Mini-Project 1 – Optimization with Calculus

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Abstract

Using methods from calculus, we are able to solve several optimization problems related to minimization and maximization. Provided with a certain constraint that includes the same variables as the problem at hand, we can attempt to minimize or maximize the function/measure in question. To do this, we define the function, say f, that we are trying to optimize in terms of a certain variable, say x. Then, we can utilize differentiation to find the derivative, $f'(x)$, and then set $f'(x) = 0$ to find the critical points of $f(x)$. Finally, these critical points will allow us to find the values needed to define our minimization or maximization functions.

1 Introduction

In this report, we will be looking at specific optimization problems, one of which relates to a 2-dimensional maximization and another that relates to a 3-dimensional minimization. The scenario in question involves a farmer who wants to build a pen, and his options include separating the pen into anywhere from 1 to 4 sections, building the pen next to his barn, and putting a roof over the pen. We will be exploring all of these scenarios for each of our optimizations: For the maximization problem, we will be attempting to maximize the amount of area in each section of the pen subject to a given total perimeter value. For the minimization, we will dive into the roof scenario, attempting to minimize the total amount of "materials" needed to build the pens with a roof over top, which will be subject to the total area of the pen. This will allow the farmer to figure out the most effective way to go about building his pen based on his wants and needs.

2 Problem Statement

We will begin with the maximization problem, which will be finding a maximum area function, A_{max}, for each section of the pen subject to the total perimeter, P, which is given (in ft.). For the scenarios that the farmer is not building against the side of his barn, our pens will be in this format:

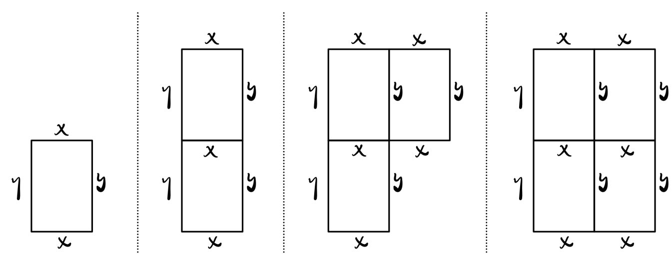

Figure 1: Pens Separated from the Barn

Then, for the other scenarios in which the farmer wants to build against the barn, the pens will look something like this:

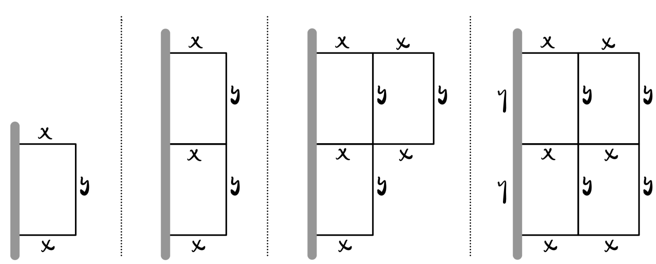

Figure 2: Pens Against the Barn

Let $s=$ the number of sections the pen is divided into, let b be a boolean variable (0 or 1) for whether or not the farmer is building against his barn, and let $P=$ perimeter. Then, we have the following constraints for our maximization problems (Note: we will always be trying to maximize $A = xy$):

Variables	s=1	s=2	s=3	s=4
b=0	$P = 2x + 2y$	$P = 3x + 4y$	$P = 5x + 5y$	$P = 6x + 6y$
b=1	$P = 2x + y$	$P = 3x + 2y$	$P = 5x + 3y$	$P = 6x + 4y$

Table 1: Perimeter Constraints for A_{max}

Moving on, we will then look at the minimization problem, which will be finding the minimum amount of "materials", M_{min}, that the farmer needs to build the pen with a roof over top, subject to the total area, A, which is given. Using the same 2-dimensional formats as we did for the maximization (e.g. Figures 1 and 2), we can define our own function for the materials needed. Assume that all materials are "worth" the same, and all of their measurements are in terms of ft. (or ft.^{2}). Also, assume that each "piece" of fencing is 1 ft. $\times$ 1 ft. $\times$ 1 ft., and we will be building the fence 5 ft. high on each side. Finally, assume the roof that the farmer is building is flat (1 ft. high), and there will be a post on every "corner" of the pen to hold the roof up that measures 2 ft. $\times$ 2 ft. $\times$ 10 ft. (for 10 ft. high roof). Then, we can set up our materials function with the format M = area of pen (to account for the roof) + 5 $\times$ perimeter of pen (to account for the 5 ft. high fence) + 40 $\times$ number of corners in the pen (to account for the posts). Using the same barn and section values as we used for the maximization, our formulas for M look like this:

Variables	b=0	b=1
s=1	$M = xy + 5(2x + 2y) + 40(4)$	$M = xy + 5(2x + y) + 40(2)$
s=2	$M = 2xy + 5(4x + 3y) + 40(6)$	$M = 2xy + 5(3x + 2y) + 40(3)$
s=3	$M = 3xy + 5(5x + 5y) + 40(8)$	$M = 3xy + 5(5x + 3y) + 40(5)$
s=4	$M = 4xy + 5(6x + 6y) + 40(9)$	$M = 4xy + 5(6x + 4y) + 40(6)$

