Mini Project 1 - Optimization with Caclulus

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Abstract

Optimization is the process of minimizing or maximizing an objective function subject to a list of constraints. In this project, we examine a problem any home owner may have: building a shed. In this scenario, the shed will be a rectangular base and a set of square sides, with a right triangular roof. The shed will be made out of wood with a wood roof support, and the homeowner can choose roof shingle materials. The cost of materials (objective function) is being minimized while the user will provide the volume their shed wants to hold and the angle of the roof (parameters). A diagram of this can be seen below in Figure 1. The cost minimizing function, $C_{min}(V, \alpha, m)$, has been generalized for any volume, angle, and material. The thorough process, solution, and tool is outlined in the rest of this report.

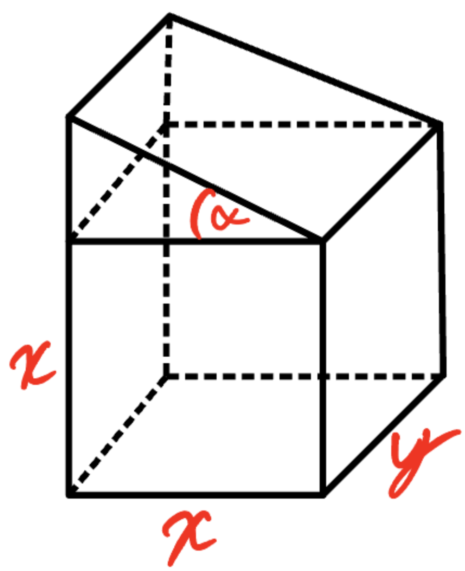

Figure 1: Rectangular based shed (x & y) with roof angle (α)

1 Introduction

In this report, we will generalize a 3-dimensional shed problem. In particular, the shed will be made of a rectangular base with one pair of walls being square. Finally, the shed will have a roof that can be made out of separate materials. This will be outlined in the methodology section. First, we will formulate the shed's surface area in terms of x and y (the shed's wall dimensions), and α (the shed's roof angle). Then, we can apply cost coefficients to each piece based on the materials of the shed and roofing shingles. This will create a cost function. Next, a shed volume function (constraint) can be rearranged to solve for y, and then put back in to the cost function to be in terms of only one variable x. Finally, we can use calculus to minimize this cost function and simplify it down to a formula for the user to input V, α, m_i. This project may then be taken further in to python to create a tool in python for the user to produce charts.

This paper is organized as follows: In Section 2, we will highlight the problem statement along with constraints for the project. In section 3, we will formulate the solution using the constraints and parameters given using calculus optimization techniques. Section 4 and 5 will be the final results of the project and a discussion about the results or other restraints respectively. Finally, Section 6 will be conclusions of the project.

2 Problem Statement

The goals of this project are to formulate and solve a 3-dimensional shed's cost minimization and present how this solution is dependent on the constraint variables (x, y) as well as parameters (V, α, m_i).

Next is the methodology section, where we will dive in to the formulation of the surface area, volume, and cost functions for the project.

3 Methodology

In this section, we will go over how the cost function is derived and minimized, and methods of simplification.

3.1 Formulating Surface Area

The first step to finding the cost function $C(x, y, \alpha)$ is by finding the total surface area. This can be broken down into smaller pieces as seen below:

3.1.1 Square Based Walls

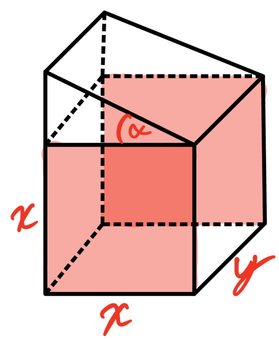

Figure 2: Rectangular based shed with square walls highlighted.

By far the simplest piece. We can find the surface area of the highlighted area by $2 \cdot x^2$.

3.1.2 Rectangular Based Walls/Floor

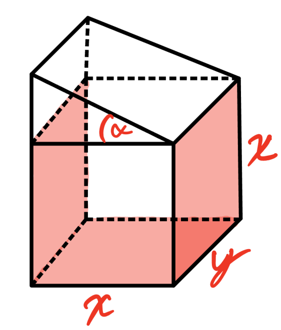

Figure 3: Rectangular based shed with square walls highlighted.

