We start with the expression for the susceptibility for the intraband transtitions,

$$\chi^{s,\ell}_{i,\mathrm{abc}} = -\frac{e^3}{\Omega\hbar^2\omega_3}\sum_{mn\mathbf{k}} \frac{\mathcal{V}^{\Sigma,\mathrm{a},\ell}_{mn}}{\omega^S_{nm}-\omega_3}\left(\frac{f_{mn}r^{\mathrm{LDA,b}}_{nm}}{\omega^S_{nm}-\omega_\beta}\right)_{;k^{\mathrm{c}}}, \tag{1}$$

where $s$ denotes *surface* and $S$ refers to the *scissors* correction. This expression diverges as $\omega_3 \to 0$. To eliminate this divergence we take the partial fraction expansion,

$$\begin{aligned} I = C&\left[-\frac{1}{2(\omega^S_{nm})^2}\frac{1}{\omega^S_{nm}-\omega} + \frac{2}{(\omega^S_{nm})^2}\frac{1}{\omega^S_{nm}-2\omega} + \frac{1}{2(\omega^S_{nm})^2}\frac{1}{\omega}\right] \\ - D&\left[-\frac{3}{2(\omega^S_{nm})^2}\frac{1}{\omega^S_{nm}-\omega} + \frac{4}{(\omega^S_{nm})^3}\frac{1}{\omega^S_{nm}-2\omega} + \frac{1}{2(\omega^S_{nm})^3}\frac{1}{\omega} - \frac{1}{2(\omega^S_{nm})^2}\frac{1}{(\omega^S_{nm}-\omega)^2}\right], \end{aligned} \tag{2}$$

where $C = f_{mn}\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}(r^{\mathrm{LDA,b}}_{nm})_{;k^{\mathrm{c}}}$, and $D = f_{mn}\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{LDA,b}}_{nm}\Delta^{\mathrm{c}}_{nm}$.

Time-reversal symmetry leads to the following relationships:

$$\begin{aligned} \mathbf{r}_{mn}(\mathbf{k})|_{-\mathbf{k}} &= \mathbf{r}_{nm}(\mathbf{k})|_{\mathbf{k}}, \\ (\mathbf{r}_{mn})_{;\mathbf{k}}(\mathbf{k})|_{-\mathbf{k}} &= (-\mathbf{r}_{nm})_{;\mathbf{k}}(\mathbf{k})|_{\mathbf{k}}, \\ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn}(\mathbf{k})|_{-\mathbf{k}} &= -\mathcal{V}^{\Sigma,\mathrm{a}}_{nm}(\mathbf{k})|_{\mathbf{k}}, \\ (\mathcal{V}^{\Sigma,\mathrm{a}}_{mn})_{;\mathbf{k}}(\mathbf{k})|_{-\mathbf{k}} &= (\mathcal{V}^{\Sigma,\mathrm{a}}_{nm})_{;\mathbf{k}}(\mathbf{k})|_{\mathbf{k}}, \\ \omega^S_{mn}(\mathbf{k})|_{-\mathbf{k}} &= \omega^S_{mn}(\mathbf{k})|_{\mathbf{k}}, \\ \Delta^a_{nm}(\mathbf{k})|_{-\mathbf{k}} &= -\Delta^a_{nm}(\mathbf{k})|_{\mathbf{k}}. \end{aligned} \tag{3}$$

For a clean cold semiconductor, $f_n = 1$ for an occupied or valence ($n = v$) band, and $f_n = 0$ for an empty or conduction ($n = c$) band independent of $\mathbf{k}$, and $f_{nm} = -f_{mn}$.

