_1 + s_2 \dot{\phi}_2 r_2\right) & 0 & 0 & 0 \\ 0 & -s_2 \dot{\phi}_2 r_1 & 0 & 0 & 0 \end{bmatrix}}_{Y_{\dot{M}}(\phi)} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}}_{\theta} \\
\dot{M}(\phi)\, r &= Y_{\dot{M}}(\phi, r)\, \theta
\end{aligned}
$$

$$
\begin{aligned}
Y\theta &= Y_M(\phi, \varphi)\, \theta + Y_C\left(\phi, \dot{\phi}\right) \theta + Y_G(\phi)\, \theta + \frac{1}{2} Y_{\dot{M}}(\phi, r)\, \theta \\
Y\theta &= \underbrace{\left(Y_M(\phi, \varphi) + Y_C\left(\phi, \dot{\phi}\right) + Y_G(\phi) + \frac{1}{2} Y_{\dot{M}}(\phi, r)\right)}_{Y} \theta
\end{aligned}
$$

substituting back in

$$
\dot{V}(\zeta(t), t) = -e^\top \alpha e + e^\top r + r^\top (Y\theta - \tau) - \widetilde{\theta}^\top \Gamma^{-1} \dot{\widehat{\theta}}
$$

Now we can design the input as

$$
\tau \triangleq Y\widehat{\theta} + e + \beta r
$$

Substituting back in we get

$$
\begin{aligned}
\dot{V}\left(\zeta\left(t\right),t\right) &= -e^{\top}\alpha e+e^{\top}r+r^{\top}\left(Y\theta-\left(Y\widehat{\theta}+e+\beta r\right)\right)-\widetilde{\theta}^{\top}\Gamma^{-1}\dot{\widehat{\theta}}\\
\dot{V}\left(\zeta\left(t\right),t\right) &= -e^{\top}\alpha e+e^{\top}r+r^{\top}\left(Y\widetilde{\theta}-e-\beta r\right)-\widetilde{\theta}^{\top}\Gamma^{-1}\dot{\widehat{\theta}}\\
\dot{V}\left(\zeta\left(t\right),t\right) &= -e^{\top}\alpha e+e^{\top}r+r^{\top}Y\widetilde{\theta}-r^{\top}e-r^{\top}\beta r-\widetilde{\theta}^{\top}\Gamma^{-1}\dot{\widehat{\theta}}\\
\dot{V}\left(\zeta\left(t\right),t\right) &= -e^{\top}\alpha e-r^{\top}\beta r+r^{\top}Y\widetilde{\theta}-\widetilde{\theta}^{\top}\Gamma^{-1}\dot{\widehat{\theta}}\\
\dot{V}\left(\zeta\left(t\right),t\right) &= -e^{\top}\alpha e-r^{\top}\beta r+\widetilde{\theta}^{\top}Y^{\top}r-\widetilde{\theta}^{\top}\Gamma^{-1}\dot{\widehat{\theta}}
\end{aligned}
$$

now we can design $\dot{\widehat{\theta}}$ as

$$\dot{\widehat{\theta}}\triangleq\Gamma Y^{\top}r$$

and substituting back in we get

$$
\begin{aligned}
\dot{V}\left(\zeta\left(t\right),t\right) &= -e^{\top}\alpha e-r^{\top}\beta r+\widetilde{\theta}^{\top}Y^{\top}r-\widetilde{\theta}^{\top}\Gamma^{-1}\Gamma Y^{\top}r\\
\dot{V}\left(\zeta\left(t\right),t\right) &= -e^{\top}\alpha e-r^{\top}\beta r+\widetilde{\theta}^{\top}Y^{\top}r-\widetilde{\theta}^{\top}Y^{\top}r\\
\dot{V}\left(\zeta\left(t\right),t\right) &= -e^{\top}\alpha e-r^{\top}\beta r
\end{aligned}
$$

and now we could use Barbalat's Lemma to show $e\rightarrow 0$ and $r\rightarrow 0$ and we get GAT. Then we can do signal chasing to show τ is bounded.