Representing Uncertainty

Chapter 13

Uncertainty in the World

- An agent can often be uncertain about the state of the world/domain since there is often ambiguity and uncertainty
- Plausible/**probabilistic inference**
  - I've got this evidence; what's the chance that this conclusion is true?
    - I've got a sore neck; how likely am I to have meningitis?
    - A mammogram test is positive; what's the probability that the patient has breast cancer?

Uncertainty

- Say we have a rule:

  if *toothache* ***then*** *problem is cavity*
- But not all patients have toothaches due to cavities, so we could set up rules like:

  if *toothache and ¬gum-disease and ¬filling and ...*
  then *problem = cavity*
- This gets complicated; better method:

  if *toothache* ***then*** *problem is cavity with 0.8 probability*

  or $P(cavity \mid toothache) = 0.8$

  the probability of cavity is 0.8 given toothache is observed

Uncertainty in the World and our Models

- True uncertainty: rules *are* probabilistic in nature
  - quantum mechanics
  - rolling dice, flipping a coin
- Laziness: too hard to determine exception-less rules
  - takes too much work to determine *all* of the relevant factors
  - too hard to use the enormous rules that result
- Theoretical ignorance: don't know all the rules
  - problem domain has no complete, consistent theory (e.g., medical diagnosis)
- Practical ignorance: do know all the rules BUT
  - haven't collected all relevant information for a particular case

Logics

Logics are characterized by what they commit to as "primitives"

Logic	What Exists in World	Knowledge States
Propositional	facts	true/false/unknown
First-Order	facts, objects, relations	true/false/unknown
Temporal	facts, objects, relations, times	true/false/unknown
Probability Theory	facts	degree of belief 0..1
Fuzzy	degree of truth	degree of belief 0..1

Probability Theory

- Probability theory serves as a formal means for
  - Representing and reasoning with uncertain knowledge
  - Modeling **degrees of belief** in a proposition (event, conclusion, diagnosis, etc.)
- *Probability is the "language" of uncertainty*
  - A key modeling method in modern AI

Sample Space

- A space of **events** in which we assign probabilities
- Events can be binary, multi-valued, or continuous
- Events are **mutually exclusive**
- Examples
  - Coin flip: {head, tail}
  - Die roll: {1,2,3,4,5,6}
  - English words: a dictionary
  - Temperature tomorrow: {-100, …, 100}

Random Variable

- A variable, X, whose domain is a sample space, and whose value is (somewhat) uncertain
- Examples:

  X = coin flip outcome

  X = first word in tomorrow's NYT newspaper

  X = tomorrow's temperature
- For a given task, the user defines a set of random variables for describing the world

Random Variable

- **Random Variables** (RV):
  - are capitalized (usually) e.g., *Sky*, *Weather*, *Temperature*
  - refer to attributes of the world whose "status" is unknown
  - have one and only one value at a time
  - have a domain of **values** that are possible states of the world:
    - Boolean: domain = <*true*, *false*>
      Cavity = true (often abbreviated as *cavity*)
      Cavity = false (often abbreviated as *¬cavity*)
    - Discrete: domain is countable (includes Boolean)
      values are ***mutually exclusive and exhaustive***
      e.g. *Sky* domain = <*clear*, *partly_cloudy*, *overcast*>
      Sky = clear abbreviated as *clear*
      Sky ≠ clear also abbreviated as *¬clear*
    - Continuous: domain is real numbers (beyond scope of CS 540)

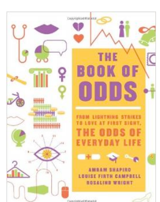
Probability for Discrete Events

- An agent's uncertainty is represented by $P(A{=}a)$ or simply $P(a)$
  - the agent's degree of belief that variable A takes on value a given no other information relating to A
  - a single probability called an unconditional or prior probability

Probability for Discrete Events

- Examples
  - *P*(head) = *P*(tail) = 0.5 fair coin
  - *P*(head) = 0.51, *P*(tail) = 0.49 slightly biased coin
  - *P*(first word = "the" when flipping to a random page in R&N) = ?

- Book: *The Book of Odds*



Probability Table

- *Weather*

sunny	cloudy	rainy
200/365	100/365	65/365

- *P*(*Weather* = sunny) = *P*(sunny) = 200/365

- **P**(*Weather*) = ⟨200/365, 100/365, 65/365⟩

- For now we'll be satisfied with obtaining the probabilities by counting frequencies from data

Probability for Discrete Events

- Probability for more complex events, A
  - $P(A$ = "head or tail") = ? fair coin
  - $P(A$ = "even number") = ? fair 6-sided die
  - $P(A$ = "two dice rolls sum to 2") = ?

