

For the intraband terms,

$$I = C \left[-\frac{1}{2\omega_{nm}^2} \frac{1}{\omega_{nm} - \omega} + \frac{2}{\omega_{nm}^2} \frac{1}{\omega_{nm} - 2\omega} + \frac{1}{2\omega_{nm}^2} \frac{1}{\omega} \right] \\ - D \left[-\frac{3}{2\omega_{nm}^3} \frac{1}{\omega_{nm} - \omega} + \frac{4}{\omega_{nm}^3} \frac{1}{\omega_{nm} - 2\omega} + \frac{1}{2\omega_{nm}^3} \frac{1}{\omega} - \frac{1}{2\omega_{nm}^2} \frac{1}{(\omega_{nm} - \omega)^2} \right], \quad (1)$$

where $C = f_{mn} \mathcal{V}_{mn}^{\Sigma,a}(r_{nm}^{\text{LDA,b}})_{;k^c}$, and $D = f_{mn} \mathcal{V}_{mn}^{\Sigma,a} r_{nm}^b \Delta_{nm}^c$. Time-reversal symmetry allow us to write, $\mathbf{r}_{mn}(\mathbf{k}) = \mathbf{r}_{nm}(-\mathbf{k})$, $\mathbf{r}_{mn;\mathbf{k}}(\mathbf{k}) = -\mathbf{r}_{nm;\mathbf{k}}(-\mathbf{k})$, $\mathcal{V}_{mn}^{\Sigma,a}(-\mathbf{k}) = -\mathcal{V}_{nm}^{\Sigma,a}(\mathbf{k})$, $\omega_{mn}^S(-\mathbf{k}) = \omega_{mn}^S(\mathbf{k})$, and $\Delta_{nm}^a(-\mathbf{k}) = -\Delta_{nm}^a(\mathbf{k})$. Also, for a clean cold semiconductor $f_n = 1$ for an occupied or valence ($n = v$) band and $f_n = 0$ for an empty or conduction ($n = c$) band independent of \mathbf{k} and $f_{nm} = -f_{mn}$.

We also use the following expression,

$$\left(\frac{f_{mn} r_{nm}^b}{\omega_{nm}^S - \omega_2} \right)_{;k^c} = \frac{f_{mn}}{\omega_{nm}^S - \omega} \left(r_{nm}^{\text{LDA,b}} \right)_{;k^c} - \frac{f_{mn} r_{nm}^b \Delta_{nm}^c}{(\omega_{nm}^S - \omega)^2}, \quad (2)$$

where LDA and scissored energies combine.