This is almost as simple as the previous pieces. We can find the surface area of the highlighted area by $3 \cdot x \cdot y$.

3.1.3 Right Triangle Side Roofing

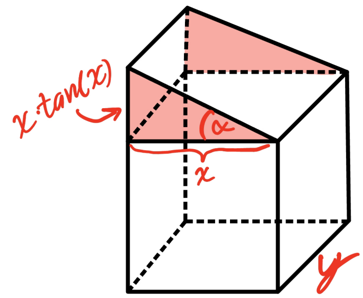

Figure 4: Rectangular based shed with right triangle side roofs highlighted

This piece is a little more complicated, but we can find that vertical edge (h) of the triangle using some trigonometry:

$$tan(\alpha) = \frac{h}{x} \quad \Longrightarrow \quad x \cdot tan(\alpha) = h$$

From here, we know the area of a triangle is $\frac{1}{2} \cdot b \cdot h$, so the total area of those pieces:

$$2 \cdot \left(\frac{1}{2} \cdot x \cdot xtan(\alpha)\right) = x^2 \tan(\alpha)$$

3.1.4 Back of Side Roofing

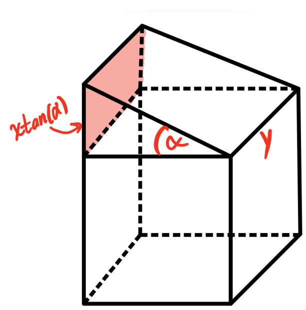

Figure 5: Rectangular based shed with back roof highlighted

Area of back rectangle: $y \cdot x \cdot \tan(\alpha) = xy \tan(\alpha)$

3.1.5 Sloped Roof

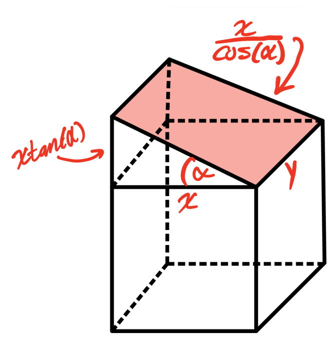

Figure 6: Rectangular based shed with sloped roof highlighted

We can solve for the hypotenuse (k) of the triangular sides using trigonometry again:

$$cos(\alpha) = \frac{x}{k} \implies k = \frac{x}{\cos(\alpha)}$$

Thus, the area of this rectangle is $\frac{x}{\cos(\alpha)} \cdot y = \frac{xy}{\cos(\alpha)}$

3.1.6 Summary of all Pieces

We can now represent the surface area of the shed by adding all of these peices together:

$$S(x, y, \alpha) = \underbrace{3xy + 2x^2}_{\text{Main Walls/Floor}} + \underbrace{x^2 \tan(\alpha)}_{\text{Triangle Side Walls}} + \underbrace{x \tan(\alpha) y}_{\text{Triangle Back Wall}} + \underbrace{\frac{xy}{cos(\alpha)}}_{\text{Slope (Wood)}} + \underbrace{\frac{xy}{cos(\alpha)}}_{\text{Slope (Roofing)}}$$

3.2 Creating the Cost Function

Now that we have the surface area function, $S(x, y, \alpha)$, we can apply cost coefficients to certain terms to estimate the cost of each piece of the shed.

3.2.1 Material Cost Coefficients

For all the wooden parts, we can apply a flat cost of $\$15.00$ to all pieces other than the roofing material shingles.

For the roofing materials, the user will have the choices below:

Material	Price per ft.^{2}	Angle
m_1 : Wood	\$0.00	$\alpha \geq 14.25^o$
m_2 : Asphalt	\$4.68	$\alpha \geq 9.5^o$
m_3 : Clay	\$9.30	$\alpha \geq 12^o$
m_4 : Metal	\$15.00	$\alpha \geq 14.25^o$
m_5 : Slate	\$27.63	$\alpha \geq 18.5^o$

Note that the wooden roof has a cost coefficient (m_1) of $\$0.00$. This is because the surface area function already accounts for a wooden roof as support for other materials.