The $\frac{1}{\omega}$ terms cancel each other out. We notice that the energy denominators are invariant under $\mathbf{k} \to -\mathbf{k}$, and then we only look at the numerators, then

$$\begin{aligned} C \to f_{mn}\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}|_{\mathbf{k}} + f_{mn}\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}&\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}|_{-\mathbf{k}} \\ &= f_{mn}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}|_{\mathbf{k}} + \left(-\mathcal{V}^{\Sigma,\mathrm{a}}_{nm}\right)\left(-r^{\mathrm{LDA,b}}_{mn}\right)_{;k^{\mathrm{c}}}|_{\mathbf{k}}\right] \\ &= f_{mn}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} + \mathcal{V}^{\Sigma,\mathrm{a}}_{nm}\left(r^{\mathrm{LDA,b}}_{mn}\right)_{;k^{\mathrm{c}}}\right] \\ &= f_{mn}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} + \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}\right)^*\right] \\ &= 2f_{mn}\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}\right], \end{aligned} \tag{4}$$

and likewise,

$$
\begin{aligned}
D \to f_{mn} \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm} \Delta^{\mathrm{c}}_{nm}|_{\mathbf{k}} + f_{mn} \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm} \Delta^{\mathrm{c}}_{nm}|_{-\mathbf{k}} \\
&= f_{mn} \left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm} \Delta^{\mathrm{c}}_{nm}|_{\mathbf{k}} + \left(-\mathcal{V}^{\Sigma,\mathrm{a}}_{nm}\right) r^{\mathrm{LDA,b}}_{mn} \left(-\Delta^{\mathrm{c}}_{nm}\right)|_{\mathbf{k}} \right] \\
&= f_{mn} \left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm} + \mathcal{V}^{\Sigma,\mathrm{a}}_{nm} r^{\mathrm{LDA,b}}_{mn} \right] \Delta^{\mathrm{c}}_{nm} \\
&= f_{mn} \left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm} + \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}\right)^* \right] \Delta^{\mathrm{c}}_{nm} \\
&= 2 f_{mn} \,\mathrm{Re} \left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm} \right] \Delta^{\mathrm{c}}_{nm}.
\end{aligned} \tag{5}
$$

The last term in the second line of (2) is dealt with as follows,

$$
\begin{aligned}
\frac{D}{2(\omega^S_{nm})^2} \frac{1}{(\omega^S_{nm}-\omega)^2} &= \frac{f_{mn}}{2} \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{(\omega^S_{nm})^2} \frac{\Delta^{\mathrm{c}}_{nm}}{(\omega^S_{nm}-\omega)^2} = \frac{f_{mn}}{2} \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{(\omega^S_{nm})^2} \left( \frac{1}{\omega^S_{nm}-\omega} \right)_{;k^{\mathrm{c}}} \\
&= -\frac{f_{mn}}{2} \left( \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{(\omega^S_{nm})^2} \right)_{;k^{\mathrm{c}}} \frac{1}{\omega^S_{nm}-\omega}.
\end{aligned} \tag{6}
$$

We use the fact that

$$
(\omega^S_{nm})_{;k^{\mathrm{c}}} = (\omega^{\mathrm{LDA}}_{nm})_{;k^{\mathrm{c}}} = \frac{p^{\mathrm{c}}_{nn} - p^{\mathrm{c}}_{mm}}{m_e} \equiv \Delta^{\mathrm{c}}_{nm}, \tag{7}
$$

and for the last line, we performed an integration by parts over the Brillouin zone, where the contribution from the edges vanishes. Using the chain rule, we obtain

$$
\left( \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{(\omega^S_{nm})^2} \right)_{;k^{\mathrm{c}}} = \frac{r^{\mathrm{LDA,b}}_{nm}}{(\omega^S_{nm})^2} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}} + \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}}{(\omega^S_{nm})^2} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} - \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{2(\omega^S_{nm})^3} \left(\omega^S_{nm}\right)_{;k^{\mathrm{c}}}. \tag{8}
$$

We will check each term of (8) over $\mathbf{k} \to -\mathbf{k}$ using the relations in (3). The first term is reduced to

$$
\begin{aligned}
\frac{r^{\mathrm{LDA,b}}_{nm}}{(\omega^S_{nm})^2} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}}|_{\mathbf{k}} + \frac{r^{\mathrm{LDA,b}}_{nm}}{(\omega^S_{nm})^2} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}}|_{-\mathbf{k}} &= \frac{r^{\mathrm{LDA,b}}_{nm}}{(\omega^S_{nm})^2} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}}|_{\mathbf{k}} + \frac{r^{\mathrm{LDA,b}}_{mn}}{(\omega^S_{nm})^2} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{nm}\right)_{;k^{\mathrm{c}}}|_{\mathbf{k}} \\
&= \frac{1}{(\omega^S_{nm})^2} \left[ r^{\mathrm{LDA,b}}_{nm} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}} + \left( r^{\mathrm{LDA,b}}_{nm} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}} \right)^* \right] \\
&= \frac{2}{(\omega^S_{nm})^2} \mathrm{Re} \left[ r^{\mathrm{LDA,b}}_{nm} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}} \right],
\end{aligned} \tag{9}
$$

