Mini-Project 3: Solving Differential Equations for the One-Body and Two-Body Problems

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Abstract

Differential equations have practical applications in several fields, including physics, engineering, chemistry, finance, and more. They describe the rate of change of a variable, say y, typically represented as $\frac{dy}{dt}$. By employing a range of mathematical methods or by utilizing software, we can solve this differential equation to find $y(t)$, a function that represents the behavior of y over time (t). Furthermore, we can provide initial conditions, which specify the value of y at a particular time, often denoted as $y(t_0) = y_0$. By setting this condition, we can tailor the general solution of the differential equation to reflect the behavior of the system at this starting point.

1 Introduction

In this project, we will be incorporating differential equations to solve what are known as the one-body and two-body problems. We will begin with the one-body problem, which involves a single object moving under the influence of forces. For our problem, we can envision this object as a satellite orbiting Earth, which is affected by a variety of forces, some of which are shown in Figure 1 below.

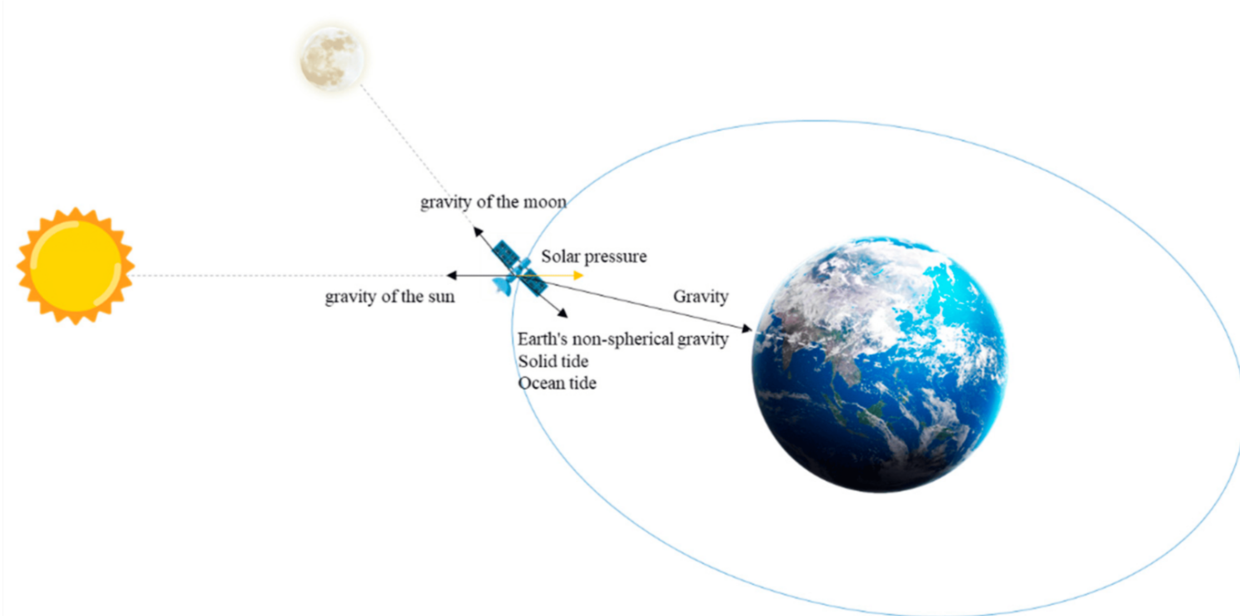

Figure 1: Satellite Orbiting Around Earth

For the sake of simplicity, our "satellite" will only be affected by a given gravitational constant and the mass of static object, but the general idea is the same. As an extension of this, the two-body problem involves the interaction between two objects, where each object exerts force on the other. Similar to our example for the one-body problem, we can now envision two planets gravitating towards each other (See Figure 2 below).

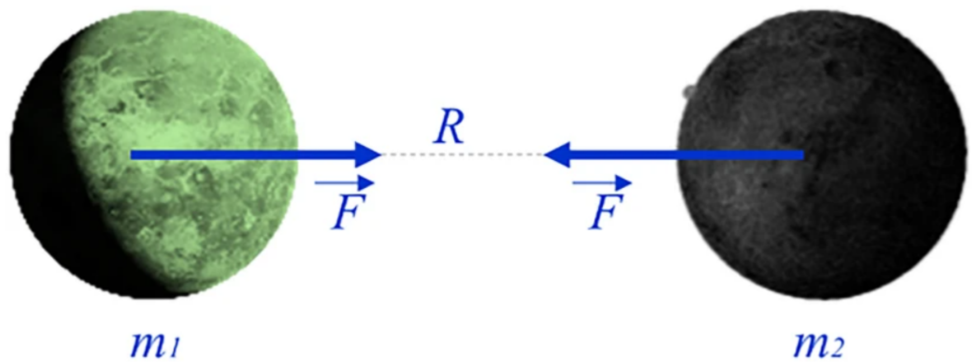

Figure 2: Force Between Two Planets

Again, for our problem in particular, we will only be accounting for the gravitational constant and the masses of our two "planets", although there are several more forces that affect this interaction in reality.

