

We start with the expression for the susceptibility for the intraband transtitions,

$$\chi_{i,abc}^{s,\ell} = -\frac{e^3}{\Omega\hbar^2\omega_3} \sum_{mn\mathbf{k}} \frac{\mathcal{V}_{mn}^{\Sigma,a,\ell}}{\omega_{nm}^S - \omega_3} \left(\frac{f_{mn}r_{nm}^b}{\omega_{nm}^S - \omega_\beta} \right)_{;k^c}, \quad (1) \quad \boxed{\text{chii}}$$

where s denotes *surface* and S refers to the *scissors* correction. This expression diverges as $\omega_3 \rightarrow 0$. To eliminate this divergence we take the partial fraction expansion,

$$\begin{aligned} I &= C \left[-\frac{1}{2(\omega_{nm}^S)^2} \frac{1}{\omega_{nm}^S - \omega} + \frac{2}{(\omega_{nm}^S)^2} \frac{1}{\omega_{nm}^S - 2\omega} + \frac{1}{2(\omega_{nm}^S)^2} \frac{1}{\omega} \right] \\ &- D \left[-\frac{3}{2(\omega_{nm}^S)^2} \frac{1}{\omega_{nm}^S - \omega} + \frac{4}{(\omega_{nm}^S)^3} \frac{1}{\omega_{nm}^S - 2\omega} + \frac{1}{2(\omega_{nm}^S)^3} \frac{1}{\omega} - \frac{1}{2(\omega_{nm}^S)^2} \frac{1}{(\omega_{nm}^S - \omega)^2} \right], \quad (2) \end{aligned}$$

where $C = f_{mn}\mathcal{V}_{mn}^{\Sigma,a}(r_{nm}^{\text{LDA},b})_{;k^c}$, and $D = f_{mn}\mathcal{V}_{mn}^{\Sigma,a}r_{nm}^b\Delta_{nm}^c$.

Time-reversal symmetry leads to the following relationships:

$$\begin{aligned} \mathbf{r}_{mn}(\mathbf{k}) &= \mathbf{r}_{nm}(-\mathbf{k}), \\ \mathbf{r}_{mn;\mathbf{k}}(\mathbf{k}) &= -\mathbf{r}_{nm;\mathbf{k}}(-\mathbf{k}), \\ \mathcal{V}_{mn}^{\Sigma,a}(-\mathbf{k}) &= -\mathcal{V}_{nm}^{\Sigma,a}(\mathbf{k}), \\ \omega_{mn}^S(-\mathbf{k}) &= \omega_{nm}^S(\mathbf{k}), \\ \Delta_{nm}^a(-\mathbf{k}) &= -\Delta_{nm}^a(\mathbf{k}). \end{aligned}$$

For a clean cold semiconductor, $f_n = 1$ for an occupied or valence ($n = v$) band, and $f_n = 0$ for an empty or conduction ($n = c$) band independent of \mathbf{k} , and $f_{nm} = -f_{mn}$.

The $\frac{1}{\omega}$ terms cancel each other out. We notice that the energy denominators are invariant under $\mathbf{k} \rightarrow -\mathbf{k}$, and then we only look at the numerators, then

$$\begin{aligned} C &\rightarrow f_{mn}\mathcal{V}_{mn}^{\Sigma,a}(r_{nm}^{\text{LDA},b})_{;k^c}|\mathbf{k} + f_{mn}\mathcal{V}_{mn}^{\Sigma,a}(r_{nm}^{\text{LDA},b})_{;k^c}|-\mathbf{k} \\ &= f_{mn} \left[\mathcal{V}_{mn}^{\Sigma,a}(r_{nm}^{\text{LDA},b})_{;k^c}|\mathbf{k} + (-\mathcal{V}_{nm}^{\Sigma,a}) \left(- (r_{mn}^{\text{LDA},b})_{;k^c} \right) |\mathbf{k} \right] \\ &= f_{mn} \left[\mathcal{V}_{mn}^{\Sigma,a}(r_{nm}^{\text{LDA},b})_{;k^c} + \mathcal{V}_{nm}^{\Sigma,a}(r_{mn}^{\text{LDA},b})_{;k^c} \right] \\ &= f_{mn} \left[\mathcal{V}_{mn}^{\Sigma,a}(r_{nm}^{\text{LDA},b})_{;k^c} + \left(\mathcal{V}_{mn}^{\Sigma,a}(r_{nm}^{\text{LDA},b})_{;k^c} \right)^* \right] \\ &= 2f_{mn} \text{Re} \left[\mathcal{V}_{mn}^{\Sigma,a}(r_{nm}^{\text{LDA},b})_{;k^c} \right], \quad (3) \end{aligned}$$

and likewise,

$$\begin{aligned}
D &\rightarrow f_{mn} \mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b \Delta_{nm}^c |_{\mathbf{k}} + f_{mn} \mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b \Delta_{nm}^c |_{-\mathbf{k}} \\
&= f_{mn} \left[\mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b \Delta_{nm}^c |_{\mathbf{k}} + (-\mathcal{V}_{nm}^{\Sigma,a}) r_{mn}^b (-\Delta_{nm}^c) |_{\mathbf{k}} \right] \\
&= f_{mn} \left[\mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b + \mathcal{V}_{nm}^{\Sigma,a} r_{mn}^b \right] \Delta_{nm}^c \\
&= f_{mn} \left[\mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b + \left(\mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b \right)^* \right] \Delta_{nm}^c \\
&= 2f_{mn} \text{Re} \left[\mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b \right] \Delta_{nm}^c.
\end{aligned} \tag{4}$$

The last term in the second line of (2) is dealt with as follows,

$$\begin{aligned}
\frac{D}{2(\omega_{nm}^S)^2} \frac{1}{(\omega_{nm}^S - \omega)^2} &= \frac{f_{mn}}{2} \frac{\mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b}{(\omega_{nm}^S)^2} \frac{\Delta_{nm}^c}{(\omega_{nm}^S - \omega)^2} = \frac{f_{mn}}{2} \frac{\mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b}{(\omega_{nm}^S)^2} \left(\frac{1}{\omega_{nm}^S - \omega} \right)_{;k^c} \\
&= -\frac{f_{mn}}{2} \left(\frac{\mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b}{(\omega_{nm}^S)^2} \right)_{;k^c} \frac{1}{\omega_{nm}^S - \omega}.
\end{aligned} \tag{5}$$

We use the fact that

$$(\omega_{nm}^S)_{;k^c} = (\omega_{nm}^{\text{LDA}})_{;k^c} = \frac{p_{nn}^c - p_{mm}^c}{m_e} \equiv \Delta_{nm}^c, \tag{6} \quad \boxed{\text{wk}}$$

and for the last line, we performed an integration by parts over the Brillouin zone, where the contribution from the edges vanishes.