Probability for Discrete Events

- Probability for more complex events, A
  - $P(A$ = "head or tail") = 0.5 + 0.5 = 1 fair coin
  - $P(A$ = "even number") = 1/6 + 1/6 + 1/6 = 0.5 fair 6-sided die
  - $P(A$ = "two dice rolls sum to 2") = 1/6 * 1/6 = 1/36

Source of Probabilities

- Frequentists
  - probabilities come from experiments
  - if 10 of 100 people tested have a cavity, $P(cavity) = 0.1$
  - probability means the fraction that would be observed in the limit of infinitely many samples
- Objectivists
  - probabilities are real aspects of the world
  - objects have a propensity to behave in certain ways
  - coin has propensity to come up heads with probability 0.5
- Subjectivists
  - probabilities characterize an agent's belief
  - have no external physical significance

Probability Distributions

Given A is a RV taking values in $\langle a_1, a_2, \ldots, a_n \rangle$

e.g., if A is Sky, then a is one of <*clear*, *partly_cloudy*, *overcast*>

- $P(a)$ represents a single probability where A=a

  e.g., if A is Sky, then $P(a)$ means any one of $P(clear)$, $P(partly\_cloudy)$, $P(overcast)$
- $\mathbf{P}(A)$ represents a probability distribution
  - the **set of values**: $\langle P(a_1), P(a_2), \ldots, P(a_n) \rangle$
  - If A takes n values, then $\mathbf{P}(A)$ is a set of n probabilities

    e.g., if A is Sky, then $\mathbf{P}(Sky)$ is the set of probabilities: $\langle P(clear), P(partly\_cloudy), P(overcast) \rangle$
  - Property: $\sum P(a_i) = P(a_1) + P(a_2) + \ldots + P(a_n) = 1$
    - sum over all values in the domain of variable A is 1 because ***domain is mutually exclusive and exhaustive***

The Axioms of Probability

1. $0 \le P(A) \le 1$
2. $P(\text{true}) = 1$, $P(\text{false}) = 0$
3. $P(A \lor B) = P(A) + P(B) - P(A \land B)$

Note: Here
$P(A)$ means $P(A{=}a)$ for some value a
and $P(A \lor B)$ means $P(A{=}a \lor B{=}b)$

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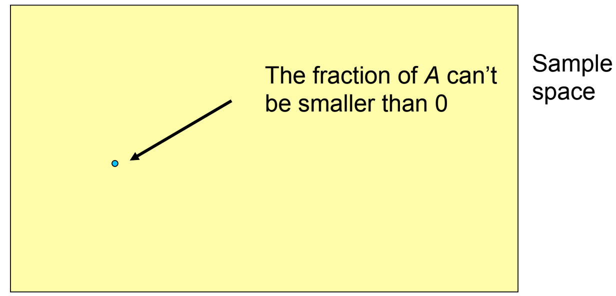

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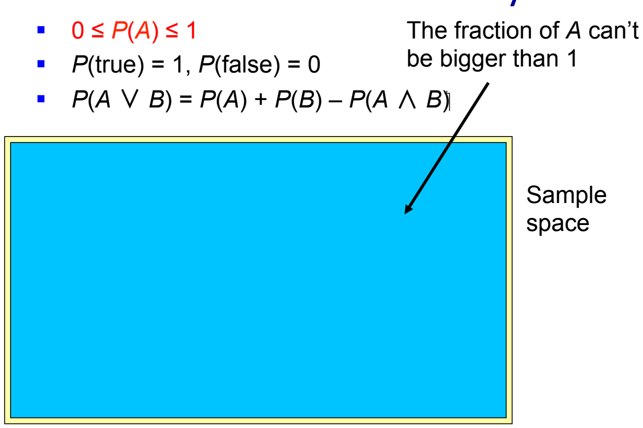

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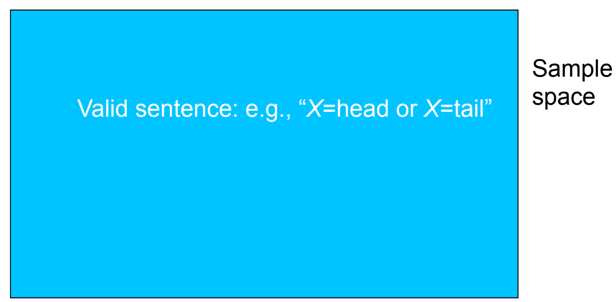

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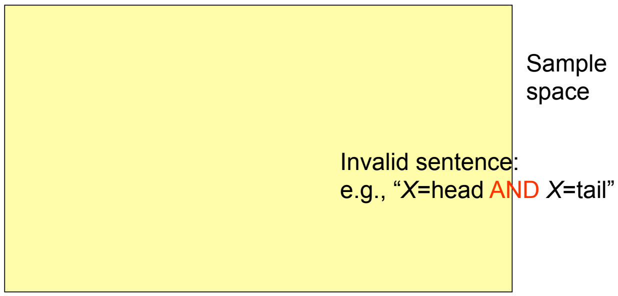

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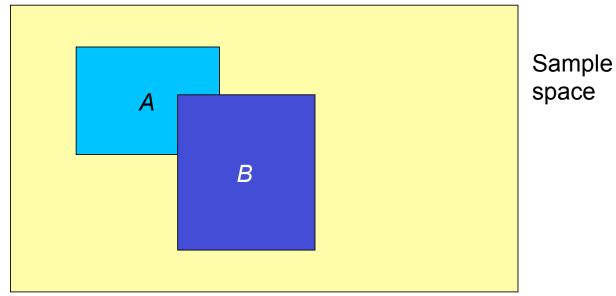