2 Problem Statement

With these ideas in mind, we now need to form systems of differential equations to model the relationships. First, we need to find a formula that represents what we are attempting to model. Since the main force we are dealing with is gravity, we can use Newton's Law of Gravitation, which is defined as $F = G\frac{m_1 m_2}{r^2}$, where F represents force, G is the gravitational constant, m_1 is the mass of our "satellite", m_2 is the mass of our static object, and r is the distance between these two objects. Now, recall Newton's Second Law of Motion, defined as $F = ma$, i.e Force = Mass $\times$ Acceleration. Substituting this into the previous formula, we find an equality of $m_1 a = G\frac{m_1 m_2}{r^2}$, which simplifies to $a = G\frac{m_2}{r^2}$. Now, we have an equation for acceleration, which we know is the second derivative of displacement, so this can be rewritten as a second-order differential equation. For the one-body problem, we can assume a 2-dimensional space with an equation in terms of x and y. For simplicity, we place our static object at the origin, $(0, 0)$, so that the distance to our satellite is just $r = \sqrt{x^2 + y^2}$. Then, multiplying by the respective unit vectors for x and y, we set up our system of second-order differential equations as follows:

$$\begin{cases} \frac{d^2x}{dt^2} = -G\frac{m_2 x}{(x^2+y^2)^{\frac{3}{2}}} \\ \frac{d^2y}{dt^2} = -G\frac{m_2 y}{(x^2+y^2)^{\frac{3}{2}}} \end{cases}$$

For the two body problem, the only alterations are that we will assume a 3-dimensional space (x, y, z), and the distance (r) formula will be slightly different. Since neither of the objects in this problem are static, we cannot assume that one of them remains located at the origin. So, in 3 dimensions, we have:

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Where (x_1, y_1, z_1) is the displacement of the first object, and (x_2, y_2, z_2) is the displacement of the second object. After making a similar change to our unit vectors, our system of differential equations is defined as follows:

$$\begin{cases} \frac{d^2x_1}{dt^2} = -G\frac{m_2(x_1-x_2)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \\ \\ \frac{d^2x_2}{dt^2} = -G\frac{m_2(x_2-x_1)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \\ \\ \frac{d^2y_1}{dt^2} = -G\frac{m_2(y_1-y_2)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \\ \\ \frac{d^2y_2}{dt^2} = -G\frac{m_2(y_2-y_1)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \\ \\ \frac{d^2z_1}{dt^2} = -G\frac{m_2(z_1-z_2)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \\ \\ \frac{d^2z_2}{dt^2} = -G\frac{m_2(z_2-z_1)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \end{cases}$$

3 Methodology

Now that we have the systems set up, we can solve them. However, in order to solve them in Python, we must first convert them to first order equations. To do this for the one-body problem, we set $\frac{dx}{dt} = v_x$ and $\frac{dy}{dt} = v_y$, as velocity is the first derivative of displacement. We then set the derivatives of these velocities equal to the equations that we formulated for acceleration, and we are left with a system of four first-order differential equations:

$$\begin{cases} \frac{dx}{dt} = v_x \\ \frac{dv_x}{dt} = -G\frac{m_2 x}{(x^2+y^2)^{\frac{3}{2}}} \\ \frac{dy}{dt} = v_y \\ \frac{dv_y}{dt} = -G\frac{m_2 y}{(x^2+y^2)^{\frac{3}{2}}} \end{cases}$$

Performing the same process for the two-body problem, our system of equations is as follows:

$$\begin{cases} \frac{dx_1}{dt} = v_{x_1} \\ \frac{dv_{x_1}}{dt} = -G\frac{m_2(x_1-x_2)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \\ \frac{dx_2}{dt} = v_{x_2} \\ \frac{dv_{x_2}}{dt} = -G\frac{m_1(x_2-x_1)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \\ \frac{dy_1}{dt} = v_{y_1} \\ \frac{dv_{y_1}}{dt} = -G\frac{m_2(y_1-y_2)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \\ \frac{dy_2}{dt} = v_{y_2} \\ \frac{dv_{y_2}}{dt} = -G\frac{m_1(y_2-y_1)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \\ \frac{dz_1}{dt} = v_{z_1} \\ \frac{dv_{z_1}}{dt} = -G\frac{m_2(z_1-z_2)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \\ \frac{dz_2}{dt} = v_{z_2} \\ \frac{dv_{z_2}}{dt} = -G\frac{m_1(z_2-z_1)}{((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2)^{\frac{3}{2}}} \end{cases}$$

From this point, we need to define specific parameters, initial conditions, and time discretizations for our systems, and we can then solve them using Python. For the one-body problem, we will set the following parameters:

- Gravitational constant: $G = 0.1$
- Mass: $m_2 = 1000$
- Initial conditions:

$$\begin{aligned} x(0) &= 8, \\ v_x(0) &= 2, \\ y(0) &= 10, \\ v_y(0) &= -1 \end{aligned}$$

- Time interval $t = [0, 100]$

Then, for the two-body problem, the parameters are as follows:

