Mass-Spring-Damper Integral Concurrent Learning Control

I. Introductory Problem

A. Dynamics

Before developing the use of ICL on the two-link arm, we will examine it on a angular rotated mass with a damper and spring

$$m\ddot{\phi}+c\dot{\phi}+k\phi=\tau$$

where the $m,c,k\in\mathbb{R}_{>0}$ are all unknown.

B. CL vs ICL

Just like with CL, we will form a data-based learning regressor using the dynamics. Recall that for the above system we would do a CL update as

$$\underbrace{\begin{bmatrix}\ddot{\phi} & \dot{\phi} & \phi\end{bmatrix}}_{\mathcal{Y}_{CL}(t)}\begin{bmatrix}m\\ c\\ k\end{bmatrix}=\underbrace{\tau}_{\mathcal{U}_{CL}(t)}$$
$$\mathcal{Y}_{CL}\theta=\mathcal{U}_{CL}.$$

Now instead, ICL takes the integral of the dynamics over a finite window and uses that for learning instead. Specifically we integrate as

$$\underbrace{\int\limits_{t-\Delta t}^{t}\begin{bmatrix}\ddot{\phi}(\iota) & \dot{\phi}(\iota) & \phi(\iota)\end{bmatrix}d\iota}_{\mathcal{Y}_{ICL}(t)}\begin{bmatrix}m\\ c\\ k\end{bmatrix}=\underbrace{\int\limits_{t-\Delta t}^{t}\tau(\iota)\,d\iota}_{\mathcal{U}_{ICL}(t)}$$
$$\mathcal{Y}_{ICL}\theta=\mathcal{U}_{ICL}$$

where Δt is the size of the integration window. Now with this integral we essentially are filtering the data and in cases where we may not be able to measure the highest order derivative or it is noisy, we take advantage of that filtering. Below are two forms of this update

$$\mathcal{Y}_{ICL}(t)\triangleq\int\limits_{t-\Delta t}^{t}\begin{bmatrix}\ddot{\phi}(\iota) & \dot{\phi}(\iota) & \phi(\iota)\end{bmatrix}d\iota$$
$$\mathcal{Y}_{ICL}(t)\triangleq\begin{bmatrix}\dot{\phi}(t)-\dot{\phi}(t-\Delta t) & \int\limits_{t-\Delta t}^{t}\dot{\phi}(\iota) & \int\limits_{t-\Delta t}^{t}\phi(\iota)\end{bmatrix}d\iota$$

where the first form is for when we can measure the acceleration and the second form is for when we cannot measure the acceleration. From here, we can design an update law just like before with the CL update to learn θ. Now lets work on the design for this system. Now let

$$\mathcal{Y}\triangleq\mathcal{Y}_{ICL}$$
$$\mathcal{U}\triangleq\mathcal{U}_{ICL}$$

and we will take a sum over a set of this data collected online. Specifically, we can determine how to augment the update law so we can use the above relationship. Here we multiply both sides by $\mathcal{Y}^{\top}$to yield

$$\mathcal{Y}^{\top}\mathcal{Y}\theta=\mathcal{Y}^{\top}\mathcal{U}$$

which is true for any time t_j

$$\mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\,\theta=\mathcal{Y}^{\top}(t_j)\,\mathcal{U}(t_j)\,.$$

The ICL approach then takes a sum over a set of these terms to get

$$\sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\,\theta = \sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{U}(t_j)$$

where $N \in \mathbb{Z}_{>0}$ is the number of data points required that resulted in

$$\lambda_{\min}\left\{\sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\right\} > \lambda_{ICL},\, t \geq T_{ICL},$$

$\lambda_{ICL} \in \mathbb{R}_{>0}$ is a user selected minimum eigenvalue condition, and $T_{ICL} \in \mathbb{R}_{>0}$ is the time the condition was satisfied. The above relationship can now be used to determine a relationship for $\widetilde{\theta}$ since

$$\widetilde{\theta}(t) = \theta - \widehat{\theta}(t),$$

we can multiple both sides by the relationship $\sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)$ to yield

$$\sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\,\widetilde{\theta}(t) = \sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\,\theta - \sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\,\widehat{\theta}(t).$$

