For the *i*ntraband terms,

$$
\begin{aligned}
I \;&=\; C\left[-\frac{1}{2\omega_{nm}^2}\frac{1}{\omega_{nm}-\omega}+\frac{2}{\omega_{nm}^2}\frac{1}{\omega_{nm}-2\omega}+\frac{1}{2\omega_{nm}^2}\frac{1}{\omega}\right]\\
&-\; D\left[-\frac{3}{2\omega_{nm}^3}\frac{1}{\omega_{nm}-\omega}+\frac{4}{\omega_{nm}^3}\frac{1}{\omega_{nm}-2\omega}+\frac{1}{2\omega_{nm}^3}\frac{1}{\omega}-\frac{1}{2\omega_{nm}^2}\frac{1}{(\omega_{nm}-\omega)^2}\right],
\end{aligned}
\tag{1}
$$

where $C = f_{mn}\mathcal{V}_{mn}^{\Sigma,\mathrm{a}}(r_{nm}^{\mathrm{LDA,b}})_{;k^{\mathrm{c}}}$, and $D = f_{mn}\mathcal{V}_{mn}^{\Sigma,\mathrm{a}} r_{nm}^{\mathrm{b}}\Delta_{nm}^{\mathrm{c}}$. Time-reversal symmetry allow us to write, $\mathbf{r}_{mn}(\mathbf{k}) = \mathbf{r}_{nm}(-\mathbf{k})$, $\mathbf{r}_{mn;\mathbf{k}}(\mathbf{k}) = -\mathbf{r}_{nm;\mathbf{k}}(-\mathbf{k})$, $\boldsymbol{\mathcal{V}}_{mn}^{\Sigma,\mathrm{a}}(-\mathbf{k}) = -\boldsymbol{\mathcal{V}}_{nm}^{\Sigma,\mathrm{a}}(\mathbf{k})$, $\omega_{mn}^S(-\mathbf{k}) = \omega_{mn}^S(\mathbf{k})$, and $\Delta_{nm}^a(-\mathbf{k}) = -\Delta_{nm}^a(\mathbf{k})$. Also, for a clean cold semiconductor $f_n = 1$ for an occupied or valence ($n = v$) band and $f_n = 0$ for an empty or conduction ($n = c$) band independent of $\mathbf{k}$ and $f_{nm} = -f_{mn}$.

We also use the following expression,

$$
\left(\frac{f_{mn} r_{nm}^{\mathrm{b}}}{\omega_{nm}^S-\omega_2}\right)_{;k^{\mathrm{c}}} = \frac{f_{mn}}{\omega_{nm}^S-\omega}\left(r_{nm}^{\mathrm{LDA,b}}\right)_{;k^{\mathrm{c}}} - \frac{f_{mn} r_{nm}^{\mathrm{b}}\Delta_{nm}^{\mathrm{c}}}{(\omega_{nm}^S-\omega)^2},
\tag{2}
$$

where LDA and scissored energies combine.