the second term is reduced to

$$
\begin{aligned}
\frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}}{(\omega^S_{nm})^2} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}|_{\mathbf{k}} + \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}}{(\omega^S_{nm})^2} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}|_{-\mathbf{k}} &= \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}}{(\omega^S_{nm})^2} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}|_{\mathbf{k}} + \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{nm}}{(\omega^S_{nm})^2} \left(r^{\mathrm{LDA,b}}_{mn}\right)_{;k^{\mathrm{c}}}|_{\mathbf{k}} \\
&= \frac{1}{(\omega^S_{nm})^2} \left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} + \left( \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} \right)^* \right] \\
&= \frac{2}{(\omega^S_{nm})^2} \mathrm{Re} \left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} \right],
\end{aligned} \tag{10}
$$

and by using (7), the third term is reduced to

$$
\begin{aligned}
\frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{2(\omega^S_{nm})^3} \left(\omega^S_{nm}\right)_{;k^{\mathrm{c}}} |_{\mathbf{k}} + \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{2(\omega^S_{nm})^3} \left(\omega^S_{nm}\right)_{;k^{\mathrm{c}}} |_{-\mathbf{k}} &= \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{2(\omega^S_{nm})^3} \Delta^{\mathrm{c}}_{nm}|_{\mathbf{k}} + \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{2(\omega^S_{nm})^3} \Delta^{\mathrm{c}}_{nm}|_{-\mathbf{k}} \\
&= \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{nm} r^{\mathrm{LDA,b}}_{mn}}{2(\omega^S_{nm})^3} \Delta^{\mathrm{c}}_{nm}|_{\mathbf{k}} + \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{2(\omega^S_{nm})^3} \Delta^{\mathrm{c}}_{nm}|_{\mathbf{k}} \\
&= \frac{1}{2(\omega^S_{nm})^3} \left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{nm} r^{\mathrm{LDA,b}}_{mn} + \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{nm} r^{\mathrm{LDA,b}}_{mn}\right)^* \right] \Delta^{\mathrm{c}}_{nm} \\
&= \frac{1}{(\omega^S_{nm})^3} \mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{nm} r^{\mathrm{LDA,b}}_{mn}\right] \Delta^{\mathrm{c}}_{nm}.
\end{aligned} \tag{11}
$$

Combining the results from (9), (10), and (11) into (8),

$$
\begin{aligned}
&-\frac{f_{mn}}{2} \left[ \left( \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{(\omega^S_{nm})^2} \right)_{;k^{\mathrm{c}}} |_{\mathbf{k}} + \left( \frac{\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}}{(\omega^S_{nm})^2} \right)_{;k^{\mathrm{c}}} |_{-\mathbf{k}} \right] \frac{1}{\omega^S_{nm} - \omega} = \\
&- f_{mn} \left( 2\mathrm{Re}\left[ r^{\mathrm{LDA,b}}_{nm} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}} \right] + 2\mathrm{Re}\left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} \right] - \frac{1}{\omega^S_{nm}} \mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{nm} r^{\mathrm{LDA,b}}_{mn}\right] \Delta^{\mathrm{c}}_{nm} \right) \frac{1}{2(\omega^S_{nm})^2} \frac{1}{\omega^S_{nm} - \omega}.
\end{aligned} \tag{12}
$$

