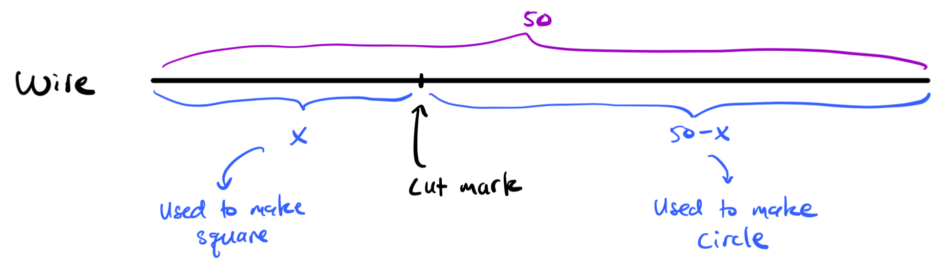
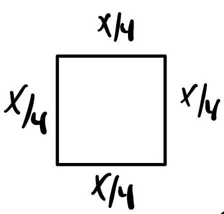
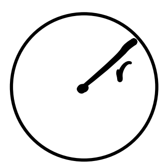
Phase 1 (part 2): Optimization on an Interval

1.] Wire Problem: A 50-inch piece of wire is to be cut into two pieces (a portion of length x and a portion of length $50 - x$, see figure below), which are then bent into a square and a circle, respectively. Where should the wire be cut (i.e. what is x?) in order to minimize the sum of the areas of these two shapes? Where should it be cut to maximize the sum of the areas? To solve both problems, formulate a function $f(x)$ that represents the sum of the two areas in terms of x, then use the Closed Interval Method to determine the absolute minimum (and maximum) on the interval $x \in [0, 50]$.





$$A_s = \frac{x^2}{16}$$



$$2\pi r = 50 - x \rightarrow r = \frac{50-x}{2\pi}$$

$$A_c = \pi r^2 \Rightarrow A_c = \pi\left(\frac{50-x}{2\pi}\right)^2$$

$$A_c = \frac{(50-x)^2}{4\pi}$$

Obj. Fun: $f(x) = \frac{x^2}{16} + \frac{(50-x)^2}{4\pi} \longrightarrow f'(x) = \frac{x}{8} - \frac{2(50-x)}{4\pi} = 0$

Constraint: $x \in [0, 50]$

$$\frac{x}{8} = \frac{100-2x}{4\pi}$$

$$4\pi x = 800 - 16x$$

$$(4\pi + 16)x = 800$$

$$x = \frac{800}{4\pi+16} \rightarrow \boxed{x = \frac{200}{\pi+4}}$$

Min

$$f''(x) = \frac{1}{8} + \frac{2}{4\pi} = \frac{1}{8} + \frac{1}{2\pi} > 0$$

CIM:

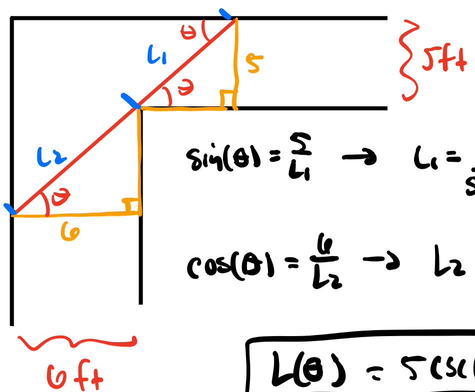
$f(0) = \frac{50^2}{4\pi} = \frac{2500}{4\pi} = \frac{625}{\pi}$ ← max

$f\left(\frac{200}{\pi+4}\right) = \frac{1}{16}\left(\frac{200}{\pi+4}\right)^2 + \frac{1}{4\pi}\left(50 - \frac{200}{\pi+4}\right)^2$ ← min

$f(50) = \frac{50^2}{16} = \frac{2500}{16} = \frac{625}{4}$

2.] Pivot Problem: Two perpendicular hallways, one of width 6 ft and the other of width 5 ft, meet at a right-angle corner. A rigid ladder (modeled as infinitely thin) is carried horizontally around the corner. What is the longest ladder that can make the turn?

$0 < \theta < \frac{\pi}{2}$



$\sin(\theta) = \frac{5}{L_1} \to L_1 = \frac{5}{\sin(\theta)} \to L_1 = 5\csc(\theta)$

$\cos(\theta) = \frac{6}{L_2} \to L_2 = \frac{6}{\cos(\theta)} \to L_2 = 6\sec(\theta)$

$\boxed{L(\theta) = 5\csc(\theta) + 6\sec(\theta)}$ ← need to minimize

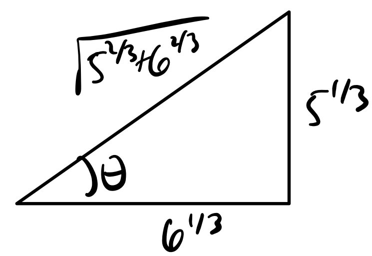
$\lim_{\theta\to 0^+} L(\theta) = \infty \qquad \lim_{\theta\to \frac{\pi}{2}^-} L(\theta) = \infty$

$L'(\theta) = -5\csc(\theta)\cot(\theta) + 6\sec(\theta)\tan(\theta) = 0$

$\Rightarrow \quad 6\sec(\theta)\tan(\theta) = 5\csc(\theta)\cot(\theta)$

$\Rightarrow \quad \frac{\sec(\theta)\tan(\theta)}{\csc(\theta)\cot(\theta)} = \frac{5}{6}$

$\Rightarrow \quad \tan^3(\theta) = \frac{5}{6} \quad \Rightarrow \quad \tan(\theta) = \frac{5^{1/3}}{6^{1/3}}$



$\csc(\theta) = \frac{\sqrt{5^{2/3}+6^{2/3}}}{5^{1/3}}$

$\sec(\theta) = \frac{\sqrt{5^{2/3}+6^{2/3}}}{6^{1/3}}$

$$L_{min} = 5\left(\frac{\sqrt{5^{2/3}+6^{2/3}}}{5^{1/3}}\right) + 6\left(\frac{\sqrt{5^{2/3}+6^{2/3}}}{6^{1/3}}\right)$$

$$= 5^{2/3}\sqrt{5^{2/3}+6^{2/3}} + 6^{2/3}\sqrt{5^{2/3}+6^{2/3}}$$

$$= \sqrt{5^{2/3}+6^{2/3}}\left(5^{2/3}+6^{2/3}\right)$$

$$\boxed{L_{min} = \left(5^{2/3}+6^{2/3}\right)^{3/2}}$$