3.2.2 Applying the Cost Coefficients

As mentioned above, we will want to apply a flat rate of $\$15.00$ per ft.^{2} for all the wooden pieces, and the roofing cost coefficient (m_i) to the roofing material piece:

$$C(x, y, \alpha) = 15.00\left(\underbrace{3xy + 2x^2 + x^2\tan(\alpha) + x\tan(\alpha)y + \frac{xy}{cos(\alpha)}}_{\text{All pieces made of wood}}\right) + m_i\left(\underbrace{\frac{xy}{cos(\alpha)}}_{\text{Roofing material}}\right)$$

$$= C(x, y, \alpha) = 45xy + 30x^2 + 15x^2\tan(\alpha) + 15x\tan(\alpha)y + \frac{15xy}{cos(\alpha)} + \frac{m_i xy}{cos(\alpha)}$$

3.3 Analyzing the Constraints

For this project, one of the main constraints is the volume of the shed. Thankfully, this is easier to formulate than the surface area or cost functions:

$$V(x, y, \alpha) = x \cdot x \cdot y + \frac{1}{2} \cdot x \cdot x \cdot tan(\alpha) \cdot y = x^2y + \frac{x^2\tan(\alpha)y}{2} = x^2y\left(1 + \frac{1}{2}\tan(\alpha)\right)$$

We can then solve for y by rearranging the equation to isolate y:

$$V = x^2y\left(1 + \frac{1}{2}\tan(\alpha)\right) \quad \Longrightarrow \quad y = \frac{V}{x^2\left(1 + \frac{1}{2}\tan(\alpha)\right)}.$$

3.4 Minimizing and Simplifying the Cost Function

Now, we can rewrite the cost function in terms of only one variable (x) by substituting y into the cost function and then use derivatives to minimize it.

3.4.1 Simplifying the Cost Function

Rearranging terms and factoring out common factors can allow us to rename complicated strings of constants.

$$C(x,y,\alpha) = 45xy + 30x^2 + 15x^2\tan(\alpha) + 15x\tan(\alpha)y + \frac{15xy}{cos(\alpha)} + \frac{m_i xy}{cos(\alpha)}$$

$$\implies C(x,y,\alpha) = xy\left(45 + 15\tan(\alpha) + \frac{15}{cos(\alpha)} + \frac{m_i}{cos(\alpha)}\right) + 30x^2 + 15x^2\tan(\alpha)$$

$$\implies C(x,y,\alpha) = x\left(\frac{1}{x^2}\cdot\frac{V}{\left(1+\frac{1}{2}\tan(\alpha)\right)}\right)\left(45 + 15\tan(\alpha) + \frac{15}{cos(\alpha)} + \frac{m_i}{cos(\alpha)}\right) + 30x^2 + 15x^2\tan(\alpha)$$

$$\implies C(x,y,\alpha) = \frac{1}{x}\left(\frac{V}{\left(1+\frac{1}{2}\tan(\alpha)\right)}\right)\left(45 + 15\tan(\alpha) + \frac{15}{cos(\alpha)} + \frac{m_i}{cos(\alpha)}\right) + 30x^2 + 15x^2\tan(\alpha)$$

$$\implies C(x,y,\alpha) = \frac{1}{x}\left(\frac{V}{\left(1+\frac{1}{2}\tan(\alpha)\right)}\right)\left(45 + 15\tan(\alpha) + \frac{15}{cos(\alpha)} + \frac{m_i}{cos(\alpha)}\right) + x^2\left(30 + 15\tan(\alpha)\right)$$

Since V, α, and m_i are parameters that will be input by the user, they can be treated as constant. This will allow us to rename pieces of our cost function to be simpler.