We have all the elements to be substituted in (2). We substitute (4), (5), and (12) in (2),

$$
\begin{aligned}
I = &\left[ -\frac{2 f_{mn} \,\mathrm{Re}\left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} \right]}{2(\omega^S_{nm})^2} \frac{1}{\omega^S_{nm} - \omega} + \frac{4 f_{mn} \,\mathrm{Re}\left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} \right]}{(\omega^S_{nm})^2} \frac{1}{\omega^S_{nm} - 2\omega} \right] \\
&- \left[ -\frac{6 f_{mn} \,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}\right] \Delta^{\mathrm{c}}_{nm}}{2(\omega^S_{nm})^2} \frac{1}{\omega^S_{nm} - \omega} + \frac{8 f_{mn} \,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}\right] \Delta^{\mathrm{c}}_{nm}}{(\omega^S_{nm})^3} \frac{1}{\omega^S_{nm} - 2\omega} \right. \\
&\left. + \frac{f_{mn} \left( 2\,\mathrm{Re}\left[ r^{\mathrm{LDA,b}}_{nm} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}} \right] + 2\,\mathrm{Re}\left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} \right] - \frac{1}{\omega^S_{nm}} \,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{nm} r^{\mathrm{LDA,b}}_{mn}\right] \Delta^{\mathrm{c}}_{nm} \right)}{2(\omega^S_{nm})^2} \frac{1}{\omega^S_{nm} - \omega} \right].
\end{aligned}
$$

If we simplify and homogenize,

$$
\begin{aligned}
I = &-\frac{2 f_{mn} \,\mathrm{Re}\left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} \right]}{2(\omega^S_{nm})^2} \frac{1}{\omega^S_{nm} - \omega} + \frac{4 f_{mn} \,\mathrm{Re}\left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} \right]}{(\omega^S_{nm})^2} \frac{1}{\omega^S_{nm} - 2\omega} \\
&+ \frac{6 f_{mn} \,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}\right] \Delta^{\mathrm{c}}_{nm}}{2(\omega^S_{nm})^2} \frac{1}{\omega^S_{nm} - \omega} - \frac{8 f_{mn} \,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA,b}}_{nm}\right] \Delta^{\mathrm{c}}_{nm}}{(\omega^S_{nm})^3} \frac{1}{\omega^S_{nm} - 2\omega} \\
&- \frac{2 f_{mn} \,\mathrm{Re}\left[ r^{\mathrm{LDA,b}}_{nm} \left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}} \right]}{2(\omega^S_{nm})^2} \frac{1}{\omega^S_{nm} - \omega} \\
&- \frac{2 f_{mn} \,\mathrm{Re}\left[ \mathcal{V}^{\Sigma,\mathrm{a}}_{mn} \left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}} \right]}{2(\omega^S_{nm})^2} \frac{1}{\omega^S_{nm} - \omega} \\
&+ \frac{f_{mn} \mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{nm} r^{\mathrm{LDA,b}}_{mn}\right] \Delta^{\mathrm{c}}_{nm}}{2(\omega^S_{nm})^3} \frac{1}{\omega^S_{nm} - \omega},
\end{aligned} \tag{13}
$$

we conveniently collect the terms in columns of $\omega$ and $2\omega$. We can now express the susceptibility in terms of $\omega$ and $2\omega$. Separating the $2\omega$ terms and substituting in (1),

$$\begin{aligned}
I_{2\omega} &= -\frac{e^3}{\hbar^2}\sum_{mn\mathbf{k}}\left[\frac{4f_{mn}\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}\right]}{(\omega^S_{nm})^2} - \frac{8f_{mn}\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{LDA,b}}_{nm}\right]\Delta^{\mathrm{c}}_{nm}}{(\omega^S_{nm})^3}\right]\frac{1}{\omega^S_{nm}-2\omega}\\
&= -\frac{e^3}{\hbar^2}\sum_{mn\mathbf{k}}\frac{4f_{mn}}{(\omega^S_{nm})^2}\left[\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}\right] - \frac{2\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{LDA,b}}_{nm}\right]\Delta^{\mathrm{c}}_{nm}}{\omega^S_{nm}}\right]\frac{1}{\omega^S_{nm}-2\omega}.
\end{aligned}\tag{14}$$