Table 2: Equations for Materials (M)

Then, to calculate M_{min}, our area constraints are as follows:

Variables	s=1	s=2	s=3	s=4
b=0	$A = xy$	$A = 2xy$	$A = 3xy$	$A = 4xy$
b=1	$A = xy$	$A = 2xy$	$A = 3xy$	$A = 4xy$

Table 3: Area Constraints for M_{min}

3 Methodology

Now that we know the functions we are trying to maximize and minimize as well as the constraints we are subjecting them to, we can use calculus to solve for A_{max} and M_{min}. Again, let's begin with A_{max}. We are trying to maximize the function $A = xy$, which refers to the area of each section of the pen, subject to certain perimeter constraints for each scenario, which are listed in Table 1. To show the process used to calculate A_{max}, we will use the scenario with 2 sections and no barn (s=2, b=0). So, we must maximize $A = xy$ subject to $P = 4x + 3y$. First, we need to solve for y in $P = 4x + 3y$ so that we can get our area function in terms of a single variable, x. We find that $y = \frac{P}{3} - \frac{4}{3}x$, and by substituting this into our area formula, we can now write $A(x) = x(\frac{P}{3} - \frac{4}{3}x)$, which then simplifies to $A(x) = (\frac{Px}{3} - \frac{4}{3}x^2)$. To find the critical point(s) of $A(x)$, we need to find the derivative and set it equal to zero: $A'(x) = \frac{P}{3} - \frac{8}{3}x = 0$. Solving for x, we find that at our critical point, $x = \frac{P}{8}$. Then, we can substitute this value in to $y = \frac{P}{3} - \frac{4}{3}x$ to find that $y = \frac{P}{6}$. Finally, we can define our A_{max} function in terms of P: $A_{\text{max}}(P) = \frac{P^2}{48}$. Repeating this process for every other scenario, we use all of the equations to define a piece-wise function in terms of P, s, and b as follows:

$$A_{\text{max}}(P, s, b) = \begin{cases} \frac{P^2(b+1)}{16} & \text{if } s = 1, \\ \frac{P^2(b+1)}{48} & \text{if } s = 2, \\ \frac{P^2}{(2s+4)^2 - b(8(s+2))} & \text{if } s = 3, 4 \end{cases}$$

Moving on to M_{min}, the process remains the same. For the scenario with 3 sections and the barn (s=3, b=1), for instance, we need to minimize $M = 3xy + 25x + 15y + 200$ subject to $A = 3xy$. Using the same steps as before, we find $y = \frac{A}{3x}$, which implies $M(x) = 3x(\frac{A}{3x}) + 25x + 15(\frac{A}{3x}) + 200$, which then simplifies to $M(x) = A + 25x + \frac{5A}{x} + 200$. To find our minimum, we set $M'(x) = 25 - \frac{5A}{x^2} = 0$ to find $x = \frac{\sqrt{5A}}{5}$ at our critical point. It follows that $y = \frac{\sqrt{5A}}{3}$ and $M_{\text{min}}(A) = 3(\frac{\sqrt{5A}}{5})(\frac{\sqrt{5A}}{3}) + 25(\frac{\sqrt{5A}}{5}) + 15(\frac{\sqrt{5A}}{3}) + 200$, which simplifies to $M_{\text{min}}(A) = A + 10\sqrt{5A} + 200$. Again, we repeat this process for every scenario, and we get a piece-wise function in terms of A, s, and b:

$$M_{\text{min}}(A, s, b) = \begin{cases} A + \frac{20}{b+1}\sqrt{A(b+1)} + \frac{160}{b+1} & \text{if } s = 1, \\ A + 10\sqrt{\frac{6A}{b+1}} + \frac{240}{b+1} & \text{if } s = 2, \\ A + \frac{50}{3}\sqrt{3A} + 320 & \text{if } s = 3 \text{ and } b = 0, \\ A + 10\sqrt{5A} + 200 & \text{if } s = 3 \text{ and } b = 1, \\ A + 30\sqrt{A} + 360 & \text{if } s = 4 \text{ and } b = 0, \\ A + 10\sqrt{6A} + 240 & \text{if } s = 4 \text{ and } b = 1 \end{cases}$$

4 Results

Using Python, we can define and plot our functions to visualize them and see the results more clearly. For $A_{\max}(P, s, b)$, the graphs for the no barn (left), and barn (right) scenarios look like this:

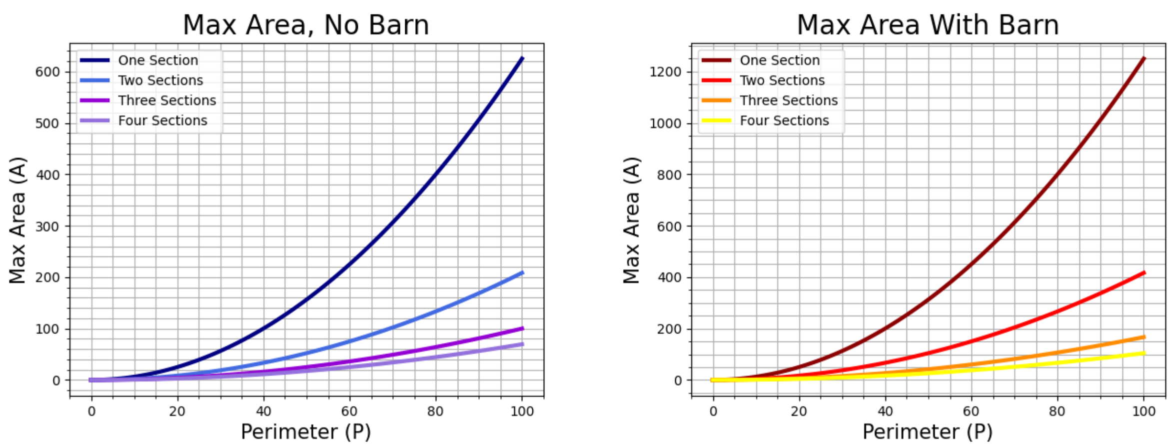

Figure 3: $A_{\max}(P, s, b)$

Here, we see that each plot follows basically the same pattern as we add more sections, which was to be expected since these functions all included the value of P^2. The one big difference we see here is that the area values (per section) for the scenarios in which the barn is included seem to be about double that of the scenarios where there is no barn. For $M_{\min}(A, s, b)$, then, our plots look like this:

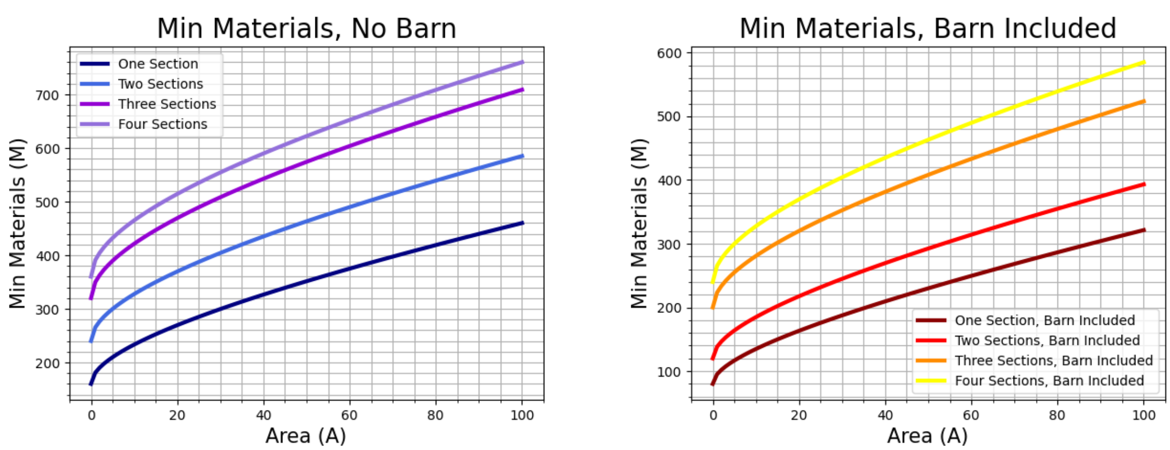

Figure 4: $M_{\min}(A, s, b)$

Again, these lines all follow the same general pattern, as they all share some square root value that contains $\sqrt{A}$. The main difference here is that the "no barn" scenarios all require about 100-200 more materials for any given section value.

5 Discussion

The results given in these plots provide very useful information for the farmer. If his priority is building a pen that provides the highest possible amount of area, then it would be smart for him to build against his barn, as he can basically double the amount of area in the pen by doing so. For the sections, though, it is important to note that we calculated the area for each individual section, so the disparity between these sections is not actually as large as it appears to be. However, if we were to adjust the functions to show the area for the entire pen, the singular sections would still yield the highest values, though not by as much. Considering the materials, then, there isn't nearly as much disparity between building against the barn and not. Obviously, building separate from the barn would cost more, but it's not like he would be even close to doubling his costs by doing so. Additionally, unless multiple sections are needed, it is evident that building one section is much more efficient in terms of the materials needed.

6 Conclusion

In summary, optimization using calculus is a great method to solve all sorts of minimization or maximization problems, and in cases like this, it allows us to fully optimize the parameters for construction. We looked at several different scenarios here, including the number of sections wanted in a pen, whether or not the farmer would want to build against his barn, and how the pen would shape out if he wanted to include a roof. If we wanted to expand on this, we can continue to add more variables such as the size of each section or the roof design. Also, one thing that I would change about this particular study would be to find the max area values for the entire pen instead of just one section of it. By doing this, we would have been able to see the area relationships between the different section values much more clearly.