$$C(x,y,\alpha) = \frac{1}{x}\cdot\underbrace{\left(\frac{V}{\left(1+\frac{1}{2}\tan(\alpha)\right)}\right)}_{\beta}\cdot\underbrace{\left(45 + 15\tan(\alpha) + \frac{15}{cos(\alpha)} + \frac{m_i}{cos(\alpha)}\right)}_{\delta} + x^2\underbrace{\left(30 + 15\tan(\alpha)\right)}_{\sigma}$$

$$\implies C(x,y,\alpha) = \frac{1}{x}\beta\delta + x^2\sigma \implies C(x,y,\alpha) = \frac{\beta\delta}{x} + \sigma x^2$$

3.4.2 Minimizing the Cost Function

First, we can take the derivative of the cost function with respect to x using the power rule. Then, we can set it equal to zero allowing us to find when the cost is minimized:

$$C(x,y,\alpha) = \frac{\beta\delta}{x} + \sigma x^2 \implies C(x,y,\alpha) = \beta\delta x^{-1} + \sigma x^2 \implies \frac{d}{dx}[C] = \frac{d}{dx}\left[\beta\delta x^{-1} + \sigma x^2\right]$$

$$\implies \frac{dC}{dx} = -\beta\delta x^{-2} + 2\sigma x \quad \text{Now setting this equal to zero yields:} \quad 0 = -\beta\delta x^{-2} + 2\sigma x$$

$$\implies (x^2)\cdot 0 = (x^2)\cdot\left(-\beta\delta x^{-2} + 2\sigma x\right) \implies 0 = -\beta\delta + 2\sigma x^3 \implies \frac{\beta\delta}{2\sigma} = x^3 \implies \sqrt[3]{\frac{\beta\delta}{2\sigma}} = x$$

Now that we know x, we can solve for y:

$$y = \frac{V}{x^2\left(1+\frac{1}{2}\tan(\alpha)\right)} \implies y = \frac{1}{x^2}\beta \implies y = \frac{\beta}{\left(\sqrt[3]{\frac{\beta\delta}{2\sigma}}\right)^2}$$

$$\implies y = \beta\cdot\left(\frac{\beta\delta}{2\sigma}\right)^{-\frac{2}{3}} \implies y = \beta^{\frac{1}{3}}\left(\frac{\delta}{2\sigma}\right)^{-\frac{2}{3}}$$

3.4.3 Finalizing the Cost Minimizing Function

Now that we know x and y, we can generalize the cost minimizing function in terms of only the parameters V, α, and m_i

$$C(x,y,\alpha) = \frac{\beta\delta}{x} + \sigma x^2 \implies C(x,y,\alpha) = \frac{\beta\delta}{\sqrt[3]{\frac{\beta\delta}{2\sigma}}} + \sigma\left(\sqrt[3]{\frac{\beta\delta}{2\sigma}}\right)^2 \implies C(x,y,\alpha) = \beta\delta\left(\frac{\beta\delta}{2\sigma}\right)^{-\frac{1}{3}} + \sigma\left(\frac{\beta\delta}{2\sigma}\right)^{\frac{2}{3}}$$

$$\implies C(x,y,\alpha) = (\beta\delta)^{\frac{2}{3}}(2\sigma)^{\frac{1}{3}} + (\beta\delta)^{\frac{2}{3}}\sigma(2\sigma)^{-\frac{2}{3}} = (\beta\delta)^{\frac{2}{3}}(2\sigma)^{\frac{1}{3}} + (\beta\delta)^{\frac{2}{3}}\cdot\frac{2\sigma}{2}(2\sigma)^{-\frac{2}{3}}$$

$$= (\beta\delta)^{\frac{2}{3}}(2\sigma)^{\frac{1}{3}} + (\beta\delta)^{\frac{2}{3}}\cdot\frac{1}{2}(2\sigma)^{\frac{1}{3}} = \frac{3}{2}(\beta\delta)^{\frac{2}{3}}(2\sigma)^{\frac{1}{3}} = C(V,\alpha,m_i)$$

4 Results

We have now formulated a cost minimizing function C_{min} only in terms of the input parameters (V, α, m_i).

$$C(V,\alpha,m_i) = \frac{3}{2}(\beta\delta)^{\frac{2}{3}}(2\sigma)^{\frac{1}{3}}$$

Where $\beta = \dfrac{V}{\left(1+\frac{1}{2}\tan(\alpha)\right)}, \qquad \delta = 45 + 15tan(\alpha) + \dfrac{15+m_i}{cos(\alpha)}, \qquad \sigma = 30 + 15\tan(\alpha)$

In the next section, we will discuss these finding further by analyzing charts made in python.