Now using the above relationship we know $\sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\,\theta = \sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{U}(t_j)$ so

$$\sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\,\widetilde{\theta}(t) = \sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{U}(t_j) - \sum_{j=1}^{N} \mathcal{Y}^{\top}(t_j)\,\mathcal{Y}(t_j)\,\widehat{\theta}(t).$$

Since the right side of this relationship only contains measurable quantities, we can use them to determine and once enough data is collected, the eigenvalue condition holds and we can use the data to improve the update law.

C. Analysis and Design

1) Error Dynamics: Let the tracking errors be defined as

$$\begin{aligned} e &= \phi_d - \phi \\ \dot{e} &= \dot{\phi}_d - \dot{\phi} \\ \ddot{e} &= \ddot{\phi}_d - \ddot{\phi} \end{aligned}$$

and the filtered tracking error defined as

$$\begin{aligned} r &= \dot{e} + \alpha e \\ \dot{r} &= \ddot{e} + \alpha \dot{e}. \end{aligned}$$

Then we multiply both sides by the mass to substitute the dynamics

$$\begin{aligned} m\dot{r} &= m\ddot{e} + m\alpha\dot{e} \\ m\dot{r} &= m\left(\ddot{\phi}_d - \ddot{\phi}\right) + m\alpha\dot{e} \\ m\dot{r} &= m\ddot{\phi}_d - m\ddot{\phi} + m\alpha\dot{e} \\ m\dot{r} &= m\ddot{\phi}_d - \left(-c\dot{\phi} - k\phi + u\right) + m\alpha\dot{e} \\ m\dot{r} &= m\ddot{\phi}_d + c\dot{\phi} + k\phi - u + m\alpha\dot{e} \\ m\dot{r} &= \underbrace{m\left(\ddot{\phi}_d + \alpha\dot{e}\right) + c\dot{\phi} + k\phi}_{Y\theta} - \tau \\ m\dot{r} &= Y\theta - \tau \end{aligned}$$

2) Analysis: Let

$$\zeta \triangleq \begin{bmatrix} e \\ r \\ \widetilde{\theta} \end{bmatrix}$$

and

$$V(\zeta, t) \triangleq \frac{1}{2}e^2 + \frac{1}{2}mr^2 + \frac{1}{2}\widetilde{\theta}^\top \Gamma^{-1} \widetilde{\theta}.$$

Taking the time derivative

$$\dot{V}(\zeta, t) = e\dot{e} + mr\dot{r} + \widetilde{\theta}^\top \Gamma^{-1} \dot{\widetilde{\theta}}$$

then we can use

$$\begin{aligned} r &= \dot{e} + \alpha e \\ \implies \dot{e} &= r - \alpha e \end{aligned}$$

and

$$m\dot{r} = Y\theta - \tau$$

$$\dot{V}(\zeta, t) = e(r - \alpha e) + r(Y\theta - \tau) - \widetilde{\theta}^\top \Gamma^{-1} \dot{\widehat{\theta}}.$$

Now we can design the input

$$\tau = Y\widehat{\theta} + e + \beta r$$

which yields

$$\begin{aligned} \dot{V}(\zeta, t) &= -\alpha e^2 + er + r\left(Y\theta - \left(Y\widehat{\theta} + e + \beta r\right)\right) - \widetilde{\theta}^\top \Gamma^{-1} \dot{\widehat{\theta}} \\ \dot{V}(\zeta, t) &= -\alpha e^2 + er + r\left(Y\widetilde{\theta} - e - \beta r\right) - \widetilde{\theta}^\top \Gamma^{-1} \dot{\widehat{\theta}} \\ \dot{V}(\zeta, t) &= -\alpha e^2 - \beta r^2 + \widetilde{\theta}^\top Y^\top r - \widetilde{\theta}^\top \Gamma^{-1} \dot{\widehat{\theta}} \end{aligned}$$