Some Theorems Derived from the Axioms

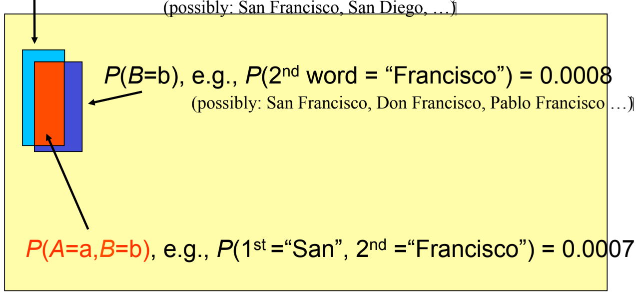
- $P(\neg A) = 1 - P(A)$
- If A can take k different values $a_1, \ldots, a_k$:
  $$P(A=a_1) + \ldots + P(A=a_k) = 1$$
- $P(B) = P(B \land \neg A) + P(B \land A)$, if A is a binary event
- $P(B) = \sum_{i=1\ldots k} P(B \land A=a_i)$, if A can take k values

Called Addition or Conditioning rule

Joint Probability

- The **joint probability** $P(A=a, B=b)$ is shorthand for $P(A=a \land B=b)$, i.e., the probability of *both* A=a and B=b happening

$P(A=a)$, e.g., $P(1^{st}$ word on a random page = "San") = 0.001
(possibly: San Francisco, San Diego, …)



Full Joint Probability Distribution

Weather

Temp		*sunny*	*cloudy*	*rainy*
	hot	150/365	40/365	5/365
	cold	50/365	60/365	60/365

- *P*(*Temp=hot*, *Weather=rainy*) = *P*(*hot*, *rainy*) = 5/365 = 0.014
- The **full joint probability distribution** table for *n* random variables, each taking *k* values, has k^n entries

Full Joint Probability Distribution

Bird	***Flier***	***Young***	**Probability**
T	T	T	0.0
T	T	F	0.2
T	F	T	0.04
T	F	F	0.01
F	T	T	0.01
F	T	F	0.01
F	F	T	0.23
F	F	F	0.5

3 Boolean random variables ⇒ $2^3 - 1 = 7$
"degrees of freedom" or "independent values"

Sums to 1

Computing from the FJPD

- **Marginal Probabilities**
  - *P*(*Bird*=T) = *P*(*bird*) = 0.0 + 0.2 + 0.04 + 0.01 = 0.25
  - *P*(*bird*, ¬*flier*) = 0.04 + 0.01 = 0.05
  - *P*(*bird* ∨ *flier*) = 0.0 + 0.2 + 0.04 + 0.01 + 0.01 + 0.01 = 0.27
- Sum over all other variables
- "**Summing Out**"
- "**Marginalization**"

Unconditional / Prior Probability

- One's uncertainty or original assumption about an event *prior* to having any data about it ***or anything else*** in the domain
- *P*(*Coin* = heads) = 0.5
- *P*(*Bird* = T) = 0.0 + 0.2 + 0.04 + 0.01 = 0.22
- Compute from the FJPD by marginalization

Marginal Probability

		Weather		
		sunny	*cloudy*	*rainy*
Temp	*hot*	150/365	40/365	5/365
	cold	50/365	60/365	60/365
	∑	200/365	100/365	65/365

P(*Weather*) = ⟨200/365, 100/365, 65/365⟩

Probability ***distribution*** for r.v. *Weather*

The name comes from the old days when the sums were written in the margin of a page

Marginal Probability

		Weather			
		sunny	*cloudy*	*rainy*	∑
Temp	*hot*	150/365	40/365	5/365	195/365
	cold	50/365	60/365	60/365	170/365

P(*Temp*) = ⟨195/365, 170/365⟩

This is nothing but $P(B) = \sum_{i=1\ldots k} P(B \wedge A=a_i)$, if A can take k values

Conditional Probability

- Conditional probabilities
  - formalizes the process of accumulating evidence and updating probabilities based on new evidence
  - specifies the belief in a proposition (event, conclusion, diagnosis, etc.) that is *conditioned on* a proposition (evidence, feature, symptom, etc.) being true
- $P(a \mid e)$: conditional probability of $A=a$ given $E=e$ evidence is all that is known true
  - $P(a \mid e) = P(a \wedge e) / P(e) = P(a,e) / P(e)$
  - conditional probability can viewed as the joint probability $P(a,e)$ normalized by the prior probability, $P(e)$

Conditional Probability

Conditional probabilities behave exactly like standard probabilities; for example:

$0 \leq P(a \mid e) \leq 1$

conditional probabilities are between 0 and 1 inclusive

$P(a_1 \mid e) + P(a_2 \mid e) + \ldots + P(a_k \mid e) = 1$

conditional probabilities sum to 1 where a_1, …, a_k are all values in the domain of random variable A