5 Discussion

Now that we have a formula for the cost minimizing function, $C_{min}(V, \alpha, m_i)$, we can recreate this function in python to graph certain input parameters for the different roof materials.

5.1 Writing the Python Code

First, we start with creating an array of numbers for V, in this case we are using all integers from 0 to 1000 inclusive:

```
# Define an array of perimeter values:
N = 1001 # if you put 5, it will make up 5 lines to build a curve
V = np.linspace(0,1000,N) # Plotting a certain # of points and connecting lines
```

Then, the angle and material parameters are defined. These values are subject to the user to analyze and output any chart desired:

```
a = np.deg2rad(30)
m1 = 0 # Wooden Roof (No material added)
m2 = 4.68 # Asphalt Roof
m3 = 9.30 # Clay Roof
m4 = 15.00 # Metal Roof
m5 = 27.63 # Slate Roof
```

Next, we can define our objective function that was formulated in Section 3. Note that β, δ, and σ were defined first, and then used in the cost minimizing function:

```
def Cost_min(V,a,m):
    beta = V / (1 + (1/2)*np.tan(a))
    delta = 45 + 15*np.tan(a) + ((15 + m)/(np.cos(a)))
    sigma = 30 + 15*np.tan(a)
    result = (3/2) * ((beta*delta)**(2/3)) * ((2*sigma)**(1/3))
    return result
```

Finally, we can use the cost minimizing function to plot 5 different lines (one for each material). Additional formatting is applied to the chart for cleanliness:

```
# Plot the max area function against A:
plt.figure(1)
plt.plot(V,Cost_min(V,a,m1), linewidth = 2, label=f'Wood (${m1}/sq ft.)') # Wood
plt.plot(V,Cost_min(V,a,m2), linewidth = 2, label=f'Asphalt (${m2}/sq ft.)') # Asphalt
plt.plot(V,Cost_min(V,a,m3), linewidth = 2, label=f'Clay (${m3}/sq ft.)') # Clay
plt.plot(V,Cost_min(V,a,m4), linewidth = 2, label=f'Metal (${m4}/sq ft.)') # Metal
plt.plot(V,Cost_min(V,a,m5), linewidth = 2, label=f'Slate (${m5}/sq ft.)') # Slate
```

```
# Plot area formatting
plt.title('Shed w/ Roof Cost (by Material)',fontsize=20) # Only formats the title
plt.xlabel('Volume (Cubic ft.)', fontsize=15) # Only formats the x-axis
plt.ylabel('Minimum Cost ($)', fontsize=15) # Only formats the y-axis
plt.gca().tick_params(labelsize=10) # Tick mark labels
plt.legend() # Adds a legend
plt.grid(True, which='both') # Adds grid lines for both x & y axes
plt.minorticks_off()
```

5.2 Displaying the Charts

In this section we will go over an example running the program wth parameters (V, α, m_i).

For example, we can input that $0 \leq V \leq 1000$ to produce a plot of the C_{min} function for all of those values. Then, all we have to do is specify some angle (α) for the roof of the shed and this chart is produced:

