Mini-Project 3 – One and Two Body Problems

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Abstract

In this project we will look at the both the one and two body problems. In the one body problem there is an object of negligible mass orbiting around a bigger object of large mass. A movie plot will be made to show the smaller object orbiting the object of larger mass. In the two body problem the first object will have comparable mass to that of the second object. Both objects will be moving and both will have some gravitational pull towards the other. We will also make a movie plot for this figure along with plotting the center of mass between the two bodies.

1 Introduction

Throughout this paper we will look at one and two body problems. We will make movie plots and interpret how different initial velocities and starting positions might change the final outcome of how long the orbit lasts and if the gravitational pull is high enough to maintain an orbit. We will then do the same thing for the two-body problem along with plotting the center of mass from the center of each object. We can observe how the center of mass changes as time passes to see how circular the orbit is.

In the following sections we will look at the each step in further detail and interpret the graphs and what they tell us about the problem.

2 Problem Statement

The goals of this project are to develop a deeper understanding of numerical methods used to solve ODE's, learn different python libraries that can help in solving ODEs, explore real-world applications of ODEs, and visualize and interpret solutions based on the context of the application.

Our first problem will be to create a movie plot for the one-body problem. In this, we will be looking at the orbit of a small object, such as a satellite, around a larger object, like a planet. Since the mass of the planet will be much larger than that of the satellite, we can remove the mass of the satellite from the equation leaving us with a one-body problem. In order to solve this we will have to find $\frac{dx}{dt}$, $\frac{dv_x}{dt}$, $\frac{dy}{dt}$, and $\frac{dv_y}{dt}$.

In the second problem we will solve the two-body problem. In this problem we will have two bodies of similar mass. This means they will both have gravitational pull towards the other. We will pick a starting position and velocity for each mass and see how they orbit around each other over time. We will also introduce the center of mass which is the distance between the two body's centers and graph that along with the orbit. To solve the two-body problem we will first need to formulate 12 formulas; 6 for each body: $\frac{dx_1}{dt}$, $\frac{dvx_1}{dt}$, $\frac{dy_1}{dt}$, $\frac{dvy_1}{dt}$, $\frac{dz_1}{dt}$, $\frac{dvz_1}{dt}$, $\frac{dx_2}{dt}$, $\frac{dvx_2}{dt}$, $\frac{dy_2}{dt}$, $\frac{dvy_2}{dt}$, $\frac{dz_2}{dt}$, and $\frac{dvz_2}{dt}$.

3 Methodology

In order to plot the graphs we have to solve for $\frac{dx}{dt}$, $\frac{dv_x}{dt}$, $\frac{dy}{dt}$, and $\frac{dv_y}{dt}$. Since $\frac{dx}{dt}$ is the integral of the acceleration, we can write it as:

$$dx/dt = v_x$$

Then we will find $\frac{dv_x}{dt}$ by Newton's Law of Gravitation and the gravitational force components. The formula for Newton's law is:

$$F = \frac{(-G * m_1 * m_2)}{r^2}$$

In this formula G is the gravitational constant, m_1 is the mass of object 1, m_2 is the mass of object 2, and r is the distance from the two centers of mass. We will also use the formula for gravitational force which is:

$$F = \frac{x(t)}{r}$$

In this formula we have x which is a subject to t and r once again. Multiplying these equations leaves us with:

$$\frac{dv_x}{dt} = \frac{-G * m_2 * x(t)}{(x^2 + y^2)^{\frac{3}{2}}}$$

The m_1 value can be removed since it will not change our numerator due to it's small size.

We can then find $\frac{dy}{dt}$ similarly to how we found $\frac{dx}{dt}$. So

$$\frac{dy}{dt} = v_y$$

The we can find $\frac{dv_y}{dt}$ by switching out the $x(t)$ for y(t) in our gravitational force equation to get:

$$\frac{dv_y}{dt} = \frac{-G * m_2 * y(t)}{(x^2 + y^2)^{\frac{3}{2}}}$$

Next we will formulate the two-body problem. We will be doing the two-body problem in 3-d so we will need to introduce a z variable in our equations. We will; however, be following the same formulas. So, our formulas will be as follows:

$$\frac{dx_1}{dt} = vx_1$$
$$\frac{dvx_1}{dt} = \frac{-G * m_2 * (x_1 - x_2)}{((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)^{\frac{3}{2}}}$$
$$\frac{dy_1}{dt} = vy_1$$
$$\frac{dvy_1}{dt} = \frac{-G * m_2 * (y_1 - y_2)}{((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)^{\frac{3}{2}}}$$
$$\frac{dz_1}{dt} = vz_1$$
$$\frac{dvz_1}{dt} = \frac{-G * m_2 * (z_1 - z_2)}{((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)^{\frac{3}{2}}}$$
$$\frac{dx_2}{dt} = vx_2$$
$$\frac{dvx_2}{dt} = \frac{-G * m_1 * (x_2 - x_1)}{((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)^{\frac{3}{2}}}$$
$$\frac{dy_2}{dt} = vy_2$$
$$\frac{dvy_2}{dt} = \frac{-G * m_1 * (y_2 - y_1)}{((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)^{\frac{3}{2}}}$$
$$\frac{dz_2}{dt} = vz_2$$
$$\frac{dvz_2}{dt} = \frac{-G * m_1 * (z_2 - z_1)}{((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2)^{\frac{3}{2}}}$$

We will put all of these formulas in python and use the function *solveivp* in order to get x and y for our graphs.