$P(\neg a \mid e) = 1 - P(a \mid e)$

negation for conditional probabilities

Conditional Probability

- *P(conjunction of events | e)*

  $P(a \wedge b \wedge c \mid e)$ or as $P(a, b, c \mid e)$
  is the agent's belief in the sentence $a \wedge b \wedge c$
  conditioned on e being true

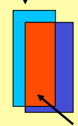
- *P(a | conjunction of evidence)*

  $P(a \mid e \wedge f \wedge g)$ or as $P(a \mid e, f, g)$
  is the agent's belief in the sentence a
  conditioned on $e \wedge f \wedge g$ being true

Conditional Probability

- The conditional probability *P*(*A*=a | *B*=b) is the fraction of time *A*=a, within the region where *B*=b

P(*A*=a), e.g. *P*(1st word on a random page = "San") = 0.001



P(*B*=b), e.g. *P*(2nd word = "Francisco") = 0.0008

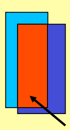
P(*A*=a | *B*=b), e.g. *P*(1st="San" | 2nd ="Francisco") = **0.875**
(possibly: San, Don, Pablo …)

Although "San" is rare and "Francisco" is rare,
given "Francisco" then "San" is quite likely!

Conditional Probability

- *P*(san | francisco)
  = #(1st=s and 2nd=f) / #(2nd=f)
  = *P*(san ∧ francisco) / *P*(francisco)
  = 0.0007 / 0.0008
  = 0.875

P(s)=0.001
P(f)=0.0008
P(s,f)=0.0007



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"degrees of freedom" or "independent values"

Sums to 1

Computing Conditional Probability

$P(\neg B \mid F)$ = ?

$P(F)$ = ?

Note: $P(\neg B \mid F)$ means $P(B\text{=false} \mid F\text{=true})$ and $P(F)$ means $P(F\text{=true})$

Computing Conditional Probability

$P(\neg B \mid F) = P(\neg B, F)/P(F)$
$= (P(\neg B, F, Y) + P(\neg B, F, \neg Y))/P(F)$
$= (0.01 + 0.01)/P(F)$

$P(F) = P(F, B, Y) + P(F, B, \neg Y) + P(F, \neg B, Y) + P(F, \neg B, \neg Y)$
$= 0.0 + 0.2 + 0.01 + 0.01$
$= 0.22$

Marginalization

Computing Conditional Probability

- Instead of using Marginalization to compute P(F), can alternatively use "**Normalization**":
- $P(B \mid F) = P(B,F)/P(F) = (0.0 + 0.2)/P(F)$
- $P(\neg B \mid F) + P(B \mid F) = 1$
- So, $0.2/P(F) + 0.02/P(F) = 1$
- Hence, $P(F) = 0.22$

Normalization

Addition rule

- In general, $P(A \mid B) = \alpha\, P(A, B)$
  where $\alpha = 1/P(B) = 1/(P(A, B) + P(\neg A, B))$
- $P(Q \mid E_1, \ldots, E_k) = \alpha\, P(Q, E_1, \ldots, E_k)$
  $= \alpha \sum_Y P(Q, E_1, \ldots, E_k, Y)$

Conditional Probability with Multiple Evidence

- $P(\neg B \mid F, \neg Y) = P(\neg B, F, \neg Y) / P(F, \neg Y)$
  $= P(\neg B, F, \neg Y) / (P(\neg B, F, \neg Y) + P(B, F, \neg Y))$
  $= .01 / (.01 + .2)$
  $= 0.048$

Conditional Probability

- $P(X_1=x_1, \ldots, X_k=x_k \mid X_{k+1}=x_{k+1}, \ldots, X_n=x_n) =$
  sum of all entries in FJPD where $X_1=x_1, \ldots, X_n=x_n$ divided by sum of all entries where $X_{k+1}=x_{k+1}, \ldots, X_n=x_n$
- But this means in general we need the entire FJPD table, requiring an *exponential number of values* to do probabilistic inference (i.e., compute conditional probabilities)

Conditional Probability

- In general, the conditional probability is

$$P(A=a \mid B) = \frac{P(A=a, B)}{P(B)} = \frac{P(A=a, B)}{\sum_{\text{all } a_i} P(A=a_i, B)}$$

- We can have everything *conditioned* on some other event(s), *C*, to get a conditionalized version of conditional probability:

$$P(A \mid B, C) = \frac{P(A, B \mid C)}{P(B \mid C)}$$

'|' has low precedence.
This should read: $P(A \mid (B,C))$

The Chain Rule

- From the definition of conditional probability we have the **chain rule**:
  $P(A, B) = P(B) * P(A \mid B) = P(A \mid B) * P(B)$
- It also works the other way around:
  $P(A, B) = P(A) * P(B \mid A) = P(B \mid A)\, P(A)$
- It works with more than 2 events too:

Called "**Product Rule**"

$P(A_1, A_2, \ldots, A_n) =$
$P(A_1) * P(A_2 \mid A_1) * P(A_3 \mid A_1, A_2) * \ldots$
$\quad * P(A_n \mid A_1, A_2, \ldots, A_{n-1})$

Probabilistic Reasoning

How do we use probabilities in AI?