Now we can design the update as

$$\dot{\widehat{\theta}}(t) \triangleq \Gamma Y^\top(t) r(t) + \Gamma k_{ICL}\left(\sum_{j=1}^{N} \mathcal{Y}^\top(t_j) \mathcal{U}(t_j) - \sum_{j=1}^{N} \mathcal{Y}^\top(t_j) \mathcal{Y}(t_j) \widehat{\theta}(t)\right)$$

which is implementable and as shown above but in the analysis we use

$$\dot{\widehat{\theta}}(t) \triangleq \Gamma Y^\top(t) r(t) + \Gamma k_{ICL} \sum_{j=1}^{N} \mathcal{Y}^\top(t_j) \mathcal{Y}(t_j) \widetilde{\theta}(t)$$

which we can use for analysis to yield

$$\begin{aligned} \dot{V}(\zeta, t) &= -\alpha e^2 - \beta r^2 + \widetilde{\theta}^\top Y^\top r - \widetilde{\theta}^\top \Gamma^{-1}\left(\Gamma Y^\top r + \Gamma k_{ICL} \sum_{j=1}^{N} \mathcal{Y}^\top(t_j) \mathcal{Y}(t_j) \widetilde{\theta}\right) \\ \dot{V}(\zeta, t) &= -\alpha e^2 - \beta r^2 - \widetilde{\theta}^\top k_{ICL} \sum_{j=1}^{N} \mathcal{Y}^\top(t_j) \mathcal{Y}(t_j) \widetilde{\theta} \\ \dot{V}(\zeta, t) &\leq \begin{cases} -\alpha e^2 - \beta r^2 & t < T_{ICL} \\ -\min\{\alpha, \beta, \lambda_{\min}\{k_{ICL}\} \lambda_{ICL}\} \|\zeta\|^2 & t \geq T_{ICL} \end{cases} \end{aligned}$$

now we can use the bounds on V

$$\begin{aligned} V(\zeta, t) &\leq \frac{1}{2}\max\left\{1, m, \lambda_{\min}\left\{\Gamma^{-1}\right\}\right\} \|\zeta\|^2 \\ -\frac{2}{\max\left\{1, m, \lambda_{\min}\left\{\Gamma^{-1}\right\}\right\}} V(\zeta, t) &\geq -\|\zeta\|^2 \end{aligned}$$

which implies

$$\dot{V}(\zeta, t) \leq \begin{cases} -\alpha e^2 - \beta r^2 & t < T_{ICL} \\ -2\underbrace{\frac{\min\{\alpha, \beta, \lambda_{\min}\{k_{ICL}\} \lambda_{ICL}\}}{\max\left\{1, m, \lambda_{\min}\left\{\Gamma^{-1}\right\}\right\}}}_{\beta_v} V(\zeta, t) & t \geq T_{ICL}. \end{cases}$$

$$\beta_v \triangleq \frac{2\min\left\{\underline{\alpha}, \underline{\beta}, k_{CL}\lambda_{CL}\right\}}{\max\left\{1, \lambda_{\max}\left\{M\left(\phi\right)\right\}, \lambda_{\max}\left\{\Gamma^{-1}\right\}\right\}}$$

Which we can use to yield the bound

$$\dot{V}\left(\zeta\left(t\right), t\right) \leq \begin{cases} 0 & \forall t < T_{CL} \\ -\beta_v V\left(\zeta\left(t\right), t\right), & \forall t \geq T_{CL} \end{cases}$$

$$V\left(\zeta\left(t\right), t\right) \leq \begin{cases} V\left(\zeta\left(0\right), 0\right) & \forall t < T_{CL} \\ V\left(\zeta\left(T_{CL}\right), T_{CL}\right)\exp\left(-\beta_v\left(t - T_{CL}\right)\right), & \forall t \geq T_{CL} \end{cases}$$

which implies

$$V\left(\zeta\left(t\right), t\right) \leq V\left(\zeta\left(0\right), 0\right)\exp\left(\beta_v\right)\exp\left(-\beta_v t\right).$$