We can express the energies in terms of transitions between bands. Therefore, $\omega^S_{nm} = \omega^S_{cv}$ for transitions between conduction and valence bands. We analyze the limit,

$$\lim_{\eta\to 0}\frac{1}{x\pm i\eta} = P\frac{1}{x} \mp i\pi\delta(x),\tag{15}$$

and can finally rewrite (14) in the desired form,

$$\mathrm{Im}[\chi^{s,\ell}_{i,\mathrm{abc},2\omega}] = -\frac{\pi e^3}{\hbar^2}\sum_{mn\mathbf{k}}\frac{4}{(\omega^S_{nm})^2}\left(\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}\right] - \frac{2\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{LDA,b}}_{nm}\right]\Delta^{\mathrm{c}}_{nm}}{\omega^S_{nm}}\right)\delta(\omega^S_{nm}-2\omega).\tag{16}$$

We do the same for the $\omega$ terms in (13) and substitute in (1),

$$\begin{aligned}
I_{\omega} = -\frac{e^3}{2\hbar^2}\sum_{mn\mathbf{k}}\Bigg[&-\frac{2f_{mn}\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}\right]}{(\omega^S_{nm})^2} + \frac{6f_{mn}\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}r^{\mathrm{LDA,b}}_{nm}\right]\Delta^{\mathrm{c}}_{nm}}{(\omega^S_{nm})^2}\\
&-\frac{2f_{mn}\,\mathrm{Re}\left[r^{\mathrm{LDA,b}}_{nm}\left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}}\right]}{(\omega^S_{nm})^2} - \frac{2f_{mn}\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA,b}}_{nm}\right)_{;k^{\mathrm{c}}}\right]}{(\omega^S_{nm})^2}\\
&+\frac{f_{mn}\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{nm}r^{\mathrm{LDA,b}}_{mn}\right]\Delta^{\mathrm{c}}_{nm}}{(\omega^S_{nm})^3}\Bigg]\frac{1}{\omega^S_{nm}-\omega},
\end{aligned}\tag{17}$$

and we reduce in the same way as (14),

**I can't seem to reduce this any further. When using $\mathcal{R}^{\mathbf{a},\ell}_{nm}$ there are $i$'s that allow two terms to cancel out leaving only three. See (A17) and (A18).**

$$
\begin{aligned}
I_\omega = \frac{e^3}{2\hbar^2}\sum_{mn\mathbf{k}} \frac{f_{mn}}{(\omega^S_{nm})^2}\Bigg[& 2\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA},\mathrm{b}}_{nm}\right)_{;k^{\mathrm{c}}}\right] \\
&-6\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA},\mathrm{b}}_{nm}\right]\Delta^{\mathrm{c}}_{nm} + 2\,\mathrm{Re}\left[r^{\mathrm{LDA},\mathrm{b}}_{nm}\left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}}\right] \\
&+2\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA},\mathrm{b}}_{nm}\right)_{;k^{\mathrm{c}}}\right] - \frac{\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{nm} r^{\mathrm{LDA},\mathrm{b}}_{mn}\right]\Delta^{\mathrm{c}}_{nm}}{(\omega^S_{nm})}\Bigg]\frac{1}{\omega^S_{nm}-\omega} \\
= \frac{e^3}{2\hbar^2}\sum_{mn\mathbf{k}} \frac{f_{mn}}{(\omega^S_{nm})^2}\Bigg[& 4\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\left(r^{\mathrm{LDA},\mathrm{b}}_{nm}\right)_{;k^{\mathrm{c}}}\right] + 2\,\mathrm{Re}\left[r^{\mathrm{LDA},\mathrm{b}}_{nm}\left(\mathcal{V}^{\Sigma,\mathrm{a}}_{mn}\right)_{;k^{\mathrm{c}}}\right] \\
&-6\,\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{mn} r^{\mathrm{LDA},\mathrm{b}}_{nm}\right]\Delta^{\mathrm{c}}_{nm} - \frac{\mathrm{Re}\left[\mathcal{V}^{\Sigma,\mathrm{a}}_{nm} r^{\mathrm{LDA},\mathrm{b}}_{mn}\right]\Delta^{\mathrm{c}}_{nm}}{(\omega^S_{nm})}\Bigg]\frac{1}{\omega^S_{nm}-\omega} \qquad (18)
\end{aligned}
$$