- You wake up with a headache
- Do you have the flu?
- *H* = headache, *F* = flu

Logical Inference: if *H* then *F*
(but the world is often not this clear cut)

Statistical Inference: compute the probability of a query/diagnosis/decision given (conditioned on) evidence/symptom/observation, i.e., $P(F \mid H)$

[Example from Andrew Moore]

Inference with Bayes's Rule: Example 1

Statistical Inference: Compute the probability of a diagnosis, *F*, given symptom, *H*, where *H* = "has a headache" and *F* = "has flu"

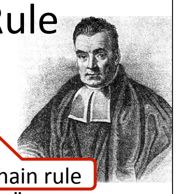
That is, compute $P(F \mid H)$

You know that

- $P(H) = 0.1$ "one in ten people has a headache"
- $P(F) = 0.01$ "one in 100 people has flu"
- $P(H \mid F) = 0.9$ "90% of people who have flu have a headache"

[Example from Andrew Moore]

Inference with Bayes's Rule



Thomas Bayes, "Essay Towards Solving a Problem in the Doctrine of Chances," 1764

$$P(F \mid H) = \frac{P(F,H)}{P(H)} = \frac{P(H \mid F)P(F)}{P(H)}$$

Def of cond. prob.

Chain rule

- $P(H) = 0.1$ "one in ten people has a headache"
- $P(F) = 0.01$ "one in 100 people has flu"
- $P(H|F) = 0.9$ "90% of people who have flu have a headache"

- $P(F|H) = 0.9 * 0.01 / 0.1 = 0.09$
- So, there's a 9% chance you have flu – much less than 90%
- But it's higher than $P(F) = 1\%$, since you have a headache

Bayes's Rule

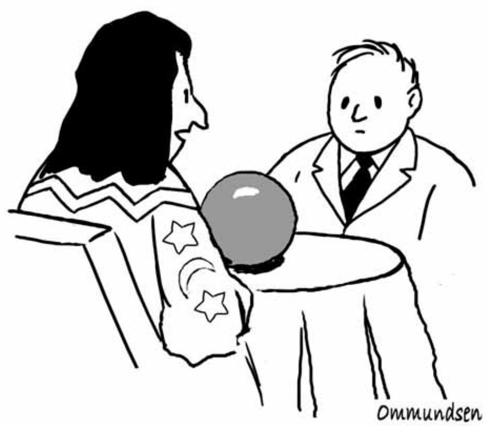
- Bayes's Rule is the basis for probabilistic reasoning given a prior model of the world, P(Q), and a new piece of evidence, E, Bayes's rule says how this piece of evidence decreases our ignorance about the world
- Initially, know P(Q) ("prior")
- Update after knowing E ("posterior"):

$$P(Q|E) = P(Q)\frac{P(E|Q)}{P(E)}$$

Inference with Bayes's Rule

- ***P*(*A*|*B*) = *P*(*B* | *A*)*P*(*A*) / *P*(*B*)** **Bayes's rule**
- Why do we make things this complicated?
  - Often $P(B|A)$, $P(A)$, $P(B)$ are easier to get
  - Some names:
    - **Prior *P*(*A*)**: probability of *A* *before* any evidence
    - **Likelihood *P*(*B*|*A*)**: assuming *A*, how likely is the evidence
    - **Posterior *P*(*A*|*B*)**: probability of *A* after knowing evidence *B*
    - **(Deductive) Inference**: deriving an unknown probability from known ones
- If we have the full joint probability table, we can simply compute $P(A|B) = P(A, B) / P(B)$

Bayes's Rule in Practice



"Is this needed for a Bayesian analysis?"

Summary of Important Rules

- **Conditional Probability**: $P(A|B) = P(A,B)/P(B)$
- **Product rule**: $P(A,B) = P(A|B)P(B)$
- **Chain rule**: $P(A,B,C,D) = P(A|B,C,D)P(B|C,D)P(C|D)P(D)$
- **Conditionalized version of Chain rule**:

  $$P(A,B|C) = P(A|B,C)P(B|C)$$
- **Bayes's rule**: $P(A|B) = P(B|A)P(A)/P(B)$
- **Conditionalized version of Bayes's rule**:

  $$P(A|B,C) = P(B|A,C)P(A|C)/P(B|C)$$
- **Addition / Conditioning rule**: $P(A) = P(A,B) + P(A,\neg B)$

  $$P(A) = P(A|B)P(B) + P(A|\neg B)P(\neg B)$$

Common Mistake

- $P(A) = 0.3$ so $P(\neg A) = 1 - P(A) = 0.7$
- $P(A|B) = 0.4$ so $P(\neg A|B) = 1 - P(A|B) = 0.6$
  because $P(A|B) + P(\neg A|B) = 1$

  but $P(A|\neg B) \neq 0.6$ (in general)
  because $P(A|B) + P(A|\neg B) \neq 1$ in general

Quiz

- A doctor performs a test that has 99% reliability, i.e., 99% of people who are sick test positive, and 99% of people who are healthy test negative. The doctor estimates that 1% of the population is sick.
- Question: A patient tests positive. What is the chance that the patient is sick?
- 0-25%, 25-75%, 75-95%, or 95-100%?