4 Results

Our results were the plot of the one-body problem plotted in two dimensions with the x-position as the x-axis and the y-position as the y-axis. The results can be seen in the figure below:

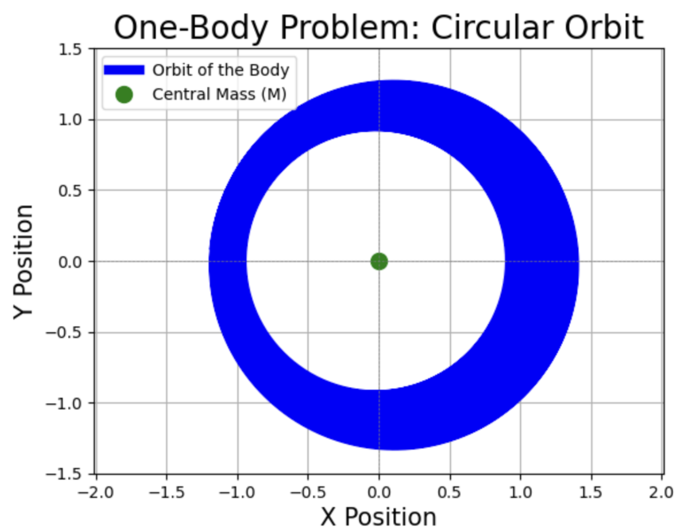

Figure 1: Graph of one-body problem.

We also have our 3-d graph for the two body problem which was plotted with respect to the x, y, and z positions. This graph can be found below:

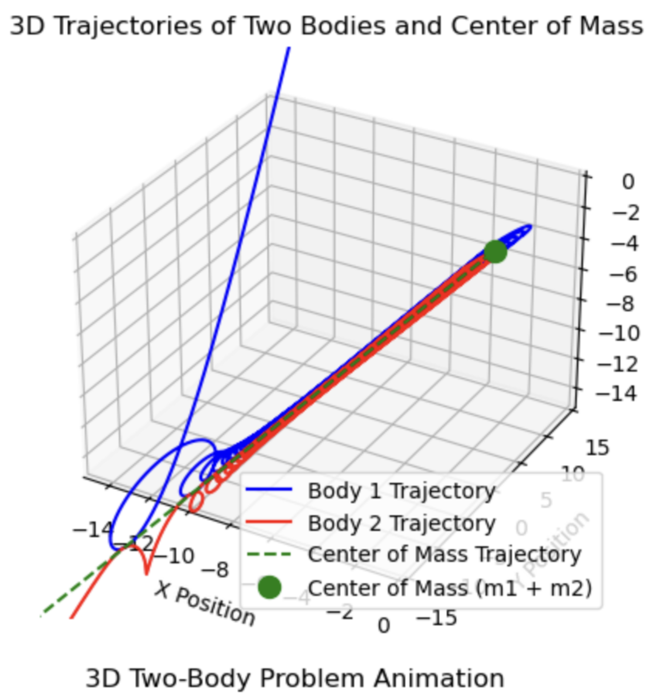

Figure 2: Graph of two-body problem.

5 Discussion

Looking at the results for our one-body problem, 1, we can see the orbiting body keeps a good distance from the central body throughout the time shown. This graph was obtained by using a gravitational force of 0.1 and a mass of object 2 of 1000. The initial velocity was found using the formula: $(G * m2/r_0)^{\frac{1}{2}}$. In this formula we have $r_0 = 1$ as our beginning position or radius. If this initial velocity is too strong the gravitational pull won't be enough for an orbit to occur, and we would instead see the outer object drift away fro the center object. If the initial velocity is too small then we would see the object get pulled in until the two bodies collide.

Moving on to the two-body problem in 3-d, 2, we can see the orbiting bodies orbit closely around the center of mass at first and then start to get further away as time goes on. For this problem we used 0.1 for gravity, 900 for mass 1, and 1000 for mass 2. The two bodies started at the points (1, 1, 1) and (-1, -1, -1) respectively with starting velocities of 2 in the x direction, -1 in the y direction, and 3 in the z direction for the first body and -2, 1, and 3 for the 2nd body. These starting positions are close to one another so the velocities are small as well in order to assure an orbit occurs. We can also see that the positions and velocities are negations of each other. We can the starting positions of each body is equidistant from the center of mass. This ensures that they have similar gravitational pulls between them. The other variable we haven't talked about yet is the mass. We know both bodies have different masses so the orbit will eventually become tilted in one direction and break. By looking at figure 2 we can see that the bodies start to diverge as time goes on meaning the initial conditions didn't create circular orbits of the two bodies around the center of mass.

6 Conclusion

Throughout this paper we have looked at the one and two body problems and studied how the bodies orbit in each problem. We also looked at the equations that were used to formulate the graph. We finally looked at the variables we used and the trend in the orbits as time progressed to see if the orbits continue forever.

To improve this project in the future, the 3 or even n-body problems can be calculated. The perfect system can be calculated so that the orbit lasts forever. The 1 body system could be graphed in 3-d. Different masses can be looked at to see how fast the orbit breaks depending on how large the difference of masses are.