We start with the expression for the susceptibility for the intraband transtitions,

$$\chi^{s,\ell}_{i,\mathrm{abc}} = -\frac{e^3}{\Omega\hbar^2\omega_3}\sum_{mn\mathbf{k}}\frac{\mathcal{V}^{\Sigma,\mathrm{a},\ell}_{mn}}{\omega^S_{nm}-\omega_3}\left(\frac{f_{mn}r^{\mathrm{b}}_{nm}}{\omega^S_{nm}-\omega_\beta}\right)_{;k^{\mathrm{c}}}, \tag{1}$$

where s denotes "surface" and S refers to the scissors correction. This expression diverges as $\omega_3 \to 0$. To eliminate this divergence, we take the partial fraction expansion

$$\begin{aligned} I &= C\left[-\frac{1}{2(\omega^S_{nm})^2}\frac{1}{\omega^S_{nm}-\omega}+\frac{2}{(\omega^S_{nm})^2}\frac{1}{\omega^S_{nm}-2\omega}+\frac{1}{2(\omega^S_{nm})^2}\frac{1}{\omega}\right] \\ &- D\left[-\frac{3}{2(\omega^S_{nm})^2}\frac{1}{\omega^S_{nm}-\omega}+\frac{4}{(\omega^S_{nm})^3}\frac{1}{\omega^S_{nm}-2\omega}+\frac{1}{2(\omega^S_{nm})^3}\frac{1}{\omega}-\frac{1}{2(\omega^S_{nm})^2}\frac{1}{(\omega^S_{nm}-\omega)^2}\right], \end{aligned} \tag{2}$$

where $C = f_{mn}\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}(r^{\mathrm{LDA,b}}_{nm})_{;k^{\mathrm{c}}}$, and $D = f_{mn}\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{b}}_{nm}\Delta^{\mathrm{c}}_{nm}$. Time-reversal symmetry allows us to write, $\mathbf{r}_{mn}(\mathbf{k}) = \mathbf{r}_{nm}(-\mathbf{k})$, $\mathbf{r}_{mn;\mathbf{k}}(\mathbf{k}) = -\mathbf{r}_{nm;\mathbf{k}}(-\mathbf{k})$, $\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}(-\mathbf{k}) = -\mathcal{V}^{\Sigma,\mathrm{a}}_{nm}(\mathbf{k})$, $\omega^S_{mn}(-\mathbf{k}) = \omega^S_{mn}(\mathbf{k})$, and $\Delta^a_{nm}(-mathbfk) = -\Delta^a_{nm}(mathbfk)$. Also, for a clean cold semiconductor $f_n = 1$ for an occupied or valence ($n = v$) band and $f_n = 0$ for an empty or conduction ($n = c$) band independent of $mathbfk$ and $f_{nm} = -f_{mn}$.

The last term in the second line of Eq. (2) is dealt with as follows.

$$\begin{aligned} \frac{D}{2(\omega^S_{nm})^2}\frac{1}{(\omega^S_{nm}-\omega)^2} &= \frac{f_{mn}}{2}\frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{b}}_{nm}}{(\omega^S_{nm})^2}\frac{\Delta^{\mathrm{c}}_{nm}}{(\omega^S_{nm}-\omega)^2} = \frac{f_{mn}}{2}\frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{b}}_{nm}}{(\omega^S_{nm})^2}\left(\frac{1}{\omega^S_{nm}-\omega}\right)_{;k^{\mathrm{c}}} \\ &= -\frac{f_{mn}}{2}\left(\frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{b}}_{nm}}{(\omega^S_{nm})^2}\right)_{;k^{\mathrm{c}}}\frac{1}{\omega^S_{nm}-\omega}, \end{aligned} \tag{3}$$

where we used Eqs. (**??**) and (**??**), and for the last line, we performed an integration by parts over the Brillouin zone, where the contribution from the edges vanishes.[**?**] Using the chain rule, we obtain

$$\left(\frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{b}}_{nm}}{(\omega^S_{nm})^2}\right)_{;k^{\mathrm{c}}} = \frac{r^{\mathrm{b}}_{nm}}{(\omega^S_{nm})^2}\left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}} + \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}}{(\omega^S_{nm})^2}\left(r^{\mathrm{b}}_{nm}\right)_{;k^{\mathrm{c}}} - \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{b}}_{nm}}{2(\omega^S_{nm})^3}\left(\omega^S_{nm}\right)_{;k^{\mathrm{c}}}, \tag{4}$$

where in the appendix **??** we show how to calculate $(\mathcal{V}^{\Sigma,a}_{nm})_{;k^b}$. For $\left(\omega^S_{nm}\right)_{;k^{\mathrm{c}}}$ we simply use Eq. (**??**), that using Eq. (**??**) gives

$$\left(\omega^S_{nm}\right)_{;k^{\mathrm{c}}} = \Delta^{\mathrm{LDA,c}}_{nm}, \tag{5}$$

and from Eq. (**??**)

$$(r^{\mathrm{b}}_{nm})_{;k^{\mathrm{a}}} \approx \frac{r^{\mathrm{a}}_{nm}\Delta^{\mathrm{LDA,b}}_{mn} + r^{\mathrm{b}}_{nm}\Delta^{\mathrm{LDA,a}}_{mn}}{\omega^{\mathrm{LDA}}_{nm}} + \frac{i}{\omega^{\mathrm{LDA}}_{nm}}\sum_{\ell}\left(\omega^{\mathrm{LDA}}_{\ell m}r^{\mathrm{a}}_{n\ell}r^{\mathrm{b}}_{\ell m} - \omega^{\mathrm{LDA}}_{n\ell}r^{\mathrm{b}}_{n\ell}r^{\mathrm{a}}_{\ell m}\right), \tag{6}$$