Quiz

- A doctor performs a test that has 99% reliability, i.e., 99% of people who are sick test positive, and 99% of people who are healthy test negative. The doctor estimates that 1% of the population is sick.
- Question: A patient tests positive. What is the chance that the patient is sick?
- 0-25%, 25-75%, 75-95%, or 95-100%?
- Common answer: 99%; Correct answer: 50%

Given:

$P(TP \mid S) = 0.99$

$P(\neg TP \mid \neg S) = 0.99$

$P(S) = 0.01$

TP = "tests positive"
S = "is sick"

Query:

$P(S \mid TP) = ?$

$P(TP \mid S) = 0.99$

$P(\neg TP \mid \neg S) = 0.99$

$P(S) = 0.01$

$P(S \mid TP) =$

$P(TP \mid S)\, P(S) / P(TP)$

$= (0.99)(0.01) / P(TP) = 0.0099/P(TP)$

$P(\neg S \mid TP) = P(TP \mid \neg S)P(\neg S) / P(TP)$

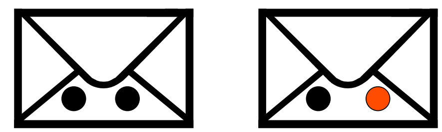
$= (1 - 0.99)(1 - 0.01) / P(TP) = 0.0099/P(TP)$

$0.0099/P(TP) + 0.0099/P(TP) = 1$, so $P(TP) = 0.0198$

So, $P(S \mid TP) = 0.0099 / 0.0198 = 0.5$

Inference with Bayes's Rule: Example 2

- In a bag there are two envelopes
  - one has a red ball (worth \$100) and a black ball
  - one has two black balls. Black balls are worth nothing



- You randomly grab an envelope, and randomly take out one ball – it's **black**
- At this point you're given the option to switch envelopes. To switch or not to switch?

Similar to the "Monty Hall Problem"

Inference with Bayes's Rule: Example 2

E: envelope, 1=(R,B), 2=(B,B)

B: the event of drawing a black ball

Given: $P(B|E{=}1) = 0.5$, $P(B|E{=}2) = 1$, $P(E{=}1) = P(E{=}2) = 0.5$

Query: Is $P(E{=}1 \mid B) > P(E{=}2 \mid B)$?

Addition rule

Use Bayes's rule: $P(E|B) = P(B|E)*P(E) / P(B)$

$P(B) = P(B|E{=}1)P(E{=}1) + P(B|E{=}2)P(E{=}2) = (.5)(.5) + (1)(.5) = .75$

$P(E{=}1|B) = P(B|E{=}1)P(E{=}1)/P(B) = (.5)(.5)/(.75) = 0.33$

$P(E{=}2|B) = P(B|E{=}2)P(E{=}2)/P(B) = (1)(.5)/(.75) = 0.67$

After seeing a black ball, the posterior probability of this envelope being #1 (thus worth \$100) is *smaller* than it being #2

Thus you should switch!

Example 3

- 1% of women over 40 who are tested have breast cancer. 85% of women who really do have breast cancer have a positive mammography test (true positive rate). 8% who do *not* have cancer will have a positive mammography (false positive rate).
- Question: A patient gets a positive mammography test. What is the chance she has breast cancer?

- Let Boolean random variable M mean "positive mammography test"
- Let Boolean random variable C mean "has breast cancer"
- Given:

  $P(C) = 0.01$

  $P(M|C) = 0.85$

  $P(M|\neg C) = 0.08$

- Compute the posterior probability: $P(C|M)$

- $P(C|M) = P(M|C)P(C)/P(M)$ by Bayes's rule

  $= (.85)(.01)/P(M)$
- $P(M) = P(M|C)P(C) + P(M|\neg C)P(\neg C)$ by the Addition rule
- So, $P(C|M) = .0085/[(.85)(.01) + (.08)(1-.01)]$

  $= 0.097$
- So, there is (only) a 9.7% chance that if you have a positive test you really have cancer!