- Gravitational constant: $G = 0.1$
- Masses:

$$\begin{aligned} m_1 &= 1000, \\ m_2 &= 800 \end{aligned}$$

- Initial conditions:
  - For body 1:

$$
\begin{aligned}
x_1(0) &= 1, \\
v_{x_1}(0) &= -1.5, \\
y_1(0) &= 1, \\
v_{y_1}(0) &= 1.5, \\
z_1(0) &= 1, \\
v_{z_1}(0) &= -1
\end{aligned}
$$

  - For body 2:

$$
\begin{aligned}
x_2(0) &= -1, \\
v_{x_2}(0) &= 1.5, \\
y_2(0) &= -1, \\
v_{y_2}(0) &= -1.5, \\
z_2(0) &= -1, \\
v_{z_2}(0) &= 2
\end{aligned}
$$

- Time interval $t = [0, 10]$

After solving these systems using Python's built-in function scipy.integrate.solve_ivp, we can then calculate the center of mass for the two-body problem (See Figure 3 below).

```
#calculate the center of mass
com_x = (m1 * x1 + m2 * x2) / (m1 + m2)
com_y = (m1 * y1 + m2 * y2) / (m1 + m2)
com_z = (m1 * z1 + m2 * z2) / (m1 + m2)
```

Figure 3: Center of Mass Calculations for Two-Body Problem

4 Results

Finally, we can plot the phase planes for each of the problems as "movie plots", including the center of mass for the two-body problem. In Figure 4 below, we cannot view the actual movement of the bodies, but in the Python file, we can see these dots moving across their trajectories, influenced by the gravitational forces that we defined. Additionally, the dot representing the center of mass in the two body problem moves steadily upwards as the two bodies climb their elliptical trajectories.

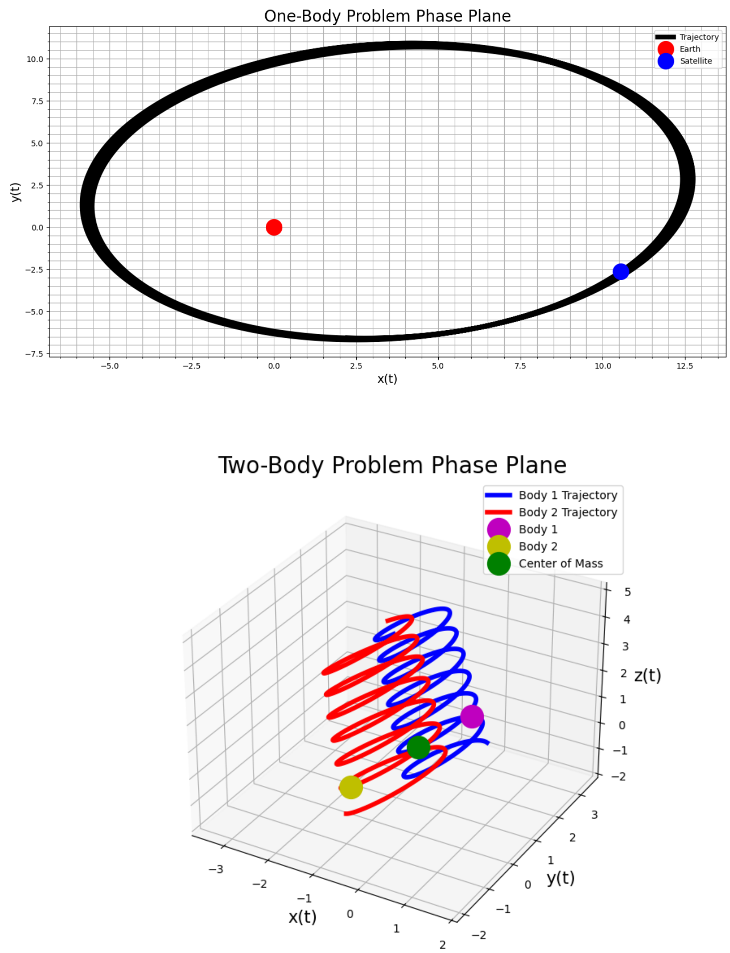

Figure 4: Phase Planes for the One-Body and Two-Body Problems

5 Discussion

In these plots, we can see the bodies orbiting around each other. In the one-body model, our “satellite” follows a fairly continuous, circular trajectory around the static object at the origin, which we labeled as “Earth”. Then, for the two-body model, we see similar trajectories, but now in a 3-dimensional space, where the bodies seem to be attracting and reflecting off of one another as time passes. We can see how the center of mass interacts with these bodies as well, shifting along with them vertically as they spiral upwards. As an extension of this project, we could attempt to model more complex versions of these problems, including some of the other outside forces mentioned earlier. Furthermore, we could add more objects or bodies into the mix, and then examine just how much of an effect this would have on the trajectories.

6 Conclusion

Although the models we looked at here were simplified, and we provided seemingly optimal parameters to output the interactions that we were looking for, this project shows how useful differential equations can be for modeling functions. Even with more complex problems, we can use similar methods as we did here to showcase the behavior of variables given their derivatives. Additionally, by altering parameters and initial conditions, we are able to examine how these variables would behave under different circumstances, which can be very useful when examining real-world processes.