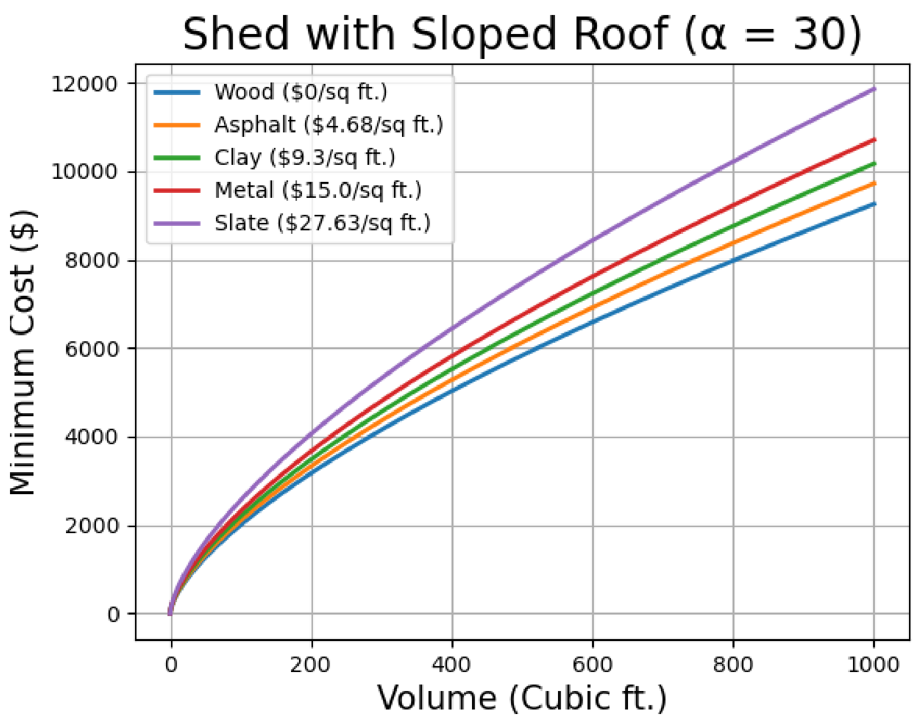

Figure 7: A volume vs. minimum cost chart for all 5 roof materials

Note: The curves have a square/cube root shape. This is consistent with our objective function as β (contains V) is to the $\frac{2}{3}$ power.

5.3 Other Restraints

From section 3.4.1, we can see that β and δ are defined as follows:

$$\beta = \frac{V}{\left(1 + \frac{1}{2}\tan(\alpha)\right)} \qquad \text{and} \qquad \delta = 45 + 15tan(\alpha) + \frac{15 + m_i}{cos(\alpha)}$$

Since we are dealing with trigonometric functions, this can limit our values of α to only where $\tan(\alpha)$ and $\cos(\alpha)$ exist. In fact, there are 3 restraints to worry about:

1. $\tan(\alpha)$ is undefined
2. $1 + \frac{1}{2}\tan(\alpha) = 0$
3. $\cos(\alpha) = 0$

5.3.1 $\tan(\alpha)$ is undefined

We know that $\alpha \neq \dfrac{(2n+1)\pi}{2}$ (radians) or in other words $\alpha \neq 90(2n+1)^{o}$ for $n \in \mathbb{Z}$.
Basically, α can not be an odd multiple of 90^{o}.

5.3.2 $1 + \frac{1}{2}\tan(\alpha) = 0$

Since this equation is in the denominator, it can not be equal to zero.
Rewriting this equation: $1 + \frac{1}{2}\tan(\alpha) \neq 0 \quad \Longrightarrow \quad \tan(\alpha) \neq -2$

5.3.3 $\cos(\alpha) = 0$

Since $\cos(\alpha)$ is in the denominator, it can not be equal to zero.
Similar to 5.3.1, $\alpha \neq \dfrac{(2n+1)\pi}{2}$ (radians) or in other words $\alpha \neq 90(2n+1)^{o}$ for $n \in \mathbb{Z}$.

This is actually because $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$. This is why both restraints are the same.

6 Conclusion

In this project, optimization through calculus was used to formulate, minimize, and model the cost of building a shed with a roof. Then, the cost minimizing function was modeled in python to be user friendly and produce a chart for any volume (V), roof angle (α), and roofing material (m_i).

In the future, improvements for this project would be made. For example, creating a python tool to produce 3-dimensional surfaces that still had Volume (V) on the x-axis, but now include angle (α) on the y-axis and cost on the z-axis. This would take away the α parameter input and already be included in the charts.

While it can be done with the solution laid out here, another improvement would be a tool in python to produce the exact optimized dimensions (x, y, α) for the cost minimizing function of certain volume.

7 Resources

For this project, angle restrictions and material prices for the roof slope were borrowed from the following websites (hyperlinked):

Roof Slope Regulations

Roof Material Costs