Bayes with Multiple Evidence

- Say the same patient goes back and gets a *second* mammography and it too is positive. Now, what is the chance she has cancer?
- Let $M1$, $M2$ be the 2 positive tests
- Compute posterior: $P(C|M1, M2)$

Bayes with Multiple Evidence

- $P(C|M1, M2) = P(M1, M2|C)P(C)/P(M1, M2)$ by Bayes's rule

  Conditionalized Chain rule

  $= P(M1|M2, C)P(M2|C)P(C)/P(M1, M2)$

Assuming ***M1* and *M2* are independent** means $P(M1, M2) = P(M1)P(M2)$ and $P(M1|M2, C) = P(M1|C)$

- From before, $P(M1) = P(M2) = 0.0877$
- So, $P(C|M1, M2) = (.85)(.85)(.01)/ (.0877)(.0877) = 0.9395$ or 93.95%

Inference Ignorance

- "Inferences about Testosterone Abuse Among Athletes," 2004
  - Mary Decker Slaney doping case
- "Justice Flunks Math," 2013
  - Amanda Knox trial in Italy

Independence

- Two events A, B are **independent** if the following hold:
  - $P(A, B) = P(A) * P(B)$
  - $P(A, \neg B) = P(A) * P(\neg B)$
  - …
  - $P(A \mid B) = P(A)$
  - $P(B \mid A) = P(B)$
  - $P(A \mid \neg B) = P(A)$
  - …

Independence

- Independence is a kind of domain knowledge
  - Needs an understanding of ***causation***
  - Very strong assumption
- Example: $P(\text{burglary}) = 0.001$, $P(\text{earthquake}) = 0.002$. Let's say they are independent. The full joint probability table = ?

Independence

- Given: $P(B) = 0.001$, $P(E) = 0.002$, $P(B|E) = P(B)$
- The full joint probability distribution table is:

Burglary	Earthquake	Prob.
B	*E*	
B	¬*E*	
¬*B*	*E*	
¬*B*	¬*E*	

- Need only 2 numbers to fill in entire table
- Now we can do anything, since we have the joint

Independence

- Given *n* independent, Boolean random variables, the joint has 2^n entries, but only need *n* numbers (degrees of freedom) to fill in entire table
- Given *n* independent random variables, where each can take *k* values, the joint probability table has:
  – k^n entries
  – Only $n(k\text{-}1)$ numbers needed

Conditional Independence

- Random variables can be dependent, but **conditionally independent**
- Example: Your house has an alarm
  – Neighbor John will call when he hears the alarm
  – Neighbor Mary will call when she hears the alarm
  – Assume John and Mary don't talk to each other
- Is *JohnCall independent* of *MaryCall*?
  – **No** – If John called, it is likely the alarm went off, which increases the probability of Mary calling
  – $P(MaryCall \mid JohnCall) \neq P(MaryCall)$

Conditional Independence

- But, if we *know* the status of the *alarm*, *JohnCall* will ***not*** affect whether or not Mary calls

  $P(MaryCall \mid Alarm, JohnCall) = P(MaryCall \mid Alarm)$
- We say *JohnCall* and *MaryCall* are **conditionally independent** given *Alarm*
- In general, "*A* and *B* are conditionally independent given *C*" means:

  $P(A \mid B, C) = P(A \mid C)$

  $P(B \mid A, C) = P(B \mid C)$

  $P(A, B \mid C) = P(A \mid C)\ P(B \mid C)$

Independence vs. Conditional Independence

- Say Alice and Bob each toss ***separate coins***. A represents "Alice's coin toss is heads" and B represents "Bob's coin toss is heads"
- A and B are **independent**
- Now suppose Alice and Bob toss the ***same coin***. Are A and B independent?
  - *No*. Say the coin may be biased towards heads. If A is heads, it will lead us to increase our belief in B beings heads. That is, $P(B|A) > P(A)$

- Say we add a new variable, C: "the coin is biased towards heads"
- The values of A and B are *dependent on* C
- But if we know *for certain* the value of C (true or false), then any evidence about A cannot change our belief about B
- That is, $P(B|C) = P(B|A, C)$
- A and B are **conditionally independent** given C

Revisiting Example 3

- Let Boolean random variable M mean "positive mammography test"
- Let Boolean random variable C mean "has breast cancer"
- Given:

  $P(C) = 0.01$

  $P(M|C) = 0.85$

  $P(M|\neg C) = 0.08$

Bayes's Rule with Multiple Evidence

- $P(C|M1, M2) = P(M1, M2|C)P(C)/P(M1, M2)$ by Bayes's rule

  $= P(M1|M2, C)P(M2|C)P(C)/P(M1, M2)$

  Conditionalized Chain rule

- $P(M1, M2) = P(M1, M2|C)P(C) +$
  $P(M1, M2|\neg C)P(\neg C)$ by Addition rule

  $= P(M1|M2, C)P(M2|C)P(C) +$
  $P(M1|M2, \neg C)P(M2|\neg C)P(\neg C)$

  by Conditionalized Chain rule

Cancer "causes" a positive test, so ***M*1 and *M*2 are conditionally independent given *C***, so

- $P(M1|M2, C) = P(M1 \mid C) = 0.85$
- $P(M1, M2) = P(M1|M2, C)P(M2|C)P(C) + P(M1|M2, \neg C)P(M2|\neg C)P(\neg C)$

$= P(M1|C)P(M2|C)P(C) + P(M1|\neg C)P(M2|\neg C)P(\neg C)$ by cond. indep.

$= (.85)(.85)(.01) + (.08)(.08)(1-.01)$

$= 0.01356$

So, $P(C|M1, M2) = (.85)(.85)(.01)/ .01356$

$= 0.533$ or 53.3%

Example 3

- Prior probability of having breast cancer:
  $P(C) = 0.01$
- Posterior probability of having breast cancer after 1 positive mammography:
  $P(C|M1) = 0.097$
- Posterior probability of having breast cancer after 2 positive mammographies (and cond. independence assumption):
  $P(C|M1, M2) = 0.533$

Bayes with Multiple Evidence

- Say the same patient goes back and gets a second mammography and it is **negative**. Now, what is the chance she has cancer?
- Let $M1$ be the positive test and $\neg M2$ be the negative test
- Compute posterior: $P(C|M1, \neg M2)$

Bayes's Rule with Multiple Evidence

- $P(C|M1, \neg M2) = P(M1, \neg M2|C)P(C)/ P(M1, \neg M2)$
  by Bayes's rule

  $= P(M1|C)P(\neg M2|C)P(C)/P(M1, \neg M2)$

  $= (.85)(1-.85)(.01)/P(M1, \neg M2)$
- $P(M1, \neg M2) = P(M1, \neg M2|C)P(C) + P(M1, \neg M2|\neg C)P(\neg C)$ by Addition rule

  $= P(M1|\neg M2, C)P(\neg M2|C)P(C) + P(M1|\neg M2, \neg C)P(\neg M2|\neg C)P(\neg C)$

  by Conditionalized Chain rule

Cancer "causes" a positive test, so ***M1* and *M2* are conditionally independent given *C***, so

$P(M1|\neg M2, C)P(\neg M2|C)P(C) +$
$\quad P(M1|\neg M2, \neg C)P(\neg M2|\neg C)P(\neg C)$

$= P(M1|C)P(\neg M2|C)P(C) +$
$\quad P(M1|\neg C)P(\neg M2|\neg C)P(\neg C)$ by cond. indep.

$= (.85)(1 - .85)(.01) + (1 - .08)(.08)(1 - .01)$

$= 0.066219 \quad (= P(M1, \neg M2))$

So, $P(C|M1, \neg M2) = (.85)(1 - .85)(.01)/ .066219$

$= 0.019$ or 1.9%

Bayes's Rule with Multiple Evidence and Conditional Independence

- Assume all evidence variables, B, C and D, are conditionally independent given the diagnosis variable, A
- $P(A|B,C,D) = P(B,C,D|A)P(A)/P(B,C,D)$

  $= \underbrace{P(B|A)P(C|A)P(D|A)}_{\text{Conditionalized Chain rule + conditional independence}}P(A)/\underbrace{P(D|B,C)P(C|B)P(B)}_{\text{Chain rule}}$

$$= P(A)\frac{P(B|A)}{P(B)}\frac{P(C|A)}{P(C|B)}\frac{P(D|A)}{P(D|B,C)}$$

Naïve Bayes Classifier

- Say we have one class/diagnosis/decision variable, A
- Goal is to find the value of A that is most likely given evidence B, C, D, … :

$$argmax_a\, P(\text{A=a})P(B|\text{A=a})P(C|\text{A=a})P(D|\text{A=a})/P(B,C,D)$$

But P(B,C,D) is a constant here for all a, so instead compute:

$$argmax_a\, P(\text{A=a})P(B|\text{A=a})P(C|\text{A=a})P(D|\text{A=a})$$

Naïve Bayes Classifier

- Find v =

  $\text{argmax}_v \underbrace{P(Y = v)}_{\text{Class variable}} \prod_{i=1}^{n} \underbrace{P(X_i = u_i|Y = v)}_{\text{Evidence variable}}$
- Assumes all evidence variables are conditionally independent of each other given the class variable
- Robust since it gives the right answer as long as the correct class is more likely than all others

Naïve Bayes Classifier

- Assume k classes and n evidence variables, each with r possible values
- k-1 values needed for computing $P(Y=v)$
- rk values needed for computing $P(X_i=u_i \mid Y=v)$ for each evidence variable X_i
- So, $(k-1) + nrk$ values needed instead of exponential size FJPD table

Naïve Bayes Classifier

- Conditional probabilities can be very, very small, so instead use logarithms to avoid underflow:

$$\mathrm{argmax}_v \log P(Y = v) + \sum_{i=1}^{n} \log P(X_i = u_i | Y = v)$$

Summary of Important Rules

- **Conditional Probability**: $P(A|B) = P(A,B)/P(B)$
- **Product rule**: $P(A,B) = P(A|B)P(B)$
- **Chain rule**: $P(A,B,C,D) = P(A|B,C,D)P(B|C,D)P(C|D)P(D)$
- **Conditionalized version of Chain rule**:

  $P(A,B|C) = P(A|B,C)P(B|C)$
- **Bayes's rule**: $P(A|B) = P(B|A)P(A)/P(B)$
- **Conditionalized version of Bayes's rule**:

  $P(A|B,C) = P(B|A,C)P(A|C)/P(B|C)$
- **Addition / Conditioning rule**: $P(A) = P(A,B) + P(A, \neg B)$

  $P(A) = P(A|B)P(B) + P(A|\neg B)P(\neg B)$